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Properties of $Q(X,\wp)$ spaces

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Theodore Roy Bambenek

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I. INTRODUCTION AND HISTORICAL SURVEY

The space $Q(X,\mathcal{P})$ (Definition 2.4), of all quasi-continuous functions on a non void set X relative to a given pre-algebra $\mathcal{P}$, has become of interest since it arises both in problems in pure functional analysis and in the application of functional analysis to signal processing. For example, in [4] J. A. Dyer used these spaces as models for certain classes of duration limited signals and R. E. Lane [17], in 1955 through 1962, used special types of $Q(X,\mathcal{P})$ spaces to study linear stationary systems. The applications of $Q(X,\mathcal{P})$ spaces in abstract functional analysis and related ideas have been considered by J. A. Dyer in [3], [6], and [7], and by W. B. Johnson in [13]. As a result of these applications, the properties of $Q(X,\mathcal{P})$ spaces have been investigated by mathematicians since the 1930's in special cases. These abstract spaces have been studied since the late 1960's. T H. Hildebrandt [12] characterized sets with compact closure in $QC([a,b])$ (Example 3.11) and this was later extended to a certain class of $Q(X,\mathcal{P})$ spaces [9] by J. A. Dyer and W. B. Johnson. The representation of linear operators on $Q(X,\mathcal{P})$ spaces when X is an interval or the real line has been considered by Kaltenborn [14], Hildebrandt [11], Lane [17] and [18], and Baker [1]. Dyer in [3] and [5] has considered the abstract operator representation problem. The problem of solving linear operator equations in special $Q(X,\mathcal{P})$ spaces has been considered in numerous papers by McNerney, Hinton, Hildebrandt, and Lane. A bibliography of these results can be found in [5]. Dyer has studied the abstract case in [5].

Up to this time almost all of the research in $Q(X,\mathcal{P})$ spaces has been devoted to properties of these spaces with the norm topology, and very little attention has been given to weak topological properties of these spaces. Recall that the weak topology $\mathcal{T}$ for a normed linear space is the smallest topology, with respect to set inclusion, such that every linear functional, continuous for the norm topology, is also continuous for $\mathcal{T}$. However, as soon as one considers the problem of the best sup norm approximation of an element of a $Q(X,\mathcal{P})$ space by elements of a given subspace of the space then one must concern oneself with the weak properties of $Q(X,\mathcal{P})$ spaces, in particular with weak sequential compactness and related matters. Since the uniform approximation problem has many important applications in signal processing it would seem that research into these areas is overdue. In this dissertation, we begin the study of some of these questions.

There are several ways to investigate $Q(X,\wp)$ spaces. One method is to take a property of the well known space $QC([a,b])$ and attempt to extend it to all $Q(X,\wp)$ spaces. A second method is to use the fact that $Q(X,\wp)$ is a closed subspace of $B(X)$, the space of all bounded complex valued functions defined on X. Because of this, some of the topological properties of $B(X)$ hold automatically for every $Q(X,\wp)$ space. In this thesis, a combination of both methods will be used to analyze several properties of these spaces. The basis for this investigation will be a new concept, that of a fundamental net of points (Definition 3.4). This is a concept of extreme importance in that it not only allows a complete characterization of $Q(X,\wp)$ but it is also neatly applicable to the study of the properties of these spaces.

The dissertation itself is divided into five chapters. Since a great deal of this work is dependent upon a firm understanding of $Q(X,\wp)$ spaces, chapter two contains a summary of all pertinent definitions and elementary properties of these spaces. Most of these results are due to J. A. Dyer [3], [5], E. M. Eltze [10], and R. A. Shive [19].

The main results of the dissertation are given in chapters three and four. In chapter three the concept of a fundamental net of points is introduced. In order to illustrate this concept, a detailed study of the fundamental nets of points is given for several important quasi-continuous function spaces. These examples are not only important in their own right, but also serve to keep the abstract results of this thesis in perspective. The rest of chapter three contains two major applications of the concept of a fundamental net of points. The first of these is a new characterization of $Q(X,\mathcal{P})$ spaces (Theorem 3.3) which is a generalization of the classical one sided limit characterization of $QC([a,b])$. The second application is to the study of the weak topology for $Q(X,\mathcal{P})$ spaces. In Theorem 3.7 necessary and sufficient conditions for a sequence $\{f_n\}_{n=1}^{\infty}$ to converge weakly to f_o in $Q(X,\mathcal{P})$ are given. This result improves those in [2,p.281], [18] and [20]. As a corollary (Corollary 3.1), one obtains some improved conditions for the interchange of limits and integration for the ψ integral (Definition 2.6).

Chapter four also contains applications of the notion of a fundamental net of points. In it, the extreme points of the closed unit ball of the adjoint space of $Q(X,\wp)$, which will be denoted by $(Q(X,\wp))^*$, are characterized. Recall that if G is a subspace of the complex normed linear space E and if f is in $E-G$ then an element g_o in G is said to be a best uniform approximation to f by G if and only if $\|f-g_o\|$ equals $\inf\|f-g\|$ where g ranges over all the elements of G. The characterization of the extreme points (Theorem 4.3) is used to give necessary and sufficient conditions for best uniform approximations by subspaces of $Q(X,\wp)$. The method used is similar to that used by I. Singer [21] in his study of best uniform approximations by subspaces in $C([a,b])$.

Chapter five contains a brief summary of the dissertation and a few notes concerning future research in $Q(X,\wp)$ spaces.

II. EXAMPLES AND PROPERTIES OF $Q(X,\mathcal{P})$ SPACES

Throughout the remainder of this work the following conventions will be used. If a and b are real numbers and $a < b$ then $[a,b]$ and (a,b) will denote the closed interval whose endpoints are a and b, and the open interval whose endpoints are a and b respectively. The empty set will be denoted by $\emptyset$. The complex conjugate of a number x will be denoted by x^*. The following definition has its origin in [23,p.17] and differs from A. C. Zaanen's definition only in that we do not require X to be in $\mathcal{P}$.

<u>Definition 2.1</u>. Let X be a nonvoid set and $\mathcal{P}$ a nonvoid collection of subsets of X. Then $\mathcal{P}$ is said to be a pre-algebra of subsets of X if and only if the following three conditions are satisfied;

(1) if A,B are in $\mathcal{P}$ then $A \cap B$ is in $\mathcal{P}$,

(2) if A,B are in $\mathcal{P}$ then there exists a finite disjoint collection of sets $\{E_i\}_{i=1}^{p}$ in $\mathcal{P}$ such that

$$A - B = \bigcup_{i=1}^{p} E_i,$$

(3) $\mathcal{P}$ contains a finite disjoint collection $\{G_k\}_{k=1}^{r}$ of sets such that $X = \bigcup_{k=1}^{r} G_k$.

In the remainder of this work $\mathcal{P}$ will always denote a pre-algebra of subsets of a nonvoid set X. The pair $(X,\mathcal{P})$ will be called a volume pair. There are many examples of volume pairs and a few of them will now be listed for future reference.

Example 2.1. Let $X = [a,b]$ and let $\mathcal{P}$ consist of all open subintervals of $[a,b]$, singleton subsets of $[a,b]$, and the empty set. Then $(X,\mathcal{P})$ is a volume pair [5].

Example 2.2. Let $X = [a,b]$ and let $\mathcal{P}$ be the collection of sets $\{(c,d] : a \leq c < d < b\} \cup \{\{a\},\emptyset\}$. Then $(X,\mathcal{P})$ is a volume pair [4].

Example 2.3. Let $X = R$ the set of all real numbers and let $\mathcal{P}$ be the collection of all sets of the form $(a,b),(a,\infty),(-\infty,b),\{a\}$ along with the empty set. It is straightforward to verify that $(X,\mathcal{P})$ is a volume pair.

<u>Example 2.4</u>. Let X be a set with an infinite number of elements and let $\mathcal{P}$ consist of all subsets A of X such that $X - A$ contains only a finite number of elements, or $A = \{a\}$, or A is the empty set. To verify that $(X,\mathcal{P})$ is a volume pair we need only show that all three conditions of Definition 2.1 are satisfied. Suppose A and B are in $\mathcal{P}$. If either A or B is a singleton, or the empty set, then $A \cap B$ is in $\mathcal{P}$. If neither A nor B is a singleton or the empty set then

$$X - (A \cap B) = (X - A) \cup (X - B)$$

which is a finite set since both $X - A$ and $X - B$ are finite sets. Thus, condition (1) is satisfied. Conditions (2) and (3) are easily verified since every singleton belongs to $\mathcal{P}$. Hence, $(X,\mathcal{P})$ is a volume pair.

<u>Example 2.5</u>. Let X be any nonvoid set and let $\mathcal{P} = \{A_1,\ldots,A_N\}$ where $\bigcup_{i=1}^{N-1} A_i = X, A_i \cap A_j = \phi$ if $i \neq j$ and $A_N = \phi$. Then $(X,\mathcal{P})$ is a volume pair.

Example 2.6. Let X be the set of all positive integers and let $\mathcal{P}$ consist of all subsets of X of the form $\{N, N+1, \ldots\}, \{N\}$ and the empty set, where N is any positive integer. Since this case is similar to Example 2.4, we see that $(X,\mathcal{P})$ is a volume pair.

Example 2.7. Let X be any nonvoid set and let $\mathcal{P}$ be the collection of all subsets of X. It is trivial to verify that $(X,\mathcal{P})$ is a volume pair.

Definition 2.3. ([4],p.6) Let $(X,\mathcal{P})$ be a volume pair. A disjoint collection $\{G_k\}_{k=1}^{r}$ of sets in $\mathcal{P}$ is said to be a $\mathcal{P}$-subdivision of X if and only if $\bigcup_{k=1}^{r} G_k = X$.

The following theorem is due to J. A. Dyer [3]. It gives the main properties of $\mathcal{P}$-subdivisions and is included for reference without proof.

Theorem 2.1. For every volume pair $(X,\mathcal{P})$ the following statements hold.

(1) the collection of all $\mathcal{P}$-subdivisions is directed by refinement and

(2) if E is in $\mathcal{P}$ then there exists a subdivision to which E belongs.

As we noted in the introduction, this dissertation is concerned with quasi-continuous function spaces. The following definition, taken from [3,p.473], explains the terminology and symbolism that will be used in the remainder of this work.

Definition 2.4. For the volume pair $(X,\mathcal{P})$, $Q(X,\mathcal{P})$, the quasi-continuous functions on X relative to $\mathcal{P}$, will denote the linear space of all complex valued functions on X which are uniformly approximatable by finite linear combinations of characteristic functions of sets in $\mathcal{P}$. $Q(X,\mathcal{P})$ will be assumed to be topologized with the sup norm topology.

As with most definitions, the definition of $Q(X,\mathcal{P})$ has several equivalent representations. In the following theorem, $Q(X,\mathcal{P})$ is shown to be a Banach space and an equivalent characterization for $Q(x,\mathcal{P})$ is given. The theorem is taken from [5] and will be used without proof.

Theorem 2.2. For every $Q(X,\mathcal{P})$ space we have the following:

(1) $Q(X,\mathcal{P})$ is a Banach space and

(2) a complex valued function f on X is an element of $Q(X,\mathcal{P})$ if and only if for every positive number ϵ there exists a $\mathcal{P}$-subdivision $\{D_j\}_{j=1}^{N}$ of X such that if p,q are in D_j then $|f(p) - f(q)| < \epsilon$ for $j = 1,\ldots,N$.

Theorem 2.2 is a very useful result and will be applied later (Theorem 3.3) to give a new characterization of $Q(X,\mathcal{P})$. Since a major purpose of this dissertation is to investigate properties of the weak topology for $Q(X,\mathcal{P})$, we will need several definitions and theorems concerning the general structure of continuous linear functionals on $Q(X,\mathcal{P})$. All of the following results are well known and an appropriate reference is given in each instance.

Definition 2.5. [4] Let $(X,\mathcal{P})$ be a volume pair. A finitely additive function u on $\mathcal{P}$ into the complex number field is said to be a p-volume; u is called a p-volume of bounded variation if and only if the net

$$\left\{ \sum_{i=1}^{p} |u(D_i)| : \{D_i\}_{i=1}^{p} \text{ is a } \mathcal{P}\text{-subdivision of } X \right\}$$

has a finite supremum. This supremum will be denoted by V_u.

Definition 2.6. [2,p.469] Let $(X,\mathcal{P})$ be a volume pair, u a p-volume on $\mathcal{P}$, and ψ a choice function (that is, $\psi(D) \in D$ for all nonvoid D in $\mathcal{P}$) for $\mathcal{P} - \{\emptyset\}$. A complex valued function f defined on X is said to be ψ-integrable with respect to u if and only if the net

$$\left\{ \sum_{i=1}^{p} f(\psi(D_i))u(D_i) : \{D_i\}_{i=1}^{p} \text{ is a } \mathcal{P}\text{-subdivision of } X \right\}$$

converges. The limit of this net, when it exists, will be denoted by $\psi \int_X f\,du$.

The above integral is known as the ψ integral and was introduced and developed by J. A. Dyer in [3], [4], and [5]. The following two theorems are very important for the remainder of this work and both results can be found in [5].

<u>Theorem 2.3</u>. Let $(X,\mathcal{P})$ be a volume pair and let ψ_1, ψ_2 be choice functions for $\mathcal{P} - \{\emptyset\}$. If u is a p-volume of bounded variation then $\psi_1 \int_X f\,du$ and $\psi_2 \int_X f\,du$ both exist and are equal for every $f \in Q(X,\mathcal{P})$.

<u>Theorem 2.4</u>. Let $(X,\mathcal{P})$ be a volume pair and let Φ be a continuous linear functional on $Q(X,\mathcal{P})$. Then, there exists a p-volume u of bounded variation such that $\Phi(f) = \psi \int_X f\,du$. Conversely, if u is a p-volume of bounded variation on $\mathcal{P}$ then $\Phi(f) = \psi \int_X f\,du$ is a continuous linear functional on $Q(X,\mathcal{P})$. Moreover, $\|\Phi\| = V_u$.

Note that ψ can be any choice function, in Theorem 2.4, because of the result given in Theorem 2.3. The fact that ψ is an arbitrary choice function will be of use later. Now that the members of $(Q(X,\mathcal{P}))^*$ have been characterized, we are ready to investigate the weak topology of $Q(X,\mathcal{P})$. We begin by introducing a new concept which will be used to characterize and analyze $Q(X,\mathcal{P})$ spaces.

III. FUNDAMENTAL NETS

Suppose (X,ρ) is a volume pair. If f is a function from X into the complex number field, then it can be difficult to apply either Definition 2.4 or Theorem 2.2 to determine whether or not f is in $Q(X,\rho)$. For the special (X,ρ) volume pair considered in Example 2.1, the task is much easier because it is well known, ([19],p.31), that f is an element of $Q(X,\rho)$ if and only if $\lim_{x\to y^+} f(x)$, $\lim_{x\to y^-} f(x)$, $\lim_{x\to a^+} f(x)$, and $\lim_{x\to b^-} f(x)$ exist for all y in (a,b). This special quasi-continuous function space is of considerable interest in that many physical systems can be modelled by using it ([3], [17]). For the rest of this dissertation this $Q(X,\rho)$ space will be denoted by $QC([a,b])$. A natural question arises; is there any way to extend the classic idea of a one sided limit in $QC([a,b])$ to an arbitrary quasi-continuous function space? To attack this problem some preliminary definitions and theorems are needed.

Definition 3.1. Let $(X,\mathcal{P})$ be a volume pair. A net of nonempty sets $\{D_\delta\}$ in $\mathcal{P}$ is said to be a fundamental net of sets if and only if

(1) $D_{\delta'}$ is contained in D_δ if δ' follows δ;

(2) if $\{F_j\}_{j=1}^p$ is a $\mathcal{P}$-subdivision of X then there exists an F_j and a δ so that D_δ is contained in F_j.

Theorem 3.1. Suppose $\{D_\delta\}$ is a fundamental net of sets for the volume pair $(X,\mathcal{P})$. Then:

(1) $D_\delta \cap D_{\delta'}$ is not empty for any δ and δ';

(2) if $\{F_j\}_{j=1}^p$ is a $\mathcal{P}$-subdivision of X then there exists a j and a $\bar{\delta}$ such that D_δ is contained in F_j for all δ which follow $\bar{\delta}$.

Proof:

(1) There exists a δ_1 such that δ_1 follows both δ and δ'. Thus, the nonempty set D_{δ_1} is contained in $D_\delta \cap D_{\delta'}$ by virtue of condition (1) of Definition 3.1.

(2) There exists a j and a $\bar{\delta}$ such that $D_{\bar{\delta}}$ is contained in F_j. If δ follows $\bar{\delta}$, D_δ is contained in $D_{\bar{\delta}}$, and so is contained in F_j.

Although Definition 3.1 is fairly easy to understand, it is not obvious that there exists a fundamental net of sets for an arbitrary volume pair $(X,\mathcal{P})$. The following theorem guarantees that every volume pair has at least one fundamental net of sets. As well as being an existence theorem, Theorem 3.2 is an important tool and will be used in the proof of many of the results of this dissertation.

<u>Theorem 3.2</u>. Let $(X,\mathcal{P})$ be a volume pair and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then there exists a fundamental net of sets $\{D_\delta\}$ such that $\{x_n\}_{n=1}^{\infty}$ is frequently in each D_δ.

Proof: Let $\mathcal{G}$ be the set $\{A \in \mathcal{P} : \{x_n\}_{n=1}^{\infty}$ is frequently in $A\}$. Note that $\mathcal{G}$ is not empty since if $\{E_j\}_{j=1}^{N}$ is a $\mathcal{P}$-subdivision of X then $\{x_n\}_{n=1}^{\infty}$ must be frequently in at least one E_j because each $\mathcal{P}$-subdivision contains only a finite number of sets. Let $\mathfrak{F}$ be the collection of all subsets M of $\mathcal{G}$ such that if A_1 and A_2 are in M then there exists an A in M such that A is a subset of $A_1 \cap A_2$. $\mathfrak{F}$ is not empty since if A is in $\mathcal{G}$ then $\{A\}$ is in $\mathfrak{F}$. Partially order $\mathfrak{F}$ by set inclusion and let $\{M_\alpha\}$ be a chain of elements in $\mathfrak{F}$. Finally, let T be the set $\bigcup_\alpha M_\alpha$. If A_1 and A_2 are elements of T then there exist sets M_1 and M_2 in $\{M_\alpha\}$ such that A_1 is in M_1 and A_2 is in M_2. Since $\{M_\alpha\}$ is a chain we see that $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$. If M_1 is contained in M_2, it follows from the definition of $\mathfrak{F}$, that there exists an A in M_2 such that A is contained in $A_1 \cap A_2$. Since $M_2 \subseteq T$ we see that A is in T and so T is an element of $\mathfrak{F}$. A similar result follows if M_2 is a subset of M_1. Thus, T is an upper bound for $\{M_\alpha\}$. It follows from Zorn's Lemma that $\mathfrak{F}$ contains a maximal element, with respect to set inclusion. This element will be denoted by $\bar{M}$. It is easily verified that $\bar{M}$ is a directed set with respect to ordering by set inclusion. Observe that no element of $\bar{M}$ is empty and that for set inclusion ordering, $\bar{M}$ satisfies condition (1) of Definition 3.1. Let $\{E_i\}_{i=1}^N$ be a $\mathcal{P}$-subdivision of X. Suppose that no A in $\bar{M}$ is contained in any of the E_i, $i = 1,\ldots,N$. Then there exists an E_i such that $E_i \cap A$ is in $\mathcal{G}$ for all A in $\bar{M}$ or there doesn't. In the first case, let $\hat{M}$ be the set

$$\overline{M} \cup \{E_i \cap A : A \text{ is in } \overline{M}\}$$

and suppose that C_1 and C_2 are elements of $\hat{M}$. Either C_1 and C_2 are both elements of $\overline{M}$, or C_1 is in $\overline{M}$ and $C_2 = E_i \cap A$ with A in $\overline{M}$, or $C_1 = E_i \cap A_1$ and $C_2 = E_i \cap A_2$ with A_1 and A_2 in $\overline{M}$. In the first case it follows from the hypothesis on $\overline{M}$ that there exists a C in $\overline{M}$, and hence in $\hat{M}$, such that C is contained in $C_1 \cap C_2$. In the second case

$$C_1 \cap C_2 = C_1 \cap E_i \cap A = (C_1 \cap A) \cap E_i \supseteq C \cap E_i$$

and this last set is in $\hat{M}$ because C_1 and A are in $\overline{M}$ and C, in $\overline{M}$ by the hypothesis on $\overline{M}$, is a subset of $C_1 \cap A$. Finally, in the third case $C_1 \cap C_2$ equals $E_i \cap (A_1 \cap A_2)$ which contains $E_i \cap A$ an element of $\hat{M}$, where A is contained in $A_1 \cap A_2$ and is an element of $\overline{M}$. Therefore, $\hat{M}$ is an elment of $\mathfrak{F}$. But, $\overline{M}$ is contained in $\hat{M}$ and since $\overline{M}$ is a maximal element of $\mathfrak{F}$ with respect to set inclusion we see that $\overline{M}$ equals $\hat{M}$. This in turn implies that $E_i \cap A$ is in $\overline{M}$ for all A in $\overline{M}$ and so

$E_i \cap A$ is a subset of E_i contrary to our basic assumption on $\{E_i\}_{i=1}^N$. Thus, it follows that for each E_i, $i = 1,\ldots,N$, there exists an A_i in $\overline{M}$ such that $E_i \cap A_i$ is not in $\mathcal{G}$. Now, since $\overline{M}$ is in $\mathfrak{F}$ we can select an A in $\overline{M}$ so that A is a subset of $\bigcap_{i=1}^N A_i$. Noting that $A \cap E_i$ is contained in $A_i \cap E_i$, which is not in $\mathcal{G}$, it follows easily that $A \cap E_i$ is not in $\mathcal{G}$ for any $i = 1,\ldots,N$. However, $\bigcup_{i=1}^N (A \cap E_i)$ equals A, which is in $\mathcal{P}$, and since $\{x_n\}_{n=1}^\infty$ is eventually not in $A \cap E_i$, $i = 1,\ldots,N$, it follows that $\{x_n\}_{n=1}^\infty$ is eventually not in A which is false. Thus, we have a contradiction and so $\overline{M}$ is a fundamental net of sets which has the desired property.

Theorem 3.2 shows that every volume pair $(X,\mathcal{P})$ has at least one fundamental net of sets associated with it. For some volume pairs it is possible to characterize all fundamental nets of sets. Many of the theorems that follow are useful only because such characterizations are possible. The following examples are extremely important and will be used extensively to illustrate our work.

Example 3.1. Let $(X,\mathcal{P})$ be as in Example 2.1. We will show that there are only three distinct types of fundamental nets of sets for this volume pair. First, let us suppose that $\{D_\delta\}$ is a fundamental net of sets. Either there exists an x in $[a,b]$ such that $D_{\delta'} = \{x\}$ for some δ' or there doesn't. In the first case it follows from Definition 3.1 (1) that $D_\delta = \{x\}$ for all $\delta \geq \delta'$. In the second case we have $D_\delta = (c_\delta, d_\delta)$ with $a \leq c_\delta < d_\delta \leq b$ for each δ. Let $\bar{D}_\delta$ denote the closure of D_δ for each δ. Theorem 3.1 (1) implies that $\{\bar{D}_\delta\}$ has the finite intersection property and since $[a,b]$ is compact we see that there exists a z in $\bar{D}_\delta$ for all δ. If z is an element of $\bigcap_\delta D_\delta$ then the $\mathcal{P}$-subdividion $\{\{a\},(a,z),\{z\},(z,b),\{b\}\}$ of $[a,b]$ would not contain any member of $\{D_\delta\}$ which is impossible in view of Theorem 3.1 (2). Therefore, there exists a δ' such that for all δ following δ', z is not in D_δ and so z must be an end-point of D_δ for all $\delta \geq \delta'$. Since $D_{\delta_1} \cap D_{\delta_2}$ is not empty for any δ_1 and δ_2 we have D_δ equal to $(z, z+\epsilon_\delta)$ with $z < z + \epsilon_\delta \leq b$ for all δ following δ'; or D_δ equals $(z-\epsilon_\delta, z)$ with $z > z - \epsilon_\delta \geq a$ for all δ following δ'. Assume the first case occurs. If $b \geq z + \eta > z$

with $\eta > o$ then $\{D_\delta\}$ is eventually contained in some element of the $\mathcal{P}$-subdivision $\{\{a\},(a,z),\{z\},(z,z+\eta),\{z+\eta\},(z+\eta,b),\{b\}\}$ and this element must be $(z,z+\eta)$. Letting η approach zero we see that $\operatorname{Lim}_\delta \epsilon_\delta$ is 0. A similar argument can be made in the second case.

In summary, if $\{D_\delta\}$ is a fundamental net of sets then either there exists a z in $[a,b]$ so that eventually D_δ is $\{z\}$; or there exists a z in $[a,b)$ so that eventually D_δ equals $(z,z+\epsilon_\delta)$ with $z < z+\epsilon_\delta \leq b$ and $\operatorname{Lim}_\delta \epsilon_\delta$ equal to 0; or there exists a z in $(a,b]$ so that eventually D_δ equals $(z-\epsilon_\delta,z)$ with $a \leq z-\epsilon_\delta < z$ and $\operatorname{Lim}_\delta \epsilon_\delta$ equal to 0.

Example 3.2. Let $(X,\mathcal{P})$ be the volume pair of Example 2.2. It can be shown by arguments similar to those given in Example 3.1, that $\{D_\delta\}$ is a fundamental net of sets if and only if one of the following three cases occurs: eventually D_δ is $\{a\}$; or eventually D_δ equals $(z-\epsilon_\delta,z]$ for some z in $(a,b]$ with $a \leq z \leq b$ and $\operatorname{Lim}_\delta \epsilon_\delta$ equal to 0; or eventually D_δ equals $(z,z+\epsilon_\delta]$ for some z in $[a,b)$

with $z < z + \epsilon_\delta \leq b$ and $\operatorname*{Lim}_\delta \epsilon_\delta$ equal to 0.

<u>Example 3.3</u>. Let $(X,\mathcal{P})$ be the volume pair of Example 2.3 and suppose $\{D_\delta\}$ is a fundamental net of sets. Either there exists a positive real constant M such that $\{D_\delta\}$ is eventually contained in the open interval $(-M,M)$ or there doesn't. In the first case it follows easily that $\{D_\delta\}$ is eventually of one of three forms discussed in Example 3.1. In the second case, let M be any fixed positive real constant. Now, the collection $\{(-\infty,-M),\{-M\},(-M,M),\{M\},(M,\infty)\}$ is a $\mathcal{P}$-subdivision of X and so $\{D_\delta\}$ is eventually contained in one of its members. By Theorem 3.1 (2) we see that either $\{D_\delta\}$ is eventually contained in $(-\infty,-M)$ or it is eventually contained in (M,∞). Assume the second case occurs. From Theorem 3.1 (2) and the above discussion it follows that eventually $D_\delta \cap (-M,M)$ is empty for any positive constant M. Also, because the pair-wise intersection of elements of a fundamental net of sets is not empty we see that D_δ equals (a_δ,∞) for all δ. Recalling that M was an arbitrary positive constant it follows that $\operatorname*{Lim}_\delta a_\delta$ is ∞. A similar argument can be made if the first case occurs.

In summary, $\{D_\delta\}$ is a fundamental net of sets if and only if either $\{D_\delta\}$ is eventually equal to $\{a\}$ for some real number a; or eventually D_δ is $(z, z+\epsilon_\delta)$ with ϵ_δ greater than 0 for all δ and $\operatorname{Lim}_\delta \epsilon_\delta$ equal to 0; or D_δ is (a_δ, ∞) for all δ with $\operatorname{Lim}_\delta a_\delta$ equal to ∞; or D_δ is $(-\infty, a_\delta)$ for all δ with $\operatorname{Lim}_\delta a_\delta$ equal to $-\infty$; or eventually D_δ is $(z-\epsilon_\delta, z)$ with ϵ_δ greater than 0 for all δ and $\operatorname{Lim}_\delta \epsilon_\delta$ equal to 0.

Example 3.4. Let $(X,\mathcal{P})$ be the volume pair of Example 2.5. It is trivial to verify that $\{D_\delta\}$ is a fundamental net of sets if and only if there exists an i with $1 \leq i \leq N-1$ such that D_δ equals A_i for all δ.

Example 3.5. Let $(X,\mathcal{P})$ be the volume pair of Example 2.6 and let $\{D_\delta\}$ be a fundamental net of sets. For any positive integer N consider the $\mathcal{P}$-subdivision $\{\{1\},\ldots,\{N\},\{N+1,N+2,\ldots\}\}$. It follows from Theorem 3.1 (2) that $\{D_\delta\}$ is eventually equal to $\{N\}$ for some positive integer N or D_δ equals $\{N_\delta, N_\delta+1,\ldots\}$ for all δ with $\operatorname{Lim}_\delta N_\delta$ equal to ∞. Conversely, every net of sets of the above two forms is a fundamental net of sets.

<u>Definition 3.2</u>. A family $\mathfrak{F}$ of subsets of a nonempty set X is said to be a filter if it possesses the following properties:

(1) the empty set is not in $\mathfrak{F}$;

(2) if A contains B and B is in $\mathfrak{F}$, then A is in $\mathfrak{F}$;

(3) if A and B are in $\mathfrak{F}$, then $A \cap B$ is in $\mathfrak{F}$.

<u>Definition 3.3</u>. If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are filters for the set X then we say that $\mathfrak{F}_1$ refines $\mathfrak{F}_2$ if $\mathfrak{F}_1$ contains $\mathfrak{F}_2$. A filter is called an ultrafiler if it is not refined by any filter but itself.

The above two definitions are taken from [2,p.30] and have been used by some authors as an alternative to nets in the study of convergence. As the following example shows, there is a relationship between the ultrafilters of a set X and the fundamental nets of sets from the volume pair of Example 2.7.

<u>Example 3.6</u>. Let $(X,\mathcal{P})$ be the volume pair of Example 2.7. Unlike the previous examples, it is not possible to simply characterize all of the fundamental nets of sets for this volume pair. However, we will show that every ultrafilter

is a fundamental net of sets and every fundamental net of sets $\{D_\delta\}$ is a subset of an ultrafilter $\mathfrak{F}$ such that if A is in $\mathfrak{F}$ then there exists a D_δ such that D_δ is a subset of A. To begin with, suppose $\mathfrak{F}$ is an ultrafilter of subsets of X. Partially order $\mathfrak{F}$ by set inclusion so that it becomes a directed set. Define the mapping S from $\mathfrak{F}$ onto $\mathfrak{F}$ by: $S(A) = A$. In this manner $\mathfrak{F}$ becomes a net and we will now verify that it is a fundamental net of sets. Only condition (2) in Definition 3.1 is nontrivial. To this end let $\{F_j\}_{j=1}^N$ be a $\mathcal{P}$-subdivision of X and assume that no element of $\mathfrak{F}$ is contained in any F_j for $j = 1,\ldots,N$. Furthermore, suppose for each j that there exists an A_j in $\mathfrak{F}$ so that $A_j \cap F_j$ is empty. Let A be the set $\bigcap_{j=1}^{N} A_j$. Then A is in $\mathfrak{F}$ by the definition of a filter and yet $A \cap F_j$ is empty for $j = 1,\ldots,N$ which is obviously impossible. Thus, there exists an F_j such that $A \cap F_j$ is not empty for every A in $\mathfrak{F}$. Let $\mathfrak{F}'$ be the collection of all subsets W of X such that there exists an A in $\mathfrak{F}$ for which $A \cap F_j$ is a subset of W. It is easy to prove that $\mathfrak{F}'$ is a filter which properly contains $\mathfrak{F}$. This is a contradiction and so $\mathfrak{F}$ is a fundamental net of sets. Let $\mathfrak{F}$ be the collection of all subsets W of X such that W contains some member of $\{D_\delta\}$. $\mathfrak{F}$ is a filter as is easily verified. If $\mathfrak{F}'$ is a filter properly containing $\mathfrak{F}$ then there exists a W' in $\mathfrak{F}'$ which contains no element of $\{D_\delta\}$. But $\{\{W'\},\{X-W'\}\}$ is a ρ-subdivision of X and by Theorem 3.1 (2) $X - W'$ must contain an element of $\{D_\delta\}$ which we will denote by $D_{\delta'}$. This implies that $W' \cap D_{\delta'}$ is empty which contradicts the fact that $\mathfrak{F}'$ is a filter. Thus, $\mathfrak{F}$ is an ultrafilter. Let A be in $\mathfrak{F}$ and consider the ρ-subdivision $\{\{A\},\{X-A\}\}$ of X. Since $\{D_\delta\}$ is a fundamental net of sets there exists a δ so that D_δ is contained in one of the members of the ρ-subdivision. But, A and D_δ are both in $\mathfrak{F}$ and so $A \cap D_\delta$ is not empty which then implies that D_δ is a subset of A.

The last example shows that for this volume pair fundamental nets of sets are essentially equivalent to ultrafilters. All of the theorems in this dissertation, when applied to this volume pair, could be stated in terms of ultrafilters. In fact, this has already been done by several authors ([2],p.280,[20]). However, for an arbitrary volume pair (X,ρ) this cannot be done because not every ultrafilter of subsets of X is a fundamental net of sets. It would be possible to introduce a new definition for a filter to circumvent the problem. For example, we could define $\mathfrak{F}$ to be a $\wp$-filter if $\mathfrak{F}$ satisfies all of the conditions of Definition 3.2 where the sets are restricted to lie in $\wp$. This is essentially what we have done in introducing the concept of a fundamental net of sets. However, there are several advantages in using fundamental nets of sets versus the concept of a $\wp$-filter. As the last example shows, we would have to work with maximal $\wp$-filters and most of the following theorems would be difficult to apply since it is usually impossible to characterize maximal filters of any type, unless one already knows what the fundamental nets of sets look like. A second advantage is that most of the theorems in chapters three and four are easier to state and prove using fundamental nets of sets. For example, Theorem 3.5 is much easier to apply than Theorem 31, page 281 in [1]. With these comments in mind, the rest of this dissertation will deal exclusively with fundamental nets of sets. The following definition is a natural extension of Definition 3.1 and is extremely important to all that follows.

Definition 3.4. Let $(X,\wp)$ be a volume pair. A net of points $\{x_\delta\}$ in X is said to form a fundamental net of points if and only if there exists a fundamental net of sets $\{D_\delta\}$ such that x_δ is in D_δ for each δ.

Since we have already characterized the fundamental nets of sets for a few volume pairs we can characterize the fundamental nets of points for them. The examples are included now for easy reference.

Example 3.7. Let $(X,\wp)$ be the volume pair of Example 3.1. It then follows that $\{x_\delta\}$ is a fundamental net of points if and only if one of the following three cases occurs. There exists a z in $[a,b]$ such that either $\{x_\delta\}$ is eventually equal to z, or eventually $\{x_\delta\}$ is less than z with $\operatorname{Lim}_\delta x_\delta$ equal to z, or eventually $\{x_\delta\}$ is greater than z with $\operatorname{Lim}_\delta x_\delta$ equal to z.

Example 3.8. Let $(X,\wp)$ be the volume pair of Example 3.2. It follows that $\{x_\delta\}$ is a fundamental net of points if and only if one of the following three cases occurs. Either x_δ is eventually equal to a, or there exists a z in $(a,b]$ such that eventually x_δ is less than or equal to z with

$\operatorname{Lim}_{\delta} x_{\delta}$ equal to z, or there exists a z in $[a,b)$ such that eventually z is less than x_{δ} with $\operatorname{Lim}_{\delta} x_{\delta}$ equal to z.

Example 3.9. Let $(X,\mathcal{P})$ be the volume pair of Example 3.3. Then $\{x_{\delta}\}$ is a fundamental net of points if and only if one of the following five cases occurs. Either there exists a real z such that eventually x_{δ} equals z, or eventually x_{δ} is less than z with $\operatorname{Lim}_{\delta} x_{\delta}$ equal to z, or eventually z is less than x_{δ} with $\operatorname{Lim}_{\delta} x_{\delta}$ equal to z, or $\operatorname{Lim}_{\delta} x_{\delta}$ is ∞, or $\operatorname{Lim}_{\delta} x_{\delta}$ is $-\infty$.

Example 3.10. Let $(X,\mathcal{P})$ be the volume pair of Example 3.5. Then $\{x_{\delta}\}$ is a fundamental net of points if and only if either there exists an integer N such that eventually x_{δ} equals N, or $\operatorname{Lim}_{\delta} x_{\delta}$ is ∞.

As we pointed out before, if $(X,\mathcal{P})$ is the volume pair of Example 3.7 then a function f is in $QC([a,b])$ if and only if all one sided limits exist for f on $[a,b]$. We are now ready to answer the question posed at the beginning of this chapter. A careful analysis of Example 3.7

leads one to suspect the following theorem.

Theorem 3.3. Let $(X,\mathcal{P})$ be a volume pair. A complex valued function f on X is an element of $Q(X,\mathcal{P})$ if and only if $\lim_{\delta} f(x_\delta)$ exists for every fundamental net of points $\{x_\delta\}$.

Proof: Suppose first that f is an element of $Q(X,\mathcal{P})$. Let ϵ, greater than 0, be given and suppose $\{x_\delta\}$ is a fundamental net of points associated with the fundamental net of sets $\{D_\delta\}$. Since f is in $Q(X,\mathcal{P})$ it follows from Theorem 2.2 (2) that there exists a $\mathcal{P}$-subdivision $\{F_j\}_{j=1}^{p}$ of X and complex numbers $\alpha_1,\ldots,\alpha_p$ such that $\|f - \sum_{j=1}^{p} \alpha_j \chi_{F_j}\|$ is less than $\epsilon/2$. Select F_j such that there exists a δ with D_δ contained in F_j. If δ_1, δ_2 both follow δ then x_{δ_1} and x_{δ_2} are both elements of F_j which implies that

$$\begin{aligned}|f(x_{\delta_1}) - f(x_{\delta_2})| &\leq |f(x_{\delta_1}) - \alpha_j| + |\alpha_j - f(x_{\delta_2})| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon.\end{aligned}$$

Thus, $\{f(x_\delta)\}$ is a Cauchy net of complex numbers and so converges.

Conversely, suppose $\lim_{\delta} f(x_\delta)$ exists for every fundamental net of points $\{x_\delta\}$ and furthermore assume the result is false. From Theorem 2.2 (2) it follows that there exists an η greater than 0 such that if $\{D_i\}_{i=1}^{N}$ is a $\mathcal{P}$-subdivision of X then there exists a D_i and x, y in D_i such that $|f(x) - f(y)| \geq \eta$. Let $\mathcal{G}$ be the set of all A in $\mathcal{P}$ such that for every $\mathcal{P}$-subdivision $\{E_i\}_{i=1}^{p}$ of A there exists an E_i and elements x,y in E_i such that $|f(x) - f(y)| \geq \eta$. We will first verify that $\mathcal{G}$ is not the empty set. Let $\{A_j\}_{j=1}^{r}$ be a $\mathcal{P}$-subdivision of X and suppose that no A_j is in $\mathcal{G}$. Then, for each A_j there exists a $\mathcal{P}$-subdivision $\{E_{ij}\}_{i=1}^{p_j}$ of A_j so that if x, y are in E_{ij} then $|f(x) - f(y)| < \eta$ for $j = 1,\ldots,r$. It follows that $\{E_{ij}\}_{j=1}^{r}{}_{i=1}^{p_j}$ is a $\mathcal{P}$-subdivision of X such that if x, y are in E_{ij} then $|f(x) - f(y)| < \eta$. This is a contradiction and so $\mathcal{G}$ is not empty. Let $\mathfrak{F}$ be the collection of all subsets M of $\mathcal{G}$ such that if A_1, A_2 are in M then there exists an A in M such that A is a subset of $A_1 \cap A_2$. $\mathfrak{F}$ is nonvoid because if A is in $\mathcal{C}$ then $\{A\}$ is in $\mathfrak{F}$. Partially order $\mathfrak{F}$ by set inclusion. Using an argument similar to that given in Theorem 3.2, it is easy to verify that $\mathfrak{F}$ has a maximal element which we will denote by $\bar{M}$. By our restrictions on $\mathfrak{F}$ we see that $\bar{M}$ is a directed set with respect to set inclusion. Define S from $\bar{M}$ into $\mathcal{P}$ by: $S(A) = A$. Suppose $\bar{M}$ is not a fundamental net of sets. It then follows that there exists a $\mathcal{P}$-subdivision $\{E_i\}_{i=1}^{p}$ of X so that no element of $\bar{M}$ is a subset of E_i for any i. Assume that there exists an E_i such that $E_i \cap A$ is in $\mathcal{C}$ for all A in $\bar{M}$. Let $\hat{M}$ be the set

$$\bar{M} \cup \{E_i \cap A : A \text{ is in } \bar{M}\}.$$

Again using an argument similar to that in Theorem 3.2, it follows that $\hat{M}$ is in $\mathfrak{F}$. However, $\bar{M}$ is properly contained in $\hat{M}$ because if $\bar{M}$ were equal to $\hat{M}$ then $E_i \cap A$ would be in $\bar{M}$ for all A in $\bar{M}$ which then implies that E_i contains an element of $\bar{M}$ which is contrary to assumption on E_i. But, since $\bar{M}$ is a maximal element of $\mathfrak{F}$ we again have a contradiction. Thus, for each E_i, $i = 1,\ldots,p$, there exists an A_i in $\bar{M}$ so that $E_i \cap A_i$

is not in $\mathfrak{G}$. Select an A in $\bar{M}$ so that A is a subset of $\bigcap_{i=1}^{p} A_i$. This is always possible since $\bar{M}$ is in $\mathfrak{F}$. We will now show that $A \cap E_i$ is not in $\mathfrak{G}$ for each $i = 1,\ldots,p$. Since $A \cap E_i$ is contained in $A_i \cap E_i$ and since $A_i \cap E_i$ is not in $\mathfrak{G}$ it suffices to show that if B_1 and B_2 are in $\mathcal{P}$ with B_1 contained in B_2 and B_2 not in $\mathfrak{G}$ then B_1 is not in $\mathfrak{G}$. Now, since B_2 is not in $\mathfrak{G}$ there exists a $\mathcal{P}$-subdivision $\{F_j\}_{j=1}^{r}$ of B_2 such that if x, y are in F_j then $|f(x) - f(y)|$ is less than η. It follows that $\{F_j \cap B_1\}_{j=1}^{r}$ is a $\mathcal{P}$-subdivision of B_1 which also has the property that if x, y are in $F_j \cap B_1$ then $|f(x) - f(y)|$ is less than η and so B_1 is not in $\mathfrak{G}$. Applying this result to our case we see that $A \cap E_i$ is not in $\mathfrak{G}$ for any $i = 1,\ldots,p$. However, $\{A \cap E_i\}_{i=1}^{p}$ is a $\mathcal{P}$-subdivision of A and since $A \cap E_i$ is not in $\mathfrak{G}$ for $i = 1,\ldots,p$ it is easy to verify that there exists a $\mathcal{P}$-subdivision $\{D_i\}_{i=1}^{M}$ of A such that if x, y are in D_i then $|f(x) - f(y)|$ is less than η. But this implies that A is not in $\mathfrak{G}$ contrary to assumption. Thus, $\bar{M}$ is a fundamental net of sets with the property that if A is in $\bar{M}$ and $\{E_i\}_{i=1}^{N}$ is a $\mathcal{P}$-subdivision of A then there exists an E_i and x, y in E_i such that $|f(x) - f(y)| \geq \eta$.

For the rest of this proof let us denote the fundamental net of sets, that was just constructed, by $\{A_\delta\}$. Now, either there exists an $\hat{A}$ in $\{A_\delta\}$ such that $\hat{A}$ is a subset of A_δ for each δ or no such $\hat{A}$ exists. In the first case consider the directed set $\{(\hat{A},i) : i = 1,2,\ldots\}$ where we define $(\hat{A},i) \leq (\hat{A},j)$ if and only if $i \leq j$. Define the function S by: $S((\hat{A},i)) = \hat{A}$. S is a fundamental net of sets because $\{A_\delta\}$ is a fundamental net of sets and because $\hat{A}$ is contained in A_δ for all δ. Select x, y in $\hat{A}$ such that $|f(x) - f(y)| \geq \eta$. Choose a fundamental net of points $\{z_i\}_{i=1}^{\infty}$ associated with S, such that z_i equals x if i is even and is y if i is odd. Clearly, $\lim_{i\to\infty} f(z_i)$ does not exist. This is a contradiction and so no $\hat{A}$ exists in $\{A_\delta\}$ such that $\hat{A}$ is a subset of A_δ for all δ. In the second case, consider the directed set $\{(A_\delta,i) : i = 0,1$ and A_δ is in $\{A_\delta\}\}$ where the order is defined by $(A_\delta,i) \leq (A_{\delta'},j)$ if and only if $A_{\delta'}$ is a subset of A_δ. This is a directed set because if A_δ does not equal $A_{\delta'}$ then there exists an $\hat{A}_\delta$ such that $\hat{A}_\delta$ is contained in $A_\delta \cap A_{\delta'}$ by the definition of $\bar{M}$. Thus, $(A_\delta, i) \leq (\hat{A}_\delta, j)$ and $(A_{\delta'}, i) \leq (\hat{A}_\delta, j)$ for $i = 0,1$ and $j = 0,1$. On the other hand, if one considers $(A_\delta, 0)$ and $(A_\delta, 1)$ then there exists a δ' such that $A_{\delta'}$ is a subset of A_δ because of the first case considered above. Therefore,

$$(A_{\delta'},)) \geq (A_\delta, 0) \quad \text{and} \quad (A_{\delta'}, 0) \geq (A_\delta, 1).$$

It follows that we have a directed set. Define S by: $S((A_\delta, i)) = A_\delta$ for all δ and $i = 0,1$. Clearly, S is a fundamental net of sets. From each $S((A_\delta, i))$ select x_{A_δ}, y_{A_δ} such that $|f(x_{A_\delta}) - f(y_{A_\delta})| \geq \eta$. The collection $\{z(\delta_i, i) : i = 0,1\}$ is a fundamental net of points where $z(\delta, 0) = x_{A_\delta}$ and $z(\delta, 1) = y_{A_\delta}$. However, $\lim\limits_{(\delta, i)} f(z(\delta, i))$ does not exist because for any (δ, i) there exists an $A_{\delta'}$ properly contained in A_δ and so $(A_\delta, i) < (A_{\delta'}, 0)$, $(A_\delta, i) < (A_{\delta'}, 1)$ which then implies that

$$|f(z(\delta', 0)) - f(z(\delta', 1))| \geq \eta.$$

This shows that $\{f(z(\delta,i)) : i = 0,1\}$ is not a Cauchy net and so does not converge. This is a contradiction and the proof is complete.

As we remarked before, the above theorem yields many corollaries each of which characterize a $Q(X,\wp)$ space in a manner well suited to applications. In Example 3.11 we obtain the result on the classical $QC([a,b])$ as a direct consequence of the last theorem and our previous examples. This points out well how the notion of a fundamental net of points can be considered as a generalization of one sided limits for $Q(X,\wp)$ spaces.

Example 3.11. Let $(X,\wp)$ be the volume pair of Example 3.7. It follows that f is in $QC([a,b])$ if and only if $\lim_{x\to y^+} f(x)$ and $\lim_{x\to y^-} f(x)$ exist for all y in (a,b) and the appropriate one sided limits exist at a and b.

Example 3.12. Let $(X,\wp)$ be the volume pair of Example 3.8. Then f is in $Q(X,\wp)$ if and only if $\lim_{x\to y^+} f(x)$ exists for all y in $[a,b)$ and $\lim_{x\to y^-} f(x)$ equals $f(y)$ for all y in $(a,b]$.

Example 3.13. Let $(X,\mathcal{P})$ be the volume pair of Example 3.9. Then f is in $Q(X,\mathcal{P})$ if and only if $\lim_{x\to y^+} f(x)$, $\lim_{x\to y^-} f(x)$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist for all real y.

Example 3.14. Let $(X,\mathcal{P})$ be as in Example 3.10. Then f is in $Q(X,\mathcal{P})$ if and only if $\lim_{N\to\infty} f(N)$ exists. Thus, we see that $Q(X,\mathcal{P})$ is just the Banach space c, consisting of all convergent complex sequences and normed with the supremum norm.

Let X be a non empty set and suppose that $\{f_n : X \to C\}_{n=1}^{\infty}$ is a sequence of functions pointwise convergent on X to a function ψ. If X is a compact Hausdorff space and each f_n is continuous then necessary and sufficient conditions for ψ to be continuous are well known ([2],p.268). Suppose now that $(X,\mathcal{P})$ is a volume pair and that f_n is in $Q(X,\mathcal{P})$. We will show that necessary and sufficient conditions for ψ to be in $Q(X,\mathcal{P})$ can be given in terms of fundamental nets of points.

Theorem 3.4. Let $(X,\mathcal{P})$ be a volume pair and suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $Q(X,\mathcal{P})$ such that $\lim_{n\to\infty} f_n(x) = \psi(x)$ exists for every x in X. Then ψ is in

$Q(X,\mathcal{P})$ if and only if for every fundamental net of points $\{x_\delta\}$ and every positive number ϵ there exists a $\bar{\delta}$ such that for each $\delta \geq \bar{\delta}$ there exists a positive integer N_δ such that if $n \geq N_\delta$ then $|f_n(x_\delta) - f_n(x_{\bar{\delta}})|$ is less than ϵ.

Proof: Suppose first that ψ is in $Q(X,\mathcal{P})$. Let $\epsilon > 0$ be given and suppose $\{x_\delta\}$ is a fundamental net of points. Since ψ is quasi-continuous $\lim_\delta \psi(x_\delta)$ exists by virtue of Theorem 3.3. Select $\bar{\delta}$ so that if $\delta_1 \geq \bar{\delta}$ we have $|\psi(x_{\delta_1}) - \psi(x_{\bar{\delta}})|$ less than $\epsilon/3$. For each fixed $\delta \geq \bar{\delta}$ choose N_δ so that if $n \geq N_\delta$ then

$$|f_n(x_\delta) - \psi(x_\delta)| < \epsilon/3 \quad \text{and} \quad |f_n(x_{\bar{\delta}}) - \psi(x_{\bar{\delta}})| < \epsilon/3.$$

Therefore, for each $n \geq N_\delta$ we have

$$|f_n(x_\delta) - f_n(x_{\bar{\delta}})| \leq$$

$$|f_n(x_\delta) - \psi(x_\delta)| + |\psi(x_\delta) - \psi(x_{\bar{\delta}})| + |\psi(x_{\bar{\delta}}) - f_n(x_{\bar{\delta}})| <$$

$$\epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

Suppose now that $\{x_\delta\}$ is a fundamental net of points. Let $\epsilon > 0$ be given and select $\bar{\delta}$ according to the hypothesis of the theorem. Let δ_1, δ_2 both follow $\bar{\delta}$ and choose N so that if $n \geq N$ then $|f_n(x_{\delta_1}) - f_n(x_{\bar{\delta}})|$ and $|f_n(x_{\delta_2}) - f_n(x_{\bar{\delta}})|$ are both less than ϵ. Thus, if $n \geq N$ we have:

$$|f_n(x_{\delta_1}) - f_n(x_{\delta_2})| \leq$$

$$|f_n(x_{\delta_1}) - f_n(x_{\bar{\delta}})| + |f_n(x_{\bar{\delta}}) - f_n(x_{\delta_2})| <$$

$$\epsilon + \epsilon = 2\epsilon.$$

This implies that $\{\psi(x_\delta)\}$ is a Cauchy net and so converges. Theorem 3.3 then shows that ψ is in $Q(X,\wp)$.

The last two theorems give very important results about $Q(X,\wp)$ spaces but they do not really depend on any particular topology that one might place on $Q(X,\wp)$. For the rest of this chapter we are going to consider a special topology for $Q(X,\wp)$, the weak topology. This particular topology is significant in duality theory and is of interest in its own right. It turns out that the concept of a fundamental net of points is applicable to the study of this topology. The significance of this concept, when applied to $Q(X,\mathcal{P})$ spaces, will become apparent as we obtain several fundamental results concerning the weak topology, which are easily stated and applied. In order to illustrate the simplicity of our theorems, we include here a short list of some known results concerning weak sequential convergence that may be adapted to $Q(X,\mathcal{P})$ spaces.

It is easy to verify, via Theorem 3.3, that $Q(X,\mathcal{P})$ is a closed subalgebra of $B(X)$ for every volume pair $(X,\mathcal{P})$. It follows from [22,p.221] and the Hahn-Banach Theorem that weak sequential convergence in $Q(X,\mathcal{P})$ can be characterized if one determines necessary and sufficient conditions for weak sequential convergence in $B(X)$. This has been done by several authors. Simons [20] has shown that a sequence $\{f_n\}_{n=1}^{\infty}$ converges weakly to 0 in $B(X)$ if and only if $\{f_n(x)\}_{n=1}^{\infty}$ converges to 0 for every x in X and $\operatorname{Lim}_m \operatorname{Lim}_k f_{n_k}(x_m)$ equals $\operatorname{Lim}_k \operatorname{Lim}_m f_{n_k}(x_m)$ whenever $\{f_{n_k}\}$ is a subset of $\{f_n\}_{n=1}^{\infty}$ (not necessarily a subsequence) and $\{x_i\}_{i=1}^{\infty}$ is such that all limits exist. In a similar vein, Pták [18] has shown that $\{f_n\}_{n=1}^{\infty}$ converges weakly to

0 in $B(X)$ if and only if whenever $x_1, x_2, \ldots$ are in X and $\lim_m f_n(x_m)$ exists for each n then $\lim_n \lim_m f_n(x_m)$ is 0. To obtain another result along these lines, we need the following definition and theorem from [2,p.281). The theorem is given without proof for reference.

Definition 3.5. A sequence $\{f_n\}_{n=1}^{\infty}$ of complex valued functions on a nonempty set X is said to be quasi-uniformly convergent on X if and only if there exists a function f_o on X such that $\{f_n(S)\}$ converges to $f_o(S)$ for every S in X, and such that for every positive number ϵ and positive integer N there exists a finite number of indices $n_1, \ldots, n_k \geq N$ such that for each S in X $\min_{1 \leq i \leq k} |f_{n_i}(S) - f_o(S)|$ is smaller than ϵ.

Theorem 3.5. Let X be an arbitrary set. A sequence $\{f_n\}_{n=1}^{\infty}$ in $B(X)$ converges weakly to f_o if and only if there exists a constant M such that $\|f_n\| \leq M$ for all n and, $\{f_n\}_{n=1}^{\infty}$ together with every subsequence, converges to f_o quasi-uniformly on X.

While the above results are perhaps adequate for $B(X)$, they are not easily applied to $Q(X,\mathcal{P})$ spaces. It would seem that one should be able to obtain much better results by using the set structure which generates $Q(X,\mathcal{P})$. The following two theorems give the major results in that direction.

Theorem 3.6. Let $(X,\mathcal{P})$ be a volume pair. Then a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $Q(X,\mathcal{P})$ converges weakly to 0 if and only if there exists a constant M greater than 0 such that $\|f_n\| \leq M$ for every n; and $\lim\limits_{n\to\infty} \lim\limits_{\delta} f_n(x_\delta)$ equals 0 for every fundamental net of points $\{x_\delta\}$.

Proof: Suppose first that $\{f_n\}_{n=1}^{\infty}$ converges weakly to 0. The existence of the constant M is a standard result and is easily verified. Now, suppose there exists a fundamental net of points $\{x_\delta\}$ for which $\lim\limits_{n\to\infty} \lim\limits_{\delta} f_n(x_\delta)$ does not equal 0. Put y_n equal to $\lim\limits_{\delta} f_n(x_\delta)$. The existence of this limit is guaranteed by Theorem 3.4. Since $\lim\limits_{n\to\infty} y_n$ is not 0 we can assume, by choosing a subsequence of $\{y_n\}_{n=1}^{\infty}$ if necessary, that there exists a positive number η such that $|y_n|$ is larger than η for all n. Define

u on $\mathcal{P}$ by the condition that $u(E)$ is 1 if $\{x_\delta\}$ is eventually in E and is 0 otherwise. Then, u is a p-volume of bounded variation as we will now show. Suppose E_1, E_2 are in $\mathcal{P}$, $E_1 \cap E_2$ is empty and $E_1 \cup E_2$ is in $\mathcal{P}$. Either $\{x_\delta\}$ is eventually in $E_1 \cup E_2$ or it isn't. In the latter case it follows that $\{x_\delta\}$ is not eventually in either E_1 or E_2 which implies that

$$u(E_1 \cup E_2) = 0 = u(E_1) + u(E_2).$$

On the other hand, suppose $\{x_\delta\}$ is eventually in $E_1 \cup E_2$. Select a $\mathcal{P}$-subdivision $\{A_i\}_{i=1}^n$ of X containing E_1 and a $\mathcal{P}$-subdivision $\{B_j\}_{j=1}^m$ of X containing E_2. Finally, choose a $\mathcal{P}$-subdivision $\{C_k\}_{k=1}^{\ell}$ which refines the previous two. Since $\{x_\delta\}$ is a fundamental net of points, there exists a $C_{k'}$ such that $\{x_\delta\}$ is eventually in $C_{k'}$. By our choice of $\{C_k\}_{k=1}^{\ell}$ and because $E_1 \cap E_2$ is empty it follows that $C_{k'}$ is a subset of E_1 or is a subset of E_2 but not both. Thus, without loss of generality, we have $u(E_1)$ equal to 1 and $u(E_2)$ equal to 0. This shows that

$$u(E_1 \cup E_2) = 1 = u(E_1) + u(E_2)$$

Conversely, if $\{x_\delta\}$ is not eventually in E_1 nor eventually in E_2 then it is not eventually in $E_1 \cup E_2$ as the previous part of the proof shows. In conclusion, $u(E_1) + u(E_2) = u(E_1 \cup E_2)$ and a similar proof will extend the result to any finite union of disjoint sets in $\mathcal{P}$. Thus, u is finitely additive and so is a p-volume. Let $\{E_i\}_{i=1}^N$ be a $\mathcal{P}$-subdivision of X. By the definition of a fundamental net of points there exists exactly one $E_{i'}$ such that $\{x_\delta\}$ is eventually in $E_{i'}$. Thus,

$$\sum_{i=1}^{N} |u(E_i)| = |u(E_{i'})| = 1$$

and so u is a p-volume of bounded variation with V_u equal to 1. From Theorems 2.3 and 2.4, it follows that $\Phi(f) = \psi \int_X f\,du$ is a continuous linear functional on $Q(X,\mathcal{P})$ which is independent of the choice of ψ. For each positive integer n choose ψ_n, a choice function on $\mathcal{P}$, such that if $\{x_\delta\}$ is eventually in D then $\psi_n(D)$ is x_δ where $|f_n(x_\delta)|$ is greater than η; whereas if $\{x_\delta\}$

is not eventually in D then $\psi_n(D)$ is an arbitrary element of D. Note that such a choice of ψ_n is possible because $|\lim_{\delta} f_n(x_\delta)|$ is larger than η and so if $\{x_\delta\}$ is eventually in D then $|f_n(x_\delta)|$ is greater than η eventually. However,

$$|\psi_n \int_X f_n \, du| = |\lim_{\mathcal{D}} \sum_{i=1}^{p} f_n(\psi_n(D_i))u(D_i)|$$

$$= |f_n(\psi_n(D_i))|$$

$$= |f_n(x_\delta)| > \eta$$

where D_i is that unique element of the $\mathcal{P}$-subdivision $\mathcal{D}$ such that $\{x_\delta\}$ is eventually in it. This contradicts the fact that $\{f_n\}_{n=1}^{\infty}$ converges weakly to 0.

Conversely, recall that $Q(X,\mathcal{P})$ is a closed subspace of $B(X)$ in both the norm and weak topologies. Thus, it suffices to show that the present hypotheses imply those of Theorem 3.5. Since every subsequence of $\{f_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 3.6, it suffices to show that $\{f_n\}_{n=1}^{\infty}$ converges quasi-uniformly to 0 on X.

Assume that this is not true. It follows that either there exists an η greater than 0 and an integer N such that for all m greater than N there exists an x_m in X with $|f_i(x_m)| \geq \eta$ for $N \leq i \leq m$ or $\lim_{n\to\infty} f_n(x_o)$ is not 0 for some x_o in X. Consider the first case. By Theorem 3.2 there exists a fundamental net of sets $\{D_\delta\}$ such that $\{x_m\}_{m=N+1}^{\infty}$ is frequently in each D_δ. Now, either $\{D_\delta\}$ contains a member $D_{\delta'}$ which is a subset of every other member or it doesn't. In the former case consider the directed set $\{D_{\delta'},i) : i = 1,2,\ldots\}$ where $(D_{\delta'},j)$ follows $(D_{\delta'},i)$ if j is greater than i. The net

$$\{A(D_{\delta'},i) : A(D_{\delta'},i) = D_{\delta'} \quad \text{for all} \quad i = 1,2,\ldots\}$$

is a fundamental net of sets because $\{D_\delta\}$ is a fundamental net of sets and because $D_{\delta'}$ is a subset of D_δ for all δ. From each $A(D_{\delta'},i)$ select z_i in $D_{\delta'}$ such that z_i equals x_m with i less than m. This is always possible since $\{x_m\}_{m=N+1}^{\infty}$ is frequently in $D_{\delta'}$. Clearly, $\{z_i\}_{i=1}^{\infty}$ is a fundamental net of points. However $\lim_{n\to\infty} \lim_{i\to\infty} f_n(z_i)$ does not equal 0 because for each fixed integer n

greater than N we have $|f_n(z_i)| \geq \eta$ if i is larger than n. This is a contradiction. In the latter case, consider the directed set $\{(D_\delta, i) : i = 1,2,\ldots$ and δ is arbitrary$\}$ where (D_{δ_1}, i) is less than (D_{δ_2}, j) if and only if D_{δ_2} is a subset of D_{δ_1}. This is a directed set because of the case just considered. Clearly,

$$\{A(D_\delta, i) : A(D_\delta, i) = D_\delta \text{ for all } \delta \text{ and } i = 1,2,\ldots\}$$

is a fundamental net of sets. From each $A(D_\delta, i)$ select $z(D_\delta, i)$ such that $z(D_\delta, i)$ is x_m, in D_δ, with i less than m. Again, this is possible because $\{x_m\}_{m=N+1}^{\infty}$ is frequently in each D_δ. $\{z(D_\delta, i)\}$ is a fundamental net of points. However, $\lim\limits_{n\to\infty} \lim\limits_{(D_\delta, i)} f_n(z(D_\delta, i))$ is not 0 because for each fixed integer n greater than N we have $|f_n(z(D_\delta, i))| \geq \eta$ if i is greater than n and because for each (D_δ, i) there exists a $D_{\delta'}$ such that $(D_{\delta'}, j)$ follows (D_δ, i) for all $j = 1,2,\ldots$. This is a contradiction. It remains only to show that $\{f_n(x_o)\}_{n=1}^{\infty}$ converges for each x_o in X. Now, the net $\{A : A \in \mathcal{P}$ and $x_o \in A\}$ is a fundamental net of sets where the ordering is by set inclusion. From each set select x_o. Thus, $\{x_o\}$ is a fundamental net of points. By hypothesis, $\lim_{n\to\infty} f_n(x_o)$ is 0 and this completes the proof.

Because of our knowledge of the fundamental nets of points in certain $Q(X,\mathcal{P})$ spaces, we have many obvious but important corollaries to the last theorem. In all of the following examples we will assume that $\{f_n\}_{n=1}^{\infty}$ is norm bounded.

Example 3.15. Let $(X,\mathcal{P})$ be the volume pair of Example 2.1. A sequence $\{f_n\}_{n=1}^{\infty}$ in $QC([a,b])$ converges weakly to 0 if and only if $\lim_{n\to\infty} f_n(x)$ equals 0 for all x in $[a,b]$ and

$$\begin{aligned}
\lim_{n\to\infty} \lim_{x\to y^+} f_n(x) &= \lim_{n\to\infty} \lim_{x\to y^-} f_n(x) \\
&= \lim_{n\to\infty} \lim_{x\to a^+} f_n(x) \\
&= \lim_{n\to\infty} \lim_{x\to b^-} f_n(x) \\
&= 0.
\end{aligned}$$

for all y in (a,b).

Example 3.16. Let $(X,\mathcal{P})$ be the volume pair of Example 2.2. A sequence $\{f_n\}_{n=1}^{\infty}$ in $Q(X,\mathcal{P})$ converges weakly to 0 if and only if $\lim_{n\to\infty} f_n(x)$ equals 0 for all x in $[a,b]$ and $\lim_{n\to\infty} \lim_{x\to y^+} f_n(x)$ equals 0 for all y in $[a,b)$.

Example 3.17. Let $(X,\mathcal{P})$ be the volume pair of Example 2.3. A sequence $\{f_n\}_{n=1}^{\infty}$ in $Q(X,\mathcal{P})$ converges weakly to 0 if and only if $\lim_{n\to\infty} f_n(x)$ is 0 for all real x and

$$\begin{aligned} \lim_{n\to\infty} \lim_{x\to y^+} f_n(x) &= \lim_{n\to\infty} \lim_{x\to y^-} f_n(x) \\ &= \lim_{n\to\infty} \lim_{x\to\infty} f_n(x) \\ &= \lim_{n\to\infty} \lim_{x\to-\infty} f_n(x) \\ &= 0 \end{aligned}$$

for all real y.

Example 3.18. Let (X,ρ) be the volume pair of Example 2.6. A sequence $\{f_n\}_{n=1}^{\infty}$ in $Q(X,\rho)$ converges weakly to 0 if and only if $\lim_{n\to\infty} f_n(m)$ is 0 for every positive integer m and $\lim_{n\to\infty} \lim_{m\to\infty} f_n(m)$ equals 0.

Recall that $Q(X,\rho)$ in Example 3.18 is the space c of all convergent complex sequences. The Banach space c_o, of all complex sequences which converge to 0, is a closed subspace of c. Applying the results from the last example to c_o we see that $\{f_n\}_{n=1}^{\infty}$ converges weakly to 0 if and only if $\{f_n\}_{n=1}^{\infty}$ is norm bounded and $\lim_{n\to\infty} f_n(m)$ is 0. Similarly, since $c([a,b])$ is a closed subspace of $Q(X,\rho)$ in Example 3.15 we obtain the following well-known result as a corollary. A sequence $\{f_n\}_{n=1}^{\infty}$ in $c([a,b])$ converges weakly to 0 if and only if $\lim_{n\to\infty} f_n(x)$ is 0 for all x in $[a,b]$ and $\{f_n\}_{n=1}^{\infty}$ is norm bounded. Although Theorem 3.6 is significant, it is not as general as Theorem 3.5 since f_o did not have to be the zero element in Theorem 3.5. The next result improves Theorem 3.6 and is an interesting and important result because of its ease in applications.

Theorem 3.7. Let $(X,\mathcal{P})$ be a volume pair. A sequence $\{f_n\}_{n=1}^{\infty}$ in $Q(X,\mathcal{P})$ converges weakly to f if and only if $\{f_n\}_{n=1}^{\infty}$ is norm bounded; $\operatorname{Lim}_{n\to\infty} f_n(x)$ equals $f(x)$ for all x in X; the iterated limits $\operatorname{Lim}_{n\to\infty} \operatorname{Lim}_{\delta} f_n(x_\delta)$ and $\operatorname{Lim}_{\delta} \operatorname{Lim}_{n\to\infty} f_n(x_\delta)$ both exist and are equal.

Proof. Suppose first that $\{f_n\}_{n=1}^{\infty}$ converges weakly to f, an element of $Q(X,\mathcal{P})$. It follows that $\operatorname{Lim}_{n\to\infty} f_n(x)$ equals $f(x)$ for all x in X because pointwise evaluation is a continuous linear functional on $Q(X,\mathcal{P})$ spaces. Therefore, $\operatorname{Lim}_{\delta} \operatorname{Lim}_{n\to\infty} f_n(x_\delta)$ equals $\operatorname{Lim}_{\delta} f(x_\delta)$ and the existence of this limit is guaranteed by Theorem 3.3. It follows that $\{f_n - f\}_{n=1}^{\infty}$ converges weakly to 0 and so $\operatorname{Lim}_{n\to\infty} \operatorname{Lim}_{\delta} (f_n - f)(x_\delta)$ is 0 by Theorem 3.6. Again, since $\operatorname{Lim}_{\delta} f(x_\delta)$ exists we see from the last statement that $\operatorname{Lim}_{n\to\infty} \operatorname{Lim}_{\delta} f_n(x_\delta)$ exists and equals $\operatorname{Lim}_{\delta} f(x_\delta)$. Combining this result with the earlier part of the proof, we obtain the desired conclusion.

Conversely, it follows from the last part of the proof of Theorem 3.6 that $\{x\}$ is a fundamental net of points for each x in X and so $\lim_{n\to\infty} f_n(x)$ exists for each x. Let $f(x)$ equal $\lim_{n\to\infty} f_n(x)$. We will show first that f is in $Q(X,\mathcal{P})$. Let $\{x_\delta\}$ be a fundamental net of points. Since $\lim_\delta \lim_{n\to\infty} f_n(x_\delta)$ exists we see that if ϵ is greater than 0 then for δ_1 and δ_2 sufficiently large

$$\left|\lim_{n\to\infty}(f_n(x_{\delta_1}) - f_n(x_{\delta_2}))\right| < \epsilon.$$

Therefore,

$$\begin{aligned} |f(x_{\delta_1}) - f(x_{\delta_2})| &= \left|\lim_{n\to\infty}(f_n(x_{\delta_1}) - f_n(x_{\delta_2}))\right| \\ &< \epsilon. \end{aligned}$$

This implies that $\lim_\delta f(x_\delta)$ exists and so f is in $Q(X,\mathcal{P})$ by Theorem 3.3. To verify that $\{f_n\}_{n=1}^\infty$ converges to f weakly it suffices to show that $\{f_n - f\}_{n=1}^\infty$ converges weakly to 0. But,

$$\left|\lim_{n\to\infty} \lim_{\delta}(f_n(x_\delta) - f(x_\delta))\right| =$$

$$\left|\lim_{n\to\infty} \lim_{\delta} f_n(x_\delta) - \lim_{\delta} f(x_\delta)\right| =$$

$$\left|\lim_{\delta} \lim_{n\to\infty} f_n(x_\delta) - \lim_{\delta} f(x_\delta)\right| =$$

$$\left|\lim_{\delta} f(x_\delta) - \lim_{\delta} f(x_\delta)\right| = 0.$$

So the hypotheses of Theorem 3.6 are satisfied which completes the proof.

The last theorem is of special interest when one considers how the weak topology is defined. Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ converges weakly to f_o if and only if $\lim_{n\to\infty} \Phi(f_n)$ equals $\Phi(f_o)$ for every continuous linear functional Φ on $Q(X,\mathcal{P})$. Since every continuous linear functional on $Q(X,\mathcal{P})$ can be represented as a ψ integral (Theorem 2.4), Theorem 3.7 yields a very general result on the interchange of limits and integration for the ψ integral.

<u>Corollary 3.1</u>. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $Q(X,\mathcal{P})$ such that $\{f_n\}_{n=1}^{\infty}$ is norm bounded and such that $\operatorname*{Lim}_{n\to\infty} f_n(x)$ exists for all x in X. Then $\operatorname*{Lim}_{n\to\infty} \psi \int_X f_n \, du$ equals $\psi \int_X \operatorname*{Lim}_{n\to\infty} f_n \, du$ for every p-volume u of bounded variation if and only if $\operatorname*{Lim}_{\delta} \operatorname*{Lim}_{n\to\infty} f_n(x_\delta)$ equals $\operatorname*{Lim}_{n\to\infty} \operatorname*{Lim}_{\delta} f_n(x_\delta)$ for every fundamental net of points $\{x_\delta\}$.

For some $Q(X,\mathcal{P})$ spaces the continuous linear functionals can be expressed as well-known concrete Stieltjes integrals. Let $(X,\mathcal{P})$ be the volume pair of Example 2.2. In [5] J. A. Dyer has shown that if u is a real valued p-volume then the function f_u on $[a,b]$ defined by: $f_u(a) = u(\{a\})$ and

$$f_u(t) = u((a,t]) + u(\{a\})$$

for t in $(a,b]$ is a function of bounded variation on $[a,b]$. Let ψ be the choice function on $\mathcal{P}$ defined by: $\psi(\{a\})$ is a and $\psi((c,d])$ is d. It is also shown in [5] that

$$\psi \int_X h\,du = h(a) f_u(a) + R \int_a^b h\, df_u$$

where this last integral is the right Cauchy integral. Since u is of bounded variation the integrals are independent of ψ. Combining these results with Corollary 3.1 we have the following.

<u>Corollary 3.2</u>. Let $(X,\mathcal{P})$ be the volume pair of Example 2.2 and let $\{h_n\}_{n=1}^{\infty}$ be a sequence in $Q(X,\mathcal{P})$. Then

$$\lim_{n\to\infty} R \int_a^b h_n\, df_u = R \int_a^b \lim_{n\to\infty} h_n\, df_u$$

for every function f_u of bounded variation on $[a,b]$ if and only if $\lim_{n\to\infty} h_n(x)$ exists for all x in $[a,b]$; $\lim_{x\to z^-} \lim_{n\to\infty} h_n(x)$ equals $\lim_{n\to\infty} h_n(x)$ for all z in $(a,b]$; $\lim_{x\to z^+} \lim_{n\to\infty} h_n(x)$ equals $\lim_{n\to\infty} \lim_{x\to z^+} h_n(x)$ for all z in $[a,b)$; $\{h_n\}_{n=1}^{\infty}$ is norm bounded.

In a similar fashion one may start from the functional representations given in [14], [8] for special $Q(X,\mathcal{P})$ spaces to obtain theorems for the interchange of limits and integration for some non-ψ type Stieltjes integrals including the mean Stieltjes integral and the interior integral.

For some $Q(X,\mathcal{P})$ spaces it is impossible to completely characterize the fundamental nets of points. For example, if X is the set $\mathfrak{n}$ of all positive integers and $\mathcal{P}$ is taken to be the pre-algebra of all subsets of $\mathfrak{n}$ then $Q(X,\mathcal{P})$ becomes the space m of all bounded complex sequences. For this quasi-continuous space, no characterization of the fundamental nets of points is apparent. Nevertheless, because they exist, we are able to give new conditions for weak sequential convergence in m.

Theorem 3.8. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in m. Then $\{f_n\}_{n=1}^{\infty}$ converges weakly to 0 if and only if $\lim_{n\to\infty} f_n(k)$ equals 0 for every positive integer k; there exists a constant M such that $\|f_n\| \leq M$ for all n; if $\{x_k\}_{k=1}^{\infty}$ is a sequence of positive integers such that $\lim_{k\to\infty} x_k$ equals $+\infty$ and if ϵ is greater than 0 then there exists a positive integer N such that for each integer $n_o \geq N$ there exists a subsequence $\{x_{k_t}\}_{t=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ such that $|f_{n_o}(x_{k_t})|$ is smaller than ϵ for all $t = 1,2,\ldots.$

Proof: Suppose first that $\{f_n\}_{n=1}^{\infty}$ converges weakly to 0. The first two conditions are standard results. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that $\operatorname{Lim}_{k\to\infty} x_k$ is $+\infty$ and let ϵ greater than 0 be given. By Theorem 3.2 there exists a fundamental net of sets $\{D_\delta\}$ such that for each δ, $\{x_k\}_{k=1}^{\infty}$ is frequently in D_δ. From each D_δ, select z_δ, so that z_δ equals x_k for some integer k. $\{z_\delta\}$ is a fundamental net of points. Consider the $\mathcal{P}$-subdivision $\{\{1\},\ldots,\{N_o\},\{N_o+1,N_o+2,\ldots\}\}$ of n where N_o is an arbitrary fixed positive integer. Since $\{D_\delta\}$ is a fundamental net of sets $\{D_\delta\}$ is eventually contained in one of the sets of the $\mathcal{P}$-subdivision. However, $\{x_k\}_{k=1}^{\infty}$ is frequently in each member of $\{D_\delta\}$ and since $\operatorname{Lim}_{k\to\infty} x_k$ is $+\infty$ it follows that D_δ is eventually a subset of $\{N_o+1,N_o+2,\ldots\}$. Thus, eventually z_δ is greater than N_o and since N_o was arbitrary it follows that $\operatorname{Lim}_{\delta} z_\delta$ is $+\infty$. By Theorem 3.6 we see that $\operatorname{Lim}_{n\to\infty} \operatorname{Lim}_{\delta} f_n(z_\delta)$ is 0. Choose an integer N such that for all n greater than N we have $|\operatorname{Lim}_{\delta} f_n(z_\delta)|$ less than ϵ. Let n_o be a fixed integer greater than N and pick δ_1 so that if δ

follows δ_1 then $|f_{n_o}(z_\delta)|$ is less than ϵ. We will now construct a subsequence $\{x_{k_t}\}_{t=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ such that for each t there exists a $\delta \geq \delta_1$ such that x_{k_t} equals z_δ. To do this let x_{k_1} equal z_{δ_1}. Note that z_{δ_1} is equal to $x_{k'}$ for some integer k'. Choose δ_2 greater than δ_1 such that z_{δ_2} is greater than $\max\{x_1,\ldots,x_{k'}\}$. Such a choice is always possible because $\operatorname{Lim}_\delta z_\delta$ is $+\infty$. Let x_{k_2} equal z_{δ_2}. Note that z_{δ_2} equals $x_{k''}$ for some integer k''. Also, k_2 is greater than k_1. Continue the process by induction to obtain a subsequence $\{x_{k_t}\}_{t=1}^{\infty}$. By our choice of $\{x_{k_t}\}_{t=1}^{\infty}$ it follows easily that $|f_{n_o}(x_{k_t})|$ is less than ϵ for all $t = 1,2,\ldots$. Thus, $\{f_n\}_{n=1}^{\infty}$ has the desired properties.

Conversely, assume the result is false and that $\{f_n\}_{n=1}^{\infty}$ does not converge weakly to 0. Then there exists a fundamental net of points $\{y_\delta\}$ such that $\operatorname{Lim}_{n\to\infty} \operatorname{Lim}_\delta f_n(y_\delta)$ does not equal 0. From this fact and from consideration of the $\mathcal{P}$-subdivision $\{\{1\},\ldots,\{N_o\},\{N_o+1,N_o+2,\ldots\}\}$ we see that $\operatorname{Lim}_\delta y_\delta$ is

$+\infty$. Since $\operatorname{Lim}_{n\to\infty} \operatorname{Lim}_{\delta} f_n(y_\delta)$ does not equal 0, we can assume, without loss of generality, that there exists an $\eta > 0$ such that $|\operatorname{Lim}_{\delta} f_n(y_\delta)|$ is greater than η for every positive integer n. Select δ_1 so that $|f_1(y_\delta)|$ is larger than η if $\delta \geq \delta_1$. This is always possible because $|\operatorname{Lim}_{\delta} f_1(y_\delta)|$ exists and is larger than η. Choose δ_2 greater than δ_1 such that y_{δ_2} follows y_{δ_1} and $|f_2(y_\delta)|$ is greater than η if $\delta \geq \delta_2$. This selection is possible because $|\operatorname{Lim}_{\delta} f_2(y_\delta)|$ exists and is larger than η and because $\operatorname{Lim}_{\delta} y_\delta$ is $+\infty$. Note also that $|f_1(y_\delta)|$ is greater than η if $\delta \geq \delta_2$ since δ_2 is greater than δ_1. We continue the process by induction. Choose δ_n such that δ_n is greater than δ_{n-1}, y_{δ_n} is greater than $y_{\delta_{n-1}}$, and $|f_n(y_\delta)|$ is greater than η if $\delta \geq \delta_n$. Note that $|f_i(y_\delta)|$ is greater than η if $i = 1,2,\ldots,n$ and $\delta \geq \delta_n$. For each positive integer n let x_n equal y_{δ_n}. Since $\{y_{\delta_i}\}_{i=1}^{\infty}$ is a strictly increasing sequence of positive integers it follows that $\operatorname{Lim}_{n\to\infty} x_n$ is $+\infty$. However, for any subsequence $\{x_{n_t}\}_{t=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ we have

$|f_m(x_{n_t})|$ greater than η for all m if t is sufficiently large. This is a contradiction and the proof is complete.

As we stated in the introduction, Chapter four of this dissertation is concerned with the problem of best uniform approximations in $Q(X,\mathcal{P})$ spaces. The following theorem will play an important part in that investigation. It is included here because it is a result about the weak topology of $Q(X,\mathcal{P})$.

Theorem 3.9. Let $(X,\mathcal{P})$ be a volume pair such that if x, y are in X with x not equal to y then there exist sets A_x and A_y in $\mathcal{P}$ such that x is in A_x, y is in A_y and $A_x \cap A_y$ is empty. (Note that all of our examples of volume pairs satisfy this condition.) Let F be a subset of $Q(X,\mathcal{P})$. Then the following conditions are equivalent.

(1) F is norm bounded and if $\{x_\delta\}$ is a fundamental net of points then for every ϵ greater than 0 and $\bar{\delta}$ there exist $\delta_1,\ldots,\delta_k \geq \bar{\delta}$ such that

$\min_{1\leq i\leq k} |f(x_{\delta_i}) - \operatorname{Lim}_{\delta} f(x_\delta)|$ is less than ϵ for all

f in F.

(2) F is norm bounded and if F_o is a denumerable subset of F and $\{x_n\}_{n=1}^{\infty}$ is a sequence in X for which $\{f(x_n)\}_{n=1}^{\infty}$ converges for each f in F_o, then for every ϵ greater than 0 and for every positive integer N there exist $n_1,\ldots,n_k \geq N$ such that $\min_{1\leq i\leq k} |f(x_{n_i}) - \lim_{n\to\infty} f(x_n)| < \epsilon$ for all f in F_o.

(3) F is weakly sequentially compact.

Proof: Suppose (1) is true and let F_o be a denumerable subset of F for which (2) is not true. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\{f(x_n)\}_{n=1}^{\infty}$ converges for each f in F_o and there exists an ϵ greater than 0, a positive integer N such that if $n_1,\ldots,n_k \geq N$ then we can find an f in F_o such that $|f(x_{n_i}) - \lim_{n\to\infty} f(x_n)| \geq \epsilon$ for $i = 1,\ldots,k$. It follows from Theorem 3.2 that one can find a fundamental net of sets $\{A_\delta\}$ such that $\{x_n\}_{n=1}^{\infty}$ is frequently in each A_δ. Either there exists an $A_{\delta'}$ such that $A_{\delta'}$ is a subset of A_δ for all δ or there doesn't. In the first case, consider the directed set $\{(A_{\delta'},i) : i = 1,2,\ldots\}$ where we define $(A_{\delta'}, i)$ to be less than $(A_{\delta'}, j)$ if and only if i is less than j. Clearly,

$$\{A(A_{\delta'}, i) : A(A_{\delta'}, i) = A_{\delta'},\ i = 1, 2, \ldots\}$$

is a fundamental net of sets. From each $A(A_{\delta'}, i)$ select $x(A_{\delta'}, i) = x_k$ where k is the smallest positive integer such that x_k is in $A_{\delta'}$ and $k \geq \max\{N, i\}$. Such a choice is possible because $\{x_n\}_{n=1}^{\infty}$ is frequently in $A_{\delta'}$. Then $\{x(A_{\delta'}, i)\}_{i=1}^{\infty}$ is a fundamental net of points and $\operatorname{Lim}_{(A_{\delta'}, i)} f(x(A_{\delta'}, i))$ is equal to $\operatorname{Lim}_{n\to\infty} f(x_n)$ for all f in F_o because $\{x(A_{\delta'}, i)\}_{i=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ by construction. Now, consider $(A_{\delta'}, 1)$ and ϵ. If $(A_{\delta'}, i_1), \ldots, (A_{\delta'}, i_t)$ are greater than or equal to $(A_{\delta'}, 1)$ then there exists an f in F_o such that

$$|f(x(A_{\delta'}, i_j)) - \operatorname{Lim}_{(A_{\delta'}, i)} f(x(A_{\delta'}, i))| \geq \epsilon$$

for $j = 1, \ldots, t$ because $x(A_{\delta'}, i_j)$ is equal to x_k for some k with $k \geq N$ and because of the assumption that (2) is false. This contradicts (1). Suppose now that no such $A_{\delta'}$ exists. That is, for each $A_{\delta'}$ there exists an A_{δ_2} not equal to A_{δ_1} such that A_{δ_2} is a subset of A_{δ_1}. Suppose there exists an x_r with $r \geq N$ such that x_r is in A_δ for all δ. Consider the directed set $\{(A_\delta, i) : i = 1,2,\ldots,$ for all $\delta\}$ where $(A_{\delta_1}, i) \leq (A_{\delta_2}, j)$ if and only if A_{δ_2} is a subset of A_{δ_1}. The collection

$$\{A(A_\delta, i) : A(A_\delta, i) = A_\delta,\ i = 1,2,\ldots, \text{ for all } \delta\}$$

is a fundamental net of sets. From each $A(A_\delta, i)$ select $x(A_\delta, i)$ such that $x(A_\delta, 1)$ equals x_r for all δ and $x(A_\delta, i)$ is x_n with x_n in A_δ and $n \geq i$ for each $i = 2,3,\ldots$ and for all δ. Such choices are always possible since $\{x_n\}_{n=1}^{\infty}$ is frequently in each A_δ and because x_r is in A_δ for all δ. Clearly, $\{x(A_\delta, i)\}$ is a fundamental net of points. Note that

$\displaystyle\lim_{(A_\delta, i)} f(x(A_\delta, i))$ exists because of Theorem 3.3. For any fixed δ, we have $\displaystyle\lim_{i\to\infty} f(x(A_\delta, i))$ equal to $\displaystyle\lim_{n\to\infty} f(x_n)$ because of the manner in which $\{x(A_\delta, i)\}$ was constructed. Moreover, for each A_δ there exists an $A_{\delta'}$ such that $A_{\delta'}$

is contained in A_δ. From these observations it follows that $\lim_{n\to\infty} f(x_n)$ is equal to $\lim_{(A_\delta,i)} f(x(A_\delta,i))$ for all f in F_o. Now, suppose $(A_{\delta'},j)$ is given. There exists an A_δ such that A_δ is not equal to $A_{\delta'}$ and A_δ is a subset of $A_{\delta'}$. Therefore, $(A_{\delta'},j)$ is less than (A_δ,i) for $i = 1,2,\ldots.$ There exists, by hypothesis, an f in F_o such that $|f(x_r) - \lim_{n\to\infty} f(x_n)|$ is greater than or equal to ϵ. But, since

$$|f(x_r) - \lim_{n\to\infty} f(x_n)| =$$

$$|f(x(A_\delta,1)) - \lim_{(A_\delta,i)} f(x(A_\delta,i))|$$

we see that last quantity is greater than or equal to ϵ. Since $(A_{\delta'},j)$ is arbitrary and $(A_{\delta'},j)$ is less than $(A_\delta,1)$ it follows from this that $\lim_{(A_\delta,i)} f(x(A_\delta,i)$ does not exist. This is a contradiction. Therefore, for every positive integer n there exists an A_{δ_n} such that x_n is not in A_{δ_n}. If $\delta \geq \delta_n$ then x_n is not in A_δ. From each A_δ select x_δ such that x_δ equals x_n with

$n \geq N$. For each f in F_o we have $\operatorname{Lim}_{n\to\infty} f(x_n)$ equal to $\operatorname{Lim}_{\delta} f(x_\delta)$ because the second limit exists and because for each positive integer n' there exists a $\delta_{n'}$ such that if $\delta \geq \delta_{n'}$ then x_δ equals x_n with n greater than n'. Now, let $\bar{\delta}$ be fixed and let $\delta_1,\ldots,\delta_k \geq \bar{\delta}$ be chosen. Since x_{δ_i} is equal to x_{n_i} with $n_i \geq N$ it follows from our assumption on $\{x_n\}_{n=1}^{\infty}$ that there exists an f in F_o such that

$$\min_{1\leq i\leq k} \left|f(x_{n_i}) - \operatorname{Lim}_{n\to\infty} f(x_n)\right| \geq \epsilon.$$

Thus,

$$\min_{1\leq i\leq k} \left|f(x_{\delta_i}) - \operatorname{Lim}_{\delta} f(x_\delta)\right| \geq \epsilon$$

because $\operatorname{Lim}_{n\to\infty} f(x_n)$ is equal to $\operatorname{Lim}_{\delta} f(x_\delta)$. Since $\bar{\delta}$ was arbitrary this contadicts (1) and so (1) implies (2).

We will now show that (2) implies (3). The proof is taken from Theorem 29 page 280 in [2]. It is included for reference. Recall that $Q(X,\mathcal{P})$ is a closed subalgebra of the complex algebra $B(X)$. Moreover, $Q(X,\wp)$ contains the unit e as well as the complex conjugate of each of its elements. Finally, $Q(X,\wp)$ distinguishes between the points of X because of the hypothesis on $\wp$. Let S_1 consist of those nonzero continuous linear functionals Φ in the closed unit sphere of $(Q(X,\wp))^*$ for which $\Phi(fg)$ equals $\Phi(f)\Phi(g)$. S_1 is not empty since it contains the evaluation functionals. It follows from [2], Theorem 18, page 275, that S_1 is a compact Hausdorff space. Define the map V from $Q(X,\wp)$ into $C(S_1)$ by $V(f)$ is f_1 where $f_1(\Phi)$ equals $\Phi(f)$ for all Φ in S_1. V is a linear isometry onto $C(S_1)$. The mapping ψ from X into S_1 defined by $\psi(S)$ equals Φ_S, where $\Phi_S(f)$ is $f(S)$ for all f in $Q(X,\wp)$, is a one to one embedding of X as a dense subset of S_1. Let F be a norm bounded subset of $Q(X,\wp)$ and let $\tilde{F}$ be the set $V(F)$. Then $\tilde{F}$ is a norm bounded subset of $C(S_1)$. Let $\tilde{F}_o$ be a denumerable subset of $\tilde{F}$ and let $\{\Phi_{S_o}, \Phi_{S_1}, \ldots\}$ be a sequence in S_1 contained in the range of ψ for which $\lim_{n\to\infty} \tilde{f}(\Phi_{S_n})$ is equal to $\tilde{f}(\Phi_{S_o})$ for all $\tilde{f}$ in $\tilde{F}_o$. Let F_o be the set $V^{-1}(\tilde{F}_o)$ and consider the sequence $\{S_i\}_{i=1}^{\infty}$ in X such that $\psi(S_i)$ equals $\bar{\varphi}_{S_i}$ for all i. Now, $\operatorname{Lim}_{n\to\infty} f(S_n)$ equals $\operatorname{Lim}_{n\to\infty} \bar{\varphi}_{S_n}(f)$ which in turn is equal to $\operatorname{Lim}_{n\to\infty} \tilde{f}(\bar{\varphi}_{S_n})$ and since this last limit exists so does the first, for all f in F_o. Therefore,

$$\min_{1\leq i\leq k} |\tilde{f}(\bar{\varphi}_{S_{n_i}}) - \operatorname{Lim}_{n\to\infty} \tilde{f}(\bar{\varphi}_{S_n})| < \epsilon.$$

It follows from Theorem 14, page 269, in [2] that $\tilde{F}$ is weakly sequentially compact in $C(S_1)$. But, $Q(X,\rho)$ is isomorphically isometric to $C(S_1)$ and so F is a weakly sequentially compact subset of $Q(X,\rho)$. Thus, (2) implies (3).

The proof will be complete once it is shown that (3) implies (1). Let $\{x_\delta\}$ be a fundamental net of points. Using the same notation as before consider the net $\{\bar{\varphi}_{x_\delta}\}$. Since S_1 is a compact Hausdorff space there exists a subnet $\{\bar{\varphi}_{x_{\delta'}}\}$ of $\{\bar{\varphi}_{x_\delta}\}$ which strongly converges. The set $\tilde{F}$, which equals $V(F)$, is weakly sequentially compact because F is and because V is a linear isometry. From Theorem 14, page 269, in [2] it follows that for every ϵ

greater than 0 and $\bar{\delta}'$ there exist $\delta_1', \ldots, \delta_k' \geq \bar{\delta}'$ such that

$$\min_{1 \leq i \leq k} |\tilde{f}(\varrho_{x_{\delta_i'}}) - \operatorname{Lim}_{\delta'} \tilde{f}(\varrho_{x_{\delta'}})| < \epsilon$$

for all $\tilde{f}$ in $\tilde{F}$. Since $\tilde{f}(\varrho_{x_\delta})$ equals $f(x_\delta)$ for all δ we see that

$$\min_{1 \leq i \leq k} |f(x_{\delta_i'}) - \operatorname{Lim}_{\delta'} f(x_{\delta'})| < \epsilon$$

for all f in F. Finally, because $\operatorname{Lim}_{\delta'} f(x_{\delta'})$ equals $\operatorname{Lim}_{\delta} f(x_\delta)$ for all f in F and because for every δ there exists a $\delta' \geq \delta$ we obtain (1).

When $(X, \mathcal{P})$ is the volume pair of Example 2.7, then $Q(X, \mathcal{P})$ is $B(X)$. Consequently, using the results of Example 3.6 we see that Theorem 3.9 reduces to Theorem 29, page 280 in [2] for this particular $Q(X, \mathcal{P})$ space. This fact points out again that many of the properties of $B(X)$ can be extended to all $Q(X, \mathcal{P})$ spaces by using fundamental nets of points. Ultrafilters for $B(X)$ are nothing more than fundamental nets of sets for this $Q(X,\mathcal{P})$ space and considering them in this way allows one to better understand the structure of $B(X)$.

IV. BEST UNIFORM APPROXIMATIONS IN $Q(X,\mathcal{P})$ SPACES

In this chapter the problem of best uniform approximations by elements of linear subspaces of $Q(X,\mathcal{P})$ will be considered. For an historical survey on the subject of best approximations and for a summary of the significance of the subject, the reader is referred to the introduction in [21]. Recall that if G is a subspace of $Q(X,\mathcal{P})$ and g_o is an element of $Q(X,\mathcal{P}) - G$ then an element f in G is said to be a best uniform approximation to g_o by G if and only if $\|f - g_o\|$ is equal to $\inf_g \|g - g_o\|$ where g ranges over all the elements of G. The problem is to determine necessary and sufficient conditions for the existence of best uniform approximations. As a final illustration of the uses of the concept of a fundamental net of points, we will give necessary and sufficient conditions for best uniform approximations in $Q(X,\mathcal{P})$ spaces in terms of these nets. We begin by introducing a definition taken from [21,p.93].

Definition 4.1. Let G be a subspace of the complex normed linear space E and let g_o be in $E - G$. The set of all best uniform approximations to g_o by G will be denoted by $p_G(g_o)$. G is said to be a proximal subspace of E if and only if $p_G(g_o)$ is not empty for each g_o in $E - G$.

Note that G must be a closed subspace of E if it is a proximal subspace. The following theorem gives sufficient conditions for a subspace G to be a proximal subspace. It is taken from [22,p.97] and is stated without proof.

Theorem 4.1. Let E be a complex normed linear space and let G be a linear subspace of E with the property that the closed unit ball of G is weakly sequentially compact. Then G is a proximal subspace of E.

Since we have already characterized, in Theorem 3.9, all of the weakly sequentially compact subsets for most $Q(X,\mathcal{P})$ spaces, the following theorem is obvious.

Theorem 4.2. Let $(X,\mathcal{P})$ be a volume pair satisfying the conditions of Theorem 3.9. Let G be a linear subspace of $Q(X,\mathcal{P})$ and let S_G be the closed unit ball of G. If for every fundamental net of points $\{x_\delta\}$ and for every ϵ greater than 0 and for every $\bar{\delta}$, there exist $\delta_1,\ldots,\delta_K \geq \bar{\delta}$ such that

$$\min_{1\le i\le k} |g(x_{\delta_i}) - \operatorname{Lim}_{\delta} g(x_\delta)| < \epsilon$$

for every g in S_G, then G is proximal.

While Theorem 4.2 yields sufficient conditions for a subspace to be proximal, it does not solve our original problem. The problem posed in the introduction to this chapter was concerned with best uniform approximations to a fixed element by a given subspace. The two questions are entirely different. That is, a subspace G of $Q(X,\mathcal{P})$ may have a best approximation to an element f, without G necessarily being a proximal subspace. For the rest of this dissertation we will be concerned only with the problem as stated in the introduction to this chapter. The methods we will use are basically the same as those used by I. Singer in [21] to analyze $C([a,b])$. The key to this method is the determination of the extreme points for the closed unit ball of $(Q(X,\mathcal{P}))^*$. As we will show, fundamental nets of points are sufficient to settle the question.

<u>Definition 4.2</u>. Let G be a normed linear space and let k be a subset of G. A point z in k is said to be an extreme point of k if and only if whenever z equals $\lambda k_1 + (1 - \lambda)k_2$ with $0 < \lambda < 1$, k_1, k_2 in k, then both k_1 and k_2 equal z.

<u>Theorem 4.3</u>. Let $\bar{\Phi}$ be a continuous linear functional on $Q(X,\mathcal{P})$. Then $\bar{\Phi}$ is an extreme point of the closed unit ball of $(Q(X,\mathcal{P}))^*$ if and only if there exists a fundamental net of points $\{x_\delta\}$ and a complex number α such that $|\alpha|$ equals 1 and $\bar{\Phi}(f)$ is equal to $\alpha \operatorname*{Lim}_{\delta} f(x_\delta)$ for all f in $Q(X,\mathcal{P})$.

Proof: We will first verify that the conditions are sufficient. Let $\{x_\delta\}$ be a fundamental net of points and define $\bar{\Phi}$ by $\bar{\Phi}(f)$ is $\alpha \operatorname*{Lim}_{\delta} f(x_\delta)$ where $|\alpha|$ equals 1. Note that $\bar{\Phi}$ is well defined by Theorem 3.3. $\bar{\Phi}$ is clearly linear and is continuous because

$$\sup_{\|f\| \leq 1} |\bar{\Phi}(f)| = \sup_{\|f\| \leq 1} |\alpha \operatorname*{Lim}_{\delta} f(x_\delta)|$$

$$\leq 1.$$

Thus, $\|\bar{\Phi}\| \leq 1$ and a straightforward argument will show that $\bar{\Phi}(\chi_E)$ is equal to 1 for every E in $\mathcal{P}$ and so $\|\bar{\Phi}\|$ is equal to 1. From Theorem 2.4 it follows that there exists a p-volume u of bounded variation such that V_u equals 1 and $\bar{\Phi}(f)$ is $\psi \int_X f\,du$ for each f in $Q(X,\mathcal{P})$. Suppose E is in $\mathcal{P}$ and suppose $\{x_\delta\}$ is not eventually in E. Thus,

$$\begin{aligned} 0 = \alpha \operatorname*{Lim}_{\delta} \chi_E(x_\delta) &= \bar{\Phi}(\chi_E) \\ &= \psi \int_X \chi_E\,du \\ &= u(E). \end{aligned}$$

If $\{x_\delta\}$ is eventually in E then

$$\begin{aligned} \alpha = \alpha \operatorname*{Lim}_{\delta} \chi_E(x_\delta) &= \bar{\Phi}(\chi_E) \\ &= \psi \int_X \chi_E\,du \\ &= u(E). \end{aligned}$$

Now, assume $\bar{\Phi}$ is not an extreme point. Then there exist continuous linear functionals $\bar{\Phi}_1$ and $\bar{\Phi}_2$ and a constant λ such that $0 < \lambda < 1$ and $\bar{\Phi}$ equals $\lambda\bar{\Phi}_1 + (1 - \lambda)\bar{\Phi}_2$ where both $\|\bar{\Phi}_1\|$ and $\|\bar{\Phi}_2\|$ are less than or equal to 1. Let u_1 and u_2 be the two p-volumes of bounded variation associated with $\bar{\Phi}_1$ and $\bar{\Phi}_2$ respectively. For every E in $\mathcal{P}$ we have:

$$
\begin{aligned}
u(E) = \psi \int_X \chi_E du = \bar{\Phi}(\chi_E) \\
&= \lambda\ \bar{\Phi}_1(\chi_E) + (1 - \lambda)\bar{\Phi}_2(\chi_E) \\
&= \lambda\ \psi \int_X \chi_E du_1 + (1 - \lambda)\psi \int_X \chi_E du_2 \\
&= \lambda\ u_1(E) + (1 - \lambda)u_2(E)
\end{aligned}
$$

and so u equals $\lambda u_1 + (1 - \lambda)u_2$ with $V_{u_1} \leq 1$ and $V_{u_2} \leq 1$. Choose E in $\mathcal{P}$ such that $\{x_\delta\}$ is eventually in E. Then

$$
\begin{aligned}
1 = |\alpha| &= |u(E)| \\
&= |\lambda\, u_1(E) + (1 - \lambda)u_2(E)| \\
&\leq \lambda\, |u_1(E)| + (1 - \lambda)\, |u_2(E)| \\
&\leq \lambda + (1 - \lambda) \\
&= 1.
\end{aligned}
$$

Therefore, both $|u_1(E)|$ and $|u_2(E)|$ are equal to 1 and since α equals $\lambda\, u_1(E) + (1 - \lambda)u_2(E)$ we must have $u_1(E)$ and $u_2(E)$ equal to α because α is an extreme point of the closed unit ball in the complex plane. If $\{x_\delta\}$ is not eventually in E then select a $\wp$-subdivision $\{E_j\}_{j=1}^N$ of X, to which E belongs, and suppose $E = E_1$. This is possible by Theorem 2.1 (2). Since $V_u \leq 1$ and since there exists an $E_{j'}$ with j' greater than 1 such that $\{x_\delta\}$ is eventually in $E_{j'}$, we see that

$$1 \geq V_{u_1} \geq \sum_{j=1}^{N} |u_1(E_j)| \geq 1$$

because $|u(E_{j'})| = |\alpha| = 1$. Thus, $u_1(E)$ equals 0 and a similar proof shows that $u_2(E)$ is 0. Therefore, u_1 and u_2 are both equal to u which implies that $\bar{\phi}_1$ and $\bar{\phi}_2$ are equal to $\bar{\phi}$. This is a contradiction.

Conversely, suppose $\bar{\phi}$ is an extreme point of the closed unit ball of $(Q(X,\mathcal{P}))^*$ and let u be the p-volume of bounded variation such that $\bar{\phi}(f)$ equals $\psi\int_X f\,du$ for all f in $Q(X,\mathcal{P})$. It is easy to verify $\|\bar{\phi}\|$ is 1 and so V_u equals 1. Suppose now that there exist two disjoint sets E_1 and E_2 in $\mathcal{P}$ such that both $u(E_1)$ and $u(E_2)$ are not zero. Define u_1 by $u_1(E)$ is $u(E \cap E_1)$. Clearly, u_1 is a p-volume of bounded variation with $V_{u_1} \leq V_u$. Since there exists a $\mathcal{P}$-subdivision of X to which E_1 belongs, we can select $\{F_j\}_{j=1}^N$ in $\mathcal{P}$ such that $X - E_1$ equals $\bigcup_{j=1}^N F_j$ where $F_j \cap F_i$ is empty if i is different than j. Define u_2 by $u_2(E)$ is $\sum_{j=1}^N u(E \cap F_j)$. Again, it is easy to show that u_2 is a p-volume of bounded variation with $V_{u_2} \leq V_u$. Let $\{G_i\}_{i=1}^k$ be any $\mathcal{P}$-subdivision of X. Note that

$$G_i = (E_1 \cap G_i) \cup (F_1 \cap G_i) \cup \ldots \cup (F_N \cap G_i)$$

for $i = 1,\ldots,k$ and each of these sets is in $\mathcal{P}$ and they are pairwise disjoint. Therefore,

$$\sum_{i=1}^{k} |u(G_i)| = \sum_{i=1}^{k} |u(E_1 \cap G_i) + \sum_{j=1}^{N} u(G_i \cap F_j)|$$

$$= \sum_{i=1}^{k} |u_1(G_i) + u_2(G_i)|$$

$$\leq \sum_{i=1}^{k} |u_1(G_i)| + \sum_{i=1}^{k} |u_2(G_i)|$$

$$\leq V_{u_1} + V_{u_2}.$$

Taking supremums we obtain $V_u \leq V_{u_1} + V_{u_2}$. To prove the reverse inequality let $\{H_j\}_{j=1}^{M}$ and $\{G_W\}_{W=1}^{k}$ be $\mathcal{P}$-subdivisions of X. By Theorem 2.1 there exists a $\mathcal{P}$-subdivision $\{J_t\}_{t=1}^{L}$ of X which refines the previous two. Then,

$$\sum_{j=1}^{M} |u_1(H_j)| + \sum_{W=1}^{k} |u_2(G_W)| \leq$$

$$\sum_{t=1}^{L} |u_1(J_t)| + \sum_{t=1}^{L} |u_2(J_t)| =$$

$$\sum_{t=1}^{L} |u(E_1 \cap J_t)| + \sum_{t=1}^{L} (\sum_{j=1}^{N} |u(J_t \cap F_j)|) \leq$$

$$V_u.$$

Taking supremums of both sides we get $V_{u_1} + V_{u_2} \leq V_u$. Therefore, V_u equals $V_{u_1} + V_{u_2}$. Now, let $\bar{u}_1 = u_1/V_{u_1}$ and $\bar{u}_2 = u_2/V_{u_2}$. Note that V_{u_1} and V_{u_2} are both non-zero because of the original assumption on u. Both $\bar{u}_1$ and $\bar{u}_2$ are p-volumes of bounded variation and each has variation 1. Moreover, for any E in $\mathcal{P}$ we have:

$$V_{u_1}\bar{u}_1(E) + (1 - V_{u_1})\bar{u}_2(E) =$$

$$V_{u_1}\bar{u}_1(E) + V_{u_2}\bar{u}_2(E) = u_1(E) + u_2(E) =$$

$$u(E \cap E_1) + \sum_{j=1}^{N} u(E \cap F_j) = u(E).$$

Also, since $u(\emptyset)$ is 0 we see that $\bar{u}_1(E_2)$ and $\bar{u}_2(E_1)$ are both 0 and so $\bar{u}_1$ and $\bar{u}_2$ both differ from u. Since $0 < V_{u_1} < 1$ we see that Φ is not an extreme point of the closed unit ball of $(Q(X,\mathcal{P}))^*$ which is a contradiction. Therefore, if E_1 and E_2 are in $\mathcal{P}$ with $E_1 \cap E_2$ empty then either $u(E_1)$ is 0 or $u_2(E)$ is 0. Let $\mathcal{A}$ be the set $\{A : A$ is in $\mathcal{P}$ with $u(A) \neq 0\}$. We claim that $\mathcal{A}$ is a directed set with respect to set inclusion. It suffices to show that $u(A_1 \cap A_2)$ is not 0. Suppose $u(A_1 \cap A_2)$ is zero. Now, there exist sets $\{D_i\}_{i=1}^k$ and $\{E_j\}_{j=1}^M$ in $\mathcal{P}$ such that

$$A - (A_1 \cap A_2) = \bigcup_{i=1}^{k} D_i \quad \text{and} \quad A_2 - (A_1 \cap A_2) = \bigcup_{j=1}^{M} E_j .$$

Since,

$$u(A_1) = u(A_1 \cap A_2) + \sum_{i=1}^{k} u(D_i)$$

and since $u(A_1)$ is not zero there exists a D_i with $u(D_i)$ not zero. Similarly, there exists an E_j with $u(E_j)$ not zero. However, $D_i \cap E_j$ is empty and this contradicts our earlier result. Thus, $u(A_1 \cap A_2)$ is not zero and so $\mathcal{G}$ is a directed set. If S is the map from $\mathcal{G}$ into $\mathcal{G}$ defined by: $S(A) = A$ then $\mathcal{G}$ forms a net. We will show that it is a fundamental net of sets. To verify (2) of Definition 3.1 let $\{E_i\}_{i=1}^N$ be a $\mathcal{P}$-subdivision of X and let A be in $\mathcal{G}$. It follows that $\{A \cap E_i\}_{i=1}^N$ is a $\mathcal{P}$-subdivision of A. Since $u(A)$ equals $\sum_{i=1}^N u(A \cap E_i)$ and since $u(A)$ is not zero it follows that $u(A \cap E_i)$ is not 0 for some i. But, $A \cap E_i$ is a subset of E_i and $A \cap E_i$ is in $\mathcal{G}$ which shows that $\mathcal{G}$ is a fundamental net of sets. Let A_1 and A_2 be in $\mathcal{G}$ with A_1 contained in A_2. Since $A_2 - A_1$ equals $\bigcup_{i=1}^M D_i$ for some collection of disjoint sets $\{D_i\}_{i=1}^M$ in $\mathcal{P}$, we see that

$$u(A_2) = u(A_1) + \sum_{i=1}^M u(D_i).$$

However, $u(A_1)$ is not zero and $A_1 \cap D_i$ is empty for $i = 1,\ldots,M$ which implies that $u(D_i)$ is 0 for each i. Thus, $u(A_1)$ equals $u(A_2)$. If A_1 and A_2 are in $\mathcal{G}$, then $A_1 \cap A_2$ is in $\mathcal{G}$ and so

$$u(A_1) = u(A_1 \cap A_2) = u(A_2)$$

because $A_1 \cap A_2$ is a subset of both A_1 and A_2. Therefore, u is constant on $\mathcal{G}$ and since V_u is 1 we see that $u(A)$ equals α with $|\alpha|$ equal to 1 for all A in $\mathcal{G}$. From each A in $\mathcal{G}$ choose x_A so that $\{x_A\}$ forms a fundamental net of points. Let E be in $\mathcal{P}$ such that $\{x_A\}$ is not eventually in E. Then $u(E)$ is 0 for otherwise E would be in $\mathcal{G}$ and so $\{x_A\}$ would eventually be in E. On the other hand, if $\{x_A\}$ is eventually in $\hat{E}$, a set in $\mathcal{P}$, and if $u(\hat{E})$ is 0 then $\hat{E}$ is not in $\mathcal{G}$ and so there exists an A in $\mathcal{G}$ with $A \cap \hat{E}$ empty. The reason for this is that there exists a $\mathcal{P}$-subdivision $\{E_i\}_{i=1}^{N}$ of X to which $\hat{E}$ belongs and u cannot vanish on every E_i because V_u is not 0. But, since A is in $\mathcal{G}$, $\{x_A\}$ must eventually be in A and in view of the fact that it is also eventually in $\hat{E}$ with $\hat{E} \cap A$ void we obtain a contradiction. Therefore, $u(\hat{E})$ is not 0 and so $u(\hat{E})$ is α. Select a choice function ψ so that $\psi(E)$ equals x_E if $\{x_A\}$ is eventually in E and is arbitrary if $\{x_A\}$ is not eventually in E. Then,

$$\bar{\Phi}(f) = \psi \int_X f\,du$$

$$= \lim_{\mathcal{D}} \sum_{i=1}^{N} f(\psi(D_i))u(D_i)$$

$$= \lim_{\mathcal{D}} \alpha\ f(\psi(D_{i'}))$$

$$= \alpha \lim_{A} f(x_A)$$

where the first limit is taken over all $\mathcal{P}$-subdivisions of X and $D_{i'}$ is that unique element of $\{D_i\}_{i=1}^{N}$ such that $\{x_A\}$ is eventually in $D_{i'}$. Note also that

$$\lim_{\mathcal{D}} \alpha\ f(\psi(D_{i'})) = \alpha \lim_{A} f(x_A)$$

because of our choice of ψ and because of Theorem 2.3.

Theorem 4.3 is interesting in its own right and it plays an important part in determining necessary and sufficient conditions for best uniform approximations by subspaces in $Q(X,\mathcal{P})$. Because of the importance for applications, we will now give a collection of examples of the application of Theorem 4.3 to those $Q(X,\mathcal{P})$ spaces for which we have determined all the fundamental nets of points. In all of the following examples α will be a complex constant with $|\alpha|$ equal to 1.

Example 4.1. Let X be the set $[a,b]$ and let $\mathcal{P}$ be the collection of all open subintervals of $[a,b]$ along with the singleton subsets of X. Φ is an extreme point of the closed unit ball of $(Q(X,\mathcal{P}))^*$ if and only if there exists a z in $[a,b]$ such that $\Phi(f)$ equals $\alpha f(z)$ for all f; or there exists a z in $[a,b)$ such that $\Phi(f)$ is $\alpha \lim\limits_{x\to z^+} f(x)$ for all f; or there exists a z in $(a,b]$ such that $\Phi(f)$ is $\alpha \lim\limits_{x\to z^-} f(x)$ for all f.

Example 4.2. Let X be as in the last example and let $\mathcal{P}$ be the collection of all subsets of X of the form $(c,d]$ along with $\{a\}$. Φ is an extreme point of the closed unit ball of $(Q(X,\mathcal{P}))^*$ if and only if there exists a z in $[a,b]$ such that $\Phi(f)$ equals $\alpha f(z)$ for all f; or there exists a z in $[a,b)$ such that $\Phi(f)$ is $\alpha \lim\limits_{x\to z^+} f(x)$ for all f.

Example 4.3. Let X be the set of all positive integers and let $\mathcal{P}$ be the collection of all subsets of X of the form $\{N, N+1, \ldots\}$ along with the singletons. Φ is an extreme point of the closed unit ball of $(Q(X,\mathcal{P}))^*$ if and only if there exists a positive integer N such that $\Phi(f)$ equals $\alpha f(N)$ for all f; or $\Phi(f)$ is $\alpha \lim_{N\to\infty} f(N)$ for all f.

Example 4.4. Let X be as in the last example and let $\mathcal{P}$ be the collection of all subsets of X. Recall that in this case $Q(X,\mathcal{P})$ is the space m. Φ is an extreme point of the closed unit ball of $(m)^*$ if and only if there exists a positive integer N such that $\Phi(f)$ equals $\alpha f(N)$ for all f; or there exists an ultrafilter $\{A_\delta\}$ of subsets of X such that $\Phi(f)$ is $\alpha \lim_{\delta} f(a_\delta)$ for all f where a_δ is in A_δ for all δ.

In order to apply Theorem 4.3 to the problem of best uniform approximations in $Q(X,\mathcal{P})$ spaces, we need some preliminary results from [21]. The following is Theorem 1.13, page 62 in [21]. It is included for reference and is stated without proof.

Theorem 4.4. Let E be a normed linear space and let G be a subspace of E. Suppose x is in $E - \bar{G}$ and g_o is in G. Then g_o is a best approximation to x by G if and only if for every g in G there exists an extreme point f^g of the closed unit ball in E^* such that

$$\mathrm{Re}(f^g(g - g_o)) \geq 0$$

and

$$f^g(x - g_o) = \|x - g_o\|.$$

By using Theorems 4.3 and 4.4 we are now able to solve the problem stated in the introduction to this chapter.

Theorem 4.5. Let $(X,\wp)$ be a volume pair. Suppose G is a linear subspace of $Q(X,\wp)$, f is an element of $Q(X,\wp) - \bar{G}$ and g_o is an element of G. Then g_o is a best uniform approximation to f by G if and only if for every g in G there exists a fundamental net of points $\{x_\delta^g\}$ such that:

$$\mathrm{Re}[(\operatorname*{Lim}_{\delta}(f(x_\delta^g)) - g_o(x_\delta^g))^*]\operatorname*{Lim}_{\delta} g(x_\delta^g) \geq 0$$

and

$$|\operatorname*{Lim}_{\delta}(f(x_\delta^g) - g_o(x_\delta^g))| = \sup_{x\in X}|f(x) - g_o(x)|.$$

Proof: From previous results we see that g_o is a best uniform approximation to f by G if and only if for every g in G there exists a fundamental net of points $\{x_\delta^g\}$ and a scalar α with $|\alpha| = 1$, such that

$$\mathrm{Re}\{\alpha \operatorname*{Lim}_{\delta}[g_o(x_\delta^g) - g(x_\delta^g)]\} \geq 0$$

and

$$\alpha \operatorname*{Lim}_{\delta}[f(x_\delta^g) - g_o(x_\delta^g)] = \sup_{x\in X}|f(x) - g(x)|.$$

Clearly,

$$\mathrm{Im}\{\alpha \operatorname*{Lim}_{\delta}[f(x_\delta^g) - g_o(x_\delta^g)]\} = 0$$

and since $|\alpha|$ is 1 we see that α equals $M^*/|M|$ where

$$M = \operatorname*{Lim}_{\delta}[f(x_\delta^g) - g_o(x_\delta^g)]$$

if this quantity is not 0. If it is 0 then the result is trivial. Thus, g_o is a best uniform approximation to f by G if and only if for every g in G there exists a fundamental net of points $\{x_\delta^g\}$ satisfying the second condition of the theorem and

$$\mathrm{Re}\{M^* \operatorname*{Lim}_{\delta}[g_o(x_\delta^g) - g(x_\delta^g)]\} \geq 0.$$

Since $g_o - G$ equals G and applying the last result to $\{x_\delta^{g_o-g}\}$ we obtain the first condition of the theorem.

Although the last theorem is a solution to our problem, in some special cases Theorem 4.5 can be improved upon. In the following theorem, we consider the problem of best uniform approximations by a subspace G of finite dimension.

<u>Theorem 4.6</u>. Let G be an n-dimensional subspace of $Q(X,\mathcal{P})$, let f be in $Q(X,\mathcal{P}) - G$ and suppose g_o is in G. The following statements are equivalent:

(1) g_o is a best uniform approximation to f by G;

(2) there exists h fundamental nets of points $\{x_{\delta_1}\},\ldots,\{x_{\delta_h}\}$ where $1 \leq h \leq n+1$ if the scalars are real and $1 \leq h \leq 2n+1$ if the scalars are complex and h scalars $\lambda_1,\ldots,\lambda_h$ all greater than 0 with $\sum_{j=1}^{h} \lambda_j$ equal to 1, such that

$$\sum_{j=1}^{h} \lambda_j \operatorname*{Lim}_{\delta_j} g(x_{\delta_j}) = 0$$

for all g in G and

$$\sum_{j=1}^{h} \lambda_j \operatorname*{Lim}_{\delta_j}(f(x_{\delta_j}) - g_o(x_{\delta_j})) = \|f - g_o\|;$$

(3) there exist h fundamental nets of points $\{x_{\delta_1}\},\ldots,\{x_{\delta_h}\}$ where $1 \leq h \leq n+1$ if the scalars are real and $1 \leq h \leq 2n+1$ if the scalars are complex, and h numbers $\lambda_1,\ldots,\lambda_h$ greater than 0 with

$\sum_{j=1}^{h} \lambda_j$ equal to 1, such that

$$\sum_{j=1}^{h} \lambda_j \operatorname*{Lim}_{\delta_j} g(x_{\delta_j}) = 0$$

for all g in G and

$$\operatorname*{Lim}_{\delta_j}(f(x_{\delta_j}) - g(x_{\delta_j})) = \|f - g_o\|$$

for each $j = 1,\ldots,h$.

Proof: The result follows from Theorem 4.3 and Theorem 1.1 of [21].

Theorems 4.5 and 4.6 are good examples of the use of fundamental nets of points. In applying these results to specific $Q(X,\mathcal{P})$ spaces, all that is necessary is a knowledge of the nets for that space. As we have shown in previous examples, this can be done in many cases.

V. CONCLUDING REMARKS

The original intent of this dissertation was to determine the weakly sequentially compact subsets of $Q(X,\mathcal{P})$, in order to use Theorem 4.1 in the determination of best uniform approximations. In considering this question, one must have some knowledge of the weak topology of $Q(X,\mathcal{P})$ spaces, especially the sequential properties of this topology. Up until this time, little has been done in this area. Apparently, the reason for this is that there has not been a characterization of $Q(X,\mathcal{P})$ spaces that is suited for such an analysis. The concept of a fundamental net of points is the major contribution to the study of the weak topology in this dissertation. This concept is based upon set theoretic properties of the volume pair $(X,\mathcal{P})$ for the space $Q(X,\mathcal{P})$ and as such is applicable to many different areas of interest concerning these spaces. For example, we were able to give a new characterization for $Q(X,\mathcal{P})$ spaces as well as necessary and sufficient conditions for best uniform approximations by subspaces of $Q(X,\mathcal{P})$. Neither of these results deals with the weak topology. This concept is also applicable to problems concerning the weak topology of $Q(X,\rho)$ spaces since we were able to give necessary and sufficient conditions for weak sequential convergence using this concept. This in turn allowed us to determine the weakly sequentially compact subsets. Thus, it is apparent that the concept of a fundamental net of points is not topologically oriented but rather is inherent in the structure of the space itself. As a result of this, the many diverse examples of $Q(X,\rho)$ spaces are more easily understood as a class of Banach spaces. For example, we have shown that several of the properties of $QC([a,b])$ which were thought to be related to the order properties of the real line are not in fact unique to this space, but are properties of general quasi-continuous function spaces. It is for this reason that fundamental nets of points are important to the study of the abstract properties of $Q(X,\rho)$.

Naturally, the usefulness of a concept depends not only on its theoretic applications but also on how easy it is to use in concrete examples. In many of the classical $Q(X,\rho)$ spaces the fundamental nets of points can be completely determined. As a result of this, the abstract results cited above are easily applied to these spaces.

For example, some limit interchange theorems for Stieltjes integrals become trivial corollaries of our more general result (Theorem 3.7) concerning the interchange of limits. This points out again the significance of fundamental nets of points.

For the reasons mentioned above, it would seem likely that this concept could be applied to settle other questions concerning $Q(X,\rho)$ spaces. For example, can sets whose closure is compact be characterized? This particular question has been settled in [9] but only for those $Q(X,\rho)$ spaces with a special ordering property on X. It is possible to characterize weak convergence in $Q(X,\rho)$ using fundamental nets of points? If it is possible, then this would improve upon the results obtained in Chapter three, since only weak sequential convergence was considered there.

In Chapter four the question of best uniform approximations by subspaces of $Q(X,\rho)$ was settled. Unfortunately, neither Theorem 4.5 nor Theorem 4.6 is especially useful in actually determining a best uniform approximation. Does there exist an algorithm which will allow one to constructively determine a best approximation? One of the major difficulties in developing such an algorithm is that best uniform approximations to an element by a subspace need not be unique in $Q(X,\rho)$ spaces. For example, let A be the subspace of $QC([a,b])$ consisting of all functions f such that $f(b)$ is zero and let g be the element of $QC([a,b])$ defined by $g(x)$ equals 2 for all x in $[a,b]$. Clearly, g is not in A. Now consider the two functions f_1 and f_2 defined by

$$f_1(x) = 1 \quad \text{for} \quad x \in [a,b)$$

and

$$f_1(b) = 0$$

while

$$f_2(x) = 0 \quad \text{for all} \quad x \quad \text{in} \quad [a,b].$$

Then f_1 and f_2 are both in A. Moreover,

$$\|f_1 - g\| = \|f_2 - g\| = 2.$$

Also, since every element in A is 0 at b whereas $g(b)$ is 2 we see that

$$\inf_{f \in A} \|f - g\| = 2.$$

Thus, we see that best uniform approximations by subspaces are not unique in $Q(X,\mathcal{P})$ spaces. This lack of uniqueness seems to make an algorithm difficult to find.

In conclusion, the results given in this dissertation point out that fundamental nets of points are very useful in the analysis of quasi-continuous function spaces. The concept is easy to apply to the classical $Q(X,\mathcal{P})$ spaces and it allows one to understand these spaces as a class of Banach spaces.

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