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Estimators for the simultaneous equation model with lagged endogenous variables and autocorrelated errors: With application to the U.S. farm labor market

by

George H. K. Wang

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## TABLE OF CONTENTS

# I. INTRODUCTION

This dissertation is composed of two parts: (1) a study of estimators for the simultaneous equation model with lagged endogenous variables and autocorrelated errors, and (2) an econometric study of the U.S. farm labor market.

## A. Estimators for Simultaneous Equation Models with Lagged Endogenous Variables and Autocorrelated Errors

Estimators for the simultaneous equation model under the assumption of independent errors were developed in the 1940's (see Goldberger (28), Johnston (49)). Since most of the economic data used in this model are of a time series nature, the assumption of independent errors may be violated. In addition, economists have been increasingly interested in specifying partial adjustment hypothesis (Nerlove and Addison (60)) in their simultaneous equation models. For this reason, lagged endogenous variables often appear in the set of predetermined variables of simultaneous equation models.

With these motivations, we shall consider in Part One the estimation of a simultaneous equation model with lagged endogenous variables and autocorrelated errors. In Chapter 2, we shall review the limited information and full information estimation methods for simultaneous equation models with autocorrelated errors. Chapter 3 is devoted to developing asymptotically efficient limited and full information estimation methods for the parameters in such models and to studying the properties of the estimators. And, we shall conclude Part One with Chapter 4, where a Monte Carlo experiment is used to evaluate the finite sample performance of our estimators.

## B. An Econometric Study of U.S. Farm Labor Market

Excess labor in agriculture and income disparities between the farm and the non-farm sector have been major problems in the farm labor market. Various reasons have been suggested to explain this phenomenon. Heady (43) suggests that these problems are mainly due to three factors. They are: (1) maladjustment in resource structure due to economic growth, (2) output-increasing technology, and (3) inelastic demand for agricultural commodities. The purpose of this study is to increase our quantitative knowledge about this market for agricultural labor. To achieve this end, we develop econometric models for U.S. farm labor market and estimate the parameters using alternative estimation methods.

We shall begin Part Two with Chapter 5, which consists of a review of previous econometric studies of the United States farm labor market along with the construction of an econometric model for the United States farm labor market. In the following Chapter 6, we shall use different estimation procedures to estimate the parameters of the farm labor market models and analyze the economic and statistical implications of the results. The last chapter uses the econometric models derived in Chapter 6 to forecast the size of the United States farm labor force in the 1980's.

# II. A REVIEW OF AUTOCORRELATION IN SIMULTANEOUS EQUATION SYSTEM: TIME DOMAIN

## A. Introduction

In general, there are two approaches to the estimation of simultaneous equation models. One approach is based on the maximum likelihood principle; the other can be interpreted as the use of intrumental variables. Based on the maximum likelihood principle, two estimators were developed: one is the full information maximum likelihood estimator (FIML) suggested by Koopmans (53) and the other is the limited information maximum likelihood estimator (LIML) suggested by Anderson and Rubin (4, 5). It is well known that the maximum likelihood estimator is the best asymptotically normal estimator in the class of consistent estimators when the errors are normally and independently distributed and certain regularity conditions are satisfied. However, the computational aspects of FIML and LIML are burdensome. For computational simplicity, Basmann (8) and Theil (71) independently developed the two stage least squares method (2SLS) and demonstrated that the two stage least squares estimator is asymptotically equivalent to the LIML estimator. The two stage least squares estimator can be interpreted as the use of instrumental variables. (See Klein (50), Madansky (54).) Later, Zellner and Theil (86) developed the three-stage least squares estimator. Sargan (68) and Rothenberg and Leenders (65) showed that three-stage least squares is, asymptotically, as efficient as the FIML estimator.

In the presence of lagged endogenous variables and autocorrelated errors, none of the estimators described above are consistent. It is for this reason that the estimation of simultaneous equation models with lagged endogenous variables and autocorrelated errors has received attention in recent econometric literature. For discussion purposes, we present the model:

$$YB + X\Gamma + Y_{-1}C = U \tag{2.1}$$

where $Y$ is an $N \times \ell$ matrix of endogenous variables, $X$ is an $N \times \Lambda$ matrix of exogenous variables, $Y_{-1}$ is an $N \times \ell$ matrix of endogenous variables lagged one period, and $U$ is an $N \times \ell$ matrix of structural disturbances; the matrices of structural coefficients, $B$, $\Gamma$. amd $C$ are of dimension $\ell \times \ell$, $\Lambda \times \ell$ and $\ell \times \ell$, respectively. The error structure of model (2.1) is assumed to be

$$U = U_{-1}R + \epsilon \tag{2.2}$$

where $U_{-1}$ is an $N \times \ell$ matrix of $U$ lagged one period and $R$ is $\ell \times \ell$ matrix and $R^N$ converges to the null matrix as $N$ goes to infinity. We further assume that the vectors $\epsilon_{t.} = (\epsilon_{t1}, \epsilon_{t2}, \ldots, \epsilon_{t\ell})$ , $t = 1, 2, \ldots, N$ are independently distributed as multivariate normal random variables with a zero mean vector and a non-singular covariance matrix $\not{\Sigma} = \{\sigma_{ij}\}$ .

## B. Full Information Estimators

We first discuss the meaning of full information estimation methods and limited information estimation methods. In general, the full information methods use all of the prior information in estimating all of the parameters in the model. On the other hand, the limited information estimation methods use only part of the prior information in estimating the parameters of a single structural equation. This section is devoted to the full information estimation methods of Sargan (66), Hendry (45), Chow and Fair (13), Fair (21), and Dhrymes (16).

Sargan (66) considered the maximum likelihood estimation of a system of dynamic simultaneous equations with errors satisfying a vector autoregressive process. Hendry (45), following Sargan's work, applied numerical methods to the log concentrated-likelihood function to obtain the maximum likelihood estimate of the structural coefficients.

Model (2.1) can also be written in a manner similar to the model of Sargan and Hendry as:

$$BY_{t.} + \Gamma X_{t.} + CY_{t-1.} = AZ'_{t.} = U_{t.} = RU_{t-1.} + \epsilon_t \qquad (2.3)$$

where

$$A = (B, \Gamma, C) \qquad Z'_{t.} = \begin{pmatrix} Y_{t.} \\ X_{t.} \\ Y_{t-1.} \end{pmatrix} .$$

$R$ is a $\ell \times \ell$ matrix of autocorrelation coefficients. In matrix form, Equation (2.3) may be written as

$$AZ' - RAZ'_{-1} = \epsilon' \tag{2.4}$$

where the subscript $-1$ is used to denote the matrix of one period lagged value of $Z$.

In our notation, Sargan writes the log likelihood function for (2.4) as

$$L = K + N \,\ell n\, |B| - \frac{1}{2} N \,\ell n\, |\Sigma| - \frac{1}{2} \operatorname{tr} \Sigma^{-1}(AZ'ZA - 2RZ'_{-1}ZA + RAZ'_{-1}Z_{-1}Z'R) \, . \tag{2.5}$$

Maximizing (2.5) with respect to $\Sigma$ and $R$,

$$\hat{\Sigma} = \frac{\epsilon'\epsilon}{N} \tag{2.6}$$

$$\hat{R} = AZ'Z_{-1}A(AZ'_{-1}Z_{-1}A)^{-1} \, . \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), Sargan obtained the concentrated log likelihood function

$$L = K + N \,\ell n\, |B| + \frac{1}{2} \,\ell n\, |AZ'ZA - A'ZZ'_{-1}(AZ_{-1}Z'_{-1}A)^{-1}AZ'_{-1}ZA| \, . \tag{2.8}$$

Hendry (45) applied the conjugate Gradient method (due to M. J. D. Powell (62)) to obtain the estimate of $A$ which maximizes (2.8). Then, based on the invariance property of the ML estimator, Hendry found the maximum likelihood estimate of $\Sigma$ and $R$ by substituting $\hat{A}_{M,L}$ into (2.6) and (2.7), respectively. The asymptotic covariance matrix of $A$ (see Hendry (46)) is

$$\mathrm{Var}\,(A) = -N^{-1} \operatorname*{plim}_{N \longrightarrow \infty} \left[ \frac{\partial^2 L'}{\partial A\, \partial A'} \right]^{-1} \tag{2.9}$$

$$= [(\Sigma^{-1} \otimes PH'QHP') - (R'\Sigma^{-1} \otimes Z'_{-1}QHP')$$

$$- (\Sigma^{-1}R \otimes PH'QZ_{-1}) + (R'\Sigma^{-1}R \otimes Z'_{-1}QZ_{-1})]^{-1}$$

where

$$P' = (\Pi' : I)\,, \qquad \Pi = B^{-1}(P,\, C_0)$$

$$H = [X,\, Y_{-1}] \quad \text{and} \quad Q = (I - Z_{-1}A'(AZ'_{-1}Z_{-1}A')^{-1}AZ'_{-1})\,.$$

Hendry discussed two test statistics for the restrictions on structural equation: one deals with the overidentification restriction on the model; the second deals with the restrictions implied by the autoregressive transformation.

Hendry's use of Powell's algorithm has the following advantages:

(1) Powell's numerical optimization method does not require the computation of first and second derivatives for the

concentrated log likelihood function (2.8) in the process of iteration,

(2) Powell (62) has proved that his algorithm converges at the $P = 2\ell + \Lambda$-th step iteration, $P$ being the number of parameters estimated at a given time,

and the following disadvantages:

(1) As Powell pointed out, his algorithm tends to be inefficient for more than ten parameters,

(2) The estimator of the asymptotic covariance matrix of $A$ requires additional computation.

Chow and Fair (13) also suggest a computing method for obtaining the full information maximum likelihood estimates of the coefficients of Model (2.1). We write (2.1) in a slightly different way as

$$YB' = X\Gamma' + Y_{-1} C' + U \tag{2.10}$$

$$= Z \bar{\Gamma}' + U \tag{2.10A}$$

$$U = U_{-1} R' + \epsilon$$

where

$$Z = [X, Y_{-1}] , \qquad \bar{\Gamma}' = [\Gamma', C']' .$$

From (2.10A), we can write (2.10)

$$YB' - Y_{-1}B'R' - Z \bar{\Gamma}' - Z_{-1} \bar{\Gamma}'R' = \epsilon . \tag{2.11}$$

The concentrated log likelihood function of (2.11) with respect to $\Sigma$ is

$$L \;=\; \text{Constant} - \ell n \left[ \; \left| \frac{1}{T} \, \epsilon'\epsilon \right| / |B|^2 \right]^{\frac{1}{2}} . \tag{2.12}$$

Based on Chow's work (11, 12), they write (2.12) as

$$L \;=\; \text{Constant} - \frac{T}{2} \, \ell n \left[ \; |S| \, / \, |W| \; \right] \tag{2.13}$$

where

$$S \;=\; \frac{1}{T} \, \epsilon'\epsilon \qquad \text{and} \qquad W \;=\; B'Y'YB \quad .$$

There are two advantages of writing the concentrated log likelihood function in the form of (2.13); (i) it is obvious that the estimates which maximize the likelihood function are also equivalent to the estimates which minimize the variance ratio $\left| \frac{1}{T} \, \epsilon'\epsilon \right| / \left| \frac{1}{T} \, B'Y'YB \right|$ , and (ii) after introducing prior restrictions on the coefficient of $B$ and $\Gamma$, it is easy to obtain a set of normal equations given the value of $\hat{\Sigma}$ and $\hat{R}$ .

Setting the partial derivatives of (2.13) with respect to the unknown coefficient $\Theta \equiv [B, \bar{\bar{B}}, \bar{\Gamma}, \bar{\bar{\Gamma}} \,]$ to zero yields the following system of normal equations

$$\begin{bmatrix} q^{11}Y_1'Y_1 & \cdots & q^{G1}Y_1'Y_G & Y_1'\xi \\ & \cdots & & \\ q^{11}Y_G'Y_1 & \cdots & q^{G1}Y_G'Y_G & Y_G'\xi \\ & \cdots & & \\ S^{11}Z_{1,-1}'Y_1 & \cdots & S^{G1}Z_{1,-1}'Y_G & Z_{1,-1}'\xi \\ & \cdots & & \\ S^{11}Z_{G,-1}'Y_1 & \cdots & S^{G1}Z_{1,-1}'Y_G & Z_{G,-1}'\xi \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_G \\ \vdots \\ \bar{\bar{B}}_1 \\ \vdots \\ \bar{\bar{B}}_G \\ \bar{\Gamma}_1 \\ \vdots \\ \bar{\Gamma}_G \\ \bar{\bar{\Gamma}}_1 \\ \vdots \\ \bar{\bar{\Gamma}}_G \end{bmatrix} = \begin{bmatrix} Y_1' \sum_h q^{h1} \frac{1}{h} \\ \vdots \\ Y_G' \sum_h q^{hG} \frac{1}{h} \\ \\ Z_{1,-1}' \sum_h S^{h1} \frac{1}{h} \\ \\ Z_{G,-1}' \sum_h S^{hG} \frac{1}{h} \end{bmatrix}, \tag{2.14}$$

where

$$\xi = \{S^{11}Y_{1,-1}, \ldots, S^{G1}Y_{G,-1}, S^{11}Z_1, \ldots, S^{G1}Z_G, S^{11}Z_{1,-1}, \ldots, S^{G1}Z_{G,-1}\},$$

$S^{ij}$ and $W^{ij}$ are the $(i,j)$-th elements of $S^{-1}$ and $W^{-1}$, respectively, defined in (2.13), $q^{ij} = (S^{ij} - W^{ij})$, $\bar{\bar{B}}_i'$ is the i-th row of the matrix $\bar{\bar{B}}' = B'R'$ defined in (2.11) and $\bar{\bar{\Gamma}}_i'$ is the i-th row of the matrix $\bar{\bar{\Gamma}}' = \bar{\Gamma}'R'$ defined in (2.11).

The system of equations can be written compactly

$$f(\theta) = \underset{\sim}{0} \ . \tag{2.15}$$

They apply the Newton method to the system of normal equations (2.15) and obtain

$$\theta^{\gamma+1} = \theta^{\gamma} + [F(\theta^{\gamma})]^{-1} f(\theta^{\gamma}) \tag{2.16}$$

where $\theta^{\gamma}$ is the value of $\theta$ in the r-th iteration and $F(\theta^{\gamma})$ is the matrix of partial derivatives of the elements of $f$ with respect to the elements of $\theta$ evaluated at the estimate obtained at the $\gamma$-th iteration.

It is clear that there are linear restrictions on the elements of $\theta$. In this case, $\bar{\bar{B}}$ is equal to $R\,B$ and $\bar{\bar{\Gamma}}$ is equal to $R\bar{\Gamma}$. Hence, the set of unknown parameters $\theta \equiv (B, \bar{\bar{B}}, R, \bar{\Gamma}, \bar{\bar{\Gamma}})$ are functions of a reduced set of parameters $\theta^{*} \equiv (B, \bar{\Gamma}, R)$.

Assuming $R$ as given, Chow and Fair derived the relationship between the unrestricted likelihood function and restricted likelihood function as follows:

$$f^{*}(\theta) = \begin{bmatrix} \frac{\alpha \ell^{*}}{\alpha B} \\ \frac{\alpha \ell^{*}}{\alpha \bar{\Gamma}} \end{bmatrix} = \begin{bmatrix} R, & I, & \underset{\sim}{0}, & \underset{\sim}{0} \\ \underset{\sim}{0}, & \underset{\sim}{0}, & R, & I \end{bmatrix} \begin{bmatrix} \frac{\alpha \ell}{\alpha B} \\ \frac{\alpha \ell}{\alpha \bar{\bar{B}}} \\ \frac{\alpha \ell}{\alpha \bar{\Gamma}} \\ \frac{\alpha \ell}{\alpha \bar{\bar{\Gamma}}} \end{bmatrix} \tag{2.17}$$

$$\equiv M\,f(\theta) \,,$$

and

$$F^*(\theta^*) = M\,F(\theta)\,M' . \tag{2.18}$$

Then the value of $(B, \bar{\Gamma})$, given the value of $R$ at the r-th iteration is

$$\theta^{*\gamma+1} = \theta^{\gamma} + [F^*(\theta^{\gamma})]^{-1} f^*(\theta^{\gamma}) . \tag{2.19}$$

From (2.19), they find the value of $B$ and $\bar{\Gamma}$ given the initial value of $R$. Now, treating $B$ and $\bar{\Gamma}$ as given, they maximize (2.13) with respect to $R$. This is equivalent to maximizing the following likelihood function

$$L_1 = \text{Constant} - \frac{T}{2} \ln \left| \frac{1}{T} \varepsilon'\varepsilon \right| .$$

since $\left| \frac{1}{N} B'Y'YB \right|$ is a constant.

In this case, the maximum likelihood estimator of $R$ is equivalent to the least squares estimator of $R$,

$$\hat{R} = (\hat{U}'_{-1} \hat{U}_{-1})^{-1} \hat{U}'_{-1}\hat{U}$$

where

$$\hat{U} = Y\hat{B}' - X\,\hat{\Gamma}' - Y_{-1}\hat{C}' . \tag{2.20}$$

Inserting $\hat{R}$ back into Equation (2.13), Chow and Fair suggest repeating the two-step iterative process until it converges.

Chow and Fair's method has the following weaknesses:

1. Their estimate will converge to the maximum likelihood estimate if the two-step process converges. However, they have not given the conditions under which the two-step process converges.
2. They recommended $F^{*}(\theta^{\gamma *})$ evaluated at $\theta^{\gamma *}$ as an asymptotic covariance matrix of $\theta^{*}$. This asymptotic covariance matrix of $\theta^{*}$ is valid only if there are no lagged endogenous variables among the set of predetermined variables.

Dhrymes (16) proposed two procedures to deal with the estimation of Model (2.1). We first discuss his linearized full information maximum likelihood estimator. Model (2.1) can also be written as

$$y_{t.} = y_{t.}B + y_{t-1.}C + X_{t.}\Gamma + U_{t.} \tag{2.21}$$

$$U_{t.} = U_{t-1.}R + \epsilon_{t.} \qquad t = 1, 2, \ldots, N$$

where

$y_{t.}$ is the row vector of observations on the $\ell$ current endogenous variables.

$y_{t-1.}$ is the row vector of observations on the $\ell$ endogenous variables lagged one period.

$X_{t.}$ is the observations vector on the $\Lambda$ exogenous variables.

$U_{t.}$ is the $\ell$-component row vector of errors.

$$\epsilon_{t.} \sim \text{MIN}(\underset{\sim}{0}, \not{\Sigma}) .$$

Neglecting the terms which vanish as $N \longrightarrow \infty$, we can write down the log likelihood function for (2.21) in our notation as

$$L = k - \frac{N}{2}\ell n |\not{\Sigma}| - \frac{N}{2}\ell n|(I-B)'(I-B)| - \frac{1}{2} \text{tr} \not{\Sigma}^{-1}[(ZA-Z_{-1}AR)'(ZA-Z_{-1}AR)] . \tag{2.22}$$

Adding and subtracting the term $\frac{N}{2} | \frac{\tilde{V}'\tilde{V}}{N} |$ to (2.22), Dhrymes obtained

$$L = k - \frac{N}{2}\ell n |\Sigma| - \frac{N}{2}\ell n \left| \frac{\tilde{V}'\tilde{V}}{N}\right| - \frac{N}{2} \ell n \left|(I-B)' \frac{\tilde{V}'\tilde{V}}{N} (I-B)\right|$$

$$- \frac{1}{2} \text{tr} \not{\Sigma}^{-1} [(ZA - Z_{-1}AR)' (ZA - Z_{-1}AR)] \tag{2.23}$$

where

$$Z_{t.} = (y_{t.}, y_{t-1.}, X_{t.}) , \qquad A = [(I-B), - C' - \Gamma'] ,$$

$$\tilde{V} = (I - N^{*})Y , \qquad N^{*} = Q (Q'Q)^{-1} Q' ,$$

$$Q = (Y_{-1}, Y_{-2}, X \;\; X_{-1}) \quad \text{and} \quad Z = (Y, Y_{-1}, X) .$$

Maximizing (2.23) with respect to $\not\Sigma$ , they obtain

$$\not\Sigma = \frac{1}{N} (ZA - Z_{-1} AR)' (ZA - Z_{-1}AR) \quad . \tag{2.24}$$

Substituting (2.24) into (2.23) yields the concentrated log likelihood function

$$L = k' - \frac{N}{2} \; \ell n \; \left| \frac{\tilde{V}'\tilde{V}}{N} \right| - \frac{1}{2} \ell n |(ZA - Z_{-1}AR)' (ZA - Z_{-1}AR)|$$

$$+ \frac{N}{2} \ell n \quad \left|(I-B)' \frac{\tilde{V}'\tilde{V}}{N} (I-B)\right| \quad . \tag{2.25}$$

Differentiating (2.25) with respect to $R$ and to the columns of $A$ , they derive the following

$$\hat{R} = (U'_{-1} U_{-1})^{-1} U'_{-1} U \tag{2.26}$$

$$\hat{\delta} = [(Z^* - (R' \otimes I) Z^*_{-1}) (\Sigma^{-1} \otimes I) (Z^* - (R' \otimes I)Z^*_{-1}) - \tilde{V}^*(S^{-1} \otimes I)\tilde{V}^*]^{-1} \times$$

$$(Z^* - (R' \otimes I) Z^*_{-1}) (\Sigma^{-1} \otimes I) \quad [y - (R' \otimes I)y_{-1}] - \tilde{V}^{*\prime}(S^{-1} \otimes I) y \; ,$$

$$\tag{2.27}$$

where

$$\delta = [\delta_1, \delta_2, \ldots, \delta_\ell] \; , \quad \delta_i = [B_i, C_i, \Gamma_i]$$

$$Z^* = \text{Diag}\,[Z_1, Z_2, \ldots, Z_\ell]\ ,$$

$$\tilde{V}^* = \text{Diag}\,[\tilde{V}_1^*, \tilde{V}_2^*, \ldots, \tilde{V}_\ell^*]\ ,$$

$$\tilde{V}_i^* = [\tilde{V}_i, \underset{\sim}{0}, \underset{\sim}{0}\,] \qquad i = 1, 2, \ldots, \ell,$$

and

$$S = (I - B)' \frac{\tilde{V}'\tilde{V}}{N} (I - B)\ .$$

We note that (2.24), (2.26) and (2.27) is a system of nonlinear equations. This system of equations can be solved by iteration methods (see Hamming (35)). Dhrymes applied the direct iteration method and suggested iterating back and forth among $\Sigma$, $R$ and $\delta$ until convergence is reached. The final estimates are the maximum likelihood estimators of $\Sigma$, $R$ and $\delta$ if the iteration converges.

Another full information estimator suggested by Dhrymes (16) is called "Full Information Dynamic Autoregressive Estimator (FIDA)." This method is mainly a combination of the Cochran-Orcutt type autoregressive transformation and three-stage least squares. FIDA is derived by minimizing

$$\text{tr}\ \Sigma^{-1}(\tilde{Z}A - Z_{-1}AR)'\ (\tilde{Z}A - Z_{-1}AR) \tag{2.28}$$

where

$$\tilde{Z} = (\tilde{Y}, Y_{-1}, X) \qquad \text{and} \qquad \tilde{Y} = N^*Y$$

subject to prior consistent estimates of $\Sigma$ and $R$ .

Differentiating (2.28) with respect to the columns of $A$ , they derive the estimator

$$[(\tilde{Z}^* - (\hat{R}' \otimes I)Z^*_{-1})'(\hat{\Sigma}^{-1} \otimes I)(\tilde{Z}^* - (\hat{R}' \otimes I)Z^*_{-1})]\hat{\delta} = (\tilde{Z}^* - (\hat{R}' \otimes I)Z^*_{-1}) \times$$

$$(\hat{\Sigma}^{-1} \otimes I)(y - (\hat{R}' \otimes I)y_{-1}), \quad \text{where} \quad \tilde{Z}^* = [I_\ell \otimes \tilde{Z}] \ . \tag{2.29}$$

Dhrymes and Erlat (18) studied the asymptotic properties of LFIML and FIDA estimators. However, they offered only a conjecture on the limiting distribution of their estimators. Based on the result that $\underset{N \to \infty}{\text{plim}} \hat{\Sigma} = \underset{N \to \infty}{\text{plim}} \hat{S} = \Sigma$ , they claim that the FIDA estimator has the same asymptotic distribution as the LFIML estimator.

In the present context, Dhrymes's estimators suffer from the following weaknesses:

1. The speed of convergence of the direct iterative method is slower than that of the Newton method (see Chow (12)).
2. The direct iterative method may not converge (see Hamming (35)), therefore, the LFIML and FIDA may never converge to the FIML estimator.
3. The asymptotic covariance matrix of $A$ has to be computed separately from the estimation of $A$ .

Fair (21) extended the work of Brundy and Jorgenson (10) to take account of autocorrelated errors in dynamic simultaneous equation models.

His procedure, called "Full Information Efficient Instrumental Variable (FIEIV)," is equivalent to the following:

1. Apply the instrumental variable method to each equation to obtain initial consistent estimates of the structural coefficients and autocorrelation coefficients.

2. Write the reduced form of Model (2.1) with independent errors as

$$Y = -X\Gamma B^{-1} - Y_{-1}CB^{-1} + Y_{-1}BRB^{-1} + X_{-1}\Gamma RB^{-1} + Y_{-2}CRB^{-1} + \varepsilon B^{-1} \ . \tag{2.30}$$

   Substituting the initial consistent estimates of $B$, $C$, $\Gamma$ and $R$ into (2.30), Fair obtained the generated value of $Y = \tilde{Y}$.

3. Define the instrumental variables $W = (\hat{\not{\Sigma}} \otimes I)\,\bar{Z}^*$ where $\bar{Z}^* = [\,\bar{Z} - (\hat{R} \otimes I)Z_{-1}]$ and $\bar{Z} = \mathrm{Diag}\,[\bar{Z}_1, \bar{Z}, \ldots, \bar{Z}_\ell]$, $\bar{Z}_i = [\,\tilde{Y}_i, Y_{-1}, X]$.

$$\hat{\delta} = (W'Z)^{-1}\, W'\, \bar{Y}^* \tag{2.31}$$

   where

$$\bar{Y}^* = (\underset{\sim}{y} - (\hat{R} \otimes I)\,\underset{\sim}{y}_{-1}) \ , \quad \underset{\sim}{y} = [y_1, y_2, \ldots, y_\ell]$$

$$Z = \mathrm{Diag}\,[Z_1, Z_2, \ldots, Z_\ell], \quad Z_i = [Y_i, Y_{i-1}, X_i] \ .$$

The final estimator is obtained when (2.30) and (2.31) converge.

The consistency of the estimator in (2.31) follows from the consistency of the initial instrumental variable estimates of $\delta$, $\Sigma$, and $R$. The purpose of the second stage instrumental variable procedure is to improve the efficiency of the first stage estimation. This estimation method has the same weaknesses as Dhrymes' LFIML and FIDA estimators.

## C. Limited Information Estimators

Now, we review the limited information methods suggested by Theil (71), Sargan (66), Madansky (54), Amemiya (1), Fair (20, 21), Fuller (22), and Dhrymes, Berner and Cummins (17). The limited information methods proposed by Theil and Madansky deal with a model containing only exogenous variables, but with autocorrelated errors.

The i-th structural equation of the model considered by Theil (71) and Madansky (54) can be expressed as

$$\begin{aligned} y_i &= Z_i \delta_i + u_i \\ u_i &= \rho_i u_{i-1} + \epsilon_i \end{aligned} \qquad (2.32)$$

where $Z_i = [Y_i, X_i]$, $\delta_i = [B_i, \Gamma_i]$. Denote $X_2$ as the exogenous variables in the system but not in the i-th equation, where there is no lagged endogenous variables in the model.

Theil's generalized two-stage least squares procedure consists of two steps: (1) given a value of $\rho$, transform all the exogenous variables by $\rho$ and obtain $\widehat{Y_i - \rho_i Y_{i-1}}$ by regressing $Y_i - \rho Y_{i-1}$ on $X_i - \rho X_{i-1}$, $X_2 - \rho X_{2-1}$ , (2) regress $y_i - \rho y_{i-1}$ on $\widehat{Y_i - \rho Y_{i-1}}$, $X_i - \rho X_{i-1}$ to obtain the final estimate $\hat{\delta}_i$ .

Madansky's generalized instrumental variable estimator consists of three steps: (1) replace the true covariance matrix of the i-th structural equation $\Sigma_{ii}$ by $S_{ii}$ where $S_{ii}$ can be obtained from the residuals of two-stage least squares applied to (2.32), and (2) multiply Equation (2.32) by the matrix $X'$ , the transpose of the matrix of observations on all the exogenous variables in the system, to obtain

$$X'y_i = X'Z_i\delta_i + X'u_i \ , \tag{2.33}$$

(3) apply generalized least squares to (2.33) to obtain

$$\hat{\delta}_i = [Z_i'X(X'S_{ii}X)^{-1}X'Z_i]^{-1} Z_i'X\,(X'S_{ii}X)^{-1}\,X'y_i \quad . \tag{2.34}$$

Wicken (85) has shown that Theil's generalized two-stage least squares with a consistent estimator of the covariance matrix and Madansky's estimator are consistent. He also demonstrated that Theil's estimator with a consistent estimate of the covariance matrix is at least as efficient as Madansky's estimator.

Sargan, in 1962, developed a limited information maximum likelihood estimator (SLI). Amemiya (1) modified Sargan's estimator and called it Sargan's two-stage least squares (S2SLS).

Sargan's limited information maximum likelihood estimator (SLI) for the i-th structure equation in Model (2.1) is

$$\begin{bmatrix} \beta_i \\ \Gamma_i \\ C_i \end{bmatrix} = \begin{bmatrix} Y_i^{*'}Y_i^{*} - \lambda_n Y^{*'}(I-\Phi(\Phi'\Phi)\Phi)Y^{*}, & Y_i^{*'}X_i^{*}, & Y_i^{*'}Y_{i-1}^{*} \\ X_i^{*'}Y_i^{*} \qquad , & X_i^{*'}X_i^{*}, & X_i^{*'}Y_{i-1}^{*} \\ Y_{i-1}^{*'}Y_i^{*} \qquad , & Y_{i-1}^{*'}X_i^{*}, & Y_{i-1}^{*}Y_{i-1}^{*} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} Y_i^{*'}y_i^{*} - \lambda_n Y^{*'}[I - \Phi(\Phi'\Phi)^{-1}\Phi']Y_i^{*} \\ X_i^{*'}y_i^{*} \\ X_{i-1}^{*}y_i^{*} \end{bmatrix} \quad . \tag{2.35}$$

$$\rho_i = \frac{(y_i - Y_i\beta_i - X_i\Gamma_i - Y_{i-1}C_i)'(y_{i-1} - Y_{i-1}\beta_i - X_{i-1}\Gamma_i - Y_{i-2}C_i)}{(y_{i-1} - Y_{i-1}\beta_i - X_{i-1}\Gamma_i - Y_{i-2}C_i)'(y_{i-1} - Y_{i-1}\beta_i - X_{i-1}\Gamma_i - Y_{i-2}C_i)} \tag{2.36}$$

where

$$\lambda_n = \min \frac{\beta_i' W_i \beta_i}{\beta_i W \beta_i}$$

$$Y_i^{*} = Y_i - \rho_i Y_{i-1}, \quad X_i^{*} = X_i - \rho_i X_{i-1}, \quad Y_{i-1}^{*} = Y_{i-1} - \rho_i Y_{i-2},$$

$$W_i = Y^{*'}[I - H_i^*(H_i^{*'} H_i^*)^{-1} H_i^{*'}] Y^*, \qquad H_i^* = [X_i^*, Y_{i-1}^*]$$

$$W = Y' [I - \Phi(\Phi'\Phi)\Phi']Y$$

$$\Phi = [X, X_{-1}, Y_{-1}, Y_{-2}] .$$

Amemiya sets $\lambda_n = 1$ in (2.35) and calls this estimator Sargan's two-stage least squares (S2SLS). Since $\operatorname*{plim}_{N \to \infty} \lambda_n = 1$ (see Theil (71)), it follows that S2SLS is asymptotically equivalent to SLI estimator.

In practice, the autocorrelation coefficient is seldom known. Consequently, Amemyia applied a direct iteration method between (2.35) and (2.36) by assuming an initial value for $\rho_i$. Amemyia also presents an asymptotic covariance matrix of $\Theta_i \equiv [B_i, \Gamma_i]$ where there are only lagged exogenous variables among the set of predetermined variables in the model.

It may be worthwhile to mention that S2SLS uses a large number of predetermined variables in the first stage regression of S2SLS. Hence, it is possible that the matrix $(\Phi'\Phi)^{-1}$ may be singular.

With the objective of reducing the number of predetermined variables in S2SLS, Fair (20) suggests an estimation procedure which amounts to the following:

1. Regress $Y_i$ on a set of predetermined variables selected from $(X, X_{-1}, Y_{-1}, Y_{-2})$. This set of instrumental variables should at least include $Y_{i-1}$, $Y_{i-2}$, $X_{-1}$ and $X$ and then obtain

the "predicted" matrix $\hat{Y}_i$ .

2. Regress $(y_{it} - \rho\, y_{it-1})$ on $(\hat{Y}_i - \rho\, \hat{Y}_{i-1})$, $(Y_{i-1} - \rho\, Y_{i-2})$, $(X_i - \rho\, X_{i-1})$ for various $\rho$'s in the interval $(-1, 1)$ and select that regression for which the sum of the squared residuals is minimized. From this equation, he obtains an estimate of $\delta_i \equiv [B_i, \Gamma_i, C_i]$, and $\rho_i$, $\tilde{\delta}_i$, $\tilde{\rho}_i$, respectively. Fair also discusses "X2SLS" which is exactly the same as the above estimation procedure except that he obtains a consistent estimate of $\rho_i$, $\hat{\rho}_i$ to replace the unknown $\rho_i$ used in the second stage regression.

Dhrymes (17) and Fair (21) independently suggested an iterative instrumental variable procedure. This procedure is the same as Fair's FIIV in (2.31) except that $\hat{\Sigma}$ and the autocorrelation matrix are diagonal matrices. Then, the single equation iterative variable estimator of $\delta_i$ becomes

$$\tilde{\delta}_i = (W_i' Z_i)^{-1} W' \bar{y}_i^* \tag{2.37}$$

where

$$W_i = [\, \bar{Z}_i - \hat{\rho}_i Z_{i-1}] \,, \qquad \bar{Z}_i = [\tilde{Y}_i, Y_{i-1}, X_i] \quad i = 1,2,\ldots,\ell$$

$$Z_i = [Y_i, Y_{i-1}, X_i] \,, \qquad \bar{y}_i^* = (y_i - \hat{\rho}_i\, y_{i-1})$$

$$\hat{\rho}_i = \hat{u}_{i-1}\hat{u}_i / \hat{u}_{i-1}' \hat{u}_{i-1} \quad \text{where } \hat{u}_i = (y_i - Y_i\hat{B}_i - X_i\hat{\Gamma}_i - Y_{i-1}\hat{C}_i)$$

and $\hat{B}_i$, $\hat{C}_i$, $\hat{\Gamma}_i$ are obtained from Step 1 of FIIV.

Dhrymes and Fair suggest iteration between (2.37) and (2.30). If the procedure converges, it is claimed that the asymptotic distribution of iterative instrumental variables will converge to that of the LIML estimator for Model (2.1). However, they did not prove that the iteration converges.

Fuller (22) proposed a single equation estimator for the model similar to Model (2.1). His procedure can be described as follows: (1) obtain initial consistent estimates of $\delta_i = [B_i, C_i, \Gamma_i]$ and of $\rho_i$ by the instrumental variable procedure, (2) transform all the variables using $\hat{\rho}_i$ and obtain $\widehat{Y_i - \hat{\rho}_i Y_{i-1}}$ by regressing $Y_i - \hat{\rho}_i Y_{i-1}$ on $X - \hat{\rho}_i X_{i-1}$, $Y_{i-1} - \hat{\rho}_i Y_{i-2}$, and $\hat{u}_{i-1}$, and (3) regress $y_i - \hat{\rho}_i y_{i-1}$ on $\widehat{Y_i - \hat{\rho}_i Y_{i-1}}$, $X_i - \hat{\rho}_i X_{i-1}$, $Y_{i-1} - \hat{\rho}_i Y_{i-2}$, $\hat{u}_{i-1}$ to obtain the one-step estimates $\hat{\delta}_i$, and $\tilde{\rho}_i = \hat{\rho}_i + \Delta\tilde{\rho}_i$ where $\hat{\rho}_i$ is obtained from (1).

The advantage of Fuller's estimator is that it estimates $\delta_i$ and $\rho_i$ simultaneously. Also, a consistent estimate of the asymptotic covariance matrix of $(\delta_i, \rho_i)$ is a by-product of the last stage regression.

# III. LIMITED INFORMATION AND FULL INFORMATION ESTIMATORS

This chapter is devoted to developing asymptotically efficient limited information and full information estimators for Model (2.1, 2.2).[1] One general approach will be as follows: First, we construct an initial estimator whose error is $O_p(N^{-\frac{1}{2}})$. Then using this initial estimator, we construct a revised estimator. The properties of the proposed estimators shall be investigated and the asymptotic covariance matrix of the derived reduced form for Model (2.1, 2.2) are also presented. To facilitate our discussion, we first introduce the assumptions needed in the rest of the chapter.

## A. The Model and Assumptions

Consider a dynamic simultaneous equation model of the following form:

$$YB + X\Gamma + Y_{-1}C = U \tag{3.1}$$

where $Y$ is an $N \times \ell$ matrix of endogenous variables, $X$ is an $N \times \Lambda$ matrix of exogenous variables, $Y_{-1}$ is an $N \times \ell$ matrix of endogenous variables lagged one period, and $U$ is an $N \times \ell$ matrix of structural

[1]After the author had finished his research for this thesis, Professor Hatanaka called the author's attention to his unpublished paper (40). In (40) Hatanaka has independently developed similar procedures from entirely different motivations.

disturbances; the matrices of structural coefficients, $B$, $\Gamma$, and $C$ are of dimension $\ell \times \ell$, $\Lambda \times \ell$ and $\ell \times \ell$. The error structure of Model (3.1) is

$$U = U_{-1}R + \epsilon \tag{3.2}$$

where $U_{-1}$ is an $N \times \ell$ matrix of $U$ lagged one period and $R = \text{diag}(\rho_1, \rho_2, \ldots, \rho_\ell)$ where $|\rho_i| < 1$ for $i = 1, 2, \ldots, \ell$. We further assume that the vectors $\epsilon'_{t.} = (\epsilon_{t1}, \epsilon_{t2}, \ldots, \epsilon_{t\ell})'$, $t = 1, 2, \ldots, N$ are independently distributed as multivariate normal random variables with a zero mean vector and a non-singular covariance matrix $\Sigma = \{\sigma_{ij}\}$.

The reduced form of Model (3.1) is

$$Y = -X\Gamma B^{-1} - Y_{-1}CB^{-1} + U_{-1}RB^{-1} + \epsilon B^{-1} \quad . \tag{3.3a}$$

Equation (3.3a) is equivalent to

$$\begin{aligned} Y &= -X\Gamma B^{-1} - Y_{-1}CB^{-1} + Y_{-1}BRB^{-1} + X_{-1}\Gamma RB^{-1} + Y_{-2}CRB^{-1} + \epsilon B^{-1} \\ &= X\pi_1 + Y_{-1}\pi_2 + X_{-1}\pi_3 + Y_{-2}\pi_4 + V \\ &= F\pi + V \end{aligned} \tag{3.3b}$$

where $V = \epsilon B^{-1}$, $F = [X, Y_{-1}, X_{-1}, Y_{-2}]$, $Y_{-2}$ is the matrix of elements of $Y$ lagged two periods, and $\pi$ is partitioned to conform to the partition of $F$.

The $i^{th}$ structural equation of Model (3.1) may be written as

$$y_i = Z_i \delta_i + u_i , \qquad i = 1, 2, \ldots, \ell \tag{3.4}$$

where

$$Z_i = [Y_i, X_i, Y_{(-1),i}] , \qquad \delta_i' = [B_i', \Gamma_i', C_i'] ;$$

$y_i$ is the $i^{th}$ column of $Y$; $Y_i$ is the $N \times \ell_i$ matrix of the explanatory endogenous variables in the $i^{th}$ equation; $X_i$ is the $N \times \Lambda_i$ matrix of exogenous variables appearing in the $i^{th}$ equation and $Y_{(-1),i}$ is the $N \times \ell_{k_i}$ matrix of observations on endogenous variables lagged one period. The coefficients of $Y_i$, $X_i$, $Y_{(-1),i}$ are the $i^{th}$ column of $B$, $\Gamma$, and $C$, respectively. The error vector $u_i$ is the $i^{th}$ column of $U$ and is assumed to satisfy

$$u_i = \rho_i u_{i-1} + \epsilon_i \tag{3.5}$$

where $\epsilon_i$ is the $i^{th}$ column of $\epsilon$ defined in (3.2).

The complete model of (3.4) may compactly be written as

$$\underset{\sim}{y} = Z\delta + \underset{\sim}{u} , \tag{3.6}$$

where

$$\underset{\sim}{y}' = [y_1', y_2', \ldots, y_\ell'] , \qquad Z = \text{Block diag } [Z_1, Z_2, \ldots, Z_\ell] ,$$

$$\delta' = [\delta_1', \delta_2', \ldots, \delta_\ell'] \quad \text{and} \quad \underset{\sim}{u} = [u_1', u_2', \ldots, u_\ell']' \; .$$

Asymptotic properties of estimators for $[\delta_i', \rho_i]$ , $i = 1, 2, \ldots, \ell$, are obtained under the following assumptions:

Assumption 3.1: Every equation is identified and the parameter matrix B is nonsingular.

Assumption 3.2: The elements of the matrices $X_i$, $i = 1, 2, \ldots, \ell$, are uniformly bounded fixed vectors.

Assumption 3.3: The matrices

$$\underset{N \to \infty}{\text{plim}} \; \frac{1}{N} \bar{Z}'\bar{Z} \quad \text{and} \quad \underset{N \to \infty}{\text{plim}} \; \frac{1}{N} \bar{Z}'\Omega^{-1}\bar{Z} \quad \text{are}$$

finite and positive definite, where

$$\bar{Z} = \text{Block diag } [\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_\ell]$$

$$\bar{Z}_i = [\bar{\bar{Y}}_i, X_i, Y_{i(-1),i}] \; ,$$

$\bar{\bar{Y}}_i$ is the $N \times \ell_i$ submatrix of $\bar{\bar{Y}} = F\pi$ defined in (3.3b) and $E(\underset{\sim}{u}\underset{\sim}{u}') = \Omega$ .

Assumption 3.4: The matrices

$$\underset{N \to \infty}{\text{plim}} \; \frac{1}{N} \bar{H}'\bar{H} \quad \text{and} \quad \underset{N \to \infty}{\text{plim}} \; \frac{1}{N} \bar{H}'(\not{\Sigma}^{-1} \otimes I)\bar{H} \quad \text{are}$$

finite and positive definite, where $\not{\Sigma} = E(\epsilon_{t.}'\epsilon_{t.})$

$$\bar{H} = \text{Block diag } [\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_\ell] \; ,$$

$$\bar{H}_{1i} = [(1-\rho_i^2)^{\frac{1}{2}} \bar{\bar{Y}}_{1i}, (1-\rho_i^2)^{\frac{1}{2}} X_{1i}, (1-\rho_i^2)^{\frac{1}{2}} Y_{(0),i}, 0],$$

$$\bar{H}_{ti} = [\ \bar{\bar{Y}}_{ti} - \rho_i Y_{t-1,i},\ X_{ti} - \rho_i X_{t-1,i}\ ,$$

$$Y_{(t-1),i} - \rho_i Y_{(t-2),i},\ u_{t-1,i}]$$

$$t = 2, 3, \cdots, N\ ,$$

$\bar{\bar{Y}}_{ti}$ is the $t^{th}$ row of $\bar{\bar{Y}}_i$ defined in Assumption 3.3.

$$T_i = \begin{bmatrix} (1-\rho_i^2)^{\frac{1}{2}} & 0 & 0 & --- & 0 & 0 & 0 \\ -\rho_i^2 & 1 & 0 & --- & 0 & 0 & 0 \\ 0 & -\rho_i & 1 & --- & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & -\rho_i & 1 & 0 \\ 0 & 0 & 0 & --- & 0 & -\rho_i & 1 \end{bmatrix}, \tag{3.7}$$

and $u_{-1,i}$ is the vector of one period lagged values of $u_i$ .

Assumption 3.5: The matrices

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N-h} X'_{t.} X_{t+h.} = D^{(h)} < \infty \text{ exist for}$$

$h = 0, 1, 2, \cdots$, where $X_{t.}$ is the $t^{th}$ row of $X$ .

Assumption 3.6: The Model (3.1) is a stable dynamic model, i.e., all roots of the polynomial equation in $q$

$$|Iq^2 - \pi_2'q - \pi_4'| = 0$$

are less than one in absolute value, where $\pi_2'$ and $\pi_4'$ are defined following (3.3b).

B. Limited Information Estimators

To estimate the unknown structural parameters of the dynamic simultaneous equation model with autocorrelated errors we proceed as follows. First, we construct an initial estimator whose error is $O_p(N^{-\frac{1}{2}})$. Then using this initial estimator, we construct a revised estimator. Using this approach, we now present three single equation estimators for the parameters of Model (3.4, 3.5). The first estimator is called autoregressive two-stage least squares I(A2SLSI).

The estimation procedure of the $i^{th}$ equation contains the following steps:

(1) Treating the lagged endogenous variables as endogenous variables, the method of instrumental variables (24) or the modified limited information maximum likelihood estimator (25) is used to estimate the parameters $\delta_i = [B_i', \Gamma_i', C_i']'$. Either the set of all exogenous variables $\psi = [X, X_{-1}]$ or a subset of these variables will serve. Using these initial estimates, we obtain the residuals,

$$\hat{u}_i = y_i - Z_i\hat{\delta}_i \tag{3.8}$$

where $\hat{\delta}_i$ are the instrumental variable (or MLIML) estimators.

The autocorrelation coefficient is estimated by

$$\hat{\rho}_i = \frac{\sum_{t=2}^{N} \hat{u}_{it}\, \hat{u}_{i,t-1}}{\sum_{t=1}^{N} \hat{u}^2_{i,t-1}} \; . \tag{3.9}$$

(2) Expanding the $i^{th}$ structural equation in a Taylor series about $[\hat{\delta}_i, \hat{\rho}_i]$ and rearranging the terms yields

$$(1-\hat{\rho}_i^2)^{\frac{1}{2}}\, Y_{1i} = (1-\hat{\rho}_i^2)^{\frac{1}{2}} Y_{ti}\, B_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}}\, X_{1i}\Gamma_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}}\, Y_{(0)i}C_i + \epsilon_{1i}$$

$$(Y_{ti} - \hat{\rho}_i\, Y_{t-1,i}) \doteq (Y_{ti} - \hat{\rho}_i Y_{t-1,i})B_i + (X_{ti} - \hat{\rho}_i\, X_{t-1,i})\Gamma_i \tag{3.10}$$

$$+ (Y_{(t-1),i} - \hat{\rho}_i Y_{(t-2),i})C_i + \Delta\rho_i\, \hat{u}_{t-1,i} + \epsilon_{ti}$$

$$t = 2, 3, \ldots, N$$

where $\hat{u}_{t-1,i}$ is the $(t-1)^{st}$ elements of $\hat{u}_i$ defined in (3.8) and $u_{oi} \equiv 0$ .

(3) Apply two-stage least squares to (3.10). At the first stage, estimate $Y_i$ by $\hat{Y}_i = F(F'F)^{-1}F'Y_i$ where $F = [X, X_{-1}, Y_{-1}, Y_{-2}]$ . The final estimator is given by

$$\tilde{W}_i = \begin{bmatrix} \tilde{\delta}_i \\ \Delta\tilde{\rho}_i \end{bmatrix} = \begin{bmatrix} \tilde{B}_i \\ \tilde{T}_i \\ \tilde{C}_i \\ \Delta\tilde{\rho}_i \end{bmatrix} = (\hat{H}_i' \hat{H}_i)^{-1} \hat{H}_i' \hat{T}_i y_i \qquad (3.11)$$

where $\hat{H}_i = [\hat{E}_i, \hat{T}_i X_i, \hat{T}_i Y_{(-1),i}, \hat{u}_{-1,i}]$ ,

$$\hat{E}_i = \begin{bmatrix} (1-\hat{\rho}_i^2)^{\frac{1}{2}} \hat{Y}_{1i} \\ \hat{Y}_{2i} - \hat{\rho}_i Y_{1i} \\ \vdots \\ \hat{Y}_{Ni} - \hat{\rho}_i Y_{N-1,i} \end{bmatrix}$$

and $\hat{T}_i$ is $T_i$ defined in (3.7) evaluated at $\rho_i = \hat{\rho}_i$ .

Hence, the improved estimator of $\rho_i$ is $\tilde{\rho}_i = \hat{\rho}_i + \Delta\tilde{\rho}_i$ where $\Delta\tilde{\rho}_i$ is the last element of $\tilde{w}_i$ defined in (3.11). The large sample covariance matrix of $(\tilde{\delta}_i, \tilde{\rho}_i)$ is estimated by $(\hat{H}_i' \hat{H}_i)^{-1} s_i^2$ where $s_i^2 = [N - \Lambda_i - \ell_i - 1]^{-1} \tilde{\epsilon}_i' \tilde{\epsilon}_i$, $\tilde{\epsilon}_i = \tilde{T}_i y_i - \tilde{T}_i Z_i \tilde{\delta}_i$ , and $\tilde{T}_i$ is $T_i$ evaluated at $\rho_i = \tilde{\rho}_i$ and $\tilde{\delta}_i$ is defined in (3.11).

The shortcoming of this estimator is the large number of predetermined variables used in the first-stage estimation. In estimating large econometric models, the number of predetermined variables may exceed the sample size.

The next estimator is proposed to reduce the number of predetermined variables used in the first stage estimation of A2SLSI. This estimator is called autoregressive two-stage least squares option II(A2SLSII). The procedure is as follows:

(1) Use the procedures outlined in Step (1) of A2SLSI on each of the $\ell$ structural equations (3.4) to obtain initial estimates for $\hat{\delta}_i$, $\hat{u}_i$, and $\hat{\rho}_i$ $i = 1, 2, \ldots, \ell$.

(2) Create the $N \times \ell$ matrix of estimated endogenous variables from the derived reduced form

$$\overset{\approx}{Y} = -X\hat{\Gamma}\hat{B}^{-1} - Y_{-1}\hat{C}\hat{B}^{-1} + \hat{U}_{-1}\hat{R}\hat{B}^{-1}, \tag{3.12}$$

where $\hat{B}$, $\hat{\Gamma}$, $\hat{C}$, $\hat{R}$ and $\hat{U}_{-1}$ are obtained from Step 1. Expanding the $i^{th}$ equation in a Taylor series about $(\hat{\delta}_i', \hat{\rho}_i)$ and substituting $\overset{\approx}{Y}_{ti}$ for $Y_{ti}$ yields

$$\hat{\epsilon}_{1i} = (1-\hat{\rho}_i^2)^{\frac{1}{2}}\overset{\approx}{Y}_{1i}\,\Delta B_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}}X_{1i}\,\Delta\Gamma_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}}Y_{(o)i}\Delta C_i + \epsilon_{1i} + \tilde{e}_{1i}\,\Delta B_i$$

$$\hat{\epsilon}_{ti} = (\overset{\approx}{Y}_{ti} - \hat{\rho}_i Y_{t-1,i})\,\Delta B_i + (X_{ti} - \hat{\rho}_i X_{t-1,i})\,\Delta\Gamma_i + (Y_{(t-1),i} - \hat{\rho}_i Y_{(t-2),i})\,\Delta C_i$$

$$+ \Delta\rho_i\hat{u}_{t-1,i} + \epsilon_{ti} + \tilde{e}_{ti}\,\Delta B_i + O_p(N^{-1})$$

$$t = 2, 3, \ldots, N$$

where

$\hat{\epsilon}_{ti}$ is the $t^{th}$ element of $\hat{\epsilon}_i = \hat{T}_i y_i - \hat{T}_i Z_i \hat{\delta}_i$ ,

$[\Delta B_i', \Delta\Gamma_i', \Delta C_i', \Delta\rho_i] \equiv [(B_i - \hat{B}_i)', (\Gamma_i - \hat{\Gamma}_i)', (C_i - \hat{C}_i)', (\rho_i - \hat{\rho}_i)]$

$\hat{u}_{t-1,i}$ is given in (3.8) and $\tilde{e}_{ti} = Y_{ti} - \tilde{\tilde{Y}}_{ti}$ .

(3) The one-step estimator of $[\delta_i', \rho_i]$ is

$$\begin{pmatrix} \tilde{\tilde{\delta}}_i \\ \tilde{\tilde{\rho}}_i \end{pmatrix} = \begin{pmatrix} \tilde{\tilde{B}}_i \\ \tilde{\tilde{\Gamma}}_i \\ \tilde{\tilde{C}}_i \\ \tilde{\tilde{\rho}}_i \end{pmatrix} = \begin{pmatrix} \hat{B}_i \\ \hat{\Gamma}_i \\ \hat{C}_i \\ \hat{\rho}_i \end{pmatrix} + \begin{pmatrix} \Delta \tilde{\tilde{B}}_i \\ \Delta \tilde{\tilde{\Gamma}}_i \\ \Delta \tilde{\tilde{C}}_i \\ \Delta \tilde{\tilde{\rho}}_i \end{pmatrix}$$

$$= (\tilde{\tilde{H}}_i' \tilde{\tilde{H}}_i)^{-1} \tilde{\tilde{H}}_i' \hat{\epsilon}_i \quad , \tag{3.13}$$

where

$$\tilde{\tilde{H}}_i = [\tilde{E}_i, \hat{T}_i X_i, \hat{T}_i Y_{(-1),i}, u_{-1,i}] ,$$

$$\tilde{E}_i = [(1-\hat{\rho}_i^2)^{\frac{1}{2}} \tilde{\tilde{Y}}_{1i}', (\tilde{\tilde{Y}}_{2i} - \hat{\rho}_i Y_{1i})', \ldots (\tilde{\tilde{Y}}_{Ni} - \hat{\rho}_i Y_{N-1,i})']' .$$

The asymptotic covariance matrix of $(\tilde{\tilde{\delta}}_i, \tilde{\tilde{\rho}}_i)$ is estimated by

$(\widetilde{\widetilde{H}}_i' \widetilde{\widetilde{H}}_i)^{-1} \overline{S}_i^2$ where $\overline{S}_i^2 = [N - (\ell_i + \Lambda_i + 1)]^{-1} \widetilde{\widetilde{\epsilon}}_i' \widetilde{\widetilde{\epsilon}}_i$ , $\widetilde{\widetilde{\epsilon}}_i = \widetilde{\widetilde{T}}y_i - \widetilde{\widetilde{T}}_i Z_i \widetilde{\widetilde{\delta}}_i$, $\widetilde{\widetilde{T}}_i$ is $T_i$ evaluated at $\rho_i = \widetilde{\widetilde{\rho}}_i$ and $\widetilde{\widetilde{\delta}}_i$ obtained from (3.13).

This estimator has two shortcomings: (1) to obtain $\widetilde{\widetilde{Y}}_i$ from the derived reduced form, we need initial estimates for the parameters of the complete model, and (2) the performance of this estimator may be affected by mis-specification of other equations in the model.

The last estimator we suggest is called the transformed instrumental variable estimator. This estimator is motivated by the article of Amemiya and Fuller (2, p. 514). The procedure consists of the following three steps:

(1) The first step is the same as Step 1 of A2SLSII.

(2) The estimated endogenous variables are constructed as per (3.12) and the instrumental variable matrix $\widetilde{\widetilde{H}}_i = [\widetilde{E}_i, \hat{T}_i X_i, \hat{T}_i Y_{(-1),i}, \hat{u}_{-1,i}]$ computed.

(3) The transformed instrumental variable estimator (TIV) is given by

$$\overline{\overline{W}}_i = (\widetilde{\widetilde{H}}_i' H_i)^{-1} \widetilde{\widetilde{H}}_i' \hat{T}_i y_i \ , \tag{3.14}$$

where

$$H_i = [\hat{T}_i Y_i, \hat{T}_i X_i, \hat{T}_i Y_{(-1),i}, \hat{u}_{-1,i}] \ ,$$

$\widetilde{\widetilde{H}}_i$ is given in (3.13) ,

$\hat{T}_i$ and $\hat{u}_{-1,i}$ are defined in (3.11), (3.8) .

## C. Full Information Estimators

In this section we suggest three full information estimators which are the generalizations of the three limited information estimators of Section B. The first estimator is referred to as autoregressive three-stage least squares I (A3SLSI). This procedure consists of the following operations:

(1) Apply two-stage least squares or modified limited information to each of the $\ell$ structural equations treating the lagged endogenous variables as endogenous to obtain initial estimates, $\hat{\delta}_i$, $\hat{u}_{-1,i}$ and $\hat{\rho}_i$ $i = 1, 2, \dots, \ell$.

(2) Expand the $i^{th}$ structural equation in a Taylor series about $[\hat{\delta}_i', \hat{\rho}_i]$ and rearrange the terms to yield

$$(1-\hat{\rho}_i^2)^{\frac{1}{2}} Y_{1i} \doteq (1-\hat{\rho}_i^2)^{\frac{1}{2}} Y_{1i}B_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}} X_{1i}\Gamma_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}} Y_{(0)i}C_i + \epsilon_{1i}$$

$$(Y_{ti} - \hat{\rho}_i Y_{t-1,i}) \doteq (Y_{ti} - \hat{\rho}_i Y_{t-1,i})B_i + (X_{ti} - \hat{\rho}_i X_{t-1,i})\Gamma_i$$

$$+ (Y_{(t-1),i} - \hat{\rho}_i Y_{(t-2),i})C_i + \Delta\rho_i \hat{u}_{t-1,i} + \epsilon_{ti}$$

$$t = 2, 3, \dots, N \qquad (3.15)$$

where $\hat{u}_{t-1,i}$ is the $(t-1)^{th}$ element of $\hat{u}_i$ defined in (3.8) and $u_{1i} \equiv 0$.

Estimate $Y_i$ by

$$\hat{Y}_i = F(F'F)^{-1} F' Y_i$$

where

$$F = [X, X_{-1}, Y_{-1}, Y_{-2}]$$

and replace $Y_i$ in (3.15) by $\hat{Y}_i$. The resulting system of equations can be written in matrix form,

$$\hat{T} \underset{\sim}{y} \doteq \hat{H}W + \tilde{\tilde{e}}B + \underset{\sim}{\epsilon} \tag{3.16}$$

where

$\hat{T} \equiv$ Block diag $[\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_\ell]$,

$\hat{T}_i$ is $T_i$ defined in (3.7) evaluated at $\rho_i = \hat{\rho}_i$ ,

$\hat{H}$ = Block diag $[\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_\ell]$ ,

$\hat{H}_i = [\hat{E}_i, \hat{T}_i X_i, \hat{T}_i Y_{(-1),i}, \hat{u}_{-1,i}]$ ,

$\hat{E}_i = [(1-\hat{\rho}_i^2)^{\frac{1}{2}} \hat{Y}'_{1i}, (\hat{Y}_{2i} - \hat{\rho}_i Y_{1i})', \ldots, (\hat{Y}_{Ni} - \hat{\rho}_i Y_{N-1,i})']'$ ,

$W = [W'_1, W'_2, \ldots, W'_\ell]'$ ,

$$W_i' = [B_i', \Gamma_i', C_i', \Delta\rho_i] = [\delta_i', \Delta\rho_i] ,$$

$$\Delta\rho_i = \rho_i - \hat{\rho}_i ,$$

$\underset{\sim}{y}$ is defined in (3.6) ,

$$\approx{e} = \text{Block diag } [\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_\ell] ,$$

$$\tilde{e}_i = Y_i - \hat{Y}_i .$$

(3) Using $\hat{\epsilon}_{ti} = \hat{u}_{ti} - \hat{\rho}_i \hat{u}_{t-1,i}$ where $\hat{\rho}_i$ was defined in (3.9) and $\hat{u}_i$ in (3.8) estimate the elements of the covariance matrix $\Sigma$ by

$$\hat{\sigma}_{ij} = (N-K_i)^{-\frac{1}{2}} (N-K_j)^{-\frac{1}{2}} \sum_{t=2}^{N} \hat{\epsilon}_{ti} \hat{\epsilon}_{tj}, \quad i,j = 1,2,\ldots,\ell . \tag{3.17}$$

Apply Aitken's generalized least squares to the system of equations (3.16). The estimator of $[W_1', W_2', \ldots, W_\ell']$ is given by

$$\tilde{W} = [\hat{H}' (\hat{\Sigma}^{-1} \otimes I) \hat{H}]^{-1} \hat{H}' (\hat{\Sigma}^{-1} \otimes I) \hat{T} \underset{\sim}{y} \tag{3.18}$$

where the elements of $\hat{\Sigma}$ are defined in (3.17). The estimates of $B_i$ , $\Gamma_i$, $C_i$ are given by the proper elements of $\tilde{W}_i$ and the improved estimator of $\rho_i$ is $\tilde{\rho}_i = \hat{\rho}_i + \Delta\tilde{\rho}_i$ where $\Delta\rho_i$ is the last element of $\tilde{W}_i$.

The second full information estimator is called autoregressive three-stage least squares II (A3SLSII). This estimator is an iterative estimator which can be used to obtain the full information maximum likelihood estimator for Models (3.4, 3.5). The procedure is as follows:

(1) Follow Step (1) of A3SLSI to obtain initial estimates, $\hat{\delta}_i$, $\hat{u}_{-1,i}$ and $\hat{\rho}_i$, $i = 1, 2, \ldots, \ell$.

(2) Create the $N \times \ell$ matrix of estimated endogenous variables from the derived reduced form

$$\approx{Y} = -X\hat{\Gamma}\hat{B}^{-1} - Y_{-1}\hat{C}\hat{B}^{-1} + \hat{U}_{-1}\hat{R}\hat{B}^{-1}, \tag{3.19}$$

where $\hat{B}$, $\hat{\Gamma}$, $\hat{C}$, $\hat{R}$ and $\hat{U}_{-1}$ are obtained from Step 1. Expanding the $i^{th}$ equation in a Taylor series about $[\hat{\delta}_i', \hat{\rho}_i]$, retaining only the first order terms, and substituting $\approx{Y}_{ti}$ for $Y_{ti}$ yields

$$\hat{\epsilon}_{1i} \doteq (1-\hat{\rho}_i^2)^{\frac{1}{2}}\approx{Y}_{1i}\Delta B_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}}X_{1i}\Delta\Gamma_i + (1-\hat{\rho}_i^2)^{\frac{1}{2}}Y_{(0)i}\Delta C_i + \tilde{V}_{1i}\Delta B_i + \epsilon_i$$

$$\hat{\epsilon}_{ti} \doteq (\approx{Y}_{ti} - \hat{\rho}_i Y_{t-1,i})\Delta B_i + (X_{ti} - \hat{\rho}_i X_{t-1,i})\Delta\Gamma_i + (Y_{(t-1),i} - \rho_i Y_{(t-2),i})\Delta C_i$$

$$+ \Delta\rho_i \hat{u}_{t-1,i} + \tilde{V}_{ti}\Delta B_i + \epsilon_{ti}, \qquad t = 2, 3, \ldots, N,$$

where

$\hat{\epsilon}_{ti}$ is the $t^{th}$ element of $\hat{\epsilon}_i = \hat{T}_i Y_i - \hat{T}_i Z_i \hat{\delta}_i$ ,

$$[\Delta B_i', \Delta\Gamma_i', \Delta C_i', \Delta\rho_i] \equiv [(B_i - \hat{B}_i)', (\Gamma_i - \hat{\Gamma}_i)', (C_i - \hat{C}_i)', (\rho_i - \hat{\rho}_i)]'$$

and $\tilde{V}_{ti} = Y_{ti} - \approx{Y}_{ti}$ . The resulting system of equations can be written in matrix form

$$\underset{\sim}{\hat{\epsilon}} = \approx{H} \Delta W + \underset{\sim}{\epsilon} + \tilde{V} \Delta B \tag{3.20}$$

where

$\underset{\sim}{\hat{\epsilon}} = [\hat{\epsilon}_1', \hat{\epsilon}_2', \ldots, \hat{\epsilon}_\ell']'$ ,

$\hat{\epsilon}_i = \hat{T}_i y_i - \hat{T}_i Z_i \hat{\delta}_i$ ,

$\approx{H} \equiv$ Block diag $[\approx{H}_1, \approx{H}_2, \ldots, \approx{H}_\ell]$ ,

$\approx{H}_i = [\tilde{E}_i, \hat{T}_i X_i, \hat{T}_i Y_{(-1),i}, \hat{u}_{-1,i}]$ ,

$\tilde{E}_i = [(1-\hat{\rho}_i^2)^{\frac{1}{2}} \approx{Y}_{1i}', (\approx{Y}_{2i} - \hat{\rho}_i Y_{1i})', \ldots, (\approx{Y}_{Ni} - \hat{\rho}_i Y_{N-1,i})']'$ ,

$\Delta W = [\Delta W_1', \Delta W_2', \ldots, \Delta W_\ell']$ ,

$$\tilde{\tilde{V}} \equiv \text{Block diag}\,[\tilde{V}_1,\ \tilde{V}_2,\ \ldots,\ \tilde{V}_\ell] \ .$$

(3) Estimate the elements of the covariance matrix $\Sigma$ , by (3.17). The estimator of $[(\Delta W_1)',\ (\Delta W_2)',\ \ldots,\ (\Delta W_\ell)']'$ is given by

$$\Delta \tilde{\tilde{W}} = [\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)\tilde{\tilde{H}}]^{-1}\, \tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)\hat{\underset{\sim}{\varepsilon}} \ , \tag{3.21}$$

where

$$\Delta W_i' = [\ \Delta B_i',\ \Delta \Gamma_i',\ \Delta C_i',\ \Delta \rho_i\ ] \ .$$

The estimator of $(\delta_i',\ \rho_i)$ is $(\hat{\delta}_i',\ \hat{\rho}_i) + \Delta\tilde{\tilde{W}}_i$ . The covariance matrix of this estimator is estimated by the inverse matrix of the last step regression , that is by $[\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)\tilde{\tilde{H}}]^{-1}$ .

The third full information estimator is named full information transformed instrumental variable (FITIV). The procedure consists of three steps:

(1) Use Step (1) of A3SLSI to obtain initial estimators of $\delta_i$, $\rho_i$ and $u_{i-1}$ , $i = 1, 2, \ldots, \ell$ .

(2) Use Step (2) of A3SLSII to obtain initial estimate of $\tilde{\tilde{Y}}$ defined in (3.19).

(3) We write the system to be estimated as

$$\begin{bmatrix} \hat{T}_1 y_1 \\ \hat{T}_2 y_2 \\ \vdots \\ \hat{T}_\ell y_\ell \end{bmatrix} = \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & H_\ell \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_\ell \end{bmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_\ell \end{pmatrix} + O_p(N^{-1}) \tag{3.22}$$

and define the estimator by

$$\bar{\bar{W}} = (\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)H)^{-1} \tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)\hat{T}\underset{\sim}{y} \tag{3.23}$$

where

$\tilde{\tilde{H}}$ is defined in (3.20),

$H = \text{Block diag}\,[H_1, H_2, \cdots, H_\ell]$ ,

$H_i$ is defined in (3.14) , and

$\hat{T}_i$ is defined in (3.16) .

Since $(\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)H)^{-1}$ is non-symmetric, two problems are created: (1) a separate computation is required to obtain a symmetric covariance matrix, (2) large rounding errors may be encountered in the inversion of the nonsymmetric matrix.

By proper modification these three full information estimators are applicable to models with vector autoregressive errors and (or) higher order autoregressive errors.

## D. The Properties of the Estimators

In this section, we show that the difference between the full estimators and the true parameters normalized by $N^{\frac{1}{2}}$ converges in distribution to the multivariate normal distribution with zero mean vector and covariance matrix $A = \underset{N \to \infty}{\text{plim}} \; [N^{-1} \bar{H}'(\not{\Sigma}^{-1} \otimes I) \bar{H}]^{-1}$. The properties of the limited information estimator follow from the properties of the full information estimator under the assumption that contemporary correlations among equations are zero.

We first prove several lemmas required in the main theorems of this section.

<u>Lemma 3.1</u>: Given Model (3.4), (3.5) and Assumptions 3.1 through 3.6, then

$$(1) \quad \frac{1}{N} \sum_{t=1}^{N} X_{t.}' \, \epsilon_{tj} = 0_p(N^{-\frac{1}{2}}) , \tag{3.24a}$$

$$(2) \quad \frac{1}{N} \sum_{t=1}^{N} Y_{t-p.}' \, \epsilon_{tj} = 0_p(N^{-\frac{1}{2}}), \quad p \geq 1 , \tag{3.24b}$$

$$(3) \quad \frac{1}{N} \sum_{t=1}^{N} u_{t-1,i} \, \epsilon_{tj} = 0_p(N^{-\frac{1}{2}}) , \quad i,j = 1, 2, \ldots, \ell, \tag{3.24c}$$

where $X_{t.}'$, $Y_{t-p.}'$ are $t^{th}$ columns of $X'$ and $Y_{-p}'$ defined in (3.3a).

<u>Proof</u>: By Assumptions 3.2 and 3.3, we have

$$E\left( \frac{1}{N} \sum_{t=1}^{N} X'_{t.}\ \epsilon_{tj} \right) = 0$$

$$\mathrm{Var}\left( \frac{1}{N} \sum_{t=1}^{N} X'_{t.}\ \epsilon_{tj} \right) = \sigma_{jj} \frac{1}{N^2} \sum_{t=1}^{N} X'_{t.}\ X_{t.}$$

$$= O(N^{-1}) \ .$$

Treating the first three observations on $Y_t$ as fixed and using Assumption 3.6, the vector $Y'_{t.}$ can be expressed as

$$Y'_{t.} = \sum_{\tau=0}^{\infty} P_\tau\ \pi'_1\ X'_{t-\tau.} + \sum_{\tau=0}^{\infty} P_\tau\ \pi'_3\ X'_{t-\tau-1.} + \sum_{\tau=0}^{\infty} P_\tau\ V'_{t-\tau} \ .$$

Where the $\ell \times \ell$ matrices $P_\tau$ satisfy the recurrence relation,

$$P_\tau - \pi'_2\ P_{\tau-1} - \pi'_4\ P_{\tau-2} = 0$$

subject to the initial conditions

$$P_o = I, \quad P_{-1} = P_{-2} = 0 \ .$$

It follows that $E(Y'_{t-p.}\ \epsilon_{tj}) = 0$ for $p \geq 1$ , $t = 1, 2, \ldots, N$ and $i,j$ 1, 2, ..., $\ell$ .

Now $Y'_{t-p.}\epsilon_{tj}$ is uncorrelated with $Y'_{t-p.}\epsilon_{t'j}$ for $t \neq t'$ . By Assumption 3.3, it follows that

$$\mathrm{Var}\left( \frac{1}{N} \sum_{t=1}^{N} Y'_{t-p.} \, \epsilon_{tj} \right) = \sigma_{jj} \frac{1}{N^2} \sum_{t=1}^{N} E(Y'_{t-p.} \, Y_{t-p.})$$

$$= O(N^{-1}) \,, \qquad p = 1, 2$$

and conclusion (2) is established. Conclusion (3) follows by similar arguments. □

Lemma 3.2: Given Models (3.4), (3.5), Assumptions 3.1 through 3.6, and initial estimators $\hat{\delta}_i$ such that $(\hat{\delta}_i - \delta_i) = O_p(N^{-\frac{1}{2}})$, $i = 1, 2, \ldots, \ell$, then

$$\hat{\rho}_i = \rho_i + O_p(N^{-\frac{1}{2}}) \,, \qquad i = 1, 2, \ldots, \ell \,, \tag{3.25}$$

where $\hat{\rho}_i$ is given by (3.9).

Proof: From (3.8) and (3.5) we have

$$\hat{u}_i = u_i - Z_i(\hat{\delta}_i - \delta_i) \,,$$

where $\hat{u}_i$ is the $N \times 1$ vector of the estimated residuals defined in (3.8). Let $\hat{u}_{-1,i} = (0, \hat{u}_{1i}, \hat{u}_{2i}, \ldots, \hat{u}_{N-1,i})$. Using Assumption 3.2, Lemma 3.1 and $(\hat{\delta}_i - \delta_i) = O_p(N^{-\frac{1}{2}})$, we have

$$\hat{\rho}_i = \frac{1}{N} \hat{u}'_i \, \hat{u}_{-1,i} \left( \frac{1}{N} \hat{u}'_{-1,i} \, \hat{u}_{-1,i} \right)^{-1}$$

$$= \rho_i + \frac{1}{N} u'_{-1,i}\, \epsilon_i \left( \frac{1}{N} u'_{-1i} u_{-1i} \right)^{-1} + O_p(N^{-\frac{1}{2}})$$

$$= \rho_i + O_p(N^{-\frac{1}{2}}) \quad . \qquad \square$$

Lemma 3.3: Given Assumptions 3.1 through 3.6 and initial estimators $\hat{\delta}_i$ , satisfying $(\hat{\delta}_i - \delta_i) = O_p(N^{-\frac{1}{2}})$ , then

$$\hat{\sigma}_{ij} = \sigma_{ij} + O_p(N^{-\frac{1}{2}}) , \qquad i,j = 1, 2, \ldots, \ell , \tag{3.26}$$

where $\hat{\sigma}_{ij}$ is defined in (3.17) .

Proof: We have

$$\hat{\epsilon}_{ti} = \hat{u}_{ti} - \hat{\rho}_i \hat{u}_{t-1,i} , \qquad t = 2, 3, \ldots, N ,$$

and

$$\hat{\epsilon}_{ti} = \epsilon_{ti} - (Z_{ti} - \rho_i Z_{t-1,i})(\hat{\delta}_i - \delta_i) - u_{t-1,i}(\hat{\rho}_i - \rho_i)$$

$$+ (\hat{\rho}_i - \rho_i) Z_{t-1,i}(\hat{\delta}_i - \delta_i) .$$

It follows that

$$\frac{\sum_{t=2}^{N} \hat{\epsilon}_{ti}\, \hat{\epsilon}_{tj}}{(N-K_i)^{\frac{1}{2}}\,(N-K_j)^{\frac{1}{2}}} = \frac{1}{(N-K_i)^{\frac{1}{2}}\,(N-K_j)^{\frac{1}{2}}}\; \epsilon_i'\, \epsilon_j + O_p(N^{-\frac{1}{2}})$$

$$= \sigma_{ij} + O_p(N^{-\frac{1}{2}}) \; . \qquad \square$$

Lemma 3.4: Given Models (3.4), (3.5) and Assumptions 3.1 through 3.6, the autoregressive three-stage least squares estimator $I$ defined in (3.18) satisfies

$$N^{\frac{1}{2}}(\tilde{W} - W) = [N^{-1}\, \bar{H}'(\not\Sigma^{-1} \otimes I)\bar{H}\,]^{-1}\, [N^{-\frac{1}{2}}\, \bar{H}'(\not\Sigma^{-1} \otimes I)\underset{\sim}{\epsilon}] + O_p(N^{-\frac{1}{2}}) \; , \tag{3.27}$$

where

$$\tilde{W} - W = [(\tilde{W}_1 - W_1)', (\tilde{W}_2 - W_2)', \ldots, (\tilde{W}_\ell - W_\ell)']'$$

and

$$(\tilde{W}_i - W_i) = [(\tilde{B}_i - B_i)', (\tilde{\Gamma}_i - \Gamma_i)', (\tilde{C}_i - C_i)', (\tilde{\rho}_i - \rho_i)\,]'$$

$$i = 1, 2, \ldots, \ell \; .$$

Proof: Using the result that $\approx\atop{e}$ is orthogonal to $F$, where $\approx\atop{e}$ is defined in (3.16) and $F = [X, Y_{-1}, X_{-1}, Y_{-2}]$, we may write

$$\tilde{W} = W + [\hat{H}'(\hat{\not\Sigma}^{-1} \otimes I)\hat{H}]^{-1}\, \hat{H}^{-1}(\hat{\not\Sigma}^{-1} \otimes I)\underset{\sim}{\epsilon} + O_p(N^{-1}) \; .$$

Expanding $\hat{H}_i$ in a Taylor series, we find

$$\hat{H}_i \doteq \overline{H}_i + \{F(\hat{\pi}_i - \pi_i) - Y_i(\hat{\rho}_i - \rho_i),\ M_i(\rho_i^*)\, X_i(\hat{\rho}_i - \rho_i)\ ,$$

$$M_i(\rho_i^*) Y_{(-1),i}(\hat{\rho}_i - \rho_i),\ Z_{-1,i}(\hat{\delta}_i - \delta_i)\} + O_p(N^{-1}) \tag{3.28}$$

where $\overline{H}_i$ is defined in Assumption 3.4, $M_i(\rho_i^*)$ is the $N \times N$ matrix of partial derivatives of the elements of the matrix $T_i$ evaluated at $\rho_i = \rho_i^*$ where $\rho_i^*$ lies between $\rho_i$ and $\rho_i^*$.

Using Lemma 3.1, Lemma 3,2, and (3.28), it can be shown that

$$\frac{1}{N}\hat{H}_i'\hat{H}_j = \frac{1}{N}\overline{H}_i'\overline{H}_j + O_p(N^{-\frac{1}{2}}) \tag{3.29}$$

and

$$\frac{1}{N}\hat{H}_i'\epsilon_j = \frac{1}{N}\overline{H}_i'\epsilon_j + O_p(N^{-1}) \quad . \tag{3.30}$$

Letting $\cancel{\Sigma}^{-1} = \{\sigma^{ij}\}$ , we have

$$\hat{\sigma}^{ij} = \sigma^{ij} + O_p(N^{-\frac{1}{2}}) \ , \tag{3.31}$$

by Lemma 3.3.

From (3.29), (3.30), and (3.31), we have

$$\frac{1}{N}\hat{H}'(\hat{\not{Z}}^{-1}\otimes I)\hat{H} = \frac{1}{N}\bar{H}'(\not{Z}^{-1}\otimes I)\bar{H} + O_p(N^{-\frac{1}{2}})$$

$$\frac{1}{N}\hat{H}'(\hat{\not{Z}}^{-1}\otimes I)\underset{\sim}{\varepsilon} = \frac{1}{N}\bar{H}'(\not{Z}^{-1}\otimes I)\underset{\sim}{\varepsilon} + O_p(N^{-1})$$

therefore,

$$N^{\frac{1}{2}}(\tilde{W}-W) = [\frac{1}{N}\bar{H}'(\not{Z}^{-1}\otimes I)\bar{H}]^{-1}[N^{-\frac{1}{2}}\bar{H}'(\not{Z}^{-1}\otimes I)\underset{\sim}{\varepsilon}] + O_p(N^{-\frac{1}{2}}) . \qquad \square$$

Theorem 3.1: Let Models (3.4), (3.5) and Assumptions 3.1 through 3.6 hold. Then

$$N^{\frac{1}{2}}(\tilde{W}-W) \xrightarrow{\mathcal{L}} N(\underset{\sim}{0}, A) , \tag{3.32}$$

where

$$A = \underset{N\to\infty}{\text{plim}} [N^{-1}\bar{H}'(\not{Z}^{-1}\otimes I)\bar{H}]^{-1} .$$

Proof: From (3.27),

$$N^{\frac{1}{2}}(\tilde{W}-W) = [\frac{1}{N}\bar{H}'(\not{Z}^{-1}\otimes I)\bar{H}]^{-1} N^{-\frac{1}{2}}\bar{H}'(\not{Z}^{-1}\otimes I)\underset{\sim}{\varepsilon} + O_p(N^{-\frac{1}{2}}) .$$

A typical subvector of $\bar{H}'(\not{Z}^{-1}\otimes I)\underset{\sim}{\varepsilon}$ is $\sum_{j=1}^{\ell}\sigma^{ij}\bar{H}_i'\varepsilon_j$ .

By arguments used in the proof of Lemma 3.1 we can express $Y'_{t.}$ as a sum of fixed and random parts

$$Y'_{t.} = \xi_{t.} + \eta_{t.} ,$$

where

$$\xi_{t.} = P_0 \pi'_1 X'_{t.} + \sum_{\tau=0}^{\infty} [P_{\tau+1} \pi'_1 + P_\tau \pi'_3] X'_{t-\tau-1.}$$

$$\eta_{t.} = \sum_{\tau=0}^{\infty} P_\tau V'_{t-\tau,.} ,$$

and $P_\tau$ was defined in Lemma 3.1.

Let

$$\eta_{t.} = \eta^{(k)}_{t.} + b_{t.}$$

$$u_{t-1.} = u^{(k)}_{t-1,i} + C_{t-1,i} ,$$

where

$$b_{t.} = \sum_{\tau=k+1}^{\infty} P_\tau V'_{t-\tau.}$$

$$C_{t-1,i} = \sum_{\tau=k+1}^{\infty} \rho^\tau e_{t-\tau,i}$$

and define the normalized sums

$$N^{-\frac{1}{2}} \sum_{t=1}^{N} b_{t-p} \,.\, \epsilon_{tj} \qquad\qquad p = 1,\, 2$$

and

$$N^{-\frac{1}{2}} \sum_{t=1}^{N} C_{t-1,i}\, \epsilon_{tj} \,.$$

Now,

$$\mathrm{Var}(N^{-\frac{1}{2}} \sum_{t=1}^{N} b_{t-p} \,.\, \epsilon_{tj}) = \frac{\sigma_j^2}{N} \sum_{\tau=k+1}^{\infty} p_\tau \ngeq p_\tau' \qquad\qquad p = 1,\, 2$$

$$\equiv \; M_k$$

and $M_k \longrightarrow 0$ as $k \longrightarrow 0$ because $p_\tau$ is an absolutely summable sequence of matrices [see Fuller (24), p. 2-92]. Similarly,

$$\mathrm{Var}(N^{-\frac{1}{2}} \sum_{t=1}^{N} C_{t-1,i}\, \epsilon_{tj}) = \frac{\sigma_j^2}{N} \sum_{h=N-1}^{N-1} (N-h)( \sum_{\tau=k+1}^{\infty} \rho_i^\tau\, \rho_i^{\tau+h}\, \sigma_i^2)$$

$$\leq \frac{\sigma_j^2\, \sigma_i^2}{N} \sum_{\tau=k+1}^{\infty} \rho_i^{2\tau} + \frac{2\sigma_j^2 \sigma_i^2}{N} \sum_{h=0}^{N-1} \sum_{\tau=k+1}^{\infty} |\rho_i^\tau\, \rho_i^{\tau+h}|$$

$$\leq \frac{2\sigma_j^2 \sigma_i^2}{N} ( \sum_{\tau=k+1}^{\infty} |\rho_i^\tau|^2 ) \equiv G_k$$

and $G_k \longrightarrow 0$ as $k \longrightarrow \infty$ because $\{\rho_i^\tau\}$ is an absolute summable sequence. Therefore, we may choose $k$ so that the distribution of $N^{-\frac{1}{2}} \sum_{t=1}^{N} \eta_{t.}\, \epsilon_{jt}$ and $N^{-\frac{1}{2}} \sum_{t=1}^{N} u_{t-1.i}\, \epsilon_{tj}$ differ little from those of $N^{-\frac{1}{2}} \sum_{t=1}^{N} \eta_{t-p}^{(k)}\, .\epsilon_{tj}$ and $N^{-\frac{1}{2}} \sum_{t=1}^{N} u_{t-1,i}^{(k)}\, \epsilon_{jt}$, respectively. The variables $(\xi_{t-p\,.} + \eta_{t-p\,.}^{(k)})\, \epsilon_{jt}$, $p = 1, 2$, and $u_{t-1,i}^{(k)}\, \epsilon_{jt}$ are $(k+1)^{th}$ order dependent time series with mean zero and finite moments by Assumptions 3.2, 3.3, and 3.5. Therefore, following the arguments of Anderson (3) and Fuller (24), we can demonstrate that an arbitrary linear combination of

$$N^{-\frac{1}{2}} \sum_{t=1}^{N} (X_{ti} - \rho_i X_{t-1,i})\epsilon_{tj},\ N^{-\frac{1}{2}} \sum_{t=1}^{N} (\xi_{t-p.} + \eta_{t-p.})\epsilon_{tj}$$

$$\text{and } N^{-\frac{1}{2}} \sum_{t=1}^{N} (u_{t-1,i}\, \epsilon_{tj}) \ ,$$

$i,j = 1, 2, \ldots, \ell$, $p = 1, 2$ converges in distribution to a normal random variable. Hence, by the multivariate central limit theorem (24, 63), we have

$$N^{-\frac{1}{2}}\, \bar{H}'(\Sigma^{-1} \otimes I)\underset{\sim}{\epsilon} \xrightarrow{\mathcal{L}} N(0, \underset{N \to \infty}{\text{plim}} [\, \tfrac{1}{N} (\bar{H}'(\Sigma^{-1} \otimes I)\bar{H})]) \ .$$

Because $N^{\frac{1}{2}}(\tilde{W} - W) - [N^{-1}\, \bar{H}'(\Sigma^{-1} \otimes I)\bar{H}]^{-1}\, N^{-\frac{1}{2}}\, \bar{H}(\Sigma^{-1} \otimes I)\underset{\sim}{\epsilon} = O_p(N^{-\frac{1}{2}})$ and

$[N^{-1}\,\bar{H}'(\not\Sigma^{-1} \otimes I)\bar{H}] = O_p(1)$ we have

$$N^{\frac{1}{2}}\,(\tilde{W} - W) \xrightarrow{\mathcal{L}} N(\underset{\sim}{0}\,, \underset{N \to \infty}{\text{plim}}\,[N^{-1}\,\bar{H}'(\not\Sigma^{-1} \otimes I)\bar{H}]^{-1}) . \quad \square$$

We note that the result of Theorem 3.1 holds under weaker conditions. The assumption of normal errors can be replaced by the assumption of independent errors with finite $4 + \delta$, $\delta > 0$ moments and the assumption of bounded $X$'s can be weakened (24). Theorem 3.1 permits us to apply the techniques of usual regression theory in large samples. The covariance matrix for the estimated coefficients is estimated by the inverse of the last step regression.

In Theorem 3.2, we demonstrate that A3SLII has the same limiting distribution of A3SLSI.

<u>Theorem 3.2</u>: Given the assumptions of Theorem 3.1, then

$$N^{\frac{1}{2}}\,(\approx\!\!\!\!W - W) = N^{\frac{1}{2}}\,(\tilde{W} - W) + O_p(N^{-\frac{1}{2}}) ,$$

and $\overset{\approx}{W}$ is given by (3.21).

<u>Proof</u>: From (3.20) and (3.21) we have

$$(\overset{\approx}{W} - W) = [\,\frac{1}{N}\,\overset{\approx}{H}'(\hat{\not\Sigma}^{-1} \otimes I)\overset{\approx}{H}]^{-1}\,[\,\frac{1}{N}\,\overset{\approx}{H}'(\hat{\not\Sigma}^{-1} \otimes I)\underset{\sim}{\varepsilon}]$$

$$+ [\,\frac{1}{N}\,\overset{\approx}{H}'(\hat{\not\Sigma}^{-1} \otimes I)\overset{\approx}{H}]^{-1}[\frac{1}{N}\,\overset{\approx}{H}'(\hat{\not\Sigma}^{-1} \otimes I)\overset{\approx}{V}\,\Delta\,B]$$

where $\tilde{\tilde{V}}$ and $\Delta B$ are defined in (3.20). The $i^{th}$ subvector of $[\frac{1}{N}\tilde{\tilde{H}}'(\hat{\Sigma}^{-1}\otimes I)\tilde{\tilde{V}}\Delta B]$ is $\frac{1}{N}\sum_{j=1}^{\ell}\hat{\sigma}^{ij}\tilde{\tilde{H}}_i'\tilde{V}_j\Delta B_j$, where the $t^{th}$ row of $\tilde{\tilde{H}}_i$ is

$$\tilde{\tilde{H}}_{ti} = [(\tilde{\tilde{Y}}_{ti} - \hat{\rho}_i Y_{t-1,i}),\ (X_{ti} - \hat{\rho}_i X_{t-1,i}),\ (Y_{(t-1),i} - \hat{\rho}_i Y_{(t-2),i}),\ \hat{u}_{t-1,i}].$$

By a Taylor's expansion with remainder about the true parameters $(B, \Gamma, C, R)$, we obtain

$$\tilde{\tilde{H}}_i - \bar{H}_i = \{g[F, (\hat{\delta} - \delta), (\hat{R} - R), \delta, R]_{\cdot \ell_i} - Y_{-1,i}(\hat{\rho}_i - \rho_i),$$

$$M_i(\rho_i^*)X_{-1,i}(\hat{\rho}_i - \rho_i), M_i(\rho_i^*)Y_{(-2),i}(\hat{\rho}_i - \rho_i), Z_{-1,i}(\hat{\delta}_i - \delta_i)\} \quad (3.33)$$

where

$$\tilde{\tilde{Y}}_i - \bar{\bar{Y}}_i \equiv g\,[F, (\hat{\delta} - \delta), (\hat{R} - R), \delta, R]_{\cdot \ell_i}$$

$$= X\,\{[\Gamma B^{-1}(\hat{B} - B)B^{-1}] - [(\hat{\Gamma} - \Gamma)B^{-1}]\}_{\cdot \ell_i} + Y_{-1}\{[CB^{-1}(\hat{B} - B)B^{-1}]$$

$$+ [(\hat{C} - C)B^{-1}] - [BRB^{-1}(\hat{B} - B)B^{-1}] + [B(\hat{R} - R)B^{-1}] + [(\hat{B} - B)RB^{-1}]\}_{\cdot \ell_i}$$

$$+ X_{-1}\{[\Gamma(\hat{R}-R)B^{-1}] - [\Gamma R B^{-1}(\hat{B}-B)B^{-1}] + [(\hat{\Gamma}-\Gamma)RB^{-1}]\}_{\cdot \ell_i}$$

$$+ Y_{-2}\{[B^{-1}(\hat{R}-R)C] - [CRB^{-1}(\hat{B}-B)B^{-1}] + [(\hat{C}-C)RB^{-1}]\}_{\cdot \ell_i}$$

the subscript $\cdot \ell_i$ identifies the appropriate $\ell_i$ columns of the matrix in the { } brackets, and $\overline{H}_i$ is defined in Assumption 3.4.

Next, consider

$$\tilde{V} = Y - \tilde{\tilde{Y}} = [\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_\ell]$$

using the results

$$U_{-1} = \hat{U}_{-1} - Y_{-1}(\hat{B} - B) - X_{-1}(\hat{\Gamma} - \Gamma) - Y_{-2}(\hat{C} - C) ,$$

$$\hat{\Gamma}\hat{B}^{-1} - \Gamma B^{-1} = (\hat{\Gamma} - \Gamma)B^{-1} - \hat{\Gamma}\hat{B}^{-1}(\hat{B} - B)B^{-1} ,$$

$$\hat{C}\hat{B}^{-1} - CB^{-1} = (\hat{C} - C)B^{-1} - \hat{C}\hat{B}^{-1}(\hat{B} - B)B^{-1} ,$$

$$\hat{R}\hat{B}^{-1} - RB^{-1} = (\hat{R} - R)B^{-1} - \hat{R}\hat{B}^{-1}(\hat{B} - B)B^{-1} ,$$

$$V = \{\epsilon + [\tilde{\tilde{Y}}(\hat{B} - B) + X(\hat{\Gamma} - \Gamma) + Y_{-1}(\hat{C} - C) - \hat{U}_{-1}(\hat{R} - R)]$$

$$- [Y_{-1}(\hat{B} - B) + X_{-1}(\hat{\Gamma} - \Gamma) + Y_{-2}(\hat{C} - C)]R\}B^{-1}$$

and

$$\tilde{V}_i = \{\epsilon + [\approx{Y}(\hat{B} - B) + X(\hat{\Gamma} - \Gamma) + Y_{-1}(\hat{C} - C) - \hat{U}_{-1}(\hat{R} - R)]$$

$$- [Y_{-1}(\hat{B} - B) + X_{-1}(\hat{\Gamma} - \Gamma) + Y_{-2}(\hat{C} - C)]R\}B^{-1}_{\cdot \ell_i} \qquad (3.34)$$

where $B^{-1}_{\cdot \ell_i}$ denotes the $\ell \times \ell_i$ submatrix of $B^{-1}$. By Lemmas 3.1, 3.2, 3.3, (3.33), and (3.34), we have

$$\frac{1}{N} \hat{\sigma}^{ij} \approx{H}'_j \tilde{v}_i \Delta B_i = \frac{1}{N} \sigma^{ij} \bar{H}'_j \tilde{v}_i \Delta B_i + O_p(N^{-3/2})$$

$$= O_p(N^{-1}) \qquad (3.35)$$

$$\left( \frac{1}{N} \approx{H}'_i \approx{H}_j \right) = \frac{1}{N} \bar{H}'_i \bar{H}_j + O_p(N^{-\frac{1}{2}})$$

and

$$\hat{\sigma}^{ij} = \sigma^{ij} + O_p(N^{-\frac{1}{2}}) \quad .$$

Hence

$$\left[ \frac{1}{N} \approx{H}'(\hat{\Sigma}^{-1} \otimes I)\approx{H} \right]^{-1} = \left[ \frac{1}{N} \bar{H}'(\Sigma^{-1} \otimes I)\bar{H} \right]^{-1} + O_p(N^{-\frac{1}{2}}) \ ,$$

$$\frac{1}{N}\,\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)\underset{\sim}{\varepsilon} = \frac{1}{N}\,\bar{H}'(\Sigma^{-1} \otimes I)\underset{\sim}{\varepsilon} + O_p(N^{-1}) \quad ,$$

$$\frac{1}{N}\,\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)\tilde{\tilde{V}}\,\Delta B = O_p(N^{-1}) \quad . \tag{3.36}$$

Therefore

$$(\tilde{\tilde{W}} - W) = (\tilde{W} - W) + O_p(N^{-1}) \quad . \qquad \square$$

Theorem 3.3: Given the assumptions of Theorem 3.1, then

$$N^{\frac{1}{2}}(\bar{\bar{W}} - W) = N^{\frac{1}{2}}(\tilde{W} - W) + O_p(N^{-\frac{1}{2}})$$

where $\bar{\bar{W}}$ is defined in (3.23).

Proof: We have

$$(\bar{\bar{W}} - W) = [\,\frac{1}{N}\,\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)H]^{-1}\,[\,\frac{1}{N}\,\tilde{\tilde{H}}\,(\hat{\Sigma}^{-1} \otimes I)\underset{\sim}{\varepsilon}] + O_p(N^{-1}).$$

A typical submatrix of $N^{-1}\,\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)H$ is $N^{-1}\,\hat{\sigma}^{ij}\,\tilde{\tilde{H}}_i'\,H_j$ where $H_i$ is

$$[\hat{T}_i\,Y_i,\ \hat{T}_i\,X_i,\ \hat{T}_i\,Y_{(-1),i},\ \hat{u}_{-1,i}] \tag{3.37}$$

and $Y_i = \bar{\bar{Y}}_i + V_i$ where $\bar{\bar{Y}}_i$ is defined in Assumption 3.4.

By (3.33) and (3.37),

$$\frac{1}{N}\,\tilde{\tilde{H}}_i'\;H_j \;=\; \frac{1}{N}\,\bar{H}_i\;\bar{H}_j + O_p(N^{-\frac{1}{2}}) \tag{3.38}$$

and it follows that

$$N^{-1}\,[\tilde{\tilde{H}}\,(\hat{\Sigma}^{-1} \otimes I)H] \;=\; N^{-1}\,[\bar{H}(\Sigma^{-1} \otimes I)\bar{H}\,] + O_p(N^{-\frac{1}{2}}) \tag{3.39}$$

by Lemma 3.3 and (3.38).

Using arguments similar to those used to obtain (3.36), we have

$$N^{-1}\,[\tilde{\tilde{H}}'(\hat{\Sigma}^{-1} \otimes I)\underset{\sim}{\varepsilon}] \;=\; N^{-1}\,\bar{H}'(\Sigma^{-1} \otimes I)\underset{\sim}{\varepsilon} + O_p(N^{-1})\ . \tag{3.40}$$

The desired result is an immediate consequence of (3.39) and (3.40). ▯

We now compare the asymptotic covariance matrix of A3SLSI with the inverse of the information matrix associated with the full information maximum likelihood estimator. The models (3.4, 3.5) can be written compactly as

$$\underset{\sim}{y} \;=\; Z\,\delta + \underset{\sim}{u}$$

---

[1]The proof of asymptotic normality of the maximum likelihood estimator is not immediate since the observations are not independent. Bar-Shalom (7) has derived a set of seven regularity conditions under which the maximum likelihood estimator obtained from dependent observations are weakly consistent and asymptotically efficient.

$$\underset{\sim}{u} = (R \otimes I)\underset{\sim}{u}_{-1} + \underset{\sim}{\varepsilon} \tag{3.41}$$

or

$$\underset{\sim}{y} = Z\,\delta + (R \otimes I)(\underset{\sim}{y}_{-1} - Z_{-1}\,\delta) + \underset{\sim}{\varepsilon}$$

Under the assumption that the first 3 observations on Y are fixed, the likelihood function for (3.41) is

$$\ell = (2\pi)^{-\frac{\ell N}{2}} \det |\not{\Sigma} \otimes I|^{-\frac{1}{2}} e^{-\frac{1}{2}\underset{\sim}{\varepsilon}'(\not{\Sigma}^{-1} \otimes I)\underset{\sim}{\varepsilon}} . \tag{3.42}$$

The logarithmic likelihood function of (3.42) is defined as

$$\ell n\ L = \frac{1}{N}\ \ell n\ \ell = k + \frac{1}{2N} \det (\not{\Sigma}^{-1} \otimes I) - \frac{1}{2N}\underset{\sim}{\varepsilon}'(\not{\Sigma}^{-1} \otimes I)\underset{\sim}{\varepsilon} . \tag{3.43}$$

We consider the transformation form $\underset{\sim}{\varepsilon}$ to $\underset{\sim}{Y}$ defined in (3.41). The resulting logarithmic likelihood function now becomes

$$\ell n\ L = k + \frac{1}{2N} \det |\not{\Sigma}^{-1} \otimes I| + \frac{1}{N}\ \ell n\ |J| - \frac{1}{2N}[\underset{\sim}{y} - Z\delta - (R \otimes I)(\underset{\sim}{y}_{-1} - Z_{-1}\,\delta)]$$

$$(\not{\Sigma}^{-1} \otimes I)\,[\underset{\sim}{y} - Z\,\delta - (R \otimes I)(\underset{\sim}{y}_{-1} - Z_{-1}\,\delta)] \tag{3.44}$$

where $|J|$ is the absolute value of Jacobian of the transformation from $\underset{\sim}{\varepsilon}$ to $\underset{\sim}{y}$, that is

$$J = \det \left[ \frac{\partial\, \epsilon_{it}}{\partial\, y_{jt'}} \right]$$

where $\epsilon_{it}$ and $y_{jt'}$ represent typical scalar elements of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{y}$. In our case, ordering the equations of (3.41) in groups from the same time period, the Jacobian transformation from $\underset{\sim}{\epsilon}$ to $\underset{\sim}{y}$ is equal to $\det^N |B|$ because $|J|$ is an upper triangular matrix. Thus, Equation (3.44) becomes

$$\ell n\ L = k + \frac{1}{2} \det \not{\Sigma}^{-1} + \frac{1}{2}\ |\det B| - \frac{1}{2N} [\underset{\sim}{y} - Z\delta - (R \otimes I)(\underset{\sim}{y}_{-1} - Z_{-1}\delta)]$$

$$(\not{\Sigma}^{-1} \otimes I)\ [\underset{\sim}{y} - Z\delta - (R \otimes I)(\underset{\sim}{y}_{-1} - Z_{-1}\delta)] \quad . \tag{3.45}$$

We evaluate the matrix $V^{-1}$ where

$$V = -\operatorname*{plim}_{N \longrightarrow \infty} N^{-1} \begin{bmatrix} \dfrac{\partial^2\ \ell n\ L}{\partial W\ \partial W'} & \dfrac{\partial^2\ \ell n\ L}{\partial W\ \partial' \mathrm{Vec}(\not{\Sigma})} \\[2ex] \dfrac{\partial^2\ \ell n\ L}{\partial\ \mathrm{Vec}\ (\not{\Sigma}) \partial W'} & \dfrac{\partial^2\ \ell n\ L}{\partial\ \mathrm{Vec}(\not{\Sigma})\ \partial' \mathrm{Vec}(\not{\Sigma})} \end{bmatrix} \tag{3.46}$$

$$= \begin{bmatrix} V_1 & V_2 \\ V_2' & V_4 \end{bmatrix}$$

where $\text{Vec}(\not{Z})$ denotes the vector composed of the columns of $\not{Z}$.

The asymptotic covariance matrix of the FIML estimator of $(\delta_i, \rho_i)$, $i = 1, 2, \ldots, \ell$ is equal to $V_1^{-1} = [V_1 - V_2^{-1} V_4 V_2']^{-1}$. Under certain regularity conditions, Koopmans and his associates (53) have shown that $V_1^{-1}$ is also equal to $\underset{N \to \infty}{\text{plim}} -[N^{-1} \frac{\partial \ \ell n \ L^*}{\partial \gamma \ \partial \gamma'}]^{-1}$ where $\gamma$ is the parameter vector containing the elements of $\delta_i$ and $\rho_i$ $i = 1, 2, \ldots, \ell$ and $\ell n \ L^*$ is a concentrated likelihood function with respect to $\Sigma^{-1}$. In our case, $\ell n \ L^*$ is

$$\ell n \ L^* = k + \ell n \ |\det B| + \frac{1}{2} \ell n \det S \qquad (3.47)$$

where the $ij^{th}$ elements of $S$ is

$$S_{ij} = N^{-1} [y_i - Z_i \delta_i - \rho_i (y_{-1,i} - Z_{-1,i} \delta_i)]' \\ [y_j - Z_j \delta_j - \rho_j (y_{-1,j} - Z_{-1,j} \delta_j)] \ .$$

We first find the second partial derivatives of (3.47) with respect to $\delta_i$ and $\rho_j$ $i = j = 1, 2, \ldots, \ell$. Following Rothenberg and Leenders (65), using Lemma 3.1 and Assumptions 3.1 to 3.6, we obtain the matrix $V_1^{-1}$. The $ij^{th}$ submatrix of $V_1^{-1}$ is denoted by $Q_{ij}$

$$Q_{ij} = \underset{N \to \infty}{\text{plim}} \; N^{-1} \begin{bmatrix} Y_i^{*\prime} Y_i^* & Y_i^{*\prime} X_i^* & Y_i^{*\prime} Y_{(-1),i}^* & Y_i^{*\prime} u_{-1,i} \\ X_i^{*\prime} Y_i^* & X_i^{*\prime} X_i^* & X_i^{*\prime} Y_{(-1),i}^* & X_i^{*\prime} u_{-1,i} \\ Y_{(-1),i}^{*\prime} Y_i^* & Y_{(-1),i}^{*\prime} X_i^* & Y_{(-1),i}^{*\prime} Y_{(-1),i}^* & Y_{(-1),i}^{*\prime} u_{-1,i} \\ u_{-1,i}' Y_i^* & u_{-1,i}' X_i^* & u_{-1,i}' Y_{(-1),i}^* & u_{-1,i}' u_{-1,i} \end{bmatrix}$$

where

$$Y_i^* = \bar{\bar{Y}}_i - \rho_i Y_{-1,i} ,$$

$$X_i^* = X_i - \rho_i X_{-1,i} ,$$

$$Y_{(-1),i}^* = Y_{(-1),i} - \rho_i Y_{(-2),i} ,$$

and

$$u_{-1,i} = y_{-1,i} - Z_{-1,i} \delta_i ,$$

which is equivalent to the submatrix of $A$ defined in (3.32). Therefore, we have shown the equivalence of the two matrices.

By arguments similar to those used to obtain the properties of full information estimators, we can demonstrate, under the assumption of diagonal covariance matrix $\not{\Sigma}$ , that the distribution of the A2SLSI estimator converges to a multivariate normal with asymptotic covariance matrix $N^{-1}(\bar{H}_i' \bar{H}_i)^{-1} \sigma_i^2$ . Further, the A2SLSII and TIV estimators have the same limiting distribution as A2SLSI.

### E. The Asymptotic Covariance Matrix of the Derived Reduced Forms

The derived reduced form of a structural model is useful for policy analysis and forecasting. Goldberger, Nagar and Odeh (29) derived the asymptotic covariance matrices of the reduced form coefficients for a structural model with independent errors. Fuller (22) independently obtained the result. The purpose of this section is to derive the asymptotic covariance matrix of the derived reduced form for a dynamic simultaneous equation model with autocorrelated errors.

We first introduce the necessary notations. The model of (3.4, 3.5) is

$$YB + X\Gamma + Y_{-1}C = U$$

$$U = U_{-1}R + \epsilon \ .$$

The reduced form of (3.4) and (3.5) with independent errors can also be written as

$$\begin{aligned} Y &= -X\Gamma B^{-1} - Y_{-1}CB^{-1} + U_{-1}RB^{-1} + \epsilon B^{-1} \\ &= G\bar{\pi} + V \end{aligned} \qquad (3.48)$$

where

$$G = [X,\ Y_{-1},\ U_{-1}] \qquad \bar{\pi}' = [-(\Gamma B^{-1})',\ -(CB^{-1})',\ (RB^{-1})'] \ .$$

We introduce $\underset{\sim}{a}$ , the $\ell(\Lambda + 3\ell) \times 1$ vector of estimated structural coefficients arranged by structural equations, that is by columns of $[B, \Gamma, C, R]$ . Therefore,

$$\underset{\sim}{a} = \begin{pmatrix} \underset{\sim}{a}_1 \\ \underset{\sim}{a}_2 \\ \vdots \\ \underset{\sim}{a}_\ell \end{pmatrix}$$

where

$$\underset{\sim}{a}_i' = (B_{1i},\ \ldots,\ B_{\ell i},\ \Gamma_{1i},\ \ldots,\ \Gamma_{\Lambda i},\ C_{1i},\ \ldots,\ C_{\ell i},\ \rho_{1i},\ \ldots,\ \rho_{\ell i})'$$

$$= (B_{1i},\ \ldots,\ B_{\ell i},\ D_{1i},\ \ldots,\ D_{\Lambda i},\ D_{\Lambda+1,i},\ \ldots,\ D_{\Lambda+\ell i},$$

$$D_{\Lambda+\ell+1,i},\ \ldots,\ D_{ki})'$$

where $k = \Lambda + 2\ell$ .

Let the asymptotic covariance matrix of $\hat{\underset{\sim}{a}}$ be denoted by

$$\bar{E}\{(\tilde{\underset{\sim}{a}} - \underset{\sim}{a})(\tilde{\underset{\sim}{a}} - \underset{\sim}{a})'\} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1\ell} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2\ell} \\ \vdots & & \ddots & \\ \Sigma_{\ell 1} & \cdots & & \Sigma_{\ell\ell} \end{pmatrix} = \Phi$$

where $\Sigma_{ij}$ is a $(\Lambda + 3\ell) \times (\Lambda + 3\ell)$ covariance matrix for the estimated coefficients of the $i^{th}$ and $j^{th}$ structural equations.

We write

$$\mathrm{Vec}(\bar{\pi}) = \begin{pmatrix} \bar{\pi}_{\cdot 1} \\ \bar{\pi}_{\cdot 2} \\ \vdots \\ \bar{\pi}_{\cdot \ell} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \Gamma \\ C \\ R \end{pmatrix} B^{-1}_{\cdot 1} \\ \begin{pmatrix} \Gamma \\ C \\ R \end{pmatrix} B^{-1}_{\cdot 2} \\ \vdots \\ \begin{pmatrix} \Gamma \\ C \\ R \end{pmatrix} B^{-1}_{\cdot \ell} \end{pmatrix}$$

where

$$\bar{\pi}'_{i\cdot} = (\bar{\pi}_{i1}, \bar{\pi}_{i2}, \ldots, \bar{\pi}_{ik})' .$$

Then the asymptotic covariance matrix of $\mathrm{Vec}(\hat{\bar{\pi}})$ is given by

$$\bar{E}\,[(\mathrm{Vec}(\hat{\bar{\pi}}) - \mathrm{Vec}(\bar{\pi}))\,(\mathrm{Vec}(\hat{\bar{\pi}}) - \mathrm{Vec}(\bar{\pi}))']$$

$$= \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1\ell} \\ \Omega_{2\ell} & \Omega_{22} & \cdots & \Omega_{2\ell} \\ \vdots & & \ddots & \\ \Omega_{\ell 1} & \cdots & & \Omega_{\ell\ell} \end{pmatrix} = \Omega$$

where

$$\hat{\bar{\pi}} = \begin{pmatrix} \hat{\Gamma}\hat{B}^{-1} \\ \hat{C}\hat{B}^{-1} \\ \hat{R}\hat{B}^{-1} \end{pmatrix}$$

and $\hat{B}, \hat{\Gamma}, \hat{C}, \hat{R}$ are estimates obtained from A3SLSI or A3SLSII and $\Omega_{ij}$ is the $k \times k$ covariance matrix of the $\hat{\bar{\pi}}_{i.}$ , with $\hat{\bar{\pi}}_{j.}$ .

We state three lemmas required in our derivation of this formula.

<u>Lemma 3.5</u>: Let $(\underset{\sim}{X}_n)$ be a sequence of a real valued k-dimensional random variable such that $\underset{N \to \infty}{\mathrm{plim}}\ \underset{\sim}{X}_n = \underset{\sim}{X}$ . Let $\underset{\sim}{g}(\underset{\sim}{X})$ be a function mapping the real k-dimensional vector $\underset{\sim}{X}$ into a real p-dimensional space. Let $\underset{\sim}{g}(\underset{\sim}{X})$ be continuous. Then

$$\underset{N \longrightarrow \infty}{\text{plim}} \; \underset{\sim}{g}(\underset{\sim}{X}_n) = \underset{\sim}{g}(\underset{\sim}{X}) \; .$$

Proof: (See Fuller (24), p. 5-17.)

Lemma 3.6: Let $\underset{\sim}{X}_n$ be a sequence of a real valued k-dimensional random variable satisfying

$$\underset{\sim}{X}_n = \underset{\sim}{a} + O_p(\gamma_n) \; ,$$

where $\underset{\sim}{X}_n = (X_{1n}, X_{2n}, \ldots, X_{kn})'$ , $\underset{\sim}{a} = (a_1, a_2, \ldots, a_{1k})'$ and $\gamma_n \longrightarrow 0$ as $n \longrightarrow \infty$ . Let $\underset{\sim}{g}(\underset{\sim}{X})$ be a function mapping the real k-dimensional space into a real p-dimensional space. Let $\underset{\sim}{g}(\underset{\sim}{X})$ have continuous partial derivatives of order three at $\underset{\sim}{a}$ then

$$\underset{\sim}{g}(\underset{\sim}{X}_n) = \underset{\sim}{g}(\underset{\sim}{a}) + \frac{\partial \, \underset{\sim}{g}(\underset{\sim}{a})}{\partial \, \underset{\sim}{X}_n} (\underset{\sim}{X}_n - \underset{\sim}{a}) + O_p(\gamma_n^2) \; .$$

Proof: This result is an immediate generalization of Corollary 5.15 of Fuller (see Fuller (24), p. 5-23, 5-24).

Lemma 3.7: Let $\{\underset{\sim}{X}_n, \underset{\sim}{Y}_n\}$ be a sequence of random variables where both $\underset{\sim}{X}_n$ and $\underset{\sim}{Y}_n$ are of dimension $k$ .

$$\underset{\sim}{Y}_n \xrightarrow{\mathcal{L}} \underset{\sim}{Y}$$

$$\underset{\sim}{X}_n \xrightarrow{\mathcal{L}} \underset{\sim}{C}$$

where $\underset{\sim}{C}$ is a fixed vector, then:

i) $\underset{\sim}{X}_n \mid \underset{\sim}{Y}_n \xrightarrow{\mathcal{L}} \underset{\sim}{C} \cdot \underset{\sim}{Y}$

ii) $\underset{\sim}{C}'\underset{\sim}{Y}_n \xrightarrow{\mathcal{L}} \underset{\sim}{C}'\underset{\sim}{Y}$ .

<u>Proof</u>: (See Fuller (24), p. 5-34, 5-35.)

By Theorem 3.2, Lemma 3.5, and expanding $\mathrm{Vec}(\hat{\bar{\pi}})$ in a Taylor series around $\underset{\sim}{a}$ , we have

$$\mathrm{Vec}(\hat{\bar{\pi}}) = \mathrm{Vec}(\bar{\pi}) + F(\underset{\sim}{a})\ (\hat{\underset{\sim}{a}} - \underset{\sim}{a}) + O_p(N^{-1}) \tag{3.49}$$

where

$$F(\underset{\sim}{a}) = \begin{pmatrix} \dfrac{\partial\ \bar{\pi}_{.1}}{\partial\ \underset{\sim}{a}} \\ \dfrac{\partial\ \bar{\pi}_{.2}}{\partial\ \underset{\sim}{a}} \\ \vdots \\ \dfrac{\partial\ \bar{\pi}_{.\ell}}{\partial\ \underset{\sim}{a}} \end{pmatrix}$$

and

$$\frac{\partial \bar{\pi}_{.i}}{\partial \underset{\sim}{a}} = \begin{pmatrix} \frac{\partial \bar{\pi}_{1i}}{\partial \underset{\sim}{a}_1} & \frac{\partial \bar{\pi}_{1i}}{\partial \underset{\sim}{a}_2} & \cdots & \frac{\partial \bar{\pi}_{1i}}{\partial \underset{\sim}{a}_\ell} \\ \frac{\partial \bar{\pi}_{2i}}{\partial \underset{\sim}{a}_1} & \frac{\partial \bar{\pi}_{2i}}{\partial \underset{\sim}{a}_2} & \cdots & \frac{\partial \bar{\pi}_{2i}}{\partial \underset{\sim}{a}_\ell} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \bar{\pi}_{ki}}{\partial \underset{\sim}{a}_1} & \frac{\partial \bar{\pi}_{ki}}{\partial \underset{\sim}{a}_2} & \cdots & \frac{\partial \bar{\pi}_{ki}}{\partial \underset{\sim}{a}_\ell} \end{pmatrix} .$$

Multiplying by $N^{\frac{1}{2}}$ on both sides of (3.45), we have

$$N^{\frac{1}{2}} \left(\mathrm{Vec}(\hat{\bar{\pi}}) - \mathrm{Vec}(\bar{\pi})\right) = F(\underset{\sim}{a})\, N^{\frac{1}{2}}(\tilde{\underset{\sim}{a}} - \underset{\sim}{a}) + O_p(N^{-\frac{1}{2}}) \quad .$$

By Theorem 3.2 and Lemma 3.7, we have

$$N^{\frac{1}{2}} \left(\mathrm{Vec}(\hat{\bar{\pi}}) - \mathrm{Vec}(\bar{\pi})\right) \xrightarrow{\mathcal{L}} N\Big(0, \underset{N \to \infty}{\mathrm{plim}} \left[N^{-1} F(\underset{\sim}{a})\, \Phi\, F'(\underset{\sim}{a})\right]\Big) \quad .$$

Now we evaluate the matrix $F(\underset{\sim}{a})$ . From (3.48), we see that

$$\bar{\pi}_{ij} = - \sum_{u=1}^{\ell} \Gamma_{iu} B^{uj} \qquad\qquad j = 1, 2, \ldots, \Lambda$$

$$\bar{\pi}_{ij} = -\sum_{d=1}^{\ell} C_{id} B^{dj} \qquad j = \Lambda+1, \ldots, \Lambda+\ell$$

$$\bar{\pi}_{ij} = \sum_{k=1}^{\ell} \rho_{ik} B^{kj} \qquad j = \Lambda+\ell+1, \ldots, k \quad .$$

Differentiating $\bar{\pi}_{ij}$ with respect to $B_{mn}$, $\Gamma_{h\ell}$, $C_{ef}$, $\rho_{pq}$, we have

$$\frac{\partial \bar{\pi}_{ij}}{\partial B_{mn}} = -\sum_{u=1}^{\ell} \Gamma_{iu} \frac{\partial B^{uj}}{\partial B_{mn}}$$

$$= -\sum_{u=1}^{\ell} \Gamma_{iu} (- B^{um} B^{nj})$$

$$= B^{nj} \sum_{u=1}^{\ell} \Gamma_{iu} B^{um}$$

$$= - B^{nj} \bar{\pi}_{im} \quad , \qquad (3.50)$$

$$\frac{\partial \bar{\pi}_{ij}}{\partial \Gamma_{h\ell}} = - B^{\ell j} \quad , \qquad \text{if} \qquad i = h$$

$$= 0 \quad , \qquad \text{otherwise} \; ,$$

$$\frac{\partial \bar{\pi}_{ij}}{\partial C_{ef}} = -B^{fj}, \qquad \text{if} \qquad i = e$$
$$= 0, \qquad \text{otherwise},$$

$$\frac{\partial \bar{\pi}_{ij}}{\partial \rho_{pq}} = +B^{qj}, \qquad \text{if} \qquad i = p$$
$$= 0, \qquad \text{otherwise}.$$

From the above result, we have

$$\frac{\partial \bar{\pi}_{1i}}{\partial \underset{\sim}{a}} = [B^{1i}(\bar{\pi}, I_{kxk}), B^{2i}(\bar{\pi}, I_{kxk}), \ldots, B^{\ell i}(\bar{\pi}, I_{kxk})]$$
$$i = 1, 2, \ldots, \ell,$$

hence

$$F(\underset{\sim}{a}) = (B^{-1})' \otimes [\bar{\pi}, I_{kxk}]$$

where $k = \Lambda + 2\ell$.

We summarize our discussion in the following theorem.

Theorem 3.4: Let Model (3.4) and (3.5) and Assumptions 3.1 through 3.6 hold. Then

$$N^{\frac{1}{2}} (\mathrm{Vec}(\hat{\pi}) - \mathrm{Vec}(\bar{\pi})) \xrightarrow{\mathfrak{L}} N(0, G)$$

where

$$G = \underset{N \to \infty}{\text{plim}} \; [N^{-1}(F(\underset{\sim}{a}) \, \Phi \, F'(\underset{\sim}{a})]$$

$$F(\underset{\sim}{a}) = (B^{-1})' \otimes [\; \bar{\pi}, I_{k \times k}]$$

and

$$k = \Lambda + 2\ell \,.$$

In practice, $B^{-1}$ and $\bar{\pi}$ are unknown, by Lemma 3.5, we can replace $B^{-1}$ and $\bar{\pi}$ by the consistent estimator $\hat{B}^{-1}$ and $\hat{\bar{\pi}}$ obtained from A3SLSI or A3SLSII. From Theorem 3.4, it is clear that the asymptotic covariance matrix of the reduced form coefficients for a structural model with independent errors is an inconsistent estimator in our case. On the other hand, the asymptotic covariance matrix of A3SLSI and A3SLSII provide all the required information for the computation of the consistent estimator for the asymptotic covariance matrix of the reduced form coefficients for a dynamic simultaneous equation model with autocorrelated errors.

# IV. A MONTE CARLO STUDY

In Chapter 3, we have investigated the limiting behavior of the proposed estimators. It is natural to ask to what extent the asymptotic results will hold in small samples. To partially answer this equation, a Monte Carlo study was undertaken in this chapter.

## A. Generating the Data

We consider the following model:

$$\begin{aligned} y_{t1} &= B_{12}\, y_{t2} + C_{11}\, y_{t-1,1} + \Gamma_{11}\, X_{t1} + \Gamma_{12}\, X_{t2} + \Gamma_{01} + u_{t1} \\ y_{t2} &= B_{21}\, y_{t1} + C_{22}\, y_{t-1,2} + \Gamma_{23}\, X_{t3} + \Gamma_{24}\, X_{t4} + \Gamma_{02} + u_{t2} \end{aligned} \tag{4.1}$$

$$u_{t1} = \rho_1\, u_{t-1,1} + \epsilon_{t1}$$

$$u_{t2} = \rho_2\, u_{t-1,2} + \epsilon_{t2} \quad .$$

The $y$'s are endogenous variables while the $X$'s are exogenous variables. $y_{t-1,i}$, $i = 1, 2$, are one period lagged endogenous variables. The vector $\epsilon'_{t\cdot} = (\epsilon_{t1}, \epsilon_{t2})$, $t = 1, 2, \ldots, N$, are independently distributed as a bivariate normal with zero mean vector and variance-covariance matrix

$$\not\Sigma = \begin{pmatrix} 1.0 & 1.21 \\ 1.21 & 2.21 \end{pmatrix} \quad . \tag{4.2}$$

We considered four sets of parameter values, with differences in the magnitudes and signs of the autocorrelations and in the magnitude of coefficients of lagged endogenous variables (see Table 4.1).

Samples of size 30 and 60 were created using the X-values of Kmenta and Gilbert (52):

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | 1, | 3, | 0, | 9, | 1, | 6, | 3, | 8, | 6, | 7, |
| $X_2$ | 0, | 1, | 1, | 1, | 0, | 1, | 0, | 1, | 0, | 0, |
| $X_3$ | 0, | 10, | 8, | 0, | 16, | 0, | 0, | 4, | 10, | 14, |
| $X_4$ | 1, | 0, | 0, | 0, | 0, | 1, | 1, | 1, | 1, | 1 . |

In a sample of size 30, the sequence of ten numbers was repeated 3 times. The $u_{ti}$ were defined by

$$u_{1i} = (1-\rho_i^2)^{\frac{1}{2}} \epsilon_{1i}$$

$$u_{ti} = \rho_i u_{t-1,i} + \epsilon_{ti} \qquad t = 2, 3, \ldots, N, \quad i = 1, 2,$$

where $(\epsilon_{t1}, \epsilon_{t2})$ are bivariate normal independent variables with zero mean vector and covariance matrix specified in (4. 2).

For a sample size $N$, $N + 20$ observations on the endogenous variables were calculated using the reduced form of structural equation (4.1 ) and initial values

$$\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = \begin{pmatrix} 1 & -B_{12} \\ -B_{21} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_{11}, & \Gamma_{12}, & 0, & 0, & 1 \\ 0, & 0, & \Gamma_{23}, & \Gamma_{24}, & 1 \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \\ \bar{X}_4 \\ 1 \end{pmatrix},$$

where $\bar{X}_1 = 4.4$, $\bar{X}_2 = 0.5$, $\bar{X}_3 = 6.2$ and $\bar{X}_4 = 0.6$. The first twenty observations were discarded, and the last N observation constituted the sample.

Table 4.1. Parameter values

| Model | Equation 1 | | | | | | Equation 2 | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\Gamma_{10}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\Gamma_{20}$ | $\rho_2$ |
| 1 | 1.0 | 0.8 | 2.0 | 1.0 | 1.0 | 0.9 | -1.0 | 0.5 | 0.9 | 4.0 | 1.0 | 0.9 |
| 2 | 1.0 | 0.2 | 2.0 | 1.0 | 1.0 | 0.9 | -1.0 | 0.5 | 0.9 | 4.0 | 1.0 | 0.3 |
| 3 | 1.0 | 0.8 | 2.0 | 1.0 | 1.0 | 0.9 | -1.0 | 0.5 | 0.9 | 4.0 | 1.0 | -0.6 |
| 4 | 1.0 | 0.8 | 2.0 | 1.0 | 1.0 | 0.9 | -1.0 | 0.5 | 0.9 | 4.0 | 1.0 | 0.0 |

## B. Alternative Estimators

For each model and sample size, 200 samples were generated. The parameters of each model were estimated by four procedures, (1) A3SLSI, (2) A3SLSII, (3) transformed instrumental variable estimator and (4) autoregressive two-stage least squares given by Fuller (22) (FA2SLS). The estimation procedure of FA2SLS was discussed in Section

C of Chapter 2. In all models, the modified limited information maximum likelihood estimator $(\alpha = 4$ MLIML(4)) was employed as the initial estimator. To shed some light on the effect of different consistent initial estimators on the performance of these four estimators, Model 1 and 4 were estimated again by these four estimators employing MLIML $(\alpha = 1)$ and MLIML $(\alpha = 0$ and $\lambda = 1$, i.e., instrumental variable estimates) as initial estimators.

In our case, we treat lagged endogenous variables in the system as endogenous. The procedure of modified limited information maximum likelihood with arbitrary $\alpha$ for the first equation of the Model (4.1) is described as follows:

$$\begin{pmatrix} \hat{\beta}_{12} \\ \hat{C}_{11} \\ \hat{\Gamma}_{11} \\ \hat{\Gamma}_{12} \\ \hat{\Gamma}_{10} \end{pmatrix} = \begin{pmatrix} Y_3'Y_3 - (\lambda^* - \frac{\alpha}{T-n-k_1})W_{22}, & Y_3'X_3 \\ \\ X_3'Y_3 & X_3'X_3 \end{pmatrix}^{-1} \begin{pmatrix} Y_3'Y_1 - (\lambda^* - \frac{\alpha}{T-n-k_1})W_{21} \\ \\ X_3'Y_1 \end{pmatrix} \tag{4.3}$$

where

$T \equiv$ sample size,

$n \equiv$ excluded exogenous variables in the i-th equation,

$k_i \equiv$ exogenous variables in the i-th equation,

$$Y_3 = \begin{bmatrix} Y_{12} & Y_{01} \\ Y_{22} & Y_{11} \\ Y_{32} & Y_{31} \\ \vdots & \vdots \\ Y_{N2} & Y_{N-1,1} \end{bmatrix} , \qquad X_3 = \begin{bmatrix} X_{11} & X_{12} & 1 \\ X_{21} & X_{22} & 1 \\ X_{31} & X_{32} & 1 \\ \vdots & \vdots & \vdots \\ X_{N1} & X_{N2} & 1 \end{bmatrix} ,$$

$\lambda^*$ is the smallest root of

$$|B - \lambda^* W| = 0 ,$$

$$W = \begin{pmatrix} W_{11}, & W_{12} \\ W_{21}, & W_{22} \end{pmatrix} ,$$

$$W = [Y'(I - Z(Z'Z)^{-1}Z')Y] ,$$

$$Z = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{01} & X_{02} & X_{03} & X_{04} & 1 \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{11} & X_{12} & X_{13} & X_{14} & 1 \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{21} & X_{22} & X_{23} & X_{24} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{N1} & X_{N2} & X_{N3} & X_{N4} & X_{N-11} & X_{N-12} & X_{N-13} & X_{N-14} & 1 \end{bmatrix} , \qquad (4.4)$$

$$B = [Y'(I - X_3(X_3'X_3)^{-1}X_3')Y] ,$$

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{01} \\ Y_{21} & Y_{22} & Y_{11} \\ Y_{31} & Y_{32} & Y_{21} \\ \vdots & \vdots & \vdots \\ Y_{N1} & Y_{N2} & Y_{N-11} \end{bmatrix} ,$$

$$X_3 = \begin{bmatrix} X_{11} & X_{12} & 1.0 \\ X_{21} & X_{22} & 1.0 \\ X_{31} & X_{32} & 1.0 \\ \vdots & \vdots & \vdots \\ X_{N1} & X_{N2} & 1.0 \end{bmatrix} .$$

The asymptotic covariance of $(\hat{\beta}_{12}, \hat{C}_{11}, \hat{\Gamma}_{11}, \hat{\Gamma}_{12}, \hat{\Gamma}_{10})$ is

$$\hat{\sigma}_1^2 \begin{pmatrix} Y_3'Y_3 - (\lambda^* - \frac{\alpha}{T-12})W_{22}, & Y_3'X_3 \\ & \\ X_3'Y_3 & X_3'X_3 \end{pmatrix}^{-1} \tag{4.5}$$

where

$$\hat{\sigma}_1^2 = \sum_{t=1}^{N} \hat{u}_{t1}^2 /T-5$$

and

$$\hat{u}_{t1} = Y_{t1} - \hat{B}_{12} Y_{t2} - \hat{C}_{11} y_{t-1,1} - \hat{\Gamma}_{11} X_{t1} - \hat{\Gamma}_{12} X_{t2} - \hat{\Gamma}_{10} .$$

The MLIML($\alpha$) was also applied to Equation 2 of the Model (4.1) to obtain the initial estimates. The value of $\alpha$ is chosen to be 4, which is based on Corollary 2 of Theorem 2 of Fuller (25).

C. Analysis of the Results

For Model 1, autocorrelation in both equations is equal to 0.9. This model provides an ideal situation for FA2SLS. Table 4.2 presents the ratio of the root mean square error to that of A3SLSI. For Equation 1 of Model 1 with sample size 30, on the basis of RMSE, A3SLSII and A3SLSII perform the best, followed by FA2SLS and then by FITIV. The initial estimator has the largest RMSE. As expected, full information estimators outperform the single equation estimators when the errors in different equations are contemporaneously correlated. As the sample size is increased, the RMSE of all estimators decrease roughly as $1/N$ . However, it seems that the RMSE of the initial estimator decreases at a slightly slower rate than that of the other estimators.

In Model 2 and Model 4, we encounter large rounding errors in inverting the last stage non-symmetric matrix of the FITIV estimator for about one half of the 200 samples. With the same degree of computational precision, no problems were encountered with the other estimators. Thus, we do not present summary statistics for the FITIV estimator.

In Model 4 the disturbances in the second equation are serially independent. In this case, A3SLSI seems to have a slight edge over A3SLSII.

Based on sample summary statistics, FA2SLS performed quite well in Models 2, 3 and 4. This was contrary to our expectations because of the differences between $\rho_1$ and $\rho_2$ in these models. The good performance of FA2SLS may be due to the use of substantially fewer predetermined variables in the first stage estimation or due to the special nature of the models we considered (i.e., there is only one explanatory endogenous variable in each equation of the model).

From Table 4.3, it is clear that the initial estimator MLIML ($\alpha = 4$) yields some estimators with significant bias in all four models. Most of the estimators display significant biases for the structural parameters at sample size 30. There is no general agreement in the direction of the biases in the estimated coefficients of the lagged endogenous variables. There are significant downward biases in the estimators of all positive autocorrelation coefficients. The absolute value of the biases decrease as sample size increases to 60. The full information estimators yield uniformly smaller estimated biases in the estimated autocorrelation coefficients than FA2SLS.

The estimated standard errors of MLIML obtained by application of the formula stated (4.5) is known to be inconsistent when the disturbances are autocorrelated. The standard errors of our proposed estimators are consistent estimators, of the standard errors of the limiting distributions. We compare the mean of the 200 estimated standard errors with the empirical standard errors.[1] From Table 4.4, we see that the agreement for MLIML ($\alpha = 4$), as expected, was poor in the equations with large positive autocorrelation coefficients. The asymptotic standard errors of the other estimators seem to slightly overestimate the corresponding empirical standard errors in most cases. As the sample size increases to 60, the agreement becomes quite good.

The theoretical percentiles of the $t$ distribution with 25 and 55 degrees of freedom, were also compared with the sample "t-statistic" computed from the regression output of the last step of the three improved estimators. The results are consistent with the results of the comparisons reported in Tables 4.5 to 4.8. Because of space limitations, we only present the sample "t-statistic" for parameters in Equation 1 of our models.

Sample summary statistics of the three initial estimators and their corresponding improved estimators are given in Table 4.9. On the basis of RMSE, we see that MLIML ($\alpha = 1$) performs best, followed by the instrumental variable estimator ($\lambda = 1$, and $\alpha = 0$) and then by MLIML ($\alpha = 4$). The large RMSE of MLIML ($\alpha = 4$) is due to relatively large bias. The rank of the initial estimators also holds for the rank of the corresponding improved estimators.

In summary, the Monte Carlo results are generally consistent with the theoretical results of Section D of Chapter 3. In view of sample

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[1]The empirical standard errors are the square roots of the sample variances computed from the 200 Monte Carlo samples.

summary statistics, MLIML ($\alpha$ = 1) can be recommended for initial estimators. For the improved estimators, we have found that A3SLSI and A3SLSII perform best followed by FA2SLS. The transformed instrumental variable estimators gave less satisfactory results. Because of cost consideration, we did not consider further iteration of these estimators. Before accepting the general validity of these findings, it is desirable to study these estimators under alternative models with different degrees of mutlicollinearity among exogenous variables and different numbers of explanatory endogenous variables in the equation. However, these sampling experiments have provided us some evidence of the small sample properties of the estimators we considered.

Table 4.2 Ratio of root mean square errors of alternative estimators to that of A3SLSI (percent)

| Model | Estimation | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Sample size 30 | | | | | |
| 1 | MLIML(4) | 119 | 182 | 132 | 208 | -- | 215 | 173 | 215 | 203 | -- |
| | A3SLSII | 99 | 104 | 100 | 102 | 99 | 91 | 90 | 89 | 94 | 96 |
| | FA2SLS | 97 | 119 | 100 | 121 | 111 | -- | -- | -- | -- | -- |
| | TIV | 188 | 139 | 109 | 197 | 162 | -- | -- | -- | -- | -- |
| 2 | MLIML(4) | 132 | 220 | 156 | 217 | -- | 104 | 125 | 114 | 120 | -- |
| | A3SLSII | 95 | 100 | 98 | 103 | 96 | 104 | 104 | 101 | 106 | 101 |
| | FA2SLS | 97 | 126 | 103 | 127 | 115 | -- | -- | -- | -- | -- |
| 3 | MLIML(4) | 163 | 149 | 158 | 192 | -- | 170 | 176 | 159 | 201 | -- |
| | A3SLSII | 96 | 105 | 100 | 105 | 101 | 110 | 103 | 109 | 112 | 98 |
| | FA2SLS | 110 | 113 | 116 | 119 | 116 | -- | -- | -- | -- | -- |
| 4 | MLIML(4) | 148 | 147 | 148 | 192 | -- | 110 | 143 | 124 | 120 | -- |
| | A3SLSII | 97 | 103 | 102 | 104 | 98 | 103 | 106 | 107 | 110 | 100 |
| | FA2SLS | 100 | 107 | 107 | 119 | 114 | -- | -- | -- | -- | -- |
| | | | | | | Sample size 60 | | | | | |
| 1 | MLIML(4) | 126 | 184 | 139 | 235 | -- | 238 | 200 | 239 | 153 | -- |
| | A3SLSII | 98 | 100 | 99 | 98 | 101 | 98 | 97 | 98 | 99 | 98 |
| | FA2SLS | 100 | 116 | 100 | 122 | 115 | -- | -- | -- | -- | -- |
| | TIV | 149 | 129 | 101 | 166 | 165 | -- | -- | -- | -- | -- |
| 2 | MLIML(4) | 145 | 210 | 146 | 218 | -- | 108 | 129 | 119 | 125 | -- |
| | A3SLSII | 98 | 100 | 99 | 100 | 100 | 102 | 106 | 102 | 102 | 99 |
| | FA2SLS | 102 | 121 | 106 | 120 | 135 | -- | -- | -- | -- | -- |

Table 4.3. Estimated bias of estimators computed from 200 replicates (estimated parameter-true parameters)

| Model | Estimator | Parameter | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
| | | | | | | Sample size 30 | | | | | |
| 1 | MLIML(4) | -0.005 | 0.078* | 0.019 | 0.416* | -- | 0.189* | -0.139* | -0.123* | -1.090* | -- |
| | A3SLSI | -0.007 | 0.006 | 0.007 | 0.003 | -0.147* | 0.047* | -0.037* | -0.022* | -0.050 | -0.110* |
| | A3SLSII | 0.005 | 0.015* | 0.009 | 0.073* | -0.151* | 0.027* | -0.016* | -0.010* | -0.010 | -0.108* |
| | FA2SLS | 0.003 | 0.023* | 0.003 | 0.138* | -0.155* | -- | -- | -- | -- | -- |
| | TIV | 0.023* | 0.029* | 0.018* | 0.151* | -0.167* | -- | -- | -- | -- | -- |
| 2 | MLIML(4) | -0.040* | 0.080* | -0.020 | -0.430* | -- | 0.020 | -0.010 | -0.010 | -0.040 | -- |
| | A3SLSI | -0.010 | 0.000 | -0.010 | 0.030 | -0.110* | 0.010 | 0.006 | -0.010 | 0.050 | -0.080* |
| | A3SLSII | 0.000 | 0.010* | -0.010 | 0.110* | -0.110* | -0.010 | 0.020* | 0.010 | 0.180* | -0.080* |
| | FA2SLS | 0.000 | 0.010* | 0.000 | 0.100* | -0.140* | -- | -- | -- | -- | -- |
| 3 | MLIML(4) | -0.061 | 0.024* | 0.043* | 0.152* | -- | 0.047* | -0.052* | -0.035* | -0.265* | -- |
| | A3SLSI | -0.015* | -0.001 | -0.016* | -0.048 | -0.116* | -0.003 | 0.025* | -0.001 | 0.094 | 0.035* |
| | A3SLSII | -0.003 | 0.007 | -0.013* | 0.108* | -0.131* | -0.042* | 0.029* | 0.023* | 0.290* | 0.045* |
| | FA2SLS | 0.001 | 0.009 | -0.003 | 0.092* | -0.144* | -- | -- | -- | -- | -- |
| 4 | MLIML(4) | -0.054* | 0.032* | -0.033* | 0.191* | -- | 0.022* | -0.016 | -0.017* | -0.084 | -- |
| | A3SLSI | -0.018* | 0.003 | -0.016* | 0.067* | -0.115* | 0.003 | 0.026* | 0.002 | 0.128* | -0.030* |
| | A3SLSII | -0.005 | 0.013* | 0.013 | 0.126* | -0.126* | 0.041* | 0.043* | 0.026* | 0.336* | -0.026* |
| | FA3SLS | -0.002 | 0.008 | -0.006 | 0.085* | -0.144* | -- | -- | -- | -- | -- |

*Significant at the 5% level.

Table 4.3 Continued

| Model | Estimator | Parameter | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
| | | Sample size 60 | | | | | | | | | |
| 1 | MLIML(4) | -0.008 | 0.047* | 0.011 | 0.240* | -- | 0.119* | -0.089* | -0.080* | -0.750* | -- |
| | A3SLS I | -0.008 | -0.004 | 0.001 | -0.034 | -0.061* | -0.020* | -0.015* | -0.013* | -0.060 | -0.065* |
| | A3SLS II | -0.001 | 0.000 | 0.002 | -0.004 | -0.063* | 0.012* | -0.007 | -0.007* | -0.046 | -0.064* |
| | FA2SLS | 0.001 | 0.008 | 0.000 | 0.040 | -0.066* | -- | -- | -- | -- | -- |
| | TIV | 0.010 | 0.007 | 0.003 | 0.040 | -0.048* | -- | -- | -- | -- | -- |
| 2 | MLIML(4) | -0.030* | 0.050* | 0.012 | 0.237* | -- | 0.009 | -0.002 | -0.007 | -0.033 | -- |
| | A3SLS I | -0.009* | 0.002 | -0.006 | 0.022 | -0.040* | 0.001 | 0.012* | 0.001 | 0.044 | -0.037* |
| | A3SLS II | -0.003 | 0.005 | -0.003 | 0.048* | -0.043* | 0.012* | 0.019* | 0.009* | 0.096* | -0.036* |
| | FA2SLS | 0.002 | 0.006 | 0.000 | 0.046* | -0.064* | -- | -- | -- | -- | -- |

Table 4.4. Ratio of the empirical standard errors of the alternative estimators to the mean of the 200 estimated standard errors (percent)

| Model | Estimators | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Sample size 30 | | | | | | | | | |
| 1 | MLIML(4) | 61 | 86 | 84 | 81 | -- | 81 | 66 | 69 | 81 | -- |
| | A3SLSI | 95 | 86 | 99 | 87 | 105 | 102 | 103 | 96 | 91 | 112 |
| | A3SLSII | 90 | 84 | 101 | 85 | 100 | 106 | 100 | 94 | 87 | 107 |
| | FA2SLS | 95 | 96 | 102 | 95 | 112 | -- | -- | -- | -- | -- |
| 2 | MLIML(4) | 62 | 22 | 80 | 66 | -- | 94 | 102 | 99 | 102 | -- |
| | A3SLSI | 94 | 82 | 95 | 83 | 103 | 109 | 111 | 107 | 102 | 96 |
| | A3SLSII | 91 | 81 | 94 | 82 | 116 | 111 | 111 | 110 | 105 | 94 |
| | FA2SLS | 94 | 94 | 99 | 91 | 117 | -- | -- | -- | -- | -- |
| 3 | MLIML(4) | 63 | 73 | 81 | 74 | -- | 119 | 92 | 102 | 101 | -- |
| | A3SLSI | 88 | 83 | 91 | 84 | 111 | 104 | 95 | 104 | 94 | 104 |
| | A3SLSII | 90 | 88 | 94 | 86 | 104 | 111 | 98 | 111 | 99 | 101 |
| | FA2SLS | 93 | 88 | 98 | 90 | 117 | -- | -- | -- | -- | -- |
| 4 | MLIML(4) | 63 | 73 | 82 | 72 | -- | 98 | 97 | 98 | 97 | -- |
| | A3SLSI | 100 | 86 | 95 | 86 | 117 | 108 | 110 | 109 | 105 | 98 |
| | A3SLSII | 95 | 88 | 99 | 86 | 109 | 111 | 113 | 112 | 108 | 99 |
| | FA2SLS | 94 | 90 | 100 | 91 | 116 | -- | -- | -- | -- | -- |
| | | Sample size 60 | | | | | | | | | |
| 1 | MLIML(4) | 52 | 69 | 70 | 62 | -- | 59 | 50 | 50 | 60 | -- |
| | A3SLSI | 98 | 89 | 103 | 79 | 97 | 97 | 94 | 89 | 83 | 101 |
| | A3SLSII | 98 | 88 | 103 | 79 | 96 | 104 | 95 | 91 | 83 | 97 |
| | FA2SLS | 102 | 96 | 104 | 85 | 100 | -- | -- | -- | -- | -- |
| 2 | MLIML(4) | 55 | 51 | 70 | 52 | -- | 92 | 95 | 93 | 100 | -- |
| | A3SLSI | 102 | 93 | 103 | 86 | 100 | 106 | 98 | 100 | 100 | 90 |
| | A3SLSII | 102 | 91 | 102 | 84 | 97 | 105 | 102 | 100 | 100 | 89 |
| | FA2SLS | 102 | 94 | 106 | 88 | 100 | -- | -- | -- | -- | - |

Table 4.5. Comparison of theoretical t distribution with computed t for $B_{12}$, $C_{11}$, Equation 1, Model 1
Sample size = 30

| Probability percent | Theoretical percentile[a] | Observed percentile | | | | | |
|---|---|---|---|---|---|---|---|
| | | $B_{12}(4)$ | | | $C_{11}(4)$ | | |
| | | A3SLSI | A3SLSII | FA2SLS | A3SLSI | A3SLSII | FA2SLS |
| 1 | -2.48 | -2.63 | -2.35 | -2.31 | -1.99 | -1.82 | -2.19 |
| 5 | -1.71 | -1.92 | -1.59 | -1.66 | -1.41 | -1.35 | -1.50 |
| 10 | -1.32 | -1.22 | -1.07 | -1.14 | -1.13 | -0.97 | -1.06 |
| 50 | 0.00 | -0.08 | 0.06 | -0.02 | 0.08 | 0.11 | 0.28 |
| 90 | 1.32 | 1.10 | 1.22 | 1.27 | 1.18 | 1.22 | 1.49 |
| 95 | 1.71 | 1.42 | 1.67 | 1.79 | 1.63 | 1.63 | 1.90 |
| 99 | 2.48 | 2.23 | 2.34 | 2.43 | 2.10 | 2.22 | 2.55 |

[a]Theoretical percentile for Student's distribution with 25 d.f.

Table 4.6. Comparison of theoretical t distribution with computed t for $\Gamma_{11}$, $\Gamma_{12}$ of Equation 1, Model 1
Sample size = 30

| Probability percent | Theoretical percentile | Observed percentile | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\Gamma_{11}(4)$ | | | $\Gamma_{12}(4)$ | | |
| | | A3SLSI | A3SLSII | FA2SLS | A3SLSI | A3SLSII | FA2SLS |
| 1 | -2.48 | -2.40 | -2.54 | -2.64 | -2.00 | -1.75 | -2.12 |
| 5 | -1.71 | -1.54 | -1.47 | -1.70 | -1.55 | -1.41 | -1.15 |
| 10 | -1.32 | -1.16 | -1.14 | -1.37 | -1.15 | -1.06 | -0.98 |
| 50 | 0.00 | 0.16 | 0.10 | 0.07 | -0.03 | -0.09 | 0.10 |
| 90 | 1.32 | 1.40 | 1.43 | 1.39 | 1.16 | 1.22 | 1.51 |
| 95 | 1.71 | 1.79 | 1.82 | 1.70 | 1.46 | 1.55 | 1.84 |
| 99 | 2.48 | 2.63 | 3.22 | 2.69 | 2.46 | 2.32 | 2.81 |

Table 4.7. Comparison of theoretical t distribution with computed t for $B_{12}$, $C_{11}$ of Equation 1, Model 4
Sample size = 30

| Probability in percent | Theoretical percentile | Observed percentile | | | | | |
|---|---|---|---|---|---|---|---|
| | | $B_{12}(4)$ | | | $C_{11}(4)$ | | |
| | | A3SLSI | A3SLSII | FA2SLS | A3SLSI | A3SLSII | FA2SLS |
| 1 | -2.48 | -2.89 | -2.54 | -2.46 | -1.69 | -1.87 | -2.06 |
| 5 | -1.71 | -1.86 | -1.43 | -1.56 | -1.50 | -1.43 | -1.62 |
| 10 | -1.30 | -1.38 | -1.18 | -1.16 | -1.23 | -1.11 | -1.08 |
| 50 | 0.00 | -0.21 | -0.03 | 0.01 | 0.02 | 0.14 | 0.15 |
| 90 | 1.30 | 0.95 | 1.19 | 1.21 | 1.18 | 1.27 | 1.25 |
| 95 | 1.71 | 1.48 | 1.59 | 1.70 | 1.64 | 1.77 | 1.68 |
| 99 | 2.48 | 1.93 | 2.22 | 2.39 | 2.34 | 2.78 | 2.21 |

Table 4.8. Comparison of theoretical t distribution with computed t for $\Gamma_{11}$, $\Gamma_{12}$ of Equation 1, Model 4
Sample size = 30

| Probability in percent | Theoretical percentile | Observed percentile | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\Gamma_{11}(4)$ | | | $\Gamma_{12}(4)$ | | |
| | | A3SLSI | A3SLSII | FA2SLS | A3SLSI | A3SLSII | FA2SLS |
| 1 | -2.48 | -2.57 | -2.82 | -2.68 | -1.74 | -1.43 | -1.94 |
| 5 | -1.71 | -1.60 | -1.69 | -1.60 | -1.20 | -1.10 | -1.41 |
| 10 | -1.30 | -1.39 | -1.39 | -1.35 | -0.93 | -0.78 | -1.00 |
| 50 | 0.00 | -0.16 | -0.21 | -0.07 | -0.05 | 0.16 | 0.09 |
| 90 | 1.30 | 1.15 | 1.11 | 1.29 | 1.22 | 1.30 | 1.34 |
| 95 | 1.71 | 1.49 | 1.72 | 1.61 | 1.79 | 1.87 | 1.83 |
| 99 | 2.48 | 2.21 | 2.48 | 2.38 | 2.64 | 2.57 | 2.55 |

Table 4.9. Estimated bias of estimators from 200 replicates (estimated parameters - true parameters)

| Model | Estimators | Parameters | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
| 1 | MLIML(4) | -0.005 | 0.078 | 0.019 | 0.416 | -- | 0.189 | -0.139 | -0.123 | -1.090 | -- |
| | MLIML(1) | 0.003 | 0.011 | 0.013 | 0.043 | -- | 0.043 | -0.032 | -0.033 | 0.270 | -- |
| | MLIML(0) | 0.003 | 0.015 | 0.013 | 0.064 | -- | 0.064 | -0.040 | -0.047 | 0.391 | -- |
| | A3SLSI(4) | -0.007 | 0.007 | 0.003 | 0.003 | -0.147 | 0.047 | -0.037 | -0.022 | -0.050 | -0.110 |
| | A3SLSI(1) | -0.010 | 0.001 | 0.003 | -0.014 | -0.144 | -0.041 | -0.031 | -0.024 | -0.090 | -0.127 |
| | A3SLSI(0) | -0.011 | 0.000 | 0.002 | -0.017 | -0.145 | 0.042 | -0.033 | -0.025 | -0.098 | -0.128 |
| | A3SLSII(4) | 0.005 | 0.015 | 0.009 | 0.073 | -0.151 | 0.027 | -0.016 | -0.010 | -0.010 | -0.108 |
| | A3SLSII(1) | 0.003 | 0.008 | 0.005 | 0.042 | -0.143 | 0.012 | 0.006 | 0.004 | 0.013 | -0.126 |
| | A3SLSII(0) | 0.003 | 0.008 | 0.007 | 0.045 | -0.146 | -0.014 | -0.007 | -0.004 | 0.013 | -0.125 |
| | FA2SLS(4) | 0.003 | 0.023 | 0.003 | 0.138 | -0.155 | -- | -- | -- | -- | -- |
| | FA2SLS(1) | 0.003 | 0.020 | 0.004 | 0.118 | -0.153 | -- | -- | -- | -- | -- |
| | FA2SLS(0) | 0.003 | 0.021 | 0.003 | 0.123 | -0.155 | -- | -- | -- | -- | -- |
| 4 | MLIML(4) | -0.054 | 0.032 | -0.033 | 0.191 | -- | 0.022 | -0.016 | 0.017 | -0.084 | -- |
| | MLIML(0) | -0.011 | 0.011 | -0.003 | 0.078 | -- | -0.017 | -0.010 | 0.013 | 0.053 | -- |
| | A3SLSI(4) | -0.018 | 0.003 | -0.016 | 0.067 | -0.115 | 0.003 | 0.026 | 0.002 | 0.128 | -0.030 |
| | A3SLSI(0) | -0.017 | 0.005 | 0.014 | 0.071 | -0.117 | 0.002 | 0.025 | 0.002 | 0.118 | -0.036 |

Table 4.9. (Continued)

| Model | Estimators | Parameters | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
| 4 | A3SLS II(4) | -0.005 | 0.013 | 0.013 | 0.126 | -0.126 | 0.041 | 0.043 | 0.026 | 0.336 | -0.026 |
| | A3SLS II(0) | -0.005 | 0.013 | -0.012 | 0.124 | -0.125 | 0.040 | 0.042 | 0.026 | 0.326 | -0.026 |
| | FA2SLS(4) | -0.002 | 0.008 | -0.006 | 0.085 | -0.144 | -- | -- | -- | -- | -- |
| | FA2SLS(0) | 0.003 | 0.010 | -0.005 | 0.089 | -0.147 | -- | -- | -- | -- | -- |

Table 4.10. The root mean square errors of alternative estimators

| Model | Estimators | Parameters | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
| 1 | MLIML(4) | 0.089 | 0.135 | 0.141 | 0.847 | -- | 0.228 | 0.187 | 0.152 | 1.477 | -- |
| | MLIML(1) | 0.093 | 0.116 | 0.152 | 0.788 | -- | 0.129 | 0.137 | 0.090 | 0.993 | -- |
| | MLIML(0) | 0.092 | 0.110 | 0.152 | 0.740 | -- | 0.133 | 0.122 | 0.088 | 1.004 | -- |
| | A3SLSI(4) | 0.075 | 0.074 | 0.107 | 0.408 | 0.213 | 0.106 | 0.108 | 0.071 | 0.728 | 0.198 |
| | A3SLSI(1) | 0.075 | 0.067 | 0.109 | 0.380 | 0.204 | 0.095 | 0.096 | 0.065 | 0.671 | 0.185 |
| | A3SLSI(0) | 0.076 | 0.069 | 0.106 | 0.388 | 0.205 | 0.098 | 0.100 | 0.067 | 0.679 | 0.189 |
| | A3SLSII(4) | 0.074 | 0.077 | 0.107 | 0.417 | 0.211 | 0.096 | 0.097 | 0.063 | 0.682 | 0.190 |
| | A3SLSII(1) | 0.073 | 0.067 | 0.103 | 0.377 | 0.197 | 0.092 | 0.095 | 0.063 | 0.684 | 0.183 |
| | A3SLSII(0) | 0.073 | 0.070 | 0.104 | 0.388 | 0.201 | 0.091 | 0.095 | 0.063 | 0.680 | 0.187 |
| | FA2SLS(4) | 0.073 | 0.088 | 0.107 | 0.494 | 0.237 | -- | -- | -- | -- | -- |
| | FA2SLS(1) | 0.073 | 0.082 | 0.105 | 0.462 | 0.224 | -- | -- | -- | -- | -- |
| | FA2SLS(0) | 0.072 | 0.084 | 0.106 | 0.475 | 0.229 | -- | -- | -- | -- | -- |
| 4 | MLIML(4) | 0.108 | 0.112 | 0.147 | 0.756 | -- | 0.129 | 0.150 | 0.103 | 0.947 | -- |
| | MLIML(0) | 0.091 | 0.114 | 0.146 | 0.364 | -- | 0.132 | 0.148 | 0.105 | 0.962 | -- |
| | A3SLSI(4) | 0.073 | 0.073 | 0.099 | 0.393 | 0.194 | 0.117 | 0.108 | 0.083 | 0.787 | 0.178 |
| | A3SLSI(0) | 0.072 | 0.074 | 0.099 | 0.395 | 0.191 | 0.108 | 0.109 | 0.083 | 0.787 | 0.179 |

Table 4.10. (Continued)

| Model | Estimators | Parameters | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $B_{12}$ | $C_{11}$ | $\Gamma_{11}$ | $\Gamma_{12}$ | $\rho_1$ | $B_{21}$ | $C_{22}$ | $\Gamma_{23}$ | $\Gamma_{24}$ | $\rho_2$ |
| 4 | A3SLSII(4) | 0.071 | 0.078 | 0.101 | 0.409 | 0.191 | 0.121 | 0.115 | 0.089 | 0.807 | 0.178 |
| | A3SLSII(0) | 0.071 | 0.077 | 0.100 | 0.408 | 0.190 | 0.120 | 0.116 | 0.090 | 0.864 | 0.179 |
| | FA2SLS(4) | 0.073 | 0.081 | 0.106 | 0.467 | 0.222 | -- | -- | -- | -- | -- |
| | FA2SLS(0) | 0.073 | 0.080 | 0.106 | 0.467 | 0.221 | -- | -- | -- | -- | -- |

# V. REVIEW OF U.S. FARM LABOR MARKET

This chapter consists of four sections. The first two sections review the historical movement and previous econometric studies of the U.S. farm labor market. The last two sections discuss the theoretical concepts on which the econometric model is based and the sources of data used in the estimation of this model.

## A. Historical Trend of Farm Labor

U.S. farm labor employment reached a peak in 1916. Since that time the trend has been downward except for 1931-35 and a brief period following World War II. Table 5.1 presents the annual rates of decline in farm labor for the period of 1941 to 1973. The annual rates of decline in family farm labor range from -8.4% to -1.5% for the period 1950 to 1969. However, this downward trend has stabilized substantially since the beginning of 1970. Hired farm labor, which constitutes about 27% of the total farm labor force, followed a pattern similar to that of family farm labor. Hired farm labor fluctuated around 1160 thousand from 1970 to 1973. The recent change in the downward trend in farm labor can be explained by the following three factors: (1) expanded demand for U.S. agricultural commodities due to poor world crops and the devaluation of the U.S. dollars, (2) high unemployment rate in the rest of the economy, and (3) an apparent decline in the growth rate of agricultural productivity.

Table 5.1. Annual rate of decline in family and hired farm workers, 1941-73

| Year | Family farm labor | Hired farm labor | Year | Family farm labor | Hired farm labor |
|---|---|---|---|---|---|
| 42 | -0.8 | -3.7 | 58 | -2.5 | 2.2 |
| 43 | 0.8 | -4.7 | 59 | -2.4 | -1.5 |
| 44 | -0.3 | -8.4 | 60 | -4.9 | -3.4 |
| 45 | -1.3 | -5.0 | 61 | -1.9 | 0.3 |
| 46 | 2.9 | 3.3 | 62 | -3.1 | -3.3 |
| 47 | 0.1 | 3.6 | 63 | -2.8 | -2.6 |
| 48 | -1.1 | 3.1 | 64 | -4.9 | -9.9 |
| 49 | -3.9 | -3.6 | 65 | -8.4 | -7.6 |
| 50 | -1.5 | 3.4 | 66 | -6.6 | -8.2 |
| 51 | -3.8 | -4.0 | 67 | -5.3 | -7.9 |
| 52 | -4.2 | -4.1 | 68 | -3.1 | -3.2 |
| 53 | -3.3 | -2.6 | 69 | -3.3 | -3.1 |
| 54 | -3.0 | -0.4 | 70 | -2.1 | -0.1 |
| 55 | -3.4 | -2.2 | 71 | -2.2 | -1.2 |
| 56 | -7.0 | -4.1 | 72 | -1.4 | -1.3 |
| 57 | -4.1 | -0.6 | 73 | -1.8 | 1.9 |

## B. Previous Studies of U.S. Farm Labor Market

Based on the nature of data used, previous econometric studies of U.S. farm labor market can be classified into three categories: (1) time series studies, (2) cross-sectional studies, and (3) pooled cross-sectional and time series studies.

Time series data have been widely used in econometric studies of the farm labor market. Such studies include Johnson (48), Tweeten (73), Heady and Tweeten (44), Schuh (70), Tyrchniewicz and Schuh (74, 75), Arcus (6), Martinos (59), and Hammonds, Yadav and Vathana (36).

Johnson (48) examined both demand and supply functions of U.S. hired and family farm labor for three time periods, 1910-57, 1929-57 and 1940-57. He specified demand for farm labor to be a function of real farm wage rate, prices of substitutable resources and prices received for farm products. The supply equation was a function of real farm wage rate, non-farm wage rate and the unemployment rate. A Nerlove-type distributed lag hypothesis was introduced into each demand and supply function to obtain long-run and short-run elasticities. Ordinary least squares and two-stage least squares were used to estimate the parameters of his models. He also applied his national models to each of the nine census regions.

For the demand function of family labor, significant coefficients were attained by using the national data. Only the farm wage rate remained significant in all of Johnson's sub-period models. The prices received variable was significant only in the two periods, 1920-57 and 1940-57. Results at the regional level were not sufficiently well defined to draw definite conclusions.

For the supply functions of hired and family farm labor, Johnson found that the signs of the farm wage rate and that of the non-farm wage adjusted for unemployment were consistent with prior expectations, but none of them were statistically significant for the time period

1929-59. Empirical estimates for regional data also had large standard errors.

Using time series data from 1929-56, Tweeten (73) estimated national demand functions for hired and family farm labor. In the hired farm labor market, he hypothesized that the demand for hired farm labor was a function of the ratio of farm wage rate to prices received by farm products, the ratio of farm wage rate to prices paid for operating inputs and machinery, the stock of product assets, an index of governmental policies and time trend. The results indicated that for the period 1926-59, the farm wage rate, the prices received by farmers, the stock of productive farm assets and time were important variables affecting national demand for hired farm labor.

Tweeten specified the national supply of hired farm labor to be a function of farm wage rate and the wage rate in manufacturing adjusted for the unemployment rate. Data for the years 1926-59, except for the years 1942-45, supported these hypotheses when a dummy variable was included to separate the two periods, 1926-41 and 1946-59.

In analyzing the national family farm labor market, Tweeten expressed demand as a function of the ratio of the average wage rate in manufacturing to the residual farm income per farm worker, the unemployment rate, the ratio of proprietor's equity to liabilities in agriculture, the percentage of forced sales through bankruptcy, an index of government policies, the stock of productive farm machinery, and time. He reported that for the period 1926-56, only the income ratio, the unemployment rate, the equity ratio and time trend had significant effects on the quantity of family farm labor demanded. Ordinary least squares and limited information maximum likelihood estimator were employed in Tweeten's study.

Heady and Tweeten (44) analyzed regional demand functions for hired and family farm labor. Data for the nine census regions were examined using the national demand model originally developed by Tweeten (73). Empirical results indicated that the farm wage rate was an important variable in demand for hired farm labor. The parity ratio was significant in the regressions for four of the nine regions while trend was significant in only one region.

Schuh (70) made an empirical study of the U.S. hired farm labor market. His basic model assumed the simultaneous determination of the hired labor wage rate and hired labor employment. The demand function expressed hired farm labor as a function of (a) real wage of hired farm labor, (b) an index of the prices of agricultural products, (c) an index of the prices of other inputs, and a measure of technology. The supply function expressed hired farm labor as a function of (a) real wage of hired farm labor, (b) non-farm income, (c) unemployment, and (d) the size of civilian labor force. Time series data from 1929-57 and two-stage least squares were used in this study. Schuh concluded that hired farm labor responded to economic stimuli with a distributed lag.

Tyrchniewicz and Schuh (74) applied Schuh's model for national hired farm labor market to each of nine census regions. The main purpose of their study was to test the hypothesis that regional supply of hired farm labor was a function of regional farm wage rate, national non-farm income, the size of civilian labor force and a time trend. The analysis was made using time series data from 1929-59. Empirical results supported their hypothesis for most of the regions studied except for New England, the mountain and the Pacific regions.

Tyrchniewicz and Schuh (75) constructed a simultaneous equation model consisting of six equations for the U.S. farm labor market. It took account of the interdependence among unpaid family, hired and operator farm labor. The model was estimated by two-stages least squares with time series data from 1929 to 1961. This study showed that the demand and supply elasticities were substantially different among the components and the interdependence among the three components was significant at 10% level. The estimated structural models were also used to evaluate a number of alternative policies that bear on labor use and labor returns.

Arcus (6) estimated equations for farm employment in the U.S. and in ten production regions using data from 1941 to 1963. Four measures of farm employment were used. They are farm population, hired farm labor, family farm labor, total farm labor and farm population. The farm employment function was specified as a function of farm income, non-farm income, the unemployment rate, stock of farm machinery, amount of land farmed, and technology. Ordinary least squares was used to estimate the parameters of the farm employment function.

Based on the estimated employment function and assumed values of exogenous variables, Arcus made projections of farm population and family, hired and total farm labor for the years 1970, 1975 and 1980.

Martinos (59) conducted an empirical analysis of the farm labor market of the U.S., the north central United States, and three subregions of the north central regions. This study dealt with three categories of farm labor; hired, family and total farm labor. Simultaneous equation models were fitted to the data from 1941 to 1969 for each of the three subregions of the north central region. National demand for hired, family and total farm labor were estimated by Ordinary least squares and Generalized least squares. Results indicated that the farm wage rate and the prices received of farm product were important factors affecting the demand for farm labor.

Hammonds, Yadov and Vathana (36) estimated the parameters of the hired labor market model developed by Schuh (70) using time series data from 1941 to 1969. Results indicated that the demand elasticity of hired labor became more elastic over time. Using this result, they attempted to reconcile the conflicting outcomes that the wage demand elasticities obtained from cross-section data of 1959 (see Wallace and Hoover (80)) were more elastic than those obtained from time series data (see Heady and Tweeten (44), Schuh (70) and Tyrchniewicz and Schuh (75)).

Wallace and Hoover (80) made a study of the effects of technology, measured by the expenditure of agricultural experimental stations extension service activity, on the agricultural labor market. Their investigation was restricted to cross-sectional state data for the agricultural census year 1959. The model was composed of demand and supply functions of agricultural labor. The demand for agricultural labor, measured by agricultural man-day requirement, was specified as a function of the farm wage rate, research and expenditure of experimental stations, the stock of land, other inputs, age and education. The supply function specified farm wage rate to be a function of agricultural labor, age, education and non-farm wage rate. All coefficients in the model had signs consistent with a prior theoretic expectations except the coefficient of the education variable in the demand function.

The primary purpose of Bauer's study (9) was to estimate the time path of the effects of technology, as measured by research and extension expenditures, on the farm labor market. To accomplish this purpose, a rational distributed lag function was incorporated into a simultaneous equation model of the farm labor market developed by Wallace and Hoover (80). The modified model was estimated by 2SLS with a pooling of cross-sectional state and time series data from 1951 to 1961. The estimated lag distribution, had an inverted V shape instead of the geometric-decline shape implied by the Koyck-Nerlove type distributed lag.

## C. The Model

The basic model postulated here contains a demand and a supply function for farm labor. Since farm labor is one of the inputs in agricultural production, marginal productivity theory provides us with a guide for the specification of the demand for farm labor. Under the assumption of perfect competition in factor and product markets and given a production function, input demand of the individual firm is obtained from the individual firm's first-order conditions for profit maximization (47). As a result, input demand is a function of input prices, price of output, and the level of technology. The aggregate demand function of input is obtained by summing the individual input demand functions.

Under the assumption of occupational immobility, an individual's supply of labor is derived from the individual's first-order conditions for utility maximization. On this basis, the individual's supply of labor is a function of wage rate. Relaxing the assumption of occupational immobility, the market supply of labor is again obtained by summing up all individual supply functions of labor. As a result, the market supply of labor becomes a function of wage rate, alternative wage rates and factors influencing labor's occupational mobility.

In economics, static analysis ignores the time required to adjust to changes in the exogenous variables in the system. It is very common that economic agents distribute their response to an economic stimuli over a period of time. Lags in their response may be due to institutional factors and to imperfect information about the future. In our case, the reasons for expecting a lag response in demand for farm labor are: (1) contractual obligations with hired labor may limit changes in demand, (2) uncertainty about the future and habit persistence may limit the extent of changes in demand.

The reasons for expecting a lag in the response of the supply of farm labor to the changing economic conditions are: (1) a lack of training or education may prevent many farm workers from competing for better non-farm jobs, and (2) previously signed contracts may prevent farm labor from seeking alternative employment, even though alternative employment opportunities are available.

For these reasons, a Nerlove-type distributed lag hypothesis is introduced into each equation of the model. The partial adjustment hypothesis can be stated as

$$y_t^* = \sum_{i=1}^{p} \alpha_i x_{ti} \quad . \tag{5.1}$$

The $y_t^*$ is long-run equilibrium quantity demanded. The observed variables may reflect a partial adjustment of economic unit from current to long-run equilibrium level. Nerlove specified the partial adjustment process as follows:

$$y_t - y_{t-1} = \gamma(y_t^* - y_{t-1}) + u_t \qquad \gamma \in (0, 1) \; . \tag{5.2}$$

This states that the change in the observed magnitude is proportional to the difference between the long-run equilibrium level and the current level. Substituting (5.1) into (5.2), we have a statistical model in which all variables are observable.

Based on the conceptual model discussed above, we derive the following statistical model:

$$\begin{aligned} y_{1t} &= \beta_{11} + \beta_{12}y_{2t} + \Gamma_{11}x_{1t} + \Gamma_{12}x_{2t} + \beta_{10}y_{1t-1} + u_{1t} \\ y_{1t} &= \beta_{21} + \beta_{22}y_{2t} + \Gamma_{25}x_{5t} + \Gamma_{26}x_{6t} + \beta_{20}y_{1t-1} + u_{2t} \end{aligned} \tag{5.3}$$

$$|\beta_{i0}| < 1 \qquad i = 1, 2,$$

$$u_{1t} = \rho_1 u_{1t-1} + \epsilon_{1t}$$

$$|\rho_i| < 1 \quad i = 1, 2,$$

$$u_{2t} = \rho_2 u_{2t-1} + \epsilon_{2t}$$

$$\begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \sim \text{iid} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11}, & \sigma_{12} \\ \sigma_{21}, & \sigma_{22} \end{pmatrix} \right)$$

where

$y_{1t}$ = farm employment in agriculture,

$y_{2t}$ = real farm wage rate or real farm income per farm,

$x_{1t}$ = real prices received by farmers for all farm products,

$x_{2t}$ = a measure of level of technology,

$y_{1t-1}$ = $y_{1t}$ lagged one period,

$x_{5t}$ = real non-farm wage rate adjusted for the unemployment rate,

$x_{6t}$ = a time trend with 1941 = 1 .

The prior constraints on the parameters in the structural demand function are: (1) $\beta_{12} < 0$, (2) $\Gamma_{11} > 0$, and (3) $0 < \beta_{10} < 1$ . For the structural supply function they are: (1) $\beta_{22} > 0$, (2) $\Gamma_{25} < 0$,

(3) $\Gamma_{26} < 0$, and (4) $0 < \beta_{20} < 1$. There is no a priori basis for placing constraints on the coefficient of technology; it can be negative or positive. The above inequalities are theoretical restrictions.

Based on the necessary condition of identification (see Johnston (49)), both equations are over-identified since the number of variables that do not appear in a given equation is larger than the number of endogenous variables in a given equation less one. Each will be identified provided at least one of the identifying variables in each equation has a coefficient that is different from zero.

## D. The Data

In this section, we discuss various measures for the variables of the structural equation (5.3) and the sources of the data. Time series data covering the period from 1940 to 1973 are used in this study.

$y_{1t}$ represent farm labor, measured by the number of workers (in thousands) on farm. Data about hired, family and total farm labor are available in various issues of Agricultural Statistics (76).

$y_{2t}$ represents the real farm wage. Three measures for real farm wage were used in this study. The first measure for real farm wage rate is defined as the index of hired labor composite hourly wage rate deflated by the consumer price index (1957-59 = 100). In the second measure of the real farm wage rate, the index of prices paid for living items in rural areas was used as deflator. One of these two measures for the real farm wage rate was treated as the price of hired farm labor and was assumed to be the "going" price of family farm labor. The index of hired labor composite hourly wage rate and the index of prices paid for living items in rural areas are both obtained from Agricultural Prices (77). Our source for the consumer price index is various issues of Survey of Current Business (79).

$x_{1t}$ is the index of real prices received by farmers for all farm products. It is defined as the ratio of the index of prices received by farmers for all commodities to the index of prices paid by farmers for production items, excluding the wage rate. These indices are available in Agricultural Statistics (76).

$x_{2t}$ stands for the level of technology. The technology index for agricultural production is not available. The agricultural productivity ratio is chosen as a proxy variable for technology on the grounds that increases in technology shift the production function upward. The productivity ratio is available in various issues of Agricultural Statistics (76).

$x_{5t}$ is the real non-farm wage rate adjusted for unemployment. It is calculated on the following three steps: (1) $K_t \equiv A_t(1 - 5 \cdot U_t)$,

$$(2)\quad \bar{K}_t \equiv K_t / \sum_{t=57}^{59} K_t/3 \cdot 100.0, \quad (3)\quad x_{5t} \equiv \bar{K}_t/\mathrm{CPI}(1957\text{-}59 = 100)$$

where $A_t$ stands for the average hourly earnings of production workers on manufacturing payrolls. $U_t$ stands for the unemployment rate and CPI represents the consumer price index. This variable, reflecting the appeal of real wage earned in non-farm sectors and the opportunities of non-farm employment, is a slight modification of the variable first suggested by Johnson (48). It is assumed that when the unemployment rate of the economy reaches 20%, there are no off-farm opportunities. Consequently, this variable has a zero effect on the supply of farm labor. It is a recognized fact that when laborers leave the farm they go to various industries other than manufacturing industry. Empirically, however, the hourly wage rate of production workers proved to be the best proxy for the alternative labor wage in the non-agricultural sector.

$x_{6t}$ denotes a trend variable which represents secular changes occurring over a period of years. These include raising levels of schooling in farm areas, gradual changes in interest for employment in agriculture and improvement in communication and transportation between the farm and the non-farm sectors.

# VI. EMPIRICAL RESULTS

The first three sections of this chapter discuss the statistical results for the family, hired labor, and total farm labor markets. The economic implications of structural elasticities are analyzed in Section D and the results of the dynamic analysis of the models are presented in Section E. Finally, Section F summarizes the main findings of this chapter.

## A. Analysis of Family Farm Labor Market

We have estimated three versions of a dynamic model for the family labor market. Model 1 is identical to the basic model described in Section C of Chapter 5. A dummy variable was introduced into each equation to capture the war effects on farm labor market for the period 1941-1945. The resulting model is called Model 2. The substitution of real farm wage and non-farm wage by real net farm operator's income and real income per manufacturing worker adjusted for unemployment in Model 2 resulted in Model 3. Each model was estimated by three estimation methods: (1) ordinary least squares (OLS), (2) two-stage least squares (2SLS) and (3) Fuller's autoregressive two-stage least squares (FA2SLS). The use of these estimation methods makes possible statistical comparisons of the estimates obtained when the same economic relationship is fitted. The empirical results are presented in Equations (5.1) through (5.18).

Time period (1941-73)

(6.1) Demand, Ordinary Least Squares (OLS)

$$y^F_{1t} = -863.24 - 2.14\, y_{2t} + 12.55\, x_{1t} + 0.55\, x_{2t} + 0.92 y^F_{1t-1}$$

(990.63) (2.78) (3.23) (6.76) (0.06)

RMS = 11062.8 , $R^2$ = 0.99 , d = 1.65

(6.2) Supply, Ordinary Least Squares (OLS)

$$y^F_{1t} = 1029.2 + 12.26\, y_{2t} - 4.5\, x_{5t} - 53.22\, x_{6t} + 0.82\, y^F_{1t-1}$$

(489.61) (2.21) (1.83) (10.67) (0.05)

RMS = 8500.8 , $R^2$ = 0.99 , d = 2.08

(6.3) Demand, Two-stage Least Squares (2SLS)

$$y^F_{1t} = -789.76 - 12.25\, y_{2t} + 20.81\, x_{1t} + 7.32\, x_{2t} + 0.82\, y^F_{1t-1}$$

(1202.56) (4.94) (4.90) (8.55) (0.08)

RMS = 16294.5 , d = 1.41

(6.4) Supply, Two-stage Least Squares (2SLS)

$$y^F_{1t} = 1048.16 + 16.55\, y_{2t} - 6.15\, x_{5t} - 63.24\, x_{6t} + 0.80\, y^F_{1t-1}$$

(521.8) (3.13) (2.10) (12.35) (3.06)

RMS = 9662.9 , d = 1.97

(6.5) Demand, Fuller's Autoregressive Two-stage Least Squares (FA2SLS)

$$y^F_{1t} = 1472.25 - 19.25y_{2t} + 20.16x_{1t} + 0.09x_{2t} + 0.70y^F_{1t-1} + 0.22\hat{u}_{1t-1}$$

(2052.5) (8.81) (6.25) (11.02) (0.15) (0.26)

RMS = 20996.0 $\quad$ d = 1.78 $\quad$ $\tilde{\rho}_1 = 0.23 + 0.22 = 0.43$

(6.6) Demand, Ordinary Least Squares (OLS)

$$y^F_{1t} = -1424.96 - 0.73y_{2t} + 11.08x_{1t} + 3.68x_{2t} + 0.96y^F_{1t-1} + 73.7x_{3t}$$

(1156.8) (3.16) (3.59) (7.54) (0.08) (78.1)

RMS = 11109.2 $\quad$ $R^2 = 0.99$ $\quad$ d = 1.83

(6.7) Supply, Ordinary Least Squares (OLS)

$$y^F_{1t} = 1912.06 + 12.76y_{2t} - 4.25x_{5t} - 75.32x_{6t} + 0.72y^F_{1t-1} - 139.1x_{3t}$$

(761.7) (2.19) (1.8) (18.1) (0.08) (92.95)

RMS = 8145.1 $\quad$ $R^2 = 0.99$ $\quad$ d = 1.84

(6.8) Demand, Two-stage Least Squares (2SLS)

$$y^F_{1t} = 147.54 - 17.39y_{2t} + 25.53x_{1t} + 4.12x_{2t} + 0.73y^F_{1t-1} - 120.41x_{3t}$$

$$(1764.5) \quad (8.04) \quad (7.7) \quad (10.7) \quad (0.15) \quad (135.7)$$

RMS = 22560.0 d = 1.27

(6.9) Supply, Two-stage Least Squares (2SLS)

$$y^F_{1t} = 2125.57 + 17.55y_{2t} - 5.99x_{5t} - 91.03x_{6t} + 0.68y^F_{1t-1} - 169.3x_{3t}$$

$$(832.18) \quad (3.19) \quad (2.10) \quad (20.86) \quad (0.09) \quad (101.7)$$

RMS = 9604.0 d = 1.71

(6.10) Demand, Two-stage Least Squares (2SLS)

$$y^F_{1t} = -788.24 - 12.46y_{2t} + 20.98x_{1t} + 7.46x_{2t} + 0.82y^F_{1t-1}$$

$$(1210.6) \quad (4.98) \quad (4.94) \quad (8.61) \quad (0.08)$$

RMS = 16512.3 d = 1.40

(6.11) Demand, Fuller's Autoregressive Two-stage Least Squares (FA2SLS)

$$y^F_{1t} = 1231.03 - 18.73y_{2t} + 20.65x_{1t} + 0.92x_{2t} + 0.71y^F_{1t-1} + 0.19\hat{u}_{1t-1}$$

(1971.6) (8.26) (6.19) (11.0) (0.14) (0.27)

RMS = 21697.3 $\tilde{\rho} = 0.20 + 0.19 = 0.39$

(6.12) Supply, Fuller's Autoregressive Two-stage Least Squares (FA3SLS)

$$y^F_{1t} = 3495.64 + 23.99y_{2t} - 4.73x_{5t} - 91.23x_{6t} + 0.48y^F_{1t-1} - 287.1x_{3t}$$

(1706.9) (10.6) (3.4) (36.9) (0.21) (190.1)

$$- 0.09\hat{u}_{2t-1}$$

(0.27)

RMS = 12409.9 $\tilde{\rho} = 0.39 - 0.09 = 0.30$

(6.13) Demand, Ordinary Least Squares (OLS)

$$y^F_{1t} = -1416.59 - 0.20y^*_{2t} + 24.63x_{1t} + 2.59x_{2t} + 0.83y^F_{1t-1} + 67.7x_{3t}$$

(994.2) (0.07) (5.66) (6.7) (0.07) (61.4)

RMS = 8777.8 $R^2 = 0.99$ d = 2.06

(6.14) Supply, Ordinary Least Squares (OLS)

$$y^F_{1t} = 2603.15 + 0.14y^*_{2t} - 0.12x_{8t} - 54.56x_{6t} + 0.71y^F_{1t-1} - 176.35x_{3t}$$

(1027.59) (0.03) (0.11) (20.92) (0.10) (115.22)

RMS = 10732.9 $R^2$ 0.99 d = 1.47

(6.15) Demand, Two-stage Least Squares (2SLS)

$$y_{1t}^{F} = -946.24 - 0.42y_{2t}^{*} + 41.06x_{1t} - 1.45x_{2t} + 0.65y_{1t-1}^{F}$$
$$\quad (1015.0) \quad (0.16) \quad (11.5) \quad (6.66) \quad (0.12)$$

RMS = 11620.8 $\qquad$ d = 1.79

(6.16) Supply, Two-stage Least Squares (2SLS)

$$y_{1t}^{F} = 2781.04 + 0.17y_{2t}^{*} - 0.13x_{8t} - 60.98x_{6t} + 0.69y_{1t-1}^{F} - 211.08x_{3t}$$
$$\quad (1047.7) \quad (0.03) \quad (0.11) \quad (21.41) \quad (0.09) \quad (117.94)$$

RMS = 11128.1 $\qquad$ d = 1.52

(6.17) Demand, Fuller's Autoregressive Two-stage Least Squares (FA2SLS)

$$y_{1t}^{F} = -1763.47 - 0.48y_{2t}^{*} + 46.76x_{1t} + 3.17x_{2t} + 0.63y_{1t-1}^{F} - 0.02\hat{u}_{1t-1}$$
$$\quad (1233.12) \quad (0.15) \quad (11.57) \quad (7.82) \quad (0.12) \quad (0.22)$$

RMS = 12882.3 $\qquad$ $\tilde{\rho}_1 = 0.03 - 0.02 = 0.01$

(6.18) Supply, Fuller's Autoregressive Two-stage Least Squares (FA2SLS)

$$y_{1t}^{F} = 5249 + 0.12\,y_{2t}^{*} - 0.39x_{8t} - 52.66x_{6t} + 0.52y_{1t-1}^{F} - 77.28x_{3t} + 0.2\hat{u}_{2t-1}$$
$$\quad (1219.9) \quad (0.033) \quad (0.14) \quad (19.2) \quad (0.11) \quad (111.9) \quad (0.18)$$

RMS = 7541.2 $\qquad$ $\tilde{\rho}_1 = 0.167 + 0.2 = 0.367$

The notation of variables used here is defined in Section C of Chapter 5. A superscript F beside $y_{1t}$ represents the family farm labor. The numbers in parentheses are the standard errors of the estimated coefficients. RMS stands for Residual Mean Square Errors (see Johnston (49), p. 128). We are reporting the Durbin-Watson d statistic (Johnston (49), p. 251) for the OLS estimates and 2SLS estimates for sake of information, bearing in mind that this statistic was designed for single equation regression models where the explanatory variables are exogenous. Also $R^2$ is only reported for OLS estimates because it is meaningless in the simultaneous equation context (Christ (14), p. 519).

The coefficient of farm wage rate $y_{2t}$ in Equation (6.1) had the expected sign but was non-significant. The high value of $R^2$ is due to the presence of the lagged dependent variable in the equation.

In Equations (6.3) and (6.4), 2SLS was employed to re-estimate Equations (6.1) and (6.2). Compared with OLS estimates, 2SLS produced an increase in the size of the coefficients of $y_{2t}$ in both the demand and supply functions. For example, the OLS estimate of $y_{2t}$ was only one-fifth the size of that of the 2SLS estimate. The coefficients of the lagged endogenous variable estimated by 2SLS were slightly smaller than those estimated by OLS. Although the asymptotic standard errors of 2SLS were uniformly larger than those of OLS estimates, all 2SLS estimates were highly significant and had the expected signs, except technology variable.

We present the final results of the re-estimation of Equation (6.3) by FA2SLS in Equation (6.5). This procedure assumes a given order of the autoregressive process in the errors and requires a set of initial consistent estimates of the parameters. The instrumental variable procedure described in Section B of Chapter 3 is used to obtain a set of initial estimates. The overfitting procedure was applied to the estimated residuals $\hat{u}_t$ to determine the order of the autoregressive process, where $\hat{u}_t$ is defined in (3.8). The improved estimate $\tilde{\rho}_1$ is $0.43 = 0.23 + 0.22$, where 0.23 is the initial estimate of autocorrelation and 0.22 is the coefficient of $\hat{u}_{1t-1}$ in Equation (6.5). The resulting "t-statistic" of 1.65 is significant at the 10% level. Except for the technology variable, coefficients of all variables in Equation (6.5) were statistically significant and had signs in accordance with a priori expectations.

In Model 2, dummy variables were introduced into each of the Equations (6.6) and (6.7) to capture the effects of war during the period of 1941 to 1945. The OLS estimate of the farm wage coefficient in the demand function was negative but statistically insignificant. The war dummy in the supply function had a negative coefficient which was significant at the 10% level.

Equations (6.8) and (6.9) present the 2SLS estimates of Model 2. The coefficients of $y_{2t}$ estimated by 2SLS were larger than the corresponding coefficients estimated by OLS. As in Model 1, most of the estimated standard errors of OLS are smaller than those of 2SLS. The war dummy was non-significant in the demand function and therefore it was dropped. The modified demand function estimated by 2SLS is presented in Equation (6.10) which indicated a strong negative relation between quantity of labor demanded and farm wage.

Equations (6.11) and (6.12) present the results of FA2SLS estimates. The improved estimates of $\tilde{\rho}_1$ and $\tilde{\rho}_2$ were 0.39 and 0.30, respectively. Based on the Monte Carlo results of Chapter 4, FA2SLS is expected to perform reasonably well in this circumstance.

In Model 3, the farm operator's net income $y^*_{2t}$ was used as an alternative price of family farm labor and adjusted real income per manufacturing worker $X_{8t}$ was employed as an alternative price of non-farm labor. The OLS estimates for Model 3 were presented in Equations (6.13) and (6.14). All variables had significant coefficients with expected signs, except the technology variable and the dummy variable in the demand function. Equations (6.15) and (6.16) report 2SLS estimates of the supply function and the demand function excluding the war dummy variable. It can be seen that the farm wage coefficients estimated by 2SLS in both equations were somewhat larger than those estimated by OLS. By the order of the standard errors, 2SLS estimates were similar to the OLS estimates in Equation (6.14). This was the only case we found in this section in which these two estimates were roughly equal.

Finally, Model 3 was re-estimated by FA2SLS and the results are reported in Equations (6.17) and (6.18). The Durbin-Watson $d$ statistic in Equation (6.16) failed to reject the null hypothesis of independent errors. However, the application of FA2SLS to Equation (6.16) resulted in a highly significant estimate of the autocorrelation coefficient $\tilde{\rho}_1 = 0.37$ with an estimated standard error of 0.18. This result provides us evidence of the low power of the $d$ statistic in the case of a dynamic simultaneous equation model with autocorrelated errors. The differences between 2SLS and FA2SLS estimates are similar to those made for previous models.

In summary, the empirical results on the annual basis between 1941-73 strongly supported our hypothesis on the family farm labor market: (1) demand for family farm labor mainly depends on the price of family farm labor and the price of farm products, (2) supply of family labor is a function of price of family farm labor and prices of non-farm labor, and (3) the response of demanders and suppliers of family farm labor to changes in exogenous variables is spread over several time periods. Further, it was found that both farm wage and farm operator's net income per farm were satisfactory measures of the price of family farm labor.

On the basis of empirical estimates, we found a considerable difference in the estimates obtained by the three methods of estimation. The OLS estimates of endogenous variable coefficients were considerably smaller than those estimated by 2SLS and FA2SLS. This is consistent with prior theoretical results that OLS estimates are inconsistent in the simultaneous equation context. The FA2SLS estimates of lagged endogenous variables were the smallest of the three sets of estimates. Again the results support theoretical results that OLS and 2SLS estimates of lagged endogenous variables are biased upward in the case of autocorrelated errors. Consequently, the resulting coefficients of adjustment are downward biased and the long-run elasticities are inflated. The estimated standard errors of the FA2SLS were the largest. This indicates that usual formulas of OLS and 2SLS for standard errors are biased in the dynamic simultaneous equation model with autocorrelated errors.

### B. Analysis of Hired Farm Labor Market

The empirical results of hired farm labor are presented in Equations (6.19) to (6.30):

Time period (1941-73)

(6.19) Demand, Ordinary Least Squares (OLS)

$$y^H_{1t} = \underset{(767.8)}{136.57} - \underset{(4.6)}{2.29}y_{2t} + \underset{(2.11)}{3.58}x_{1t} - \underset{(3.19)}{0.31}x_{2t} + \underset{(0.19)}{0.85}y^H_{1t-1}$$

RMS = 3640.9 $\qquad R^2 = 0.98 \qquad$ d = 1.43

(6.20) Supply, Ordinary Least Squares (OLS)

$$y^H_{1t} = \underset{(544.6)}{1228.39} - \underset{(2.7)}{0.04}y_{2t} - \underset{(1.1)}{5.15}x_{5t} - \underset{(3.97)}{7.35}x_{6t} + \underset{(0.13)}{0.69}y^H_{1t-1}$$

RMS = 1654.8 $\qquad R^2 = 0.99 \qquad$ d = 2.33

(6.21) Demand, Two-stage Least Squares (2SLS)

$$y_{1t}^{H} = 1968.78 - 16.85y_{2t} + 7.73x_{1t} + 4.39x_{2t} + 0.21y_{1t-1}^{H}$$

$$(1106.8)\quad(7.77)\qquad(3.76)\qquad(4.8)\qquad(0.34)$$

RMS = 8317.4 $\qquad$ d = 0.47

(6.22) Supply, Two-stage Least Squares (2SLS)

$$y_{1t}^{H} = 88.11 + 5.0y_{2t} - 5.91x_{5t} - 7.0x_{6t} + 0.98y_{1t-1}^{H}$$

$$(578.17)\quad(2.99)\quad(1.11)\qquad(3.5)\qquad(0.10)$$

RMS = 2690.5 $\qquad$ d = 1.94

(6.23) Demand, Ordinary Least Squares (OLS)

$$y_{1t}^{H} = 846.65 - 6.08y_{2t} + 3.81X_{1t} - 0.38X_{2t} + 0.68y_{1t-1}^{H} - 130.4x_{3t}$$

$$(538.64)\quad(2.45)\qquad(1.69)\qquad(2.6)\qquad(0.12)\qquad(35.2)$$

RMS = 3317.7 $\qquad$ $R^2$ = 0.98 $\qquad$ d = 1.16

(6.24) Supply, Ordinary Least Squares (OLS)

$$y_{1t}^{H} = 1014.78 + 0.36y_{2t} - 4.39x_{5t} - 7.21x_{6t} + 0.74y_{1t-1}^{H} - 106.65x_{3t}$$

$$(327.96)\quad(1.5)\qquad(0.81)\qquad(3.01)\qquad(0.09)\qquad(26.77)$$

RMS = 1602.4 $\qquad$ d = 2.24

(6.25) Demand, Two-stage Least Squares (2SLS)

$$y^H_{1t} = \underset{(1245.8)}{2696.37} - \underset{(9.0)}{23.65}y_{2t} + \underset{(4.35)}{10.95}x_{1t} + \underset{(5.25)}{5.71}x_{2t} - \underset{(0.39)}{0.04}y^H_{1t-1} - \underset{(65.67)}{189.1}x_{3t}$$

RMS = 9668.8 d = 0.81

(6.26) Supply, Two-stage Least Squares (2SLS)

$$y^H_{1t} = \underset{(518.8)}{502.6} + \underset{(2.6)}{3.18}y_{2t} - \underset{(0.98)}{5.03}x_{5t} - \underset{(3.41)}{5.65}x_{6t} + \underset{(0.14)}{0.87}y^H_{1t-1} - \underset{(30.2)}{93.8}x_{3t}$$

RMS = 1806.3 d = 2.36

(6.27) Demand, Two-stage Least Squares (2SLS)

$$y^H_{1t} = \underset{(506.66)}{2575.12} - \underset{(2.67)}{22.73}y_{2t} + \underset{(2.82)}{10.61}x_{1t} + \underset{(4.69)}{5.52}x_{2t} - \underset{(56.84)}{186.12}x_{3t}$$

RMS = 0686.2 d = 0.81

(6.28) Demand, Theil's Autoregressive Two-stage Least Squares (TA2SLS)

$$y^H_{1t} = \underset{(393.98)}{3103.67} - \underset{(2.97)}{24.18}y_{2t} + \underset{(2.23)}{7.48}x_{1t} + \underset{(4.24)}{5.13}x_{2t} - \underset{(59.81)}{74.68}x_{3t}$$

RMS = 3660.3 $\hat{u}_{1t} = \underset{(0.16)}{0.768}\,\hat{u}_{1t-1} - \underset{(0.16)}{0.395}\,\hat{u}_{1t-2}$ d = 1.92

(6.29) Supply, Fuller's Autoregressive Two-stage Least Squares (FA2SLS) (first iteration)

$$y_{1t}^{H} = \underset{(1112.1)}{-70.29} + \underset{(4.7)}{4.69}y_{2t} - \underset{(1.0)}{5.17}x_{5t} - \underset{(4.7)}{0.60}x_{6t} + \underset{(0.3)}{1.05}y_{1t-1}^{H} - \underset{(31.02)}{74.39}x_{3t}$$

$$- \underset{(0.41)}{0.72}\,\hat{u}_{2t-1}$$

RMS = 1646.74 $\qquad \tilde{\rho}_2 = 0.12 - 0.72 = -0.60$

(6.30) Supply, Fuller's Autoregressive Two-stage Least Squares (FA2SLS) (second iteration)

$$y_{1t}^{H} = \underset{(529.8)}{756.78} + \underset{(2.54)}{1.74}y_{2t} - \underset{(0.84)}{5.32}x_{5t} - \underset{(3.27)}{5.46}x_{6t} + \underset{(0.13)}{0.81}y_{1t-1}^{H} - \underset{(30.11)}{103.26}x_{3t}$$

$$+ \underset{(0.17)}{0.07}\,\hat{\hat{u}}_{2t-1}$$

RMS = 1650.8 $\qquad \tilde{\rho}_2(2) = -0.24 + 0.07 = -0.14$ .

The models for hired labor markets are similar to those for family farm labor markets. The variable $y_{1t}^{H}$ denotes hired farm labor employment. Equations (6.19) and (6.20) report the OLS estimates of Model 1 and their related statistics. The results were discouraging because the coefficient of farm wage rates were non-significant in both demand and supply functions. Furthermore, the coefficient of farm wage was negative in the supply function.

The application of 2SLS to each equation significantly improved the results. In Equation (6.21), the coefficient of $y_{2t}$ was negative and highly significant. The farm wage rate regression coefficient in Equation (6.22) became positive and significant at the 10% level. As before, the standard errors of 2SLS were larger than those of OLS. The price of farm product variable and non-farm wage had the expected signs and were significant at the 5% level. These results suggested that the demand for hired farm labor increased as prices of farm product increased and the supply of hired farm labor decreased when non-farm wage rate increased. The coefficient of lagged endogenous variable in the Equation (6.21) did not statistically differ from zero, hence the corresponding coefficient of adjustment in the demand equation was not significantly different from one.

The OLS estimates of Model 2 are reported in Equations (6.23) and (6.24). The addition of a dummy variable to Model 1 improved the results. In the supply equation, the farm wage variable regression coefficient was positive but the coefficient was only one-fourth the size of its standard error.

When Model 2 was re-estimated by 2SLS, the estimate of $y^H_{1t-1}$ in Equation (6.25) was not statistically significant and therefore $y^H_{1t-1}$ was dropped from the demand function. The modified demand function estimated by 2SLS is presented in Equation (6.27). The d statistic of 0.8 rejected the null hypothesis of independence in the residuals. To determine the order of autoregressive process, the second-order autoregressive process was fitted using the residuals computed from Equation (6.27). The resulting equation was

$$\hat{u}_{1t} = \underset{(0.16)}{0.77}\ \hat{u}_{1t-1} - \underset{(0.17)}{0.39}\ \hat{u}_{1t-2} \quad .$$

The asymptotic t statistic indicated that the coefficients of $\hat{u}_{1,t-1}$ and $\hat{u}_{1,t-2}$ were both significant. Therefore, the error term was assumed to follow a second-order autoregressive process. Since the modified demand function was a static model, Theil's autoregressive two-stage least squares (TA2SLS) was used in this case. Equation (6.28) reports the results of TA2SLS. Comparisons of these estimates revealed that TA2SLS increased the size of coefficients of farm wage rate.

FA2SLS was employed to estimate the supply function. The farm wage variable regression coefficient in Equation (6.29) was positive but non-significant. The improved estimate of $\tilde{\rho}_2$ , the autocorrelation coefficient, was $\tilde{\rho}_2 = 0.12 - 0.72 = -0.6$. The value of -0.72 was the estimated coefficient of $\hat{u}_{1,t-1}$ and four times larger than the magnitude of $N^{-\frac{1}{2}} = 0.18$. Thus, a second iteration was carried out with initial estimate of $\hat{\rho}_2(2) = 0.12 - \frac{1}{2}(-0.72) = -0.24$ and $\hat{\hat{u}}_{1,t-1}$ , calculated residuals from the Equation (6.29). The result of the second iteration is shown in Equation (6.30). The coefficient of $\hat{\hat{u}}_{1,t-1}$ was 0.07 and the second step estimate of $\tilde{\rho}_2(2)$ was -0.14. The coefficient of the lagged endogenous variable in (6.29) had a coefficient of 1.05, while the second iteration gave a value of 0.81.

In all previous equations, the coefficients of war dummy variable had negative signs and were statistically significant at the 5% level. This result demonstrates two points: (1) during the war period, the supply of hired farm labor was reduced, and (2) the estimates obtained by Hammonds, et al. (36) and Martinos (59) studies were biased due to the omission of the war dummy variable in their models.

In summary, in the demand for hired farm labor, farm wage rate and prices of farm product were significant at the 5% level and had the expected signs, but the technology variable was inconclusive. A static model is preferred over a dynamic model for the demand for hired farm labor because the coefficient of lagged endogenous variables was not significantly different from zero. For the supply function, the non-farm wage rate variable had a negative coefficient, which was highly significant. The coefficient of farm wage rate was positive but not significant. This result is similar to the result of the recent study by Hammonds, et al.(36) for the period of 1941-69, but different from the previous studies by Schuh (70) and Heady and Tweeten (44) for the period of 1929-57.

The empirical estimates of this section provide us a basis for further comparisons of the effects of different estimation methods. In terms of significance of coefficients and correct signs, FA2SLS performed the best, followed by 2SLS. Ordinary least squares gave less satisfactory results.

## C. Analysis of Total Farm Labor Market

In this section we report the empirical results for the total farm labor market. Total farm labor is defined to be the sum of family farm labor and hired farm labor. Total farm labor market was specified in accordance with the Model 2 of family and hired farm labor markets. As before, three estimation procedures were employed to estimate the parameters of total farm labor market. The estimated equations are presented in the following:

Time period 1941-73

(6.31) Demand, Ordinary Least Squares (OLS)

$$y^T_{1t} = -959.43 - 3.76y_{2t} + 14.97x_{1t} + 1.52x_{2t} + 0.92y^T_{1t-1} - 69.8x_{3t}$$
$$(1681.2) \quad (5.1) \quad (5.01) \quad (9.81) \quad (0.08) \quad (106.58)$$

RMS = 22861.4 $\quad R^2 = 0.99 \quad d = 1.57$

(6.32) Supply, Ordinary Least Squares (OLS)

$$y^T_{1t} = 2846.04 + 13.0y_{2t} - 8.62x_{5t} - 80.44x_{6t} + 0.73y^T_{1t-1} - 236.9x_{3t}$$
$$(1095.8) \quad (2.85) \quad (2.37) \quad (21.2) \quad (0.08) \quad (113.1)$$

RMS = 14227.7 $\quad R^2 = 0.99 \quad d = 2.03$

(6.33) Demand, Two-stage Least Squares (2SLS)

$$y_{1t}^{T} = -546.31 - 14.69y_{2t} + 22.95x_{1t} + 7.68x_{2t} + 0.83y_{1t-1}^{T}$$
$$(1742.01) \quad (8.1) \quad (7.01) \quad (10.47) \quad (0.11)$$

RMS = 28832.0 $\qquad$ d = 1.31

(6.34) Supply, Two-stage Least Squares (2SLS)

$$y_{1t}^{T} = 2277.03 + 18.08y_{2t} - 10.45x_{5t} - 80.5x_{6t} + 0.76y_{1t-1}^{T} - 199.1x_{3t}$$
$$(1191.5) \quad (3.91) \quad (2.66) \quad (22.41) \quad (0.09) \quad (121.0)$$

RMS = 15898.7 $\qquad$ d = 2.04

(6.35) Demand, Fuller's Autoregressive Two-stage Least Squares (FA2SLS)

$$y_{1t}^{T} = 3180.96 - 29.24y_{2t} + 26.52x_{1t} - 2.24x_{2t} + 0.62y_{1t-1}^{T} + 0.16\hat{u}_{1t-1}$$
$$(4023.3) \quad (19.2) \quad (10.85) \quad (15.46) \quad (0.25) \quad (0.25)$$

RMS = 41697.6 $\qquad$ $\tilde{\rho}_1 = 0.21 + 0.16 = 0.37$

(6.36) Supply, Fuller's Autoregressive Two-stage Least Squares (FA2SLS)

$$y_{1t}^{T} = 2479.4 + 17.68y_{2t} - 7.82x_{5t} - 60.48x_{6t} + 0.73y_{1t-1}^{T} - 183.87x_{3t}$$
$$(1919.79) \quad (7.84) \quad (3.45) \quad (24.74) \quad (0.14) \quad (138.35)$$
$$- 0.32\hat{u}_{2t-1}$$
$$(0.17)$$

RMS = 15800.5 $\qquad$ $\tilde{\rho}_2 = 0.359 - 0.32 = 0.03$

(6.37) Supply, Fuller's Autoregressive Two-stage Least Squares (FA2SLS)

$$y_{1t}^{T} = \underset{(1902.2)}{3021.43} + \underset{(6.67)}{17.78}y_{2t} - \underset{(3.5)}{11.25}x_{5t} - \underset{(28.9)}{76}x_{6t} + \underset{(0.15)}{0.71}y_{1t-1}^{T} - \underset{(139.5)}{191.9}x_{3t}$$

$$- \underset{(0.25)}{0.07}\,\tilde{\tilde{u}}_{2t-1}$$

$$RMS = 16770.3 \qquad \tilde{\rho}_2(2) = 0.20 - 0.07 = 0.13$$

The parameters estimated by OLS were not very satisfactory because the farm wage coefficient in the demand function was insignificant. The application of 2SLS to the same structural equations excluding the dummy variable in the demand function resulted in improved results. The farm wage regression coefficients in both equations were highly significant and had expected signs. The coefficient of the technology variable was positive but insignificant. Finally, FA2SLS was used to estimate the model and the result is reported in Equations (6.35) through (6.37). Since the estimated coefficient of $\hat{u}_{2t-1}$ in the supply function is larger than $N^{-\frac{1}{2}} = 0.18$, the second iteration of the Equation (6.36) was performed with initial estimates of $\hat{\rho}_2(2) = 0.359 + \frac{1}{2}(-0.32) \doteq 0.20$ and $\approx u_{2t} = y_{1t}^{T} - 2479.4 - 17.68y_{2t} + 7.82x_{5t} + 60.48x_{6t} - 0.73y_{1t-1}^{T} + 183.87x_{3t}$. The resulting equation is reported in (6.37). The improved estimate of $\hat{\rho}_2(2)$ is 0.13 with estimated standard of 0.25. Consequently, the estimates of Equation (6.37) are similar to those of Equation (6.34) estimated by 2SLS.

The estimates of demand for and supply of total farm labor are relatively similar to those of the family farm labor market, compared to those of hired farm labor market. This is reasonable because family farm labor accounts for about 73% of the total farm labor force.

## D. Structural Elasticities

Having obtained the estimates of the models for farm labor markets, we now compute and analyze the structural elasticities. Two types of elasticities are computed: one at the mean and the other at 1973 levels.

The structural elasticities of family farm labor are summarized in Table 6.1. The short-run demand elasticities with respect to the farm wage rate were in the range -0.22 to -0.33. This means that a ten percent increase in farm wage would result in decrease of 2.2 to 3.3 percent in the quantity of family farm labor demanded, other things being equal. The price of farm products has an elasticity of 0.39, somewhat larger than that for farm wage rate. This indicates that the demand for family labor is somewhat more responsive to farming profitability than to changes in the farm wage rate. The technology variable has a short-run elasticity of 0.02 in Model 2, though the parameter estimate is not significant at the 50% level.

The coefficient of adjustment in the demand function (6.11) of Model 2 is 0.29, indicating that about 30% of the discrepancy between equilibrium and actual employment is eliminated in a given period of time by the demanders of labor. The coefficient of adjustment also implies long-run elasticies that are slightly more than three times as large as the short-run elasticities. The long-run elasticities of the farm wage rate and price of farm product in Equation (6.11) were -1.12 and 1.306, respectively. Since the coefficient of the lagged endogenous variable estimated by 2SLS are upward biased in the dynamic models with autocorrelated errors, we found that in both Model 2 and Model 3, the long-run elasticities computed from 2SLS estimates were more elastic than those computed from FA2SLS.

Evaluating the elasticities at the 1973 level indicated each of the relevant elasticities has increased over time. This is a reflection of the secular increase in each of the independent variables concurrent with a decline in the family farm labor employment.

In the supply side, the short-run farm wage elasticity at the mean was 0.428, considerably larger than the demand elasticity. The adjusted non-farm wage had an elasticity of -0.08, considerably less than the farm wage elasticity. This indicates that suppliers of family farm labor are somewhat less responsive to non-farm income incentives than to farm-income incentives. This is reasonable because the major portion of family farm labor is farm operators who have a much stronger commitment to agriculture. The coefficient of adjustment in Model 2 is 0.52, somewhat larger than the corresponding adjustment coefficient in the demand function. This means that 52 percent of the discrepancy between equilibrium and employment is eliminated in a given period of time by the suppliers of family farm labor. The long-run elasticities of farm wage and adjusted non-farm wage were 0.82 and -0.16, respectively.

Table 6.1. The structural elasticities and coefficients of adjustment of family farm labor market 1941-73

| Equation | Coefficient of adjustment | Price of farm labor | | Price of farm product | | Price of non-farm labor | | Index of technology | |
|---|---|---|---|---|---|---|---|---|---|
| | | Short run | Long run | Short run | Long run | Short run | Long run | Short run | Long run |
| Demand (6.10) | 0.18 | | | | | | | | |
| at mean | | -0.22 | -1.26 | 0.39 | 2.23 | --- | --- | 0.12 | 0.69 |
| at 1973 | | -0.57 | -3.17 | 0.73 | 4.05 | --- | --- | 0.28 | 1.56 |
| Supply (6.9) | 0.32 | | | | | | | | |
| at mean | | 0.31 | 0.98 | --- | --- | -0.11 | -0.33 | --- | --- |
| at 1973 | | 0.81 | 2.53 | --- | --- | -0.25 | -0.78 | --- | --- |
| Demand (6.11) | 0.29 | | | | | | | | |
| at mean | | -0.33 | -1.12 | 0.39 | 1.31 | --- | --- | 0.02 | 0.05 |
| at 1973 | | -0.86 | -2.96 | 0.72 | 2.48 | --- | --- | 0.04 | 0.14 |
| Supply (6.12) | 0.52 | | | | | | | | |
| at mean | | 0.43 | 0.82 | --- | --- | -0.08 | -0.16 | --- | --- |
| at 1973 | | 1.10 | 2.10 | --- | --- | -0.19 | -0.37 | --- | --- |
| Demand (6.15) | 0.35 | | | | | | | | |
| at mean | | -0.24[a] | -0.69[a] | 0.76 | 2.17 | --- | --- | -0.02 | -0.06 |
| at 1973 | | -0.83[a] | -2.37[a] | 1.49 | 4.26 | --- | --- | -0.06 | -0.17 |
| Supply (6.16) | 0.31 | | | | | | | | |
| at mean | | 0.1[a] | 0.33[a] | --- | --- | -0.1[b] | -0.33[b] | --- | --- |
| at 1973 | | 0.33[a] | 1.06[a] | --- | --- | -0.25[b] | -0.81[b] | --- | --- |

[a]Price of farm labor is measured by $y^{*}_{2t}$, real net farm operator income per farm.

[b]Price of non-farm labor is measured by $X_{8t}$, real income per manufacturing worker.

Table 6.2. The structural elasticities and coefficients of adjustment of hired farm labor market 1941-73

| Equation | Coefficient of adjustment | Farm wage | | Price of farm product | | Adjusted non-farm wage | | Index of technology | |
|---|---|---|---|---|---|---|---|---|---|
| | | Short run | Long run | Short run | Long run | Short run | Long run | Short run | Long run |
| Demand (6.27) | --- | | | | | | | | |
| at mean | | -1.25 | --- | 0.61 | --- | --- | --- | 0.29 | --- |
| at 1973 | | -2.84 | --- | 1.03 | --- | --- | --- | 0.57 | --- |
| Supply (6.26) | 0.13 | | | | | | | | |
| at mean | | 0.17 | 1.34 | --- | --- | -0.28 | -2.14 | --- | --- |
| at 1973 | | 0.40 | 3.05 | --- | --- | -0.55 | -4.23 | --- | --- |
| Demand (6.28) | --- | | | | | | | | |
| at mean | | -1.33 | --- | 0.43 | --- | --- | --- | 0.27 | --- |
| at 1973 | | -3.02 | --- | 0.72 | --- | --- | --- | 0.53 | --- |
| Supply (6.30) | 0.19 | | | | | | | | |
| at mean | | 0.13 | 0.53 | --- | --- | -0.30 | -1.58 | --- | --- |
| at 1973 | | 0.22 | 1.16 | --- | --- | -0.59 | -3.11 | --- | --- |

Table 6.3. The structural elasticities and coefficients of adjustment of total farm labor market 1941-73

| Equation | Coefficient of adjustment | Farm wage | | Price of farm product | | Adjusted non-farm wage | | Index of technology | |
|---|---|---|---|---|---|---|---|---|---|
| | | Short run | Long run | Short run | Long run | Short run | Long run | Short run | Long run |
| Demand (6.33) | 0.17 | | | | | | | | |
| at mean | | -0.19 | -1.12 | 0.32 | 1.88 | --- | --- | 0.10 | 0.58 |
| at 1973 | | -0.30 | -2.94 | 0.59 | 3.51 | --- | --- | 0.21 | 1.24 |
| Supply (6.34) | 0.24 | | | | | | | | |
| at mean | | 0.24 | 1.00 | --- | --- | -0.14 | -0.59 | --- | --- |
| at 1973 | | 0.61 | 2.54 | --- | --- | -0.31 | -1.29 | --- | --- |
| Demand (6.35) | 0.38 | | | | | | | | |
| at mean | | -0.39 | -1.03 | 0.37 | 0.98 | --- | --- | | |
| at 1973 | | -0.99 | -2.61 | 0.69 | 1.82 | --- | --- | -0.06 | -0.16 |
| Supply (6.37) | 0.29 | | | | | | | | |
| at mean | | 0.24 | 0.82 | --- | --- | -0.15 | -0.52 | --- | --- |
| at 1973 | | 0.60 | 2.07 | --- | --- | -0.34 | -1.17 | --- | --- |

Similar to the demand relation, the long-run and the short-run elasticities of supply at 1973 levels were three times the corresponding elasticities at the means.

The structural elasticities of hired farm labor market are shown in Table 6.2. The coefficient of adjustment in the demand side was not significantly different from one, indicating almost all the discrepancy between equilibrium and actual employment is eliminated in a given period of time by the demanders of labor. Evaluated at the means the elasticity of demand for hired farm labor with respect to the real farm wage was -1.25, compared to -0.33, the short-run wage demand elasticity of family farm labor. Price of farm products had an elasticity of 0.61, while the corresponding short-run elasticity of family farm labor was 0.39. This result supports the hypothesis that hired labor is the marginal labor input in the production process and hence is the one that farm operators manipulate most readily.

In the supply side, the short-run wage elasticities taken at the means ranged from 0.174 computed from 2SLS estimates to 0.13, computed from FA2SLS, both are computed from parameter estimates that were not significant at the 20% level. This result is consistent with the recent study by Hammonds, et al.(36). Using time series data 1941-1969 and a similar model, they found that the farm wage variable was non-significant in their hired labor supply function and had a short-run elasticity of 0.24. These results are inconsistent with the results of the earlier studies of Heady and Tweeten (44) and Schuh (70). Using data from 1919-1957, they found that farm wage rate was highly significant in the hired labor supply function and had short-run elasticities ranging from 0.48 to 0.65. One of the possible explanations for this situation is that the portion of hired farm labor that had greater mobility and alternatives to non-farm employment had immigrated out of the farm sector in this period. Those remaining in the farm sector either have fewer alternative employment opportunities or have much stronger commitment to agriculture. This tentative explanation can be investigated by examining the demographic factors of hired farm labor. We will not pursue this topic of research further here. The adjusted non-farm wage had a short-run elasticity of -0.28, somewhat larger than for the farm wage rate. This suggests that suppliers of hired farm labor are more responsive to non-farm wage incentives than to farm wage incentives. The coefficient of adjustment was 0.17, indicating long-run elasticities that are approximately five times as large as their corresponding short-run elasticities, the short-run referring to the response within one year. Hence, the long-run elasticities of the farm wage and the adjusted non-farm wage were 1.23 and -2.13, respectively, the latter suggests that given a sufficient time period, suppliers of hired farm labors are highly responsive to change in non-farm wage.

Table 6.3 presents the structural elasticities of the total farm labor market. Total farm labor is defined to be the sum of family farm labor and hired farm labor. Hence, the structural elasticities of the total farm market reflect its two components. The structural elasticities in Table 6.3 lie between those of their corresponding variables in the models for family and hired farm labor market.

For comparison purposes, we summarize the demand and supply elasticities of previous studies in Tables 6.4 and 6.5. On examining these estimates, we found that, in the family farm labor market, our estimates of the demand and supply elasticities with respect to the farm wage were more elastic than those obtained by earlier studies using data from 1929 to 1961, while the elasticities of prices of farm product and the non-farm wage obtained by previous studies were somewhat larger than the corresponding estimates of this study.

In the hired farm labor market, both short-run and long-run farm wage elasticities of demand have increased over time. For example, the short-run farm wage elasticity estimated by Schuh (70) for the period of 1929-57 was -0.12. His long-run elasticity was -0.4. These elasticities are considerably less than those obtained by Hammonds et al. study (36) and our study. The farm wage elasticities of supply estimated by previous studies ranged from 0.13 to 0.66. Our estimates were 0.13 and 0.17, respectively. These results indicate that the responsiveness of the supply of hired farm labor to farm wage has still been inelastic.

Finally, our estimates of the non-farm wage elasticities were inelastic in the short-run but highly elastic in the long-run. These results are similar to the results obtained by Schuh (70) and by Tyrchniewicz and Schuh (75).

Table 6.4. A summary of demand elasticity studies of family and hired farm labor market prior to 1973

| Study | Time period | Methods of estimation | Farm wage | | Prices of farm product | |
|---|---|---|---|---|---|---|
| | | | Short run | Long run | Short run | Long run |
| | | | | Family farm labor | | |
| 1. Heady and Tweeten (44) | 1910-57 | OLS | -0.14 | --- | --- | --- |
| 2. Tyrchniewicz and Schuh (75) | 1929-61 | 2SLS | | | | |
| a. unpaid family labor | | | -0.42 | -3.0 | 0.56 | 3.99 |
| b. farm operator | | | -0.069 | --- | 0.66 | --- |
| 3. Martinos (59) | 1941-69 | GLS | -0.20 | --- | 0.60 | --- |
| | | | | Hired farm labor | | |
| 1. Heady and Tweeten (44) | 1929-57 | OLS | -0.26 | -0.37 | 0.20 | 0.26 |
| | 1940-57 | | -0.46 | -0.60 | 0.10 | 0.13 |
| 2. Schuh (70) | 1929-57 | 2SLS | -0.12 | -0.4 | 0.15 | 0.52 |
| 3. Tyrchniewicz and Schuh (75) | 1929-61 | 2SLS | -0.26 | -0.49 | 0.31 | 0.58 |
| 4. Hammonds, et al.(36) | 1941-69 | 2SLS | -0.85 | -1.05 | --- | --- |
| 5. Martinos (59) | 1941-69 | GLS | -0.55 | --- | --- | --- |

Table 6.5. A summary of supply elasticity studies of family and hired farm labor prior to 1973

| Study | Time period | Methods of estimation | Farm wage | | Prices of non-farm labor | |
|---|---|---|---|---|---|---|
| | | | Short run | Long run | Short run | Long run |
| | | | | *Family farm labor* | | |
| 1. Tyrchniewicz and Schuh (75) | 1929-61 | 2SLS | | | | |
| a. unpaid family labor | | | 0.68 | 1.51 | -1.47 | -3.26 |
| b. farm operator | | | 0.005 | --- | -0.09 | --- |
| | | | | *Hired farm labor* | | |
| 1. Johnson (48) | 1929-57 | 2SLS | 0.13 | 0.71 | -0.06 | -0.31 |
| 2. Schuh (70) | 1929-57 | 2SLS | 0.25 | 0.78 | -0.36[c] | -1.11[a] |
| 3. Tyrchniewicz and Schuh (75) | 1929-61 | 2SLS | 0.65 | 1.55 | -1.42[c] | -3.38 |
| 4. Hammonds, et al. (36) | 1941-69 | 2SLS | 0.24 | 0.82 | --- | --- |

[a]Indicates supply elasticity with respect to the non-farm income adjusted for unemployment.

## E. Dynamic Analysis of the Models

The derived reduced form has two major uses: (1) it can be used to evaluate the impacts of exogenous variables on endogenous variables during the sample period, and (2) given a set of estimated exogenous variables, it can be used to predict future values of endogenous variables. In this section, derived reduced forms are employed to analyze the impacts of exogenous variables such as prices of farm product, the non-farm wage rate, etc. on farm labor employment.

If we use the estimated structural equations and express current endogenous variables in terms of predetermined variables, we obtain a set of derived reduced form equations. The derived reduced form for family farm labor, hired farm labor are as follows:

Family farm labor market (derived from Equations (6.9) and (6.10)):

$$y^F_{1t} = 1131.96 + 12.82x_{1t} + 2.06x_{2t} - 2.98x_{5t} - 45.3x_{6t} + 0.7y^F_{t-1} - 144.7x_{3t}$$

$$y_{2t} = -56.62 + 0.73x_{1t} + 0.12x_{2t} + 0.17x_{5t} + 2.6x_{6t} + 0.001y^F_{t-1} + 1.40x_{3t} \tag{6.38}$$

Family farm labor market (derived from Equations (6.11) and (6.12))

$$y^F_{1t} = 2244.89 + 11.41x_{1t} + 0.51x_{2t} - 2.12x_{5t} - 40.84x_{6t} + 0.60y^F_{1t-1} - 128.5x_{3t}$$

$$y_{2t} = -54.15 + 0.49x_{1t} + 0.02x_{2t} + 0.11x_{5t} + 2.18x_{6t} + 0.05y^F_{1t-1} + 6.86x_{3t} \tag{6.39}$$

Family farm labor market (derived from Equations (6.15) and (6.16)):

$$y^F_{1t} = 170.78 + 11.83x_{1t} - 0.42x_{2t} - 0.093x_{8t} - 43.41x_{6t} + 0.68y^F_{1t-1} - 150.3x_{3t}$$

$$y^*_{2t} = -6317.42 + 69.59x_{1t} - a.46x_{2t} + 0.22x_{8t} + 103.36x_{6t} - 0.06y^F_{1t-1} + 357.76x_{3t}$$

(6.40)

Hired farm labor market (derived from Equations (6.25) and (6.26)):

$$y^H_{1t} = 747.97 + 1.26x_{1t} + 0.65x_{2t} - 4.44x_{5t} - 4.98x_{6t} + 0.77y^H_{1t-1} - 104.12x_{3t}$$

$$y_{2t} = 77.26 + 0.40x_{1t} + 0.20x_{2t} + 0.19x_{5t} + 0.21x_{6t} - 0.03y^H_{1t-1} - 3.47x_{3t}$$

(6.41)

Hired farm labor market (derived from Equations (6.28) and (6.30)):

$$y^H_{1t} = 914.33 + 0.51x_{1t} + 0.34x_{2t} - 4.96x_{5t} - 5.09x_{6t} + 0.75y^H_{1t-1} - 101.34x_{3t}$$

$$y_{2t} = 90.54 + 0.29x_{1t} + 0.20x_{2t} + 0.21x_{5t} + 0.21x_{6t} - 0.03y^H_{1t-1} - 1.10x_{3t} .$$

(6.42)

Most of the coefficients of the predetermined variables have signs in accordance with our a priori expectation. The coefficients of these equations, which measure the effects of a one unit change in the exogenous variables in the current period, are called impact multipliers.

According to our estimates of Equation (6.39) for family farm labor market, the impact of one percent increase in real farm prices is an increase in family farm labor employment by 11.41 thousand persons, while the impact of one percent increase in the adjusted non-farm wage is a decrease in family farm labor employment by 2.12 thousands. For the hired farm labor market, the effect of one percent increase in prices of farm product $x_{1t}$ is an increase in hired farm labor employment by 1.26 thousands while an increase is one percent of the non-farm wage rate reduce 4.44 thousand hired farm labor. These results suggest that one unit increase in the farm product prices has a larger impact than one unit change in the adjusted non-farm wage rate on family farm labor employment. The situation is, however, reversed for hired farm labor employment.

The derived reduced form discussed above presents a clear picture of the immediate response of endogenous variables to changes in the predetermined variables. It enables us to estimate the effects of exogenous variables when the immediate past history of all endogenous variables is given. However, it does not answer the questions such as: what are the impacts of previous exogenous variables on current endogenous variables. By successive substitution and a given set of initial conditions of endogenous variables, the general form of the derived reduced equation in our case can be written as

$$y_{1t} = \sum_{j=0}^{N-1} \gamma_1^j \beta_0 + \sum_{i=1}^{6} \sum_{j=0}^{N-1} \gamma_1^j \beta_i x_{i,t-j} + \gamma_1^N y_{1t-N} . \qquad (6.43)$$

Since $|\gamma_1| < 1$ and when $N \longrightarrow \infty$ the Equation (6.43) is

$$y_{1t} = \sum_{j=0}^{\infty} \gamma_1^j \beta_0 + \sum_{i=1}^{6} \beta_i x_{ti} + \sum_{i=1}^{6} \sum_{j=1}^{\infty} \gamma^j \beta_i x_{it-j} . \qquad (6.44)$$

The coefficients attached to the lagged exogenous variables in Equation (6.44) are called interim multipliers or dynamic multipliers. A dynamic multiplier measures the effect of a one unit change in an exogenous variable on the endogenous variables in the current period, given that the increase is maintained in all intervening periods. The sum of all dynamic multipliers attached to a specific exogenous variable gives the value of long-run equilibrium multiplier for that variable. It measures the long-run effect of a permanent change in an exogenous variable.

The estimates of the dynamic multipliers and long-run multipliers for the two components of farm labor are presented in Tables 6.6 and 6.7. The main features of the results are the following: (1) As expected, all dynamic multipliers exhibit a geometric decline and converge to zero as the length of the time lag increases. (2) In the family farm labor market, changes in prices of farm product at time $t$ have more far-reaching effects than changes in non-farm wage rate at time $t$ . The

Table 6.6. Dynamic multipliers for the time path of family farm labor employment, 1941-73

| Lag period | Equation (6.38) | | Equation (6.39) | | Equation (6.40) | |
|---|---|---|---|---|---|---|
| | $x_{1t}$ [a] | $x_{5t}$ [b] | $x_{1t}$ | $x_{5t}$ | $x_{1t}$ | $x_{8t}$ [c] |
| 0 | 12.82 | -2.98 | 11.41 | -2.12 | 11.83 | -0.09 |
| 1 | 8.97 | -2.09 | 6.85 | -1.27 | 8.04 | -0.06 |
| 2 | 6.28 | -1.46 | 4.11 | -0.76 | 5.57 | -0.04 |
| 3 | 4.40 | -1.02 | 2.46 | -0.46 | 3.72 | -0.03 |
| 4 | 3.09 | -0.72 | 1.48 | -0.27 | 2.53 | -0.02 |
| 5 | 2.15 | -0.50 | 0.89 | -0.16 | 1.72 | -0.01 |
| 6 | 1.51 | -0.35 | 0.53 | -0.10 | 1.17 | 0.0 |
| 7 | 1.06 | -0.25 | 0.32 | 0.0 | 0.80 | 0.0 |
| 8 | 0.74 | -0.17 | 0.20 | 0.0 | 0.54 | 0.0 |
| 9 | 0.52 | -0.12 | 0.12 | 0.0 | 0.37 | 0.0 |
| 10 | 0.36 | -0.08 | 0.07 | 0.0 | 0.25 | 0.0 |
| 11 | 0.25 | -0.05 | 0.02 | 0.0 | 0.17 | 0.0 |
| 12 | 0.18 | -0.04 | 0.01 | 0.0 | 0.12 | 0.0 |
| 13 | 0.12 | -0.03 | 0.0 | 0.0 | 0.08 | 0.0 |
| 14 | 0.08 | -0.02 | 0.0 | 0.0 | 0.05 | 0.0 |
| 15 | 0.06 | -0.0 | 0.0 | 0.0 | 0.04 | 0.0 |
| 16 | 0.04 | -0.0 | 0.0 | 0.0 | 0.03 | 0.0 |
| 17 | 0.03 | 0.0 | 0.0 | 0.0 | 0.02 | 0.0 |
| 18 | 0.02 | 0.0 | 0.0 | 0.0 | 0.01 | 0.0 |
| 19 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| Total multipliers | 42.73 | -9.9 | 28.52 | -5.3 | 36.97 | -0.29 |

[a] $x_{1t}$ for real prices received by farmers for all farm products,

[b] $x_{5t}$ for real non-farm wage rate adjusted for the unemployment rate,

[c] $x_{8t}$ for real non-farm income per manufacturer worker adjusted for unemployment.

Table 6.7. Dynamic multipliers for the time path of hired labor employment, 1941-73

| Lag period | Equation (6.41) | | Equation (6.42) | |
|---|---|---|---|---|
| | $x_{1t}$ [a] | $x_{5t}$ [a] | $x_{1t}$ | $x_{5t}$ |
| 0 | 1.26 | -4.44 | 0.51 | -4.96 |
| 1 | 0.97 | -3.42 | 0.38 | -3.72 |
| 2 | 0.75 | -2.63 | 0.29 | -2.79 |
| 3 | 0.58 | -2.03 | 0.22 | -2.09 |
| 4 | 0.44 | -1.56 | 0.16 | -1.57 |
| 5 | 0.34 | -1.20 | 0.12 | -1.18 |
| 6 | 0.26 | -0.93 | 0.09 | -0.88 |
| 7 | 0.20 | -0.71 | 0.07 | -0.66 |
| 8 | 0.16 | -0.55 | 0.05 | -0.50 |
| 9 | 0.12 | -0.42 | 0.04 | -0.37 |
| 10 | 0.09 | -0.33 | 0.03 | -0.28 |
| 11 | 0.07 | -0.25 | 0.02 | -0.21 |
| 12 | 0.05 | -0.19 | 0.01 | -0.16 |
| 13 | 0.04 | -0.15 | 0.0 | -0.12 |
| 14 | 0.03 | -0.11 | 0.0 | -0.09 |
| 15 | 0.02 | -0.09 | 0.0 | -0.06 |
| 16 | 0.01 | -0.06 | 0.0 | -0.05 |
| 17 | 0.0 | -0.05 | 0.0 | -0.04 |
| 18 | 0.0 | -0.04 | 0.0 | -0.03 |
| 19 | 0.0 | -0.02 | 0.0 | -0.02 |
| Total multipliers | 5.48 | -19.30 | 2.04 | -19.84 |

[a] $x_{1t}$ and $x_{5t}$ are defined in Table 6.6.

opposite situation occurs in the hired farm labor market. (3) Dynamic multipliers obtained from the structural equations estimated by FA2SLS damp more rapidly than those obtained from the structural equations estimated by 2SLS.

The above analysis does not answer the question as to which exogenous variables were more influential in causing the change in farm labor employment during this sample period. The reason for this is that in any given year, the actual effects of each exogenous variable depends not only on the magnitude of impact multipliers but also on the actual values of exogenous variables. In order to examine the effects of current changes in exogenous variables on current changes in endogenous variables, a simple marginal analysis is performed. To do so, a first difference for each exogenous variable is multiplied by an appropriate impact multiplier in the derived reduced forms. Table 6.8 presents results of the simple marginal analysis for the two components of farm labor. Of the five exogenous variables, prices of farm product has both the largest and most variable effect on family farm labor employment. In 32 years, the marginal impact of prices of farm product has changed signs (direction) 21 times. The largest positive increase in farm labor, 160.0 thousand, occurred between 1972-1973; the largest decrease, 130 thousand, occurred between 1948 and 1949. In the period of 1942 to late 1950's, the marginal impact of the non-farm wage was a reduction in family farm labor employment, but the effect has been moderate since 1960 except for 1963 to 1964. The marginal impact of the non-farm wage ranged from plus 36.4 thousands in 1958 to minus 58.5 thousands in 1942. The trend

Table 6.8. Simple marginal analysis of family and hired farm labor employment functions, 1942-73

| Year | Equation (6.38) | | | Equation (6.39) | | | Equation (6.41) | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\Delta \tilde{y}^F_{1t}$ | $12.82 \cdot \Delta x_{1t}$ | $-2.98 \cdot \Delta x_{5t}$ | $\Delta \tilde{y}^F_{1t}$ | $11.41 \cdot \Delta x_{1t}$ | $-2.12 \cdot \Delta x_{3t}$ | $\Delta \tilde{y}^H_{1t}$ | $1.26 \cdot \Delta x_{1t}$ | $-4.44 \cdot \Delta x_{5t}$ |
| 42 | -159.77 | 151.72 | -82.24 | -130.6 | 135.0 | -58.5 | -128.9 | 14.9 | -122.54 |
| 43 | -5.28 | 150.66 | -58.32 | 9.8 | 134.1 | -41.5 | -153.2 | 14.8 | -86.9 |
| 44 | -62.3 | -41.23 | -20.83 | -55.2 | -36.7 | -14.8 | -130.9 | -4.1 | -31.04 |
| 45 | -7.80 | 34.47 | 16.06 | -11.4 | 30.7 | 11.4 | -134.8 | 3.4 | 23.9 |
| 46 | 142.70 | 74.46 | 36.60 | 117.5 | 66.3 | 26.0 | 76.97 | 7.3 | 54.5 |
| 47 | 122.9 | 11.60 | 3.84 | 106.1 | 10.3 | 2.7 | 54.3 | 1.1 | 5.7 |
| 48 | -154.9 | -121.95 | -5.69 | -145.1 | -108.5 | -4.1 | 38.3 | -12.0 | -8.4 |
| 49 | -239.7 | -146.13 | 18.71 | -212.1 | -130.1 | 13.3 | 61.0 | -14.4 | 27.9 |
| 50 | -272.6 | 6.59 | -14.04 | -233.4 | 5.8 | -10.1 | -90.7 | 0.7 | -20.9 |
| 51 | -76.9 | 82.37 | -33.44 | -60.3 | 73.3 | -23.8 | 12.6 | 8.1 | -49.8 |
| 52 | -332.2 | -78.13 | -14.99 | -291.5 | -69.5 | -10.7 | -104.4 | -7.7 | -22.3 |
| 53 | -354.6 | -81.49 | -16.63 | -307.6 | -72.5 | -11.8 | -107.8 | -8.0 | -24.8 |
| 54 | -202.6 | -41.70 | 43.00 | -184.7 | -37.1 | 30.6 | 13.4 | -4.1 | 64.1 |
| 55 | -272.7 | -54.32 | -34.30 | -235.4 | -48.3 | -24.4 | -66.1 | -5.3 | -51.1 |
| 56 | -214.2 | 0.0 | -16.10 | -186.1 | 0.0 | -11.5 | -62.1 | 0.0 | -23.9 |
| 57 | -385.1 | -29.38 | -1.19 | -334.3 | -26.2 | -0.9 | -73.6 | -2.9 | -1.8 |
| 58 | -94.8 | 53.18 | 51.14 | -97.6 | 47.3 | 36.4 | 71.7 | 5.2 | 76.2 |
| 59 | -230.5 | -54.07 | -36.18 | -197.5 | -48.1 | -25.7 | -31.1 | -5.3 | -53.9 |
| 60 | -134.4 | 0.29 | -4.80 | -120.8 | 0.3 | -3.4 | -32.9 | 0.1 | -7.2 |

Table 6.8. (Continued)

| Year | Equation (6.38) | | | Equation (6.39) | | | Equation (6.41) | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\tilde{\Delta y}^F_{1t}$ | $12.82 \cdot \Delta x_{1t}$ | $-2.98 \cdot \Delta x_{5t}$ | $\tilde{\Delta y}^F_{1t}$ | $11.41 \cdot \Delta x_{1t}$ | $-2.12 \cdot \Delta x_{3t}$ | $\tilde{\Delta y}^H_{1t}$ | $1.26 \cdot \Delta x_{1t}$ | $-4.44 \cdot \Delta x_{5t}$ |
| 61 | -218.6 | -13.59 | 21.99 | -194.5 | -12.1 | 15.6 | -24.9 | -1.3 | 32.8 |
| 62 | -130.4 | 13.02 | -31.85 | -110.1 | 11.6 | -22.7 | -46.6 | 1.3 | -47.5 |
| 63 | -162.3 | -13.30 | -1.58 | -145.6 | -11.8 | -1.1 | -54.9 | -1.3 | -2.4 |
| 64 | -231.4 | -26.18 | -60.70 | -189.49 | -23.3 | -43.2 | -135.6 | -2.6 | -90.4 |
| 65 | -137.7 | 38.93 | 24.0 | -126.6 | 34.6 | 17.0 | -98.7 | 3.8 | 35.7 |
| 66 | -286.3 | 49.2 | -20.86 | -239.8 | 43.8 | -14.8 | -126.7 | 4.8 | -31.1 |
| 67 | -309.6 | -74.97 | -4.62 | -273.5 | -66.7 | -3.3 | -110.9 | -7.4 | -6.9 |
| 68 | -188.3 | 12.13 | -14.42 | -162.1 | 10.8 | -10.3 | -106.9 | 1.2 | -21.5 |
| 69 | -72.1 | 11.21 | -3.55 | -60.9 | 10.0 | -2.5 | -34.9 | 1.1 | -5.3 |
| 70 | -163.5 | -23.34 | 34.75 | -148.5 | -20.8 | 24.7 | 10.3 | -2.3 | 51.8 |
| 71 | -89.9 | -32.26 | 18.44 | -94.4 | -28.7 | 13.1 | 24.5 | -3.2 | 27.5 |
| 72 | -44.2 | 72.83 | -18.30 | -33.4 | 64.8 | -13.0 | -36.6 | 7.2 | -27.3 |
| 73 | 81.6 | 179.72 | -17.55 | 77.9 | 160.0 | -12.5 | -25.7 | 17.7 | -26.2 |

variable, has had a constant negative effect on family farm labor employment.

In the hired farm labor market, the non-farm wage had the largest effect on labor employment. The range of its marginal impact was from negative 90.4 thousand in 1964 to plus 64.1 thousand in 1954. Similar to the family labor market, farm product prices had the most variable effect on hired farm labor employment.

If the goals of agricultural policy are to achieve higher farm income and to eliminate surplus farm labor, then these results have some policy implications: (1) The impact of price-support programs have larger positive effects on family farm income than on hired farm labor income. However, they will have adverse effects on the adjustment of surplus labor. (2) Increase in labor mobility, which is measured by the trend variable in this study, has the desirable effect of eliminating surplus farm labor and of raising farm income. Therefore, greater emphasis should be put on education, training and labor market information programs.

## F. Summary and Conclusions

In this chapter we have obtained some new quantitative information about the family farm labor market and the hired farm labor market. The empirical results support our hypotheses about the family farm labor market: (1) demand for family farm labor depends on the price of family farm labor and price of farm products, (2) supply of family farm labor is a function of the price of family farm labor and the price of non-farm labor. Further, the demanders and suppliers of family farm labor spread their response to changes in economic stimuli over several time periods. In general, the structural elasticities of the family farm labor market have increased over time. The demand elasticities with respect to the farm wage and to the price of farm products are inelastic in the short-run but elastic in the long-run. In the supply function farm wage and non-farm wage elasticities are inelastic in both short-run and long-run.

A static model was preferred to a dynamic model for the demand for hired farm labor. The coefficients of the farm wage and the price of farm products were statistically significant and had expected signs. The technology variable was inconclusive in both the family and the hired farm labor market. This does not necessarily mean that technology has no impact on the demand for farm labor. This inconclusive result may be due to the poor proxy for technology or due to the fact that the effect of technology is transmitted to the farm labor market through the relative decline in farm product prices. In the supply function, the non-farm wage had a negative coefficient which was highly significant. The farm wage regression coefficient was positive but insignificant. In comparison with the elasticities of previous studies, it was found that the demand elasticity with respect to the farm wage has increased substantially. However, the farm wage supply elasticity has still remained inelastic.

There are two kinds of empirical evidence of the small-sample properties of various estimators. One kind is Monte Carlo studies and the other kind is econometric studies of real world data in which two or more estimation procedures have been used for the same structural equations. This chapter provided us the latter kind of the empirical evidence of the small-sample properties of the following three estimators: (1) OLS, (2) 2SLS, and (3) FA2SLS. The major differences are: (1) compared to the rest of the estimates, OLS estimates are often unreasonable in signs and magnitude, (2) in most cases, OLS estimates of endogenous variables differ from those estimated by 2SLS and FA2SLS, (3) FA2SLS estimates of the lagged endogenous variables are less than those estimated by 2SLS and OLS, especially in the case where the errors appear to be autocorrelated, and (4) OLS, 2SLS, and FA2SLS differ considerably in the magnitude of estimated standard errors. These empirical results are consistent with the theoretical results on the properties of these estimators.

# VII. FORECASTING THE SIZE OF FARM LABOR EMPLOYMENT

## A. Introduction

The purpose of this chapter is to predict possible levels of farm employment in the next decade under different assumptions on future levels of exogenous variables. The projections of family, hired and total farm labor employment provide useful information for governmental long-term policy planning purposes, especially for the planning of educational investment in agricultural sector.

In general, three approaches have been devised to project future farm employment. These are (1) time trend extrapolation (Heady and Tweeten (44)), (2) derived demand for farm labor based on the projections of consumer demand for agricultural products (Daly and Egbert (15)), and (3) projections derived from an estimated single equation model and assumed behavior of independent variables (Arcus (6) and Martinos (59)). The procedure undertaken here is closely related to the third approach, except that the estimated single equation is replaced by the estimated reduced forms of simultaneous equation models developed in Chapter 6. By inserting the future values of exogenous variables and 1973 observations of endogenous variables, the reduced forms (restricted or unrestricted) recursively generate time paths of farm labor employment from 1974 to 1985.

The validity of our projections rests on two key assumptions: (1) the estimated dynamic structure relationship within the sample period (1941-73) holds in the future, and (2) the assumed values of exogenous variables are their future true values. These are strong assumptions since in the next ten years, many events such as war, strong increase in foreign demand and technology innovation may happen. Therefore, three sets of projections of farm labor employment are provided under three possible conditions for the future values of exogenous variables.

Section B discusses in detail the projection of exogenous variables. In Section C, three time paths of hired, family and total farm labor employment are presented.

## B. Projection of the Exogenous Variables

The reduced form equations of Section D of Chapter 6 contain four exogenous variables. They are (1) the ratio of the index of prices received for all farm products to the index of the prices paid for production items excluding hired farm labor wage rate, (2) the technology index, (3) real non-farm wage rate adjusted for the umployment rate, and (4) real non-farm income per worker adjusted for the unemployment rate.

We first assume that the exogenous variables can be adequately approximated by the following autoregressive trend model:

$$y_t = f(\theta : T) + u_t \tag{7.1}$$

$$u_t = \sum_{j=1}^{p} \rho_j u_{t-j} + e_t$$

$$e_t \sim \text{i.i.d. } (0, \sigma_e^2)$$

where $y_t$ is the dependent variable, $f(\theta: T)$ represents the mean function of time and $u_t$ follows a $p^{th}$ order stationary autoregressive process.

For example, if $f(\theta: T) = \alpha + \beta T$ and $u_t$ follows a second-order autoregressive process, then Model (7.1) is algebraically equivalent to the following equation:

$$y_t = \beta_o + \beta_1 T + \beta_2 y_{t-1} + \beta_3 y_{t-2} + e_t \quad . \tag{7.2}$$

For forecasting purposes, Equation (7.2) is adopted and is estimated by ordinary least squares. In practice, researchers must decide on the order of the autoregressive process. In our case, the overfitting procedure was used both for determining the order of polynomial trend and the order of the autoregressive process. Following the procedure mentioned above, various forms of Equation (7.2) are fitted to the exogenous variables. The estimated equations are as follows:

$$x_{5t} = \underset{(13.2)}{49.40} + \underset{(0.34)}{1.05T} + \underset{(0.18)}{0.62x_{5t-1}} - \underset{(0.16)}{0.26x_{5t-2}} \tag{7.3}$$

$$\text{RMS} = 62.4 \qquad\qquad R^2 = 0.78$$

$$x_{2t} = 28.55 + 0.66T + 0.33x_{2t-1} + 0.28x_{2t-2}$$
$$(11.8) \quad (0.29) \quad (0.17) \qquad (0.16) \tag{7.4}$$
$$\text{RMS} = 5.48 \qquad R^2 = 0.98$$

$$x_{1t} = 77.99 - 2.45T + 0.054T^2 + 0.65x_{1t-1} - 0.17x_{1t-2}$$
$$(20.24) \quad (0.6) \quad (0.01) \quad (0.21) \qquad (0.15) \tag{7.5}$$
$$\text{RMS} = 17.38 \qquad R^2 = 0.89$$

$$x_{8t} = 1484.45 + 55.51T + 0.76x_{8t-1} + 0.29x_{8t-2}$$
$$(357.88) \quad (13.44) \quad (0.19) \qquad (0.15) \tag{7.6}$$
$$\text{RMS} = 12753.2 \qquad R^2 = 0.98$$

The numbers in parentheses are the estimated standard errors of the coefficients. The projected values of exogenous variables from these equations are shown in Table 7.1. On examining the projected values of the exogenous variables, we found that projected real prices received for farm products grow rapidly after the 1980's due to the influence of the second-degree trend term. This situation is considered less probable. Therefore, this data is extrapolated by the grafted polynomials developed by Fuller (23).

The independent variable of the grafted polynomials for $x_{1t}$, real prices received for farm products, are specified as:

$Z_{1t} = T \qquad T = 1$ for 1941, ..., $T = 33$ for 1973

$Z_{2t} = (T-18)^2 \qquad t < 1958$
$= 0 \qquad$ otherwise

$Z_{3t} = (T-11)^2 \qquad t < 1951$
$= 0 \qquad$ otherwise

The resulting estimated grafted polynomial equation is

$$x_{1t} = \underset{(4.5)}{95.36} + \underset{(0.18)}{0.15}Z_{1t} + \underset{(0.05)}{0.40}Z_{2t} - \underset{(0.13)}{1.01}Z_{2t} \tag{7.7}$$

$$\text{RMS} = 22.09 \qquad R^2 = 0.86$$

The numbers in parentheses are the estimated standard errors of the coefficients. The data and the fitted grafted polynomial (7.7) is plotted in Figure 7.1. The projected values of $x_{1t}$ based on Equation (7.7) are also reported in Table 7.1.

The second set of the assumed values of the exogenous variables is generated based on the assumption that recent observations have greater influences on the future behavior of exogenous variables. Various forms of Equation (7.2) are again fitted to exogenous variables from 1959 to 1973 and the resulting estimated equations are as follows:

Table 7.1. The projected exogenous variables from 1975 to 1985[a]

| Year | Technology index ($x_{2t}$) | Adjusted real non-farm wage rate ($x_{5t}$) | The index of real price received for all farm products ($x_{1t}$) | The index of real price received for all products ($x^*_{1t}$)[b] |
|---|---|---|---|---|
| 1975 | 127.7 | 134.1 | 110.9 | 100.7 |
| 76 | 129.3 | 135.1 | 113.4 | 100.8 |
| 77 | 130.9 | 136.5 | 116.1 | 100.9 |
| 78 | 132.6 | 138.1 | 119.1 | 101.1 |
| 79 | 134.3 | 139.8 | 122.2 | 101.3 |
| 80 | 135.9 | 141.4 | 125.5 | 101.4 |
| 81 | 137.6 | 143.1 | 129.1 | 101.6 |
| 82 | 139.3 | 144.7 | 132.9 | 101.7 |
| 83 | 140.9 | 146.3 | 136.8 | 101.8 |
| 84 | 142.5 | 147.9 | 140.9 | 102.0 |
| 85 | 144.2 | 149.6 | 145.4 | 102.2 |

[a]The projected exogenous variables were computed from Equations (7.4), (7.3), and (7.5).

[b]$x^*_{1t}$ was calculated from Equation (7.7).

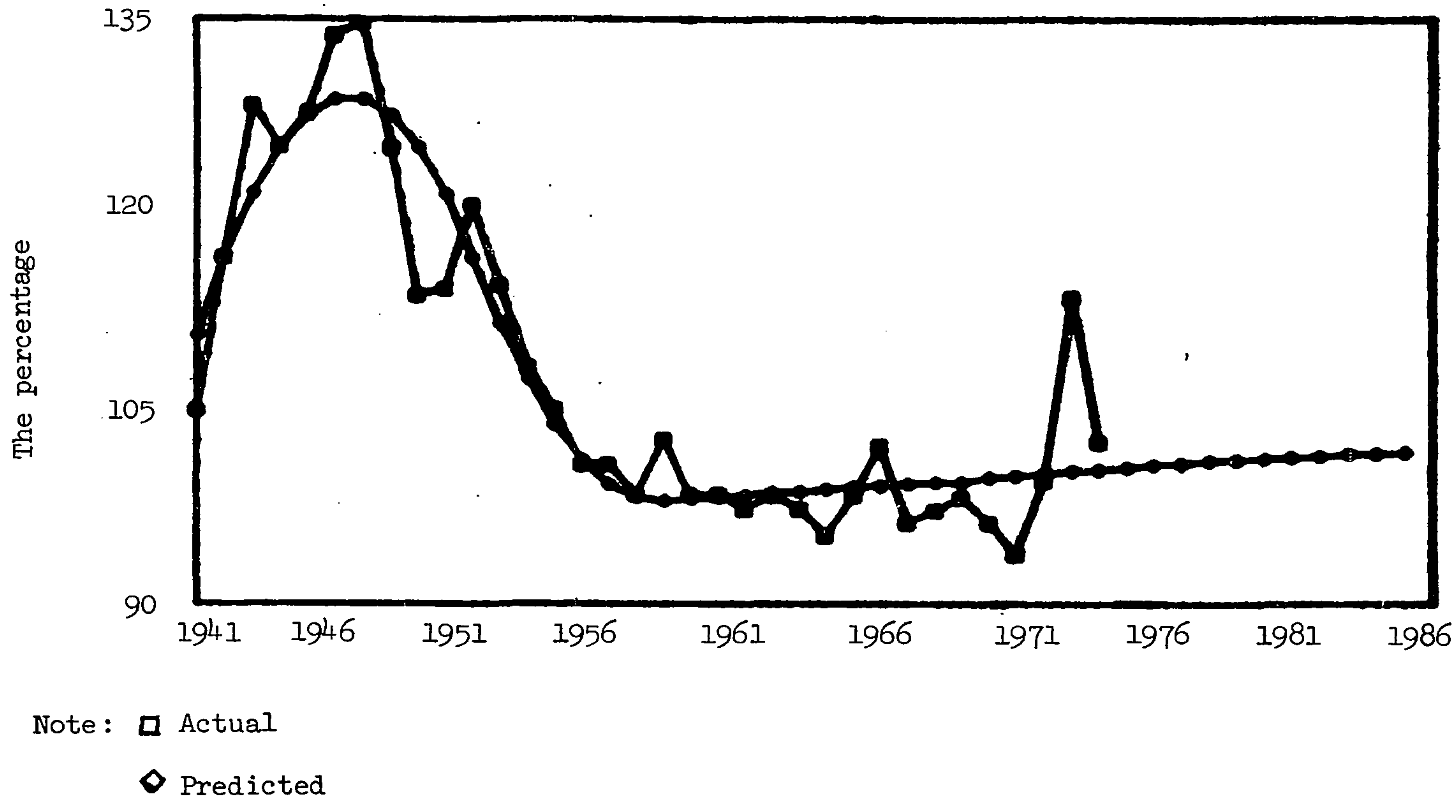



Figure 7.1. Actual and predicted real price received for all farm products, 1941-1985

$$x_{5t} = 39.42 + 0.74T + 0.52x_{5t-1} \qquad (7.8)$$
$$(17.4) \quad (0.72) \quad (0.22)$$
$$RMS = 59.3 \qquad R^2 = 0.61$$

$$x_{2t} = 92.17 + 1.46T - 0.15x_{2t-1} \qquad (7.9)$$
$$(2.17) \quad (0.38) \quad (0.26)$$
$$RMS = 4.4 \qquad R^2 = 0.88$$

Again, the numbers in parentheses are standard errors of estimated coefficients. The projected values of $x_{2t}$ , the technology index, and $x_{5t}$ , real non-farm wage rate adjusted for unemployment, are presented in Table 7.2.

In the period 1959 to 1973, there are no discernible trend patterns in real prices received for all farm products. Hence, the ratio of the index of price received for all farm products to the index of prices paid for production purposes, excluding hired farm labor wage is assumed to follow a simple growth formula

$$x_{1,1973+D} = \frac{\bar{\bar{X}}_1(1+\gamma_1)^D}{\bar{\bar{X}}_2(1+\gamma_2)^D} \qquad (7.10)$$

where

$\bar{\bar{X}}_1$ stands for the average of the last three observations on the index of price received for all farm products,

$\bar{\bar{X}}_2$ stands for the average of the last three observations on the index of price paid for production purposes excluding hired labor wage rate,

$\gamma_i$ , $i = 1, 2$ represents compound annual growth rates of the $i^{th}$ variables, and

$$D = (t - 1973) \qquad t \leq 1973 .$$

Two pairs of growth rates are assigned to (7.10). In the first pair, $\gamma_1$ is assumed to be 3.0% for the index of price received for farm product and $\gamma_2$ is equal to 2.0% for the index of prices paid for production purposes excluding hired farm labor wage rate. In the second pair, $\gamma_1$ is equal to 4.0% and $\gamma_2$ remains the same as before. These two pairs of growth rates are used to reflect relative farming profitability due to an increase in foreign and domestic demand. The projected values based on these two pairs of growth rates are shown in Table 7.2 .

C. Projection Results

In this section, we present projected time paths of family, hired and total farm labor employment from 1974 to 1985. They are calculated from the unrestricted reduced form and restricted reduced form of the dynamic econometric models developed in Chapter 6. The reduced form of

Table 7.2. The projected exogenous variables from 1975 to 1985

| Year | Technology index ($x_{2t}$)[b] | Adjusted real non-farm wage rate ($x_{5t}$)[b] | The index of real price received for all farm products ($x_{1t}(1)$)[a] | The index of real price received for all farm products ($x_{1t}(2)$)[a] |
|---|---|---|---|---|
| 1975 | 125.7 | 127.9 | 105.5 | 105.3 |
| 76 | 126.1 | 132.3 | 106.5 | 107.4 |
| 77 | 127.5 | 135.3 | 107.6 | 110.0 |
| 78 | 128.7 | 137.6 | 108.6 | 111.6 |
| 79 | 130.0 | 140.0 | 109.7 | 113.8 |
| 80 | 131.3 | 141.3 | 110.8 | 116.1 |
| 81 | 132.5 | 142.9 | 111.8 | 118.3 |
| 82 | 133.8 | 144.5 | 112.9 | 120.7 |
| 83 | 135.1 | 146.1 | 114.1 | 123.0 |
| 84 | 136.4 | 147.6 | 115.2 | 125.4 |
| 85 | 137.6 | 149.2 | 116.3 | 128.0 |

[a] $x_{1t}(1)$ represents the projected value of $x_{1t}$ under the first assumption (i.e., $\gamma_1 = 0.3$, $\gamma_2 = 0.2$). $x_{1t}(2)$ represents the projected value of $x_{1t}$ under the second assumption (i.e., $\gamma_1 = 0.4$, $\gamma_2 = 0.2$).

[b] The projected exogenous variables are calculated from Equations (7.9) and (7.8), respectively.

a simultaneous equation estimated by ordinary least squares is called unrestricted reduced form. The unrestricted reduced form equations used in this section are as follows:

Family farm labor market

$$y^F_{1t} = 604.94 + 12.71x_{1t} + 6.64x_{2t} - 2.57x_{5t} - 49.3x_{6t} + 0.72y^F_{1t-1} - 143.73x_{3t}$$

$$\quad (1076.9)\ (2.06)\quad (6.62)\quad (1.69)\quad (16.81)\quad (0.08)\qquad (89.66)$$

$$\text{RMS} = 7482.3 \qquad R^2 = 0.99 \tag{7.11}$$

$$y^F_{2t} = -26.58 + 0.73x_{1t} - 0.12x_{2t} + 0.16x_{5t} + 2.74x_{6t} - 0.0073y^F_{1t-1} + 0.93x_{3t}$$

$$\quad (67.44)\ (0.13)\quad (0.42)\quad (0.11)\quad (1.05)\quad (0.005)\qquad (5.62)$$

$$\text{RMS} = 29.38 \qquad R^2 = 0.94 \tag{7.12}$$

Hired farm labor market

$$y^H_{1t} = 732.2 + 1.36x_{1t} + 0.91x_{2t} - 4.49x_{5t} - 5.04x_{6t} + 0.77y^H_{1t-1} - 104.2x_{3t}$$

$$\quad (334.39)\ (1.09)\ (2.8)\quad (0.77)\quad (7.13)\quad (0.07)\qquad (28.36)$$

$$\text{RMS} = 1565.8 \qquad R^2 = 0.99 \tag{7.13}$$

$$y^H_{2t} = 83.04 + 0.44x_{1t} + 0.08x_{2t} + 0.16x_{5t} + 0.62x_{6t} - 0.03y^H_{1t-1} - 2.65x_{3t}$$

$$\quad (34.59)\ (0.11)\quad (0.28)\quad (0.07)\quad (0.74)\quad (0.01)\qquad (2.93)$$

$$\text{RMS} = 16.73 \qquad R^2 = 0.97 \tag{7.14}$$

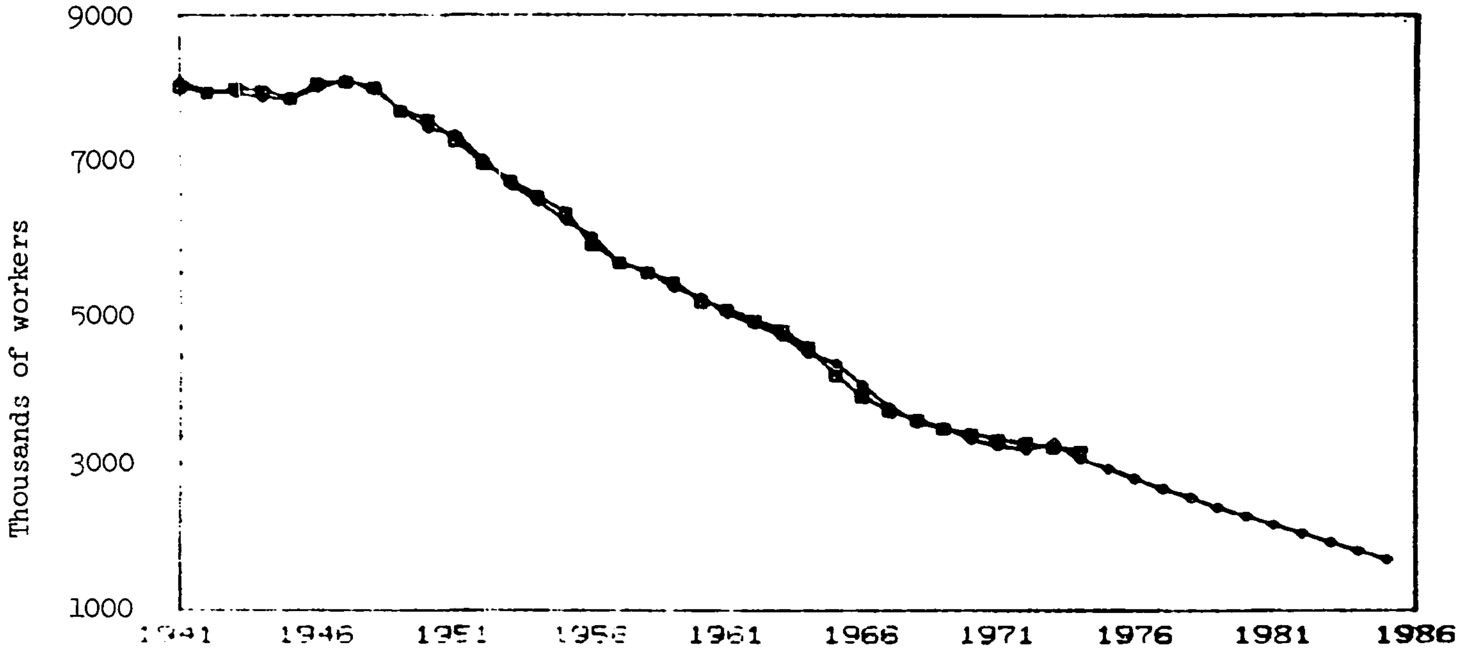
As before, the number in parentheses are standard errors of the estimated coefficients. The restricted reduced forms are obtained from Section D of Chapter 6. We found that the coefficients of the restricted reduced form and the unrestricted reduced form are generally consistent in signs but are somewhat different in their magnitudes. Hence, it is interesting to observe the prediction performance of these two types of reduced forms.

In projecting farm labor employment up to 1985, three sets of assumed values of exogenous variables are used and the corresponding three sets of projected family, hired and total farm labor employment are obtained. Under Case 1, real non-farm wage rate adjusted for the unemployment rate, and the technology index are assumed to grow according to their first-order autoregressive trend functions from 1959 to 1973. The index of price received for farm product is assumed to increase 3.0% annually and the index of prices paid for production purposes is to increase 2.0%. The projections of farm labor employment from 1974 to 1985 are given in Table 7.3 and the actual and predicted family and hired farm labor employment under Case 1 are displayed from Figures 7.2 to 7.5. The projected number of family farm labor in 1985 is slightly over 1660 (thousands). This number is approximately 40.0% below the 1973 level. Hired farm labor is projected to be around 600 (thousands), compared with 1168 (thousands) in 1973.

In Case 2, the future values of exogenous variables are the same as those in Case 1 except that the index of the price of farm product received is assumed to grow at 4.0% annually. The projected farm labor

Table 7.3. Projected farm labor employment in the U.S., 1974-1985 (under Case 1)

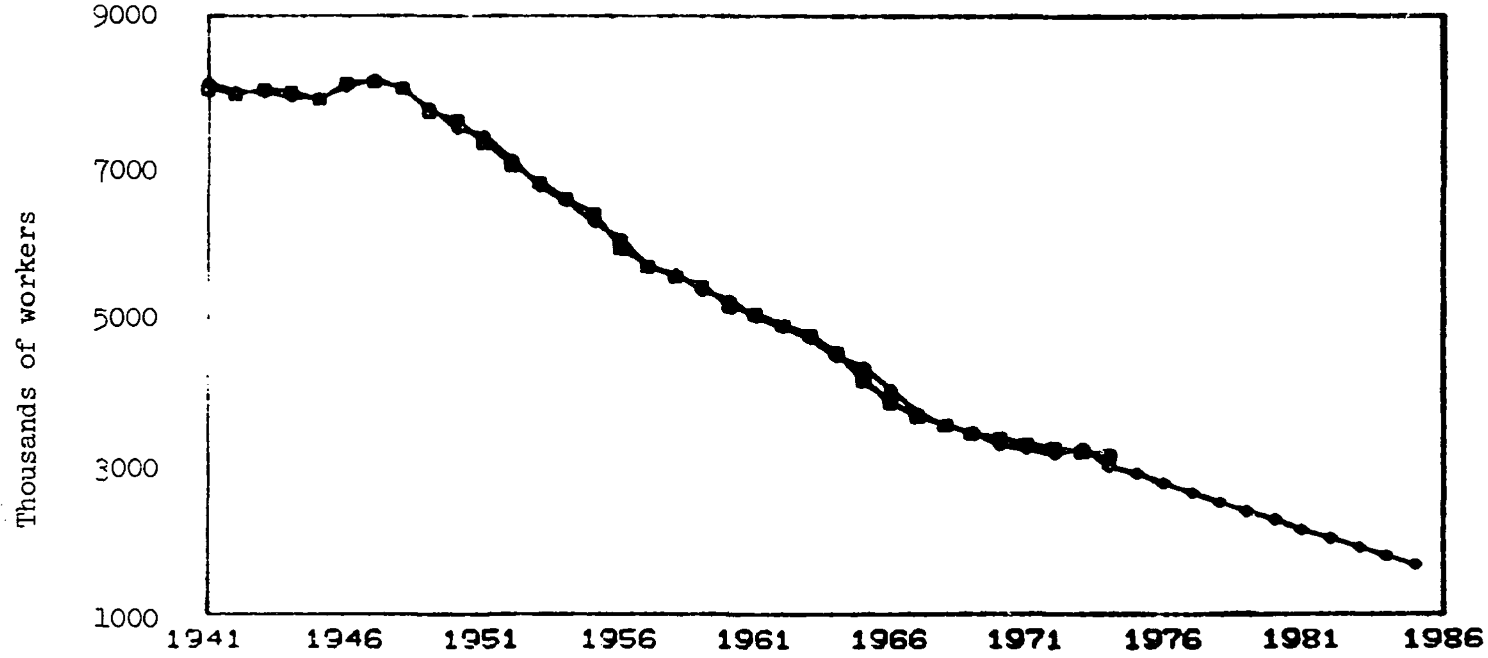
| Year | Family farm labor calculated from Equation (6.38) | Hired farm labor calculated from Equation (6.44) | Total farm labor | Family farm labor calculated from Equation (7.11) | Hired farm labor calculated from Equation (7.13) | Total farm labor |
|---|---|---|---|---|---|---|
| 1974 | 3008.8 | 1147.2 | 4156.0 | 2983.1 | 1159.6 | 4097.7 |
| 75 | 2883.0 | 1103.8 | 3986.8 | 2868.6 | 1127.6 | 3996.2 |
| 76 | 2750.1 | 1047.4 | 3797.5 | 2740.2 | 1080.1 | 3820.2 |
| 77 | 2619.8 | 987.9 | 3607.7 | 2613.2 | 1027.9 | 3641.1 |
| 78 | 2491.7 | 929.0 | 3420.7 | 2486.3 | 975.1 | 3461.4 |
| 79 | 2366.4 | 870.2 | 3236.6 | 2361.4 | 921.5 | 3282.9 |
| 80 | 2246.3 | 816.4 | 3062.7 | 2240.6 | 872.3 | 3112.9 |
| 81 | 2127.5 | 764.9 | 2892.4 | 2120.1 | 824.8 | 2944.9 |
| 82 | 2011.0 | 715.4 | 2726.4 | 2000.7 | 779.0 | 2779.7 |
| 83 | 1897.5 | 667.6 | 2565.1 | 1886.2 | 734.5 | 2620.7 |
| 84 | 1785.0 | 621.4 | 2406.0 | 1771.6 | 691.3 | 2462.9 |
| 85 | 1672.8 | 575.9 | 2248.7 | 1657.0 | 648.6 | 2305.6 |




Note: □ Actual

◇ Predicted

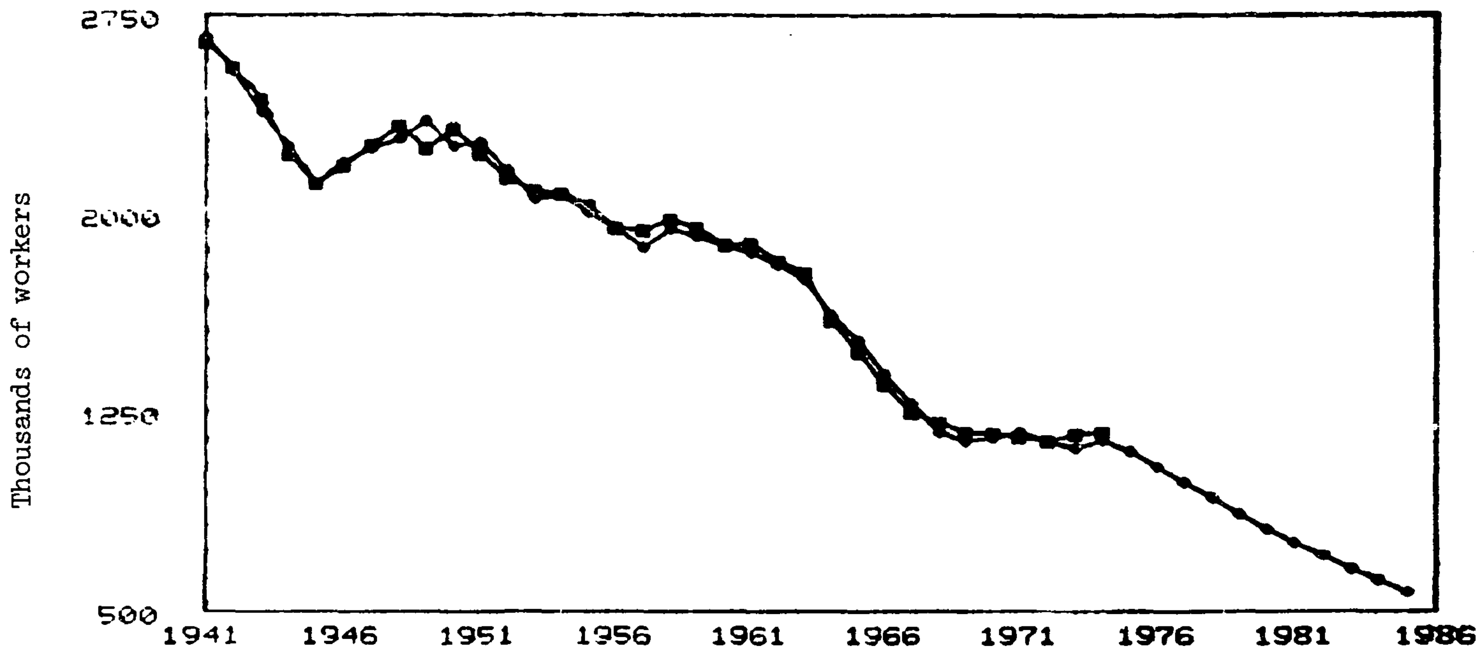
Figure 7.2. Actual and predicted numbers of family farm workers in the U.S., 1941-1985 (predicted estimates from Equation (6.38) and projected exogenous variables are calculated under Case 1)




Note: Actual

Predicted

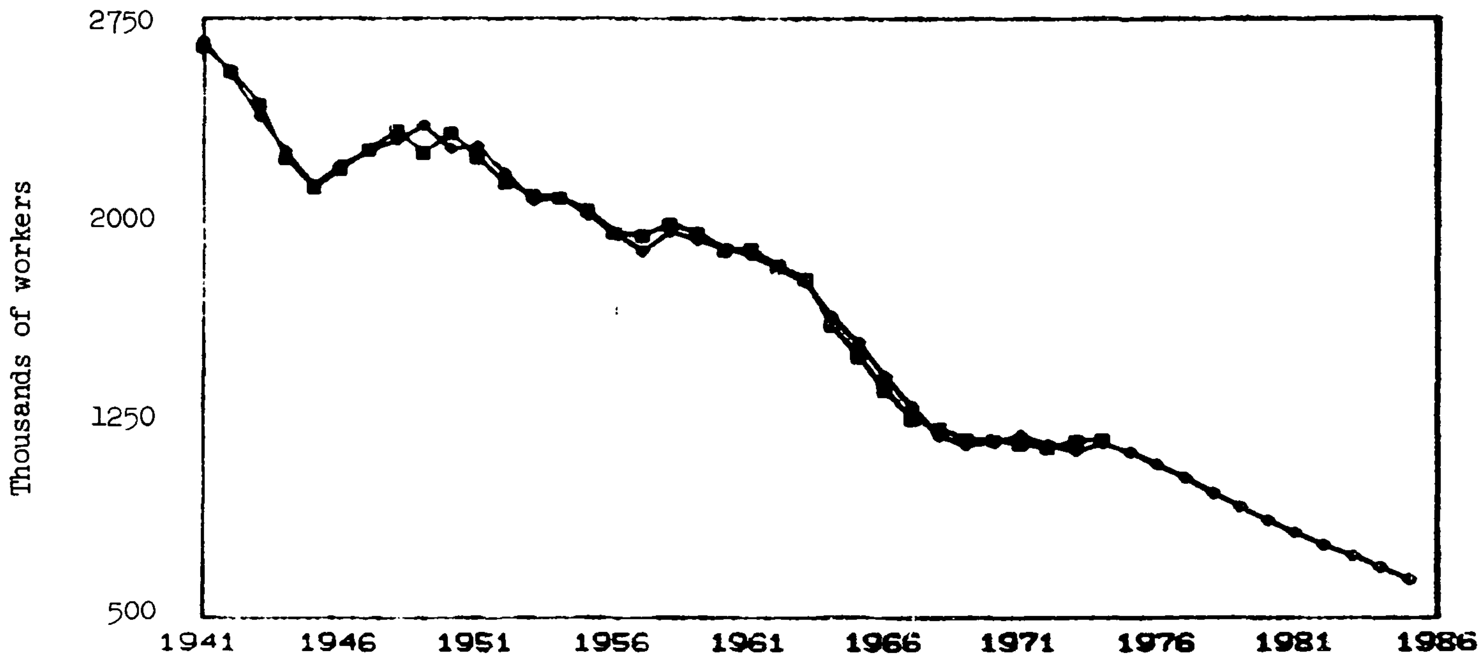
Fiture 7.3. Actual and predicted numbers of family farm workers in the U.S., 1941-1985 (predicted estimates from Equation (7.11) and projected exogenous variables are calculated under Case 1)




Note: □ Actual

◇ Predicted

Figure 7.4. Actual and predicted numbers of hired farm workers in the U.S., 1941-1985 (predicted estimates from Equation (6.44 ) and predicted exogenous variables are calculated under Case 1)




Note: ◻ Actual

◇ Predicted

Figure 7.5. Actual and predicted numbers of hired farm workers in the U.S., 1941-1985 (predicted estimates from Equation (7.13) and predicted exogenous variables are calculated under Case 1)

employment under Case 2 is shown in Table 7.4 and the corresponding observed and predicted farm labor employment are plotted from Figures 7.6 to 7.9. Family farm labor employment under Case 2 is projected to be 2046 thousands or 540 thousands more than that under Case 1. Hired farm labor employment in 1985 is projected to be around 460 thousands below the 1973 level, but 46 thousands more than that under Case 1. The projected farm labor employment under Case 2 is higher than those under Case 1. This phenomenon reflects the impact of stronger demand for farm commodities on farm labor employment.

In the third case, future exogenous variables are assumed to behave according to trend function from 1941 to 1973. The projected farm labor employment under the third case is presented in Table 7.5 and the observed and predicted farm labor employment are plotted in Figures 7.10 to 7.13. Family farm labor is projected at 1200 thousands, compared with 3196 in 1973, a 60 percent decline. Hired farm labor employment in 1985 also remains below the 1973 level at 597 thousands, compared with 1168 thousands in 1973.

As expected, there is a difference between the projected farm labor employment generated from two different reduced forms, but the difference is not large. In this study, there is no definite conclusion on prediction performance of these two reduced forms. Comparing the 1974 observed values with the 1974 predicted values, the restricted reduced form performs better in predicting family labor employment, but slightly worse in hired farm labor employment.

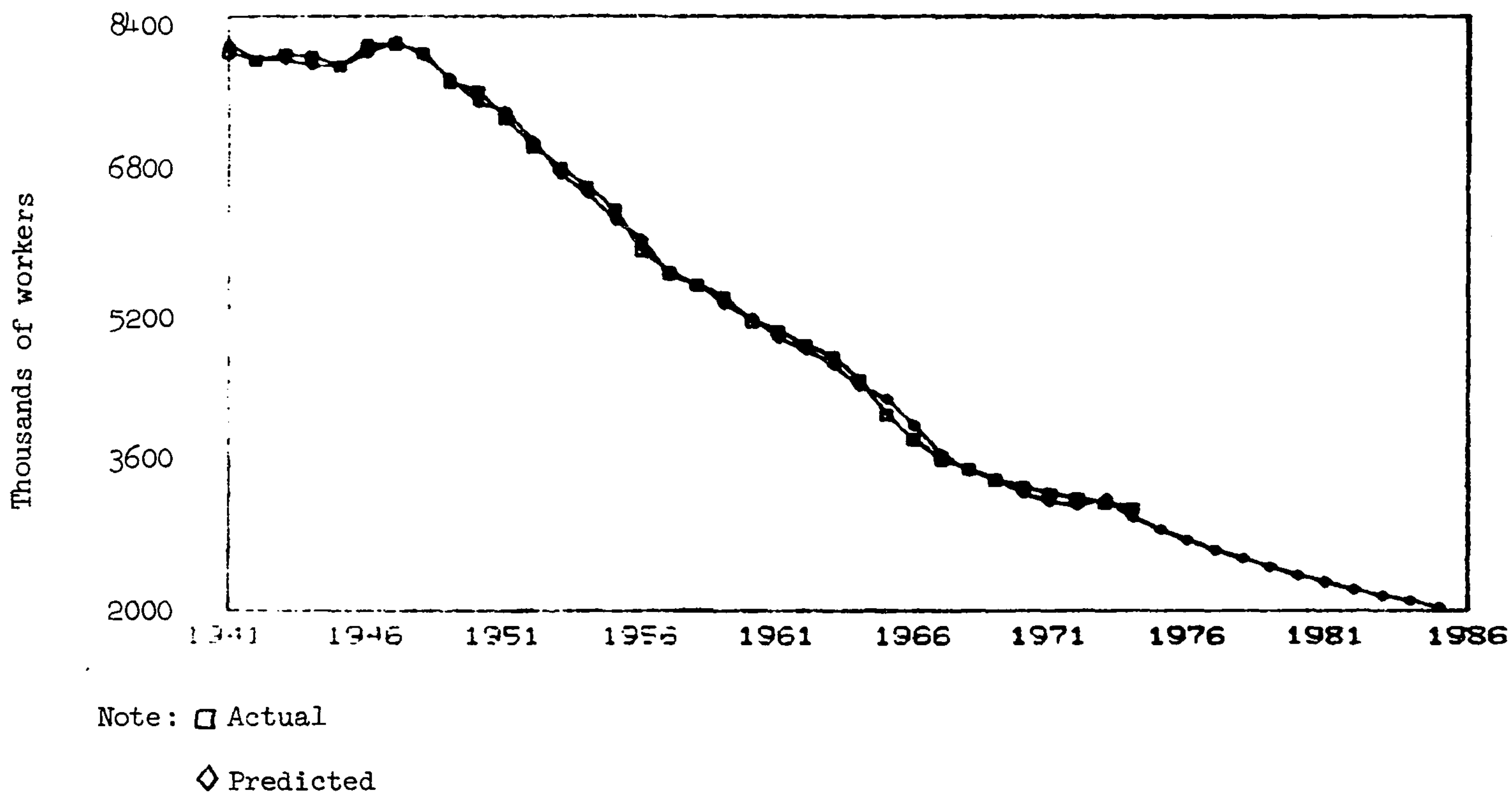



Figure 7.6. Actual and predicted number of family farm workers in the U.S., 1941-1985 (predicted estimates from Equation (6. 38) and predicted exogenous variables are calculated under Case 2)

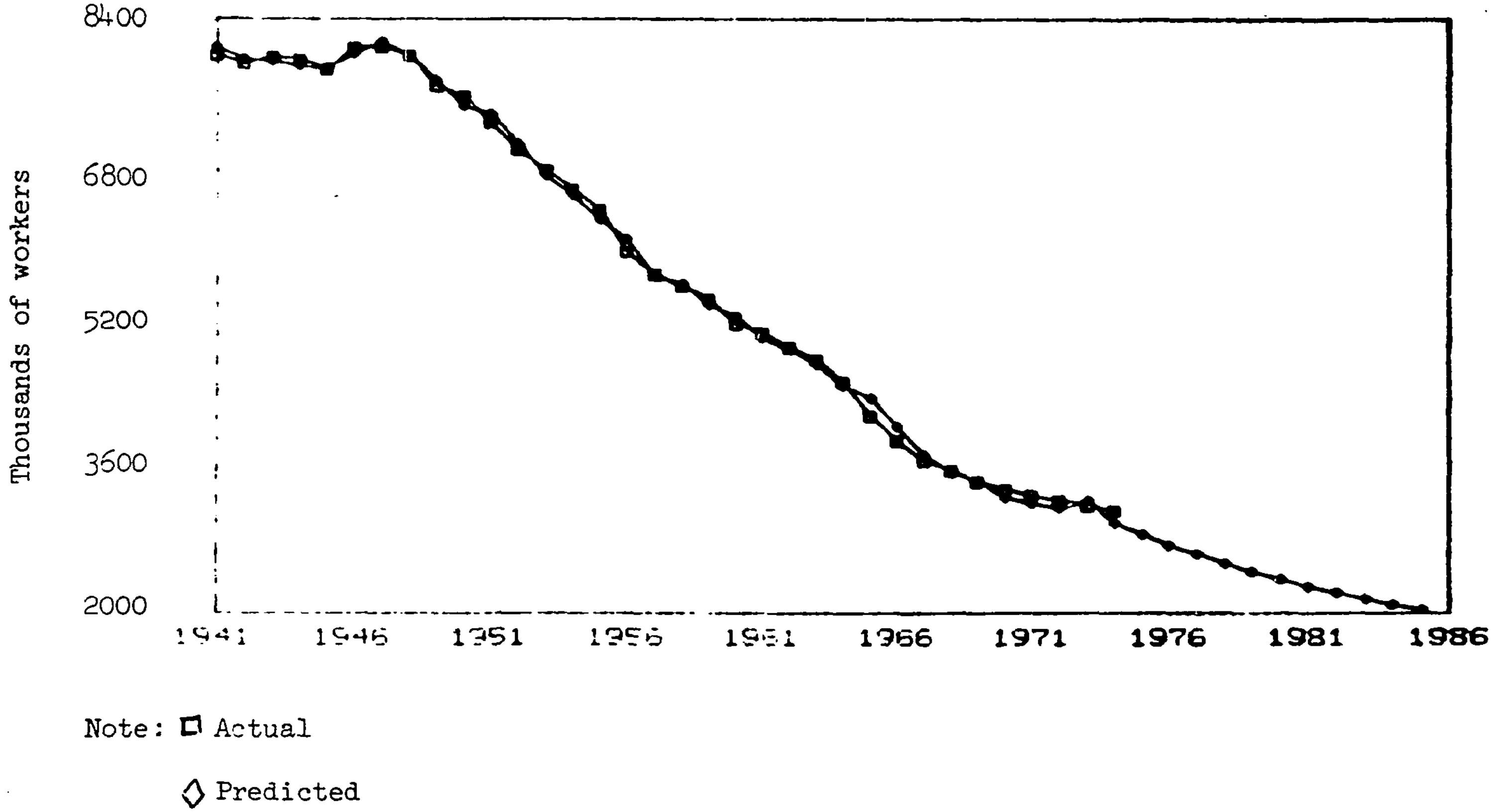



Figure 7.7. Actual and predicted number of family farm workers in the U.S., 1941-1985 (predicted estimates from Equation (7.11) and predicted exogenous variables are calculated under Case 2)

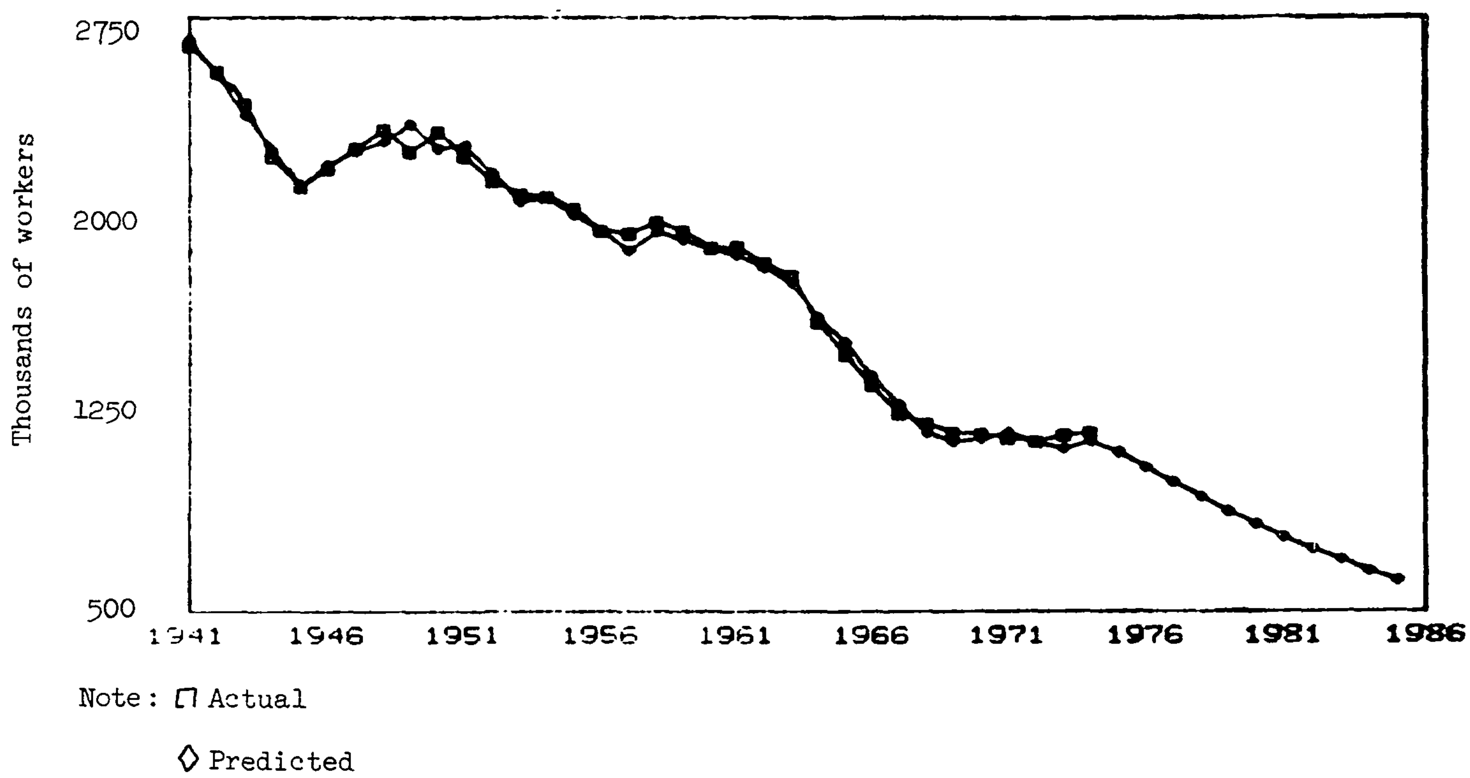
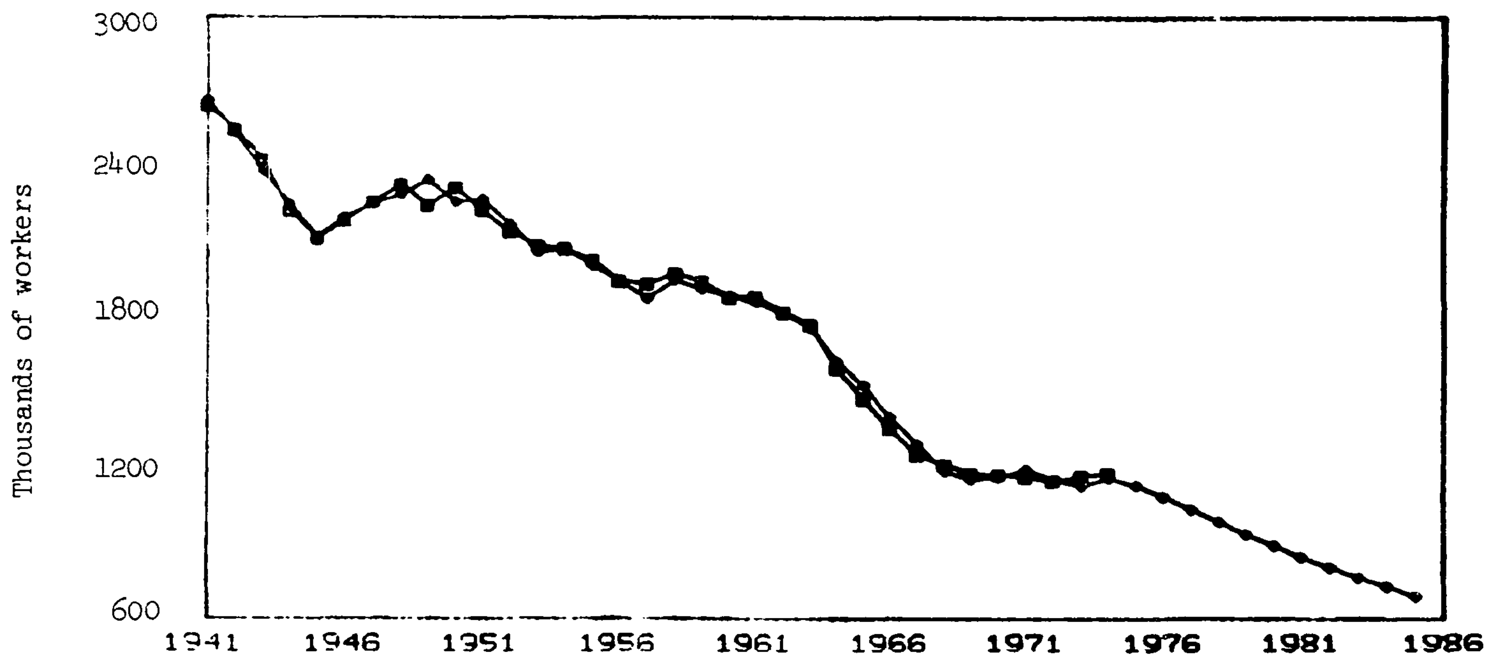



Figure 7.8. Actual and predicted number of hired farm workers in the U.S., 1941-1985 (predicted estimates from Equation (6.44) and projected exogenous variables are calculated under Case 2)




Note: ◻ Actual

◊ Predicted

Figure 7.9. Actual and predicted number of hired farm workers in the U.S., 1941-1985 (predicted estimates from Equation (7.13) and projected exogenous variables are calculated under Case 2)

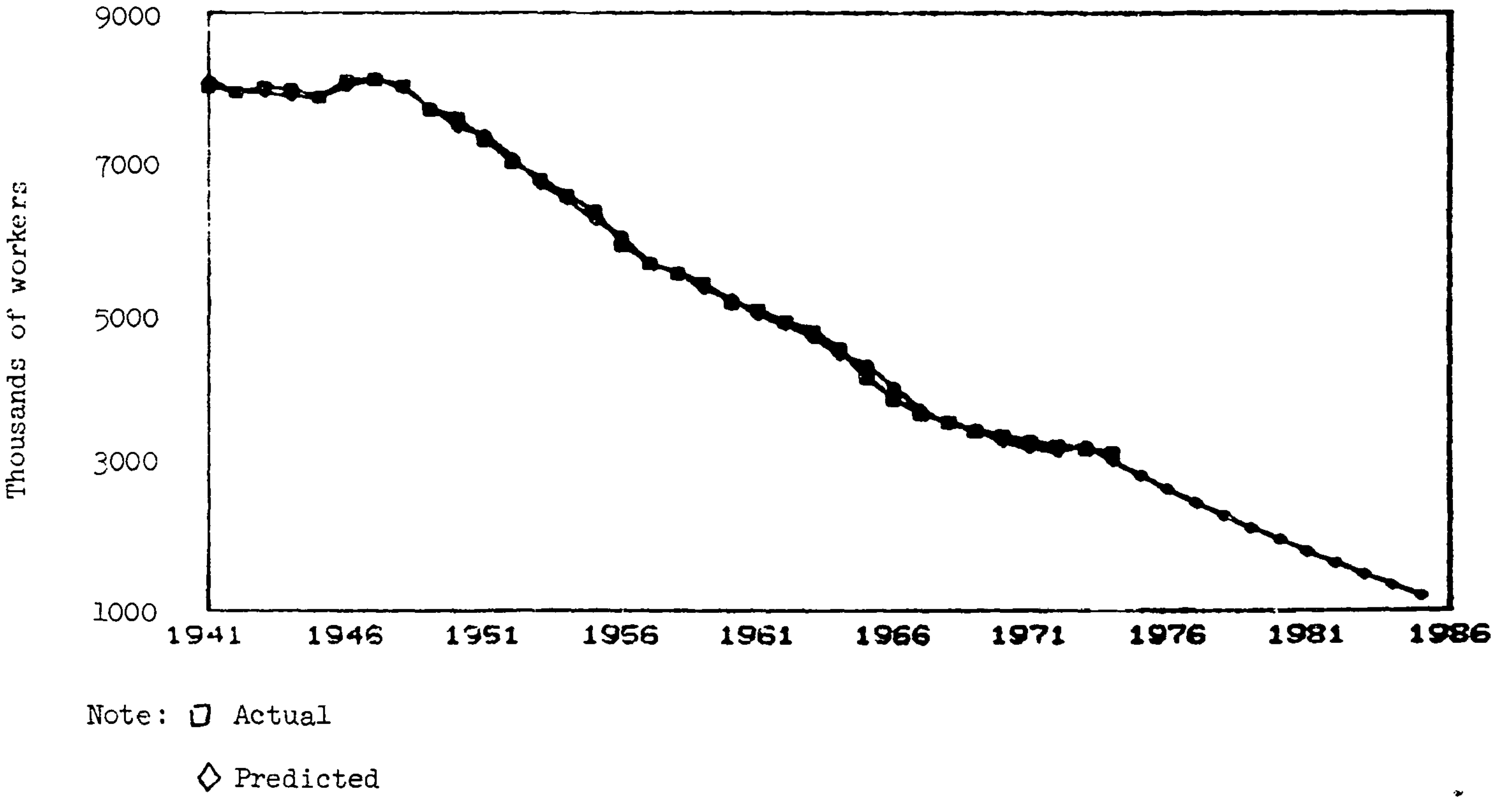



Figure 7.10. Actual and predicted number of family farm workers in the U.S., 1941-1985 (predicted estimates from equation (6.38) and predicted exogenous variables calculated under Case 3)

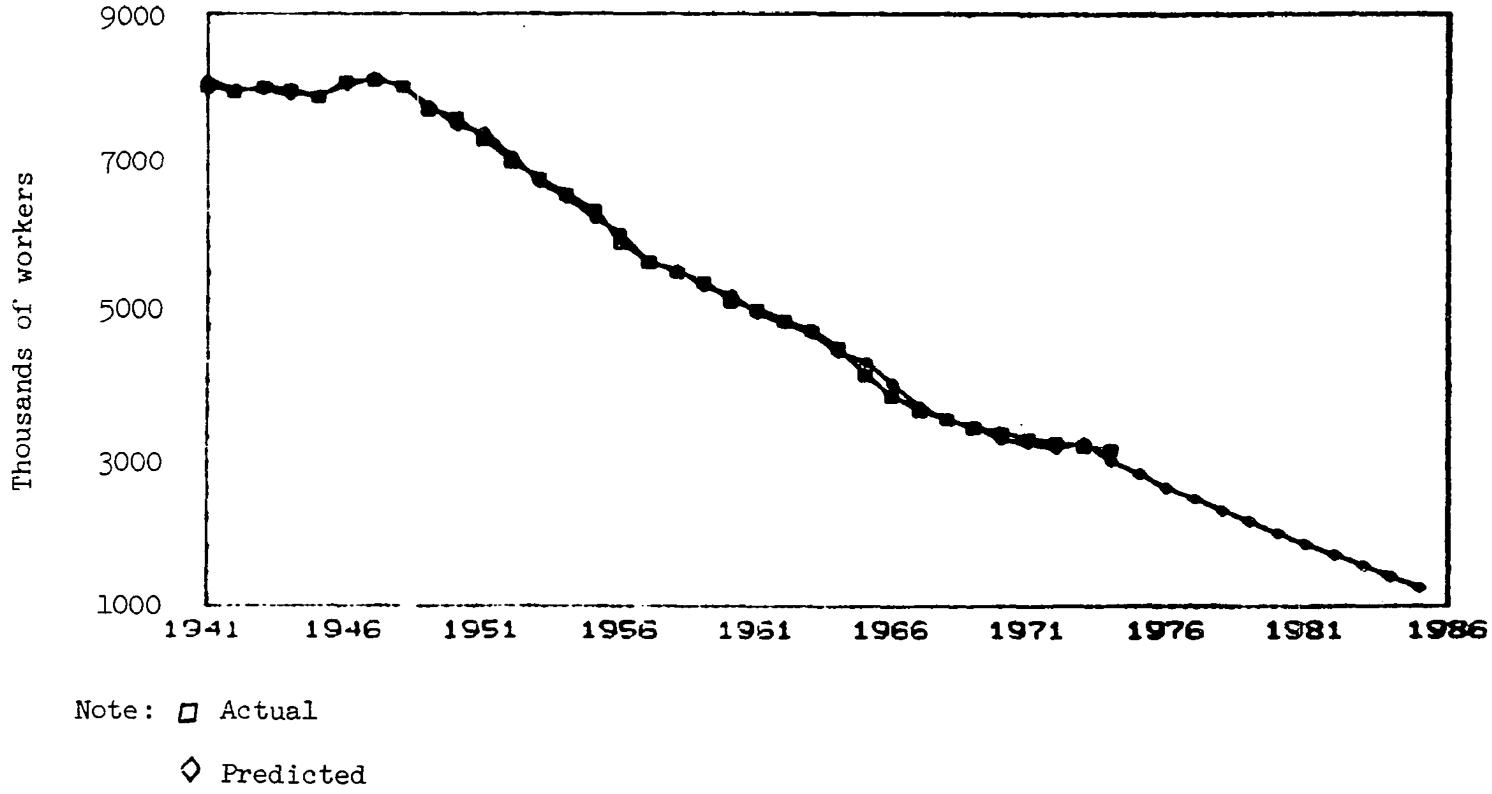
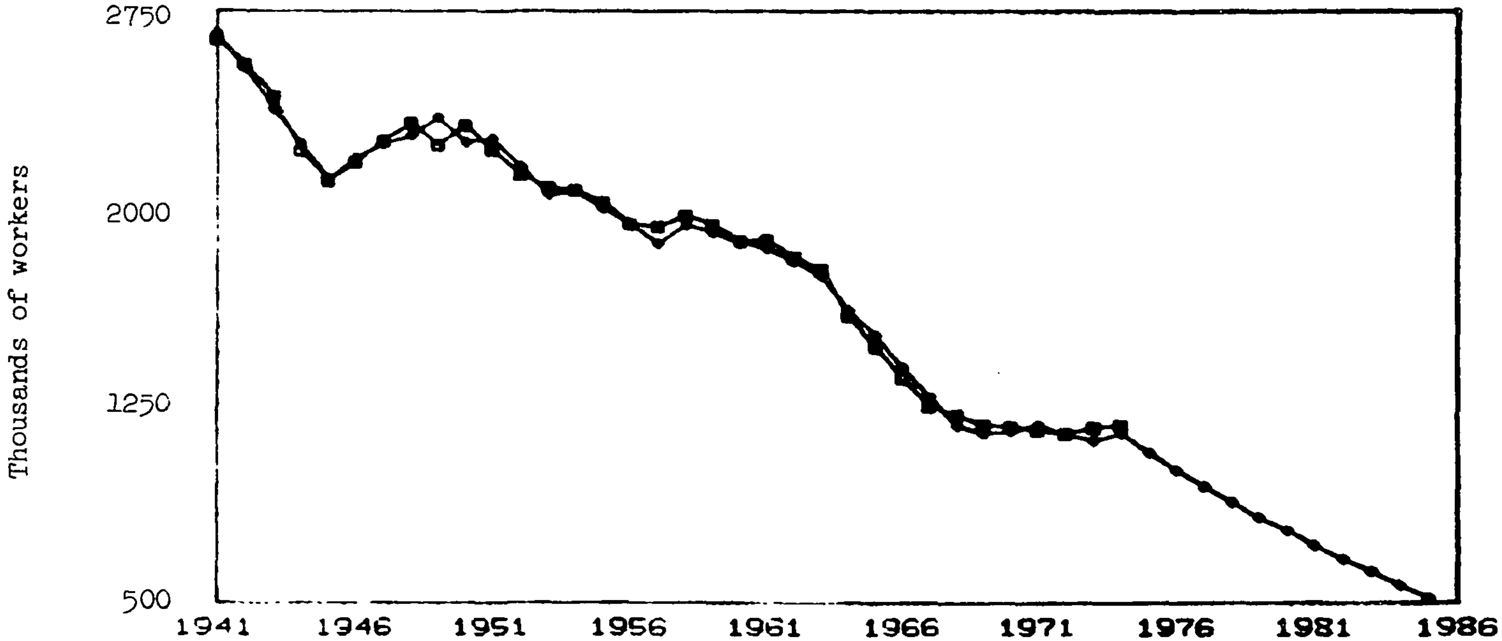



Figure 7.11. Actual and predicted number of family farm workers in the U.S., 1941-1985 (predicted estimates from Equation (7.11) and predicted exogenous variables are calculated under Case 3)




Note: □ Actual

◇ Predicted

Figure 7.12. Actual and predicted number of hired farm workers in the U.S., 1941-1985 (predicted estimates from Equation (6.44) and projected exogenous variables are calculated under Case 3)

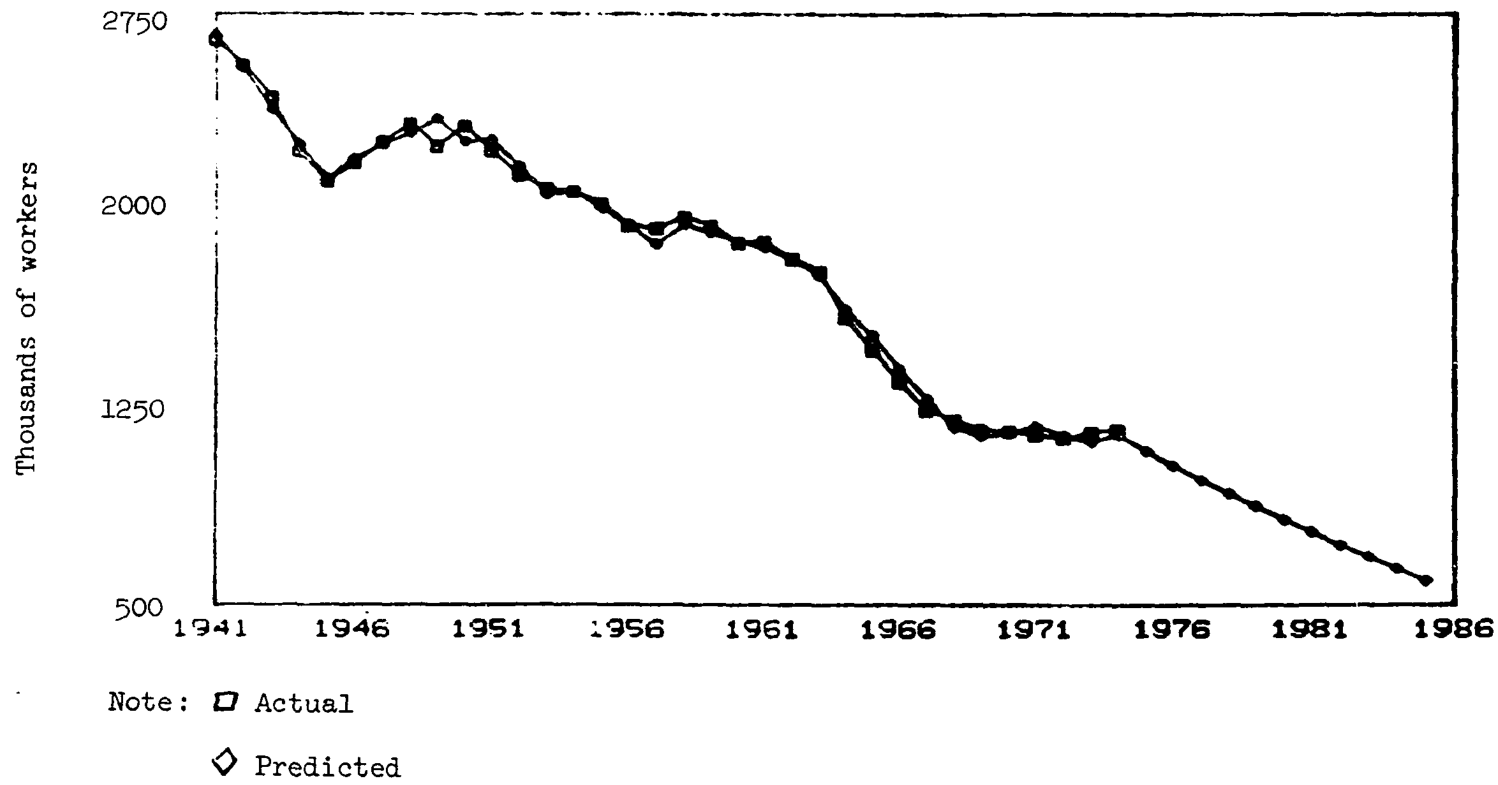



Figure 7.13. Actual and predicted number of hired farm workers in the U.S., 1941-1984 (predicted estimates from Equation (7.13) and projected exogenous variables are calculated under Case 3)

Table 7.4. Projected farm labor employment in the U.S., 1974-1985 (under Case 2)

| Year | Family farm labor calculated from Equation (6.38) | Hired farm labor calculated from Equation (6.41) | Total farm labor | Family farm labor calculated from Equation (7.11) | Hired farm labor calculated from Equation (7.13) | Total farm labor |
|---|---|---|---|---|---|---|
| 1974 | 3008.9 | 1147.2 | 4156.1 | 2983.1 | 1159.6 | 4142.7 |
| 75 | 2880.4 | 1103.5 | 3983.9 | 2866.1 | 1127.3 | 3993.4 |
| 76 | 2760.0 | 1048.3 | 3808.3 | 2749.8 | 1081.1 | 3830.9 |
| 77 | 2657.4 | 991.7 | 3649.1 | 2650.6 | 1032.0 | 3682.6 |
| 78 | 2556.5 | 935.6 | 3492.1 | 2551.5 | 982.3 | 3533.8 |
| 79 | 2464.4 | 880.5 | 3344.9 | 2460.6 | 932.6 | 3393.2 |
| 80 | 2382.8 | 831.0 | 3213.8 | 2379.7 | 888.0 | 3267.7 |
| 81 | 2306.3 | 784.4 | 3090.7 | 2303.2 | 845.7 | 3148.9 |
| 82 | 2236.2 | 740.2 | 2976.4 | 2233.3 | 805.5 | 3038.8 |
| 83 | 2169.2 | 697.9 | 2867.1 | 2166.7 | 766.9 | 2933.6 |
| 84 | 2106.0 | 657.6 | 2763.6 | 2104.1 | 730.1 | 2834.2 |
| 85 | 2047.4 | 618.5 | 2665.9 | 2046.1 | 694.1 | 2740.2 |

Table 7.5 Projected farm labor employment in the U.S., 1974-1985 (under Case 3)

| Year | Family farm labor calculated from Equation (6.38) | Hired farm labor calculated from Equation (6.44) | Total farm labor | Family farm labor calculated from Equation (7.11) | Hired farm labor calculated from Equation (7.13) | Total farm labor |
|---|---|---|---|---|---|---|
| 1974 | 3008.8 | 1147.2 | 4156.2 | 2983.1 | 1159.6 | 4142.7 |
| 75 | 2807.1 | 1071.5 | 3878.6 | 2805.0 | 1095.1 | 3900.1 |
| 76 | 2622.1 | 1005.0 | 3627.1 | 2635.8 | 1037.8 | 3673.6 |
| 77 | 2447.8 | 943.7 | 3391.5 | 2472.1 | 984.1 | 3456.2 |
| 78 | 2281.7 | 885.9 | 3167.6 | 2313.6 | 932.7 | 3246.3 |
| 79 | 2121.2 | 830.2 | 2951.4 | 2158.8 | 882.5 | 3041.3 |
| 80 | 1963.3 | 776.2 | 2739.5 | 2005.0 | 833.4 | 2838.4 |
| 81 | 1808.5 | 723.5 | 2532.0 | 1853.5 | 784.9 | 2638.4 |
| 82 | 1654.9 | 672.2 | 2327.1 | 1702.7 | 737.3 | 2440.0 |
| 83 | 1501.8 | 621.7 | 2123.5 | 1551.7 | 690.2 | 2241.9 |
| 84 | 1350.5 | 572.0 | 1922.5 | 1401.8 | 643.6 | 2045.4 |
| 85 | 1200.3 | 522.6 | 1722.9 | 1253.2 | 597.1 | 1850.3 |

In summary, farm labor employment will continue to decline in the next decade but at a slower rate. Total farm labor in 1985 will be 34-57 percent below the 1973 level. As disparities between farm income and non-farm income become smaller, and if the increase in technology slows down, the demand for farm commodities and labor mobility will be dominant factors affecting the level of future farm labor employment. Further, the transfer of labor from farm to non-farm sectors will make a smaller net contribution to the growing non-farm labor force in the future.

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# X. APPENDIX

Table A1. Actual values for the endogenous variables used in econometric analysis, 1941-1973

| Year | Family farm labor $y^F_{1t}$ | Hired farm labor employment $y^H_{1t}$ | Total farm labor employment $y^T_{1t}$ | Real farm wage rate[a] $Y^a_{2t}$ | Real income per farm operator $Y^{*b}_{2t}$ |
|---|---|---|---|---|---|
| | (000) | (000) | (000) | | |
| 1941 | 8,017 | 2,652 | 10,669 | 51.16 | 1883.39 |
| 42 | 7,949 | 2,555 | 10,504 | 59.08 | 2488.62 |
| 43 | 8,010 | 2,436 | 10,446 | 75.02 | 3216.82 |
| 44 | 7,988 | 2,231 | 10,219 | 90.39 | 3295.81 |
| 45 | 7,881 | 2,119 | 10,000 | 97.68 | 3412.46 |
| 46 | 8,106 | 2,189 | 10,295 | 98.57 | 3734.11 |
| 47 | 8,115 | 2,267 | 10,382 | 91.82 | 3750.29 |
| 48 | 8,026 | 2,337 | 10,363 | 90.41 | 3276.87 |
| 49 | 7,712 | 2,252 | 9,964 | 89.54 | 2875.03 |
| 50 | 7,597 | 2,329 | 9,926 | 86.93 | 2718.19 |
| 51 | 7,310 | 2,236 | 9,546 | 88.62 | 3016.87 |
| 52 | 7,006 | 2,144 | 9,149 | 93.03 | 2942.62 |
| 53 | 6,775 | 2,089 | 8,864 | 95.46 | 2956.04 |
| 54 | 6,570 | 2,081 | 8,651 | 93.43 | 2676.19 |

[a]Real farm wage rate is defined as the index of hired farm labor composite hourly wage rate deflated by consumer price index (1957 - 59 = 100).

[b]Real income per farm operator is defined as current income per farm operator deflated by consumer price index (1957 - 59 = 100).

Table A1. (Continued)

| Year | Family farm labor $y^{F}_{1t}$ | Hired farm labor employment $y^{Y}_{1t}$ | Total farm labor employment $y^{T}_{1t}$ | Real farm wage rate[a] $Y^{a}_{2t}$ | Real income per farm operator $Y^{*b}_{2t}$ |
|---|---|---|---|---|---|
| 55 | 6,345 | 2,036 | 8.381 | 95.34 | 2593.91 |
| 56 | 5,900 | 1,952 | 7,852 | 97.02 | 2787.23 |
| 57 | 5,660 | 1,940 | 7,600 | 98.14 | 2500.42 |
| 58 | 5,521 | 1,982 | 7,503 | 98.43 | 2975.68 |
| 59 | 5,390 | 1,952 | 7,342 | 103.39 | 2733.93 |
| 60 | 5,127 | 1,885 | 7,012 | 104.58 | 2874.18 |
| 61 | 5,029 | 1,890 | 6,919 | 106.33 | 3178.63 |
| 62 | 4,873 | 1,827 | 6,700 | 107.92 | 3247.10 |
| 63 | 4,738 | 1,780 | 6,518 | 109.36 | 3305.76 |
| 64 | 4,506 | 1,604 | 6,110 | 110.65 | 3507.66 |
| 65 | 4,128 | 1,482 | 5,610 | 114.08 | 3797.10 |
| 66 | 3,854 | 1,360 | 5,214 | 119.94 | 4440.75 |
| 67 | 3,650 | 1,253 | 4,903 | 125.36 | 3871.43 |
| 68 | 3,536 | 1,213 | 4,749 | 129.93 | 3950.80 |
| 69 | 3,419 | 1,176 | 4,595 | 135.86 | 4388.16 |
| 70 | 3,348 | 1,175 | 4,523 | 137.97 | 4207.29 |
| 71 | 3,275 | 1,161 | 4,436 | 138.48 | 3713.15 |
| 72 | 3,228 | 1,146 | 4,374 | 142.06 | 4709.46 |
| 73 | 3,169 | 1,168 | 4,337 | 145.98 | 6272.57 |
| 74 | 3,116 | 1,178 | 4,294 | ---- | 5968.14 |

Table A2. Actual values for the exogenous variables used in the econometric analysis, 1941-1973

| Year<br>$t$ | Real price received by farmers<br>$x_{1t}$[a] | Technology index<br>$x_{2t}$ | Adjusted real non-farm wage index<br>$x_{5t}$[b] | Time trend<br>$x_{6t}$ | Family farm labor lagged one period<br>$y^F_{1t-1}$<br>(000) | Hired farm labor lagged one period<br>$y^H_{1t-1}$<br>(000) | Total farm labor lagged one period<br>$y^T_{1t-1}$<br>(000) |
|---|---|---|---|---|---|---|---|
| 1940 | 87.51 | 69.84 | 14.73 | 0 | 8,611 | 2,727 | 11,338 |
| 41 | 105.07 | 72.13 | 47.13 | 1 | 8,300 | 2,679 | 10,979 |
| 42 | 116.90 | 79.00 | 74.73 | 2 | 8,017 | 2,652 | 10,669 |
| 43 | 128.65 | 76.71 | 94.30 | 3 | 7,949 | 2,555 | 10,504 |
| 44 | 125.44 | 77.86 | 101.29 | 4 | 8,010 | 2,436 | 10,446 |
| 45 | 128.13 | 79.00 | 95.90 | 5 | 7.988 | 2,231 | 10,219 |
| 46 | 133.94 | 82.44 | 83.62 | 6 | 7,881 | 2,119 | 10,000 |
| 47 | 134.84 | 80.15 | 82.33 | 7 | 8,106 | 2,189 | 10,295 |
| 48 | 125.33 | 85.87 | 84.24 | 8 | 8,115 | 2,267 | 10,382 |
| 49 | 113.93 | 83.58 | 77.96 | 9 | 8,026 | 2,337 | 10,363 |
| 50 | 114.44 | 83.58 | 82.67 | 10 | 7,712 | 2,252 | 9,964 |
| 51 | 120.87 | 83.58 | 93.89 | 11 | 7,597 | 2,329 | 9,926 |
| 52 | 114.77 | 87.02 | 98.92 | 12 | 7,310 | 2,236 | 9,546 |
| 53 | 108.42 | 88.16 | 104.50 | 13 | 7,005 | 2,144 | 9,149 |
| 54 | 105.16 | 89.31 | 90.07 | 14 | 6,775 | 2,089 | 8,864 |

[a] $x_{1t}$ is defined as the ratio of the index of prices received by farmers to the index of prices paid by farmers for production items, excluding the wage rate.

[b] The construction of variable $x_{5t}$ is discussed in Chapter V, Section 4.

Table A2. (Continued)

| Year<br>t | Real price received by farmers<br>$x_{1t}^{a}$ | Technology index<br>$x_{2t}$ | Adjusted real non-farm wage index<br>$x_{5t}^{b}$ | Time trend<br>$x_{6t}$ | Family farm labor lagged one period<br>$y_{1t-1}^{F}$ | Hired farm labor lagged one period<br>$y_{1t-1}^{H}$ | Total farm labor lagged one period<br>$y_{1t-1}^{T}$ |
|---|---|---|---|---|---|---|---|
| 55 | 100.93 | 91.60 | 101.58 | 15 | 6,570 | 2,081 | 8,651 |
| 56 | 100.93 | 93.89 | 106.98 | 16 | 6,345 | 2,036 | 8.381 |
| 57 | 98.64 | 95.04 | 107.38 | 17 | 5,900 | 1,952 | 7,852 |
| 58 | 102.78 | 101.91 | 90.22 | 18 | 5,660 | 1,940 | 7,600 |
| 59 | 98.57 | 103.05 | 102.36 | 19 | 5,521 | 1,982 | 7,503 |
| 60 | 98.58 | 106.48 | 103.97 | 20 | 5,390 | 1,952 | 7,342 |
| 61 | 97.53 | 107.63 | 96.59 | 21 | 5,127 | 1,885 | 7,012 |
| 62 | 98.54 | 108.77 | 107.28 | 22 | 5,029 | 1,890 | 6,919 |
| 63 | 97.50 | 112.21 | 107.81 | 23 | 4,873 | 1,827 | 6,700 |
| 64 | 95.46 | 109.92 | 128.18 | 24 | 4,738 | 1,780 | 6,518 |
| 65 | 98.50 | 113.35 | 120.14 | 25 | 4,506 | 1,604 | 6,110 |
| 66 | 102.34 | 111.06 | 127.14 | 26 | 4,128 | 1,482 | 5,610 |
| 67 | 96.49 | 114.50 | 128.69 | 27 | 3,854 | 1,360 | 5,214 |
| 68 | 97.44 | 115.64 | 133.53 | 28 | 3,650 | 1,250 | 4,903 |
| 69 | 98.31 | 115.64 | 134.72 | 29 | 3,536 | 1,213 | 4,749 |
| 70 | 96.49 | 114.50 | 123.06 | 30 | 3,419 | 1,176 | 4,595 |
| 71 | 93.97 | 123.66 | 116.87 | 31 | 3,348 | 1,175 | 4,523 |
| 72 | 99.65 | 122.51 | 123.01 | 32 | 3,275 | 1,161 | 4,436 |
| 73 | 113.67 | 121.36 | 128.90 | 33 | 3,228 | 1,146 | 4,374 |