SUBSTITUTION RELATIONSHIPS BETWEEN FORAGE AND GRAIN IN MILK PRODUCTION

by

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INTRODUCTION AND THE PROBLEM

Efficiency in production is one of the major goals of an economic system. Where the main effort in economic organization is to adjust output and costs to effectuate a desirable net income level, the attainment of resource efficiency and subsequent lower costs are paramount to most agricultural producers. Farmers generally assume that prices received for their products are fixed or given. This is another way of saying that most farmers, unlike many business firms, do not have a selling or pricing policy. Prices of fluid milk are in the main either negotiated or determined by an inanimate formula. Once the price is determined, the producer generally strives to attain an economic level of output, treating the price per unit of milk relatively fixed in the short run at least.

Knowledge and recognition of relevant economic principles are important in attaining efficient use of resources. Recommendations to dairy farmers are usually couched in such terms as: "One pound of grain will take the place of four pounds of hay in milk production;" "Feed a pound of grain for each four (or three) pounds of milk produced;" or "one pound of grain will produce five pounds of milk." While these statements vary quantitatively, they have one thing in common: The main underlying assumptions are the same. All of these statements assume a constant rate of substitution between factors, a constant physical product being returned to a single factor, and fail to recognize differences in the inherent ability of cows to produce milk. All of these statements are probably accurate provided the production of milk is carried out under the "correct" (assumed) conditions.

It is probable that there is some range of roughage-grain feeding, with output of milk and inherent ability given, where "one pound of grain will take the place of four pounds of hay". It is just as probable that there is some other range of roughage-grain feeding, still with output of milk and inherent ability given, where "one pound of grain will take the place of two pounds of hay". Generally, these feeding suggestions are attempts to give some average substitution rates or values to be used as feeding standards.

The extreme range in quantities of various feeds which may be used to produce a given amount of milk points up the social implications of the problem as well as the importance to the individual dairy farmer. The total quantity of feed resources used by dairy cattle in the United States is very great. It has been estimated that about 66 million tons of roughage (expressed as feed units) and about 18 million tons of grains are consumed annually by dairy cattle. 1/ It has also been estimated that 31% of all feed produced in the United States is consumed by dairy cattle and that 42% and 17% of all roughage and grain respectively is consumed by dairy cattle. At current prices, a 1% change in the total amount of grain fed, if it is not accompanied by a change in hay feeding, would amount to over 10,000,000 dollars annually.

With the recent increased emphasis on grassland farming, a thorough appraisal is needed of the forage utilization problem. Farm managers are faced with the prospect of additional quantities of roughage resulting from diversion of acreages taken out of controlled crops and/or from increased

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1/ Jennings, R. D. Consumption of Feed by Livestock 1909-47. USDA Circ. 836. Washington, D. C. 1949. p. 20.

productivity and fertilization. Dairy farmers need to know whether additional roughage can be utilized economically in milk production. Particularly they need to know to what extent one feed will substitute for another in the ration. The substitution rates between the two primary categories of feeds--roughage and grain--are essential in arriving at a minimum cost ration. Although the rate of milk production from total feed input is an important problem, it will not be given major emphasis in this presentation. By limiting the problem mainly to consideration of the rates of substitution of grain and forage in milk production, it will be possible to simplify the problem and perhaps arrive at a more complete solution.

Objectives

The primary objective of this study is to make estimates of the rate at which grain and forage substitute in milk production under specified conditions of inherent ability and level of annual output. The secondary objective is to illustrate a method of analysis which might be useful for other problems similar in character.

REVIEW OF LITERATURE

Much has been written in the past ten years on feeding dairy cows. Nearly every experiment station in the United States has had one or more publications during that period. There have also been many articles in the professional journals. Generally, these bulletins and articles have added to our store of knowledge for the most economical ways to feed dairy cows.

Undoubtedly the cooperative work between the USDA and ten agricultural experiment stations in 1938-41 is the most monumental and contributed most to our knowledge of input-output relationships in milk production. 1/ The input-output experiments reported in United States Department of Agriculture Technical Bulletin 815 were aimed at ascertaining how much feed (grain) it takes to increase total output of milk and whether these additional increases of feed (grain) increased as production went to higher levels. The following is quoted from Technical Bulletin 815:

> The law of diminishing physical output applies to milk production. There was a consistent stepping up of production with every increase in grain allowance, but the additional milk produced for each additional unit of feed decreased the average response as represented by a curve instead of a straight line. The response to increased feeding was less at the high levels than at the low levels--0.6 pound of 4-percent fat-corrected milk 2/ for each additional pound of digestible nutrients at the highest level and 1.7 pounds at the lowest level. 3/

This study gave extremely important information on total production of milk as related to total feed consumed. However, it did not answer the question of how grain and forage substitute for each other.

1/ Jensen, Einar, and others. Input-Output Relationships in Milk Production, USDA Tech. Bul. 815, Washington D. C., 1942.

2/ Another measure of output would be solids not fat.

3/ Ibid, p. 86.

Redman, in 1951, made some estimates of the marginal rates of substitution along the forage expansion path or as he refers to it the stomach capacity line. 1/ It appears as though these marginal rates of substitution are in reality a hypothesis subject to empirical test and that the stomach capacity line is not defined precisely.

Fellows, Frick and Weeks discuss the relationship between concentrate feeding, prices, and total milk production for cows with different basic production capacities. 2/ As in the Redman study, the basis for these estimates is the Jensen study. These authors do not dismiss the forage feed problem, but do indicate that for any short run period the quantity of forage is relatively fixed for each farm. They have recognized that feed concentrates can be varied over a wide range with relative ease. Likewise, they recognize that within certain physical limits the cow can substitute these concentrates for certain other feeds. If in fact roughage is surplus or is on hand and has no other use in the short run, the problem facing the dairy farmer would be that of determining what the production response would be to additional grain feeding. However, the problem of surplus forage does not face all farmers alike; and in the long run adjustments must be made to the cost involved of producing the roughage.

1/ Redman, John C. Economic Aspects of Feeding for Milk Production. Journal of Farm Economics. 34(No.3): 333-345. 1952.

2/ Fellows, I. F., Frick, G. E., and Weeks, S. B. Production Efficiency on New England Dairy Farms. Storrs Agri. Exp. Sta. Bul. 283. 1952.

Fellows, in 1952, suggests, since no price exists for forage, that a solution may be arrived at by substituting marginal cost of producing forage for price. 1/

Hoglund and Wright give another adaption from Technical Bulletin 815 on total milk production and grain consumption using either (1) good hay and pasture or (2) poor hay and pasture with some extrapolations. 2/ They also give the additional pounds of milk produced for each 100 pounds of additional grain fed for average cows, good cows, and very good cows. However, the quantity of roughage in the form of hay and pasture remains unspecified.

Of the above studies, the only one dealing with substitution relationships *per se* was the one by Redman. As stated previously, it appears that this is a hypothesis only. Fellows 3/ does give the functional relationship between grain and forage. However, this seems to be an overall function for the entire Jensen study. It is not broken down for different levels of production at different stations and for cows of different inherent ability.

The one study that appears to be closely related to the problem at hand is the one by Heady. 4/ The work by Heady cites some advantages and limitations of several different production functions. The one function

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1/ Fellows, I. F. The Economics of Grassland Farming in the Northeast. Journal of Farm Economics. 34(No. 5): 759-767, 1952.

2/ Hoglund, C. R., and Wright, K. T. Reducing Dairy Costs on Michigan Dairy Farms. Mich. Agri. Exp. Sta. Special Bul. 376. 1952.

3/ Fellows. The Economics of Grassland Farming in the Northeast.

4/ Heady, Earl O. Utilization of Feed Resources by Dairy Cows. Journal of Farm Economics. 33(No. 4 Pt 1): 485-498. 1951.

selected for illustration (Cobb-Douglas) gives some reasonable contours. It was selected because, of the seven tested, it more nearly conformed to production logic and was statistically acceptable. There are three points which might be considered in connection with these derived contours. First, the very nature of the function forced the curves to the same asymptote. Second, a rather narrow range of milk production is represented--in other words, between 8500 and 9500 pounds. It is recognized, however, that the bulk of the observations were in this area. The third point to consider is that these functions are derived with a sample of 34 cows. Heady attempted to provide homogeneity of the inherent ability by selecting "heavy breed" cows between 300 and 400 pounds expected butterfat production (when fed the standard Haecker ration). However, the error involved in assuming an average expected butterfat production over this range and that between different experiment stations might account for some of the irregularities in the different functions. 1/ Heady recognized all these limitations and others and even suggests that perhaps other functions should be fitted to available data. 2/

Some economists have questioned the feasibility of continuing the study of substitution relationships in milk production. Mighell 3/ raises a mild objection to the extensive use of the iso-product contours in economic research. His objection seems to be more nearly that of how they were derived in this particular case.

1/ As will be shown later, a significant difference in roughage and grain consumed and milk produced existed between experiment stations.

2/ Heady, op. cit. p. 496.

3/ Mighell, Ronald L. What is the Place of the Equal Product Function? Journal of Farm Economics. 35(No. 1): 29-43. 1953.

Rauchenstein [1] questions very seriously the importance of the entire forage-grain substitution in the economics of milk production. He bases his arguments on Heady's derived function for the 8500 pound contour. While he questions the derivation of this function, he accepts it and then tries to show that it would make very little difference in the total feed bill for a 20-cow dairy herd over a period of one year moving along the milk contour. Evidently, Rauchenstein feels that the contours are important pieces of information for us to know. He has selected a very "flat" portion of the contour for his illustration as well as a narrow range of forage-grain feeding. What would be of more importance would be to show what would happen in the steeper portions of the contour and also between the 8500 and 9500 and 10500 pound contours.

Other economists have pointed out the advisability of obtaining and the usefulness of the substitution rates between factors in production. Johnson [2] recognizes the importance, but points out the difficulty in obtaining such information. Nesius [3] pleads a strong case for obtaining

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1/ Rauchenstein, Emil. Forage-Grain Substitution: Its Importance in the Economics of Milk Production. Journal of Farm Economics. 35 (No. 4): 562-571. 1953.

2/ Johnson, Glenn. Needed Developments in Economic Theory as Applied to Farm Management Research. Journal of Farm Economics. 32 (No. 4 Pt. 2): 1140-1156. 1950.

3/ Nesius, Earnest J. Some Problems of Joint Use of Theory and Empirical Data in Farm Management Research. Journal of Farm Economics. 32 (No. 4 Pt. 2): 1169-1181. 1950.

substitution rates between forage and grain in milk production. Fellows [1/] points out the importance of this type of analysis and shows diagrammatically a model to work from and how one might arrive at an optimum combination of factors when price is considered.

1/ Fellows, I. F. The Application of Static Economic Theory to Farm Management Problems. Journal of Farm Economics. 32(No. 4 Pt 2): 1100-1112. 1950.

PRODUCTION FUNCTIONS AND FACTOR-FACTOR SUBSTITUTION RELATIONSHIPS

The production function relates total input and total output. The production function, sometimes referred to as the input-output ratio, serves in showing us the factor-product transformation. It also serves as a basis for deriving contour functions and substitution rates between factors. The general form of a production function is usually written $Y = F(X_1,X_2,X_3,X_4,\ldots,X_n)$ where Y is the product and the X's are the resources all of which can be varied. When the input-output relationship is to be estimated for a single variable resource factor, the above equation is written as follows: $Y = F(X_1 | X_2,X_3,X_4,\ldots,X_n)$ where X_1 is the variable that is changing and the X_2's, X_3's, X_4's etc. are fixed in quantity. If Y is the product (milk), X_1 is the expected production, X_2 is the forage consumed, and X_3 is the grain consumed, then the relationship of product to varying amounts of grain being fed is written $Y = F(X_3 | X_1,X_2)$.

If a number of cows are to be fed at a constant rate of forage and all had the same expected ability, these two variables could be considered fixed; then, if these same cows were fed varied quantities of grain and the output of milk measured, one should be able to derive empirically the relationship between milk production and grain feeding with the other factors held constant. This relationship can be shown on a two dimensional diagram and is likely to be of the general shape as drawn in figure 1.

The production function for dairy cows is more likely to be of the nature of $Y = F(X_2, X_3, | X_1)$ 1/, since it is not always possible to vary one feed with the other being held constant. To increase one feed without decreasing the other, the animal must have been on a limited diet before the experiment started. This becomes obvious when one considers the limited capacity of an animal's stomach. This is particularly true of ruminants which are under consideration in this study. The relationship between two variable input factors and output cannot be shown on the two dimensional diagram as Figure 1.

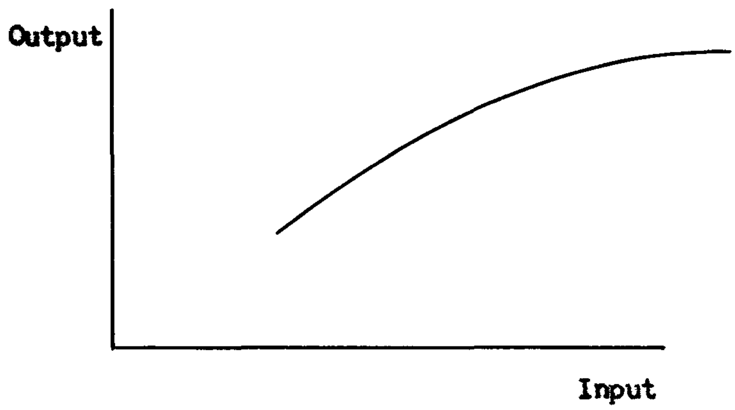

Figure 1. General Relationship between Product and One Variable Input Factor

Relationships between the factors and between the factors and total product can be illustrated in a three dimensional diagram similar to that shown in Figure 2. The shape of the production surface as shown in Figure 2 will depend on the type of production. As drawn here with either factor held constant, there is a range of increasing returns and then decreasing

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1/ Using the same notation as previously:
Y = Milk production (4% fat-corrected)
X_1 = Expected butterfat production if fed standard ration
X_2 = Pounds hay equivalent consumed
X_3 = Pounds grain equivalent consumed

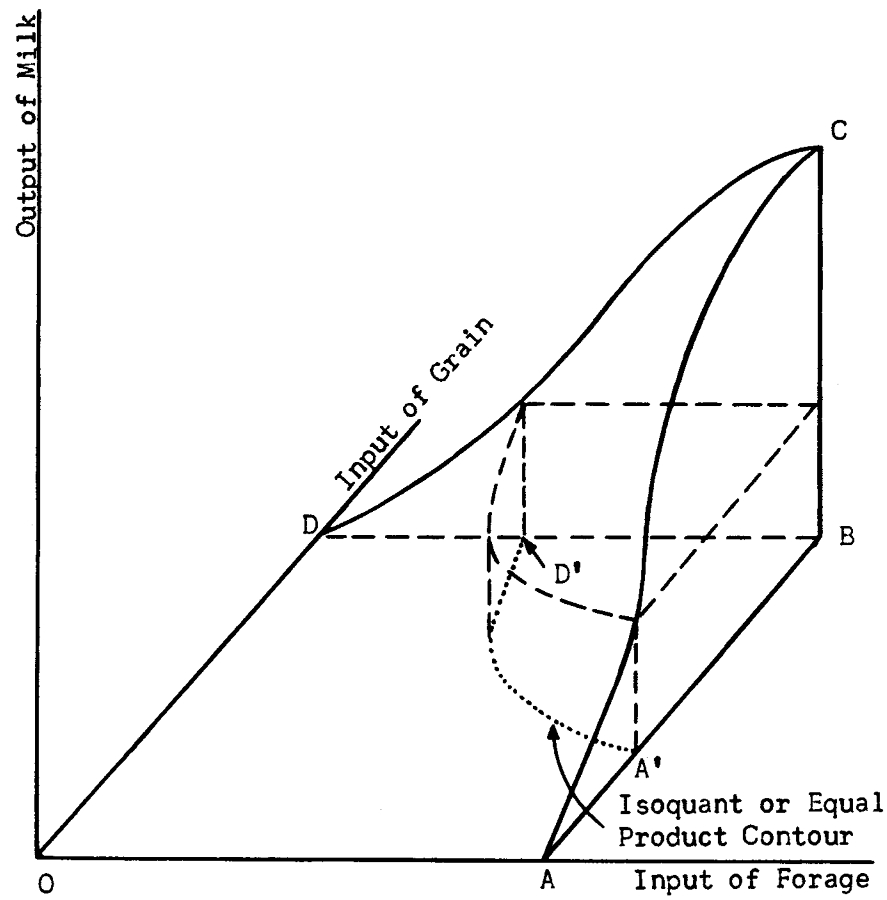

Figure 2. Production Surface Relating Two Variable Feed Inputs to Milk Output

returns to the other factor. It would be possible to draw a surface with increasing returns throughout to both factors or with increasing returns and then decreasing returns to one factor with increasing to the other factor throughout. As drawn here, AC is the response to varying quantities of grain with roughage held constant at OA hay. DC is the response to varying quantities of roughage with grain held constant at OD. D'A' shows the derivation of the iso-product contour. By dropping these vertical lines down to the ODAB plane, use of the cumbersome three dimensional diagram can be avoided.

There are many different forms or shapes which these equal product lines or iso-quants may take. For purposes of illustration, it might be well to discuss three of the more important ones. These are (1) the factors do not substitute for each other; in other words, they must be confined in fixed proportions (Figure 3a) (hydrogen and oxygen for water); (2) the factors substitute for each other at a constant marginal rate (Figure 3b) (number 1 corn for number 2 corn in producing pork); or (3) the factors substitute for each other at a diminishing marginal rate (Figure 3c).

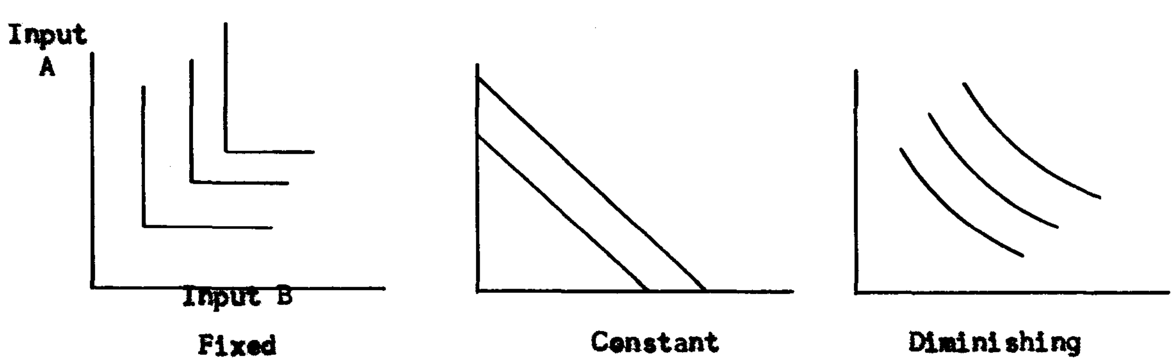

Figure 3. Three General Relationships between Two Input Factors

Derivation of equations representing the contour D'A' of Figure 2 from various production functions is the object of this study. With these equal milk product contours established for cows of different inherent ability and for stations (location) representing different conditions, the marginal rates of substitution of one feed for another can be determined. With this information, one needs only to compare it with the price ratio of grain and forage to arrive at the least cost method of feeding for a given milk output.

PROCEDURE

The source of data and procedure for coding and stratification are discussed in this section. In three immediately following sections are detailed explanations of three separate, but related, production functions derived from the experimental data. The main results, with explanations of the advantages and limitations of each, are included in the respective sections. Following the presentation of the 17-variate regression function section, an analysis of the results is presented. The methods and assumptions used in fitting the various regression functions are explained in Appendix A .

Source of Data

An experiment on feeding dairy cows, conducted cooperatively by the United States Department of Agriculture and ten State Agricultural Experiment stations from 1938 to 1941 provide the basic data for the analysis in this study. 1/ The basic data includes the expected butterfat production, the amount of feed consumed, and the actual quantity of milk produced (expressed as 4% fat corrected milk) by the cows in the experiment. All cows in the Series II experiments 2/ receiving a comparable group of feeds were included in the present analysis. These feeds were hay, corn silage, grain, and some pasture. To reduce the problem to 2-variable feeds, the corn

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1/ Jensen and others. op. cit. The individual cow record sheets were made available to the writer and data were taken directly from them.

2/ In the Series II experiments, the grain ration was varied with hay free choice.

silage was divided into grain and forage components.[1] Pasture was converted to a forage equivalent.

The 167 [2] cows used in the present study included 45 from Maryland (Sta 1), 39 from South Dakota (Sta 2), 15 from Virginia (Sta 3), 32 from Indiana (Purdue) (Sta 4), 22 from Michigan (Sta 5) and 14 from New Jersey (Sta 6). Various breeds are represented also.

Appendix table 3 shows the milk yields (Y_{sg}) and the first three regression variates--expected butterfat production (X_{s1g}), forage [3] (X_{s2g}), and grain (X_{s3g})--for the one hundred and sixty seven (167) cows and six (6) stations. Each of these variables is in coded form as explained in the following section on coding. All other variates used in the study are functions of those shown in Appendix table 3. For example X_4 is the product of X_2 and X_3, and X_5 is X_2 squared.

Coding of Data

To reduce the work involved in the calculations, the data were coded using the following method:

1/ As pointed out by Heady in the article on utilization of feed resources by dairy cows, Journal of Farm Economics 33:489, 1951, this assumes a constant rate of substitution between hay and forage from silage and between grain as such and grain from silage. These conversions were made in terms of their energy replacement value as given by The Morrison Standards.

2/ This figure might more properly be referred to as lactation periods, since the experiments ranged over a three year period and some individual cows are included more than once.

3/ When the terms grain and forage are used, they refer to the grain equivalent and hay equivalent quantities.

Observed X_1 is used in original form

(Observed X_2 - 8000) / 100 gives coded value for X_2

(Observed X_3 - 4000) / 100 gives coded value for X_3

(Observed Y) / 10 gives coded value for Y

The following procedure is required to decode the data to its original form:

(Coded X_2 + 80). 100 gives X_2 in original magnitude

(Coded X_3 + 40). 100 gives X_3 in original magnitude

(Coded Y) . 10 gives Y in original magnitude

Except when noted, the estimated statistics are in the coded data. Obviously when the statistics are quoted with pounds as the unit of measurement the data have been decoded.

Selecting the Dependent Variable

The relationship between grain and forage consumed in producing a hundred pounds of milk--without regard to total annual production per cow--was investigated as a first approximation in determining whether grain and forage are substitutes. The most elementary production logic would lead to the conclusion that a regression of grain on forage, or forage on grain, would give a negative correlation coefficient. In other words, it is not reasonable to suppose that as grain consumption increases per unit of milk produced, forage consumption would also increase. However, this could conceivably happen within narrow ranges. The stimulating effect of small amounts of grain might encourage the animal to eat larger quantity of feed than with no grain in the ration.

This preliminary investigation showed a marked negative correlation between grain and forage. Such a result indicates that, without regard to total output, grain and forage are substitutes for each other. With only the two variables in the analysis a linear substitution relationship is assumed. But, as will be shown later, there is little reason to expect that the true relationship between grain and forage is linear. In a further extension of this preliminary investigation, a term allowing a curved relationship was added to the regression equation. The linear equation was of the form:

$$Y = a + bX$$

where

$Y =$ forage per 100 pounds of milk

$X =$ grain per 100 pounds of milk.

This was changed to a second degree polynomial making the equation of the form:

$$Y = a + bX + cX^2.$$

An indication of deviation from linearity was obtained by fitting this simple polynomial. The reduction in residual sums of squares due to the squared term was significant at 1% level (as measured by F-test) for the group of cows with expected butterfat production of less than 300 pounds. The reduction was not significant for the group with an expected butterfat production of 300 pounds or more.

However, by using only the two variables which ignore total milk production, a very serious limitation is imposed upon this type of analysis. It would be valid only in case the production unit (cow) were free or had no limit to output. This would be an analogous situation to a plant that is producing chemicals having an unlimited capacity with the output not being related to fixed cost in any way. The statistical limitation is more serious, however. It has to do with the determination of the appropriate dependent variable. In this analysis, milk production was considered to be a function of the other variables. Obviously, this is the appropriate procedure in fitting the input-output relationship or the production function. However, in fitting the functions to derive factor-factor substitution relationships, other likely possibilities exist. For example, one might attempt to fit a regression of grain on all of the other variables including milk production as an independent variable on the right hand side of the equation along with forage and other variables. However, it seems far more reasonable to expect milk production to act as the normally distributed dependent variable. For these reasons, milk production is treated as the dependent variable in both the production functions and the factor-factor functions.

Use of Expected Butterfat Production As an Independent Variable Instead of Stratification

The quality of the cow, as well as the substitution ratio between feeds, is an important factor in determining total milk output. A suitable indicator of the cow's inherent producing ability was available from the original experiment. This indicator was in the form of "pounds of butterfat expected" if fed a standard ration. Two alternative approaches are available in taking this factor into consideration: (1) the data could be stratified, i.e., could put cows into expected "output" groups or (2) the "expected butterfat" could be introduced as an additional independent variable in the regression equation. Stratifying appeared undesirable since, if the grouping into classes is refined sufficiently to avoid large variations in anticipated production within any stratum, the numbers of observations in any stratum become fairly small. In the present work, the expected butterfat production is used as an independent variable in the regression equations. The main difference in these two approaches is this: In the latter the same function will be fitted to each stratum, but the average level of the function in each stratum can go up or down in order to fit the average results of the stratum as well as possible. Using expected butterfat production as an independent variable imposes more restrictions on the factors fitted, but makes use of all the data in each stratum simultaneously to get an average overall fit.

Within Station Analysis

A preliminary analysis of variance of milk production per cow (annual) gave the breakdown for between and within stations shown in Table 1.

Table 1. Analysis of Variance for Stations

	S.S.	d.f.	M.S.	M.S. Ratio
Between	3,842,534	5	768,507	
Within	9,572,460	161	59,456	12.9

The between station mean square is very large relative to the within station mean square and is significant at the 1% level (F test). Since the fitting of any regression of annual milk production on grain, forage, and other variables would only serve to increase this ratio, it was decided to allow for the variation between stations by fitting constants for the respective stations. This was achieved by working with the within-station sums of squares and sums of products when solving all of the normal equations referred to in the following sections.

INTRODUCTION TO VARIOUS REGRESSION (PRODUCTION) FUNCTIONS

As mentioned earlier, milk production must be some function of the factors surrounding or associated with the production process. Among these measurable factors and available in this study are the inherent ability of the cow, the quantity of grain and forage consumed, and the total feed consumed. Other factors likely to influence milk production are daily rate of feeding, breeds of cows and geographical location with climatic implications.

Using expected butterfat production (inherent ability) as X_1, forage consumed as X_2 and grain consumed as X_3 numerous polynomial functions of the form

$$Y(\text{milk production}) = \sum_{ijk} b_{ijk} X_1^i X_2^j X_3^k \quad i = 0,1,2;\ j,k = 0,1,\ldots,4$$

were studied. Three of these functions have been selected for detailed discussion here. These three should serve to illustrate the evolution of the study and the main results obtained. The least squares method of fitting the regression functions was used and is explained in Appendix A.

4-VARIATE REGRESSION FUNCTION 1/

The first function studied and presented here is of the form

$$Y = b_{os} + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4$$

where Y represents milk production (coded), X_1 represents expected pounds of butterfat if fed the Haecker ration, X_2 represents pounds of roughage consumed by each individual cow (coded), X_3 represents pounds of grain consumed by each individual cow (coded), X_4 is X_2 times X_3. The symbol b_{os} represents the intercept of the sth station at the origin of the respective X's.

If the function is rewritten in the form

$$Y = b_{os} + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_2X_3$$

it will be seen that the partial derivative of Y with respect to X_2 is

$$DY / DX_2 = b_2 + b_4X_4.$$

This allows the partial derivative of Y with respect to X_2 to change linearly with respect to X_3. In the same manner, it is also apparent that the function allows the partial derivative of Y with respect to X_3 to change linearly with respect to X_2.

The function was fitted by the method explained in Appendix A giving the results as shown below:

$b_{01} = 633.5694$	$b_{05} = 763.7390$	$b_1 = .55654290$
$b_{02} = 673.3484$	$b_{06} = 663.1057$	$b_2 = 4.2930872$
$b_{03} = 784.4488$	$b_{0.} = 686.3571$ (weighted average)	$b_3 = 11.956529$
$b_{04} = 687.4361$		$b_4 = .03549493$

1/ In the term p-variate as used here, p refers to the number of independent regression variates involved.

The correlation coefficients r_{ij}, the elements A_{ij} and B_{ij} in the abbreviated Doolittle solution, and the standardized regression coefficients b_i' ; $i = 1,\ldots,4$; $j = 1,\ldots,4$, Y; obtained in this process are shown in Appendix tables 1b and 2.

The square of the multiple correlation was $R^2 = .74644815$ giving $R = .863972$, and the analysis of variance testing its significance was as shown in Table 2.

Table 2. Analysis of Variance for Regression: 4-variate Function

	s.s.	d.f.	M.S.	F
Regression	7,145,345	4	1,786,336	
Residual (within stations)	2,427,115	157	15,459	115.55
TOTAL (within stations)	9,572,460	161		

The $b_{01},\ldots,b_{06}$ measures the differences between the six stations corrected for regression on $X_1,\ldots,X_4$. This variation is quite considerable. The highest (b_{03}) is about 4 times greater than the lowest (b_{01}). The $b_1 = .55654290$ and indicates a positive trend. A positive trend is as one would anticipate if the measure of expected butterfat production were a reliable indicator of actual production forthcoming.

The trend in milk production with grain (X_3) held constant is

$$DY / DX_2 = b_2 + b_4X_3.$$

Since $b_2 = 4.2930872$ and $b_4 = .03549493$, DY / DX_2 is always positive and because b_4 is positive the derivative of Y (milk production) with regard to X_2 increases as X_3 increases.

The trend in milk production with hay (X_2) held constant is

$$DY / DX_3 = b_3 + b_4X_2.$$

Again, since $b_3 = 11.956529$ and $b_4 = .63549493$, DY / DX_3 is always positive and because b_4 is positive the derivative of Y with regard to X_3 increases as X_2 increases.

Marginal Rates of Substitution of Factors

Holding the other variables constant, X_3 may be expressed as a function of X_2. This gives

$$X_3 = (Y - b_o - b_1X_1 - b_2X_2) \;/\; (b_3 + b_4X_2). \tag{1}$$

Then, the partial derivative of X_3 with regard to X_2 is found to be

$$\frac{DX_3}{DX_2} = -(b_2 + b_4X_3) \;/\; (b_3 + b_4X_2). \tag{2}$$

In graphing the results of equation (2) (X_3 plotted against X_2) the product contours are convex toward the origin as sketched in Figure 4.

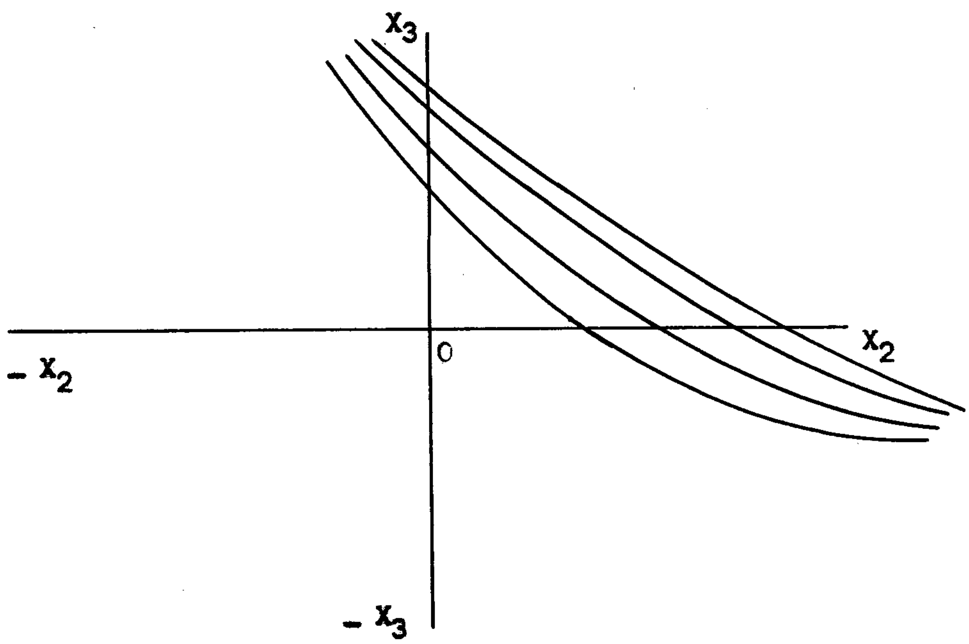

Figure 4. General Shape of Product Contour for 4-Variate Function

By dividing equation (1) through by X_2, it will be seen that X_3 approaches $-b_2 / b_4 = -120.95$ as X_2 goes to infinity. Likewise it can be shown that X_2 approaches $-b_3 / b_4 = -336.85$ as X_3 becomes infinitely large.

From equation (2) it can be shown that the derivative of X_3 with regard to X_2 is uniformly negative. It decreases from very large negative values for small values of X_2 and approaches zero for large values of X_2. This is also indicated in Figure 4.

Economic Aspects

The marginal rate of substitution of hay (X_2) for grain (X_3) is negative throughout all ranges of combinations. As the amount of hay is increased, the amount of grain needed to substitute for one unit of hay becomes less and less.

Despite the high multiple correlation given by this function, it is not entirely acceptable because of restrictions it imposes on the substitution rates. The main restriction is that the contours are not permitted adequate freedom of movement. This will be fairly obvious by noting that the asymptotes $-b_3 / b_4$ and $-b_2 / b_4$ of all of the milk product contours are independent of the fixed values of Y and X used in deriving the contours. It is clear that the asymptotes are well beyond the observed feeding rates. Moreover, it does not conform to production logic to require contours for different levels of output and different values of X_1 to converge to the same asymptotes. Reasonably, it would be expected that individual cows of different inherent ability would utilize feed in varous degrees of efficiency.

Figure 5 illustrates what one might expect a logical hypothesis to be with respect to marginal substitution rates between grain and forage for a given output of milk from cows of different inherent ability.

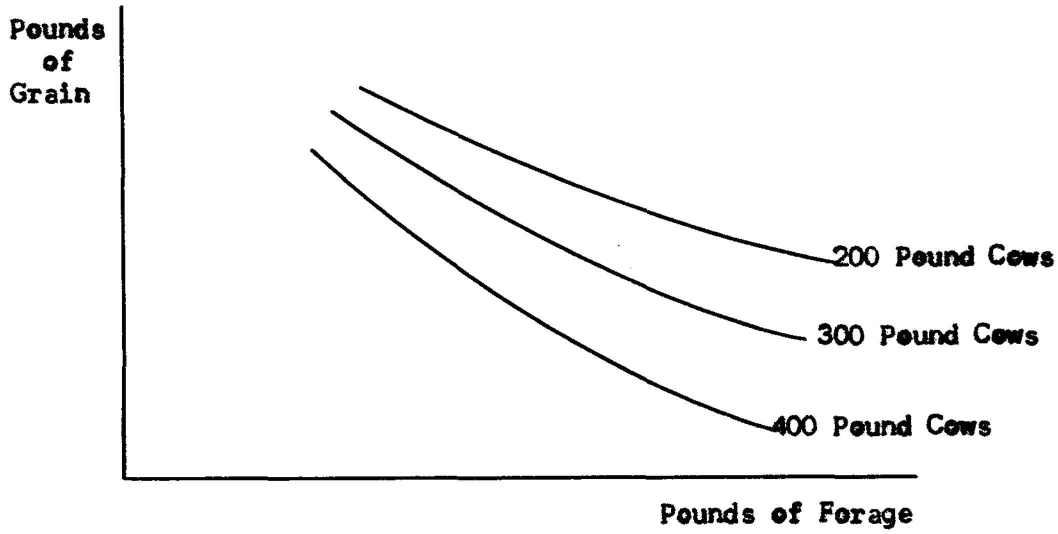

Figure 5. A Hypothesis for a Product Contour of a Given Value in Feeding Different Grade Cows

The very nature of this 4-variate function will not permit a family of contours as sketched in Figure 5. Another factor tending to limit the validity of this function is the constant returns (in the form of pounds of milk) to either grain or forage when the other is held constant. Line AC in Figure 1 would be a straight line with input of forage fixed at OA. Similarly line OC the response in milk production to additional forage feeding with grain input constant at OD would also be linear.

Using only the interaction term (X_4) in addition to the first three regression variates restricted the analysis unduly. The use of additional regression variates to allow greater flexibility is discussed in the next section.

6-VARIATE REGRESSION FUNCTION

In order to add more flexibility to the production surface, several additional functions were considered which involve a limited number of extra polynomial terms of the general form

$$X_1^i \, X_2^j \, X_3^k, \quad i = 0, 1, 2; \; j, k = 0, 1, 2, 3, 4.$$

A very simple extension of this type is the function

$$Y = b_{os} + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4 + b_5X_5 + b_6X_6 \qquad (1)$$

where the first four X variates are defined as the 4-variate function and X_5 equals X_2^2 and X_6 equals X_3^2. The seventeen-variate function discussed in the next section is a more complex extension of the same form. In this section, the results of fitting the six-variate function (1) will be discussed. This will serve as an example of several similar functions which were also considered.

This function is similar to Heady's function 2 in the work referred to earlier. 1/ The main difference in the function fitted here and the one computed by Heady is that he used only cows that had an expected butterfat production between 300 and 400 pounds. This made it unnecessary to bring the expected butterfat production in as a regression variable for each individual cow as is done in the present study. The function seemed a logical hypothesis. It allows for a diminishing rate of transformation of each feed and for the productivity of one feed to depend on the level at which the other is fed. The nature of it allows for a declining elasticity of production as well as diminishing returns for each or both feeds. Although the coefficients of this function tested significant, Heady

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1/ Heady. Utilization of Feed Resources by Dairy Cows.

rejected it in favor of another one on the basis of it giving increasing returns in part of the range of observation.

This function was fitted by the same method as the 4-variate regression function giving the results as shown below.

$b_{01} = 617.9700$	$b_1 = .544561$
$b_{02} = 659.7895$	$b_2 = 4.0613047$
$b_{03} = 758.2255$	$b_3 = 11.611245$
$b_{04} = 675.9436$	$b_4 = .02929022$
$b_{05} = 753.6768$	$b_5 = -.00283679$
$b_{06} = 650.8393$	$b_6 = .06569777$
$b_{0.} = 672.0758$	

The correlation coefficients r_{ij}, the elements A_{ij} and B_{ij} in the abbreviated Doolittle solution and the standardized regression coefficients b_i' ; $i = 1,...,6$; $j = 1,...,6$, Y; obtained in this process are shown in Appendix tables 1band 2.

The square of the multiple correlation was $R^2 = .75642739$ giving $R = .86972835$. This represents a reduction in the residual sums of squares of 95,526. The analysis of variance testing significance of this reduction of the error sum of squares remaining after fitting the 2 additional regression variates is given in Table 3.

Table 3. Analysis of Variance for the 6-Variate Regression

Source	S.S.	d.f.	M.S.	F
Regression (4-variate function)	7,145,345	4		
Difference (4-v.f. vs. 6-v.f.)	95,526	2	47,763	3.18
Regression (6-variate function)	7,240,871	6	1,206,812	
Residual (after regression)	2,331,589	155	15,042	
Total (within station)	9,572,460	161		

The F value is just significant at the 5% level of probability. The b_1 value is changed very slightly from that in the 4-variate analysis. It will be readily seen that the partial derivative of Y with regard to X_2 is

$$DY / DX_2 = b_2 + b_4X_3 + 2b_5X_2.$$

Since b_2 and b_4 are practically unchanged and since b_5 is small, this derivative is much the same as the one in the 4-variate function. It is generally positive and gets more positive as X_3 increases. However, since b_5 is negative, the derivative gets less as X_2 increases. But the b_5 value is small so the curvature produced in this way is small.

Looking at the partial derivative

$$DY / DX_3 = b_3 + b_4X_2 + 2b_6X_3$$

of Y with regard to X_3 in a similar manner, it is noted that the first two coefficients b_3 and b_4 have not changed very much from their values in the 4-variate function. The coefficient b_6 which is giving the derivative an upward curvature (by being positive) is having more effect in this function than did b_5 in the derivative of Y with regard to X_2.

Production Function Contours

On expressing X_3 as a function of X_2 with the other variables held constant an implicit quadratic function is obtained of the form

$$aX_3^2 + bX_3 + c = 0$$

where

$$a = b_6$$

$$b = b_3 + b_4X_2$$

$$c = -Y + b_o + b_1X_1 + b_2X_2 + b_5X_2^2$$

The contours of the 6-variate production function are graphed in Figure 6

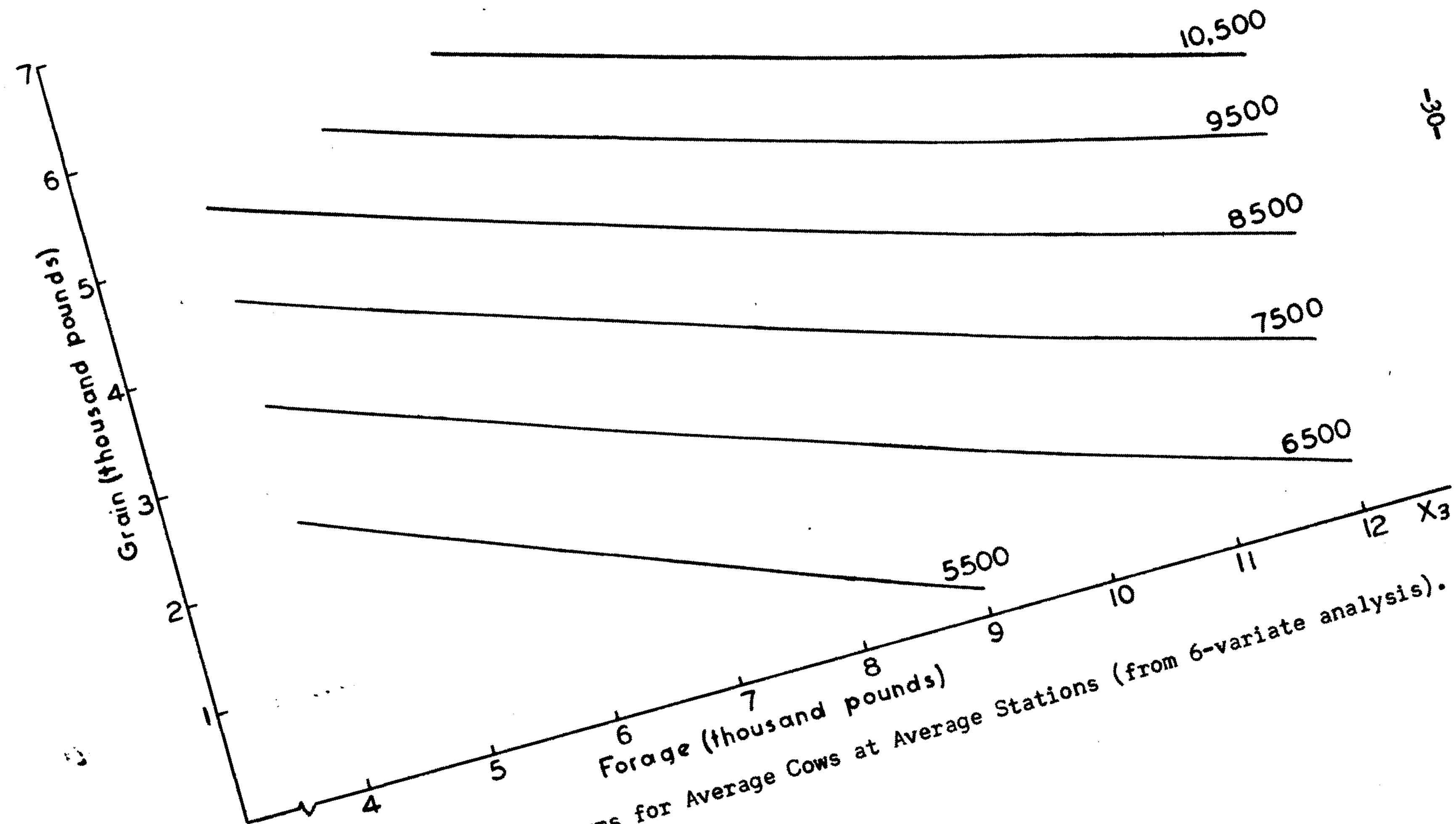

Figure 6. Milk Contours for Average Cows at Average Stations (from 6-variate analysis).

(X_3 plotted against X_2) for 6 different levels of output of milk, with X_2 varied by 1000 pound increments and for an "average" cow at an "average" station. 1/ A "map" of the production surface such as figure 6 shows visually the relationship between grain and forage at different levels of milk production. It is more useful in some respects than the precise marginal rates of substitution between grain and hay (the slope of the contour at a point). The marginal rates of substitution between points (between 4000 and 5000 pounds of forage) may be obtained directly from the contour lines. Had the point (or exact) substitution rates been desired, they could have been obtained by computing the partial derivative of X_3 with regard to X_2 as

$$\frac{DX_3}{DX_2} = -(b_2 + b_4X_3 + 2b_5X_2) \quad / \quad (b_3 + b_4X_2 + 2b_6X_3)$$

Economic Aspects

The milk product contours from this analysis are convex to the origin. They have a negative first derivative with a positive second derivative. This indicates a diminishing marginal rate of substitution of one feed for the other. The higher the level of output from a given cow, the lower the marginal rate of substitution of forage for grain. This is obvious from the graph of the results. The 10500 pound contour has less curvature than the ones below it. In terms of total output, this function gives increasing returns to additional grain feeding at any level of forage feeding (constant). It gives decreasing returns to additional forage feeding at any level of grain feeding (constant).

The contour lines are not far different from linear. If this were the

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1/ This "average" cow and "average" station situation will be referred to again and explained in the next section on the 17-variate regression function.

final answer and the milk product contours as shown were accepted, the implications would be (1) that the most profitable ration or the proportion of grain and forage fed would shift greatly with a very small change in relative prices, (2) that if it were profitable to feed grain at all, the maximum amount the cow could be induced to eat would be the most profitable ration, and (3) that (1) and (2) would hold without regard to differences in inherent ability. After this function was fitted and these implications became apparent, it was believed that it was also too restrictive. Additional flexibility was desired. To achieve this other variables were brought into the analysis as explained in the next section.

17-VARIATE REGRESSION FUNCTION

The third regression function discussed here has 17 independent variables. The number of variables was extended to allow more flexibility to the function in the expectation of obtaining better estimates of the substitution rates between grain and forage. The reasoning behind selecting this particular function should emerge as the nature of it is explained. The function fitted is of the form

$$Y = b_o + b_1X_1 + \dots + b_{17}X_{17} \qquad (1)$$

where $X_1, \dots, X_6$ are as used in the 6-variate analysis

$$X_7 = X_2L,\ X_8 = X_3L, \dots, X_{11} = X_6L$$

$$X_{12} = Q, X_{13} = X_2Q, \dots, X_{17} = X_6Q\ .$$

The quantity L is a linear function of Y obtained from the 6-variate analysis and

$$Q = L^2 - 10\ .$$

A function of this nature permits the parameters of each contour to be quadratic functions of Y, which can be seen from the following discussion. The initial aim was to fit a function of the form

$$Y = b_o + b_1X_1 + b_2X_2 + \dots + b_6X_6 \qquad (2)$$

where each b except b_1 is a quadratic function of Y. In other words

$$b_i = c_{io} + c_{i1}Y + c_{i2}Y^2 \qquad i = 0,2,3,\dots,6. \qquad (3)$$

On substituting (3) for the b's in the function (2) it has the form

$$Y = k_o + k_1X_1 + \dots + k_{17}X_{17} \qquad (4)$$

where $X_1, \dots, X_6$ are defined as before

$$X_7 = X_2Y,\ X_8 = X_3Y, \dots, X_{11} = X_6Y$$

$$X_{12} = Y^2,\ \ X_{13} = X_2Y^2,\ \ X_{14} = X_3Y^2, \dots, X_{17} = X_6Y^2$$

and $k_1 \ldots, k_{17}$ are linear functions of the c's. To reduce the complexities of fitting a function of the form (4) it was decided:

(a) to replace the Y's on the right hand side by the least squares estimates Y obtained in the 6-variate fit, and (b) to make this substitution in a simple approximate form by using the variable L instead of Y, and Q instead of Y^2 where L and Q are functions of Y as shown in Table 4.

Table 4. Values of L and Q in Terms of Y

Y	L	Q
449.5 - 572.6	-5	15
572.7 - 695.7	-4	6
695.8 - 818.8	-3	-1
818.9 - 941.9	-2	-6
942.0 - 1065.0	-1	-9
1065.1 - 1188.1	0	-10
1188.2 - 1311.2	1	-9
1311.3 - 1434.3	2	-6
1434.4 - 1557.4	3	-1
1557.5 - 1680.5	4	6
1680.6 - 1803.6	5	15

Decision (a) means in effect that instead of fitting a function involving Y, on the right hand side, we are fitting a polynomial in X_1, X_2, X_3 of a higher degree than before, which has certain restrictions built into it. Using the notation ijk for the polynomial term $X_1^i X_2^j X_3^k$, all of the polynomial terms involved are as listed in Table 5.

Table 5. Polynomial Terms Implicit in the 17-Variate Function*

001	020	050	120
002	021	051	121
003	022		122
004	023	060	
005	024		130
006		100	131
	030	101	
010	031	102	140
011	032	103	
012	033	104	200
013			201
014	040	110	202
015	041	111	
	042	112	210
		113	211
			220

* The first term (001) of column 1 denotes $X_1^0 X_2^0 X_3^1$ which is X_3. To take another example (015) of column 1 denotes $X_2 X_3^5$.

To illustrate the type of restrictions present, the variate $X_8 = X_3 Y$ has become

$$X\ (b_o + b_1 X_1 + \ldots + b_6 X_6) = b_o X_3 + b_1 X_1 X_3 + \ldots + b_6 X_3 X_6$$

where $b_o, b_1, \ldots, b_6$ have the values found in the previous section. Any regression coefficients found with respect to these individual terms will be of the form

$$k_8 b_o,\ k_8 b_1, \ldots,\ k_8 b_6$$

where k_8 alone is allowed to vary in the least squares solution. Thus, although the complete polynomial represented by (3) has far more than 17 terms, it has in effect only 17 degrees of freedom.

Decision (b) simplified the work considerably without sacrificing, it is considered, any appreciable degree of accuracy. The relations between L, Q and Y shown in Table 4 may be written as

$$L = c_0 + c_1 Y$$

$$Q = L^2 - 10$$

where

$$c_0 = -9.1514$$

$$c_1 = .008,123$$

The difficult task of calculating the variates $X_7, \ldots, X_{17}$ was considerably accelerated by the replacement of Y by the simple L and Q values. Nonetheless, the work involved in the determination of these variables and sums of squares and sums of products is not to be minimized.

The solution of the normal equations is shown in Appendix A and the resulting coefficients are as shown below.

$b_{01} \doteq 551.0886$	$b_1 = .65733655$		$b_{12} = -1.09024613$
$b_{02} = 589.4715$	$b_2 = 9.04735965$	$b_7 = 2.52964380$	$b_{13} = .10862979$
$b_{03} = 685.1358$	$b_3 = 4.18481326$	$b_8 = -.75138998$	$b_{14} = -.79349964$
$b_{04} = 618.6778$	$b_4 = -.00163921$	$b_9 = .07275811$	$b_{15} = .00451635$
$b_{05} = 676.2130$	$b_5 = -.26407307$	$b_{10} = -.06958444$	$b_{16} = -.01469859$
$b_{06} = 588.4633$	$b_6 = .05430234$	$b_{11} = .07728117$	$b_{17} = -.00169846$
$b_{6'} = 604.6604$			

The correlation coefficients r_{ij}, the elements A_{ij} and B_{ij} in the abbreviated Doolittle solution, and the standardized regression coefficients b'_i ; $i = 1,\ldots,17$; $j = 1,\ldots,17$, Y; obtained in this process are shown in Appendix tables 1b and 2.

The square of the multiple correlation was $R^2 = .782985$ giving $R = .8848646$. This represents a further reduction in the residual sum of squares of 254, 218. A pertinent analysis of variance is shown in Table 6.

Table 6. Analysis of Variance for the 17-Variate Regression

Source	S.S.	d.f.	M.S.	F
Regression (6-variate function)	7,240,871	6		
Difference (due to terms in L)	82,225	5	16,445	1.10
Regression (11-variate function)	7,323,096			
Residual after 11-variate function	2,249,364	150	14,996	
Difference (due to terms in Q)	171,994	6	28,666	1.99
Regression (17-variate function)	7,495,090	17		
Residual after 17-variate function	2,077,370	144	14,426	
Total (within station)	9,572,460	161		

It is found from the analysis of variance that the reduction in error sum of squares due to fitting the X's involving L after $X_1, \ldots, X_6$ is not significant ($F = 1.10$ where $F_{5\%} = 2.27$). The further reduction due to fitting X's involving Q, however, is almost significant ($F = 1.99$ where $F_{5\%} = 2.16$) and seems worthwhile retaining. Since the X's involving Q were retained, it was decided also to retain those involving L since Q is a function of L and the final result would be a function of L anyway. Further explanation of the effect of retaining these terms will be given in discussing the contour functions.

On working with the 17-variate function, it will be seen that the partial derivative of Y with respect to X_2 is also a lengthy polynomial function of the form

$$DY / DX_2 = \sum k_{ijk} X_1^i X_2^j X_3^k$$

The actual terms on the right hand side in the partial derivative of Y with respect to X_2 can be determined from Table 5 by

(1) deleting all terms with zero in the second place, that is by deleting all terms with X_2^0 and

(2) reducing the second digit in all other terms by unity. Thus, the list will start with 000,001 and finish with 210. The determination of the partial derivative DY / DX_3 of Y with respect to X_3 can be approached in the same way. These functions involve too many terms to be of much use beyond showing that there is considerable flexibility in the marginal productivity rates which can be measured by the function.

Production Function Contours

In graphing the production function contours of (1), it would be extremely difficult to solve for X_3 in terms of X_2 with Y held constant. The task was accomplished relatively simply, however, by a method which it is considered gives good approximate contours. In the 17-variate function, there are two types of Y values. One is the set of values given by the left hand side of the equation which is denoted by Y_{17}. The others are the Y values implicit in $X_7,\ldots,X_{17}$ which were obtained from the previous 6-variate regression problem and may be denoted Y_6. Now for the purpose of expressing X_3 as a function of X_2 with Y_{17} held constant, Y_6 and Y_{17} are close enough to be considered the same. This means that when Y_{17} is fixed in getting the required function, Y_6, and consequently L and Q may be considered fixed at the values

$$Y_6 = Y_{17}, \qquad L = c_0 + c_1Y_{17}, \qquad Q = L^2 - 10 \quad .$$

On adopting this approximate approach, the appropriate function is of the form

$$aX_3^2 + bX_3 + c = 0 \tag{5}$$

where

$$a = b_6 + b_{11} + b_{17}$$
$$b = N_0 + N_1X_2$$

$$N_o = b_3 + b_3L + b_{14}Q$$

$$N_1 = b_4 + b_9L + b_{15}Q$$

$$c = P_o + P_1X_2 + P_2X_2^2 + h - Y$$

$$P_o = b_{12}Q$$

$$P_1 = b_2 + b_7L + b_{13}Q$$

$$P_2 = b_5 + b_{10}L + b_{16}Q$$

$$h = b_o + b_1X_1$$

Various contours obtained from (5) are graphed in Figure 7. Since the constants appropriate to each station, b_{os}, $s = 1,\ldots,6$ and the effect of X_1 as measured by b_1X_1 play a similar role in this function, it was convenient to consider the effect of these together by holding $h = b_{os} + b_1X_1$ fixed rather than fixing b_{os} and b_1X_1 separately.[1/] Figure 7 shows seven product contours for $h = 850$. The grain and forage combinations for all these contours are shown in Appendix table 5.

Any value of h represents many combinations of conditions. For example, a value of 850 for h represents

(a) $X_1 = 350$, $b_o = 619.93$

or

(b) $X_1 = 450$, $b_o = 554.20$

just to take two of many combinations. Combination (a) represents average inherent ability cows at average stations, while combination (b) represents high inherent ability cows at a low station.

[1/] The b_{os} and X_1 values acting together were viewed as a "handicap" imposed on the cow and are designated as h.

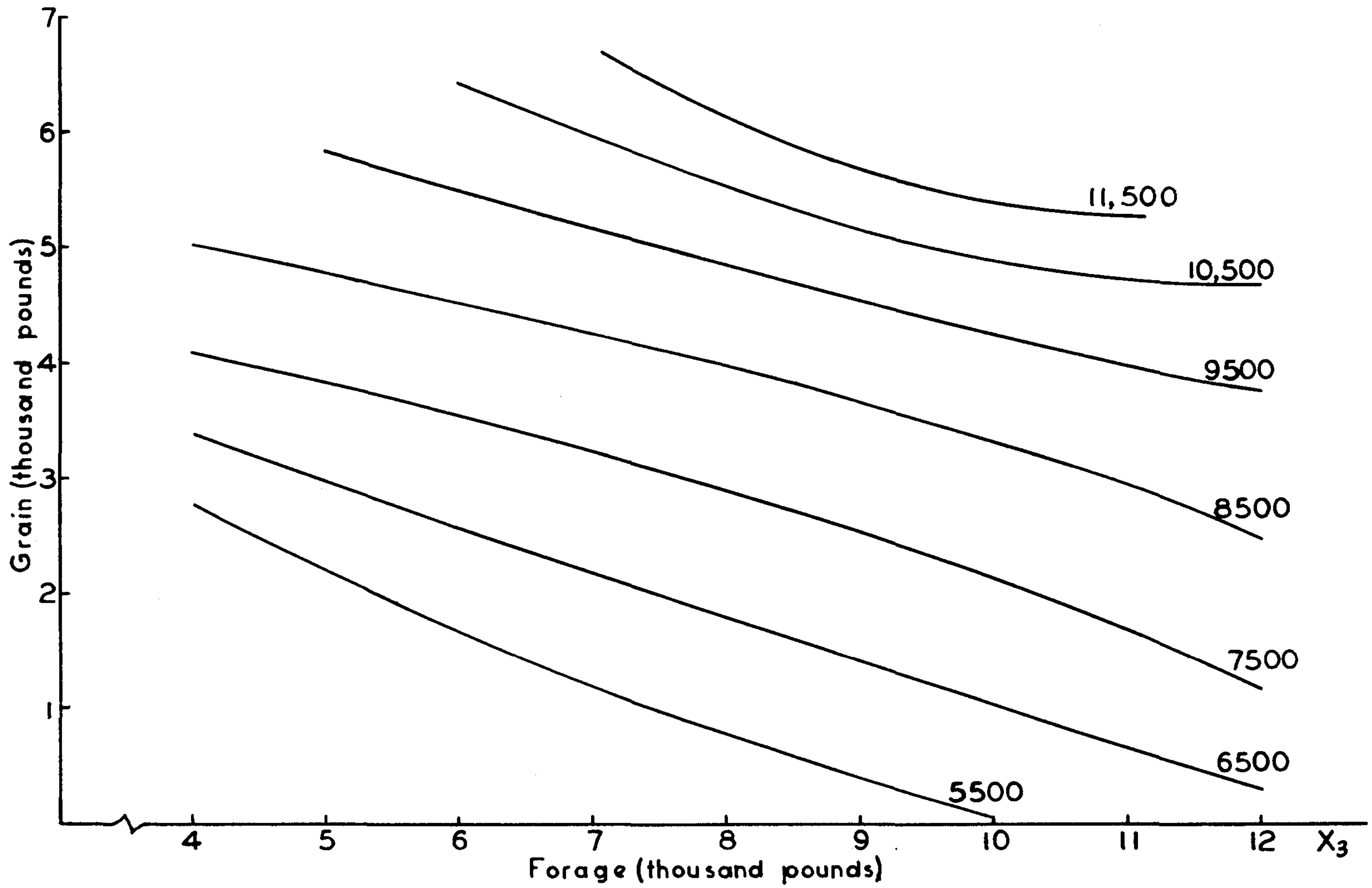

Figure 7. Milk Contours for 350 Pound Cows at Average Station or 450 Pound Cows at Low Station. h = 850

Marginal Rates of Factor Substitution

If an attempt is made to get the partial derivative DX_3 / DX_2 for the fitted production function from $-(DY / DX_2) / (DY / DX_3)$, it becomes a very involved ratio. Its principal use here perhaps is in illustrating the flexibility that exists in the function. The derivative can be expressed more simply, however, by assuming L and Q are fixed when Y_{17} is fixed as was done in graphing the product contours. In doing this, the derivative becomes

$$DX_3 / DX_2 = -(DY / DX_2) / (DY / DX_3)$$

where

$$DY / DX_2 = -(b_2 + b_4X_3 + 2b_5X_2 + b_7L + b_9LX_3$$
$$+ 2b_{10}LX_2 + b_{13}Q + b_{15}QX_3 + 2b_{16}QX_2)$$

$$DY / DX_3 = (b_3 + b_4X_2 + 2b_6X_3 + b_8L + b_9LX_2 + 2b_{11}LX_3$$
$$+ b_{14}Q + b_{15}QX_2 + 2b_{17}QX_3).$$

Product Contour Flexibility

The increased flexibility of the 17-variate function relative to the 6-variate function used in the previous section can be illustrated by discussing the respective contour maps. One of the main features of the 17-variate function is that the curvature of a contour can vary quadratically (or approximately so) with respect to the contour value. For example, in Figure 7 for h = 850, it can be seen that the contour for Y = 10500 has positive curvature;[1/] this changes to slight negative curvature for the contour at $Y = 8500$ and back to positive curvature for the contour at $Y = 6500$. In the 6-variate function, the curvature of all contours is more rigidly fixed being allowed to vary only slightly about a common value. The quadratic relationship between degree of curvature and height of contour in the 17-variate function is due mainly to the terms $X_{12},\ldots,X_{17}$ involving Q. The near significance at the 5% level of the sum of squares due to adding these terms (see Table 6) supports the hypothesis that a quadratic change of curvature in the contours reflects some effect of this nature in the data. In contour maps of the 17-variate function for other values of h (Figures 8, 9, 10 and 11), it will be seen that similar quadratic relationships are in evidence between contour curvature and contour height. Further clarification with examples of contours for other values of h will be presented in the next section.

1/ The term positive curvature in used here to measure curvature convex to the origin. Correspondingly, negative curvature denotes concavity to the origin.

INTERPRETATION AND EVALUATION

The principal results obtained in this study are discussed in this final section. First, there will be an interpretation of the findings with respect to substitution rates between grain and forage. Second, there will be comments regarding the methods of analysis used as applied to other problems of a similar nature.

Substitution Rates between Grain and Forage

Each of the three production functions analysed have their advantages as well as limitations. Considered independently, each function would be statistically significant at an acceptable level of probability on the basis of a comparison between the error sum of squares and the sum of squares due to regression. The acceptability of one regression function relative to another is measured by the ratio of the mean square due to fitting additional variates to the mean square due to error after regression.

Usually more significant variables are included in the first attempt at fitting a regression equation because of a priori information regarding the production process. This occurred in the 4-variate function. If X_1 had been left out of the 4-variate analysis but included with X_5 and X_6 in the 6-variate analysis, the improvement in the latter would have been more significant as measured by the decrease in the error sum of squares.

Perhaps the greatest merit in the 4-variate function is the relative ease of computing the coefficients and solving the contour equations. Despite the relative inflexibility of the contours from the 4-variate function, they are more acceptable than the linear concepts of substitution between grain and forage over all ranges as implied by usual feeding standards. This function is not ruled out entirely, but its greatest usefulness would be in making approximate overall estimates without extrapolations.

Likewise, the coefficients and the contour equations of the 6-variate function are relatively easy to compute. The main advantage and principal use made of this function was to provide fairly good preliminary estimates of the production function to use as a basis for the 17-variate analysis. The primary objection to the 6-variate function is that it implies increasing returns to grain feeding. While production logic would prevent general acceptance of this conclusion, there is no conclusive evidence that increasing returns to grain feeding are not present to some degree in the data. The near linear relationship between grain and forage for any level of output suggests the existence of an average tendency of this nature. As will be noted later, this linear tendency in the middle of the data also shows up in the 17-variate function.

The 17-variate analysis gives a more rational family of contours for different situations than either the 4 or 6-variate analyses, all things considered. The contours illustrated in this section were computed from the 17-variate function. Contours representing four situations are shown in Figures 8, 9, 10, and 11. Figure 8 is representative of a low (X_1 of 200) inherent ability cow at "low" stations. 1/ Figure 9 is representative of a little below (X_1 of 300) average cow at "average" stations or for a little above (X_1 of 400) average cow at "low" stations. Figure 10 is representative of a decidedly above (X_1 of 450) average cow at an "average" station or for an average (X_1 of 350) cow at "high" stations. Figure 11 is representative of a high (X_1 of 500) inherent ability cow at "high"

1/ The stations were ranked "low", "average", and "high" on the basis of the b_{3s} value relative to the overall weighted average $b_{o.}$.

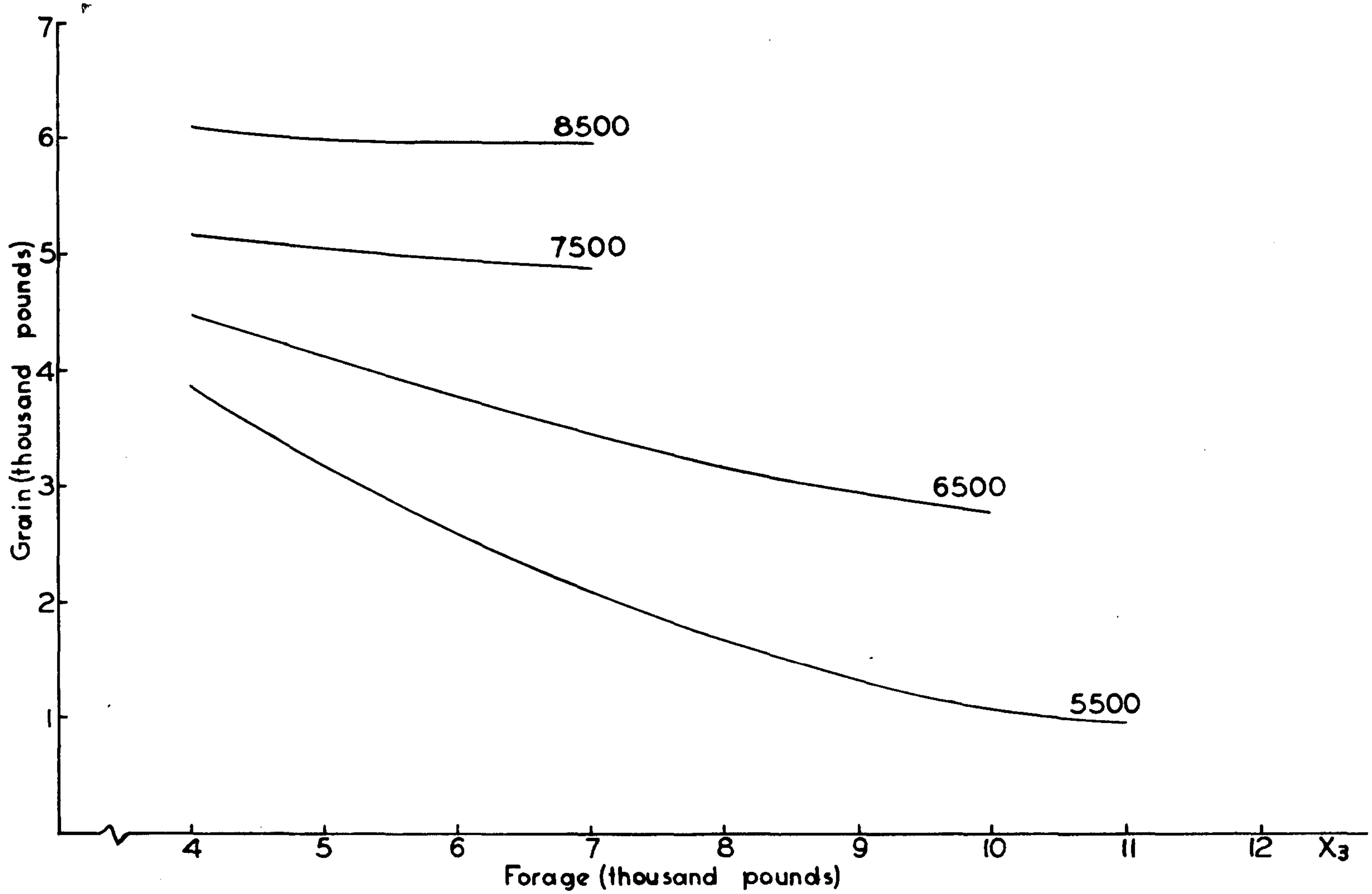

Figure 8. Milk Contours for 200 Pound Cow at Low Station. h = 700

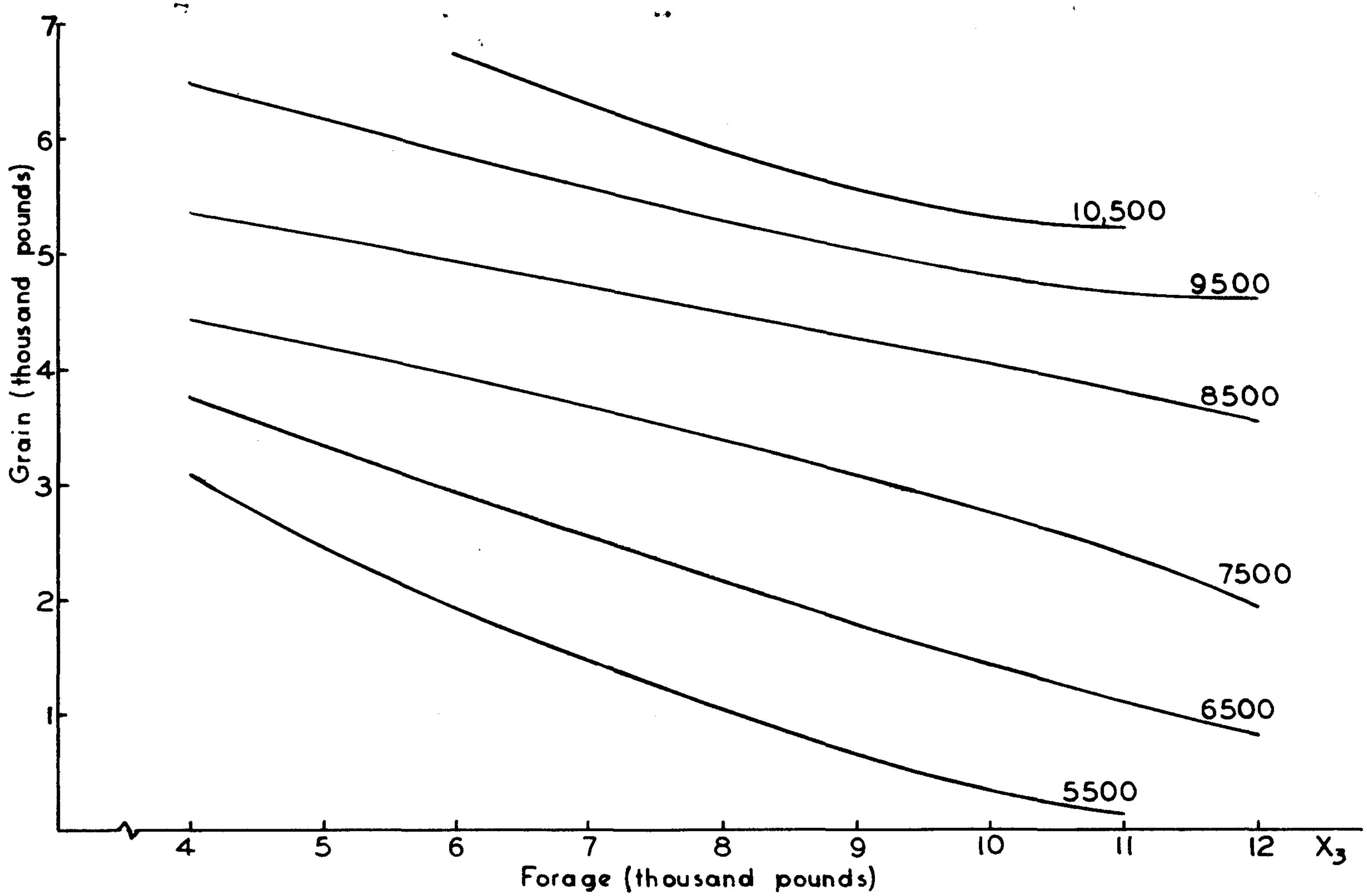

Figure 9. Milk Contours for 300 Pound Cows at Average Station or 400 Pound Cows at Low Stations. h = 800

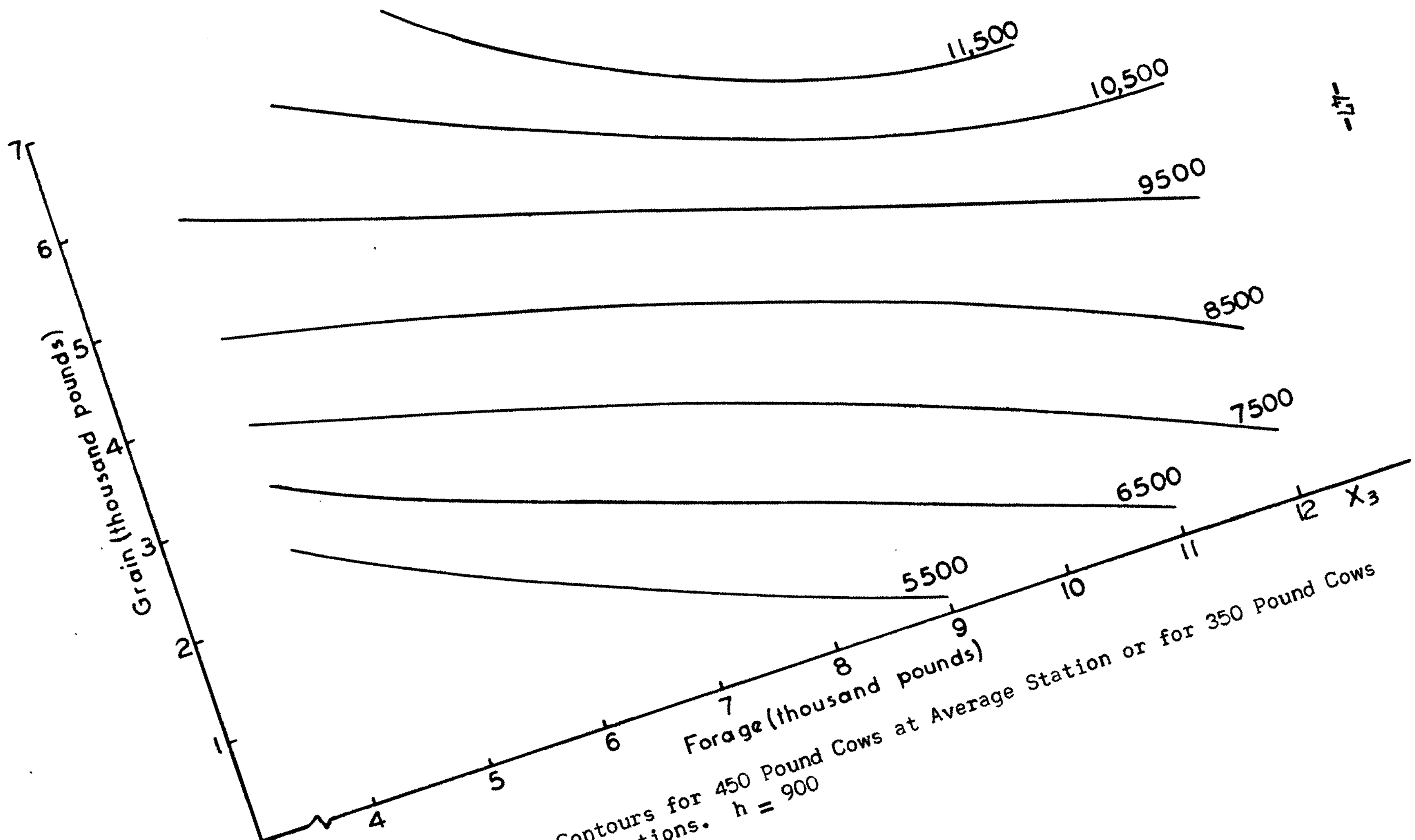

Figure 10. Milk Contours for 450 Pound Cows at Average Station or for 350 Pound Cows at High Stations. h = 900

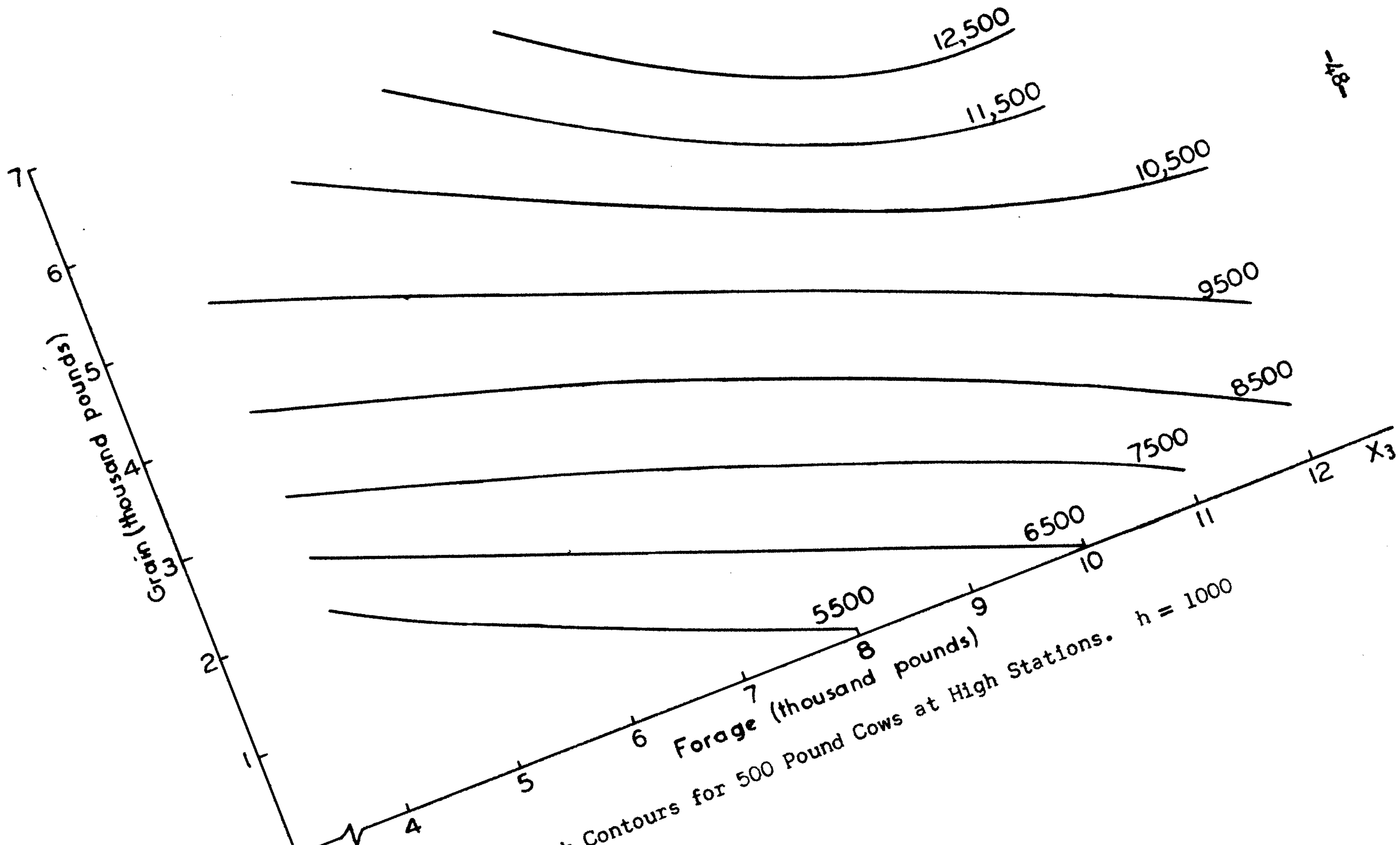

Figure 11. Milk Contours for 500 Pound Cows at High Stations. h = 1000

stations. Contours for the average (X_1 of 350) inherent ability cow at "average" stations are shown in Figure 7. 1/

The four sets of contours shown in this section are for successively higher h- values of 700, 800, 900, and 1,000. The practical extremes of the data are represented by these h- values. Appendix tables 6, 7, 8, and 9 show the various combinations of grain and forage for the different values. Figures 8, 9, 10, and 11 are based on these Appendix tables.

The flexibility of the 17-variate function is evident in the differences in the slopes of the contours on the various contour "maps". From the viewpoint of production logic, there are two unfavorable features of the 17-variate function contours: (1) the negative curvature of some of the contours and (2) the element of increasing returns to grain feeding in certain ranges. As mentioned above in discussing the 6-variate function, there is a tendency for a linear relationship between grain and forage in some ranges of the data. This linear tendency is likely present in the results of the 17-variate function where certain of the contours exhibit negative curvature. It is likely that the true relationship between grain and forage in the middle area 2/ of the data is near linear or perhaps has slight positive curvature.

The negative curvature of the contours could be a result of the nature of the function fitted. For example, strong tendencies in the data toward positive curvature at both low and high contour heights could combine and

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1/ Figure 7 is on page 40.

2/ The middle area of the data refers to a narrow band of contours near the average expected milk production for the various situations.

cause this negative curvature. It is noted in passing that the inclusion of variables $X_7,\ldots,X_{11}$ involving linear relationship of Y would permit a change in curvature of the contours from positive to negative but not back to positive. But, the inclusion of variables $X_{12},\ldots,X_{17}$ involving quadratic relationships of Y would permit the curvature to return to positive.[1/]

While increasing returns to grain are clearly evident in the 6-variate analysis, this condition is not present over all ranges and situations when the 17-variate function is used. For example, in Figure 8 for h-value of 700, with forage constant at 4000 pounds, the contours indicate decreasing returns to additional grain. But, with forage constant at 7000 pounds, the contours indicate first decreasing and then increasing returns. Of the five situations shown in Figures 7 through 11, only Figure 8 does not include negative contours.

The milk contours shown in Figures 7, 9, 10 and 11, representing various production situations, generally become negative near the average level of expected output for the respective situations. These contours become positive at both the upper and lower levels of production. Milk contours representing the output of an average cow at an average station are shown in Figure 7. The 8500 pound contour is negative, and the 9500 pound contour is positive. If a cow with an X_1 value of 350 pounds produced exactly her expected production, her output would amount to 8750 pounds of 4% fat corrected milk. A contour representing this particular output would be drawn

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1/ Two other variables not examined, but which could affect the curvature of the contours, are daily rates of feeding and proportion of total feed consumed going to maintenance.

between the negative contour (8500 pounds) and the positive contour (9500 pounds). Such a contour would likely have slight negative curvature or be approximately linear.

Evaluation of Methods Used in the Analysis

There is nothing particularly unique about the methods used in the 4 and 6-variate functions. The computational procedure employed the usual methods of least squares regression analysis applied to a series of data consisting of one dependent and several independent variables. The production functions were extended by use of polynomials of some of the independent variables. Faced with non-linear regression, many investigators have used various theoretical equations in an attempt to find a better fitting line. 1/ These equations usually take the form of a second degree or higher polynomial of one or more of the independent variables; other forms such as $\sqrt{X}$, log X, or 1/X are sometimes used. 2/

In the attempts to achieve a better fit for the 4 and 6-variate production functions, only the independent variables were extended. But in going from the 6-variate to the 17-variate function, a new concept was introduced: An expression of the dependent variable was included on the right hand side of the equation.

1/ See for example Snedecor, G. W. Statistical Methods, 4th Ed., Collegiate Press, Inc., Iowa State College, Ames, Iowa 1946. p. 379 or Heady, Earl O. Use and Estimation of Input-Output Relationships or Productivity Coefficients, Journal of Farm Economics 34 (Pt 5): 775-786 1952.

2/ Combining this method with production logic seems to be the only alternative where the fundamental biological relationships have not been determined.

By introducing an expression of the dependent variable (Y) on the right hand side of the equation, it affects the solution in two ways: (1) the linear effect of $X_7, \ldots, X_{11}$ and (2) the quadratic effect of $X_{12}, \ldots X_{17}$. The estimates of Y used to obtain $X_7, \ldots, X_{17}$ are based on the results of the 6-variate analysis. The reason for making $X_7, \ldots X_{17}$ related to Y was to permit greater flexibility in the slopes of the product contours. This technique permitted the slope of the contours to vary more according to the contour height. This procedure appears logical, since the original hypothesis was that the slope of the contours would vary with (1) cows of different inherent ability and (2) the level of the contours for a cow of given ability. Apparently, the special method used here to provide flexibility accomplished its purpose. 1/ The contours for various values of Y assumed many different shapes, and the reduction in error sum of squares due to $X_7, \ldots, X_{17}$ was considered significant.

This special method merits consideration in deriving production functions in cases where the combination of factors are affected by the level of output. Agricultural production is characterized by production processes in which factor substitution rates are dependent on the level of output. A few examples are (1) phosphorous and nitrogen (or other nutrients) in crop production, (2) grain and forage in meat production, (3) carbohydrates and protein in meat production, and (4) labor and

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1/ The special method refers to the introduction of estimates of Y (the dependent variable) in the right hand side of the equation. Dr. David B. Duncan of the Department of Statistics, Virginia Polytechnic Institute, suggested this approach.

capital (in various forms) in most production processes. Although contour equations were the end product of the special method in the present study, such equations would not be a requirement for its use. The production function is a basic tool in studying the economics of agricultural production and resource use. Therefore, it is desirable to derive the best mathematical expression of the input-output relationships attainable in the form of a production function whether the final use is to be contour equations, marginal factor substitution equations, or single factor-product equations.1/ The special method of analysis introduced in this study promisesto be beneficial in deriving better production functions.

1/ Other factors held constant.

SUMMARY

Three distinct but related production functions (regression equations) were studied in order to derive estimates of the substitution rates between forage and grain in producing milk. Contour (equal product) equations were derived from two of the production functions to indicate the various combinations of grain and forage required to produce a given quantity of milk under specified conditions of inherent ability and geographic location. Contour "maps" of milk production computed from the more complex production function are shown for five situations ranging from (1) a low inherent ability cow at a low station to (2) a high inherent ability cow at a high station. These contour maps indicate that the rate of substitution between forage and grain is different for various points on any given contour and that the rate of substitution varies between contours.

The majority of the contours indicate a diminishing marginal rate of substitution of forage for grain in milk production. A portion of the contours in the middle area of the data have a negative slope (concave to the origin) and do not conform to production logic. The conclusion is that the negative contours were due to the nature of the production function fitted and that the true relationship between grain and forage in the area of the negative contours is approximately linear.

The first two production functions (regression equations) extended the number of independent variables by interaction terms and polynomials of the three original independent variables--inherent ability, forage consumed, and grain consumed. The third and more complex production function introduced a new concept by bringing estimates of the dependent variable from an earlier analysis into the right hand side of the equation as independent variables. Both linear and quadratic expressions of this estimate of the dependent variable were included to allow the contour value to vary with contour height. The reduction in the error sum of squares for this special method was considered significant. The method merits consideration in other problems where the rate of substitution between factors is a function of level of output.

It is not argued that the estimated forage-grain combinations (and subsequent substitution rates) are the final answers. Perhaps the precise substitution rates between forage and grain await data from further experiments designed specifically for the problem.

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To Dr. Earl O. Heady I owe an extreme debt of gratitude. As professor in charge of my major subject, Dr. Heady suggested the area of work, and perhaps the greatest credit is due Dr. Heady for the inspiration and stimulation I received from him during my 3 years at Iowa State College and for the resultant contribution to my thinking in the field of Production Economics.

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Finally, I wish to thank my wife, Marion, and daughters, Sharon Joy and Leslie Ann, for their generous attitude toward my spending time on this study, which was rightly their time.

VITA

Carl Wendel Allen: Born August 26, 1916, in Henderson, Kentucky, of native born American parents--Anna J. (nee Marx) and Elva D. Allen. Lived on livestock and grain farm from 1919 to 1939. Was graduated from Weaverton High School 1933. Worked on home farm 1933 to 1936. Entered the University of Kentucky in 1936 and was graduated with B.S. degree in 1939. Worked for the Agricultural Extension Service in Kentucky until 1940. Reentered University of Kentucky in 1940 and served as Graduate Assistant in Farm Management. Received the M.S. degree in 1941 with a major in Agricultural Economics. Was called to active duty with the U. S. Air Force in 1941 and was released to inactive status in 1946. Was employed by Iowa State College as Research Associate in Production Economics in 1946 and 1947 and Graduate Assistant in 1948 and 1949. Entered Graduate School at Iowa State College in 1946. Was employed as Associate Professor of Agricultural Economics at Virginia Polytechnic Institute in 1949.

Was married to Marion Helen Kuebler of Dumont, New Jersey, in May 1939 and has two daughters, Sharon Joy and Leslie Ann, ages 11 and 7.

Blacksburg, Virginia, May 1954.

APPENDIX

APPENDIX A

METHODS AND ASSUMPTIONS USED IN FITTING REGRESSION FUNCTIONS

The methods employed for fitting the 4-, 6-, and 17-variate regression functions discussed in the main part of this study all follow the same general pattern. The pattern is described below for the general problem of fitting a p-variate regression function; that is, the special cases are obtained by putting p = 4, 6 and 17.

The problem is as follows: There are given a set of n_s observed milk yields

$$Y_{s1}, Y_{s2}, \ldots, Y_{sn_s}$$

and p sets of n_s corresponding observations for p regression variables

$$X_{s11}, X_{s12}, \ldots, X_{s1n_s}$$

$$X_{s21}, X_{s22}, \ldots, X_{s2n_s}$$

$$X_{sp1}, X_{sp2}, \ldots, X_{spn_s}$$

at each of s = 1, 2,..., 6 stations.

The model assumed is

$$Y_{sg} = \beta_{os} + \beta_1 X_{s1g} + \beta_2 X_{s2g} + \ldots + \beta_p X_{spg} + e_{sg}$$

where Y_{sg} represents the milk yield of the gth cow at the sth station; β_{os} is a constant associated with the sth station; $\beta_1, \beta_2, \ldots, \beta_p$ are unknown regression parameters and e_{sg} is the 'error' or departure of Y_{sg} from its expectation. The absolute validity and optimum properties of the following estimation methods depend on assuming that the errors e_{sg}

$s = 1, \ldots, 6; \; g = 1, \ldots, n_s$

are normally and independently distributed and have homogenous variances. It is realized that these assumptions are not likely to be met precisely in the data involved. However, it is considered that the departures from these assumptions are not likely to seriously impair the methods.

Using the method of least squares 1/ the first objective is to get an estimation equation of the form

$$Y_{sg} = b_{os} + b_1 X_{s1g} + b_2 X_{s2g} + \ldots + b_p X_{spg}$$

In doing this it is convenient to get the coefficients $b_1, b_2, \ldots, b_p$ in two stages. First, the standardized coefficients $b'_1, b'_2, \ldots, b'_p$ are obtained and these are then converted to the required coefficients.

Step 1 consists of calculating the correlation coefficients between the X variates from the equation

$$r_{ij} = S_{ij} \Big/ \sqrt{S_{ii} S_{jj}} \; ; \quad i, j = 1, \ldots, p$$

where S_{ij} is the 'pooled' sum of squares and/or products between X_i and X_j within stations obtained from

$$S_{ij} = \sum_{s=1}^{p} S_{sij}$$

where S_{sij} is the pooled sum of squares and/or products within the sth

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1/ See for example Anderson, R. L. and Bancroft, T. A. Statistical Theory in Research. McGraw-Hill Book Co., Inc. 1952. Chapters 13, 14, 15, 16. or Snedecor, G. S. Statistical Methods. 4th Ed. Collegiate Press, Inc. Iowa State College, Ames, Iowa. 1946. Chapters 13, 14.

station obtained from

$$S_{sij} = \sum_{g=1}^{n_s} X_{sig}X_{sjg} - \frac{1}{n_s}\left(\sum_{g=1}^{n_s} X_{sig}\right)\left(\sum_{g=1}^{n_s} X_{sjg} \cdot\right)$$

Step 2 consists of getting correlation r_{iy} between the X variates and the dependent variate Y, from

$$r_{iy} = S_{iy} / \sqrt{S_{ii}S_{yy}}$$

where

$$S_{iy} = \sum_{s=1}^{6} S_{siy}$$

$$S_{yy} = \sum_{s=1}^{6} S_{syy}$$

$$S_{siy} = \sum_{g=1}^{n_s} X_{sig}Y_{sg} - \frac{1}{n_s}\left(\sum_{g=1}^{n_s} X_{sig}\right)\left(\sum_{g=1}^{n_s} Y_{sg}\right)$$

$$S_{syy} = \sum_{g=1}^{n_s} Y^2_{sg} - \frac{1}{n_s}\left(\sum_{g=1}^{n_s} Y_{sg}\right)^2$$

To obtain the coefficients $b'_1, b'_2, \ldots, b'_p$ it is now necessary to solve the normal equations

$$b'_1r_{11} + b'_2r_{12} + \ldots + b'_pr_{1p} = r_{1y} \tag{1}$$

$$b'_1r_{21} + b'_2r_{22} + \ldots + b'_pr_{2p} = r_{2y}$$

$$b'_1r_{p1} + b'_2r_{p2} + \ldots + b'_pr_{pp} = r_{py}$$

where of course the elements r_{ii}, $i = 1, \ldots, p$ are unity.

This may be done conveniently by the abbreviated Doolittle Method outlined for example by Anderson and Bancroft [1] (1950, S 15-3, I, II, III, IV, V, and VI)

Briefly, this method [2] consists of factorizing the matrix of correlation coefficients $R = \left[r_{ij}\right]_{p \times p}$ into the product of two triangular matrices $A = \left[A_{ij}\right]_{p \times p}$ and $B = \left[B_{ij}\right]_{p \times p}$ such that

$$A_{ij} = B_{ij} = 0, \ i \quad j, \ B_{ii} = 1, \text{ and } R = A'B$$

At the same time the column matrix of $G = \left[r_{iy}\right]_{p \times 1}$ is factored into the product of A and a new column matrix $J = \left[B_{iy}\right]_{p \times 1}$ such as that $G = A'J$.

Now the original equations (1) written in matrix form are

$$R \left[b'_i\right]_{p \times 1} = G$$

and since $B = A'^{-1}R$ and $J = A'^{-1}G$ we may now write

$$B \left[b'_i\right] = J$$

or more fully

$$b'_1 + b'_2B_{12} + \ldots + b'_pB_{1p} = B_{1y} \qquad (2)$$

$$b'_2 + \ldots + b'_pB_{2p} = B_{2y}$$

$$b'_p = B_{py}$$

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[1] Anderson and Bancroft. Op. Cit.

[2] Duncan, D. B. and Kenney, John F. On the Solution of Normal Equations and Related Topics. Edwards Brothers, Inc. Ann Arbor, Michigan. 1946

Since the B matrix has ones down the principal diagonal and zeros below, these equations (2) can be solved by the usual method of back solution working from the bottom member to the top, getting $b'_p, b'_{p-1}, \ldots, b'_1$ in that order.

In the process of getting the column matrix J, it is convenient to get the column matrix $K = A_{iy\ p \times 1}$ such that $G = K'B$. K bears the same relation to G as A does to R, and J is obtained from K using the same steps by which B is obtained from A.

The standardized coefficients b'_i are next converted to the required coefficients b_i using the relation

$$b_i = b'_i \sqrt{S_{yy} / S_{ii}} \qquad i = 1, \ldots, p$$

and finally the constants b_{os} for each station are then obtained from

$$b_{os} = \bar{Y}_s - b_1\bar{x}_{s1} - b_2\bar{x}_{s2} - \cdots - b_p\bar{x}_{sp}, \quad s = 1, \ldots, 6.$$

Analysis of Variance

The general analysis of variance for each of the problems can be put in the form shown in Table 1a.

Table 1a. General Analysis of Variance for Six Stations

Source of Variation	Sums of Squares	Degrees of Freedom
Regression	S_R	p
Error (Residual Within Stations)	S_e	N-p-6
Total Within Stations	S_{yy}	N-6
Between Stations	S_B	5
Total	S_T	N-1

The analysis shown in the last three rows of Table A1 for variation between and within stations is common to the fitting of all three functions. The terms involved are defined as

$$S_T = \sum_{sg} Y^2_{sg} - \left(\sum_{sg} Y_{sg}\right)^2 / N$$

$$S_B = \sum_{s} \left(Y^2_{s.}/n_s\right) - \left(\sum_{sg} Y_{sy}\right)^2 / N$$

$$S_{yy} = S_T - S_B \text{ , } N = \sum_{s} n_s$$

It will be noted that the definition given here for S_{yy} differs in form from the one previously given. However, as is well known, these are mathematically equivalent. The one given here is the simpler to use given that S_T is required as well as S_{yy}.

The next term to be obtained is the sum of squares for regression S_R using the relation

$$S_R = b_1 S_{1y} + b_2 S_{2y} + \ldots + b_p S_{py}$$

and finally the error sum of squares from

$$S_e = S_{yy} - S_R$$

Multiple Correlations

In defining a multiple correlation coefficient for these problems, there is the option of using the square root of the ratio S_R / S_{yy} or the square root of the ratio $(S_R + S_B) / S_T$. The latter ratio measures the proportion of variation explained, so to speak, by fitting all constants (including the station constants). The former measures the proportion of the within station variation measured by fitting the regression functions. Since the analysis for between and within stations is the same for the fitting of all functions investigated and since the definition

$$R = \sqrt{S_R / S_{yy}}$$

is more similar to the customary usage of multiple correlation coefficients, this is the definition which has been used throughout the study.

In the actual work, it was often a little more convenient to calculate R^2 before S_R from the relation

$$R^2 = b'_1 r_{1y} + b'_2 r_{2y} + \ldots + b'_p r_{py}$$

and then get the regression sum of squares from $S_R = R^2 S_{yy}$.

Appendix table 1b. Correlation Coefficients for the 4, 6 and 17-Variate Problems

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	1.00000000	.32594296	.28089474	.03196865	-.06938677	.13270544
X_2		1.00000000	.02518835	-.30021586	.05248603	.14247129
X_3			1.00000000	.07202162	-.09448609	.21956637
X_4				1.00000000	-.35850349	.02554078
X_5					1.00000000	-.01590528
X_6						1.00000000

Appendix table 1b (Cont'd)

	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
X_1	-.17483365	-.01088904	.13168074	.21505216	.27383731	-.26232262
X_2	-.75800079	.10367585	.45675253	.31664186	.13141262	-.31162240
X_3	.06754372	-.30674556	.32008195	.28365508	.76505574	-.46130543
X_4	.59369978	.32920558	-.59366770	.26315958	.17916884	.40329943
X_5	.22549732	-.06342717	.23945304	-.77501712	-.02997148	.10677245
X_6	-.06122296	.71510237	.26313637	.06390534	.23932856	.32301328
X_7	1.00000000	.06844710	-.44229685	-.34660732	.07640724	.46497825
X_8		1.00000000	-.03657067	-.01357678	-.02164351	.76375044
X_9			1.00000000	-.00144893	.39873885	-.22311631
X_{10}				1.00000000	.20545601	-.33095918
X_{11}					1.00000000	-.15314316
X_{12}						1.00000000

Appendix table 1b(Cont'd)

	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	Y
X_{1}	.04204054	.15063301	-.00286612	-.13257046	-.08570686	.43713173
X_{2}	.09965934	.18878660	-.12899763	-.53938563	-.09943815	.33217307
X_{3}	.29843216	.26526781	-.09726979	-.18692350	-.24169352	.79449269
X_{4}	-.32550668	-.18461683	.20646983	.46232645	.46087744	.01754156
X_{5}	-.44538731	.20204036	.28272966	-.05735822	-.05902337	-.08444454
X_{6}	.20749662	-.10444660	.43201456	.09639077	.37847850	.31930712
X_{7}	-.51108034	-.07439586	.31111665	.56910116	.30062060	-.14616846
X_{8}	-.06882394	-.19900226	.48049518	.34628595	.71960815	-.08901968
X_{9}	.53838112	.56635192	-.29854756	-.35559186	-.04806286	.41709691
X_{10}	.48988316	-.05134950	-.28725155	-.42553490	-.02319701	.33228277
X_{11}	.23088879	.68896325	.06753929	-.08948334	.06975705	.67168022
X_{12}	-.29508999	-.17236686	.41507785	.63555646	.72559040	-.37717263
X_{13}	1.00000000	.21457361	-.36593988	-.34855486	-.06268575	.24615314
X_{14}		1.00000000	-.05007081	-.19379004	.07298764	.24628782
X_{15}			1.00000000	.22282756	.45800298	-.08697914
X_{16}				1.00000000	.36759592	-.27343962
X_{17}					1.00000000	-.15195082
Y						1.00000000

Appendix table 2. Matrices A_{ij} , B_{ij} and Standard Regression Efficients*

	X_1	X_2	X_3	X_4
	1.00000000	.32594296	.28089474	.03196865
	1.00000000	.32594296	.28089474	.03196865
		.89376119	-.06636731	-.31063582
		1.00000000	-.07425620	-.34756020
			.91616996	.03997516
			1.00000000	.04363291
				.88926916
				1.00000000

*The ith row of A_{ij} and B_{ij} appear together with the former on top; $i = 1,\ldots,p$.

$b'_i, i = 1\ldots4$	.13370328	.28314787	.74659904	.04450145
$b'_i, i = 1\ldots6$	.13082483	.26786080	.72503877	.03672235
$b'_i, i = 1\ldots17$	.15791782	.59671291	.26131142	-.00205515

Appendix table 2 (cont'd)

X_5	X_6	X_7	X_8	X_9	X_{10}
-.06938677	.13270544	-.17483365	-.01088904	.13168074	.21505216
-.06938677	.13270544	-.17483365	-.01088904	.13168074	.21505216
.07510216	.09921689	-.70101499	.10722506	.41383212	.24654712
.08402934	.11101052	-.78434262	.11997059	.46302315	.27585346
-.06941891	.18965758	.06459886	-.29572476	.31382312	.24155571
-.07577078	.20701135	.07050969	-.32278373	.34253810	.26365819
-.32715382	.04750690	.35282543	.37972418	-.46773879	.33143484
-.36789066	.05342241	.39675887	.42700703	-.52598112	.37270475
.86325793	.01681343	.40696789	.04409691	.06551779	-.64057789
1.00000000	.01947672	.47143255	.05108196	.07589596	-.74204692
	.92924847	-.00034943	.74471803	.15846862	-.04723704
	1.00000000	-.00037603	.80141969	.17053417	-.05083359
		.08319775	.00032836	.03793586	.03780822
		1.00000000	.00394674	.45597218	.45443801
			.13033199	.08574390	-.03393877
			1.00000000	.65788837	-.26040245
				.33182372	.00942348
				1.00000000	.02839905
					.19449928
					1.00000000
-.00610775	.10355428				
-.56856068	.08559255	.56487692	-.14800510	.31577459	-.49349534

Appendix table 2 (Cont'd)

X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}
.27383731	-.26232262	.04204054	.15063301	-.00286612	-.13257046
.27383731	-.26232262	.04204054	.15063301	-.00286612	-.13257046
.04215728	-.22612019	.08595652	.13968883	-.12806344	-.49617522
.04716839	-.25299844	.09617392	.15629324	-.14328597	-.55515413
.69126672	-.40441121	.29300600	.23332855	-.10597422	-.18652924
.75451799	-.44141505	.31981621	.25467824	-.11567092	-.20359676
.15490485	.35074079	-.30976030	-.15106289	.16667566	.30225260
.17419344	.39441466	-.34833132	-.16987308	.18742993	.33988877
.09485262	.20596317	-.54144975	.16285920	.34658054	.07219886
.10987750	.23858822	-.62721665	.18865648	.40147971	.08363533
.04508609	.44389544	.15881377	-.18334683	.45289470	.19012467
.04851887	.47769295	.17090560	-.19730657	.48737740	.20460047
.00244843	.03418429	-.07875374	.02814131	-.01170694	.01601733
.02942904	.41087998	-.94658497	.33824605	-.14071222	.19252119
.09227707	.14131316	.04886164	.06420654	.00983284	.05897530
.70801551	1.08425537	.37490136	.49263838	.07544456	.45250057
.11113972	.03924821	.24752623	.28629130	-.21957431	.03047328
.33493603	.11828030	.74595701	.86278130	-.66171975	.09183575
-.01261719	-.04071835	.14310138	-.02016458	.00882929	-.25317202
-.06487011	-.20934962	.73574247	-.10367432	.04539498	-1.30166045
.25840953	.01238209	-.00590993	.33890546	.13273382	-.02770195
1.00000000	.04791654	-.02287040	1.31150527	.51365683	-.10720174
	.11491701	.02285940	-.02171024	-.01401801	.03984803
	1.00000000	.19892094	-.18892103	-.12198377	.34675485
		.03413261	.01695880	.02677955	.03049700
		1.00000000	.49685037	.78457376	.89348573
			.05624165	.04909478	-.03183371
			1.00000000	.87292567	-.56601664
				.29642542	.00389001
				1.00000000	.01312306
					.09710650
					1.00000000
.49992713	-.03382118	.06664158	-.44270642	.04852825	-.28511626

Appendix table 2 (Cont'd)

X_{17}	Y	R^{2*}
-.08570686	.43713173	.19108415
-.08570686	.43713173	
-.07150260	.18969306	.23134486
-.08000191	.21224133	
-.22292843	.68579057	.74468706
-.24332650	.74854077	
.44849293	.03957377	.74644815
.50433879	.04450145	
.08914293	-.00353146	.74646259
.10326338	-.00409085	
.41824274	.09622766	.75642738
.45008709	.10355428	
.02546108	.01668614	.75977395
.30603087	.20055999	
.12394373	.02044224	.76298027
.95098471	.15684745	
.13734979	.02040925	.76423556
.41392396	.06150630	
.01079332	-.01018259	.76476865
.05549285	-.05235284	
.02278436	-.00804674	.76501922
.08817152	-.03113947	
.02873525	-.00570672	.76530262
.25005219	-.04965949	
.02178757	-.01332111	.77050151
.63832124	-.39027516	
.07340618	-.01577245	.77492475
1.30519250	-.28044074	
.09329225	.01031126	.77528343
.31472419	.03478534	
-.02052902	-.02703427	. 78280972
-.21140727	-.27839815	
.17332234	-.00550784	.78298475
1.00000000	-.03177802	

*Progressive values of R^2

-.03177802

Appendix table 3. Basic Data (Coded)

Station 1	(1)	(2)	(3)	(4)
(s = 1)	463($Y_{1\ 1}$)	186($X_{11\ 1}$)	-23($X_{21\ 1}$)	-16($X_{31\ 1}$)
	780($Y_{1\ 2}$)	189($X_{11\ 2}$)	-23($X_{21\ 2}$)	1($X_{31\ 2}$)
	570	177	5	-17
	564	188	-29	- 8
	657	188	-17	- 1
	332	169	0	-21
	1116	232	-17	40
	597	211	-40	9
	843	246	- 6	7
	578	227	-29	- 3
	818	249	6	5
	828	210	- 4	10
	546	233	2	-17
	598	216	-15	-24
	972	225	-17	29
	569	203	-36	3
	859	238	-34	21
	574	226	-32	- 3
	726	203	-11	3
	872	249	9	2
	659	249	- 9	- 9
	631	204	-19	- 6
	617	239	- 3	-21
	548	270	- 7	-20
	669	274	-16	-11
	586	268	-29	8
	913	268	-20	25
	871	285	19	11
	548	256	-34	-10
	816	300	4	- 6
	516	274	34	-19
	567	273	-22	-12
	928	267	-39	7
	662	256	-27	- 7
	713	267	-19	18
	776	250	-14	8
	815	289	- 8	8
	591	289	-23	-10
	667	276	7	-20
	1363	326	- 7	37
	947	338	- 9	16
	672	303	10	-16
	961	313	- 5	17
	624	314	-16	3
	635($Y_{1\ 45}$)	304($X_{11\ 45}$)	6($X_{21\ 45}$)	-20($X_{31\ 45}$)

Appendix table 3 (Cont'd)

Station 2	(1)	(2)	(3)	(4)
(s = 2)	501($Y_{2\ 1}$)	288($X_{12\ 1}$)	-16($X_{22\ 1}$)	- 8($X_{32\ 1}$)
	683($Y_{2\ 2}$)	278($X_{12\ 2}$)	-27($X_{22\ 2}$)	- 3($X_{32\ 2}$)
	473	252	-10	-31
	867	298	-21	6
	552	273	-10	-31
	608	325	-33	- 8
	608	303	59	-17
	914	310	-12	2
	697	326	-16	-12
	624	304	0	-20
	709	302	- 2	-19
	945	331	25	- 7
	440	318	- 6	-27
	548	314	-15	- 6
	810	304	- 8	- 1
	765	326	- 3	-17
	939	329	30	- 6
	770	395	0	- 6
	986	399	23	- 6
	1003	386	0	17
	852	366	17	- 3
	767	368	33	-22
	780	350	-39	4
	463	386	9	- 4
	789	425	16	-14
	1412	407	24	34
	1198	414	10	14
	1208	402	11	16
	1033	411	13	4
	845	438	0	- 9
	926	407	32	-21
	562	406	-17	-29
	903	425	11	- 6
	1153	442	7	13
	1176	418	15	13
	820	416	27	-26
	652	427	-18	-25
	918	451	7	16
	835($Y_{2\ 39}$)	463($X_{12\ 39}$)	31($X_{22\ 39}$)	- 9($X_{32\ 39}$)

Appendix Table 3 (Cont'd)

	(1)	(2)	(3)	(4)
Station 3 (s = 3)	1025($Y_{3\ 1}$)	294($X_{13\ 1}$)	40($X_{23\ 1}$)	-33($X_{33\ 1}$)
	867($Y_{3\ 2}$)	286($X_{13\ 2}$)	26($X_{23\ 2}$)	-27($X_{33\ 2}$)
	630	291	17	-28
	1104	316	33	- 3
	816	308	33	-14
	1288	322	25	4
	904	304	29	-11
	1003	378	15	- 5
	692	358	48	-31
	505	367	20	-33
	741	375	18	-15
	1027	422	44	-26
	958	416	52	-28
	855	466	28	-13
	690($Y_{3\ 15}$)	456($X_{13\ 15}$)	14($X_{23\ 15}$)	- 7($X_{33\ 15}$)
Station 4 (s = 4)	704($Y_{4\ 1}$)	294($X_{14\ 1}$)	- 2($X_{24\ 1}$)	-12($X_{34\ 1}$)
	974($Y_{4\ 2}$)	320($X_{14\ 2}$)	-41($X_{24\ 2}$)	7($X_{34\ 2}$)
	712	328	-36	0
	704	340	-22	- 1
	879	336	-25	7
	717	332	-27	- 7
	522	333	-28	-15
	434	335	-31	-16
	603	305	-19	-19
	573	301	-23	-21
	721	337	-41	1
	408	305	-26	-18
	909	397	- 8	- 3
	539	377	-36	-17
	904	359	-14	3
	964	372	2	- 4
	644	392	-19	-11
	724	372	-21	-11
	884	396	-8	- 4
	869	377	- 4	-14
	836	396	-12	2
	804	367	-21	4
	655	358	-24	- 5
	723	399	-20	- 6
	321	419	-38	-25
	941	443	-16	5
	626	433	-23	-20
	782	446	-19	- 2
	762	401	-20	- 8
	654	436	-24	-14
	1461	514	16	25
	1288($Y_{4\ 32}$)	523($X_{14\ 32}$)	0($X_{24\ 32}$)	36($X_{34\ 32}$)

Appendix table 3 (Cont'd)

Station 5	(1)	(2)	(3)	(4)
(s = 5)	1097($Y_{5\ 1}$)	396($X_{15\ 1}$)	- 5($X_{25\ 1}$)	17($X_{35\ 1}$)
	1406($Y_{5\ 2}$)	435($X_{15\ 2}$)	8($X_{25\ 2}$)	18($X_{35\ 2}$)
	766	429	9	-13
	1254	444	6	14
	740	454	0	-14
	1001	466	5	9
	844	479	21	7
	1003	468	-14	- 1
	1573	478	10	16
	740	452	- 3	-12
	1689	526	14	29
	1646	502	9	41
	1071	513	29	- 5
	1768	540	7	30
	1752	535	19	44
	1439	516	35	13
	1123	532	22	- 4
	1155	553	2	14
	1103	588	41	3
	1121	583	- 8	-10
	934	550	11	8
	1097($Y_{5\ 22}$)	603($X_{15\ 22}$)	- 1($X_{25\ 22}$)	- 8($X_{35\ 22}$)
Station 6				
(s = 6)	613($Y_{6\ 1}$)	276($X_{16\ 1}$)	13($X_{26\ 1}$)	-19($X_{36\ 1}$)
	726($Y_{6\ 2}$)	346($X_{16\ 2}$)	-24($X_{26\ 2}$)	- 3($X_{36\ 2}$)
	834	389	- 5	10
	691	397	- 4	1
	642	356	- 5	- 8
	694	357	-12	- 9
	765	396	13	- 6
	799	372	2	-11
	1327	462	20	24
	832	455	-16	-14
	1678	540	13	41
	1458	545	23	1
	580	552	6	-24
	1166($Y_{6\ 14}$)	622($X_{16\ 14}$)	7($X_{26\ 14}$)	15($X_{36\ 14}$)

Appendix table 4. Forage and Grain Combinations to Produce 6 Different Levels of Milk Output*

X_2 \ Y	5500	6500	7500	8500	9500	10,500
	X_3 for Various Levels of Milk Output					
4000	2454	3550	4504	5362		
5000	2001	3131	4108	4982	5779	
6000	1555	2720	3721	4610	5419	6166
7000	1116	2318	3341	4246	5067	5823
8000	686	1925	2970	3890	4723	5488
9000	264	1541	2608	3543	4386	5160
10000		1168	2256	3205	4059	4841
11000		805	1914	2877	3741	4530
12000		455	1583	2559	3431	4228
13000		117	1263	2250	3131	3934

* Represents an "average" cow at an "average" station computed with the 6-variate function

Appendix table 5. Forage and Grain Combinations to Produce 7 Different Levels of Milk Output. h = 850*

	X_3 for Various Levels of Milk Output						
X_2 \ Y	5500	6500	7500	8500	9500	10500	11500
4000	2795	3380	4093	5011			
5000	2202	2972	3817	4760	5800		
6000	1668	2569	3522	4501	5463	6392	
7000	1188	2171	3213	4231	5133	5923	6656
8000	760	1779	2883	3945	4814	5501	6102
9000	384	1395	2526	3638	4509	5145	5668
10000	63	1018	2134	3298	4223	4868	5384
11000		650	1695	2910	3965	4700	5273
12000		294	1187	2447	3752	4674	

* Represents "350 pound" cow at an "average" station or "450 pound" cow at a low station computed with the 17-variate function.

Appendix table 6. Forage and Grain Combinations to Produce 4 Different Levels of Milk Output. h = 700*

X_2 \ Y	X_3 for Various Levels of Milk Output: 5500	6500	7500	8500
4000	3850	4451	5182	6093
5000	3169	4098	5041	5970
6000	2590	3761	4920	5875
7000	2092	3444	4853	5827
8000	1669	3155		
9000	1322	2912		
10000	1063	2757		
11000	924			

* Represents a "200 pound" cow at a "low" station computed with the 17-variate function.

Appendix table 7. Forage and Grain Combinations to Produce 6 Different Levels of Milk Output. h = 800*

	X_3 for Various Levels of Milk Output					
X_2 \ Y	5500	6500	7500	8500	9500	10500
4000	3095	3694	4423	5352		
5000	2484	3298	4179	5135	6154	
6000	1940	2908	3921	4919	5848	6721
7000	1456	2525	3655	4700	5557	6275
8000	1029	2151	3376	4481	5283	5886
9000	659	1787	3078	4259	5036	5562
10000	353	1435	2752	4034	4827	5325
11000	117	1099	2382	3802	4676	5202
12000		801	1928	3553	4625	

* Represents "350 pound" cow at an "average" station or "450 pound" cow at a low station computed with the 17-variate function.

Appendix table 8. Forage and Grain Combinations to Produce 7 Different Levels of Milk Output. h = 900*

	X_3 for Various Levels of Milk Output						
X_4 \ Y	5500	6500	7500	8500	9500	10500	11500
4000	2526	3193	3786	4686	5820		
5000	1946	2678	3485	4406	5453	6594	
6000	1419	2263	3163	4113	5088	6062	7028
7000	943	1855	2823	3801	4724	5565	6355
8000	513	1451	2460	3467	4363	5114	5775
9000	132	1052	2067	3102	4006	4722	5316
10000		659	1639	2695	3653	4405	5008
11000		273	1168	2230	3308	4190	4884
12000			673	1688	2976	4111	

* Represents a "450 pound" cow at an "average" station or a "350 pound" cow at a "high" station computed with the 17-variate function.

Appendix table 9. Forage and Grain Combinations to Produce 8 Different Levels of Milk Output. h = 1000*

	X_3 for Various Levels of Milk Output							
X_2 \ Y	5500	6500	7500	8500	9500	10500	11500	12500
4000	2050	2579	3226	4079	5193			
5000	1488	2150	2888	3751	4780	5968		
6000	973	1724	2527	3403	4361	5388	6450	
7000	501	1300	2145	3031	3936	4836	5721	6601
8000	70	879	1739	2628	3502	4319	5080	5815
9000		462	1304	2187	3057	3852	4560	5218
10000		49	836	1700	2599	3446	4197	4872
11000			331	1154	2121	3127	4032	4815
12000				540	1616	2926		

* Represents a "500 pound" cow at a "high" station computed with the 17-variate function.