74-9115

GRIMES, Joseph Edward, 1941-
REGRESSION TYPE ESTIMATORS BASED ON
PRELIMINARY TEST OF SIGNIFICANCE.

Iowa State University, Ph.D., 1973
Statistics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

Regression type estimators based on preliminary test of significance

by

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A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Department: Statistics
Major: Statistics (Survey Sampling)

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1973

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I. INTRODUCTION

Sampling techniques are being used with increased frequency, and it is now widely recognized that a properly conducted sample survey can often be a good substitute for a complete census when it is desirable to gain information about some characteristics of a population.

The errors associated with a sample survey may be classified into two types of errors, namely sampling errors and nonsampling errors. The sampling error is determined by the sampling design and the estimation procedure. One of the widely used methods in survey sampling for lowering the sampling error is the judicious use of supplementary information.

If data on an auxiliary characteristic X correlated with the characteristic Y under study is available, then it is customary to use this data to provide a more efficient estimate of $\overline{Y}$, the population mean. This can be done either (i) by selecting the sample with probability proportional to size or (ii) using ratio or regression methods of estimation after selecting the sample by simple random sampling. In order to select the sample with probability proportional to size, it is necessary to have in advance the data in regard to size for all the units in the population. This may not always be the case and therefore cannot be used in the selection process, but usually it can be collected in the course of the survey itself and used in a variety of ways at the estimation stage to provide more efficient estimates of the population characteristics under study. We do not consider the sample design aspect but refer to Sampford (1967) and his bibliography in this subject matter.

Frequently the situation is such that the auxiliary characteristic X is positively correlated with Y. In that case if an estimate of the population mean $\bar{Y}$ is sought and population mean $\bar{X}$ is known, then, under certain conditions, the classical ratio estimator of $\bar{Y}$, given by

$$\bar{y}_R = \frac{\bar{y}}{\bar{x}} \bar{X},$$

is more efficient than the mean per unit estimate $\bar{y}$.

An example of the use of the ratio estimator is the estimation of the average surface area of the leaves in a basket. The average weight of all the leaves would be easy to determine. Hence weight would be the auxiliary variable X. Since surface area is difficult to determine, an estimate of this average could be found by determining the average surface area of a small sample and then using the ratio estimator.

The estimator $\bar{y}_R$ is in general biased and the bias is given by $-\mathrm{cov}(\frac{\bar{y}}{\bar{x}}, \bar{x})$. This bias is negligible if the relationship between Y and X is linear and passes through the origin or if the sample size is sufficiently large.

Several lines of research have been considered for reducing the amount of bias. Hartley and Ross (1954), considered the estimator $\bar{r} = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{x_i}$ and obtained its bias. Estimating the bias and then adjusting the estimator $\bar{r}$ for bias, they developed an unbiased ratio type estimator of $\bar{Y}$. Another method of reducing bias is the technique developed by Quenouille (1956), Murthy and Nanjamma (1959), Durbin (1959) for estimating the bias of a ratio estimator unbiasedly to the second order of approximation based on interpenetrating sub-samples. This estimator may then be used to correct the ratio estimator for its bias, thereby getting a ratio estimator, which is unbiased to a second order of approximation. Another technique consists in modifying the sampling procedure so that the ratio estimator $\bar{y}_R$ becomes unbiased. The procedures suggested by Lahiri (1951) and Midzuno (1950) are of this type.

No matter what type of ratio-estimator is used, its usefulness is somewhat restricted. The ratio estimator is at its best when the relationship between Y and X is linear and the line passes through the origin. Cochran (1963) discusses conditions under which the ratio estimator is optimum.

Even though Y and X are correlated and the relationship between the two variables is linear, it is often the case that the relationship does not pass through the origin or the correlation between Y and X is not sufficiently high to recommend the use of a ratio type estimator.

In such situations the use of regression type estimators is usually recommended. A frequently used estimator of this type is the so-called difference estimator suggested by Hansen, Hurwitz and Madow (1953), defined as

$$\bar{y}_d = \bar{y} + \beta_0(\bar{X}-\bar{x}),$$

where β_0 is a fixed constant, assumed to be known. The value of β_0 that minimize $V(\bar{y}_d)$ is easily shown to be $\beta_0=\beta_2= S_{12}/S_1^{\,2}$, where β_2 is the regression coefficient of Y on X. If no reliable guess can be made about the value of the regression coefficient, the usual practice is to estimate it from the sample by $\hat{\beta}_2 = \frac{s_{12}}{s_1^{\,2}}$ where s_{12} and $s_1^{\,2}$ are unbiased estimates of S_{12} and $S_1^{\,2}$ respectively and we use as an estimator of $\bar{Y}$, the regression estimator $\bar{y}_\ell$ defined as

$$\bar{y}_\ell = \bar{y} + \hat{\beta}_2(\bar{X}-\bar{x}) \ .$$

The difference estimator $\bar{y}_d$ is an unbiased estimator of the population mean $\bar{Y}$ and its variance is given by

$$V(\bar{y}_d) = \frac{\sigma_2^{\,2}(1-\rho^2)}{n}[1+\delta^2]$$

where

$$\delta = \frac{(\rho - \frac{\beta_0\sigma_1}{\sigma_2})}{(1-\rho^2)^{1/2}} .$$

The regression estimator on the other hand is generally biased, the bias vanishing when the relationship between Y and X is linear. Further its variance to terms of order $\frac{1}{n^2}$ is given by

$$V(\bar{y}_\ell) = \frac{\sigma_2{}^2(1-\rho^2)}{n}(1 + \frac{1}{n}) .$$

If we are fairly certain about the guessed value β_0 of β_2, it would be desirable to use $\bar{y}_d$ as an estimator of $\bar{Y}$. If however, this is not the case and our guessed value β_0 is likely to differ from β_2, it may be preferable to use the regression estimator $\bar{y}_\ell$. Sometimes, however, it may not be obvious whether our guessed value is close enough to β_2 or not, and in such situations it would be desirable to choose between the two estimators on the basis of a test of significance of the relative closeness of β_0 to β_2. The problem of estimation subsequent to tests of significance has been considered by several authors.

Although inference procedures involving preliminary tests of significance have been extensively used in the past by statisticians, only recently, have attempts been made to evaluate properties of such inference procedures. Bancroft

(1944) was the first to study the overall properties of inference procedures which incorporated a preliminary test of significance with subsequent inference of prime interest. He calls such procedures "Inferences for incompletely specified models" (1965). Many authors have covered various areas concerning aspects of preliminary testing.

Kitagawa (1959) discusses biased estimation of linear regression coefficients under an incompletely specified model and gives sequential designs of experiments in two and three stages where preliminary tests of significance are used to decide whether to perform further experiments in order to obtain a better fit.

Larson (1957) and Larson and Bancroft (1963a) derived the bias and mean square error of the predicant Y in multiple regression with k coefficients for the case when a preliminary test is made to see if the last m, where $m<k$, regression coefficients considered jointly are all equal to zero.

The consequences of the use of two common sequential decision rules for determining, in situations of uncertainty, the number of predictors to be included in a final fitted regression model with regard to the bias and mean square error of the predicant Y were developed by Larson (1960) and Larson and Bancroft (1963b).

Johnson (1967) studied the effect of pooling two regression lines.

Kennedy (1971) considered the technique of model building for prediction in regression analysis, based on repeated significance tests.

A survey of the work done in this area reveals that not much has been done in the field of survey sampling. Ruhl and Sedransk (1967), considered the problem of estimating the mean of a population based on a preliminary test of significance. Their work dealt primarily with the consideration of pooling information from two or more sample survey mean estimates. Carrillo (1969) considered the problem of estimation of variance in stratified sampling subsequent to a preliminary test of significance of the homogeneity of strata variances. Tang (1971) considered in detail the problem of allocation of sample sizes to the different strata based on a preliminary test of significance and investigated its efficiency with respect to proportional allocation and modified Neyman allocation.

We shall consider the problem of choosing between the difference estimator $\bar{y}_d$ and the regression estimator $\bar{y}_\ell$. We develop a sometimes regression estimator which chooses between $\bar{y}_d$ and $\bar{y}_\ell$ on the basis of a preliminary test of significance of the relative closeness of β_0 to β_2. We also study the efficiency of the new estimator with respect to the difference and regression estimators. Finally, an analogue of the sometimes regression estimator has been developed for the case of stratified sampling and its efficiency investigated with respect to current estimators in use.

II. REGRESSION TYPE ESTIMATOR $\bar{y}_s$ BASED ON PRELIMINARY TEST OF SIGNIFICANCE

Consider a population of N distinct units $U_i (i = 1,2,...N)$ from which a sample of n units has been drawn by simple random sampling without replacement. Let Y be the characteristic under study and X an auxiliary characteristic which is correlated with Y. Let (Y_i, X_i), $i = 1,2,...N$, denote the values of the characteristics Y and X respectively, corresponding to the ith unit U_i of the population. Consider the problem of estimating the population total $\sum_{i=1}^{N} Y_i$ or the mean

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i \tag{2.1}$$

when data on the auxiliary characteristic X is also available.

A frequently used estimator is the difference estimator $\bar{y}_d$ given by

$$\bar{y}_d = \bar{y} + \beta_0(\bar{X}-\bar{x}) \tag{2.2}$$

where $\bar{y}$ and $\bar{x}$ are the sample means, $\bar{X}$ is the population mean of the characteristic X and β_0 is some fixed constant which is assumed to be known. This is an unbiased estimator and its variance is

$$V(\bar{y}_d) = \frac{N-n}{N-1} \cdot \frac{1}{n}[\sigma_2{}^2+\beta_0{}^2\sigma_1{}^2-2\beta_0\sigma_{12}] \qquad (2.3)$$

$$\simeq \frac{1}{n}[\sigma_2{}^2+\beta_0{}^2\sigma_1{}^2-2\beta_0\sigma_{12}]$$

if the population is infinite or the finite correction factor $\frac{(N-n)}{N-1}$ can be assumed to be unity,

where

$$\sigma_2{}^2 = E(Y-\bar{Y})^2, \qquad (2.4)$$

$$\sigma_1{}^2 = E(X-\bar{X})^2, \qquad (2.5)$$

and

$$\sigma_{12} = E[(Y-\bar{Y})(X-\bar{X})]. \qquad (2.6)$$

If β_0 is in fact equal to $\beta_2 = \frac{\sigma_{12}}{\sigma_1{}^2}$, the regression coefficient of Y on X, then $\bar{y}_d$ is the minimum variance unbiased estimator of $\bar{Y}$.

When β_2 the regression coefficient of Y on X is not known, it is customary to estimate it from the sample with a consistent estimator of β_2 given by

$$\hat{\beta}_2 = \frac{s_{12}}{s_1{}^2} \qquad (2.7)$$

where

$$s_{12} = \frac{1}{n-1}\sum_i^n (x_i-\bar{x})(y_i-\bar{y}) \qquad (2.8)$$

with $\sum_i^n$ indicating the sum over all the units in the sample

and

$$s_1^{\,2} = \frac{1}{n-1} \sum_{i}^{n} (x_i - \bar{x})^2 \; . \tag{2.9}$$

The estimator of $\bar{Y}$, so obtained, is known as the regression estimator and is given by

$$\bar{y}_\ell = \bar{y} + \hat{\beta}_2 (\bar{X} - \bar{x}) \; . \tag{2.10}$$

In general, this estimator will be biased and the bias is given by

$$\text{Bias}(\bar{y}_\ell) = -\text{Cov}(\hat{\beta}_2, \bar{x}) \; . \tag{2.11}$$

The variance of this estimator to the first order of approximation is

$$V(\bar{y}_\ell) \simeq \frac{N-n}{N-1} \frac{1}{n} \sigma_2^{\,2} (1-\rho^2) \tag{2.12}$$

$$\simeq \frac{\sigma_2^{\,2} (1-\rho^2)}{n}$$

where $\rho = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$ is the correlation coefficient between Y and X. If the joint distribution of Y and X is bivariate normal, then the bias of the regression estimator reduces to zero and its variance is given by

$$V(\bar{y}_\ell) = \frac{\sigma_2 (1-\rho^2)}{n} [1 + \frac{1}{n-3}] \; . \tag{2.13}$$

It is known (see for example, Hasel (1942)) that if sampling is with replacement, $E(y_{ij}|x_i) = \alpha + \beta_2 x_i$ and $V(y_{ij}|x_i)$ is a constant that is independent of x_i, then the regression estimator is optimum in the sense that it is unbiased and has minimum variance.

From past experience, we are often able to make an intelligent guess about β_2 the regression coefficient of Y on X. Let β_0 denote the guessed value of β_2. If β_0 is relatively close to β_2, it would appear from the above that $\bar{y}_d$ is more appropriate than $\bar{y}_\ell$ as an estimator of $\bar{Y}$. If however, β_0 is not relatively close to β_2, the regression estimator $\bar{y}_\ell$ would appear to be the most appropriate of the two estimators. We therefore propose an estimator which chooses between $\bar{y}_\ell$ and $\bar{y}_d$, based on a preliminary test of significance of the relative closeness of β_0 to β_2. This estimator to be called sometimes regression estimator, may be defined as

$$\begin{aligned} \bar{y}_s &= \bar{y} + \beta_0(\bar{X}-\bar{x}) \text{ if the preliminary test indicates that } \beta_2 \text{ is relatively close to } \beta_0, \\ &= \bar{y} + \hat{\beta}_2(\bar{X}-\bar{x}) \text{ otherwise.} \end{aligned} \tag{2.14}$$

A common method of making a test of the relative closeness of β_2 to β_0 is the usage of the statistic

$$t = \frac{\sqrt{n-2}\,(\hat{\beta}_2-\beta_0)\,s_1}{(s_2{}^2-\hat{\beta}_2{}^2 s_1{}^2)^{1/2}} \tag{2.15}$$

where

$$s_2^{\ 2} = \frac{1}{n-1}\sum_{i}^{n}(y_i-\bar{y})^2 \ . \tag{2.16}$$

Using the test statistic t defined in (2.15) for testing the relative closeness of β_2 to β_0, the sometimes regression estimator $\bar{y}_s$ now takes the form

$$\begin{aligned} \bar{y}_s &= \bar{y} + \beta_0(\bar{X}-\bar{x}) \quad \text{if} \quad t\varepsilon A \\ &= \bar{y} + \hat{\beta}_2(\bar{X}-\bar{x}) \quad \text{if} \quad t\varepsilon A^c, \end{aligned} \tag{2.17}$$

where A is the acceptance region in the sample space and A^c is the rejection region.

Now we need to look at a criterion for deciding whether or not the proposed estimator $\bar{y}_s$ has any advantages over $\bar{y}_d$ and $\bar{y}_\ell$. Commonly, statisticians use squared error as a loss function. This then leads to considering the variance of the estimator $\bar{y}_s$ if it is unbiased, or the mean square error of $\bar{y}_s$ if it is biased. We then have the expected value of $\bar{y}_s$ given by

$$E(\bar{y}_s) = E(\bar{y}_d|A)P(A) + E(\bar{y}_\ell|A^c)P(A^c) \ , \tag{2.18}$$

and the mean square error of $\bar{y}_s$ is given by

$$\begin{aligned} \text{M.S.E.}(\bar{y}_s) &= E(\bar{y}_s-\bar{Y})^2 \\ &= E[(\bar{y}_d-\bar{Y})^2|A]P(A)+E[(\bar{y}_\ell-\bar{Y})^2|A^c]P(A^c) \ . \end{aligned} \tag{2.19}$$

The acceptance and rejection regions will depend upon the *a priori* information available concerning the possible range of values of β_2. In order to evaluate fully the expected value and mean square error of the estimator $\bar{y}_s$, it is necessary to define precisely the acceptance and rejection regions and make suitable assumptions about the joint distribution of Y and X. This will be done in the next section.

III. EXPECTED VALUE AND VARIANCE OF $\overline{y}_s$

As stated in the previous section it is necessary to make suitable assumptions about the joint distribution of X and Y in order to obtain a closed form for the expected value and the variance of $\overline{y}_s$. A widely used and quite often closely fitted distribution for large populations is the normal distribution. In what follows, we assume that the population is infinite and that X and Y have a bivariate normal distribution function. Under the hypothesis that $\beta_2 = \beta_0$ and the assumption that X and Y have a bivariate normal distribution, the test statistic t defined in (2.15) has a central "Student's t" distribution with n-2 degrees of freedom. We shall consider three different cases, depending upon the *a priori* information available concerning the range of values of β_2. These are:

Case I:

From past experience, it is hypothesized that β_2 is β_0, but nothing further is known about β_2. The estimator to be used here is

$$\begin{aligned} \overline{y}_s &= \overline{y}_d \quad \text{if} \quad |t| \leq t_0 \\ &= \overline{y}_\ell \quad \text{if} \quad |t| > t_0 \end{aligned} \tag{3.1}$$

where t_0 is a fixed positive constant.

Case II:

From past experience, it is hypothesized that β_2 is β_0 and also it is further known that $\beta_2 \leq \beta_0$. The estimator in this case is defined as

$$\begin{aligned} \bar{y}_s &= \bar{y}_d \quad \text{if} \quad t \geq t_0 \\ &= \bar{y}_\ell \quad \text{if} \quad t < t_0 \end{aligned} \tag{3.2}$$

where t_0 is some fixed constant.

Case III:

From past experience, it is hypothesized that β_2 is β_0 and also it is further known that $\beta_2 \geq \beta_0$. The estimator now reduces to

$$\begin{aligned} \bar{y}_s &= \bar{y}_d \quad \text{if} \quad t \leq t_0 \\ &= \bar{y}_\ell \quad \text{if} \quad t > t_0 \end{aligned} \tag{3.3}$$

where t_0 is a fixed constant.

Theorem 3.1:

$\bar{y}_s$ is an unbiased estimator of the population mean $\bar{Y}$, that is

$$E(\bar{y}_s) = \bar{Y} \; . \tag{3.4}$$

Proof:

Consider the estimator $\bar{y}_s$ as defined in its most general form in (2.17). Then, we have

$$
\begin{aligned}
E(\bar{y}_s) &= E(\bar{y}_d|A)P(A) + E(\bar{y}_\ell|A^c)P(A^c) \\
&= E(\bar{y}|A)P(A) + E(\bar{y}|A^c)P(A^c) \\
&\quad + \beta_0 E[(\bar{X}-\bar{x})|A]P(A) \\
&\quad + E[\hat{\beta}_2(\bar{X}-\bar{x})|A^c]P(A^c) .
\end{aligned}
$$

Under the assumption of bivariate normality, we have that $\bar{x}$ and (s_1^2, s_2^2, s_{12}) are statistically independent, and therefore

$$E[\hat{\beta}_2(\bar{X}-\bar{x})|A^c]P(A^c) = E(\bar{X}-\bar{x})E[\hat{\beta}_2|A^c]P(A^c) .$$

Also

$$E[(\bar{X}-\bar{x})|A]P(A) = E(\bar{X}-\bar{x}) = 0.$$

Hence

$$
\begin{aligned}
E(\bar{y}_s) &= E(\bar{y}|A)P(A) + E(\bar{y}|A^c)P(A^c) \\
&= E(\bar{y}) \\
&= \bar{Y}.
\end{aligned}
$$

Hence the theorem is proved.

Since $\bar{y}_s$ is an unbiased estimator, we now obtain the variance of $\bar{y}_s$.

$$V(\bar{y}_s) = E(\bar{y}_s-\bar{Y})^2$$

$$= E(\bar{y}_s)^2-\bar{Y}^2$$

$$= E(\bar{y}_d{}^2|A)P(A) + E(\bar{y}_\ell{}^2|A^c)P(A^c)-\bar{Y}^2$$

$$= E(\bar{y}_d{}^2)-\bar{Y}^2 + E(\bar{y}_\ell{}^2|A^c)P(A^c) - E(\bar{y}_d{}^2|A^c)P(A^c)$$

$$= V(\bar{y}_d) + E[(\bar{y} + \hat{\beta}_2(\bar{X}-\bar{x}))^2|A^c]P(A^c)$$

$$-E[(\bar{y} + \beta_0(\bar{X}-\bar{x}))^2|A^c]P(A^c)$$

$$= V(\bar{y}_d) + 2E[(\hat{\beta}_2-\beta_0)(\bar{X}-\bar{x})\bar{y}|A^c]P(A^c)$$

$$+2\beta_0E[(\hat{\beta}_2-\beta_0)(\bar{X}-\bar{x})^2|A^c]P(A^c)$$

$$+ E[(\hat{\beta}_2-\beta_0)^2(\bar{X}-\bar{x})^2|A^c]P(A^c) .$$

Under the assumption of bivariate normality, we have that $(\bar{x}, \bar{y})$ and $(s_1{}^2, s_2{}^2, s_{12})$ are statistically independent, and therefore

$$V(\bar{y}_s) = V(\bar{y}_d) - \frac{2\sigma_{12}}{n} E[(\hat{\beta}_2-\beta_0)|A^c]P(A^c)$$

$$+\frac{2\beta_0\sigma_1{}^2}{n}E[(\hat{\beta}_2-\beta_0)|A^c]P(A^c)+\frac{\sigma_1{}^2}{n}E[(\hat{\beta}_2-\beta_0)^2|A^c]P(A^c) . \tag{3.5}$$

In order to further evaluate this, we need an expression for $E[(\hat{\beta}_2-\beta_0)^h|A^c]P(A^c)$ for $h = 0,1,2$. This is obtained for general h and is given in Lemma 3.1. Although in practice t_0 will always be negative in Case II and positive in Case III, for completeness, the variance will be developed allowing

t_0 to be any real number. It will be assumed that the sample size is $n \geq 4$.

Lemma 3.1:

Case I:

$$KP(|t|>t_0)E[(\hat{\beta}_2-\beta_0)^h \mid |t|>t_0]$$

$$= \frac{1}{2}\sum_{i=0}^{\infty}(1+(-1)^{h+i})\left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2},\frac{h+i+1}{2}\right) \tag{3.6}$$

$$= \begin{cases} \displaystyle\sum_{i=0}^{\infty}\left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{2i} \frac{\Gamma(\frac{h+2i+1}{2})\Gamma(\frac{n+2i-h-1}{2})}{\Gamma(2i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+2i+1}{2}\right), & \text{if } h \text{ is even;} \quad (3.7) \\[2ex] \displaystyle\sum_{i=0}^{\infty}\left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{2i+1} \frac{\Gamma(\frac{h+2i+2}{2})\Gamma(\frac{n+2i-h}{2})}{\Gamma(2i+2)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+2i+2}{2}\right), & (3.8) \\ \text{if } h \text{ is odd.} \end{cases}$$

<u>Case II</u>:

$KP(t<t_0)E[(\hat{\beta}_2-\beta_0)^h | t<t_0]$

$$= \sum_{i=0}^{\infty} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{2i} \frac{\Gamma(\frac{h+2i+1}{2})\Gamma(\frac{n+2i-h-1}{2})}{\Gamma(2i+1)}$$

$$-\frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{i} \frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+i+1}{2}\right),$$

if h is even and $t_0>0$; (3.9)

$$= \sum_{i=0}^{\infty} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{2i+1} \frac{\Gamma(\frac{h+2i+2}{2})\Gamma(\frac{n+2i-h}{2})}{\Gamma(2i+2)}$$

$$-\frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{i} \frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+i+1}{2}\right),$$

if h is odd and $t_0>0$; (3.10)

$$= \frac{1}{2} \sum_{i=0}^{\infty} (-1)^{h+i} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{i} \frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+i+1}{2}\right),$$

if $t_0 \le 0$. (3.11)

<u>Case III</u>:

$KP(t>t_0)E[(\hat{\beta}_2-\beta_0)^h|t>t_0]$

$$= \sum_{i=0}^{\infty} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{2i} \frac{\Gamma(\frac{h+2i+1}{2})\Gamma(\frac{n+2i-h-1}{2})}{\Gamma(2i+1)}$$

$$+ \frac{1}{2} \sum_{i=0}^{\infty} (-1)^{h+i+1} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{i} \frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \left(\frac{h+i+1}{2}\right)\right),$$

if h is even and $t_0<0$; (3.12)

$$= \sum_{i=0}^{\infty} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{2i+1} \frac{\Gamma(\frac{h+2i+2}{2})\Gamma(\frac{n+2i-h}{2})}{\Gamma(2i+2)}$$

$$= \frac{1}{2} \sum_{i=0}^{\infty} (-1)^{h+i+1} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{i} \frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+i+1}{2}\right),$$

if h is odd and $t_0<0$; (3.13)

$$= \frac{1}{2} \Sigma \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{i} \frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+i+1}{2}\right),$$

if $t_0 \geq 0$. (3.14)

where

$$\delta = \frac{(\rho - \frac{\beta_0 \sigma_1}{\sigma_2})}{(1-\rho^2)^{1/2}} \tag{3.15}$$

$$m_0 = \frac{1}{1 + \frac{t_0^2}{n-2}}, \tag{3.16}$$

$$I_{m_0}(a,b) = \frac{1}{B(a,b)} \int_0^{m_0} z^{a-1}(1-z)^{b-1} dz, \tag{3.17}$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{3.18}$$

and

$$K = \left(\frac{\sigma_1}{\sigma_2\sqrt{1-\rho^2}}\right)^h \sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)(1+\delta^2)^{\frac{n-h-1}{2}}. \tag{3.19}$$

<u>Proof</u>:

In order to find the expectation, the joint density function for the variables over which the expectation is taken is needed. It is well known that the joint density function for s_1, s_2 and $r = \frac{s_{12}}{s_1 s_2}$ is given by

$$f(s_1,s_2,r) = \begin{cases} K_1(s_1^2 s_2^2)^{\frac{n-2}{2}}(1-r^2)^{\frac{n-4}{2}} \\ \times \exp[-\frac{n-1}{2(1-\rho^2)}(\frac{s_1^2}{\sigma_1^2} - \frac{2\rho s_1 s_2 r}{\sigma_1\sigma_2} + \frac{s_2^2}{\sigma_2^2})] \\ \qquad \text{if } 0<s_1<\infty,\ 0<s_2<\infty, \text{ and } r^2<1, \\ 0 \quad \text{otherwise,} \end{cases}$$

with

$$K_1 = \frac{(n-1)^{n-1}}{\pi\Gamma(n-2)[(1-\rho^2)\sigma_1^2\sigma_2^2]^{\frac{n-1}{2}}} .$$

Making the transformation

$$u = \frac{(n-1)s_1^2}{2\sigma_1^2(1-\rho^2)} ,$$

$$v = \frac{(n-1)rs_1s_2}{2\sigma_1\sigma_2(1-\rho^2)}$$

and

$$w = \frac{(n-1)s_2^2}{2\sigma_2^2(1-\rho^2)}$$

and noting that the Jacobian of the transformation is

$$J = \left| \frac{\partial (u,v,w)}{\partial (s_1, s_2, r)} \right|^{-1}$$

$$= \frac{\sigma_1 \sigma_2 (1-\rho^2)}{2(n-1)uw},$$

the joint density of u, v, and w is

$$f(u,v,w) = \begin{cases} K_2 (uw-v^2)^{\frac{n-4}{2}} \exp[-(u-2\rho v+w)] \\ \qquad \text{for } 0<u<\infty,\ 0<w<\infty, \text{ and } v^2<wu; \\ 0 \qquad \text{otherwise;} \end{cases}$$

where

$$K_2 = \frac{2^{n-3}(1-\rho^2)^{\frac{n-1}{2}}}{\pi\Gamma(n-2)} .$$

In terms of the transformed variables u, v, and w

$$\hat{\beta}_2 = \frac{rs_2}{s_1}$$

$$= \frac{v\sigma_2}{u\sigma_1}$$

and with

$$t' = \frac{t}{\sqrt{n-2}} ,$$

we have

$$t' = \frac{(\hat{\beta}_2-\beta_0)s_1}{(s_2^{\,2}-\hat{\beta}_2^{\,2}s_1^{\,2})^{1/2}}$$

$$= \frac{(v- \frac{u\sigma_1}{\sigma_2}\,\beta_0)}{(uw-v^2)^{1/2}}\,.$$

Consider the transformation

$$u = u$$

$$v = v$$

and

$$t' = \frac{v\sigma_2-\beta_0 u\sigma_1}{(uw-v^2)^{1/2}\sigma_2}\,.$$

Noting that the Jacobian of the transformation is

$$J = \left|\frac{\partial(u,v,t')}{\partial(u,v,w)}\right|^{-1}$$

$$= \frac{-2(v- \frac{u\beta_0\sigma_1}{\sigma_2})^2}{ut'^3},$$

and the fact that if $t'>0$ then

$$v > \frac{u\sigma_1}{\sigma_2}\,\beta_0,$$

and if $t'<0$ then

$$v < \frac{u\sigma_1\beta_0}{\sigma_2},$$

the joint density function for u, v and t' is

$$f(u,v,t') = K_3 \frac{\left(v - \frac{\beta_0 u\sigma_1}{\sigma_2}\right)^{n-2}}{u(t')^{n-1}} \exp\left[-\left(u-2\rho v+ \frac{\left(v-\frac{\beta_0\sigma_1 u}{\sigma_2}\right)^2}{ut'^2} + \frac{v^2}{u}\right)\right],$$

in R_1; (3.20)

$$K_3 \frac{\left|v-\frac{\beta_0 u\sigma_1}{\sigma_2}\right|^{n-2}}{u|t'|^{n-1}} \exp\left[-\left(u-2\rho v+ \frac{\left(v-\frac{\beta_0\sigma_1 u}{\sigma_2}\right)^2}{ut'^2} + \frac{v^2}{u}\right)\right],$$

in R_2; (3.21)

0 otherwise;

with

$$K_3 = \frac{2^{n-2}(1-\rho^2)^{\frac{n-1}{2}}}{\pi\Gamma(n-2)}, \tag{3.22}$$

$$R_1 = \{0\le u<\infty,\ 0\le t'<\infty,\ v\ge \frac{u\sigma_1\beta_0}{\sigma_2}\}, \tag{3.23}$$

and

$$R_2 = \{0\le u<\infty,\ -\infty<t'<0,\ v< \frac{u\sigma_1\beta_0}{\sigma_2}\}. \tag{3.24}$$

In order that $f(u,v,t')$ be in a useful form, it is necessary to work with the exponent and express it in a more convenient form. Now

$$\exp[-(u-2\rho v+\frac{(v-\frac{\beta_0 u\sigma_1}{\sigma_2})^2}{ut'^2}+\frac{v^2}{u})]$$

$$=\exp[-(u(1-\rho^2)(1+\delta^2)+(v-\frac{\beta_0 u\sigma_1}{\sigma_2})^2(\frac{1+t'^2}{ut'^2})$$

$$-2\delta\sqrt{1-\rho^2}(v-\frac{\beta_0 u\sigma_1}{\sigma_2}))].$$

Then expanding the last term of the exponent as an infinite series, the joint density f(u,v,t') can be expressed in the form

$$f(u,v,t')=\frac{K_3}{ut'^{n-1}}\exp[-u(1-\rho^2)(1+\delta^2)-\frac{1+t'^2}{ut'^2}(v-\frac{\beta_0 u\sigma_1}{\sigma_2})^2]\times$$

$$\sum_{i=0}^{\infty}\frac{2^i(v-\frac{\beta_0 u\sigma_1}{\sigma_2})^{n+i-2}\delta^i(1-\rho^2)^{\frac{i}{2}}}{\Gamma(i+1)}$$

in R_1; (3.25)

$$=\frac{K_3}{u|t'|^{n-1}}\exp[-u(1-\rho^2)(1+\delta^2)-\frac{1+t'^2}{ut'^2}(v-\frac{\beta_0 u\sigma_1}{\sigma_2})^2]\times$$

$$\sum_{i=0}^{\infty}\frac{(-1)^i 2^i|v-\frac{\beta_0 u\sigma_1}{\sigma_2}|^{n+i-2}\delta^i(1-\rho^2)^{\frac{i}{2}}}{\Gamma(i+1)},$$

in R_2; (3.26)

$= 0, \quad$ otherwise . (3.27)

Let

$$t_0' = \frac{t_0}{\sqrt{n-2}} ,$$

where t_0 is a positive number.

$$P(t<t_0)E((\hat{\beta}_2-\beta_0)^h|t<t_0)$$

$$= P(t'<t_0')E((\hat{\beta}_2-\beta_0)^h|t'<t_0')$$

$$= P(t'<t_0')E((\frac{v\sigma_2}{u\sigma_1} - \beta_0)^h|t'<t_0').$$

The region of integration can now be split into two mutually exclusive regions namely R_2 and

$$R_3 = \{0<u<\infty,\ 0<t'<t_0',\ \frac{\beta_0 u\sigma_1}{\sigma_2} <v<\infty\}.$$

Then

$$P(t'<t_0')E((\hat{\beta}_2-\beta_0)^h|t'<t_0')$$

$$= \int_{R_2} [(v- \frac{\beta_0 u\sigma_1}{\sigma_2})\frac{\sigma_2}{u\sigma_1}]^h \; f(u,v,t')du\ dv\ dt'$$

$$+\int_{R_3} [(v- \frac{\beta_0 u\sigma_1}{\sigma_2})\frac{\sigma_2}{u\sigma_1}]^h \; f(u,v,t')du\ dv\ dt'$$

$$= I_2 + I_3 .$$

Now,

$$I_2 = \int_{R_2} [(v - \frac{\beta_0 u\sigma_1}{\sigma_2})\frac{\sigma_2}{u\sigma_1}]^h \; f(u,v,t')du\,dv\,dt'$$

$$= K_3\int_{-\infty}^{0}\int_{0}^{\infty}\int_{-\infty}^{\frac{\beta_0 u\sigma_1}{\sigma_2}} (\frac{\sigma_2}{\sigma_1})^h (\frac{1}{u})^{h+1} \frac{1}{|t'|^{n-1}}\exp[-u(1-\rho^2)(1+\delta^2)] \times$$

$$\sum_{i=0}^{\infty} (-1)^{h+i}\exp[-\frac{1+t'^2}{ut'^2}(v-\frac{\beta_0 u\sigma_1}{\sigma_2})^2]$$

$$\times \frac{|v - \frac{\beta_0 u\sigma_1}{\sigma_2}|^{n+h+i-2}}{\Gamma(i+1)}(2\delta\sqrt{1-\rho^2})^i \; dv\,du\,dt' \; .$$

Making the change of variables

$$u = u,$$

$$t' = t',$$

and

$$\frac{x}{\sqrt{2}} = \sqrt{1+t'^2}\;\left(\frac{v - \frac{\beta_0 u\sigma_1}{\sigma_2}}{\sqrt{u}\;|t'|}\right),$$

we get

$$I_2 = K_4\int_{-\infty}^{0}\int_{0}^{\infty}\int_{-\infty}^{0} \exp[-u(1-\rho^2)(1+\delta^2)] \sum_{i=0}^{\infty} [\frac{(-1)^{h+i}(\sqrt{2}\sqrt{1-\rho^2}\delta)^i}{\Gamma(i+1)}$$

$$\times \frac{u^{\frac{n+i-h-3}{2}} |t'|^{h+i}}{(1+t'^2)^{\frac{n+h+i-1}{2}}} \exp(-\frac{x^2}{2}) |x|^{n+h+i-2}] dx\, du\, dt'$$

where

$$K_4 = (\frac{\sigma_2}{\sigma_1})^h \frac{2^{\frac{n-h-3}{2}} (1-\rho^2)^{\frac{n-1}{2}}}{\pi\Gamma(n-2)} .$$

In order to evaluate the above, it is necessary that we have the following lemma which is stated without proof.

Lemma 3.2:

$$\int_{-\infty}^{0} |x|^n e^{-\frac{1}{2}x^2} dx = \int_0^{\infty} |x|^n e^{-\frac{1}{2}x^2} dx$$

$$= (-1)^n \int_{-\infty}^{0} x^n e^{-\frac{1}{2}x^2}$$

$$= 2^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2}) .$$

Because the series is the result of expanding the exponential, the order of summation and integration may be interchanged. Hence upon evaluating the integrals in x and u,

$$I_2 = K_5 \sum_{i=0}^{\infty} (-1)^{h+i} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{n+i-h-1}{2})\Gamma(\frac{n+h+i-1}{2})}{\Gamma(i+1)} \int_0^{\infty} \frac{t'^{h+i}}{(1+t'^2)^{\frac{n+h+i-1}{2}}} dt'$$

where

$$K_5 = \left(\frac{\sigma_2}{\sigma_1}\right)^h \frac{2^{n-3}(1-\rho^2)^{\frac{h}{2}}}{\pi\Gamma(n-2)(1+\delta^2)^{\frac{n-h-1}{2}}} .$$

Making the variable change

$$y = \frac{1}{1+t'^2} ,$$

we have

$$\int_0^{\infty} \frac{(t')^{h+i}}{(1+t'^2)^{\frac{n+h+i-1}{2}}} dt' = \frac{1}{2} \int_0^1 (1-y)^{\frac{h+i-1}{2}} y^{\frac{n-4}{2}} dy$$

$$= \frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{h+i+1}{2})}{2\Gamma(\frac{n+h+i-1}{2})} .$$

Hence, we have

$$I_2 = K_6 \sum_{i=0}^{\infty} (-1)^{h+i} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{n+i-h-1}{2})\Gamma(\frac{h+i+1}{2})}{\Gamma(i+1)}$$

with

$$K_6 = \left(\frac{\sigma_2}{\sigma_1}\right)^h \frac{2^{n-4}\Gamma\left(\frac{n-2}{2}\right)(1-\rho^2)^{\frac{h}{2}}}{\pi\Gamma(n-2)(1+\delta^2)^{\frac{n-h-1}{2}}} .$$

Again without proof the following useful lemma is given.

Lemma 3.3:

$$\Gamma\left(\frac{j-2}{2}\right)\Gamma\left(\frac{j-1}{2}\right)2^{j-3} = \Gamma(j-2)\sqrt{\pi} .$$

Using Lemma 3.3, we have

$$I_2 = \frac{1}{2K} \sum_{i=0}^{\infty} (-1)^{h+i} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma\left(\frac{h+i+1}{2}\right)\Gamma\left(\frac{n+i-h-1}{2}\right)}{\Gamma(i+1)} .$$

The same steps used in finding I_2 can also be used to find I_3. These are given below without explanation.

$$I_3 = \int_{R_3} \left[\left(v - \frac{\beta_0 u\sigma_1}{\sigma_2}\right)\frac{\sigma_2}{u\sigma_1}\right]^h f(u,v,t')\,du\,dv\,dt'$$

$$= K_3\int_0^{t_0'}\int_0^{\infty}\int_{\frac{u\beta_0\sigma_1}{\sigma_2}}^{\infty} \left(\frac{\sigma_2}{\sigma_1}\right)^h \left(\frac{1}{u}\right)^{h+1} \frac{1}{(t')^{n-1}}\exp[-u(1-\rho^2)(1+\delta^2)]\times$$

$$\sum_{i=0}^{\infty} \frac{\exp\left[-\frac{1+t'^2}{ut'^2}\left(v-\frac{\beta_0 u\sigma_1}{\sigma_2}\right)^2\right]\left(v-\frac{\beta_0 u\sigma_1}{\sigma_2}\right)^{n+h+i-2} (2\delta\sqrt{1-\rho})^i\,dv\,du\,dt'}{\Gamma(i+1)}$$

$$=K_4\int_0^{t'_0}\int_0^{\infty}\int_0^{\infty}\exp[-u(1-\rho^2)(1+\delta^2)]\sum_{i=0}^{\infty}[\frac{(\sqrt{2}\sqrt{1-\rho}\delta)^i}{\Gamma(i+1)}\times$$

$$\frac{u^{\frac{n+i-h-3}{2}}(t')^{h+i}}{(1+t'^2)^{\frac{n+h+i-1}{2}}}\exp(-\frac{x^2}{2})(x)^{n+h+i-2}]dx\,du\,dt'$$

$$=K_5\sum_{i=0}^{\infty}(\frac{2\delta}{\sqrt{1+\delta^2}})^i\frac{\Gamma(\frac{n+i-h-1}{2})\Gamma(\frac{n+h+i-1}{2})}{\Gamma(i+1)}\int_0^{t'_0}\frac{(t')^{h+i}}{(1+t'^2)^{\frac{n+h+i-1}{2}}}dt'$$

$$=K_6\sum_{i=0}^{\infty}(\frac{2\delta}{\sqrt{1+\delta^2}})^i\frac{\Gamma(\frac{n+i-h-1}{2})\Gamma(\frac{h+i+1}{2})}{\Gamma(i+1)}(1-I_{m_0}(\frac{n-2}{2},\frac{h+i+1}{2}))$$

$$=\frac{1}{2K}\sum_{i=0}^{\infty}(\frac{2\delta}{\sqrt{1+\delta^2}})^i\frac{\Gamma(\frac{h+i+1}{2})\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)}(1-I_{m_0}(\frac{n-2}{2},\frac{h+i+1}{2}))$$

where

$$m_0=\frac{1}{1+t_0'^2}$$

$$=\frac{1}{1+\frac{t_0^2}{n-2}}\;.$$

Considering the cases of h being even and odd separately and combining I_2 and I_3 gives the result for $t_0>0$ in Case II.

Now consider Case II where t_0 is negative. Again, the same steps used in finding I_2 and I_3 can be used. For this case

$$P(t<t_0)E((\hat{\beta}_2-\beta_0)^h|t<t_0)$$

$$= \int_{R_4} [(v-\frac{\beta_0 u\sigma_1}{\sigma_2})\frac{\sigma_2}{u\sigma_1}]^h \; f(u,v,t')du\,dv\,dt',$$

where

$$R_4 = \{0<u<\infty,\ -\infty\ t'<t_0',\ v<\frac{u\sigma_1\beta_0}{\sigma_2}\}.$$

Hence

$$P(t<t_0)E((\hat{\beta}_2-\beta_0)^h|t<t_0)$$

$$= K_3\int_{-\infty}^{t_0'}\int_0^{\infty}\int_{-\infty}^{\frac{u\beta_0\sigma_1}{\sigma_2}} (\frac{\sigma_2}{\sigma_1})^h (\frac{1}{u})^{h+1} \frac{1}{|t'|^{n-1}}\exp[-u(1-\rho^2)(1+\delta^2)]\times$$

$$\sum_{i=0}^{\infty}(-1)^{h+i}\frac{\exp[-\frac{1+t'^2}{ut'^2}(v-\frac{\beta_0 u\sigma_1}{\sigma_2})^2]\,|v-\frac{\beta_0 u\sigma_1}{\sigma_2}|^{n+h+i-2}\quad(2\delta\sqrt{1-\rho^2})^i dvdudt'}{\Gamma(i+1)}$$

$$= K_4\int_{-\infty}^{t_0'}\int_0^{\infty}\int_{-\infty}^{0}\exp[-u(1-\rho^2)(1+\delta^2)]\sum_{i=0}^{\infty}(-1)^{h+i}[\frac{(\sqrt{2}\delta\sqrt{1-\rho^2})^i}{\Gamma(i+1)}\times$$

$$\frac{u^{\frac{n+i-h-3}{2}}\;|t'|^{h+i}}{(1+t'^2)^{\frac{n+h+i-1}{2}}}\exp(-\frac{x^2}{2})|x|^{n+h+i-2}\ dx\ du\ dt'$$

$$= K_5 \sum_{i=0}^{\infty} (-1)^{h+i} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{n+i-h-1}{2})\,\Gamma(\frac{n+h+i-1}{2})}{\Gamma(i+1)} \int_{-\infty}^{t_0'} \frac{|t'|^{h+i}}{(1+t'^2)^{\frac{n+h+i-1}{2}}} dt$$

$$= K_6 \sum_{i=0}^{\infty} (-1)^{h+i} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{n+i-h-1}{2})\,\Gamma(\frac{h+i+1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+i+1}{2}\right)$$

$$= \frac{1}{2K} \sum_{i=0}^{\infty} (-1)^{h+i} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{h+i+1}{2})\,\Gamma(\frac{n+i-h-1}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{h+i+1}{2}\right).$$

This gives the desired result for $t_0' \leq 0$ for Case II.

The result for Case III can be obtained by using techniques analogous to those used in deriving Case II.

The result for Case I can be obtained as follows. We have

$$P(|t'|>t_0')E((\hat{\beta}_2-\beta_0)^h \,|\, |t'|>t_0')$$

$$= \int_{R_4} \left[\left(v- \frac{\beta_0 u\sigma_1}{\sigma_2}\right)\frac{\sigma_2}{u\sigma_1}\right]^h f(u,v,t')\,du\,dv\,dt'$$

$$+ \int_{R_5} \left[\left(v- \frac{\beta_0 u\sigma_1}{\sigma_2}\right)\frac{\sigma_2}{u\sigma_1}\right]^h f(u,v,t')\,du\,dv\,dt'$$

$$= I_4 + I_5$$

where

$$R_5 = \{0 \le u \le \infty,\ v \ge \frac{\beta_0 u \sigma_1}{\sigma_2},\ t_0' < t' < \infty\}.$$

We obtain the desired result immediately by recognizing that I_4 is the same as Case II with $t_0' \le 0$, and I_5 is Case III with $t_0' \ge 0$.

Remark 3.1:

Upon setting h=0 in Case II and using Lemma 3.3 we get the distribution function of the t statistic and hence the power function of the t test is given by

$$P(t<t_0) = \sum_{i=0}^{\infty} \left(\frac{\delta^2}{1+\delta^2}\right)^i \frac{\Gamma(\frac{n+2i-1}{2})}{\Gamma(i+1)\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}}$$

$$-\frac{1}{2}\sum_{i=0}^{\infty} \left(\frac{\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{n+i-1}{2})}{\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}} I_{m_0}\left(\frac{n-2}{2}, \frac{i+1}{2}\right),$$

for $t_0 > 0$;

$$=\frac{1}{2}\Sigma(-1)^i \left(\frac{\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{n+i-1}{2})}{\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}} I_{m_0}\left(\frac{n-2}{2}, \frac{i+1}{2}\right),$$

for $t_0 \le 0$. (3.28)

Upon recognizing that

$$\frac{\Gamma(\frac{n+2i-1}{2})}{\Gamma(i+1)\Gamma(\frac{n-1}{2})} (\frac{\delta^2}{1+\delta^2})^i (\frac{1}{1+\delta^2})^{\frac{n-1}{2}} \qquad i = 0,1,2\ \ldots$$

is the density function of the negative binomial, we get upon simplification

$$P(t<t_0) = 1-\frac{1}{2}\sum_{i=0}^{\infty}(\frac{\delta}{\sqrt{1+\delta^2}})^i \frac{\Gamma(\frac{n+i-1}{2})}{\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}} I_{m_0}(\frac{n-2}{2}, \frac{i+1}{2}),$$

for $t_0>0$;

$$= \frac{1}{2}\sum_{i=0}^{\infty}(-1)^i(\frac{\delta}{\sqrt{1+\delta^2}})^i \frac{\Gamma(\frac{n+i-1}{2})}{\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}} I_{m_0}(\frac{n-2}{2},\frac{i+1}{2}),$$

for $t_0 \leq 0$. (3.29)

<u>Remark 3.2</u>:

Referring to (3.25) and (3.26), if we integrate out u and v and make the transformation $t = \sqrt{n-2}\ t'$, we get the density of the t statistic,

$$f(t/\delta) = \frac{1}{\sqrt{n-2}\ \pi\Gamma(n-2)} \sum_{i=0}^{\infty} \frac{\Gamma^2(\frac{n+i-1}{2})\delta^i 2^{n+i-3}}{\Gamma(i+1)(1+\delta^2)^{\frac{n+i-1}{2}}} \frac{(\frac{t}{\sqrt{n-2}})^i}{(1+\frac{t^2}{n-2})^{\frac{n+i-1}{2}}},$$

for $-\infty<t<\infty$

$= 0$, otherwise . (3.30)

When $\delta=0$ this reduces to the central "Student's t" density function with n-2 degrees of freedom.

It is possible to define a sometimes estimator of the regression coefficient as

$$\hat{\beta}_{2s} = \begin{cases} \beta_0 & \text{if} \quad t \varepsilon A \\ \hat{\beta}_2 & \text{if} \quad t \varepsilon A^c. \end{cases}$$

There are three different cases for this estimator, analogous in nature to those of the sometimes regression estimator. They are:

<u>Case I</u>:

$$\hat{\beta}_{2s} = \begin{cases} \beta_0 & \text{if} \quad |t| \leq t_0 \\ \hat{\beta}_2 & \text{if} \quad |t| > t_0 \end{cases} ,$$

<u>Case II</u>:

$$\hat{\beta}_{2s} = \begin{cases} \beta_0 & \text{if} \quad t \geq t_0 \\ \hat{\beta}_2 & \text{if} \quad t < t_0, \end{cases}$$

<u>Case III</u>:

$$\hat{\beta}_{2s} = \begin{cases} \beta_0 & \text{if} \quad t \leq t_0 \\ \hat{\beta}_2 & \text{if} \quad t > t_0 \end{cases} .$$

Using the results of Lemma 3.1 we get the expectation of the sometimes estimator of the regression coefficient. In general $\hat{\beta}_{2s}$ is a biased estimator of β_2. Noting that

$$E(\hat{\beta}_{2s}) = \beta_0 + E[(\hat{\beta}_2-\beta_0)|t\varepsilon A^c]P(t\varepsilon A^c),$$

we obtain from Lemma 3.1 the following corollary:

Corollary:

Case I:

$$E(\hat{\beta}_{2s}) = \beta_0 + \frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_1(1+\delta^2)^{\frac{n-2}{2}}\Gamma(\frac{n-1}{2})}\sum_{i=0}^{\infty}\left(\frac{\delta}{\sqrt{1+\delta^2}}\right)^{2i+1}$$

$$\times \frac{\Gamma(\frac{n+2i-1}{2})}{\Gamma(\frac{2i+2}{2})} I_{m_0}\left(\frac{n-2}{2}, \frac{2i+3}{2}\right).$$

Case II:

$$E(\hat{\beta}_{2s}) = \beta_0 + \frac{\sigma_2\sqrt{1-\rho^2}\,\delta}{\sigma_1}$$

$$- \frac{\sigma_2\sqrt{1-\rho^2}}{2\sigma_1(1+\delta^2)^{\frac{n-2}{2}}\Gamma(\frac{n-1}{2})}\sum_{i=0}^{\infty}\left(\frac{\delta}{\sqrt{1+\delta^2}}\right)^{i}\frac{\Gamma(\frac{n+i-2}{2})}{\Gamma(\frac{i+1}{2})}I_{m_0}\left(\frac{n-2}{2}, \frac{i+2}{2}\right),$$

if $t_0>0$;

$$= \beta_0 + \frac{\sigma_2\sqrt{1-\rho^2}}{2\sigma_1(1+\delta^2)^{\frac{n-2}{2}}\Gamma(\frac{n-1}{2})}\sum_{i=0}^{\infty}(-1)^{i+1}\left(\frac{\delta}{\sqrt{1+\delta^2}}\right)^{i}$$

$$\times \frac{\Gamma(\frac{n+i-2}{2})}{\Gamma(\frac{i+1}{2})}I_{m_0}\left(\frac{n-2}{2}, \frac{i+2}{2}\right),$$

if $t_0\leq 0$.

Case III:

$$E(\hat{\beta}_{2s}) = \beta_0 + \frac{\sigma_2\sqrt{1-\rho^2}\,\delta}{\sigma_1}$$

$$+ \frac{\sigma_2\sqrt{1-\rho^2}}{2\sigma_1(1+\delta^2)^{\frac{n-2}{2}}\,\Gamma(\frac{n-1}{2})} \sum_{i=0}^{\infty} (-1)^{i+2} \left(\frac{\delta}{\sqrt{1+\delta^2}}\right)^{i}$$

$$\times \frac{\Gamma(\frac{n+i-2}{2})}{\Gamma(\frac{i+1}{2})} I_{m_0}\left(\frac{n-2}{2}, \frac{i+2}{2}\right),$$

if $t_0 \leq 0$;

$$= \beta_0 + \frac{\sigma_2\sqrt{1-\rho^2}}{2\sigma_1(1+\delta^2)^{\frac{n-2}{2}}\,\Gamma(\frac{n-1}{2})} \sum_{i=0}^{\infty} \left(\frac{\delta}{\sqrt{1+\delta^2}}\right)^{i} \frac{\Gamma(\frac{n+i-2}{2})}{\Gamma(\frac{i+1}{2})} I_{m_0}\left(\frac{n-2}{2}, \frac{i+2}{2}\right),$$

if $t_0 > 0$.

From Lemma 3.1 for Case II with $t_0 > 0$

$$E[(\hat{\beta}_2-\beta_0)^2 \mid t<t_0]P(t<t_0) = \left(\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_1}\right)^2 \frac{1}{\Gamma(\frac{n-1}{2})\sqrt{\pi}(1+\delta^2)^{\frac{n-3}{2}}} \times$$

$$\left[\sum_{i=0}^{\infty} \left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^{2i} \frac{\Gamma(\frac{2i+3}{2})\Gamma(\frac{n+2i-3}{2})}{\Gamma(2i+1)}\right.$$

$$-\frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{2\delta}{\sqrt{1+\delta^2}}\right)^i \frac{\Gamma(\frac{i+3}{2})\Gamma(\frac{n+i-3}{2})}{\Gamma(i+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{i+3}{2}\right)].$$

Using Lemma 3.3

$$E[(\hat{\beta}_2-\beta_0)^2 | t<t_0]P(t<t_0)=\left(\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_1}\right)^2 \left[\sum_{i=0}^{\infty}\frac{(2i+1)\Gamma(\frac{n+2i-3}{2})}{(n-3)\Gamma(\frac{2i+2}{2})\Gamma(\frac{n-3}{2})}\right.$$

$$\times\left(\frac{\delta^2}{1+\delta^2}\right)^i\left(\frac{1}{1+\delta^2}\right)^{\frac{n-3}{2}} - \sum_{i=0}^{\infty}\frac{(i+1)\Gamma(\frac{n+i-3}{2})\delta^i}{4\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+i-3}{2}}}$$

$$\times I_{m_0}\left(\frac{n-2}{2}, \frac{i+3}{2}\right)] .$$

Since

$$\frac{\Gamma(\frac{n+2i-3}{2})}{\Gamma(\frac{2i+2}{2})\Gamma(\frac{n-3}{2})}\left(\frac{\delta^2}{1+\delta^2}\right)^i\left(\frac{1}{1+\delta^2}\right)^{\frac{n-3}{2}} \qquad i = 0,1,2\ldots$$

is the negative binomial density function,

$$E[(\hat{\beta}_2-\beta_0)^2 | t<t_0]P(t<t_0)= \left(\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_1}\right)^2 [\delta^2 + \frac{1}{n-3}$$

$$- \sum_{i=0}^{\infty}\frac{(i+1)\Gamma(\frac{n+i-3}{2})\delta^i}{4\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+i-3}{2}}} I_{m_0}\left(\frac{n-2}{2}, \frac{i+3}{2}\right)]. \qquad (3.31)$$

From (3.5) and using the relationship of δ

$$V(\bar{y}_s) = V(\bar{y}_d) + \frac{\sigma_1^{\,2}}{n} E[(\hat{\beta}_2-\beta_0)^2|A^c]P(A^c)$$

$$- \frac{2\delta\sigma_1\sigma_2}{n}(1-\rho^2)^{\frac{1}{2}} E[(\hat{\beta}_2-\beta_0)|A^c]P(A^c). \qquad (3.32)$$

Using the result of the corollary and (3.31) and upon substituting into (3.32) we get the result in the following theorem for Case II with $t_0>0$. The other results of the following theorem may be obtained by using analogous techniques on Lemma 3.1 to those used in obtaining (3.31), using the corollary, and substituting the results in (3.32).

<u>Theorem 3.2</u>:

<u>Case I</u>:

$$V(\bar{y}_s)=V(\bar{y}_d) - \frac{2\sigma_2^{\,2}(1-\rho^2)}{n} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{n+2i-1}{2})\,\delta^{2i+2}}{\Gamma(i+1)\Gamma(\frac{n-1}{2})\,(1+\delta^2)^{\frac{n+2i-1}{2}}}$$

$$\times\, I_{m_0}(\tfrac{n-2}{2}, \tfrac{2i+3}{2}) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n} \sum_{i=0}^{\infty} \frac{(2i+1)\Gamma(\frac{n+2i-3}{2})\,\delta^{2i}}{2\Gamma(i+1)\Gamma(\frac{n-1}{2})\,(1+\delta^2)^{\frac{n+2i-3}{2}}}$$

$$\times\, I_{m_0}(\tfrac{n-2}{2}, \tfrac{2i+3}{2}).$$

Case II:

$$V(\bar{y}_s) = V(\bar{y}_d) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n(n-3)}\,(1-(n-3)\delta^2)$$

$$+ \frac{\sigma_2^{\,2}(1-\rho^2)}{n} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{n+i-2}{2})\,\delta^{i+1}}{\Gamma(\frac{i+1}{2})\,\Gamma(\frac{n-1}{2})\,(1+\delta^2)^{\frac{n+i-2}{2}}}\, I_{m_0}(\tfrac{n-2}{2}, \tfrac{i+2}{2})$$

$$- \frac{\sigma_2^{\,2}(1-\rho^2)}{n} \sum_{i=0}^{\infty} \frac{(i+1)\,\Gamma(\frac{n+i-3}{2})\,\delta^{i}}{4\Gamma(\frac{i+2}{2})\,\Gamma(\frac{n-1}{2})\,(1+\delta^2)^{\frac{n+i-3}{2}}}\, I_{m_0}(\tfrac{n-2}{2}, \tfrac{i+3}{2}),$$

if $t_0>0$;

$$= V(\bar{y}_d) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n} \sum_{i=0}^{\infty} \frac{(-1)^{i}\,\Gamma(\frac{n+i-2}{2})\,\delta^{i+1}}{\Gamma(\frac{i+1}{2})\,\Gamma(\frac{n-1}{2})\,(1+\delta^2)^{\frac{n+i-2}{2}}}\, I_{m_0}(\tfrac{n-2}{2}, \tfrac{i+2}{2})$$

$$+ \frac{\sigma_2^{\,2}(1-\rho^2)}{n} \sum_{i=0}^{\infty} \frac{(-1)^{i}(i+1)\,\Gamma(\frac{n+i-3}{2})\,\delta^{i}}{4\Gamma(\frac{i+2}{2})\,\Gamma(\frac{n-1}{2})\,(1+\delta^2)^{\frac{n+i-3}{2}}}\, I_{m_0}(\tfrac{n-2}{2}, \tfrac{i+3}{2}),$$

if $t_0 \le 0$.

Case III:

$$V(\bar{y}_s) = V(\bar{y}_d) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n(n-3)}(1-(n-3)\delta^2)$$

$$-\frac{\sigma_2^{\,2}(1-\rho^2)}{n}\sum_{i=0}^{\infty}\frac{(-1)^{i+2}\Gamma(\frac{n+i-2}{2})\,\delta^{i+1}}{\Gamma(\frac{i+1}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+i-2}{2}}}I_{m_0}(\tfrac{n-2}{2},\tfrac{i+2}{2})$$

$$-\frac{\sigma_2^{\,2}(1-\rho^2)}{n}\sum_{i=0}^{\infty}\frac{(-1)^{i+2}(i+1)\Gamma(\frac{n+i-3}{2})\,\delta^{i}}{4\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+i-3}{2}}}I_{m_0}(\tfrac{n-2}{2},\tfrac{i+3}{2}),$$

if $t_0<0$;

$$= V(\bar{y}_d) - \frac{\sigma_2^{\,2}(1-\rho^2)}{n}\sum_{i=0}^{\infty}\frac{\Gamma(\frac{n+i-2}{2})\,\delta^{i+1}}{\Gamma(\frac{i+1}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+i-2}{2}}}I_{m_0}(\tfrac{n-2}{2},\tfrac{i+2}{2})$$

$$+\frac{\sigma_2^{\,2}(1-\rho^2)}{n}\sum_{i=0}^{\infty}\frac{(i+1)\Gamma(\frac{n+i-3}{2})\,\delta^{i}}{4\Gamma(\frac{i+2}{2})\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+i-3}{2}}}I_{m_0}(\tfrac{n-2}{2},\tfrac{i+3}{2}),$$

if $t_0\geq 0$.

Remark 3.3:

As $t_0 \to\infty$ in Case II, as $t_0\to-\infty$ in Case III, and as $t_0\to 0$ in Case I, then $V(\bar{y}_s)$ becomes

$$\lim_{t_0} V(\bar{y}_s) = V(\bar{y}_\ell)$$

$$= \frac{\sigma_2^{\,2}(1-\rho^2)}{n}\ (1+ \frac{1}{n-3}),$$

which is to be expected since the estimator $\bar{y}_s$ in those cases reduces to $\bar{y}_\ell$.

Remark 3.4:

As $t_0 \to -\infty$ in Case II, as $t_0 \to \infty$ in Case III and as $t_0 \to \infty$ in Case I then $V(\bar{y}_s)$ reduces to

$$\lim_{t_0} V(\bar{y}_s) = V(\bar{y}_d) = \frac{\sigma_2^{\,2}}{n} - \frac{2\sigma_{12}\beta_0}{n} + \frac{\sigma_1^{\,2}\beta_0^{\,2}}{n}$$

$$= \frac{\sigma_2^{\,2}(1-\rho^2)}{n}\ (1+\delta^2)$$

which is to be expected since the estimator $\bar{y}_s$ in those cases reduces to $\bar{y}_d$.

IV. COMPARISON OF DIFFERENT ESTIMATORS

In this chapter a comparison is made of the sometimes regression estimator with both the difference estimator and the regression estimator for all the three cases. It will be assumed that the sample size is $n \geq 4$.

A. Comparison of the Sometimes Regression Estimator with the Difference Estimator for Case I

For this case we have

$$V(\bar{y}_s) = V(\bar{y}_d) - \frac{\sigma_2{}^2(1-\rho^2)}{n}\Big[\sum_{i=0}^{\infty} \frac{2\Gamma(\frac{n+2i-1}{2})\,\delta^{2i+2}}{\Gamma(i+1)\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+2i-1}{2}}}$$

$$\times\, I_{m_0}(\frac{n-2}{2},\ \frac{2i+3}{2})$$

$$- \sum_{i=0}^{\infty} \frac{(2i+1)\Gamma(\frac{n+2i-3}{2})\,\delta^{2i}}{2\Gamma(i+1)\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n+2i-3}{2}}}\, I_{m_0}(\frac{n-2}{2},\ \frac{2i+3}{2})\Big].$$

Let

$$\theta = \frac{\delta}{\sqrt{1+\delta^2}} \tag{4.1.1}$$

and

$$D_2^*(\theta,\ m_0) = \frac{n\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-3}{2}}}{\sigma_2{}^2(1-\rho^2)}\,(V(\bar{y}_s)-V(\bar{y}_d)). \tag{4.1.2}$$

Since

$$\frac{n\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-3}{2}}}{\sigma_2{}^2(1-\rho^2)} > 0,$$

$D_2^*(\theta, m_0)$ has the same sign as $V(\bar{y}_s)-V(\bar{y}_d)$, and we have

$$D_2^*(\theta, m_0) = \sum_{i=0}^{\infty} \frac{(2i+1)\Gamma(\frac{n+2i-3}{2})\theta^{2i}}{2\Gamma(i+1)} I_{m_0}(\frac{n-2}{2}, \frac{2i+3}{2})$$

$$- \sum_{i=0}^{\infty} \frac{2\Gamma(\frac{n+2i-1}{2})\theta^{2i+2}}{\Gamma(i+1)} I_{m_0}(\frac{n-2}{2}, \frac{2i+3}{2}) \tag{4.1.3}$$

$$= \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+2j-3}{2})\theta^{2j}}{2\Gamma(j+1)} I_{m_0}(\frac{n-2}{2}, \frac{2j+3}{2})[(2j+1)-2(n+2j-3)\theta^2]. \tag{4.1.4}$$

Let $j=i-1$ in the first summation of (4.1.3), then we have

$$D_2^*(\theta, m_0) = \frac{\Gamma(\frac{n-3}{2})I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{2}$$

$$+ 2\sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+2j-1}{2})\theta^{2j+2}}{\Gamma(j+1)} (\frac{(2j+3)}{4(j+1)}I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})-I_{m_0}(\frac{n-2}{2},\frac{2j+3}{2})) \tag{4.1.5}$$

$$= \frac{\Gamma(\frac{n-3}{2})I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{2}$$

$$+ 2\sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+2j-1}{2})\theta^{2j+2}}{\Gamma(j+1)} I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2}) (\frac{2j+3}{4(j+1)} - \frac{I_{m_0}(\frac{n-2}{2}, \frac{2j+3}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})}). \tag{4.1.6}$$

Consider first the effect of variation in θ. θ will vary over the interval (-1, 1) since δ may vary over the interval $(-\infty, \infty)$.

Using (4.1.6), $D_2^*(\theta, m_0)$ can be expressed in the form

$$D_2^*(\theta, m_0) = \sum_{j=0}^{\infty} C_j^*(m_0)\theta^{2j}, \tag{4.1.7}$$

where

$$C_0^*(m_0) = \frac{\Gamma(\frac{n-3}{2}) I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{2}, \tag{4.1.8}$$

$$C_{j+1}^*(m_0) = \frac{2\Gamma(\frac{n+2j-1}{2}) I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})}{\Gamma(j+1)} \left(\frac{2j+3}{4(j+1)} - \frac{I_{m_0}(\frac{n-2}{2}, \frac{2j+3}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})}\right),$$

$$j = 0,1,2\ldots . \tag{4.1.9}$$

<u>Lemma 4.1</u>:

For $a>1$, $b>1$ and $h>0$,

$$B(a, b+h) < B(a,b). \tag{4.1.10}$$

<u>Proof</u>:

The lemma follows from the fact that

$$\begin{aligned} B(a,b+h) &= \int_0^1 x^{a-1}(1-x)^{b+h-1}\,dx \\ &< \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \\ &= B(a,b) . \qquad \text{Q.E.D.} \end{aligned}$$

Define

$$R_x(a,b,b+h) = \frac{I_x'(a,b)}{I_x'(a,b+h)} \tag{4.1.12}$$

$$= \frac{B(a,b+h)}{B(a,b)(1-x)^h}, \quad 0<x<1,$$

where

$$I_x'(a,b) = \frac{d\, I_x(a,b)}{dx}. \tag{4.1.13}$$

<u>Lemma 4.2</u>:

There exists an $x^* \ni 0<x^*<1$ and for which

$$\begin{aligned} R_x(a,b,b+h) &< 1 \quad 0<x<x^* \\ &= 1 \quad x=x^* \\ &> 1 \quad x^*<x<1, \text{ where } h>0. \end{aligned}$$

<u>Proof</u>:

$$R_0(a,b,b+h) < 1.$$

Also

$$\lim_{x\to 1^-} R_x(a,b,b+h) = +\infty.$$

The result now follows on noting that $R_x(a,b,b+h)$ is a strictly increasing continuous function of x for $0<x<1$.

Q.E.D.

Lemma 4.3:

$$I_x(a,b) \leq I_x(a,b+h) \quad \text{for} \quad a>1,\ b>1 \quad \text{and } h>0.$$

Proof:

From Lemma 4.1,

$$\frac{B(a,b)}{B(a,b+h)} = \ell$$
$$> 1.$$

Since

$$\lim_{x\to 0} I_x(a,b) = \lim_{x\to 0} I_x(a,b+h)$$
$$= 0,$$

we have

$$\lim_{x\to 0+} \frac{I_x(a,b+h)}{I_x(a,b)} = \lim_{x\to 0+} \frac{I'_x(a,b+h)}{I'_x(a,b)}$$
$$= \frac{B(a,b)}{B(a,b+h)}$$
$$= \ell$$
$$> 1.$$

Hence for given

$$\varepsilon = \ell-1$$
$$> 0,$$

$\exists\,\delta>0$, such that for $0<x<\delta$

$$\ell-\varepsilon < \frac{I_x(a,b+h)}{I_x(a,b)} < \ell+\varepsilon$$

that is

$$1 < \frac{I_x(a,b+h)}{I_x(a,b)} < 2\ell-1 \ .$$

Hence for $0<x<\delta$

$$I_x(a,b) < I_x(a,b+h) \ . \tag{4.1.14}$$

From Lemma 4.2, $\exists$ x^* $\ni$ for $0<x<x^*$

$$I'_x(a,b) < I'_x(a,b+h) \ .$$

Let x_1 be an arbitrary but fixed number such that $0<x_1<x^*$.

Also let

$$\varepsilon = \frac{I'_{x_1}(a,b+h) - I'_{x_1}(a,b)}{2} \tag{4.1.15}$$

$$> 0$$

be given. Then since $I'_x(a,j)$ is a continuous function in x, $\exists$ $\delta_1>0$, $\delta_2>0$ and $\delta_3>0$; such that

i) for $|x-x_1| < \delta_1$, $0<I'_x(a,b)<1$,

ii) for $|x-x_1| < \delta_2$, $I'_x(a,b+h)>I'_{x_1}(a,b+h)-\varepsilon$,

and

iii) for $|x-x_1| < \delta_3$, $I'_x(a,b) < I'_{x_1}(a,b)+\varepsilon$.

Let

$$\Delta I_{x_1}(a,j,q) = I_{x_1+q}(a,j) - I_{x_1-q}(a,j) \tag{4.1.16}$$

and

$$\delta = \mathrm{Min}(\delta_1, \delta_2, \delta_3).$$

Then, we have by the Mean Value Theorem that

$$\frac{\Delta I_{x_1}(a,b+h,\delta)}{\Delta I_{x_1}(a,b,\delta)} > \frac{I'_{x_1}(a,b+h)-\varepsilon}{I'_{x_1}(a,b)+\varepsilon}$$

$$= 1 . \tag{4.1.17}$$

Hence by (4.1.17)

$$\Delta I_x(a,b,\delta) < \Delta I_x(a,b+h,\delta) \tag{4.1.18}$$

$$\text{for every } x\varepsilon(0,x^*).$$

Therefore by 4.1.14 and 4.1.18 for $x\varepsilon(0,x^*]$,

$$I_x(a,b) < I_x(a,b+h) . \tag{4.1.19}$$

Now let x be such that

$$x^* < x < 1.$$

Then we have from Lemma 4.2

$$I'_x(a,b) > I'_x(a,b+h) .$$

Let x_2 be an arbitrary but fixed number such that $x^*<x_2\leq 1$. Then using the Mean Value Theorem, and proceeding in a manner analogous to that used in obtaining (4.1.17), we find

that $\exists\ \delta > 0 \ni$

$$\frac{\Delta I_{x_2}(a,b,\delta)}{\Delta I_{x_2}(a,b+h,\delta)} > 1.$$

Hence

$$\frac{\Delta I_x(a,b,\delta)}{\Delta I_x(a,b+h,\delta)} > 1 \tag{4.1.20}$$

for every $x \varepsilon (x^*, 1]$.

Hence if for some $x_3 \ni x^* < x_3 < 1$

$$I_{x_3}(a,b) > I_{x_3}(a,b+h),$$

then by (4.1.20)

$$I_x(a,b) > I_x(a,b+h)$$

for every $x \varepsilon [x_3, 1]$.

But this contradicts the hypothesis that

$$I_1(a,b) = I_1(a,b+h) = 1.$$

Hence for all $x \ni 0 \leq x \leq 1$

$$I_x(a,b) \leq I_x(a,b+h),$$

and the lemma is proved.

Q.E.D.

Lemma 4.4:

For $a = 1, \frac{3}{2}, 2, \frac{5}{2} \ldots$ and $c = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

$$\frac{I_x(a,c)}{I_x(a,c+1)} \Big/ \frac{I_x(a,c+\frac{1}{2})}{I_x(a,c+\frac{3}{2})} \leq 1, \quad \text{for} \quad 0<x\leq 1.$$

Proof:

Since

$$\lim_{x\to 0^+} I_x(a,c) = \lim_{x\to 0^+} I_x(a,c+1) = \lim_{x\to 0^+} I_x(a,c+\tfrac{1}{2})$$

$$= \lim_{x\to 0^+} I_x(a,c+\tfrac{3}{2}) = 0,$$

and

$$\lim_{x\to 0^+} \frac{I'_x(a,c)}{I'_x(a,c+1)} = \frac{c}{a+c} < \frac{c+\frac{1}{2}}{a+c+\frac{1}{2}} = \lim_{x\to 0^+} \frac{I'_x(a,c+\frac{1}{2})}{I'_x(a,c+\frac{3}{2})},$$

we have by L'Hospital's rule

$$\lim_{x\to 0^+} \frac{I_x(a,c)}{I_x(a,c+1)} = \frac{c}{a+c} < \frac{c+\frac{1}{2}}{a+c+\frac{1}{2}} = \lim_{x\to 0^+} \frac{I_x(a,c+\frac{1}{2})}{I_x(a,c+\frac{3}{2})}.$$

Let

$$\varepsilon_1 = \left(\frac{c+\frac{1}{2}}{a+c+\frac{1}{2}} - \frac{c}{a+c}\right)/2 > 0$$

and let ε_2 be such that

$$0<\varepsilon_2<\frac{c}{a+c}\ .$$

Then for

$$\varepsilon = \mathrm{Min}(\varepsilon_1,\ \varepsilon_2)$$

there exists a $\delta>0 \ni$ for $0<x<\delta$

$$0 < \frac{c}{a+c} - \varepsilon < \frac{I_x(a,c)}{I_x(a,c+1)} < \frac{c}{a+c} + \varepsilon \le \frac{c+\frac{1}{2}}{a+c+\frac{1}{2}} - \varepsilon$$

$$< \frac{I_x(a,c+1)}{I_x(a,c+2)} < \frac{c+\frac{1}{2}}{a+c+\frac{1}{2}} + \varepsilon.$$

Hence for $0<x<\delta$,

$$\frac{I_x(a,c)}{I_x(a,c+1)} \Big/ \frac{I_x(a,c+\frac{1}{2})}{I_x(a,c+\frac{3}{2})} < 1. \tag{4.1.21}$$

Now consider

$$\phi(x) = I_x(a,c+1)I_x(a,c+\tfrac{1}{2})-I_x(a,c)I_x(a,c+\tfrac{3}{2})\ . \tag{4.1.22}$$

By (4.1.21) $\phi(x)>0$ for $0<x<\delta$. We must show that $\phi(x)\ge 0$ for $0\le x\le 1$.

$$\phi'(x)= \frac{x^{a-1}(1-x)^{c-1}}{B(a,c)B(a,c+\frac{3}{2})}[B(1-x)^{\frac{1}{2}}\int_0^x y^{a-1}(1-y)^c dy$$

$$+B(1-x)\int_0^x y^{a-1}(1-y)^{c-\frac{1}{2}}dy-(1-x)^{\frac{3}{2}}\int_0^x y^{a-1}(1-y)^{c-1}dy$$

$$-\int_0^x y^{a-1}(1-y)^{c+\frac{1}{2}}dy] \tag{4.1.23}$$

for $0\le x\le 1$,

where

$$B = \frac{(a+c)(c+\frac{1}{2})}{(a+c+\frac{1}{2})c} \ge 1.$$

Since $\phi(0) = \phi(1) = 0$ then in order to show that $\phi(x)\ge 0$ for $0\le x\le 1$, it will be sufficient to prove that there exists an x' such that

$$\begin{aligned} \phi'(x) &\ge 0 \qquad 0\le x\le x' \\ &\le 0 \qquad x'<x\le 1 \quad . \end{aligned}$$

Now consider

$$\phi_1(x) = B(1-x)^{\frac{1}{2}}\int_0^x y^{a-1}(1-y)^c dy + B(1-x)\int_0^x y^{a-1}(1-y)^{c-\frac{1}{2}}dy$$

$$-(1-x)^{\frac{3}{2}}\int_0^x y^{a-1}(1-y)^{c-1}dy - \int_0^x y^{a-1}(1-y)^{c+\frac{1}{2}}dy \qquad (4.1.24)$$

and recognize that $\phi'(x)$ and $\phi_1(x)$ have the same sign in the interval $0<x<1$.

We shall first consider the case when $a\ge 4$, we have

$$\phi_1'(x) = 2(B-1)x^{a-1}(1-x)^{c+\frac{1}{2}} - \frac{B}{2}(1-x)^{-\frac{1}{2}}\int_0^x y^{a-1}(1-y)^c dy$$

$$-B\int_0^x y^{a-1}(1-y)^{c-\frac{1}{2}}dy + \frac{3}{2}(1-x)^{\frac{1}{2}}\int_0^x y^{a-1}(1-y)^{c-1}dy \ , \qquad (4.1.25)$$

for $0\le x<1$.

$$\phi_1''(x) = [-(B-1)(2c+\frac{5}{2})x^{a-1}(1-x)^{c+1}-\frac{B}{4}\int_0^x y^{a-1}(1-y)^c dy$$

$$+2(a-1)(B-1)x^{a-2}(1-x)^{c+2}-\frac{3}{4}(1-x)\int_0^x y^{a-1}(1-y)^{c-1}dy](1-x)^{-\frac{3}{2}},$$

(4.1.26)

for $0\leq x<1$.

Let

$$\phi_2(x) = -(B-1)(2c+\frac{5}{2})x^{a-1}(1-x)^{c+1}-\frac{B}{4}\int_0^x y^{a-1}(1-y)^c dy$$

$$+2(a-1)(B-1)x^{a-2}(1-x)^{c+2}-\frac{3}{4}(1-x)\int_0^x y^{a-1}(1-y)^{c-1}dy ,$$

for $0\leq x\leq 1$.

Note that $\phi_2(x)$ and $\phi_1''(x)$ have the same sign in the interval $0\leq x<1$. Then, we have

$$\phi_2'(x) = [(B-1)(c+1)(2c+\frac{5}{2})-\frac{B}{4}\frac{3}{4}]x^{a-1}(1-x)^c$$

$$-(a-1)(B-1)(4c+\frac{13}{2})x^{a-2}(1-x)^{c+1}+2(a-1)(a-2)(B-1)x^{a-3}(1-x)^{c+2}$$

$$+\frac{3}{4}\int_0^x y^{a-1}(1-y)^{c-1}dy,$$

for $0\leq x\leq 1$.

$$\phi_2''(x) = \{-[(B-1)c(c+1)(2c+\tfrac{5}{2})-\frac{Bc}{4} - \frac{3c}{4} -\frac{3}{4}]x^3$$

$$+2(a-1)(a-2)(a-3)(B-1)(1-x)^3$$

$$+(a-1)[3(B-1)(c+1)(2c+3)-\frac{B}{4} - \frac{3}{4}]x^2(1-x)$$

$$-(a-1)(a-2)(B-1)(6c+\frac{21}{2})x(1-x)^2\}x^{a-4}(1-x)^{c-1},$$

for $0\leq x\leq 1$.

Let

$$\phi_3(x) = -[(B-1)c(c+1)(2c+\tfrac{5}{2})-\frac{Bc}{4} - \frac{3c}{4} - \frac{3}{4}]x^3$$

$$+2(a-1)(a-2)(a-3)(B-1)(1-x)^3$$

$$+(a-1)[3(B-1)(c+1)(2c+3)-\frac{B}{4} - \frac{3}{4}]x^2(1-x)$$

$$-(a-1)(a-2)(B-1)(6c+\frac{21}{2})x(1-x)^2,$$

for $0\leq x\leq 1$.

Note that $\phi_2''(x)$ and $\phi_3(x)$ have the same sign for $0<x<1$.

For $a\geq 4$, $\phi(x)$, $\phi'(x)$, $\phi_1(x)$, $\phi_1'(x)$, $\phi_1''(x)$, $\phi_2(x)$, $\phi_2'(x)$, $\phi_2''(x)$ and $\phi_3(x)$ are all continuous functions for $0\leq x\leq 1$ with the exception that $\phi_1'(x)$ and $\phi_1''(x)$ are discontinuous at 1. Since $\phi(0) = \phi'(0) = 0$ and $\phi(x)$ is positive in a neighborhood of zero, it must be true that $\phi'(x)$ is positive in a neighborhood of zero. Thus $\phi_1(x)$ must also be positive in a neighborhood of zero.

By using the same reasoning as used on $\phi'(x)$, it can be shown that $\phi_1'(x)$, $\phi_1''(x)$, $\phi_2(x)$, $\phi_2'(x)$, $\phi_2''(x)$ and $\phi_3(x)$ are all positive in a neighborhood of zero.

As $\phi_3(x)$ is a cubic function of x it can have at most two stationary points. Since $\phi_3(x)$ must be positive in a neighborhood of zero and $\phi_3(1) < 0$ then it is allowable that $\phi(x)$ be of the form

$$\phi_3(x) \begin{cases} > 0 & 0 \le x < x_1 \\ = 0 & x = x_1 \\ < 0 & x_1 < x \le x_2 \\ = 0 & x = x_2 \\ > 0 & x_2 < x \le x_3 \\ = 0 & x = x_3 \\ < 0 & x_3 < x \le 1 \end{cases}$$

for some x_1, x_2 and x_3. But $\phi_2(1) < 0$ and $\phi_2'(1) > 0$. It follows that $\phi_3(x)$ can have no other possible form. This implies that $\phi_2'(x)$ be of the form

$$\phi_2'(x) \begin{cases} = 0 & x = 0 \\ > 0 & 0 < x < x_4 \\ = 0 & x = x_4 \\ < 0 & x_4 < x < x_5 \end{cases}$$

$$= 0 \qquad x=x_5$$

$$> 0 \qquad x_5<x\leq 1$$

for some x_4 and x_5.

Hence it is necessary that $\phi_2(x)$ be of the form

$$\begin{aligned}\phi_2(x) &= 0 && x=0\\ &> 0 && 0<x<x_6\\ &= 0 && x=x_6\\ &< 0 && x_6<x\leq 1\end{aligned}$$

for some x_6. It can now be seen that $\phi_1''(x)$, $\phi_1'(x)$, $\phi_1(x)$ and $\phi'(x)$ have necessarily the same form as $\phi_2(x)$ with the exception that $\phi_1'(x)$ and $\phi_1''(x)$ are discontinuous at $x=1$. Hence the lemma is proved for $a = 4, \frac{9}{2}, 5 \ldots$.

For $a = \frac{7}{2}$ the only real change is that $\phi_2''(x)$ is discontinuous at $x=0$ at which point $\lim_{x\to 0^+} \phi_2''(x) = \infty$. It can be argued that all of the functions retain the same form as before.

When $a=3$ we have

$$\phi_2'(x) = [(B-1)(c+1)(2c+\tfrac{5}{2}) - \tfrac{B}{4} - \tfrac{3}{4}]x^{a-1}(1-x)^c$$

$$-(a-1)(B-1)(4c+\tfrac{13}{2})x^{a-2}(1-x)^{c+1}$$

$$+2(a-1)(a-2)(B-1)(1-x)^{c+2}+\tfrac{3}{4}\int_0^x y^{a-1}(1-y)^{c-1}dy,$$

$$\phi_2''(x) = \{[-(B-1)c(c+1)(2c+\tfrac{5}{2})+\tfrac{Bc}{4} + \tfrac{3c}{4} + \tfrac{3}{4}]x^2$$

$$+(a-1)[2(B-1)(c+1)(c+2)-\tfrac{B}{4} - \tfrac{3}{4}]x(1-x)$$

$$-(a-1)(a-2)(B-1)(6c+\tfrac{21}{2})(1-x)^2\}(1-x)^{c-1},$$

$$\phi_3(x) = [-(B-1)c(c+1)(2c+\tfrac{5}{2})+\tfrac{Bc}{4} + \tfrac{3c}{4} + \tfrac{3}{4}]x^2$$

$$+(a-1)[2(B-1)(c+1)(c+2)-\tfrac{B}{4} - \tfrac{3}{4}]x(1-x)$$

$$-(a-1)(a-2)(B-1)(6c+\tfrac{21}{2})(1-x)^2,$$

and hence $\phi_3(x)$ is now a quadratic function. It can be reasoned that the form of $\phi_3(x)$ is

$$\begin{aligned} \phi_3(x) &< 0 \quad 0\le x<x_1 \\ &= 0 \quad x=x_1 \\ &> 0 \quad x<x<x_2 \\ &= 0 \quad x=x_2 \\ &< 0 \quad x_2<x\le 1 \end{aligned}$$

for some x_1 and x_2.

$\phi_2'(0) > 0$, but otherwise it can be reasoned that the functions are of the same form as before.

When $a = \frac{5}{2}$ then $\phi_2'(x)$ and $\phi_2''(x)$ are discontinuous at $x=0$. The necessary form of $\phi_3(x)$ is

$$\begin{aligned} \phi_3(x) &< 0 \quad 0 \le x < x_1 \\ &= 0 \quad x = x_1 \\ &> 0 \quad x_1 < x < x_2 \\ &= 0 \quad x = x_2 \\ &< 0 \quad x_2 < x \le 1 \end{aligned}$$

for some x_1 and x_2.

Here $\lim_{x \to 0^+} \phi_2''(x) = -\infty$ and $\lim_{x \to 0^+} \phi_2'(x) = \infty$. With the exception of these changes the functions remain the same.

When a=2 we have

$$\phi_2(x) = -(B-1)(2c+\tfrac{5}{2})x(1-x)^{c+1} - \tfrac{B}{4}\int_0^x y(1-y)^c dy$$

$$+2(B-1)(1-x)^{c+2} - \tfrac{3}{4}(1-x)\int_0^x y(1-y)^{c-1}dy,$$

$$\phi_2'(x) = [(B-1)(c+1)(2c+\tfrac{5}{2}) - \tfrac{B}{4} - \tfrac{3}{4}]x(1-x)^c - (B-1)(4c+\tfrac{13}{2})(1-x)^{c+1}$$

$$+ \tfrac{3}{4}\int_0^x y(1-y)^{c-1}dy,$$

$$\phi_2''(x) = \{[3(B-1)(c+1)(2c+3) - \tfrac{B}{4} - \tfrac{3}{4}](1-x)$$

$$+ [-c((B-1)(c+1)(2c+\tfrac{5}{2}) - \tfrac{B}{4} - \tfrac{3}{4}) + \tfrac{3}{4}]x\}(1-x)^{c-1},$$

and

$$\phi_3(x) = [3(B-1)(c+1)(2c+3)-\frac{B}{4} - \frac{3}{4}](1-x)$$

$$+[-c((B-1)(c+1)(2c+\frac{5}{2})-\frac{B}{4} - \frac{3}{4})+\frac{3}{4}]x.$$

$\phi_3(x)$ is linear in this case. Further $\phi_3(0) > 0$ and $\phi_3(1) < 0$.

Then the form of $\phi_3(x)$ is

$$\phi_3(x) \begin{cases} > 0 & 0 \le x < x_1 \\ = 0 & x = x_1 \\ < 0 & x_1 < x \le 1 \end{cases}$$

for some x_1. Also

$$\phi_2'(x) \begin{cases} < 0 & 0 \le x < x_2 \\ = 0 & x = x_2 \\ > 0 & x_2 < x \le 1 \end{cases}$$

for some x_2, and

$$\phi_2(x) \begin{cases} > 0 & 0 \le x < x_3 \\ = 0 & x = x_3 \\ < 0 & x_3 < x \le 1 \end{cases}$$

for some x_3.

The rest of the functions are of the same form as before.

For $a = \frac{3}{2}$, $\phi_1''(x)$, $\phi_2(x)$, $\phi_2'(x)$, $\phi_2''(x)$ are discontinuous at $x=0$, and

$$\phi_3(x) = [-(B-1)c(c+1)(2c+\tfrac{5}{2}) + \tfrac{Bc}{4} + \tfrac{3c}{4} + \tfrac{3}{4}]x^3$$

$$+ \tfrac{3}{4}(B-1)(1-x)^3 + \tfrac{1}{2}[3(B-1)(c+1)(2c+3) - \tfrac{B}{4} - \tfrac{3}{8}]x^2(1-x)$$

$$+ \tfrac{1}{4}(B-1)(6c+\tfrac{21}{2})x(1-x)^2,$$

$$\phi_3'(x) = [-3(B-1)c(c+1)(2c+\tfrac{5}{2}) - \tfrac{3}{2}(B-1)(c+1)(2c+3) + \tfrac{3Bc}{4} + \tfrac{9c}{4} + \tfrac{B}{8} + \tfrac{21}{8}]x^2$$

$$+ [3(B-1)(c+1)(2c+3) - \tfrac{B}{4} - \tfrac{3}{4} - \tfrac{1}{2}(B-1)(6c+\tfrac{21}{2})]x(1-x)$$

$$+ \tfrac{1}{4}(B-1)(6c+\tfrac{3}{2})(1-x)^2,$$

and

$$\phi_3''(x) = [-6(B-1)c(c+1)(2c+\tfrac{5}{2}) - 6(B-1)(c+1)(2c+3)$$

$$+ \tfrac{1}{2}(B-1)(6c+\tfrac{21}{2}) + \tfrac{3Bc}{2} + \tfrac{B}{2} + \tfrac{9}{2}c + 6]x$$

$$+ [3(B-1)(c+1)(2c+3) - 6(B-1)(c+1) - \tfrac{B}{4} - \tfrac{3}{4}](1-x).$$

For $\phi_3''(x)$,

Coefficient of $x < 0$,

and the constant term is negative for c small and positive for c large. Since $\phi_3''(x)$ is linear in x then either

$\phi_3''(x) < 0$ for all x,

or

$$\begin{aligned}\phi_3''(x) &> 0 && 0\le x\le x_1\\ &= 0 && x=x_1\\ &< 0 && x_1<x\le 1\end{aligned}$$

for some x_1.

For either case the required form of $\phi_3'(x)$ is

$$\begin{aligned}\phi_3'(x) &> 0 && 0\le x\le x_2\\ &= 0 && x=x_2\\ &< 0 && x_2<x\le 1\end{aligned}$$

for some x_2.

Hence it can be shown that it is necessary that

$$\begin{aligned}\phi_3(x) &> 0 && 0\le x\le x_3\\ &= 0 && x=x_3\\ &< 0 && x_3<x\le 1\end{aligned}$$

for some x_3,

$$\begin{aligned}\phi_2'(x) &< 0 && 0<x\le x_4\\ &= 0 && x=x_4\\ &> 0 && x_4<x\le 1\end{aligned}$$

for some x_4,

$$\begin{aligned}\phi_2(x) &> 0 && 0<x<x_5\\ &= 0 && x=x_5\\ &< 0 && x_5<x\le 1\end{aligned}$$

for some x_5.

The rest of the functions are the same as when $a = 4, \frac{9}{2}$, 5 ... .

When $a = 1$

$$\phi_1''(x) = [-(B-1)(2c+\tfrac{5}{2})(1-x)^{c+1} - \tfrac{B}{4}\int_0^x (1-y)^{c-1}dy - \tfrac{3}{4}(1-x)\int_0^x (1-y)^{c-1}dy] \times (1-x)^{-\frac{3}{2}} \quad 0\le x\le 1,$$

$$\phi_2(x) = -(B-1)(2c+\tfrac{5}{2})(1-x)^{c+1} - \tfrac{B}{4}\int_0^x (1-y)^{c}dy - \tfrac{3}{4}(1-x)\int_0^x (1-y)^{c-1}dy$$

$$0\le x\le 1,$$

$$\phi_2'(x) = [(B-1)(c+1)(2c+\tfrac{5}{2}) - \tfrac{B}{4}\tfrac{3}{4}](1-x)^{c} + \tfrac{3}{4}\int_0^x (1-y)^{c-1}dy$$

$$0\le x\le 1,$$

$$\phi_2''(x) = [-((B-1)(c+1)(2c+\tfrac{5}{2}) - \tfrac{B}{4}\tfrac{3}{4})c + \tfrac{3}{4}](1-x)^{c-1}$$

$$0\le x\le 1,$$

and

$$\phi_3(x) = -((B-1)(c+1)(2c+\tfrac{5}{2}) - \tfrac{B}{4}\tfrac{3}{4})c + \tfrac{3}{4}$$

$$< 0$$

$0\le x\le 1$. Q.E.D.

<u>Lemma 4.5</u>:

For $0<m_0\le 1$ $C_0^*(m_0)$, $C_1^*(m_0)$, $C_2^*(m_0)$, ... is a sequence of numbers such that $\exists$ $K > 0$ such that

$$C_j^*(m_0) \geq 0 \qquad j \leq K$$
$$< 0 \qquad j > K .$$

Proof:

$\frac{2j+3}{4(j+1)}$ is a decreasing function of j. By Lemma 4.4

$$\frac{I_{m_0}(\frac{n-2}{2}, \frac{2j+3}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})} \qquad j = 0,1,2 \ldots$$

is an increasing function of j. We therefore have that

$$\frac{2j+3}{4(j+1)} - \frac{I_{m_0}(\frac{n-2}{2}, \frac{2j+3}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})} \qquad j = 0,1,2 \ldots$$

is a decreasing function of j. We have

$$C_0^*(m_0) = \frac{\Gamma(\frac{n-3}{2}) I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{2} > 0.$$

Now suppose

$$C_j^*(m_0) \geq 0 \qquad j = 0,1,2 \ldots . \tag{4.1.27}$$

This implies that

$$D_2^*(\theta, m_0) \geq 0 \qquad 0 \leq \theta < 1 . \tag{4.1.28}$$

But by (4.1.4) for $\theta = \frac{1}{\sqrt{2}} + \varepsilon$ with $\varepsilon > 0$

$$[(2j+1)-2(n+2j-3)\theta^2]=2j+1-2(n+2j-3)[\tfrac{1}{2}+\varepsilon^2+\sqrt{2}\varepsilon]$$

$$< 0 \qquad j = 0,1,2 \ldots,$$

and hence

$$D_2^*(\frac{1}{\sqrt{2}} + \varepsilon,\ m_0) < 0 \qquad 0<m_0\leq 1.$$

This contradicts (4.1.28) and also (4.1.27). Hence there exists a K such that

$$C_j^*(m_0) < 0 \qquad j = K,\ K+1,\ K+2,\ \ldots$$

and therefore the lemma is proved.

Q.E.D.

<u>Theorem 4.1</u>:

For m_0 fixed such that $0<m_0\leq 1$, there exists a θ_0 where $0<\theta_0<1$ and

$$\begin{aligned} D_2^*(\theta, m_0) &< 0 && -1<\theta<-\theta_0 \\ &= 0 && \theta=-\theta_0 \\ &> 0 && -\theta_0<\theta<\theta_0 \\ &= 0 && \theta=\theta_0 \\ &< 0 && \theta_0<\theta<1, \end{aligned}$$

and hence

$$\begin{aligned} V(\bar{y}_s) &< V(\bar{y}_d) && -1<\theta<-\theta_0 \\ &= V(\bar{y}_d) && \theta=-\theta_0 \\ &> V(\bar{y}_d) && -\theta_0<\theta<\theta_0 \end{aligned}$$

$$= V(\bar{y}_d) \qquad \theta=\theta_0$$
$$< V(\bar{y}_d) \qquad \theta_0<\theta<1.$$

Proof:

Since from (4.1.6) $D_2^*(\theta, m_0)$ is symmetric in θ, it is necessary only to consider $D_2^*(\theta, m_0)$ for θ positive.

From (4.1.5)

$$D_2^*(0, m_0) = \frac{\Gamma(\frac{n-3}{2}) I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{2}$$
$$> 0 \qquad\qquad 0<m_0\leq 1.$$

From (4.1.4) for $\theta = \frac{1}{\sqrt{2}} + \varepsilon$ with $\varepsilon > 0$

$$[(2j+1) - 2(n+2j-3)\theta^2]\Big|_{\theta=\frac{1}{\sqrt{2}}+\varepsilon} < 0 \qquad j = 0,1,2,\ldots,$$

and we have

$$D_2^*(\frac{1}{\sqrt{2}} + \varepsilon, m_0) < 0 \qquad 0<m_0\leq 1 \,.$$

Since $D_2^*(\theta, m_0)$ is continuous then there exists a θ_0 such that $0<\theta_0<1$ and

$$D_2^*(\theta_0, m_0) = 0 \,.$$

We now show that $D_2^*(\theta, m_0) < 0$ for $\theta > \theta_0$. By Lemma 4.5 there exists a K such that

$$C_j^*(m_0) > 0 \quad \text{for} \quad j < K$$
$$\leq 0 \quad \text{for} \quad j \geq K.$$

Hence

$$\sum_{j=0}^{\infty} C_j^*(m_0)\,\theta_0^{2j} = 0$$

i.e.

$$\sum_{j=0}^{\infty} C_j^*(m_0)\,\theta_0^{2j-1} = 0.$$

Since $D_2^*(\theta, m_0)$ is a power series in θ which converges for $-1<\theta<1$, we get

$$\frac{\partial D_2^*(\theta, m_0)}{\partial\theta} = \sum_{j=0}^{\infty} 2j\; C_j^*(m_0)\,\theta^{2j-1} \qquad \text{for } 0\leq\theta<1.$$

$$\left.\frac{\partial D_2^*(\theta, m_0)}{\partial\theta}\right|_{\theta=\theta_0} = \sum_{j=1}^{\infty} 2j\; C_j^*(m_0)\,\theta_0^{2j-1}$$

$$\leq 2K \sum_{j=1}^{\infty} C_j(m_0)\,\theta_0^{2j-1}$$

$$= \frac{-2K\; C_0(m_0)}{\theta_0}$$

$$< 0.$$

It can be similarly shown that if

$$D_2^*(\theta^*, m_0) < 0$$

then

$$\left.\frac{\partial D_2^*(\theta,\ m_0)}{\partial\theta}\right|_{\theta=\theta^*} < 0.$$

Therefore we have that with m_0 fixed, as θ increases, $D_2^*(\theta,\ m_0)$ becomes negative in sign and stays negative. Hence the theorem is proved.

Q.E.D.

Next consider the variation of $D_2^*(\theta,\ m_0)$ due to m_0 with θ fixed.

<u>Lemma 4.6</u>:

$$Q_{m_0}(a,c,c+1) = \frac{I_{m_0}(a,c+1)}{I_{m_0}(a,c)}$$

is a decreasing function of m_0 for $0<m_0\leq 1$.

<u>Proof</u>:

$$\frac{\partial Q_{m_0}(a,c,c+1)}{\partial_{m_0}}$$

$$= \frac{\frac{{m_0}^{a-1}(1-m_0)^{c-1}}{B(a,c)B(a,c+1)}\left[(1-m_0)\int_0^{m_0} x^{a-1}(1-x)^{c-1}dx-\int_0^{m_0} x^{a-1}(1-x)^{c}dx\right]}{I_{m_0}^{\ 2}(a,c)}\ .$$

But

$$\int_0^{m_0} x^{a-1}(1-x)^c dx \geq (1-m_0)\int_0^{m_0} x^{a-1}(1-x)^{c-1} dx.$$

Hence

$$\frac{\partial Q_{m_0}(a,c,c+1)}{\partial m_0} \leq 0 .$$

Hence $Q_{m_0}(a,c,c+1)$ is a decreasing function of m_0.

Q.E.D.

<u>Lemma 4.7</u>:

If for fixed θ, there exists an $m_0^* \varepsilon (0,1)$ such that

$$\left.\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0}\right|_{m_0=m_0^*} = 0$$

then

$$\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0} \begin{cases} > 0 & 0 \leq m_0 < m_0^* \\ = 0 & m_0 = m_0^* \\ < 0 & m_0^* < m_0 < 1 . \end{cases}$$

<u>Proof</u>:

Let

$$H_j^*(\theta) = \frac{\Gamma(\frac{n+2j-3}{2})\,\theta^{2j}}{2\Gamma(j+1)}[(2j+1)-2(n+2j-3)\theta^2] \quad j = 0,1,2...$$

then from (4.1.4)

$$D_2^*(\theta, m_0) = \sum_{j=0}^{\infty} H_j^*(\theta) I_{m_0}\left(\frac{n-2}{2}, \frac{2j+3}{2}\right), \tag{4.1.29}$$

and

$$\frac{\partial D_2^*(\theta, m_0)}{\partial m_0} = \sum_{j=0}^{\infty} H_j^*(\theta) I'_{m_0}\left(\frac{n-2}{2}, \frac{2j+3}{2}\right). \tag{4.1.30}$$

For $\theta^2 \geq \frac{1}{2}$

$$(2j+1) - 2(n+2j-3)\theta^2$$

is a decreasing function of j. Since

$$H_0(\theta) < 0$$

then

$$H_j^*(\theta) < 0 \qquad j = 0,1,2, \ldots .$$

Hence for $\theta^2 \geq \frac{1}{2}$ we have

$$\frac{\partial D_2^*(\theta, m_0)}{\partial m_0} \leq 0 .$$

For $\theta^2 < \frac{1}{2}$

$$(2j+1) - 2(n+2j-3)\theta^2$$

is an increasing function of j. Let

$$L = \left[\frac{2(n-3)\theta^2-1}{2-4\theta^2}\right],$$

where [ ] is the greatest integer function.

Then

$$H_j^*(\theta) \le 0 \qquad j \le L$$
$$\qquad > 0 \qquad j > L\ .$$

Suppose that m_{0_2} is such that $0<m_{0_2}<1$ and

$$\left.\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0}\right|_{m_0=m_{0_2}} = 0.$$

Then noting that

$$R_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2j+3}{2},\ \frac{2L+3}{2}\right) = \frac{I'_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2j+3}{2}\right)}{I'_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2L+3}{2}\right)} \quad j = 0,1,2\ \ldots,$$

we have

$$\left.\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0}\right|_{m_0=m_{0_2}}$$

$$= I'_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2L+3}{2}\right) \sum_{j=0}^{\infty} H_j^*(\theta) R_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2j+3}{2},\ \frac{2L+3}{2}\right)$$

$$= 0.$$

Since $I'_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2L+3}{2}\right) > 0,$

then

$$\sum_{j=0}^{\infty} H_j^*(\theta)\, R_{m_{0_2}}\left(\frac{n-2}{2}, \frac{2j+3}{2}, \frac{2L+3}{2}\right) = 0. \tag{4.1.31}$$

Now

$$R_{m_0}\left(\frac{n-2}{2}, \frac{2j+3}{2}, \frac{2L+3}{2}\right) = \frac{B\left(\frac{n-2}{2}, \frac{2L+3}{2}\right)}{B\left(\frac{n-2}{2}, \frac{2j+3}{2}\right)(1-m_0)^{L-j}} \,. \tag{4.1.32}$$

Hence for $\varepsilon>0$

$$\frac{R_{m_0}\left(\frac{n-2}{2}, \frac{2j+3}{2}, \frac{2L+3}{2}\right)}{R_{m_0}+\varepsilon\left(\frac{n-2}{2}, \frac{2j+3}{2}, \frac{2L+3}{2}\right)} = \left(\frac{1-m_0-\varepsilon}{1-m_0}\right)^{L-j}$$

$$= \begin{cases} > 1 & j=0,1,2\ldots L-1 \\ = 1 & j=L \\ < 1 & j=L+1,\ L+2\ \ldots\ . \end{cases}$$

Let $m_{0_3} = m_{0_2} + \varepsilon$ for $\varepsilon>0$ such that $m_{0_3} \leq 1$, then

$$\left.\frac{\partial D_2^*(\theta, m_0)}{\partial m_0}\right|_{m_0=m_{0_3}} = \sum_{j=0}^{\infty} H_j(\theta) I'_{m_{0_3}}\left(\frac{n-2}{2}, \frac{2j+3}{2}\right).$$

By (4.1.32) for b_0^*, b_1^*, b_2^*, ... such that

$$b_j^* = \frac{R_{m_{0_3}}\left(\frac{n-2}{2}, \frac{2j+3}{2}, \frac{2L+3}{2}\right)}{R_{m_{0_2}}\left(\frac{n-2}{2}, \frac{2j+3}{2}, \frac{2L+3}{2}\right)} \qquad j = 0,1,2\ \ldots,$$

we have

$$b_0^*>b_1^*>b_2^*> \ldots >b_{L-1}^*>b_L^* = 1>b_{L+1}^*>b_{L+2}^*\cdots,$$

and

$$\left.\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0}\right|_{m_0=m_{0_3}}$$

$$= I'_{m_{0_3}}\left(\frac{n-2}{2},\ \frac{2L+3}{2}\right)\sum_{j=0}^{\infty} H_j^*(\theta) b_j^* R_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2j+3}{2},\ \frac{2L+3}{2}\right)$$

$$< I'_{m_{0_3}}\left(\frac{n-2}{2},\ \frac{2L+3}{2}\right)\sum_{j=0}^{\infty} H_j^*(\theta) R_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{2j+3}{2},\ \frac{2L+3}{2}\right)$$

$$= 0.$$

Hence we have that if $\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0}$ is zero at m_{0_2} then it is negative for $m_0>m_{0_2}$. Hence the lemma is proved.

Q.E.D.

<u>Theorem 4.2</u>:

There exists $\theta_1^*>0$ and $\theta_2^*>0$ defined as

$$D_2(\theta_1^*,\ 1) = 0,$$

and

$$\theta_2^* = \operatorname*{Inf}_{U_2} \theta,$$

where

$$U_2 = \{\theta:\ \theta>0,\ D_2(\theta,\ m_0) \le 0 \text{ for all } m_0 \ni 0<m_0\le 1\};$$

such that

a) for θ fixed and $\varepsilon[-\theta_1^*, \theta_1^*]$

$$D_2^*(\theta, m_0) \geq 0 \quad \text{for} \quad 0<m_0\leq 1$$

and hence

$$V(\bar{y}_s) \geq V(\bar{y}_d) \quad \text{for} \quad 0<m_0\leq 1;$$

b) for θ fixed and $\varepsilon\{(-\theta_2^*, -\theta_1^*)\cup(\theta_1^*, \theta_2^*)\}$, $\exists$ $m_0^* \ni 0<m_0^*<1$, and

$$\begin{aligned} D_2^*(\theta, m_0) &\geq 0 \qquad 0<m_0\leq m_0^* \\ &< 0 \qquad m_0^*<m_0\leq 1, \end{aligned}$$

and hence

$$\begin{aligned} V(\bar{y}_s) &\geq V(\bar{y}_d) \qquad 0<m_0\leq m_0^* \\ &< V(\bar{y}_d) \qquad m_0^*<m_0\leq 1; \end{aligned}$$

c) for θ fixed and $\varepsilon\{(-1, -\theta_2^*]\cup[\theta_2^*, 1)\}$

$$D_2^*(\theta, m_0) \leq 0 \quad \text{for} \quad 0<m_0\leq 1$$

and hence

$$V(\bar{y}_s) \leq V(\bar{y}_d) \quad \text{for} \quad 0<m_0\leq 1.$$

Proof:

Since $D_2^*(\theta, m_0)$ is symmetric in θ, it is necessary only to consider $D_2^*(\theta, m_0)$ for θ positive.

Suppose that for θ fixed $\ni 0<\theta<1$, $\exists$ $m_0^* \ni m_0^* \in (0,1)$ and

$$D_2^*(\theta, m_0^*) = 0.$$

Since

$$\lim_{m_0 \to 0} D_2^*(\theta, m_0) = \lim_{m_0 \to 0} \frac{\partial D_2^*(\theta, m_0)}{\partial m_0} = 0,$$

it follows from Lemma 4.7 that if

$$\frac{\partial D_2^*(\theta, m_0)}{\partial m_0} < 0$$

in a neighborhood of $m_0=0$, then

$$\frac{\partial D_2^*(\theta, m_0)}{\partial m_0} < 0 \qquad 0<m_0\leq 1.$$

Hence there could not be a point $m_0^* \ni 0<m_0^*\leq 1$ and

$$D_2^*(\theta, m_0) = 0.$$

Hence in order that $D_2^*(\theta, m_0^*) = 0$ it follows that there must exist an m_0^{**} such that

$$0<m_0^{**}<m_0^*\leq 1$$

and

$$\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0} \begin{array}{ll} > 0 & 0<m_0<m_0^{**} \\ = 0 & m=m_0^{**} \\ < 0 & m_0^{**}<m_0\leq 1. \end{array}$$

Hence if

$$D_2^*(\theta,\ m_0^*) = 0$$

then for $m_0>m_0^*$

$$D_2^*(\theta,\ m_0) < 0.$$

By above if for $\theta=\theta_1$

$$D_2^*(\theta_1,\ 1) \geq 0$$

then

$$D_2^*(\theta_1, m_0) \geq 0 \qquad 0<m_0\leq 1.$$

If further for $\theta=\theta_2$

$$D_2^*(\theta_2,\ 1) < 0,$$

then by Theorem 4.1

$$\theta_2>\theta_1 .$$

Hence

$$\theta_1^* = \{\theta:\ \theta>0 \ \text{ and } \ D_2^*(\theta,\ 1) = 0\}.$$

If

$$D_2^*(\theta,\ 1) < 0$$

then either $\theta=\theta_3$

and

$$D_2^*(\theta_3, m_0) \leq 0 \qquad 0<m_0\leq 1,$$

or

$$\theta=\theta_4 \text{ and } \exists\ m_0^* \ni$$

$$\begin{aligned} D_2^*(\theta_4, m_0) &> 0 \qquad 0<m_0<m_0^* \\ &= 0 \qquad m_0=m_0^* \\ &< 0 \qquad m_0^*<m_0\leq 1. \end{aligned}$$

Now for

$$m_{0_1}<m_0^*$$

$$D_2^*(\theta_3, m_{0_1}) \leq 0$$

and

$$D_2^*(\theta_4, m_{0_1}) \geq 0,$$

then by Theorem 4.1

$$\theta_4\leq\theta_3.$$

Hence

$$\theta_2^* = \inf_{U_2} \theta$$

and the theorem is proved.

Q.E.D.

Now let us look at the efficiency of $\bar{y}_s$ with respect to $\bar{y}_d$. We have

$$V(\bar{y}_d) = \frac{1}{n}[\sigma_2^{\ 2}+\beta_0^{\ 2}\sigma_1^{\ 2}-2\beta_0\sigma_{12}]$$
$$=\frac{\sigma_2^{\ 2}}{n}(1-\rho^2)(1+\delta^2).$$

Therefore

relative efficiency of $\bar{y}_s$ with respect to $\bar{y}_d$

$$= e_2^*(\delta,\ m_0)$$
$$= \frac{V(\bar{y}_d)}{V(\bar{y}_s)}$$
$$= \frac{1}{1+\dfrac{D_2^*(\theta,\ m_0)}{\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}}}\ .$$

Immediately from Theorems 4.1 and 4.2 we have Theorems 4.3 and 4.4.

<u>Theorem 4.3</u>:

For m_0 fixed such that $0<m_0\leq 1$ there exists a θ_0 with $0<\theta_0<1$ $\ni$

$$e_2^*(\delta,\ m_0) \begin{cases} >1 & -1<\theta<-\theta_0 \\ =1 & \theta=-\theta_0 \\ <1 & -\theta_0<\theta<\theta_0 \\ =1 & \theta=\theta_0 \\ >1 & \theta_0<\theta<1\ . \end{cases}$$

Theorem 4.4:

Let θ_1^* and θ_2^* be as in Theorem 4.2 then

a) for θ fixed and $\varepsilon[-\theta_1^*, \theta_1^*]$

$e_2^*(\delta, m_0) \leq 1$ for $0 \leq m_0 \leq 1$,

b) for θ fixed and $\varepsilon\{(-\theta_2^*, -\theta_1^*) \cup (\theta_1^*, \theta_2^*)\}$

$\exists\, m_0^* \ni 0 < m_0^* < 1$ and

$$e_2^*(\delta, m_0) = \begin{cases} \leq 1 & \text{for } 0 < m_0 \leq m_0^* \\ > 1 & \text{for } m_0^* < m_0 \leq 1, \end{cases}$$

c) for θ fixed and $\varepsilon\{(-1, -\theta_2^*] \cup [\theta_2^*, 1)\}$

$e_2^*(\delta, m_0) \geq 1$ for $0 \leq m_0 \leq 1$.

Theorem 4.5:

For e_0 fixed such that $0 < e_0 < 1$, there exists an m_0^* such that for $m_0 \leq m_0^*$

$$e_2^*(\delta, m_0) \geq e_0.$$

Proof:

By Lemma 4.7 for fixed θ or equivalently for fixed δ, $\exists\, m_0(\theta) \ni$

$$e_2^*(\delta, m_0) = \cfrac{1}{1 + \cfrac{D_2^*(\theta, m_0)}{\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}}}$$

$$\geq e_0 \qquad\qquad 0 < m_0 \leq m_0(\theta).$$

Here $m_0(\theta)$ may be 1.

Pick

$$m_0^* = \operatorname*{Inf}_{0 \le \theta < 1} m_0(\theta).$$

Hence

$$e_2(\delta, m_0) \ge e_2(m_0^*, \delta)$$
$$\ge e_0 \qquad \text{for } 0 < m_0 \le m_0^*$$

and for any $\delta \in [\theta, \infty)$.

B. Comparison of the Sometimes Regression Estimator with the Regression Estimator for Case I

In this case we have

$$V(\bar{y}_s) = V(\bar{y}_\ell) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n}\left(\delta^2 - \frac{1}{n-3}\right)$$
$$+ \frac{\sigma_2^{\,2}(1-\rho^2)}{n\Gamma\left(\frac{n-1}{2}\right)(1+\delta^2)^{\frac{n-3}{2}}} D_2^*(\theta, m_0). \qquad (4.2.1)$$

Let

$$D_1^*(\theta, m_0) = \frac{n}{\sigma_2^{\,2}(1-\rho^2)(1-\theta^2)^{\frac{n-3}{2}}}(V(\bar{y}_s) - V(\bar{y}_\ell)). \qquad (4.2.2)$$

Then

$$D_1^*(\theta, m_0) = \frac{\theta^2}{(1-\theta^2)^{\frac{n-1}{2}}} - \frac{1}{(n-3)(1-\theta^2)^{\frac{n-3}{2}}} + \frac{I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{n-3}$$

$$+ \sum_{j=0}^{\infty} \frac{2\Gamma(\frac{n+2j-1}{2})\,\theta^{2j+2}}{\Gamma(j+1)\Gamma(\frac{n-1}{2})}\left(\frac{(2j+3)I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})}{4(j+1)}\right.$$

$$\left. - I_{m_0}(\frac{n-2}{2}, \frac{2j+3}{2})\right)$$

$$= \frac{\theta^2}{(1-\theta^2)^{\frac{n-1}{2}}} - \frac{1}{(n-3)(1-\theta^2)^{\frac{n-3}{2}}} + \frac{I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{n-3}$$

$$+ \sum_{j=0}^{\infty} \frac{2\Gamma(\frac{n+2j-1}{2})\,\theta^{2j+2}}{\Gamma(j+1)\Gamma(\frac{n-1}{2})} I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})$$

$$\times \left(\frac{2j+3}{4(j+1)} - \frac{I_{m_0}(\frac{n-2}{2}, \frac{2j+3}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{2j+5}{2})}\right). \qquad (4.2.3)$$

Consider first the effect of variation of θ.

<u>Theorem 4.6</u>:

For m_0 fixed such that $0<m_0\leq 1$, there exists a θ_0 such that $0<\theta_0<1$ and

$$D_1^*(\theta,\ m_0) \begin{cases} > 0 & -1<\theta<-\theta_0 \\ = 0 & \theta=-\theta_0 \\ < 0 & -\theta_0<\theta<\theta_0 \\ = 0 & \theta=\theta_0 \\ > 0 & \theta_0<\theta<1, \end{cases}$$

and hence

$$V(\bar{y}_s) \begin{cases} > V(\bar{y}_\ell) & -1<\theta<-\theta_0 \\ = V(\bar{y}_\ell) & \theta=-\theta_0 \\ < V(\bar{y}_\ell) & -\theta_0<\theta<\theta_0 \\ = V(\bar{y}_\ell) & \theta=\theta_0 \\ > V(\bar{y}_\ell) & \theta_0<\theta<1\ . \end{cases}$$

Proof:

$$\frac{\partial D_1^*(\theta,\ m_0)}{\partial\theta} = \frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)^{\frac{n+1}{2}}}$$

$$+ \frac{1}{\Gamma(\frac{n-1}{2})} \sum_{j=0}^{\infty} \frac{(4j+4)\Gamma(\frac{n+2j-1}{2})\theta^{2j+1}}{\Gamma(j+1)}$$

$$\times \left(\frac{(2j+3)I_{m_0}(\frac{n-2}{2},\ \frac{2j+5}{2})}{4(j+1)} - I_{m_0}(\frac{n-2}{2},\ \frac{2j+3}{2})\right). \qquad (4.2.4)$$

Let

$$M^*(\theta,\ m_0) = \frac{1}{\Gamma(\frac{n-1}{2})} \sum_{j=0}^{\infty} \frac{4(j+1)\Gamma(\frac{n+2j-1}{2})}{\Gamma(j+1)} \Big(\frac{(2j+3)\, I_{m_0}(\frac{n-2}{2},\ \frac{2j+5}{2})}{4(j+1)}$$

$$- I_{m_0}(\frac{n-2}{2},\ \frac{2j+3}{2}) \Big) \theta^{2j+1}$$

$$= \frac{1}{\Gamma(\frac{n-1}{2})} \sum_{j=0}^{\infty} \frac{4(j+1)\Gamma(\frac{n+2j-1}{2})}{\Gamma(j+1)} I_{m_0}(\frac{n-2}{2},\ \frac{2j+5}{2})$$

$$\times \Big[\frac{2j+3}{4(j+1)} - \frac{I_{m_0}(\frac{n-2}{2},\ \frac{2j+3}{2})}{I_{m_0}(\frac{n-2}{2},\ \frac{2j+5}{2})}\Big] \theta^{2j+1} \ . \qquad (4.2.5)$$

If $M^*(\theta,\ m_0)$ is nonnegative then $\frac{\partial D_1^*(\theta,\ m_0)}{\partial \theta} > 0$ and $D_1^*(\theta,\ m_0)$ is an increasing function of θ.

Next we look at $M^*(\theta,\ m_0)$ and determine that for fixed θ if $M^*(\theta,\ m_0)$ is ever negative then its smallest value is when $m_0 = 1$.

Let

$$h^*(j,\ m_0) = I_{m_0}(\frac{n-2}{2},\ \frac{2j+5}{2}) \Big[\frac{2j+3}{4(j+1)} - \frac{I_{m_0}(\frac{n-2}{2},\ \frac{2j+3}{2})}{I_{m_0}(\frac{n-2}{2},\ \frac{2j+5}{2})}\Big] \qquad (4.2.6)$$

where $j = 0, 1, \ldots$.

By Lemma 4.5 $h^*(j, m_0)$ is a sequence of numbers in j which start out positive and become negative as j increases and stays negative.

Also,

$$I_{m_0}\left(\frac{n-2}{2}, \frac{2j+5}{2}\right)$$

is an increasing function of m_0. Hence if

$$h^*(j, m_0) \leq 0,$$

then for $\varepsilon>0$

$$h^*(j, m_0 + \varepsilon) < h^*(j, m_0) \leq 0 .$$

But at $m_0=1$ $h^*(0, 1) < 0$. Hence for all j,

$$h^*(j, 1) < h^*(j, m_0) \qquad 0 \leq m_0 < 1.$$

Therefore

$$M^*(\theta, 1) < M^*(\theta, m_0) \quad \text{for} \quad 0 \leq m_0 < 1.$$

Since for θ fixed

$$\frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)^{\frac{n+1}{2}}}$$

is a positive fixed constant in m_0, we have

$$\frac{\partial D_1^*(\theta,\ m_0)}{\partial\theta} \geq \frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)^{\frac{n+1}{2}}} + M^*(\theta,\ 1)$$

$$= \frac{1}{(1-\theta^2)^{\frac{n-1}{2}}}\left[\frac{\theta(1-\theta^2)+(n-1)\theta^3}{1-\theta^2}\right.$$

$$\left. - \theta \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+2j-1}{2})\theta^{2j}}{\Gamma(\frac{n-1}{2})\Gamma(j+1)} (1-\theta^2)^{\frac{n-1}{2}} (2j+1)\right].$$

But

$$\frac{\Gamma(\frac{n+2j-1}{2})}{\Gamma(\frac{n-1}{2})\Gamma(j+1)} \theta^{2j} (1-\theta^2)^{\frac{n-1}{2}}$$

is negative binomial for $j=0,1\ \ldots$.

Hence

$$\frac{\partial D_1^*(\theta,\ m_0)}{\partial\theta} \geq \frac{1}{(1-\theta^2)^{\frac{n-1}{2}}}\left[\frac{\theta(1-\theta^2)+(n-1)\theta^3}{1-\theta^2} - \frac{\theta^3(n-1)}{1-\theta^2}\right.$$

$$\left. - \frac{\theta(1-\theta^2)}{1-\theta^2}\right]$$

$$= 0.$$

Hence $D_1^*(\theta,\ m_0)$ is a nondecreasing function of θ for fixed m_0.

We have

$$D_1^*(0,\ m_0) \leq 0. \tag{4.2.7}$$

By Remark 3.3, for $t_0=0$ and hence $m_0=1$, the sometimes regression estimator becomes the regression estimator and we have

$$D_1^*(\theta,\ 1) = 0.$$

By Theorem 4.2 for θ in a neighborhood of 1.

$$D_1^*(\theta,\ m_0) > D_1^*(\theta,\ 1),$$

therefore

$$\lim_{\theta\to 1} D_1^*(\theta,\ m_0) > 0 \qquad 0<m_0<1\ . \tag{4.2.8}$$

By (4.2.7) and (4.2.8) and the fact that $D_2(\theta,\ m_0)$ is a nondecreasing function in θ, the theorem is proved. Q.E.D.

Next we consider the effect of m_0 with θ fixed. This result is given in Theorem 4.7.

Theorem 4.7:

With θ fixed $D_1^*(\theta,\ m_0)$ varies with $D_2^*(\theta,\ m_0)$ as a function of m_0. For θ fixed such that $0\leq\theta<1$ then $D_1^*(\theta,\ m_0)$ falls in one of the following three categories:

a) $D_1^*(\theta,\ m_0)$ is always increasing as a function of m_0 for $0<m_0\leq 1$;

b) $\exists\, m_0^*$ such that $0<m_0^*<1$ and $D_1^*(\theta,\ m_0)$ is increasing as a function of m_0 for $m_0<m_0^*$ and decreasing for $m_0>m_0^*$;

c) $D_1^*(\theta, m_0)$ is always decreasing as a function of m_0 for $0<m_0\leq 1$.

Proof:

The proof follows from the fact that

$$D_1^*(\theta, m_0) = \frac{\theta}{(1-\theta^2)^{\frac{n-1}{2}}} - \frac{1}{(n-3)(1-\theta^2)^{\frac{n-3}{2}}}$$

$$+ \frac{D_2^*(\theta, m_0)}{\Gamma(\frac{n-1}{2})} ,$$

and Lemma 4.7. Q.E.D.

If we define by $e_1^*(\delta, m_0)$ the relative efficiency of $\bar{y}_s$ with respect to $\bar{y}_\ell$, we have

$$e_1^*(\delta, m_0) = \frac{V(\bar{y}_\ell)}{V(\bar{y}_s)} \tag{4.2.9}$$

then the following two theorems are direct consequences of Theorems 4.6 and 4.7.

Theorem 4.8:

For m_0 fixed such that $0<m_0\leq 1$, there exists a θ_0 such that

$$
\begin{array}{lll}
e_1^*(\delta,\ m_0) & < 1 & -1<\theta<-\theta_0 \\
 & = 1 & \theta=-\theta_0 \\
 & > 1 & -\theta_0<\theta<\theta_0 \\
 & = 1 & \theta=\theta_0 \\
 & < 1 & \theta_0<\theta<1 .
\end{array}
$$

Theorem 4.9:

For δ fixed and contained in $(-1, 1)$, $e_1^*(\delta, m_0)$ and $e_2^*(\delta, m_0)$ vary in the same manner as a function of m_0 for $0<m_0\leq 1$.

C. Comparison of the Sometimes Regression Estimator with the Difference Estimator and the Regression Estimator for Case II with $t_0\leq 0$

In this case the range of δ will be $(-\infty, 0]$ since it is assumed that $\beta_2\leq\beta_0$. For this case we have

$$
V(\bar{y}_s)=V(\bar{y}_d)-\frac{\sigma_2^2(1-\rho^2)}{n\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-3}{2}}}\Big[\sum_{i=0}^{\infty}\frac{(-1)^{i+1}\Gamma(\frac{n+i-2}{2})}{\Gamma(\frac{i+1}{2})}\Big(\frac{\delta}{\sqrt{1+\delta^2}}\Big)^{i+1}
$$

$$
\times I_{m_0}\Big(\frac{n-2}{2},\ \frac{i+2}{2}\Big)-\sum_{i=0}^{\infty}\frac{(-1)^{i}(i+1)\Gamma(\frac{n+i-3}{2})}{4\Gamma(\frac{i+2}{2})}\Big(\frac{\delta}{\sqrt{1+\delta^2}}\Big)^{i}
$$

$$
\times I_{m_0}\Big(\frac{n-2}{2},\ \frac{i+3}{2}\Big)\Big] \tag{4.3.1}
$$

for $-\infty<\delta\leq 0$.

But (4.3.1) is the same as

$$V(\bar{y}_s)=V(\bar{y}_d)-\frac{\sigma_2{}^2(1-\rho^2)}{n\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-3}{2}}}\Big[\sum_{i=0}^{\infty}\frac{\Gamma(\frac{n+i-2}{2})}{\Gamma(\frac{i+1}{2})}\Big(\frac{\delta}{\sqrt{1+\delta^2}}\Big)^{i+1} I_{m_0}\Big(\frac{n-2}{2},\frac{i+2}{2}\Big)$$

$$-\sum_{i=0}^{\infty}\frac{(i+1)\Gamma(\frac{n+i-3}{2})}{4\Gamma(\frac{i+2}{2})}\Big(\frac{\delta}{\sqrt{1+\delta^2}}\Big)^{i} I_{m_0}\Big(\frac{n-2}{2},\frac{i+3}{2}\Big)\Big] \tag{4.3.2}$$

for $0\leq\delta<\infty$.

It will be noted that $V(\bar{y}_s)$ as given in (4.3.2) is exactly the same as $V(\bar{y}_s)$ for Case III given by (4.4.1) when $t_0\geq 0$ and $0\leq\delta<\infty$. Hence the two cases are symmetric in δ. In the next section, we compare $V(\bar{y}_s)$ with $V(\bar{y}_d)$ for Case III. The findings of that investigation will apply to Case II in a symmetric manner. Likewise, the results of the comparison of $V(\bar{y}_s)$ and $V(\bar{y}_\ell)$ in Section E apply to Case II in a symmetric manner.

D. Comparison of the Sometimes Regression Estimator with the Difference Estimator for Case III with $t_0\geq 0$

For this case we have

$$V(\bar{y}_s)=V(\bar{y}_d)-\frac{\sigma_2{}^2(1-\rho^2)}{n\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-3}{2}}}\Big[\sum_{i=0}^{\infty}\frac{\Gamma(\frac{n+i-2}{2})\,\delta^{i+1}}{\Gamma(\frac{i+1}{2})(1+\delta^2)^{\frac{i+1}{2}}} I_{m_0}\Big(\frac{n-2}{2},\frac{i+2}{2}\Big)$$

$$-\sum_{i=0}^{\infty}\frac{(i+1)\Gamma(\frac{n+i-3}{2})\,\delta^{i}}{4\Gamma(\frac{i+2}{2})(1+\delta^2)^{\frac{i}{2}}} I_{m_0}\Big(\frac{n-2}{2},\frac{i+3}{2}\Big)\Big]. \tag{4.4.1}$$

With $\theta = \dfrac{\delta}{\sqrt{1+\delta^2}}$

and let

$$D_2(\theta,\ m_0) = \frac{n\ (\frac{n-1}{2})\,(1+\delta^2)^{\frac{n-3}{2}}}{\sigma_2{}^2(1-\rho^2)}(V(\bar{y}_s)-V(\bar{y}_d)\,. \qquad (4.4.2)$$

Since

$$\frac{n\Gamma(\frac{n-1}{2})\,(1+\delta^2)^{\frac{n-3}{2}}}{\sigma_2{}^2(1-\rho^2)} \geq 0,$$

$D_2(\theta,\ m_0)$ has the same sign as $V(\bar{y}_s)-V(\bar{y}_d)$, and we have

$$D_2(\theta,\ m_0) = \sum_{i=0}^{\infty} \frac{(i+1)\Gamma(\frac{n+i-3}{2})\,\theta^i}{4\Gamma(\frac{i+2}{2})}\, I_{m_0}(\frac{n-2}{2},\ \frac{i+3}{2})$$

$$- \sum_{i=0}^{\infty} \frac{\Gamma(\frac{n+i-2}{2})\,\theta^{i+1}}{\Gamma(\frac{i+1}{2})}\, I_{m_0}(\frac{n-2}{2},\ \frac{i+2}{2})\ . \qquad (4.4.3)$$

Letting j=i-1 in the first summation and j=i in the second summation of (4.4.3) we get

$$D_2(\theta,\ m_0) = \frac{\Gamma(\frac{n-3}{2})\,I_{m_0}(\frac{n-2}{2},\ \frac{3}{2})}{4}$$

$$+ \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+j-2}{2})\,\theta^{j+1}}{\Gamma(\frac{j+1}{2})}\left\{\frac{(j+2)}{2(j+1)} I_{m_0}(\frac{n-2}{2},\ \frac{j+4}{2}) - I_{m_0}(\frac{n-2}{2},\ \frac{j+2}{2})\right\}$$

(4.4.4)

$$= \frac{\Gamma(\frac{n-3}{2}) I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{4} + \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+j-2}{2}) \theta^{j+1}}{\Gamma(\frac{j+1}{2})} I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2}) \{\frac{(j+2)}{2(j+1)}$$

$$- \frac{I_{m_0}(\frac{n-2}{2}, \frac{j+2}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})}\}. \qquad (4.4.5)$$

Letting j=i-1 in the second summation and j=i in the first summation of (4.4.3)

$$D_2(\theta, m_0) = - \frac{\Gamma(\frac{n-2}{2})\theta}{\Gamma(\frac{1}{2})} I_{m_0}(\frac{n-2}{2}, 1)$$

$$+ \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+j-3}{2})\theta^j}{2\Gamma(\frac{j+2}{2})} I_{m_0}(\frac{n-2}{2}, \frac{j+3}{2}) (\frac{j+1}{2} - (n+j-3)\theta^2). \qquad (4.4.6)$$

Consider first the effect of variation in θ. θ will vary only over the interval $[0, 1)$ since it is known that $\beta_2 \geq \beta_0$.

Using (4.4.5), $D_2(\theta, m_0)$ can be expressed in the form

$$D_2(\theta, m_0) = \sum_{j=0}^{\infty} C_j(m_0)\theta^j \qquad (4.4.7)$$

where

$$C_0(m_0) = \frac{\Gamma(\frac{n-3}{2})}{4} I_{m_0}(\frac{n-2}{2}, \frac{3}{2}) \qquad (4.4.8)$$

$$C_{j+1}(m_0) = \frac{\Gamma(\frac{n+j-2}{2}) I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})}{\Gamma(\frac{j+1}{2})} \left\{ \frac{(j+2)}{2(j+1)} - \frac{I_{m_0}(\frac{n-2}{2}, \frac{j+2}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})} \right\} \tag{4.4.9}$$

$j = 0,1,2,3 \ldots .$

Lemma 4.8:

For $0<m_0\leq 1$ $C_0(m_0)$, $C_1(m_0)$, $C_2(m_0)$ ... is a sequence of numbers which starts out positive and becomes negative and stays negative.

Proof:

$\frac{(j+2)}{2(j+1)}$ is a decreasing function of j, and by Lemma 4.4

$$\frac{I_{m_0}(\frac{n-2}{2}, \frac{j+2}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})}$$

is an increasing function of j. Hence

$$\frac{j+2}{2(j+1)} - \frac{I_{m_0}(\frac{n-2}{2}, \frac{j+2}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})}$$

is a decreasing function of j.

$$C_0(m_0) > 0$$

and

$$C_1(m_0) \geq 0.$$

Now suppose $C_j(m_0) \geq 0$ for all j, but then $D_2(\theta, m_0) \geq 0$ for all θ and m_0. But by (4.4.6) with $\varepsilon > 0$, $D_2(\frac{1}{\sqrt{2}} + \varepsilon, m_0) <$ 0 for $0<m_0<1$. Hence there exists a j such that $C_j(m_0) < 0$ and therefore the lemma is proved.

Q.E.D.

Theorem 4.10:

For m_0 fixed such that $0<m_0\leq 1$, there exists a θ_0 where $0<\theta_0<1$ and

$$\begin{aligned} D_2(\theta, m_0) &> 0 \qquad 0\leq\theta<\theta_0 \\ &= 0 \qquad \theta=\theta_0 \\ &< 0 \qquad \theta_0<\theta<1 , \end{aligned}$$

and hence

$$\begin{aligned} V(\overline{y}_s) &> V(\overline{y}_d) \qquad 0\leq\theta<\theta_0 \\ &= V(\overline{y}_d) \qquad \theta=\theta_0 \\ &> V(\overline{y}_d) \qquad \theta_0<\theta<1 . \end{aligned}$$

Proof:

From (4.4.5)

$$D_2(0, m_0) = \frac{\Gamma(\frac{n-3}{2})}{4} I_{m_0}(\frac{n-2}{2}, \frac{3}{2}) > 0.$$

From (4.4.6) if for $\varepsilon > 0$, $\theta = \frac{1}{\sqrt{2}} + \varepsilon < 1$, then since

$$\frac{j+1}{2} - (n+j-3)(\frac{1}{\sqrt{2}} + \varepsilon)^2 < 0$$

for all j, we have

$$D_2\left(\frac{1}{\sqrt{2}} + \varepsilon \quad , m_0\right) < 0.$$

So there exists a θ_0 such that $0<\theta_0<1$ and

$$D_2(\theta_0, m_0) = 0.$$

We shall now show that $D_2(\theta, m_0) < 0$ for $\theta>\theta_0$. By Lemma 4.8 there exists a K such that

$$\begin{aligned} C_j(m_0) &> 0 \quad \text{for } j < K \\ &\leq 0 \quad \text{for } j \geq K. \end{aligned}$$

Hence by the fact that $D_2(\theta_0, m_0) = 0$,

$$\sum_{j=0}^{\infty} C_j(m_0)\theta_0{}^j = 0 \ ,$$

i.e. since $\theta_0>0$

$$\sum_{j=0}^{\infty} C_j(m_0)\theta_0{}^{j-1} = 0.$$

$$\frac{\partial D_2(\theta, m_0)}{\partial\theta} = \sum_{j=1}^{\infty} jC_j(m_0)\theta^{j-1} \ .$$

$$\left.\frac{\partial D_2(\theta, m_0)}{\partial\theta}\right|_{\theta=\theta_0} = \sum_{j=1}^{\infty} jC_j(m_0)\theta_0{}^{j-1} \ ,$$

$$\leq K\sum_{j=1}^{\infty} C_j(m_0)\theta_0{}^{j-1}$$

$$= -\frac{KC_0}{\theta_0} < 0 \ .$$

It can be similarly shown that if

$D_2(\theta^*, m_0) < 0$

then

$$\left.\frac{\partial D_2(\theta, m_0)}{\partial\theta}\right|_{\theta=\theta^*} < 0.$$

Therefore we have that with m_0 fixed, as θ increases, $D_2(\theta, m_0)$ becomes negative in sign and stays negative. Hence the theorem is proved.

Q.E.D.

Now consider the variation of $D_2(\theta, m_0)$ due to m_0. It will be assumed that $t_0 \varepsilon [0, \infty)$ and hence $m_0 \varepsilon (0, 1]$, because if t_0 were allowed to be less than one, then β_0 could be rejected when $\hat{\beta}_2 < \beta_0$.

<u>Lemma 4.9</u>:

If for fixed θ, there exists an $m_0^* \varepsilon (0, 1)$ such that

$$\left.\frac{\partial D_2(\theta, m_0)}{\partial m_0}\right|_{m_0=m_0^*} = 0$$

then

$$\frac{\partial D_2(\theta,\, m_0)}{\partial m_0} \begin{cases} > 0 & 0<m_0<m_0^* \\ = 0 & m_0=m_0^* \\ < 0 & m_0^*<m_0\leq 1. \end{cases}$$

<u>Proof</u>:

Let

$$H_0(\theta) = \frac{-\Gamma(\frac{n-2}{2})}{\Gamma(\frac{1}{2})}\,\theta\,,$$

$$H_j(\theta) = \frac{\Gamma(\frac{n+j-4}{2})\,\theta^{j-1}}{2\Gamma(\frac{j+1}{2})}\left(\frac{j}{2} - (n+j-4)\,\theta^2\right) \qquad j = 1,2,3,\ldots,$$

then from (4.4.6)

$$D_2(\theta,\, m_0) = \sum_{j=0}^{\infty} H_j(\theta) I_{m_0}\left(\frac{n-2}{2},\, \frac{j+2}{2}\right) \tag{4.4.10}$$

and

$$\frac{\partial D_2(\theta,\, m_0)}{\partial m_0} = \sum_{j=0}^{\infty} H_j(\theta) I'_{m_0}\left(\frac{n-2}{2},\, \frac{j+2}{2}\right). \tag{4.4.11}$$

For $\theta^2 \geq \frac{1}{2}$

$$\frac{j}{2} - (n+j-4)\,\theta^4$$

is a nonincreasing function of j. Since $H_0(\theta)$ and $H_1(\theta)$ are nonpositive then $H_j(\theta) < 0$ for all j. Hence for $\theta^2 \geq \frac{1}{2}$ we have

$$\frac{\partial D_2(\theta,\, m_0)}{\partial m_0} \leq 0.$$

For $\theta^2 < \frac{1}{2}$, $\frac{j}{2} - (n+j-4)\theta^2$ is an increasing function of j. Let

$$L = [\frac{2\theta^2(n-4)}{1-2\theta^2}]$$

where [ ] is the greatest integer function. Then

$$\begin{aligned} H_j(\theta) &\leq 0 \quad j \leq L \\ &> 0 \quad j > L . \end{aligned}$$

Suppose that m_{0_2} is such that $0<m_{0_2}<1$ and

$$\left.\frac{\partial D_2(\theta, m_0)}{\partial m_0}\right|_{m_0=m_{0_2}} = 0.$$

Then noting that

$$R_{m_{0_2}}(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2}) = \frac{I'_{m_{0_2}}(\frac{n-2}{2}, \frac{j+2}{2})}{I'_{m_{0_2}}(\frac{n-2}{2}, \frac{L+2}{2})} \quad \text{where } j = 0,1,2 \ldots,$$

we have

$$\left.\frac{\partial D_2(\theta, m_0)}{\partial m_0}\right|_{m_0=m_{0_2}} = I'_{m_{0_2}}(\frac{n-2}{2}, \frac{L+2}{2}) \sum_{j=0}^{\infty} H_j(\theta) R_{m_{0_2}}(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2})$$

$$= 0.$$

Since

$$I'_{m_{0_2}}(\frac{n-2}{2}, \frac{L+2}{2}) > 0$$

then

$$\sum_{j=0}^{\infty} H_j(\theta)\, R_{m_{0_2}}\left(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2}\right) = 0. \tag{4.4.12}$$

Now

$$R_{m_0}\left(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2}\right) = \frac{B\left(\frac{n-2}{2}, \frac{L+2}{2}\right)}{B\left(\frac{n-2}{2}, \frac{j+2}{2}\right)(1-m_0)^{\frac{L-j}{2}}}. \tag{4.4.13}$$

Hence for $\varepsilon>0$

$$\frac{R_{m_0}\left(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2}\right)}{R_{m_0+\varepsilon}\left(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2}\right)} = \frac{(1-m_0-\varepsilon)^{\frac{L-j}{2}}}{(1-m_0)^{\frac{L-j}{2}}}$$

$$= \begin{cases} > 1 & j=0,1,2,\ldots L-1 \\ = 1 & j=L \\ < 1 & j=L+1,\ L+2,\ L+3\ldots\ . \end{cases}$$

Let $m_{0_3}=m_{0_2}+\varepsilon$ for $\varepsilon>0$ such that $m_{0_3}\le 1$, then

$$\left.\frac{\partial D_2(\theta,\ m_0)}{\partial m_0}\right|_{m_0=m_{0_3}} = \sum_{j=0}^{\infty} H_j(\theta)\ I'_{m_{0_3}}\left(\frac{n-2}{2}, \frac{j+2}{2}\right).$$

By (4.4.13) for b_0, b_1, $b_2\ldots$ such that

$$b_j = \frac{R_{m_{0_3}}\left(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2}\right)}{R_{m_{0_2}}\left(\frac{n-2}{2}, \frac{j+2}{2}, \frac{L+2}{2}\right)} \qquad j = 0,1,2\ldots,$$

we have $b_0>b_1>b_2\ \ldots b_{L-1}>b_L = 1>b_{L+1}>b_{L+2}\ldots$ and

$$\left.\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0}\right|_{m_0=m_{0_3}} = I'_{m_{0_3}}\left(\frac{n-2}{2},\ \frac{L+2}{2}\right)\sum_{j=0}^{\infty} H_j(\theta)b_j R_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{j+2}{2},\ \frac{L+2}{2}\right)$$

$$< I'_{m_{0_3}}\left(\frac{n-2}{2},\ \frac{L+2}{2}\right)\sum_{j=0}^{\infty} H_j(\theta) R_{m_{0_2}}\left(\frac{n-2}{2},\ \frac{j+2}{2},\ \frac{L+2}{2}\right)$$

$$= 0\ .$$

Hence we have that if $\frac{\partial D_2^*(\theta,\ m_0)}{\partial m_0}$ is zero at m_{0_2} then it is negative for $m_0 > m_{0_2}$. Hence the lemma is proved.

Q.E.D.

<u>Theorem 4.11</u>:

There exists $\theta_1 > 0$ and $\theta_2 > 0$ defined as

$$D_2(\theta_1,\ 1) = 0,$$

and

$$\theta_2 = \operatorname*{Inf}_{U_2}\ (\theta)$$

where

$$U_2 = \{\theta:\ D_2(\theta,\ m_0) \leq 0 \text{ for all } m_0 \ni 0 < m_0 \leq 1\},$$

such that

a) for θ fixed and $\varepsilon[0,\ \theta_1^*]$

$$D_2(\theta,\ m_0) \geq 0 \qquad 0 < m_0 \leq 1,$$

and hence

$$V(\bar{y}_s) > V(y_d) \qquad 0 < m_0 \leq 1;$$

b) for θ fixed and $\varepsilon(\theta_1, \theta_2)$, then $\exists\ m_0^* \ni 0<m_0^*\leq 1$,

and

$$
\begin{aligned}
D_2(\theta, m_0) &> 0 && 0<m_0<m_0^* \\
&= 0 && m_0=m_0^* \\
&< 0 && m_0^*<m_0\leq 1,
\end{aligned}
$$

and hence

$$
\begin{aligned}
V(y_s) &> V(\bar{y}_d) && 0<m_0<m_0^* \\
&= V(\bar{y}_d) && m_0=m_0^* \\
&> V(\bar{y}_d) && m_0^*<m_0\leq 1;
\end{aligned}
$$

c) for θ fixed and $\varepsilon[\theta_2, 1]$,

$$D_2(\theta, m_0) \leq 0 \qquad 0<m_0\leq 1,$$

and hence

$$V(\bar{y}_s) \leq V(\bar{y}_d) \qquad 0<m_0\leq 1.$$

Proof:

Suppose that for θ fixed $\ni 0<\theta<1$, $\exists\ m_0^* \ni m_0^*\varepsilon(0, 1)$ such that

$$D_2(\theta, m_0^*) = 0 \ .$$

Since

$$
\begin{aligned}
\lim_{m_0\to 0} D_2(\theta, 0) &= \lim_{m_0\to 0} \frac{\partial D_2(\theta, m_0)}{\partial m_0} \\
&= 0,
\end{aligned}
$$

it follows from Lemma 4.9 that if

$$\frac{\partial D_2(\theta,\ m_0)}{\partial m_0} < 0 \text{ in a neighborhood of } m_0=0,$$

then

$$\frac{\partial D_2(\theta,\ m_0)}{\partial m_0} < 0 \qquad 0<m_0\leq 1,$$

and hence there couldn't be a point $m_0^* \ni 0<m_0^*\leq 1$ and

$$D_2(\theta,\ m_0^*) = 0 .$$

Hence in order that $D_2(\theta,\ m_0^*) = 0$ it follows that there must exist an m_0^{**} such that

$$0<m_0^{**}<m_0^*\leq 1$$

and

$$\begin{aligned} \frac{\partial D_2(\theta,\ m_0)}{\partial m_0} &> 0 \qquad 0<m_0<m_0^{**} \\ &= 0 \qquad m_0=m_0^{**} \\ &< 0 \qquad m_0^{**}<m_0\leq 1. \end{aligned}$$

Hence if $D_2(\theta,\ m_0^*) = 0$ then for $m_0>m_0^*$

$$D_2(\theta,\ m_0^*)< 0.$$

By the above if for $\theta=\theta_1$

$$D_2(\theta_1,\ 1) \geq 0$$

then

$$D_2(\theta_1,\ m_0) \geq 0 \qquad 0<m_0\leq 1.$$

If further for $\theta=\theta_2$

$$D_2(\theta_2, 1) < 0$$

then by Theorem 4.10

$$\theta_2 > \theta_1.$$

Hence

$$\theta_1^* = \{\theta: D_2(\theta, 1) = 0\}.$$

If

$$D_2(\theta, 1) < 0$$

then either

$$\theta=\theta_3 \quad \text{and} \quad D_2(\theta_3, m_0) \leq 0 \qquad 0<m_0\leq 1$$

or

$$\theta=\theta_4 \quad \text{and} \ \exists \ m_0^* \ni$$

$$\begin{aligned} D_2(\theta_4, m_0) &> 0 \qquad 0<m_0<m_0^* \\ &= 0 \qquad m_0=m_0^* \\ &< 0 \qquad m_0^*<m_0\leq 1 \ . \end{aligned}$$

Now for

$$m_{0_1}<m_0^*$$

$$D_2(\theta_3, m_{0_1}) \leq 0$$

and

$$D_2(\theta_4, m_{0_1}) \geq 0,$$

then by Theorem 4.1

$$\theta_4 \leq \theta_3.$$

Hence

$$\theta_2^* = \operatorname*{Inf}_{U_2} \theta$$

and the theorem is proved.

Q.E.D.

Now let us look at the efficiency of $\bar{y}_s$ with respect to $\bar{y}_d$. We have

$$V(\bar{y}_d) = \frac{1}{n}[\sigma_2{}^2+\beta_0{}^2\sigma_1{}^2-2\beta_0\sigma_{12}]$$
$$= \frac{\sigma_2{}^2}{n}(1-\rho^2)(1+\delta^2) .$$

Therefore

$$\text{relative efficiency of } \bar{y}_s \text{ with respect to } \bar{y}_d = e_2(\delta, m_0)$$
$$= \frac{V(\bar{y}_d)}{V(\bar{y}_s)}$$
$$= \frac{1}{1+ \dfrac{D_2(\theta, m_0)}{\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}}} . \qquad (4.4.14)$$

Immediately from Theorems 4.10 and 4.11 we have Theorems 4.12 and 4.13.

Theorem 4.12:

For m_0 fixed such that $0<m_0\leq 1$ there exists a θ_0 with $0<\theta_0<1$

$$\begin{aligned} e_2(\delta, m_0) &< 1 & 0\leq\theta<\theta_0 \\ &= 1 & \theta=\theta_0 \\ &> 1 & \theta_0<\theta<1. \end{aligned}$$

Theorem 4.13:

Let θ_1 and θ_2 be as in Theorem 4.10 then

a) for θ fixed and $\varepsilon[0, \theta_1]$,

$$e_2(\delta, m_0) \leq 1 \qquad 0<m_0\leq 1;$$

b) for θ fixed and $\varepsilon(\theta_1, \theta_2)$

$\exists\, m_0^* \ni 0<m_0^*<1$ and

$$\begin{aligned} e_2(\delta, m_0) &< 1 & 0<m_0<m_0^* \\ &= 1 & m_0=m_0^* \\ &> 1 & m_0^*<m_0\leq 1; \end{aligned}$$

c) for θ fixed and $\varepsilon[\theta_2, 1]$,

$$e_2(\delta, m_0) \geq 1 \qquad 0<m_0\leq 1.$$

Theorem 4.14:

For e_0 fixed such that $0<e_0\leq 1$, there exists an m_0^* such that for $m_0\leq m_0^*$

$$e_2(\delta,\ m_0) \geq e_0 \ .$$

Proof:

By Lemma 4.7 for fixed θ or equivalently for fixed δ, $\exists\ m_0(\theta)\ \ni$

$$e_2(\delta,\ m_0) = \cfrac{1}{1+\cfrac{D_2(\theta,\ m_0)}{\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-1}{2}}}}$$

$$\geq e_0 \qquad 0<m_0\leq m_0(\theta).$$

Here $m_0(\theta)$ may be 1.

Pick

$$m_0^* = \operatorname*{Inf}_{0\leq\theta<1} m_0(\theta) \ .$$

Hence

$$e_2(\delta,\ m_0) \geq e_2(m_0^*,\ \delta)$$

$$\geq e_0 \qquad \text{for} \qquad 0<m_0\leq m_0^*$$

and for any $\delta\varepsilon[0,\ \infty)$.

E. Comparison of the Sometimes Regression Estimator with the Regression Estimator for Case III with $t_0 \geq 0$

Now a comparison shall be made between the sometimes regression estimator $\bar{y}_s$ and the regression estimator $\bar{y}_\ell$ in Case III. Consider first the effect of variation in θ. θ will vary over the interval $[0, 1)$.

Noting that

$$V(\bar{y}_d) = V(\bar{y}_\ell) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n}\left(\delta^2 - \frac{1}{n-3}\right) .$$

Then

$$V(\bar{y}_s) = V(\bar{y}_d) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-3}{2}}} D_2(\theta, m_0)$$

$$= V(\bar{y}_\ell) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n}\left(\delta^2 - \frac{1}{n-3}\right) + \frac{\sigma_2^{\,2}(1-\rho^2)}{n\Gamma(\frac{n-1}{2})(1+\delta^2)^{\frac{n-3}{2}}} D_2(\theta, m_0). \tag{4.5.1}$$

Let

$$D_1(\theta, m_0) = \frac{n}{\sigma_2^{\,2}(1-\rho^2)(1-\theta^2)^{\frac{n-3}{2}}}\left(V(\bar{y}_s) - V(\bar{y}_\ell)\right)$$

$$= \frac{\theta^2}{(1-\theta^2)^{\frac{n-1}{2}}} - \frac{1}{(n-3)(1-\theta^2)^{\frac{n-3}{2}}} + \frac{I_{m_0}(\frac{n-2}{2}, \frac{3}{2})}{2(n-3)}$$

$$+ \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+j-2}{2})\,\theta^{j+1}}{\Gamma(\frac{n-1}{2})\Gamma(\frac{j+1}{2})}\left(\frac{j+2}{2(j+1)} I_{m_0}\left(\frac{n-2}{2}, \frac{j+4}{2}\right) - I_{m_0}\left(\frac{n-2}{2}, \frac{j+2}{2}\right)\right). \tag{4.5.2}$$

Then

$$\frac{\partial D_1(\theta,\ m_0)}{\partial\theta} = \frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)^{\frac{n+1}{2}}}$$

$$+\sum_{j=0}^{\infty}\frac{(j+1)\Gamma(\frac{n+j-2}{2})\theta^j}{\Gamma(\frac{n-1}{2})\Gamma(\frac{j+1}{2})} I_{m_0}(\frac{n-2}{2},\ \frac{j+4}{2})(\frac{j+2}{2(j+1)}$$

$$-\frac{I_{m_0}(\frac{n-2}{2},\ \frac{j+2}{2})}{I_{m_0}(\frac{n-2}{2},\ \frac{j+4}{2})}) \ . \tag{4.5.3}$$

<u>Lemma 4.10</u>:

For $n = 1,2,3,\ldots$ and $k = 1,2,3,\ldots$

$$\frac{B(\frac{n}{2},\ \frac{k}{2})}{B(\frac{n}{2},\ \frac{k+1}{2})} \geq \frac{B(\frac{n}{2},\ \frac{k+1}{2})}{B(\frac{n}{2},\ \frac{k+2}{2})}$$

<u>Proof</u>:

$$\frac{\dfrac{B(\frac{n}{2},\ \frac{k}{2})}{B(\frac{n}{2},\ \frac{k+1}{2})}}{\dfrac{B(\frac{n}{2},\ \frac{k+1}{2})}{B(\frac{n}{2},\ \frac{k+2}{2})}} = \frac{\Gamma(\frac{k}{2})\Gamma(\frac{n+k+1}{2})\Gamma(\frac{n+k+1}{2})\Gamma(\frac{k+2}{2})}{\Gamma(\frac{n+k}{2})\Gamma(\frac{k+1}{2})\Gamma(\frac{k+1}{2})\Gamma(\frac{n+k+2}{2})}$$

$$= \frac{[(n+k-1)(n+k-3)(n+k-5)\ldots(k+1)]^2}{(n+k-2)(n+k-4)\ldots(k)(n+k)(n+k-2)\ldots(k+2)}$$

$$> 1.$$

Theorem 4.15:

For m_0 fixed such that $0<m_0<1$, there exists a θ_0 such that $0<\theta_0<1$ and

$$D_1(\theta, m_0) \begin{array}{ll} < 0 & 0\leq\theta<\theta_0 \\ = 0 & \theta=\theta_0 \end{array}$$

and $D_1(\theta, m_0)$ is strictly increasing for

$$0 \leq \theta \leq \frac{B(\frac{n-3}{2}, \frac{3}{2})}{2B(\frac{n-3}{2}, 1)}$$

and hence

$$V(\bar{y}_s) \begin{array}{ll} < V(\bar{y}_\ell) & 0\leq\theta<\theta_0 \\ = V(\bar{y}_\ell) & \theta=\theta_0 \ . \end{array}$$

Proof:

$$\frac{\partial D_1(\theta, 1)}{\partial\theta} = \frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)^{\frac{n+1}{2}}}$$

$$- \sum_{j=0}^{\infty} \frac{(j+1)\Gamma(\frac{n+j-1}{2})\theta^{j+1}}{2\Gamma(\frac{n-1}{2})\Gamma(\frac{j+2}{2})}$$

$$= \frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)^{\frac{n+1}{2}}} - \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+2j-1}{2})\theta^{2j+1}(2j+1)}{2\Gamma(\frac{n-1}{2})\Gamma(j+1)}$$

$$- \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+2j}{2})\theta^{2j+2}(2j+2)}{2\Gamma(\frac{n-1}{2})\Gamma(\frac{2j+3}{2})} \ .$$

But

$$\frac{\Gamma(\frac{n+2j-1}{2})\,\theta^{2j}(1-\theta^2)^{\frac{n-1}{2}}}{\Gamma(j+1)\,\Gamma(\frac{n-1}{2})} \qquad j = 0,1,2, \ldots$$

is the negative binomial density function, and therefore

$$\sum_{j=0}^{\infty} \frac{\Gamma(\frac{n+2j-1}{2})\theta^{2j+1}(1-\theta^2)^{\frac{n-1}{2}}(2j+1)}{\Gamma(j+1)\,\Gamma(\frac{n-1}{2})} = \frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)} .$$

Hence,

$$\frac{\partial D_1(\theta,\ 1)}{\partial\theta} = \sum_{j=0}^{\infty} \frac{\theta^{2j+1}}{n-3}\left[\frac{\Gamma(\frac{n+2j-1}{2})(2j+1)}{\Gamma(j+1)\,\Gamma(\frac{n-3}{2})} - \frac{\Gamma(\frac{n+2j}{2})(2j+2)\theta}{\Gamma(\frac{2j+3}{2})\,\Gamma(\frac{n-3}{2})}\right] . \tag{4.5.4}$$

For

$$\theta(j) < \frac{\Gamma(\frac{n+2j-1}{2})\,\Gamma(\frac{2j+3}{2})\,\Gamma(\frac{n-3}{2})\,(2j+1)}{\Gamma(\frac{n-3}{2})\,\Gamma(j+1)\,\Gamma(\frac{n+2j}{2})\,(2j+2)}$$

$$= \frac{B(\frac{n-3}{2},\ \frac{2j+3}{2})\,(2j+1)}{B(\frac{n-3}{2},\ j+1)\,(2j+2)} \qquad j = 0,1,2\ldots,$$

the jth term is positive.

By Lemma 4.10

$$\theta(j) < \theta(j+1), \qquad j = 0,1,2\ldots .$$

Therefore for

$$0 \le \frac{B(\frac{n-3}{2}, \frac{3}{2})}{2B(\frac{n-3}{2}, 1)},$$

$D_1(\theta, 1)$ is an increasing function of θ.

Since $I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})$ is an increasing function of m_0 and by Lemma 4.6

$$\frac{I_{m_0}(\frac{n-2}{2}, \frac{j+2}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})}$$

is an increasing function of m_0. Hence if

$$I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})\left(\frac{j+2}{2(j+1)} - \frac{I_{m_0}(\frac{n-2}{2}, \frac{j+2}{2})}{I_{m_0}(\frac{n-2}{2}, \frac{j+4}{2})}\right)$$

becomes negative as m_0 increases, then it continues to decrease. But

$$I_1(\frac{n-2}{2}, \frac{j+4}{2})\left(\frac{j+2}{2(j+1)} - \frac{I_1(\frac{n-2}{2}, \frac{j+2}{2})}{I_1(\frac{n-2}{2}, \frac{j+4}{2})}\right) \le 0 \quad j=0,1,2\dots . \tag{4.5.5}$$

Hence since

$$\frac{\partial D_1(\theta, 0)}{\partial\theta} = \frac{\theta(1-\theta^2)+(n-1)\theta^3}{(1-\theta^2)^{\frac{n+1}{2}}},$$

and applying (4.5.5) to (4.5.3), we have

$$\frac{\partial D_1(\theta,\ 1)}{\partial\theta} < \frac{\partial D_1(\theta,\ m_0)}{\partial\theta} \qquad 0<m_0<1.$$

Hence for fixed m_0, $D_1(\theta,\ m_0)$ must continue to increase as θ increases as long as $D_1(\theta,\ 1)$ increases. Hence we have that $D_1(\theta,\ m_0)$ is strictly increasing for

$$\theta \le \frac{B(\frac{n-3}{2},\ \frac{3}{2})}{2B(\frac{n-3}{2},\ 1)}.$$

For m_0 fixed such that $0<m_0\le 1$

$D_1(\theta,\ m_0) < 0.$

Also

$$(1-\theta^2)^{\frac{n-1}{2}} D_1(\theta,\ m_0)\Big|_{\theta=1} = 1$$

and $D_1(\theta,\ m_0)$ converges for θ in a neighborhood of 1. Hence

$D_1(\theta,\ m_0) > 0$

in a neighborhood of $\theta=1$.

Hence the theorem is proved.

Q.E.D.

The theorem does not give any idea concerning the efficiency of $\bar{y}_s$ with respect to $\bar{y}_\ell$ for $\theta>\theta_0$. On the basis of the results obtained so far and the numerical evidence, it is strongly felt that the following result is true.

Conjecture:

For m_0 fixed such that $0<m_0<1$, there exists a θ_0 such that $0<\theta_0<1$ and

$$D_1(\theta, m_0) \begin{cases} < 0 & 0\leq\theta<\theta_0 \\ = 0 & \theta=\theta_0 \\ > 0 & \theta_0<\theta<1, \end{cases}$$

and hence

$$V(\bar{y}_s) \begin{cases} < V(\bar{y}_\ell) & 0\leq\theta<\theta_0 \\ = V(\bar{y}_\ell) & \theta=\theta_0 \\ > V(\bar{y}_\ell) & \theta_0<\theta<1\ . \end{cases}$$

Theorem 4.16:

For θ fixed and $\varepsilon[0, 1)$, $D_1(\theta, m_0)$ and $D_2(\theta, m_0)$ vary in the same manner as a function of m_0 with $0<m_0\leq 1$.

Proof:

This theorem follows by observing that

$$D_1(\theta, m_0) = \frac{\theta^2}{(1-\theta^2)^{\frac{n-1}{2}}} - \frac{1}{(n-3)(1-\theta^2)^{\frac{n-3}{2}}} + \frac{D_2(\theta, m_0)}{\Gamma(\frac{n-1}{2})}\ .$$

Q.E.D.

If $e_1(\delta, m_0)$, the relative efficiency of the sometimes regression estimator with respect to the regression estimator, is defined as

$$e_1(\delta,\ m_0) = \frac{V(\overline{y}_\ell)}{V(\overline{y}_s)}, \qquad (4.5.6)$$

then we have the following results concerning the efficiency of $\overline{y}_s$ with respect to $\overline{y}_\ell$.

Theorem 4.17:

For m_0 fixed such that $0<m_0<1$, there exists a θ_0 such that $0<\theta_0<1$ and

$$\begin{aligned} e_1(\delta,\ m_0) &> 1 \qquad 0<\theta<\theta_0 \\ &= 1 \qquad \theta=\theta_0 \ . \end{aligned}$$

Theorem 4.18:

For θ fixed and $\varepsilon[0,\ 1)$, $e_1(\delta,\ m_0)$ and $e_2(\delta,\ m_0)$ vary in the same manner as a function of m_0 for $0<m_0\leq 1$.

V. STRATIFIED REGRESSION TYPE ESTIMATOR $\bar{y}_{w_s}$ BASED ON A PRELIMINARY TEST OF SIGNIFICANCE

In this section we shall extend some of the results obtained so far and develop a sometimes regression estimator appropriate to the case of stratified sampling.

Consider a population classified into k strata, the ith stratum having a proportion W_i of the units in the population so that $\sum_{i=1}^{k} W_i = 1$. Let Y be the characteristic under study and consider the problem of estimating the population mean

$$\bar{Y} = \sum_{i=1}^{k} W_i \bar{Y}_i$$

based on a stratified random sample of size $\sum_{i=1}^{k} n_i$, where n_i units are drawn at random from the ith stratum, i = 1,2,3...k. An unbiased estimate of $\bar{Y}$ is

$$\bar{y}_w = \sum_{i=1}^{k} W_i \bar{y}_i, \tag{5.1}$$

where $\bar{y}_i$ is the sample mean based on n_i units drawn at random in the ith stratum. The sample mean $\bar{y}_i$ in the ith stratum is an unbiased estimate of the ith stratum mean $\bar{Y}_i$. Let X be the auxiliary characteristic on which information is available for all units in the population. If X and Y are correlated and the relationship between them is linear, within each stratum, regression type estimators may be used to estimate the population mean $\bar{Y}$. A commonly used estimator is the difference estimator defined as

$$\bar{y}_{w_d} = \sum_{i=1}^{k} W_i \bar{y}_{d_i}\,, \tag{5.2}$$

where

$$\bar{y}_{d_i} = \bar{y}_i + \beta_{0_i}(\bar{X}_i - \bar{x}_i)\,,$$

β_{0_i} is some fixed constant that is assumed to be known, and $\bar{X}_i$ and $\bar{x}_i$ are population and sample means respectively in the ith stratum. This is an unbiased estimator and its variance is given by

$$V(\bar{y}_{w_d}) = \sum_{i=1}^{k} \frac{W_i^{\,2}}{n_i} \frac{N_i - n_i}{N_i - 1}[\sigma_{2_i}^2 + \sigma_{1_i}^2 \beta_{0_i}^2 - 2\beta_{0_i}\sigma_{12_i}]$$

$$\doteq \sum_{i=1}^{k} \frac{W_i^{\,2}}{n_i}[\sigma_{2_i}^2 + \beta_{0_i}^2 \sigma_{1_i}^2 - 2\beta_{0_i}\sigma_{12_i}]$$

where $\sigma_{2_i}^2$, $\sigma_{1_i}^2$ and σ_{12_i} are the variance of Y, the variance of X and the covariance of Y and X respectively in the ith stratum. When β_{0_i} is in fact the regression coefficient of Y on X in the ith stratum, $i = 1,2,\ldots,k$, the estimator $\bar{y}_{w_d}$ is the minimum variance unbiased estimator of $\bar{Y}$.

When β_{2_i}, the regression coefficient of Y on X in the ith stratum is not known, it is customary to estimate it from the sample with a consistent estimator of β_{2_i} given by

$$\hat{\beta}_{2_i} = \frac{s_{12_i}}{s^2_{1_i}} \qquad i = 1,2\ldots,k,$$

where

$$s_{12_i} = \frac{1}{n_i-1} \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)(y_{ij}-\bar{y}_i),$$

and

$$s^2_{1_i} = \frac{1}{n_i-1} \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)^2 \ .$$

This estimator of $\bar{Y}$, so obtained is known as the regression estimator and is given by

$$\bar{y}_{w_\ell} = \sum_{i=1}^{k} W_i \bar{y}_{\ell_i}, \tag{5.3}$$

where

$$\bar{y}_{\ell_i} = \bar{y}_i + \hat{\beta}_{2_i}(\bar{X}_i-\bar{x}_i).$$

In general this estimator will be biased and the bias is given by

$$\text{Bias}(\bar{y}_{w_\ell}) = -\sum_{i=1}^{k} W_i \ \text{Cov}(\hat{\beta}_{2_i}, \bar{x}_i) \ .$$

The variance of this estimator to the first order of approximation is

$$V(\bar{y}_{w_\ell}) = \sum_{i=1}^{k} w_i^{\ 2} \left(\frac{N_i - n_i}{N_i - 1}\right) \left(\frac{\sigma_{2_i}^2 (1-\rho_i^2)}{n_i}\right)$$

$$\doteq \sum_{i=1}^{k} \frac{W_i^{\ 2}}{n_i} \sigma_{2_i}^2 (1-\rho_i^2) ,$$

where

$$\rho_i = \frac{\sigma_{12_i}}{\sigma_{1_i} \sigma_{2_i}}$$

is the correlation coefficient between Y and X in the ith stratum.

It would appear that if β_{0_i} is an intelligent guess for β_{2_i} then the difference estimator $\bar{y}_{d_i}$ would be more appropriate. On the other hand if there is little or no knowledge of β_{2_i} then regression estimator $\bar{y}_{\ell_i}$ would have an advantage. We therefore propose an estimator based on a preliminary test of significance of the relative closeness of β_{0_i} to β_{2_i}.

This estimator to be called "sometimes regression estimator" may be defined as

$$\bar{y}_{w_s} = \sum_{i=1}^{k} W_i \bar{y}_{s_i} , \tag{5.4}$$

where,

$$\bar{y}_{s_i} = \bar{y}_{d_i} \quad \text{if} \quad t_i \varepsilon A$$
$$= \bar{y}_{\ell_i} \quad \text{if} \quad t_i \varepsilon A^c$$

is an estimate of the ith stratum mean, and

$$t_i = \frac{\sqrt{n_i - 2}\ (\hat{\beta}_{2_i} - \beta_{0_i}) s_{1_i}}{(s_{2_i}^2 - \beta_{2_i}^2 s_{1_i}^2)^{1/2}} .$$

Again the acceptance and rejection regions will be dependent on the a priori information available concerning the possible values of β_{2_i}. In order to evaluate fully the expected value and mean square error of the estimator $\bar{y}_{w_s}$, it is necessary to define exactly the acceptance and rejection regions and make suitable assumptions about the joint distributions of Y and X. It will be assumed as before that (Y, X) has a bivariate normal distribution within each stratum. Under the assumption of bivariate normality and since sampling is carried out independently in each stratum, the tests t_i are independent. Now the expectation and mean square error can be determined.

$$E(\bar{y}_{w_s}) = \sum_{i=1}^{k} W_i E(\bar{y}_{s_i}) .$$

By the work of Chapter III

$$E(\bar{y}_{w_s}) = W_i \bar{Y}_i .$$

Hence

$$E(\bar{y}_{w_s}) = \bar{Y} .$$

Further,

$$V(\bar{y}_{w_s}) = \sum_{i=1}^{k} W_i^2 \, V(\bar{y}_{s_i}) .$$

Now the expression for $V(\bar{y}_{w_s})$ is dependent on the case into which $\bar{y}_{s_i}$ falls. There are three cases given as

<u>Case I</u>:

$$\begin{aligned} \bar{y}_{s_i} &= \bar{y}_{d_i} \quad \text{if} \quad |t_i| \leq t_{0_i} \\ &= \bar{y}_{\ell_i} \quad \text{if} \quad |t_i| > t_{0_i} ; \end{aligned}$$

<u>Case II</u>:

$$\begin{aligned} \bar{y}_{s_i} &= \bar{y}_{d_i} \quad \text{if} \quad t_i \geq t_{0_i} \\ &= \bar{y}_{\ell_i} \quad \text{if} \quad t_i < t_{0_i} ; \end{aligned}$$

<u>Case III</u>:

$$\begin{aligned} \bar{y}_{s_i} &= \bar{y}_{d_i} \quad \text{if} \quad t_i \leq t_{0_i} \\ &= \bar{y}_{\ell_i} \quad \text{if} \quad t_i > t_{0_i} . \end{aligned}$$

In general, depending upon the <u>a priori</u> information available concerning the range of values of β_{2_i}, the estimator $\bar{y}_{s_i}$ of $\bar{Y}_i$ for the ith stratum (i = 1,2,...k) may belong to any one of the three cases mentioned above.

It is possible that $\bar{y}_{w_s}$ is such that $\bar{y}_{s_i}$ is of Case I for k_1 of the strata, is of Case II for k_2 of the strata and is of Case III for the remaining strata. We shall consider the problem when no a priori information is available concerning the range of values of β_{2_i} so that $\bar{y}_{s_i}$ is necessarily of Case I. The estimator $\bar{y}_{w_s}$ now takes the form

$$\bar{y}_{w_s} = \sum_{i=1}^{k} W_i \, \bar{y}_{s_i} \, , \tag{5.5}$$

where

$$\begin{aligned} \bar{y}_{s_i} &= \bar{y}_{d_i} \quad \text{if} \quad |t_i| \leq t_{0_i} \\ & \bar{y}_{\ell_i} \quad \text{if} \quad |t_i| > t_{0_i} \, . \end{aligned}$$

The following theorem gives the result regarding the expected value of the general estimator $\bar{y}_{w_s}$ and its variance. The variance result follows by using the result of Theorem 3.2. The corresponding results when $\bar{y}_{s_i}$ for the ith stratum is not necessarily of Case I can be obtained in a similar manner.

Theorem 5.1:

The sometimes regression estimator $\bar{y}_{w_s}$ is an unbiased estimator of the population mean, i.e.

$$E(\bar{y}_{w_s}) = \bar{Y} \, .$$

The variance of $\bar{y}_{w_s}$, when $\bar{y}_{s_i}$ $(i = 1,2,\ldots k)$ are of Case I is given by

$$V(\bar{y}_{w_s}) = \sum_{j=1}^{k} W_j^{\,2}\Big[V(\bar{y}_{d_j}) - \frac{\sigma_{2_j}^2(1-\rho_j^{\,2})}{n_j} \sum_{i=0}^{\infty} \frac{2\Gamma\left(\frac{n_j+2i-1}{2}\right)\delta_j^{\,2i+2}}{\Gamma(i+1)\Gamma\left(\frac{n_j-1}{2}\right)(1+\delta_j^{\,2})^{\frac{n_j+2i-1}{2}}}$$

$$\times I_{m_{0_j}}\left(\frac{n_j-2}{2}, \frac{2i+3}{2}\right)$$

$$+ \frac{\sigma_{2_j}^2(1-\rho_j^{\,2})}{n_j} \sum_{i=0}^{\infty} \frac{(2i+1)\Gamma\left(\frac{n_j+2i-3}{2}\right)\delta_j^{\,2i}}{2\Gamma(i+1)\Gamma\left(\frac{n_j-1}{2}\right)(1+\delta_j^{\,2})^{\frac{n_j+2i-3}{2}}}$$

$$\times I_{m_{0_j}}\left(\frac{n_j-2}{2}, \frac{2i+3}{2}\right)\Big],$$

where

$$\delta_j = \frac{\rho_j - \frac{\beta_{0_j}\sigma_{1_j}}{\sigma_{2_j}}}{(1-\rho_j^{\,2})^{1/2}},$$

$$m_{0_j} = \frac{1}{1 + \frac{t_{0_j}^2}{n_j-2}},$$

$\sigma_{2_j}^2$ is the variance of Y in the jth stratum, and ρ_j is the correlation coefficient between X and Y in the jth stratum.

Now we make a comparison of $V(\bar{y}_{w_s})$ with $V(\bar{y}_{w_d})$.

Define

$$D^*_{2w}(\tilde{\theta}_s, \tilde{m}_{0_s}) = V(\bar{y}_{w_s}) - V(\bar{y}_{w_d})$$

$$= \sum_{j=1}^{k} \frac{W_j^2 \sigma_{2_j}^2 (1-\rho_j^2)}{n_j (1+\delta_j^2)^{\frac{n_j-3}{2}} \Gamma(\frac{n_j-1}{2})} D^*_{2j}(\theta^{(j)}, m_{0_j}), \qquad (5.6)$$

where

$$D^*_{2j}(\theta^{(j)}, m_{0_j}) = \frac{n_j \Gamma(\frac{n_j-1}{2})(1+\delta_j^2)^{\frac{n_j-3}{2}}}{\sigma_{2_j}^2 (1-\rho_j^2)} (V(\bar{y}_{s_j}) - V(\bar{y}_{d_j})),$$

$$\tilde{\theta}_s = (\theta^{(1)}, \theta^{(2)}, \ldots \theta^{(k)}),$$

$$\theta^{(j)} = \frac{\delta_j}{\sqrt{1+\delta_j^2}} \qquad j = 1,2,\ldots,k,$$

and

$$\tilde{m}_{0_s} = (m_{0_1}, m_{0_2}, \ldots, m_{0_k}) .$$

Define

$$\tilde{\theta}_{s_0} \le \tilde{\theta}_s,$$

if

$$\tilde{\theta}_{s_0} = (\theta_0^{(1)}, \theta_0^{(2)}, \ldots \theta_0^{(k)}),$$

$$\tilde{\theta}_s = (\theta^{(1)}, \theta^{(2)}, \ldots \theta^{(k)}),$$

and

$$\theta_0^{(i)} \leq \theta^{(i)} \qquad i = 1,2,\ldots k \quad .$$

Let

$$\tilde{0} = (0, 0, 0, \ldots 0)$$

and

$$\tilde{1} = (1, 1, 1, \ldots 1) \; .$$

We have

<u>Theorem 5.2</u>:

For $\tilde{m}_{0_s}$ fixed such that $\tilde{0}<\tilde{m}_{0_s}\leq 1$, there exists $\tilde{\theta}_{s_0}$ where $\tilde{0}<\tilde{\theta}_{s_0}<\tilde{1}, \ni D^*_{2j}(\theta_0^{(j)}, m_{0_j}) = 0$ $(j = 1,2,\ldots k)$ and

$$
\begin{aligned}
D^*_{2w}(\tilde{\theta}_s, \tilde{m}_{0_s}) &< 0 & -\tilde{1}<\tilde{\theta}_s<-\tilde{\theta}_{s_0} \\
&= 0 & \tilde{\theta}_s=-\tilde{\theta}_{s_0} \\
&> 0 & -\tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{\theta}_{s_0} \\
&= 0 & \tilde{\theta}_s=\tilde{\theta}_{s_0} \\
&< 0 & \tilde{\theta}_{s_0}<\tilde{\theta}_s<1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
V(\bar{y}_{w_s}) &< V(\bar{y}_{w_d}) && -1<\tilde{\theta}_s<-\tilde{\theta}_{s_0} \\
&= V(\bar{y}_{w_d}) && \tilde{\theta}_s=-\tilde{\theta}_{s_0} \\
&> V(\bar{y}_{w_d}) && -\tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{\theta}_{s_0} \\
&= V(\bar{y}_{w_d}) && \tilde{\theta}_s=\tilde{\theta}_{s_0} \\
&< V(\bar{y}_{w_d}) && \tilde{\theta}_{s_0}<\tilde{\theta}_s\leq 1,
\end{aligned}
$$

where

$$-\tilde{\theta}_s = (-\theta^{(1)}, -\theta^{(2)}, \ldots -\theta^{(k)}).$$

Proof:

Using Theorem 4.1 we know that for fixed m_{0_j} there exists a $\theta_0{}^{(j)}$ where $0<\theta_0{}^{(j)}<1$ and such that

$$
\begin{aligned}
D^*_{2j}(\theta^{(j)}, m_{0_j}) &< 0 && -1<\theta^{(j)}<-\theta_0{}^{(j)} \\
&= 0 && \theta^{(j)}=-\theta_0{}^{(j)} \\
&> 0 && -\theta_0{}^{(j)}<\theta^{(j)}<\theta_0{}^{(j)} \\
&= 0 && \theta^{(j)}=\theta_0{}^{(j)} \\
&< 0 && \theta_0{}^{(j)}<\theta^{(j)}<1.
\end{aligned}
$$

Applying this result to (5.6) the theorem is proved.

Q.E.D.

The following theorem gives the result regarding the effect of variation of $\tilde{m}_{0_s}$ with $\tilde{\theta}_s$ fixed.

Theorem 5.3:

There exists $\tilde{\theta}_1^*>0$ and $\tilde{\theta}_2^*>0$, defined as

$$\tilde{\theta}_1^* = (\theta_1^{(1)}, \theta_1^{(2)}, \theta_1^{(3)}, \ldots \theta_1^{(k)}),$$

where

$$D_{2j}^*(\theta_1^{(j)}, 1) = 0, \; j = 1,2,3 \ldots k, \text{ and}$$

$$\tilde{\theta}_2 = (\theta_2^{(1)}, \theta_2^{(2)}, \theta_2^{(3)} \ldots \theta_2^{(k)}),$$

where

$$\theta_2^{(j)} = \operatorname*{Inf}_{U_{2_j}} \theta^{(j)}, \qquad j = 1,2,\ldots k$$

with

$$U_{2_j} = \{\theta^{(j)}: \theta^{(j)} \geq 0 \text{ and } D_{2j}^*(\theta^{(j)}, m_{0_j}) \leq 0 \text{ for}$$

$$\text{all } m_{0_j} \ni 0<m_{0_j} \leq 1\} \quad j = 1,2,\ldots,k,$$

such that

a) for $\tilde{\theta}_s$ fixed and $\varepsilon[-\tilde{\theta}_1^*, \tilde{\theta}_1^*]$

$$D_{2w}^*(\tilde{\theta}_s, \tilde{m}_{0_s}) \geq 0 \quad \text{for} \quad \tilde{0}<\tilde{m}_{0_s} \leq 1,$$

and hence

$$V(\bar{y}_{w_s}) \geq V(\bar{y}_{w_d}) \quad \text{for} \quad \tilde{\delta} < \tilde{m}_{0_s} \leq 1;$$

b) for $\tilde{\delta}_s$ fixed and $\varepsilon\{(-\tilde{\delta}_2^*, -\tilde{\delta}_1^*) \cup (\tilde{\delta}_1^*, \tilde{\delta}_2^*)\}$

$$\exists\ \tilde{m}_0^* = (m_{0_1}^*, \ldots m_{0_k}^*) \ni \tilde{\delta} < \tilde{m}_0^* < 1, \text{ and}$$

$$\begin{aligned} D_{2w}^*(\tilde{\delta}_s, \tilde{m}_{0_s}) &\geq 0 \qquad \tilde{\delta} < \tilde{m}_{0_s} \leq \tilde{m}_0^* \\ &< 0 \qquad \tilde{m}_0^* < \tilde{m}_{0_s} \leq 1, \end{aligned}$$

and hence

$$\begin{aligned} V(\bar{y}_{w_s}) &\geq V(\bar{y}_{w_d}) \qquad \tilde{\delta} < \tilde{m}_{0_s} \leq \tilde{m}_0^* \\ &< V(\bar{y}_{w_d}) \qquad \tilde{m}_0^* < \tilde{m}_{0_s} \leq 1; \end{aligned}$$

c) for $\tilde{\delta}_s$ fixed and $\varepsilon\{(-\tilde{1}, -\tilde{\delta}_2^*] \cup [\tilde{\delta}_2^*, \tilde{1})\}$,

$$D_{2w}^*(\tilde{\delta}_s, \tilde{m}_{0_s}) \leq 0 \qquad \tilde{\delta} < \tilde{m}_{0_s} \leq 1,$$

and hence

$$V(\bar{y}_{w_s}) \leq V(\bar{y}_{w_d}) \qquad \tilde{\delta} < \tilde{m}_{0_s} \leq 1 .$$

If we define

Relative Efficiency of $\bar{y}_{w_s}$ with respect to $\bar{y}_{w_d}$

$$= e_{2_s}^*(\tilde{\delta}_s, \tilde{m}_{0_s}) = \frac{V(\bar{y}_{w_d})}{V(\bar{y}_{w_s})}$$

then the following two theorems are a direct consequence of Theorems 5.2 and 5.3.

<u>Theorem 5.4</u>:

For $\tilde{m}_{0_s}$ fixed such that $\tilde{0}<\tilde{m}_{0_s}\leq\tilde{1}$, there exists $\tilde{\theta}_{s_0}$ where $\tilde{0}<\tilde{\theta}_{s_0}<\tilde{1}, \ni D_{2j}(\theta_0{}^{(j)}, m_{0_j}) = 0$, $(j = 1,2\ldots k)$ and

$$
\begin{aligned}
e^*_{2s}(\tilde{\delta}_s, \tilde{m}_{0_s}) &> 1 \qquad -\tilde{1}<\tilde{\theta}_s<-\tilde{\theta}_{s_0} \\
&= 1 \qquad \tilde{\theta}_s=-\tilde{\theta}_{s_0} \\
&< 1 \qquad -\tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{\theta}_{s_0} \\
&= 1 \qquad \tilde{\theta}_s=\tilde{\theta}_{s_0} \\
&> 1 \qquad \tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{1}\ .
\end{aligned}
$$

<u>Theorem 5.5</u>:

For $\tilde{\theta}^*_1$ and $\tilde{\theta}^*_2$, defined as in Theorem 5.3

a) for $\tilde{\theta}_s$ fixed and $\varepsilon[-\tilde{\theta}^*_1, \tilde{\theta}^*_1]$

$e^*_{2s}(\tilde{\delta}_s, \tilde{m}_{0_s}) \leq 1$ for $\tilde{0}<\tilde{m}_{0_s}\leq\tilde{1}$;

b) for $\tilde{\theta}_s$ fixed and $\varepsilon\{(-\tilde{\theta}^*_2, \tilde{\theta}^*_1)\cup(\tilde{\theta}^*_1, \tilde{\theta}^*_2)\}$

$\exists\ \tilde{m}^*_0=(m^*_{0_1}, m^*_{0_2},\ldots m^*_{0_k}) \ni \tilde{0}<\tilde{m}^*_0<\tilde{1}$, and

$$
\begin{aligned}
e^*_{2s}(\delta_s, \tilde{m}_{0_s}) &\leq 1 \quad \text{for } \tilde{0}<\tilde{m}_{0_s}\leq\tilde{m}^*_0 \\
&> 1 \quad \text{for } \tilde{m}^*_0<\tilde{m}_{0_s}\leq\tilde{1};
\end{aligned}
$$

c) for $\tilde{\theta}_s$ fixed and $\varepsilon\{(-\tilde{1}, -\tilde{\theta}_s^*]\cup[\tilde{\theta}_s^*, \tilde{1})\}$,

$e_{2s}^*(\tilde{\theta}_s, \tilde{m}_{0_s}) \geq 1$ for $\tilde{0}<\tilde{m}_{0_s}\leq\tilde{1}$.

Next we look at a comparison of the variance of the sometimes regression estimator $\bar{y}_{w_s}$ and the regression estimator $\bar{y}_{w_\ell}$.

Let

$$D_{1s}^*(\tilde{\theta}_s, \tilde{m}_{0_s}) = V(\bar{y}_{w_s})-V(\bar{y}_{w_\ell})$$

$$= \sum_{j=1}^{k} \frac{W_j^2\sigma_{2_j}^2(1-\rho_j^2)(1-\theta_j^2)^{\frac{n_j-3}{2}}}{n_j}[D_{1j}^*(\theta^{(j)}, m_{0_j})], \quad (5.7)$$

where

$$D_{1j}^*(\theta^{(j)}, m_{0_j})= \frac{n_j}{\sigma_{2_j}^2(1-\rho_j^2)(1-\theta_j^2)^{\frac{n_j-3}{2}}}[V(\bar{y}_{s_j})-V(\bar{y}_{\ell_j})].$$

We then have the following theorem regarding the effect of variation of $\tilde{\theta}_s$ with $\tilde{m}_{0_s}$ fixed.

<u>Theorem 5.6:</u>

For $\tilde{m}_{0_s}$ fixed such that $\tilde{0}<\tilde{m}_{0_s}\leq\tilde{1}$, there exists a $\tilde{\theta}_{s_0}$ where $\tilde{0}<\tilde{\theta}_{s_0}<\tilde{1}, \ni D_{1j}^*(\theta_0^{(j)}, m_{0_j}) = 0$, $(j = 1,2,3...k)$ and

$$
\begin{aligned}
D^*_{1s}(\tilde{\theta}_s, \tilde{m}_{0_s}) &> 0 && -\tilde{1}<\tilde{\theta}_s<-\tilde{\theta}_{s_0} \\
&= 0 && \tilde{\theta}_s=-\tilde{\theta}_{s_0} \\
&< 0 && -\tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{\theta}_{s_0} \\
&= 0 && \tilde{\theta}_s=\tilde{\theta}_{s_0} \\
&> 0 && \tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{1}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
V(\bar{y}_{w_s}) &> V(\bar{y}_{w_\ell}) && -\tilde{1}<\tilde{\theta}_s<-\tilde{\theta}_{s_0} \\
&= V(\bar{y}_{w_\ell}) && \tilde{\theta}_s=-\tilde{\theta}_{s_0} \\
&< V(\bar{y}_{w_\ell}) && -\tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{\theta}_{s_0} \\
&= V(\bar{y}_{w_\ell}) && \tilde{\theta}_s=\tilde{\theta}_{s_0} \\
&> V(\bar{y}_{w_\ell}) && \tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{1}\ .
\end{aligned}
$$

<u>Proof</u>:

The theorem is proved by applying the results of Theorem 4.6 to $D^*_{1j}(\theta^{(j)}, m_{0_j})$ for $j = 1,2, \ldots k$.

Q.E.D.

The following theorem which is a direct consequence of Theorem 4.7 is concerned with the effect of varying $\tilde{m}_{0_s}$ while holding $\tilde{\theta}_s$ fixed.

Theorem 5.7:

For $\tilde{\theta}_s$ fixed and $\varepsilon(-1, 1)$, $D^*_{1_s}(\tilde{\theta}_s, \tilde{m}_{0_s})$ will vary as a function of $\tilde{m}_{0_s}$ in the same manner as $D^*_{2s}(\theta, \tilde{m}_{0_s})$ for $\gamma<\tilde{m}_{0_s}\leq\tilde{1}$.

Let us now consider the efficiency of $\bar{y}_{w_s}$ with respect to $\bar{y}_{w_\ell}$. Then we have

$$\text{Relative efficiency of } \bar{y}_{w_s} \text{ with respect to } \bar{y}_{w_\ell} = e^*_{1s}(\tilde{\theta}_s, \tilde{m}_{0_s})$$
$$= \frac{V(\bar{y}_{w_\ell})}{V(\bar{y}_{w_s})},$$

then we immediately have the following two theorems from Theorems 5.6 and 5.7.

Theorem 5.8:

For $\tilde{m}_{0_s}$ fixed such that $\gamma<\tilde{m}_{0_s}\leq\tilde{1}$, there exists $\tilde{\theta}_{s_0}$ where $\gamma<\tilde{\theta}_{s_0}<\tilde{1}$, and

$$
\begin{aligned}
e^*_{1s}(\tilde{\theta}_s, \tilde{m}_{0_s}) &< 1 \qquad -\tilde{1}<\tilde{\theta}_s<-\tilde{\theta}_{s_0} \\
&= 1 \qquad \tilde{\theta}_s= -\tilde{\theta}_{s_0} \\
&> 1 \qquad -\tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{\theta}_{s_0} \\
&= 1 \qquad \tilde{\theta}_s=\tilde{\theta}_{s_0} \\
&< 1 \qquad \tilde{\theta}_{s_0}<\tilde{\theta}_s<\tilde{1}.
\end{aligned}
$$

<u>Theorem 5.9</u>:

For $\tilde{\gamma}_s$ fixed and $\varepsilon(-\tilde{1}, \tilde{1})$, $e^*_{1s}(\tilde{\gamma}_s, \tilde{m}_{0_s})$ will vary as a function of $\tilde{m}_{0_s}$ in the same manner as $\tilde{e}_{2s}(\tilde{\gamma}_s, \tilde{m}_{0_s})$ for $\gamma < \tilde{m}_{0_s} \leq \tilde{1}$.

VI. CONCLUSIONS AND RECOMMENDATIONS REGARDING THE USE OF THE SOMETIMES REGRESSION ESTIMATOR

If conditions are such that the use of regression type estimators is warranted, the question arises as to when the sometimes regression estimator would be most appropriate. Actually, the sometimes regression estimator includes both the difference estimator $\bar{y}_d$ and the regression estimator $\bar{y}_\ell$ as special cases. Hence the sometimes regression estimator may be used whenever it is appropriate to use regression type estimators.

Consider the effect of change in the relative closeness of β_0 to β_2. Theorems 4.1 and 4.10 for Cases I and III respectively give the result that for fixed m_0, i.e. fixed level of significance, $V(\bar{y}_s)$ is greater than $V(\bar{y}_d)$ for β_0 close to β_2, but this relationship reverses itself as the distance of β_2 from β_0 increases and it remains reversed. Theorems 4.6 and 4.15 for Cases I and III respectively illustrate that the situation is reversed for the relationship of the variance of the sometimes regression estimator to the variance of the regression estimator with the exception that Theorem 4.15 does not give a result for larger values of δ. Analogous results hold for the relative efficiencies. These results are illustrated in Figures 6.1, 6.3, 6.5 and 6.7 for Case III with n equal to 6 and 12 respectively. Figures 6.3

Figure 6.1. Graphs of $e_2(\delta, m_0)$ vs δ for fixed levels of m_0 with n=6 for Case III

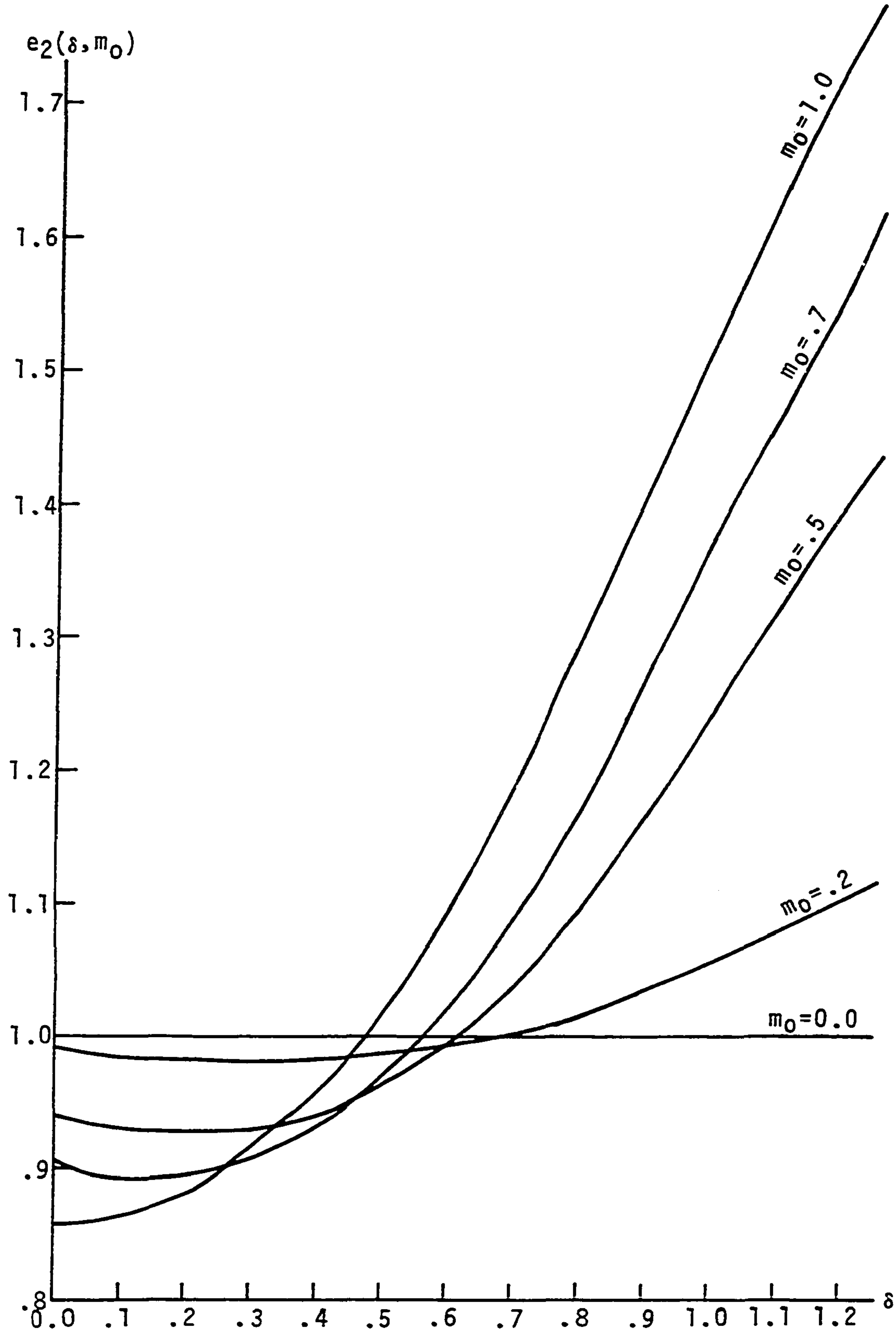
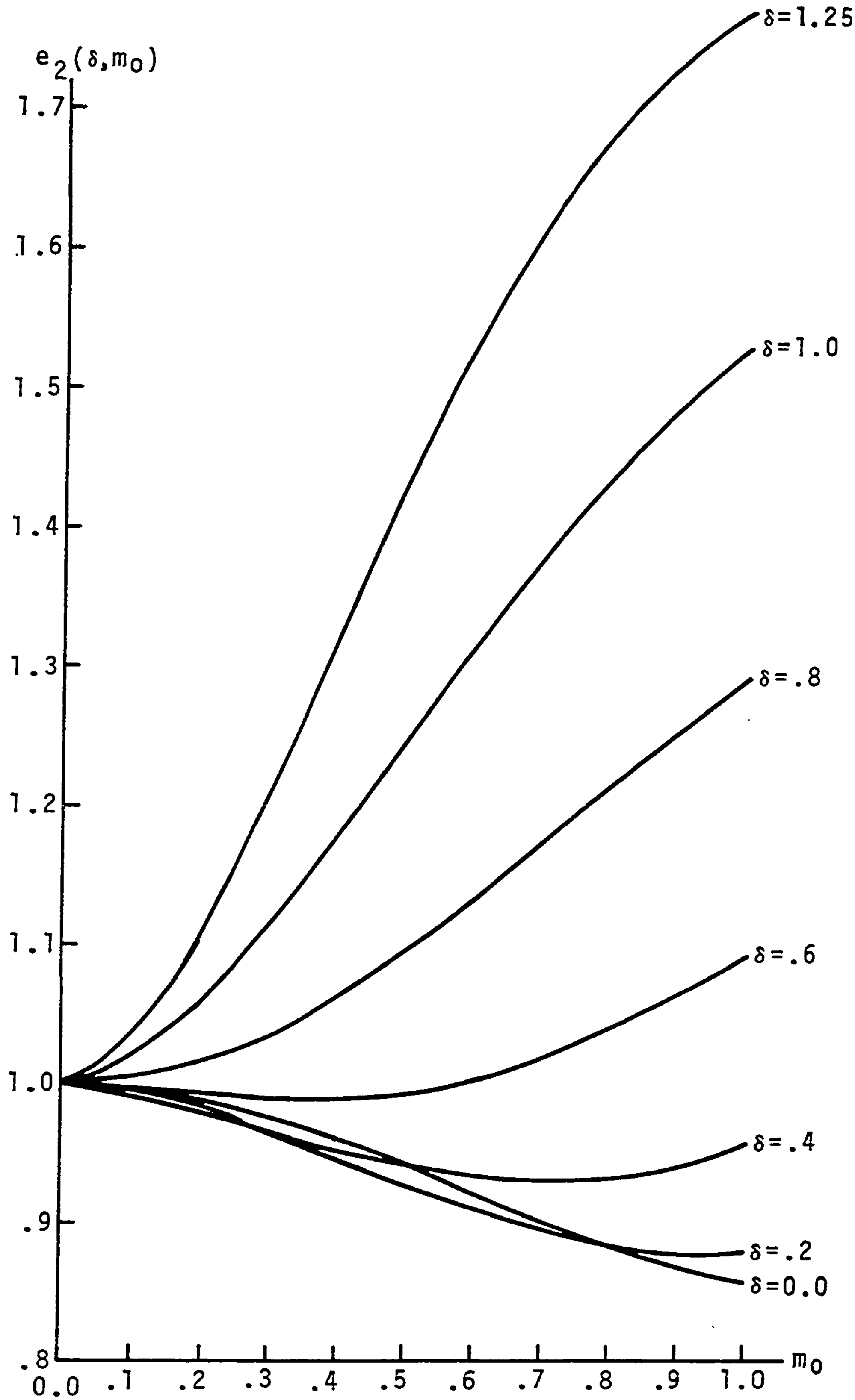

$e_2(\delta, m_o)$
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
δ
m_o=1.0
m_o=.7
m_o=.5
m_o=.2
m_o=0.0

Figure 6.2. Graphs of $e_2(\delta, m_0)$ vs m_0 for fixed levels of δ with n=6 for Case III



$e_2(\delta, m_0)$
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
m_0
δ=1.25
δ=1.0
δ=.8
δ=.6
δ=.4
δ=.2
δ=0.0

Figure 6.3. Graphs of $e_1(\delta, m_0)$ vs δ for fixed levels of m_0 with n=6 for Case III

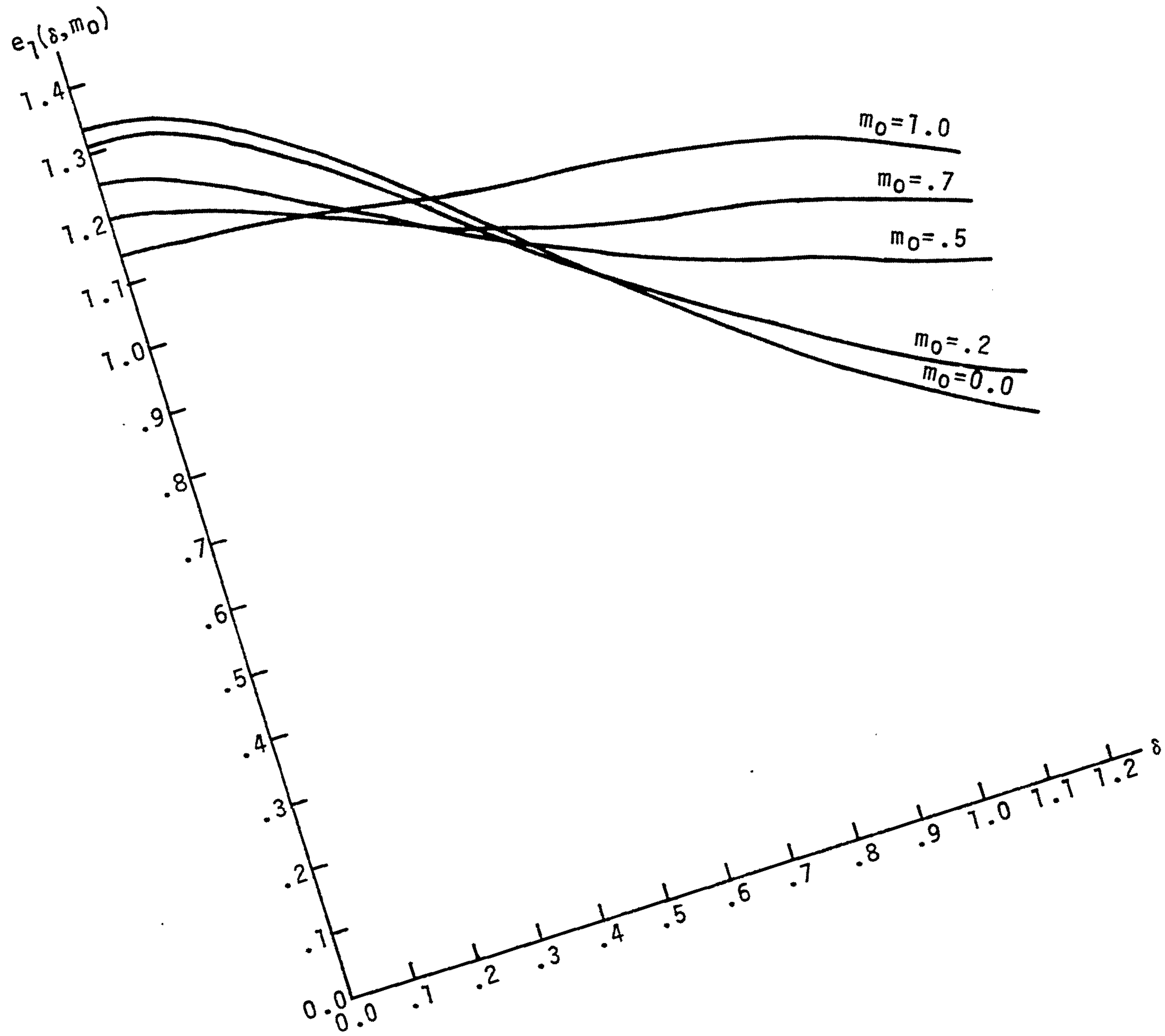

$e_1(\delta, m_0)$
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0.0
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
δ
m_0=1.0
m_0=.7
m_0=.5
m_0=.2
m_0=0.0

Figure 6.4. Graphs of $e_1(\delta, m_0)$ vs m_0 for fixed levels of δ with n=6 for Case III

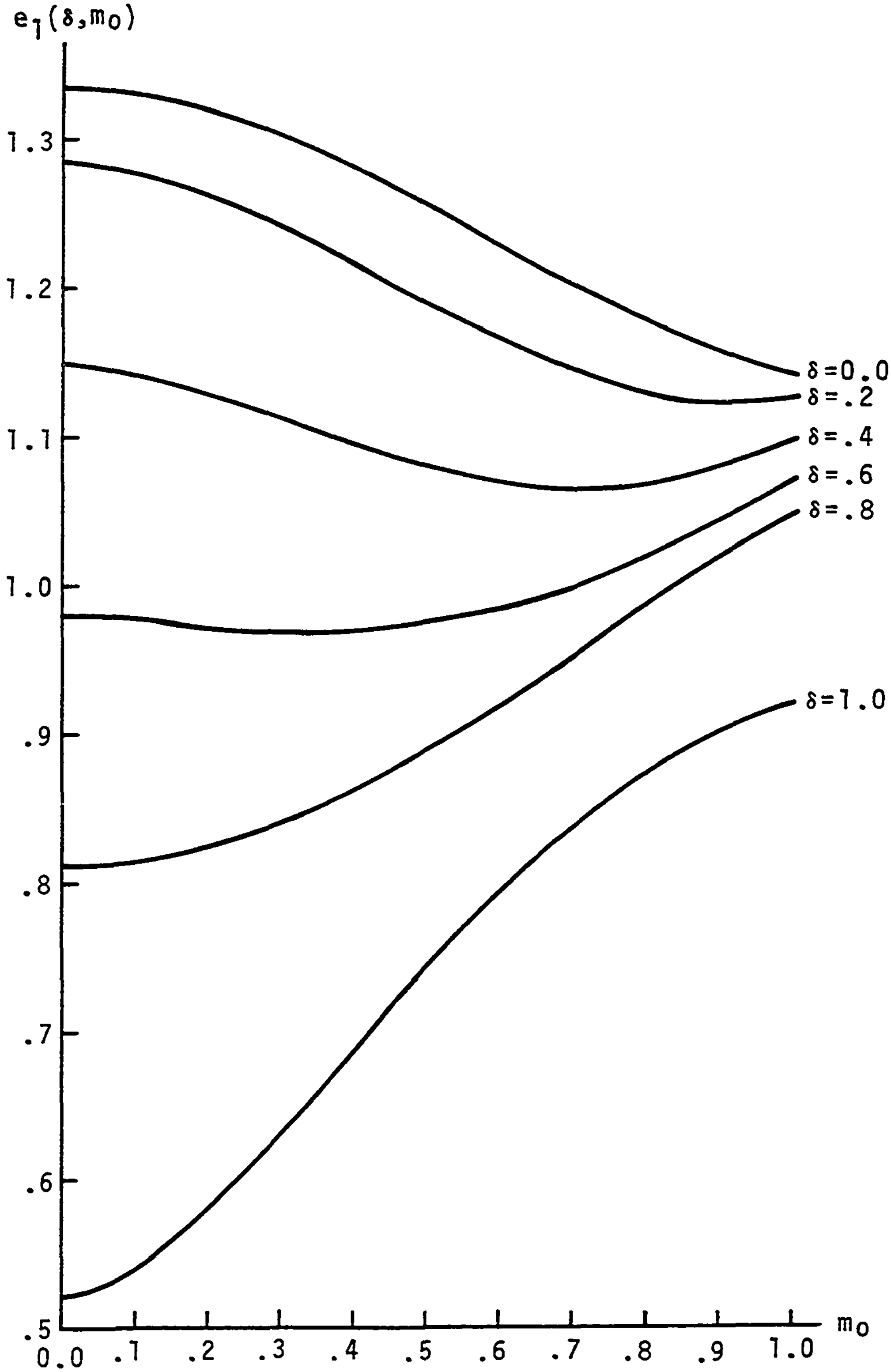
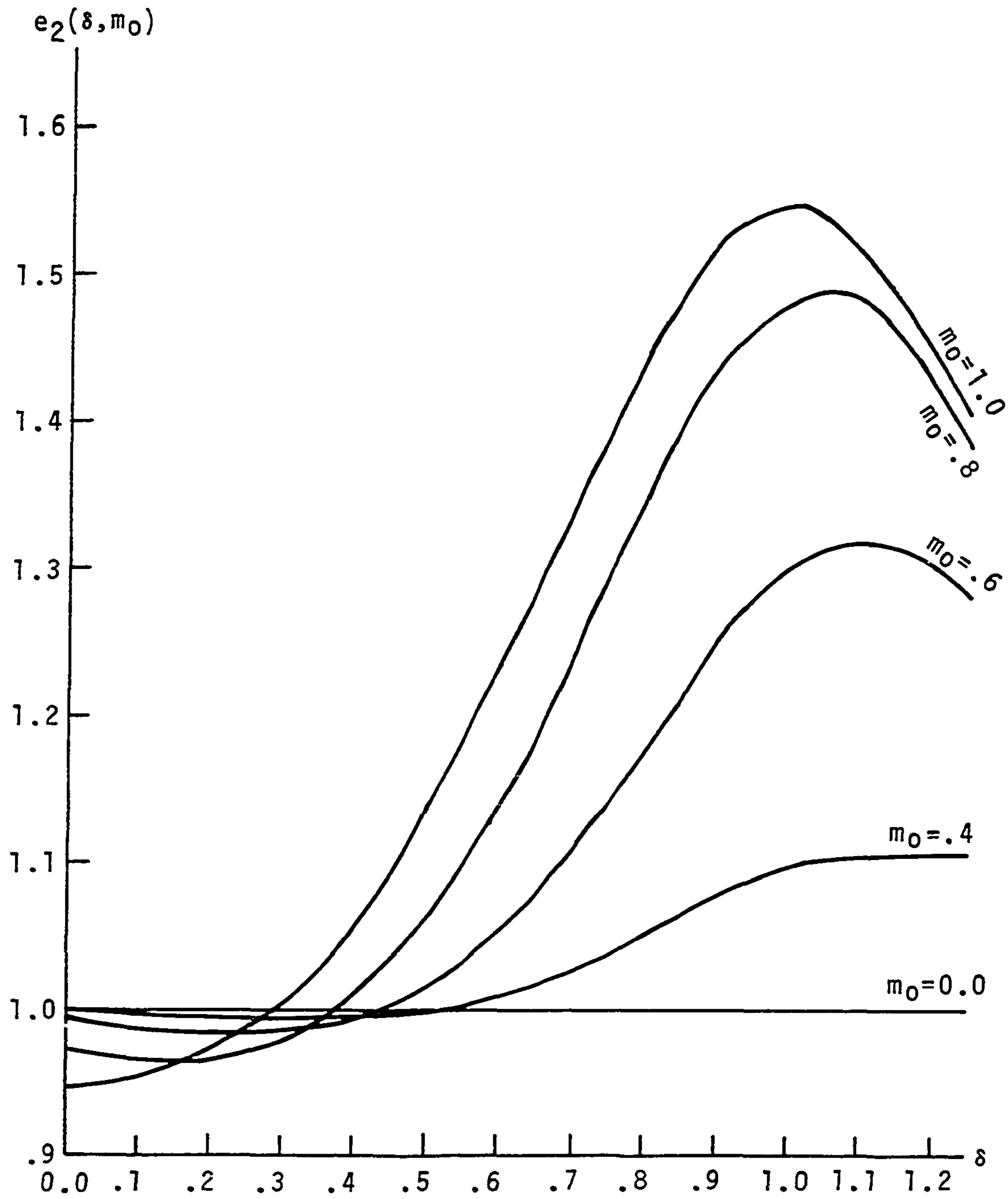
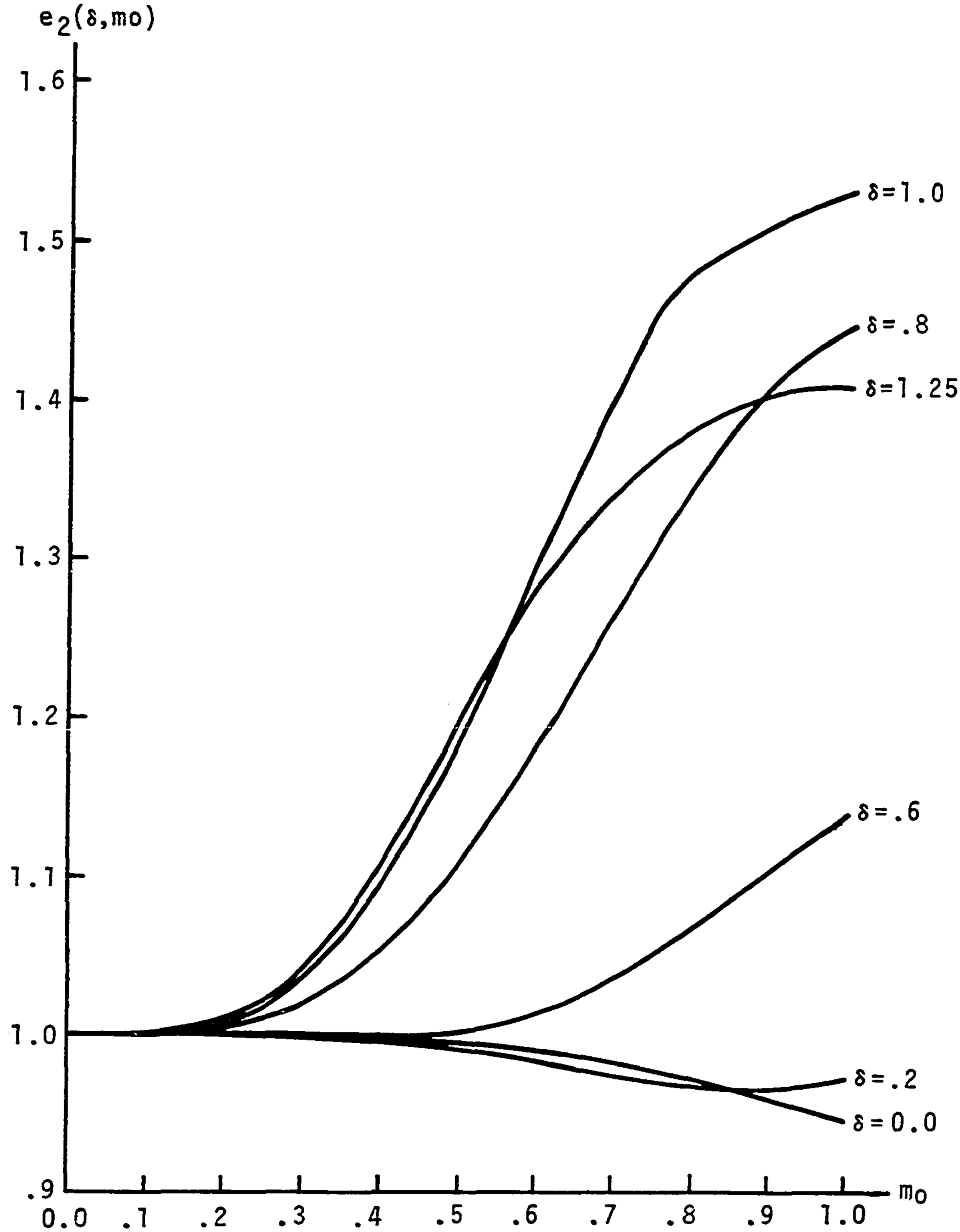

e1(δ,mo)
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
mo
δ=0.0
δ=.2
δ=.4
δ=.6
δ=.8
δ=1.0

Figure 6.5. Graphs of $e_2(\delta, m_0)$ vs δ for fixed levels of m_0 with n=12 for Case III



$e_2(\delta, m_o)$
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
δ
m_o=1.0
m_o=.8
m_o=.6
m_o=.4
m_o=0.0

Figure 6.6. Graphs of $e_2(\delta, m_0)$ vs m_0 for fixed levels of δ with n=12 for Case III



e2(δ,mo)
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
mo
δ=1.0
δ=.8
δ=1.25
δ=.6
δ=.2
δ=0.0

Figure 6.7. Graphs of $e_1(\delta, m_0)$ vs δ for fixed levels of m_0 with n=12 for Case III

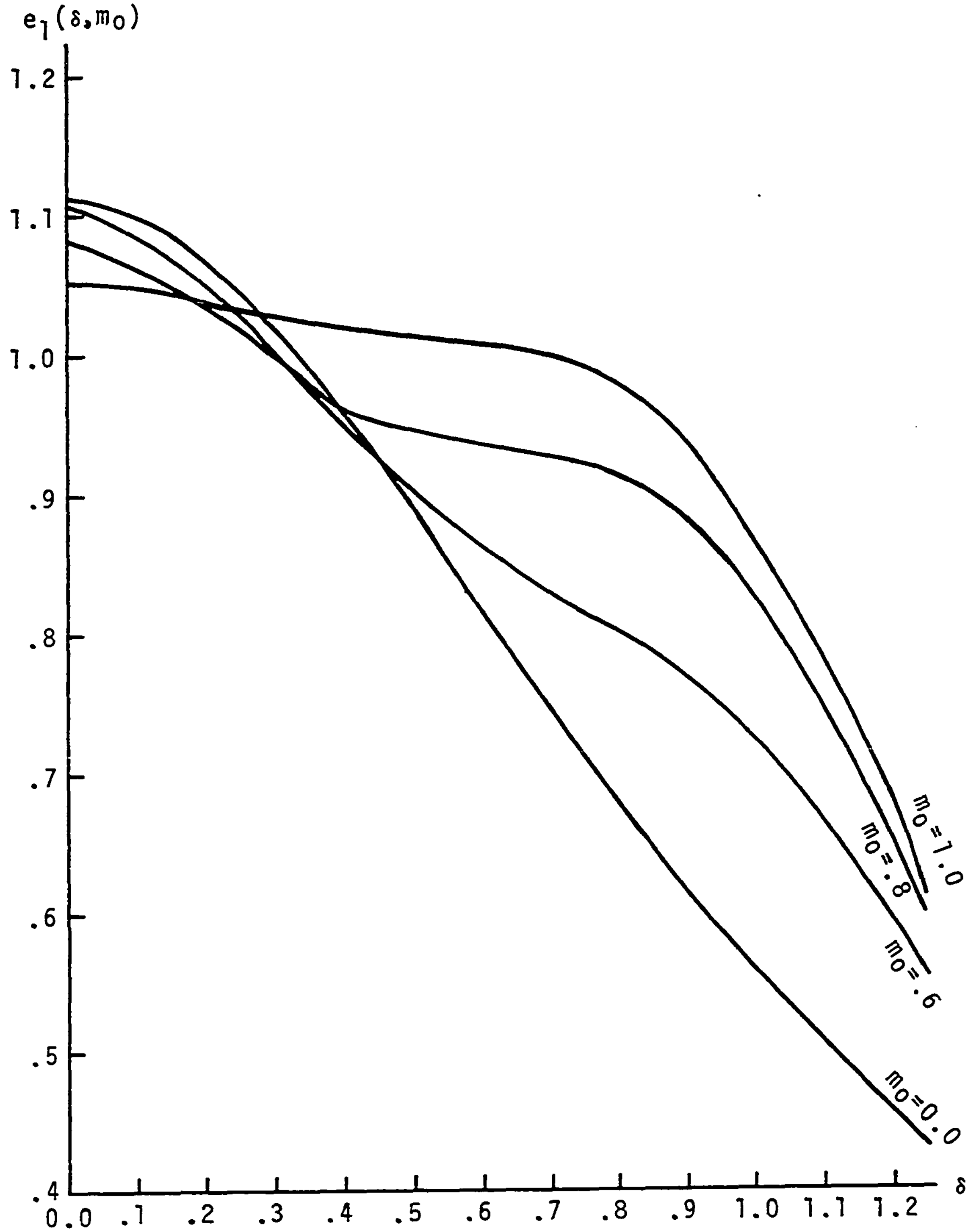
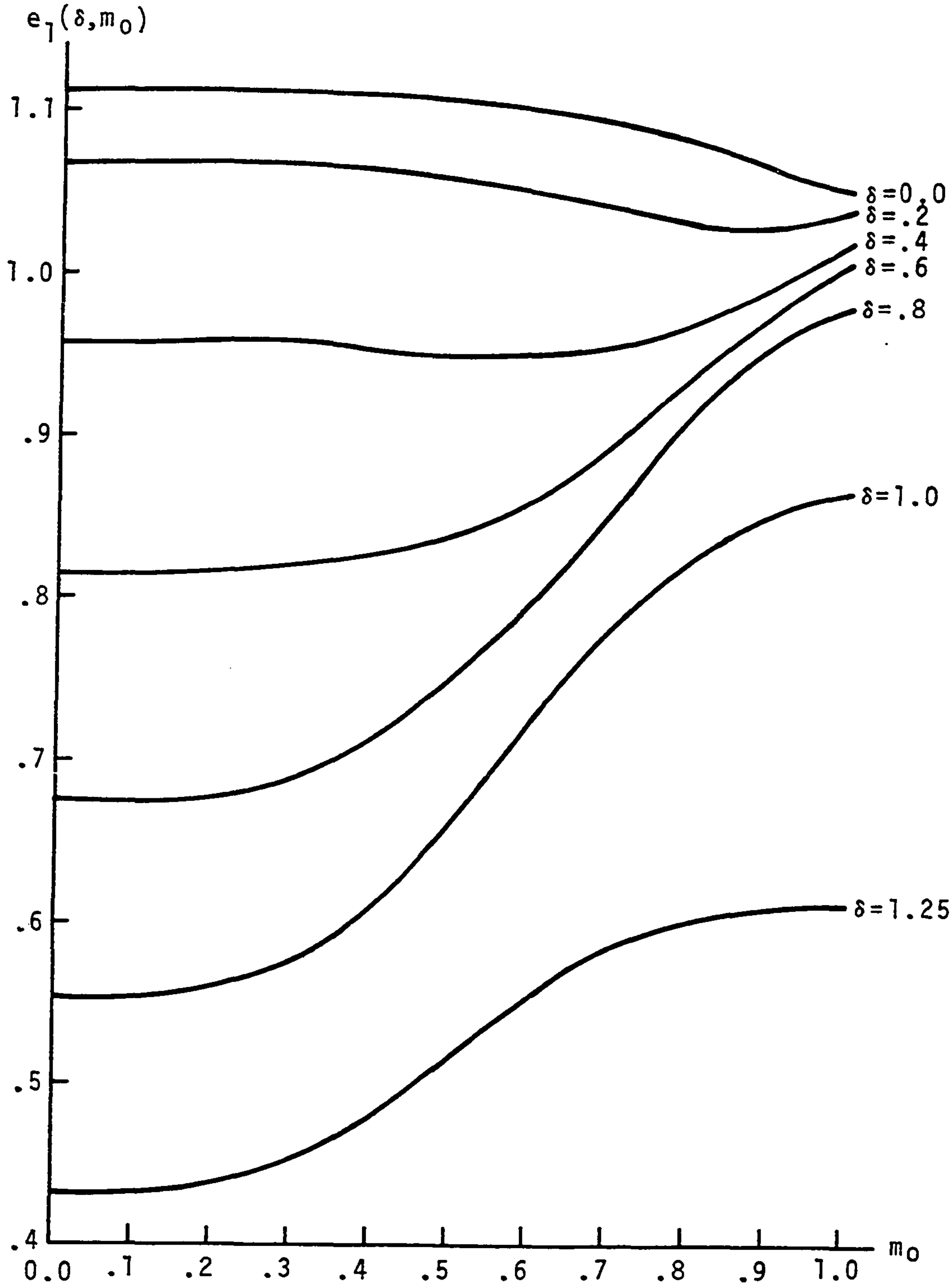

$e_1(\delta, m_0)$
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
δ
$m_0=1.0$
$m_0=.8$
$m_0=.6$
$m_0=0.0$

Figure 6.8. Graphs of $e_1(\delta, m_0)$ vs m_0 for fixed levels of δ with n=12 for Case III



$e_1(\delta, m_0)$
1.1
1.0
.9
.8
.7
.6
.5
.4
0.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
m_0
δ=0.0
δ=.2
δ=.4
δ=.6
δ=.8
δ=1.0
δ=1.25

and 6.7 also give empirical justification to the conjecture following Theorem 4.15. The relative distance between β_2 and β_0 is a fixed unknown quantity. However on the basis of past experience, it may be possible to have some idea about the likely range of values it can take on.

The level of significance of the preliminary test and hence m_0 can be fixed in any manner we please. If m_0 is fixed such that the probability of using $\bar{y}_d$ is very high, then the relative efficiency of $\bar{y}_s$ with respect to $\bar{y}_d$ is close to 1. On the other hand if the level of significance is such that the probability of using $\bar{y}_\ell$ is high, then the relative efficiency of $\bar{y}_s$ with respect to $\bar{y}_\ell$ is close to 1. The effect of changing the level of significance of the test when the relative distance between β_0 and β_2 is fixed is illustrated in Figures 6.2, 6.4, 6.6 and 6.8 for Case III when n equals 6 and 12 respectively.

The guidelines for using the sometimes regression estimator may be stated as follows:

1. If there is *a priori* information that β_0 is the actual value of β_2 and β_0 is a very strong guess for β_2 then t_0 may be chosen so that the likelihood that $\bar{y}_s$ results in using $\bar{y}_d$ is high. This would tend to minimize the loss in efficiency of $\bar{y}_s$ with respect to $\bar{y}_d$.

2. If there is *a priori* information that β_0 is the actual value of β_2 and β_0 is not considered to be a strong

choice for β_2 then t_0 may be chosen so that the likelihood that $\bar{y}_s$ results in using $\bar{y}_\ell$ is very high. This would tend to minimize the loss in efficiency of $\bar{y}_s$ with respect to $\bar{y}_\ell$.

3. If there is *a priori* information that β_0 is the actual value of β_2 and the strength of the β_0 choice is unknown then a middle range value for the level of significance of the test may be used.

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VIII. ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to Dr. B. V. Sukhatme for his suggestions, guidance, and patience throughout the course of this research and preparation of the manuscript.

The author also wishes to thank his wife, Mary Rae, for her encouragement and understanding while the author's graduate work was being completed.

Thanks are also due to Dr. T. A. Bancroft and Dr. H. A. David for their encouragement and assistance to the author during his stay at Iowa State University.

Acknowledgment is also made to the Mathematics Department of Iowa State University and to the U.S. Office of Education, Department of Health, Education and Welfare under contract No. OEC-0-70-3655 for their support of the author's graduate work and research.