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STRESSES IN MODERATELY THICK RECTANGULAR PLATES

By

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TABLE OF CONTENTS

I. INTRODUCTION

Problems in "moderately thick" plates may be distinguished from problems in "thick" plates by the manner in which the boundary conditions at an edge of the plate are given (10, p.458-9). In thick plates the stresses have prescribed values at every point of the edge while in moderately thick plates the stresses are represented by their force- and couple- resultants taken along a vertical element of an edge. It is true that moderately thick plate solutions may not be exact but Saint-Venant's principle states that they will sufficiently approximate the exact solutions for all points which are not too close to the edge of the plate (10, p.131-2).

Although solutions for the displacements of thin plates were obtained in the first half of the ninteenth century, Saint-Venant (2, p.337) found the first solution for a moderately thick plate in 1883. It was after 1900 before A. E. H. Love (10, p.465) found the correct solution for a moderately thick circular plate under a uniform load. He used a method developed by J. H. Mitchell (11) in 1899.

C. A. Garabedian (4) was the first to solve successfully problems in moderately thick circular plates by a method involving the assumption that the displacements can be expanded in a series of rational integral powers of a small parameter. This method of solution was introduced by G. D. Birkhoff (1) in 1922. During the years 1924 - 32 Garabedian (5, 6, 7) published results for a moderately thick rectangular plate subject to a uniform load for several types of edge conditions. He never published his method for rectangular plates except to say (5) that it is similar to his method for circular plates. In 1930, B. G. Galerkin (3) published an entirely different method for solving problems in moderately thick plates. In 1931-2 he (3) published general, but no specific, results for both circular and rectangular plates. He expressed the stresses and displacements in terms of three functions ϕ_1, ϕ_2 and ϕ_3 where $\Delta_4\phi_1 = C_1$, $\Delta_4\phi_2 = C_2$ and $\Delta_4\phi_3 = C_3$ and where C_1, C_2 and C_3 are arbitrary constants and $\Delta_4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial z^4}$.

In 1931, H. W. Sibert (12) solved problems in moderately thick circular plates by a method similar to that used by Garabedian. However, he succeeded in eliminating the parameter used by Garabedian and in expanding the displacements directly in positive integral powers of z. He also succeeded in finding the general terms for his series. It seems desirable to extend Sibert's method of analysis to problems in rectangular plates.

This investigation leads to a solution for the displacements in an elastic isotropic moderately thick rectangular plate under the action of a given distribution of load and with prescribed boundary conditions at the edges. The method, analogous to that used by Sibert (12) for a circular plate, is based on the fundamental assumption that the components of displacement can be developed in positive integral powers of z. In this type of problem the displacements must satisfy (a) the stress equations of equilibrium throughout the body, (b) the surface traction conditions on the upper and lower faces, (c) the boundary conditions at the edges.

In chapter II, the first two of these conditions are satisfied for any normal load which can be expressed as a polynomial in x,y continuous over the entire plate. The result is a set of differential equations which define the displacements. In chapter III, these differential equations are solved for the normal load $P\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$ on the upper face subject to three different sets of edge conditions: pinned-pinned, pinned-free and pinned-clamped where these terms are defined in the following manner. Let T, S, N be the components of the stress-resultant belonging to an edge-line s, and H, G be the components of the stress-couple belonging to the same line. T is a tension, S and N are shearing forces tangential and normal to the middle plane, G is a flexural couple, and H a torsional couple (10, p.455). Then at a free edge T, S, $N-\frac{\partial H}{\partial s}$, G vanish. At a pinned edge the displacement w of a point on the middle plane at right angles to this plane vanishes, and T, S, G also vanish. At a clamped edge, the displacements u, v, w of a point on the middle plane vanish, and $\frac{\partial w}{\partial n}$ also vanishes, n denoting the direction of the normal to the edge-line (10,p. 462).

II. GENERAL THEORY

A. Form of the Displacements Necessary to Satisfy the Equations of Equilibrium

From the fundamental assumption of expansion in positive integral powers of z, the displacements u, v, w are given by

$$(1)\quad u = \sum_{k=0}^{\infty} U_k \frac{z^k}{k!}, \qquad v = \sum_{k=0}^{\infty} V_k \frac{z^k}{k!}, \qquad w = \sum_{k=0}^{\infty} W_k \frac{z^k}{k!},$$

where U_k, V_k, W_k are continuous and continuously differentiable functions of x and y.

The equations of equilibrium in terms of displacements, with body forces zero, are (10, p.126, 134)

$$(2.1)\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} + (1 - 2\rho)\left(\Delta_2 u + \frac{\partial^2 u}{\partial z^2}\right) = 0,$$

$$(2.2)\quad \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial z} + (1 - 2\rho)\left(\Delta_2 v + \frac{\partial^2 v}{\partial z^2}\right) = 0,$$

$$(2.3)\quad \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + (1 - 2\rho)\ \Delta_2 w + 2(1 - \rho)\frac{\partial^2 w}{\partial z^2} = 0$$

where throughout this paper $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and the number ρ is "Poisson's ratio". When equations (1) are substituted in the equilibrium equations (2) and the coefficients of like powers of z are equated to zero, there results

$$(3.1)\quad U_k = \frac{-1}{1-2\rho}\left[(1-2\rho)\Delta_2 U_{k-2} + \frac{\partial^2 U_{k-2}}{\partial x^2} + \frac{\partial^2 V_{k-2}}{\partial x \partial y} + \frac{\partial W_{k-1}}{\partial x}\right],$$

$$(3.2)\quad V_k = \frac{-1}{1-2\rho}\left[(1-2\rho)\ \Delta_2 V_{k-2} + \frac{\partial^2 V_{k-2}}{\partial y^2} + \frac{\partial^2 U_{k-2}}{\partial x \partial y} + \frac{\partial W_{k-1}}{\partial y}\right],$$

$$(3.3)\quad W_k = \frac{-1}{2(1-\rho)}\left[(1-2\rho)\ \Delta_2 W_{k-2} + \frac{\partial U_{k-1}}{\partial x} + \frac{\partial V_{k-1}}{\partial y}\right].$$

By successive applications of the recurrence relations (3), it is possible to express U_k, V_k, W_k, directly in terms of U_0, U_1, V_0, V_1, W_0, W_1. A material simplification of these formulas is obtained by the introduction of two new functions defined by

$$(4.1)\quad (1-2\rho)\overline{W}_1 = \frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} + W_1\ ,$$

$$(4.2)\quad 2(1-\rho)\overline{W}_0 = \frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} - \Delta_2 W_0.$$

The final results are

$$(5.1)\quad U_{2n} = (-1)^n \Delta_{2n-2}\left[\Delta_2 U_0 + n\frac{\partial \overline{W}_1}{\partial x}\right]\ ,$$

$$(5.2)\quad V_{2n} = (-1)^n \Delta_{2n-2}\left[\Delta_2 V_0 + n\frac{\partial \overline{W}_1}{\partial y}\right]\ ,$$

$$(5.3)\quad W_{2n+1} = (-1)^n \Delta_{2n} \left[(1 - 2\rho - n)\overline{W}_1 - \left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right) \right] ,$$

$$(6.1)\quad U_{2n+1} = (-1)^n \Delta_{2n-2} \left[\Delta_2 U_1 + n \frac{\partial \overline{W}_o}{\partial x} \right] ,$$

$$(6.2)\quad V_{2n+1} = (-1)^n \Delta_{2n-2} \left[\Delta_2 V_1 + n \frac{\partial \overline{W}_o}{\partial y} \right] ,$$

$$(6.3)\quad W_{2n} = (-1)^n \Delta_{2n-2} \left[(n - 2 + 2\rho)\overline{W}_o + \frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} \right] .$$

Formulas (5) involve only U_o, V_o, $\overline{W}_1$ while formulas (6) involve only U_1, V_1, $\overline{W}_o$. When formulas (5) and (6) are substituted in equations (1) there results

$$(7.1)\quad u = \sum_{n=o}^{\infty} (-1)^n \Delta_{2n-2} \left[\Delta_2 U_o + n \frac{\partial \overline{W}_1}{\partial x} \right] \frac{z^{2n}}{(2n)!}$$

$$+ \sum_{n=o}^{\infty} (-1)^n \Delta_{2n-2} \left[\Delta_2 U_1 + n \frac{\partial \overline{W}_o}{\partial x} \right] \frac{z^{2n+1}}{(2n+1)!} ,$$

$$(7.2)\quad v = \sum_{n=o}^{\infty} (-1)^n \Delta_{2n-2} \left[\Delta_2 V_o + n \frac{\partial \overline{W}_1}{\partial y} \right] \frac{z^{2n}}{(2n)!}$$

$$+ \sum_{n=o}^{\infty} (-1)^n \Delta_{2n-2} \left[\Delta_2 V_1 + n \frac{\partial \overline{W}_o}{\partial y} \right] \frac{z^{2n+1}}{(2n+1)!} ,$$

$$(7.3)\quad w = \sum_{n=o}^{\infty} (-1)^n \Delta_{2n-2} \left[(n - 2 + 2\rho)\overline{W}_o + \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right) \right] \frac{z^{2n}}{(2n)!}$$

$$+ \sum_{n=0}^{\infty} (-1)^n \Delta_{2n} \left[(1 - 2\rho - n) \overline{W}_1 - \left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y} \right) \right] \frac{z^{2n+1}}{(2n+1)!} \ .$$

These expressions for the displacements satisfy formally the stress equations of equilibrium throughout the body.

B. Surface Traction Equations Imposed on the Displacements

A right-handed coordinate system with its origin at a corner of the middle plane of the plate will be used. Thus the equations of the faces of the plate are $z = \pm h$, $x = 0$, $x = a$, $y = 0$ and $y = b$. The x,y and z components of the surface tractions will be designated by L_1, J_1 and P_1 respectively on the upper face and by L_2, J_2 and P_2 respectively on the lower face. Using the notation of Love (10, p.77) where the capital letter indicates the direction of the component stress and the small letter the direction of the normal to the plane on which the stress acts, the surface traction conditions on the upper and lower faces may be written as

$$(8) \quad \begin{cases} (X_z)_{z=h} = L_1 \ , & (X_z)_{z=-h} = L_2 \ , \\ (Y_z)_{z=h} = J_1 \ , & (Y_z)_{z=-h} = J_2 \ , \\ (Z_z)_{z=h} = P_1 \ , & (Z_z)_{z=-h} = P_2 \ . \end{cases}$$

The stresses in terms of displacements are (10, p.101-2, 104)

$$(9.1)\quad Z_z = \frac{2G}{1-2\rho}\left[(1-\rho)\frac{\partial w}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right],$$

$$(9.2)\quad X_z = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

$$(9.3)\quad Y_z = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

where the quantity G is the "modulus of rigidity". Replace the displacements in (9) by their values from (7) and substitute the results in the surface traction conditions (8). Then take the sum and difference of the two resulting values of Z_z, X_z and Y_z. Finally, eliminate W_1 and W_o by means of relations (4). The resulting equations are

$$(10)\quad \sum_{n=o}^{\infty} (-1)^n \Delta_{2n-2}\left[\Delta_2 U_1 + \frac{n}{1-\rho}\left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial x \partial y}\right) - \frac{n-1+\rho}{1-\rho}\Delta_2\left(\frac{\partial W_o}{\partial x}\right)\right]\frac{h^{2n}}{(2n)!} = \frac{L_1+L_2}{2G},$$

$$(11)\quad \sum_{n=o}^{\infty} (-1)^n \Delta_{2n-2}\left[\Delta_2 V_1 + \frac{n}{1-\rho}\left(\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 U_1}{\partial x \partial y}\right) - \frac{n-1+\rho}{1-\rho}\Delta_2\left(\frac{\partial W_o}{\partial y}\right)\right]\frac{h^{2n}}{(2n)!} = \frac{J_1+J_2}{2G},$$

$$(12)\quad \sum_{n=0}^{\infty} (-1)^n \Delta_{2n} \left[\frac{n-1+\rho}{2(1-\rho)} \Delta_2 W_0 - \frac{n+1-\rho}{2(1-\rho)} \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)\right] \frac{h^{2n}}{(2n+1)!}$$

$$= \frac{P_1-P_2}{4Gh} ,$$

$$(13)\quad \sum_{n=0}^{\infty} (-1)^{n+1} \Delta_{2n} \left[\Delta_2 U_0 + \frac{2n+1}{1-2\rho} \left(\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial x \partial y}\right)\right.$$

$$\left. + \frac{2(n+\rho)}{1-2\rho} \frac{\partial W_1}{\partial x}\right] \frac{h^{2n}}{(2n+1)!} = \frac{L_1-L_2}{2Gh} ,$$

$$(14)\quad \sum_{n=0}^{\infty} (-1)^{n+1} \Delta_{2n} \left[\Delta_2 V_0 + \frac{2n+1}{1-2\rho} \left(\frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 U_0}{\partial x \partial y}\right)\right.$$

$$\left. + \frac{2(n+\rho)}{1-2\rho} \frac{\partial W_1}{\partial y}\right] \frac{h^{2n}}{(2n+1)!} = \frac{J_1-J_2}{2Gh} ,$$

$$(15)\quad \sum_{n=0}^{\infty} (-1)^{n+1} \Delta_{2n} \left[\frac{n-\rho}{1-2\rho} \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) + \frac{n-1+\rho}{1-2\rho} W_1\right] \frac{h^{2n}}{(2n)!}$$

$$= \frac{P_1+P_2}{4G} .$$

Equations (10), (11) and (12) involve only W_0, U_1 and V_1 and equations (13), (14) and (15) involve only U_0, V_0 and W_1. These two simultaneous systems of equations can be solved by an indirect process due to Sibert (12, p.337). This process requires that U_0, V_0, W_0, U_1, V_1 and W_1 be expressed as infinite sequences of terms of ascending order of magnitude. Let s represent a first degree function of x, y. Order of magnitude may then be defined as follows: If r and t are two functions of s which contain the same number of terms, t is defined to be of the nth order of magnitude as compared to r if each term of t is proportional to $\left(\frac{h}{s}\right)^n$ times the corresponding term in r. It then follows that $h^{2n}\Delta_{2n}r$ is of the 2nth order of magnitude as compared to r. It is necessary to assume that U_o, V_o, W_o, U_1, V_1 and W_1 are expressions in x,y which involve h in such a manner that their terms can be grouped and arranged in ascending order of magnitude.

Since equations (10), (11), (12), (13), (14) and (15) have been arranged so that only even powers of h occur in their left members, it is only necessary to provide for even orders of magnitude. Therefore, W_o, U_o, V_o, W_1, U_1 and V_1 may be written

$$(16)\quad \begin{cases} W_o = \sum_{n=o}^{\infty} W_{2n,o} \,, & W_1 = \sum_{n=o}^{\infty} W_{2n,1} \,, \\ U_o = \sum_{n=o}^{\infty} U_{2n,o} \,, & U_1 = \sum_{n=o}^{\infty} U_{2n,1} \,, \\ V_o = \sum_{n=o}^{\infty} V_{2n,o} \,, & V_1 = \sum_{n=o}^{\infty} V_{2n,1} \,, \end{cases}$$

where $W_{2n,o}$, $U_{2n,o}$, $V_{2n,o}$, $W_{2n,1}$, $U_{2n,1}$ and $V_{2n,1}$ are of the 2nth order of magnitude as compared to W_{oo}, U_{oo}, V_{oo}, W_{o1}, U_{o1} and V_{o1} respectively. It is assumed that W_{oo}, U_{oo}, V_{oo}, W_{o1}, U_{o1} and V_{o1}, being the terms of lowest order of magnitude, do not vanish identically unless W_0, U_0, V_0, W_1, U_1 and V_1 respectively are identically zero.

For simplicity, the problem will now be restricted to the case of a normal surface load only. This means that $L_1 = L_2 = J_1 = J_2 = 0$. In Chapter IV the case of a shearing load will be solved. By superposing these solutions, the solutions for more complicated problems can be obtained.

C. General Solutions for W_0, U_1 and V_1

In order to solve equations (10), (11) and (12) it is necessary to write each one as an infinite system of equations by equating terms of the same order of magnitude. Before this can be done the order of magnitude of the right member of equation (12) must be determined. Assume it to be of the same order of magnitude as $\Delta_2 W_{00}$. Let $P_1 - P_2 = -p_1$ where p_1 is a function of x,y. Then the equations of lowest order of magnitude in (10), (11) and (12) may be written

(10.0) $U_{01} + \dfrac{\partial W_{00}}{\partial x} = 0$,

(11.0) $V_{01} + \dfrac{\partial W_{00}}{\partial y} = 0$,

(12.0) $\Delta_2 W_{00} + \dfrac{\partial U_{01}}{\partial x} + \dfrac{\partial V_{01}}{\partial y} = \dfrac{p_1}{2Gh}$.

However $\frac{\partial(10.0)}{\partial x} + \frac{\partial(11.0)}{\partial y}$ gives $\Delta_2 W_{00} + \frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y} = 0$.
This result is inconsistent with (12.0). Therefore $\frac{p_1}{2Gh}$ must be of the fourth or higher order of magnitude as compared to W_{00}. At present, it will be assumed that $\frac{p_1}{2Gh}$ is of the fourth order of magnitude as compared to W_{00}. It will be shown later that this assumption is correct.

The infinite systems of equations from (10), (11) and (12) will now be given.

(10.0) $U_{01} + \frac{\partial W_{00}}{\partial x} = 0$,

(10.1) $$U_{21} + \frac{\partial W_{20}}{\partial x} - \left[\Delta_2 U_{01} + \frac{1}{1-\rho}\left(\frac{\partial^2 U_{01}}{\partial x^2} + \frac{\partial^2 V_{01}}{\partial x \partial y}\right) - \frac{\rho}{1-\rho}\Delta_2\left(\frac{\partial W_{00}}{\partial x}\right)\right]\frac{h^2}{2!} = 0 ,$$

(10.2) $$U_{41} + \frac{\partial W_{40}}{\partial x} - \left[\Delta_2 U_{21} + \frac{1}{1-\rho}\left(\frac{\partial^2 U_{21}}{\partial x^2} + \frac{\partial^2 V_{21}}{\partial x \partial y}\right) - \frac{\rho}{1-\rho}\Delta_2\left(\frac{\partial W_{20}}{\partial x}\right)\right]\frac{h^2}{2!} + \Delta_2\left[\Delta_2 U_{01} + \frac{2}{1-\rho}\left(\frac{\partial^2 U_{01}}{\partial x^2} + \frac{\partial^2 V_{01}}{\partial x \partial y}\right) - \frac{1+\rho}{1-\rho}\Delta_2\left(\frac{\partial W_{00}}{\partial x}\right)\right]\frac{h^4}{4!} = 0 ,$$

(10.3) $$U_{61} + \frac{\partial W_{60}}{\partial x} - \left[\Delta_2 U_{41} + \frac{1}{1-\rho}\left(\frac{\partial^2 U_{41}}{\partial x^2} + \frac{\partial^2 V_{41}}{\partial x \partial y}\right) - \frac{\rho}{1-\rho}\Delta_2\left(\frac{\partial W_{40}}{\partial x}\right)\right]\frac{h^2}{2!} + \Delta_2\left[\Delta_2 U_{21} + \frac{2}{1-\rho}\left(\frac{\partial^2 U_{21}}{\partial x^2}\right.\right.$$

$$+ \frac{\partial^2 V_{21}}{\partial x \partial y}\Big) - \frac{1+\rho}{1-\rho}\, \Delta_2 \left(\frac{\partial W_{20}}{\partial x}\right)\Big] \frac{h^4}{4!} - \Delta_4 \Big[\Delta_2 U_{01}$$

$$+ \frac{3}{1-\rho}\left(\frac{\partial^2 U_{01}}{\partial x^2} + \frac{\partial^2 V_{01}}{\partial x \partial y}\right) - \frac{2+\rho}{1-\rho}\, \Delta_2 \left(\frac{\partial W_{00}}{\partial x}\right)\Big] \frac{h^6}{6!} = 0 \;,$$

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$$(11.0)\quad V_{01} + \frac{\partial W_{00}}{\partial y} = 0 \;,$$

$$(11.1)\quad V_{21} + \frac{\partial W_{20}}{\partial y} - \Big[\Delta_2 V_{01} + \frac{1}{1-\rho}\left(\frac{\partial^2 V_{01}}{\partial y^2} + \frac{\partial^2 U_{01}}{\partial x \partial y}\right)$$

$$- \frac{\rho}{1-\rho}\, \Delta_2 \left(\frac{\partial W_{00}}{\partial y}\right)\Big] \frac{h^2}{2!} = 0 \;,$$

$$(11.2)\quad V_{41} + \frac{\partial W_{40}}{\partial y} - \Big[\Delta_2 V_{21} + \frac{1}{1-\rho}\left(\frac{\partial^2 V_{21}}{\partial y^2} + \frac{\partial^2 U_{21}}{\partial x \partial y}\right)$$

$$- \frac{\rho}{1-\rho}\, \Delta_2 \left(\frac{\partial W_{20}}{\partial y}\right)\Big] \frac{h^2}{2!} + \Delta_2 \Big[\Delta_2 V_{01} + \frac{2}{1-\rho}\Big(\frac{\partial^2 V_{01}}{\partial y^2}$$

$$+ \frac{\partial^2 U_{01}}{\partial x \partial y}\Big) - \frac{1+\rho}{1-\rho}\, \Delta_2 \left(\frac{\partial W_{00}}{\partial y}\right)\Big] \frac{h^4}{4!} = 0 \;,$$

$$(11.3)\quad V_{61} + \frac{\partial W_{60}}{\partial y} - \Big[\Delta_2 V_{41} + \frac{1}{1-\rho}\left(\frac{\partial^2 V_{41}}{\partial y^2} + \frac{\partial^2 U_{41}}{\partial x \partial y}\right)$$

$$- \frac{\rho}{1-\rho}\, \Delta_2 \left(\frac{\partial W_{40}}{\partial y}\right)\Big] \frac{h^2}{2!} + \Delta_2 \Big[\Delta_2 V_{21} + \frac{2}{1-\rho}\Big(\frac{\partial^2 V_{21}}{\partial y^2}$$

$$+ \frac{\partial^2 U_{21}}{\partial x \partial y}\Big) - \frac{1+\rho}{1-\rho}\, \Delta_2 \left(\frac{\partial W_{20}}{\partial y}\right)\Big] \frac{h^4}{4!} - \Delta_4 \Big[\Delta_2 V_{01}$$

$$+ \frac{3}{1-\rho} \left(\frac{\partial^2 V_{01}}{\partial y^2} + \frac{\partial^2 U_{01}}{\partial x \partial y} \right) - \frac{2+\rho}{1-\rho} \Delta_2 \left(\frac{\partial W_{00}}{\partial y} \right) \Bigg] \frac{h^6}{6!} = 0 \; ,$$

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$$(12.0) \quad \Delta_2 W_{00} + \frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y} = 0 \; ,$$

$$(12.1) \quad \Delta_2 W_{20} + \frac{\partial U_{21}}{\partial x} + \frac{\partial V_{21}}{\partial y}$$

$$+ \Delta_2 \left[\frac{\rho}{1-\rho} \Delta_2 W_{00} - \frac{2-\rho}{1-\rho} \left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y} \right) \right] \frac{h^2}{3!} = \frac{p_1}{2Gh} \; ,$$

$$(12.2) \quad \Delta_2 W_{40} + \frac{\partial W_{41}}{\partial x} + \frac{\partial V_{41}}{\partial y} + \Delta_2 \left[\frac{\rho}{1-\rho} \Delta_2 W_{20} - \frac{2-\rho}{1-\rho} \left(\frac{\partial U_{21}}{\partial x} \right.\right.$$

$$\left.\left. + \frac{\partial V_{21}}{\partial y} \right) \right] \frac{h^2}{3!} - \Delta_4 \left[\frac{1+\rho}{1-\rho} \Delta_2 W_{00} - \frac{3-\rho}{1-\rho} \left(\frac{\partial U_{01}}{\partial x} \right.\right.$$

$$\left.\left. + \frac{\partial V_{01}}{\partial y} \right) \right] \frac{h^4}{5!} = 0 \; ,$$

$$(12.3) \quad \Delta_2 W_{60} + \frac{\partial U_{61}}{\partial x} + \frac{\partial V_{61}}{\partial y} + \Delta_2 \left[\frac{\rho}{1-\rho} \Delta_2 W_{40} - \frac{2-\rho}{1-\rho} \left(\frac{\partial U_{41}}{\partial x} \right.\right.$$

$$\left.\left. + \frac{\partial V_{41}}{\partial y} \right) \right] \frac{h^2}{3!} - \Delta_4 \left[\frac{1+\rho}{1-\rho} \Delta_2 W_{20} - \frac{3-\rho}{1-\rho} \left(\frac{\partial U_{21}}{\partial x} \right.\right.$$

$$\left.\left. + \frac{\partial V_{21}}{\partial y} \right) \right] \frac{h^4}{5!} + \Delta_6 \left[\frac{2+\rho}{1-\rho} \Delta_2 W_{00} - \frac{4-\rho}{1-\rho} \left(\frac{\partial U_{01}}{\partial x} \right.\right.$$

$$\left.\left. + \frac{\partial V_{01}}{\partial y} \right) \right] \frac{h^6}{7!} = 0 \; ,$$

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Since $\frac{\partial(10.0)}{\partial x} + \frac{\partial(11.0)}{\partial y} = (12.0)$, it is necessary to form another system of equations by subtracting $\frac{\partial(10.n)}{\partial x} + \frac{\partial(11.n)}{\partial y}$ from (12.n). The resulting system may be written as follows:

(17.0) $0 = 0$,

(17.1) $$\Delta_2\left[(2-\rho)\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) - \rho\Delta_2 W_{00}\right] = \frac{2p_1}{D} ,$$

(17.2) $$\Delta_2\left[(2-\rho)\left(\frac{\partial U_{21}}{\partial x} + \frac{\partial V_{21}}{\partial y}\right) - \rho\Delta_2 W_{20}\right]$$

$$- \Delta_4\left[(3-\rho)\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) - (1+\rho)\Delta_2 W_{00}\right]\frac{6\cdot 2\cdot h^2}{5!} = 0 ,$$

(17.3) $$\Delta_2\left[(2-\rho)\left(\frac{\partial U_{41}}{\partial x} + \frac{\partial V_{41}}{\partial y}\right) - \rho\Delta_2 W_{40}\right]$$

$$- \Delta_4\left[(3-\rho)\left(\frac{\partial U_{21}}{\partial x} + \frac{\partial V_{21}}{\partial y}\right) - (1+\rho)\Delta_2 W_{20}\right]\frac{6\cdot 2\cdot h^2}{5!}$$

$$+ \Delta_6\left[(4-\rho)\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) - (2+\rho)\Delta_2 W_{00}\right]\frac{6\cdot 3\cdot h^4}{7!} = 0 ,$$

(17.4) $$\Delta_2\left[(2-\rho)\left(\frac{\partial U_{61}}{\partial x} + \frac{\partial V_{61}}{\partial y}\right) - \rho\Delta_2 W_{60}\right]$$

$$- \Delta_4\left[(3-\rho)\left(\frac{\partial U_{41}}{\partial x} + \frac{\partial V_{41}}{\partial y}\right) - (1+\rho)\Delta_2 W_{40}\right]\frac{6\cdot 2\cdot h^2}{5!}$$

$$+ \Delta_6\left[(4-\rho)\left(\frac{\partial U_{21}}{\partial x} + \frac{\partial V_{21}}{\partial y}\right) - (2+\rho)\Delta_2 W_{20}\right]\frac{6\cdot 3\cdot h^4}{7!}$$

$$- \Delta_8\left[(5-\rho)\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) - (3+\rho)\Delta_2 W_{00}\right]\frac{6\cdot 4\cdot h^6}{9!} = 0,$$

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where $D = \frac{4Gh^3}{3(1-\rho)}$.

Systems (10), (11) and (17) can now be solved simultaneously for $W_{2n,0}$, $U_{2n,1}$ and $V_{2n,1}$. Equations (17.1) and the Laplacian of $\frac{\partial(10.0)}{\partial x} + \frac{\partial(11.0)}{\partial y}$ become $\Delta_2\left[(2-\rho)\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) - \rho\Delta_2 W_{00}\right] = \frac{2p_1}{D} b_0$, and $\Delta_2\left[\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) + \Delta_2 W_{00}\right] = \frac{2p_1}{D} d_0$ respectively where $b_0 = 1$ and $d_0 = 0$. The simultaneous solution is

$$\Delta_4 W_{00} = \frac{p_1}{D}\left[(2-\rho)d_0 - b_0\right], \quad \Delta_2\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) = \frac{p_1}{D}\left[\rho d_0 + b_0\right].$$

Equations (17.2) and the Laplacian of $\frac{\partial(10.1)}{\partial x} + \frac{\partial(11.1)}{\partial y}$ become $\Delta_2\left[(2-\rho)\left(\frac{\partial U_{21}}{\partial x} + \frac{\partial V_{21}}{\partial y}\right) - \rho\Delta_2 W_{20}\right] = \frac{2h^2}{D} b_1 \Delta_2 p_1$, and $\Delta_2\left[\left(\frac{\partial U_{21}}{\partial x} + \frac{\partial V_{21}}{\partial y}\right) + \Delta_2 W_{20}\right] = \frac{2h^2}{D} d_1 \Delta_2 d_1$ respectively where $b_1 = \frac{1}{5}$ and $d_1 = \frac{1}{2(1-\rho)}$. The simultaneous solution is

$$\Delta_4 W_{20} = \frac{h^2\Delta_2 p_1}{D}\left[(2-\rho)d_1 - b_1\right], \quad \Delta_2\left(\frac{\partial U_{21}}{\partial x} + \frac{\partial V_{21}}{\partial y}\right) = \frac{h^2\Delta_2 p_1}{D}\left[\rho d_1 + b_1\right].$$

Continuing in the manner indicated, by solving (17.n+1) with the Laplacian of $\frac{\partial(10.n)}{\partial x} + \frac{\partial(11.n)}{\partial y}$, one obtains the general solutions. They are

$$(18)\quad \Delta_4 W_{2n,0} = \frac{h^{2n}\Delta_{2n}p_1}{D}\left[(2-\rho)d_n - b_n\right] \quad ,$$

$$(19)\quad \Delta_2\left(\frac{\partial U_{2n,1}}{\partial x} + \frac{\partial V_{2n,1}}{\partial y}\right) = \frac{h^{2n}\Delta_{2n}p_1}{D}\left[\rho d_n + b_n\right] \quad , \quad \text{where}$$

$$(20)\quad b_n = 6\sum_{i=0}^{n-1}(-1)^i\,\frac{(i+2)\{(i+2)b_{n-1-i} - (i+1)(1-\rho)d_{n-1-i}\}}{(2i+5)!} \qquad (n = 1,2,3,\ldots),$$

$$(21)\quad d_n = \sum_{i=0}^{n-1}(-1)^i\,\frac{(i+1)b_{n-1-i} - i(1-\rho)d_{n-1-i}}{(2i+2)!\qquad\qquad (1-\rho)} \qquad (n = 1,2,3,\ldots).$$

For clarity, the first few equations from each of the general formulas (18), (20) and (21) will now be listed

$$(18.0)\quad \Delta_4 W_{00} = -\frac{p_1}{D} \quad ,$$

$$(18.1)\quad \Delta_4 W_{20} = -\frac{8-3\rho}{10(1-\rho)}h^2\Delta_6 W_{00} \quad ,$$

$$(18.2)\quad \Delta_4 W_{40} = -\frac{227-157\rho}{2^3\cdot 3\cdot 5^2\cdot 7(1-\rho)}\,h^4\Delta_8 W_{00} \quad ,$$

$$(18.3)\quad \Delta_4 W_{6\,0} = -\frac{26-791\rho}{2^4\cdot 3^3\cdot 5^3\cdot 7(1-\rho)}\, h^6\, \Delta_{10} W_{00} \quad ,$$

$$(18.4)\quad \Delta_4 W_{8\,0} = +\frac{296{,}519+43{,}821\rho}{2^7\cdot 3^3\cdot 5^4\cdot 7^2\cdot 11(1-\rho)}\, h^8\, \Delta_{12} W_{00} \quad ,$$

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As (18.0) is the differential equation which defines the normal displacement in thin plate theory (10, p.488), W_{00} is the vertical displacement of the corresponding thin plate.

$(20.0)\ b_0 = 1$,

$(20.1)\ b_1 = \frac{1}{5}$,

$(20.2)\ b_2 = -\dfrac{29}{2^3\cdot 5^2\cdot 7}$,

$(20.3)\ b_3 = -\dfrac{389}{2^2\cdot 3^3\cdot 5^3\cdot 7}$,

$(20.4)\ b_4 = -\dfrac{384{,}161}{2^7\cdot 3^3\cdot 5^4\cdot 7^2\cdot 11}$.

$(21.0)\ d_0 = 0$,

$(21.1)\ d_1 = \dfrac{1}{2(1-\rho)}$,

$(21.2)\ d_2 = \dfrac{1}{60(1-\rho)}$,

$(21.3)\ d_3 = -\dfrac{17}{2^4\cdot 3\cdot 5^2\cdot 7(1-\rho)}$,

$(21.4)\ d_4 = -\dfrac{221}{2^5\cdot 3^3\cdot 5^3\cdot 7(1-\rho)}$.

Sibert (12, p.344) has given the upper bounds of these sequences as

$$|\, b_n \,| < \frac{1}{3^n} \quad , \qquad |\, d_n \,| < \frac{1}{3^{n-1}\cdot 4\cdot (1-\rho)} \qquad (n = 2,3,4,\ldots).$$

It is desirable to combine (18), (19) and (10) to obtain $U_{2n,1}$ in terms of $W_{2n,o}$ and to combine (18), (19) and(11) to obtain $V_{2n,1}$ in terms of $W_{2n,o}$ instead of listing equations (19) in their present form. The first few resulting relations for each case together with the general form are given here.

$$(22.0)\quad U_{01} + \frac{\partial W_{oo}}{\partial x} = 0 \quad ,$$

$$(22.1)\quad U_{21} + \frac{\partial W_{2o}}{\partial x} + \frac{h^2}{1-\rho}\,\Delta_2 \left(\frac{\partial W_{oo}}{\partial x}\right) = 0 \quad ,$$

$$(22.2)\quad U_{41} + \frac{\partial W_{4o}}{\partial x} + \frac{h^2}{1-\rho}\,\Delta_2 \left(\frac{\partial W_{2o}}{\partial x}\right) + \frac{h^4(5-2\rho)}{6(1-\rho)^2}\,\Delta_4 \left(\frac{\partial W_{oo}}{\partial x}\right) = 0 \quad ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$(22.n)\quad U_{2n,1} + \frac{\partial W_{2n,o}}{\partial x} + \frac{h^2}{1-\rho}\,\Delta_2 \left(\frac{\partial W_{2n-2,o}}{\partial x}\right) + h^{2n}\Delta_{2n} \left(\frac{\partial W_{oo}}{\partial x}\right)\left[2d_n + \frac{(2-\rho)d_{n-1} - b_{n-1}}{1-\rho}\right] = 0$$

$$(n = 1,2,3,\ldots).$$

$$(23.0)\quad V_{01} + \frac{\partial W_{oo}}{\partial y} = 0 \quad ,$$

$$(23.1)\quad V_{21} + \frac{\partial W_{2o}}{\partial y} + \frac{h^2}{1-\rho}\,\Delta_2 \left(\frac{\partial W_{oo}}{\partial y}\right) = 0 \quad ,$$

$$(23.2)\quad V_{41} + \frac{\partial W_{4o}}{\partial y} + \frac{h^2}{1-\rho}\,\Delta_2 \left(\frac{\partial W_{2o}}{\partial y}\right) + \frac{h^4(5-2\rho)}{6(1-\rho)^2}\,\Delta_4 \left(\frac{\partial W_{oo}}{\partial y}\right) = 0 \quad ,$$

. . . . ,

$$(23.n)\quad V_{2n,1} + \frac{\partial W_{2n,0}}{\partial y} + \frac{h^2}{1-\rho}\Delta_2\left(\frac{\partial W_{2n-2,0}}{\partial y}\right) + h^{2n}\Delta_{2n}\left(\frac{\partial W_{00}}{\partial y}\right)\left[2d_n + \frac{(2-\rho)d_{n-1} - b_{n-1}}{1-\rho}\right] = 0$$

$$(n = 1,2,3,\ldots).$$

It is necessary to complete the proof that $\frac{p_1}{2Gh}$ is of the fourth order of magnitude as compared to W_{00}. It has already been shown to be either of the fourth or of a higher order of magnitude. Assume that its order of magnitude as compared to W_{00} is greater than the fourth. Then equations (17.1) and the Laplacian of $\frac{\partial(10.0)}{\partial x} + \frac{\partial(11.0)}{\partial y}$ become

$$\Delta_2\left[(2-\rho)\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) - \rho\Delta_2 W_{00}\right] = 0 \quad \text{and}$$

$$\Delta_2\left[\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) + \Delta_2 W_{00}\right] = 0 \quad \text{respectively.}$$

The simultaneous solution is $\Delta_4 W_{00} = \Delta_2\left(\frac{\partial U_{01}}{\partial x} + \frac{\partial V_{01}}{\partial y}\right) = 0$.

Since the trivial solution $\Delta_4 W_{2n,0} = 0$ does not depend upon the load, p_1 must occur on the right hand side of some one of equations (18). But since (18.0) is the equation of lowest order of magnitude in the system (18), its right member cannot be zero unless the right members of all equations of the system are zero. From this contradiction it follows that the term

$\frac{p_1}{2Gh}$ must be of the fourth order of magnitude as compared to W_{oo}.

If $W_{2n,o}$ is eliminated from equations (22) and (23) the relation $\frac{\partial U_{2n,1}}{\partial y} = \frac{\partial V_{2n,1}}{\partial x}$ is obtained. This relation combined with equations (19) gives

$$(24)\quad \Delta_4 U_{2n,1} = \frac{h^{2n}\Delta_{2n}\left(\frac{\partial p_1}{\partial x}\right)}{D}\left[\rho d_n + b_n\right] \quad \text{and}$$

$$(25)\quad \Delta_4 V_{2n,1} = \frac{h^{2n}\Delta_{2n}\left(\frac{\partial p_1}{\partial y}\right)}{D}\left[\rho d_n + b_n\right] \ .$$

Finally, equations (18), (19), (24) and (25) substituted in (16) give the general solutions for W_o, U_1 and V_1. They are

$$(26)\quad \Delta_4 W_o = \sum_{n=0}^{\infty} \frac{h^{2n}\Delta_{2n}p_1}{D}\left[(2-\rho)d_n - b_n\right] \ ,$$

$$(27)\quad \Delta_4 U_1 = \sum_{n=0}^{\infty} \frac{h^{2n}\Delta_{2n}\left(\frac{\partial p_1}{\partial x}\right)}{D}\left[\rho d_n + b_n\right] \ ,$$

$$(28)\quad \Delta_4 V_1 = \sum_{n=0}^{\infty} \frac{h^{2n}}{D}\,\Delta_{2n}\left(\frac{\partial p_1}{\partial y}\right)\left[\rho d_n + b_n\right] \ ,$$

$$(29)\quad \Delta_2\left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right) = \sum_{n=0}^{\infty} \frac{h^{2n}}{D}\,\Delta_{2n}p_1\left[\rho d_n + b_n\right] \ .$$

D. General Solutions for W_1, U_0 and V_0.

The simultaneous solution of equations (13), (14) and (15) will now be obtained in essentially the same manner as that used to obtain the simultaneous solution of (10), (11) and (12).

Assume that the order of magnitude of the right member of equation (15) is the same as that of W_{01}. It will be shown later that this assumption is correct. Let $P_1 + P_2 = -p_2$ where p_2 is a function of x,y. Then the three systems of equations which result from equating all terms of the same order of magnitude in (13), (14) and (15) respectively may be written as follows:

$$(13.0)\quad (1-2\rho)\Delta_2 U_{00} + \frac{\partial^2 U_{00}}{\partial x^2} + \frac{\partial^2 V_{00}}{\partial x \partial y} + 2\rho \frac{\partial W_{01}}{\partial x} = 0 \quad ,$$

$$(13.1)\quad (1-2\rho)\Delta_2 U_{20} + \frac{\partial^2 U_{20}}{\partial x^2} + \frac{\partial^2 V_{20}}{\partial x \partial y} + 2\rho \frac{\partial W_{21}}{\partial x}$$

$$- \Delta_2 \left[(1-2\rho)\Delta_2 U_{00} + 3\left(\frac{\partial^2 U_{00}}{\partial x^2} + \frac{\partial^2 V_{00}}{\partial x \partial y}\right) + 2(1+\rho)\frac{\partial W_{01}}{\partial x}\right] \frac{h^2}{3!} = 0 \quad ,$$

$$(13.2)\quad (1-2\rho)\Delta_2 U_{40} + \frac{\partial^2 U_{40}}{\partial x^2} + \frac{\partial^2 V_{40}}{\partial x \partial y} + 2\rho \frac{\partial W_{41}}{\partial x}$$

$$- \Delta_2 \left[(1-2\rho)\Delta_2 U_{20} + 3\left(\frac{\partial^2 U_{20}}{\partial x^2} + \frac{\partial^2 V_{20}}{\partial x \partial y}\right)\right.$$

$$+ 2(1+\rho)\,\frac{\partial W_{21}}{\partial x}\Big]\frac{h^2}{3!} + \Delta_4\Big[(1-2\rho)\Delta_2 U_{00}$$

$$+ 5\left(\frac{\partial^2 U_{00}}{\partial x^2} + \frac{\partial^2 V_{00}}{\partial x \partial y}\right) + 2(2+\rho)\,\frac{\partial W_{01}}{\partial x}\Big]\frac{h^4}{5!} = 0 \ ,$$

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$$(14.0)\quad (1-2\rho)\Delta_2 V_{00} + \frac{\partial^2 V_{00}}{\partial y^2} + \frac{\partial^2 U_{00}}{\partial x \partial y} + 2\rho\,\frac{\partial W_{01}}{\partial y} = 0 \ ,$$

$$(14.1)\quad (1-2\rho)\Delta_2 V_{20} + \frac{\partial^2 V_{20}}{\partial y^2} + \frac{\partial^2 U_{20}}{\partial x \partial y} + 2\rho\,\frac{\partial W_{21}}{\partial y}$$

$$- \Delta_2\Big[(1-2\rho)\Delta_2 V_{00} + 3\left(\frac{\partial^2 V_{00}}{\partial y^2} + \frac{\partial^2 U_{00}}{\partial x \partial y}\right)$$

$$+ 2(1+\rho)\,\frac{\partial W_{01}}{\partial y}\Big]\frac{h^2}{3!} = 0 \ ,$$

$$(14.2)\quad (1-2\rho)\Delta_2 V_{40} + \frac{\partial^2 V_{40}}{\partial y^2} + \frac{\partial^2 U_{40}}{\partial x \partial y} + 2\rho\,\frac{\partial W_{41}}{\partial y}$$

$$- \Delta_2\Big[(1-2\rho)\Delta_2 V_{20} + 3\left(\frac{\partial^2 V_{20}}{\partial y^2} + \frac{\partial^2 U_{20}}{\partial x \partial y}\right)$$

$$+ 2(1+\rho)\,\frac{\partial W_{21}}{\partial y}\Big]\frac{h^2}{3!} + \Delta_4\Big[(1-2\rho)\Delta_2 V_{00}$$

$$+ 5\left(\frac{\partial^2 V_{00}}{\partial y^2} + \frac{\partial^2 U_{00}}{\partial x \partial y}\right) + 2(2+\rho)\,\frac{\partial W_{01}}{\partial y}\Big]\frac{h^4}{5!} = 0 \ ,$$

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$$(15.0)\quad \rho\left(\frac{\partial U_{00}}{\partial x}+\frac{\partial V_{00}}{\partial y}\right)+(1-\rho)W_{01}=-\frac{p_2(1-2\rho)}{4G}\ ,$$

$$(15.1)\quad \rho\left(\frac{\partial U_{20}}{\partial x}+\frac{\partial V_{20}}{\partial y}\right)+(1-\rho)W_{21}$$

$$+\Delta_2\left[(1-\rho)\left(\frac{\partial U_{00}}{\partial x}+\frac{\partial V_{00}}{\partial y}\right)+\rho W_{01}\right]\frac{h^2}{2!}=0\ ,$$

$$(15.2)\quad \rho\left(\frac{\partial U_{40}}{\partial x}+\frac{\partial V_{40}}{\partial y}\right)+(1-\rho)W_{41}$$

$$+\Delta_2\left[(1-\rho)\left(\frac{\partial U_{20}}{\partial x}+\frac{\partial V_{20}}{\partial y}\right)+\rho W_{21}\right]\frac{h^2}{2!}$$

$$-\Delta_4\left[(2-\rho)\left(\frac{\partial U_{00}}{\partial x}+\frac{\partial V_{00}}{\partial y}\right)+(1+\rho)W_{01}\right]\frac{h^4}{4!}=0\ ,$$

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The Laplacian of (15.0) and $\frac{\partial(13.0)}{\partial x}+\frac{\partial(14.0)}{\partial y}$ become

$$\rho\Delta_2\left(\frac{\partial U_{00}}{\partial x}+\frac{\partial V_{00}}{\partial y}\right)+(1-\rho)\Delta_2 W_{01}=-\frac{(1-2\rho)\Delta_2 p_2}{4G}\,c_0\qquad\text{and}$$

$$(1-\rho)\Delta_2\left(\frac{\partial U_{00}}{\partial x}+\frac{\partial V_{00}}{\partial y}\right)+\rho\Delta_2 W_{01}=\frac{(1-2\rho)\Delta_2 p_2}{4G}\,a_0\qquad\text{respectively}$$

where $a_0 = 0$ and $c_0 = 1$. The simultaneous solution is

$$\Delta_2 W_{01}=-\frac{1}{4G}\,\Delta_2 p_2\left[(1-\rho)c_0+\rho a_0\right]\quad\text{and}$$

$$\Delta_2\left(\frac{\partial U_{00}}{\partial x}+\frac{\partial V_{00}}{\partial y}\right)=\frac{1}{4G}\,\Delta_2 p_2\left[\rho c_0+(1-\rho)a_0\right].$$

The Laplacian of (15.1) and $\frac{\partial(13.1)}{\partial x} + \frac{\partial(14.1)}{\partial y}$ become

$$\rho\Delta_2\left(\frac{\partial U_{20}}{\partial x} + \frac{\partial V_{20}}{\partial y}\right) + (1-\rho)\Delta_2 W_{21} = -\frac{(1-2\rho)h^2\Delta_4 p_2}{4G}\, c_1 \quad \text{and}$$

$$(1-\rho)\Delta_2\left(\frac{\partial U_{20}}{\partial x} + \frac{\partial V_{20}}{\partial y}\right) + \rho\Delta_2 W_{21} = \frac{(1-2\rho)h^2\Delta_4 p_2}{4G}\, a_1 \quad \text{respec-}$$

tively where $a_1 = -\frac{1}{6}$ and $c_1 = 0$. The simultaneous solution is

$$\Delta_2 W_{21} = -\frac{h^2}{4G}\Delta_4 p_2\left[(1-\rho)c_1 + \rho a_1\right] \quad \text{and}$$

$$\Delta_2\left(\frac{\partial U_{20}}{\partial x} + \frac{\partial V_{20}}{\partial y}\right) = \frac{h^2}{4G}\Delta_4 p_2\left[\rho c_1 + (1-\rho)a_1\right] \; .$$

Continuing in the manner indicated, by solving the Laplacian of (15.n) with $\frac{\partial(13.n)}{\partial x} + \frac{\partial(14.n)}{\partial y}$, one obtains the general solutions. They are

$$(30)\quad \Delta_2 W_{2n,1} = -\frac{h^{2n}}{4G}\Delta_{2n+2}\, p_2\left[(1-\rho)c_n + \rho a_n\right] \; ,$$

$$(31)\quad \Delta_2\left(\frac{\partial U_{2n,0}}{\partial x} + \frac{\partial V_{2n,0}}{\partial y}\right) = \frac{h^{2n}}{4G}\Delta_{2n+2}\, p_2\left[(1-\rho)a_n + \rho c_n\right] \; ,$$

where

$$(32)\quad a_n = \sum_{i=0}^{n-1}(-1)^i\,\frac{(i+2)a_{n-1-i} - (i+1)c_{n-1-i}}{(2i+3)!} \qquad (n = 1,2,3,\ldots),$$

$$(33)\quad c_n = \sum_{i=0}^{n-1}(-1)^i\,\frac{(i+1)a_{n-1-i} - i c_{n-1-i}}{(2i+2)!} \qquad (n = 1,2,3,\ldots).$$

Sibert (12, p.344) has given the upper bounds of these sequences of constants as

$$| a_n | < \frac{1}{2^n} , \qquad | c_n | < \frac{5}{9 \cdot 2^n} \qquad (n = 1,2,3,\ldots).$$

In order to complete the proof that the right member of equation (15) is of the same order of magnitude as W_{01} assume that it is of a higher order of magnitude than W_{01}. Then the Laplacian of (15.0) and $\frac{\partial(13.0)}{\partial x} + \frac{\partial(14.0)}{\partial y}$ become

$$\rho\Delta_2 \left(\frac{\partial U_{00}}{\partial x} + \frac{\partial V_{00}}{\partial y}\right) + (1-\rho)\Delta_2 W_{01} = 0 \qquad \text{and}$$

$$(1-\rho)\Delta_2\left(\frac{\partial U_{00}}{\partial x} + \frac{\partial V_{00}}{\partial y}\right) + \rho\Delta_2 W_{01} = 0 \qquad \text{respectively.}$$

The simultaneous solution is $\Delta_2 W_{01} = \Delta_2\left(\frac{\partial U_{00}}{\partial x} + \frac{\partial V_{00}}{\partial y}\right) = 0$.

Since the trivial solution $\Delta_2 W_{2n,1} = 0$ is of no particular interest, p_2 must occur on the right hand side of some one of equations (30). But since (30.0) is the equation of lowest order of magnitude in the system of equations (30), its right member cannot be zero unless the right members of all equations of the system are zero. From this contradiction it is evident that p_2 is of the same order of magnitude as W_{01}.

When $W_{2n,1}$ is eliminated from (13) and (14) the relation $\Delta_2\left(\frac{\partial U_{2n,0}}{\partial y}\right) = \Delta_2\left(\frac{\partial V_{2n,0}}{\partial x}\right)$ is obtained. This relation combined with (31) gives

$$(34)\quad \Delta_4 U_{2n,o} = \frac{h^{2n}}{4G}\,\Delta_{2n+2}\left(\frac{\partial p_2}{\partial x}\right)\left[(1-\rho)a_n + \rho c_n\right] \quad \text{and}$$

$$(35)\quad \Delta_4 V_{2n,o} = \frac{h^{2n}}{4G}\,\Delta_{2n+2}\left(\frac{\partial p_2}{\partial y}\right)\left[(1-\rho)a_n + \rho c_n\right] \;.$$

Finally, equations (30), (31), (34) and (35) substituted in (16) give the general solutions for W_1, U_o and V_o. They are

$$(36)\quad \Delta_2 W_1 = -\sum_{n=o}^{\infty} \frac{h^{2n}}{4G}\,\Delta_{2n+2}\, p_2 \left[(1-\rho)c_n + \rho a_n\right] \quad ,$$

$$(37)\quad \Delta_4 U_o = \sum_{n=o}^{\infty} \frac{h^{2n}}{4G}\,\Delta_{2n+2}\left(\frac{\partial p_2}{\partial x}\right)\left[\rho c_n + (1-\rho)a_n\right] \quad ,$$

$$(38)\quad \Delta_4 V_o = \sum_{n=o}^{\infty} \frac{h^{2n}}{4G}\,\Delta_{2n+2}\left(\frac{\partial p_2}{\partial y}\right)\left[\rho c_n + (1-\rho)a_n\right] \quad ,$$

$$(39)\quad \Delta_2\left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right) = \sum_{n=o}^{\infty} \frac{h^{2n}}{4G}\,\Delta_{2n+2}\, p_2 \left[\rho c_n + (1-\rho)a_n\right] .$$

The displacements u, v, w are given by relations (7) when the six functions U_o, V_o, W_o, U_1, V_1 and W_1 are known. Therefore one can say that the differential equations (26) - (29) and (36) - (39) define the displacements. Furthermore the displacements defined by these differential equations satisfy the equilibrium equations and the surface traction conditions for any normal load which can be expressed as a function of x,y continuous over the entire plate.

In 1899, J. H. Michell (11, p.119) published the differential equation defining W_o in the form

$$\Delta_4 W_o = -\frac{1-\rho^2}{E}\left[\left(\Delta_2 + \frac{\partial^2}{\partial z^2}\right)\frac{\partial Z_z}{\partial z}\right]_{z=o} - \frac{1+\rho}{E}\,\Delta_2\left(\frac{\partial Z_z}{\partial z}\right)_{z=o} .$$

It is desirable to show that this solution becomes identical with equation (26) when the stress Z_z is expressed in terms of the load. When the displacements (7) are substituted in relation (9.1), there results

$$Z_z = \frac{2G}{1-2\rho}\left\{\sum_{k=o}^{\infty}(-1)^k\Delta_{2k}\left[(\rho-k)\left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right) + (1-\rho-k)W_1\right]\frac{z^{2k}}{(2k)!}\right.$$
$$+ \frac{1-2\rho}{2(1-\rho)}\sum_{k=o}^{\infty}(-1)^k\Delta_{2k}\left[(\rho-1-k)\left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)\right.$$
$$\left.\left. + (k-1+\rho)\Delta_2 W_o\right]\frac{z^{2k+1}}{(2k+1)!}\right\} .$$

For values of $k \gtreqless 1$ this equation may be expressed in terms of the load by use of solutions (26), (29), (36) and (39). By retaining the terms for $k=0$ one gets

$$Z_z = \frac{2G}{1-2\rho}\left\{\rho\left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right) + (1-\rho)W_1 - \frac{1}{4G}\sum_{k=1}^{\infty}(-1)^k\Delta_{2k-2} \cdot\right.$$

$$\left[\sum_{n=o}^{\infty} h^{2n}\Delta_{2n+2}\, p_2\left\{k(a_n + c_n) + (1-2\rho)c_n\right\}\right]\frac{z^{2k}}{(2k)!}$$

$$- \frac{1-2\rho}{2}\left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} + \Delta_2 W_0\right) z + \frac{1-2\rho}{D(1-\rho)} \sum_{k=1}^{\infty} (-1)^k \Delta_{2k-2} \cdot$$

$$\left[\sum_{n=0}^{\infty} h^{2n} \Delta_{2n}\, p_1 \left\{(k-1)(1-\rho)d_n - kb_n\right\}\right] \frac{z^{2k+1}}{(2k+1)!} \Bigg\} .$$

This value of Z_z substituted in the result given by Michell yields equation (26). Therefore the differential equation defining W_0 obtained in this investigation agrees with that previously given by Michell.

In the next chapter the displacements will be made to satisfy the edge conditions for a particular load. This will complete the determination of the displacements as they will then satisfy all the required conditions, as given in the introduction, for this type of problem.

III. COMPLETE SOLUTION FOR A SPECIAL CASE OF NORMAL LOAD

$$p_1(x,y) = p_2(x,y) = P\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right) .$$

A. Preliminary Relations

The displacements will now be found for a normal load $P\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$ on the top surface of the plate where a and b are the horizontal dimensions of the plate, α and β are arbitrary constants, and P is a uniform load per unit area. In this case the differential equations which define the displacements reduce to

(40) $\Delta_4 W_0 = - \frac{P}{D}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$,

(41) $\Delta_4 U_1 = \frac{P\alpha}{aD}$,

(42) $\Delta_4 V_1 = \frac{P\beta}{bD}$,

(43) $\Delta_2 \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right) = \frac{P}{D}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$,

(44) $\Delta_2 W_1 = 0$,

(45) $\Delta_4 U_0 = 0$,

(46) $\Delta_4 V_o = 0$

(47) $\Delta_2 \left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right) = 0$.

The load is a linear function of x,y and it is of the fourth order of magnitude as compared with W_{oo}. Hence all terms, whose orders of magnitude are equal to or greater than six, vanish. Therefore all infinite systems of equations from the previous chapter reduce to finite systems. Likewise all infinite sequences of terms of increasing order of magnitude become finite sequences. The sums of the equations remaining in each of the systems (22), (23) and (15) give respectively

(48) $U_1 = -\frac{\partial W_o}{\partial x} - \frac{h^2}{1-\rho}\Delta_2\left(\frac{\partial W_o}{\partial x}\right) - \frac{h^4(5-2\rho)}{6(1-\rho)^2}\Delta_4\left(\frac{\partial W_o}{\partial x}\right)$,

(49) $V_1 = -\frac{\partial W_o}{\partial y} - \frac{h^2}{1-\rho}\Delta_2\left(\frac{\partial W_o}{\partial y}\right) - \frac{h^4(5-2\rho)}{6(1-\rho)^2}\Delta_4\left(\frac{\partial W_o}{\partial y}\right)$,

(50) $W_1 = -\frac{\rho}{1-\rho}\left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right) - \frac{(1-2\rho)P}{4G(1-\rho)}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$.

The infinite series (7) which give the displacements in terms of W_o, U_o, V_o, W_1, U_1 and V_1 reduce to the following series which have a finite number of terms.

(51) $u = U_o + U_1 z - \left[\Delta_2 U_o + \frac{1}{1-2\rho}\frac{\partial}{\partial x}\left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y} + W_1\right)\right]\frac{z^2}{2!}$

$$+ \frac{2-\rho}{1-\rho} \Delta_2 \left(\frac{\partial W_0}{\partial x}\right) \frac{z^3}{3!} + \frac{3-2\rho}{2(1-\rho)^2} h^2 \Delta_4 \left(\frac{\partial W_0}{\partial x}\right) \frac{z^3}{3!}$$

$$- \frac{3-\rho}{1-\rho} \Delta_4 \left(\frac{\partial W_0}{\partial x}\right) \frac{z^5}{5!} \quad ,$$

$$(52)\ v = V_0 + V_1 z - \left[\Delta_2 V_0 + \frac{1}{1-2\rho} \frac{\partial}{\partial y} \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} + W_1\right)\right] \frac{z^2}{2!}$$

$$+ \frac{2-\rho}{1-\rho} \Delta_2 \left(\frac{\partial W_0}{\partial y}\right) \frac{z^3}{3!} + \frac{3-2\rho}{2(1-\rho)^2} h^2 \Delta_4 \left(\frac{\partial W_0}{\partial y}\right) \frac{z^3}{3!}$$

$$- \frac{3-\rho}{1-\rho} \Delta_4 \left(\frac{\partial W_0}{\partial y}\right) \frac{z^5}{5!} \quad ,$$

$$(53)\ w = W_0 + W_1 z + \frac{\rho z^2}{2(1-\rho)} \Delta_2 W_0 + \frac{h^2 z^2}{4(1-\rho)^2} \Delta_4 W_0 - \frac{1+\rho}{1-\rho} \Delta_4 W_0 \frac{z^4}{4} \ .$$

The problem of this investigation is restricted to moderately thick plates because the tractions applied to the edges are represented by their force-and couple-resultants taken along a vertical element of an edge. These classical conditions at an edge have been defined, by Love, in terms of T, S, G and $N - \frac{\partial H}{\partial s}$. Other writers do not mention S as an edge condition and it will not be used in this analysis. The remaining quantities T, G and $N - \frac{\partial H}{\partial s}$ must be expressed in terms of the displacements before they can be used here. They may be written (10, p.101-4, 456)

$$T_1 = \int_{-h}^{h} X_x dz = \frac{2G}{1-2\rho} \int_{-h}^{h} \left[(1-\rho) \frac{\partial u}{\partial x} + \rho \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] dz \quad ,$$

$$T_2 = \int_{-h}^{h} Y_y dz = \frac{2G}{1-2\rho}\int_{-h}^{h} \left[(1-\rho)\ \frac{\partial v}{\partial y} + \rho\ \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right]\ dz \quad ,$$

$$G_1 = \int_{-h}^{h} zX_x dz = \frac{2G}{1-2\rho}\int_{-h}^{h} \left[(1-\rho)\ \frac{\partial u}{\partial x} + \rho\ \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] zdz \quad ,$$

$$G_2 = \int_{-h}^{h} zY_y dz = \frac{2G}{1-2\rho}\int_{-h}^{h} \left[(1-\rho)\ \frac{\partial v}{\partial y} + \rho\ \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] zdz \quad ,$$

$$N_2 - \frac{\partial H_2}{\partial x} = \int_{-h}^{h} Y_z dz + \frac{\partial}{\partial x}\int_{-h}^{h} zX_y dz$$

$$= G\int_{-h}^{h} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) dz + G\ \frac{\partial}{\partial x}\int_{-h}^{h} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\ zdz \quad .$$

When u, v and w are replaced by their values from equations (51), (52) and (53) respectively and the indicated integrations performed, there results

$$(54)\quad T_1 = \frac{4Gh}{1-\rho}\left[\frac{\partial U_o}{\partial x} + \rho\ \frac{\partial V_o}{\partial y} - \frac{P\rho}{4G}\ \left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)\right.$$
$$\left. - \frac{h^2}{6}\left\{(1-\rho)\Delta_2\ \left(\frac{\partial U_o}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\ \left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right)\right\}\right] \quad ,$$

$$(55)\quad T_2 = \frac{4Gh}{1-\rho}\left[\frac{\partial V_o}{\partial y} + \rho\ \frac{\partial U_o}{\partial x} - \frac{P\rho}{4G}\ \left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)\right.$$
$$\left. - \frac{h^2}{6}\left\{(1-\rho)\Delta_2\ \left(\frac{\partial V_o}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\ \left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right)\right\}\right] \quad ,$$

$$(56)\quad G_1 = -\ \frac{4Gh^3}{3(1-\rho)}\left[\frac{\partial^2 W_o}{\partial x^2} + \rho\ \frac{\partial^2 W_o}{\partial y^2} + \frac{8-3\rho}{10(1-\rho)}\ h^2\Delta_4 W_o\right.$$

$$- \frac{8+\rho}{10} h^2 \Delta_2 \left(\frac{\partial^2 W_0}{\partial y^2} \right) \Bigg] ,$$

$$(57)\quad G_2 = - \frac{4Gh^3}{3(1-\rho)} \left[\frac{\partial^2 W_0}{\partial y^2} + \rho \frac{\partial^2 W_0}{\partial x^2} + \frac{8-3\rho}{10(1-\rho)} h^2 \Delta_4 W_0 \right.$$

$$\left. - \frac{8+\rho}{10} h^2 \Delta_2 \left(\frac{\partial^2 W_0}{\partial x^2} \right) \right] ,$$

$$(58)\quad N_2 - \frac{\partial H_2}{\partial x} = - \frac{4Gh^3}{3(1-\rho)} \left[(2-\rho) \frac{\partial^3 W_0}{\partial x^2 \partial y} + \frac{\partial^3 W_0}{\partial y^3} \right.$$

$$\left. + \frac{8+\rho}{10} h^2 \Delta_2 \left(\frac{\partial^3 W_0}{\partial x^2 \partial y} \right) + \frac{8-3\rho}{10(1-\rho)} \Delta_4 \left(\frac{\partial W_0}{\partial y} \right) \right] .$$

Perhaps it is well to mention that in equations (54) to (58) inclusive

T_1 = the tensile force per unit of edge x = constant,
T_2 = the tensile force per unit of edge y = constant,
G_1 = the flexural moment per unit of edge x = constant,
G_2 = the flexural moment per unit of edge y = constant,
H_2 = the torsional moment per unit of edge y = constant,
N_2 = the shearing force, normal to the middle plane, per unit of edge y = constant.

Also, since W_0, U_0, V_0, W_1, U_1 and V_1 have been written as infinite sequences of terms of increasing order of magnitude, it is necessary to write these forces and moments in the same way before they can be used as boundary conditions. Each of equations (54) to (58) is equivalent to an infinite system of equations when terms of the same order of magnitude are equated to zero for use as edge conditions. Only those equations from each infinite system which impose conditions on the quantities that survive in this problem will be listed. They are

$$(54.0)\quad \frac{\partial U_{00}}{\partial x} + \rho\,\frac{\partial V_{00}}{\partial y} = \frac{P\rho}{4G}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right) \quad ,$$

$$(54.1)\quad \frac{\partial U_{20}}{\partial x} + \rho\,\frac{\partial V_{20}}{\partial y} = \frac{h^2}{6}\left[(1-\rho)\Delta_2\left(\frac{\partial U_{00}}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial U_{00}}{\partial x} + \frac{\partial V_{00}}{\partial y}\right)\right] \;;$$

$$(55.0)\quad \frac{\partial V_{00}}{\partial y} + \rho\,\frac{\partial U_{00}}{\partial x} = \frac{P\rho}{4G}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right) \quad ,$$

$$(55.1)\quad \frac{\partial V_{20}}{\partial y} + \rho\,\frac{\partial U_{20}}{\partial x} = \frac{h^2}{6}\left[(1-\rho)\Delta_2\left(\frac{\partial V_{00}}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial U_{00}}{\partial x} + \frac{\partial V_{00}}{\partial y}\right)\right] \;;$$

$$(56.0)\quad \frac{\partial^2 W_{00}}{\partial x^2} + \rho\,\frac{\partial^2 W_{00}}{\partial y^2} = 0 \quad ,$$

$$(56.1)\quad \frac{\partial^2 W_{20}}{\partial x^2} + \rho\,\frac{\partial^2 W_{20}}{\partial y^2} + \frac{8-3\rho}{10(1-\rho)}\,h^2\Delta_4 W_{00}$$

$$- \frac{8+\rho}{10}\,h^2\Delta_2\left(\frac{\partial^2 W_{00}}{\partial y^2}\right) = 0 \quad ,$$

$$(56.2)\quad \frac{\partial^2 W_{40}}{\partial x^2} + \rho\,\frac{\partial^2 W_{40}}{\partial y^2} = \frac{8+\rho}{10}\,h^2\Delta_2\left(\frac{\partial^2 W_{20}}{\partial y^2}\right) \;;$$

$$(57.0)\quad \frac{\partial^2 W_{00}}{\partial y^2} + \rho \frac{\partial^2 W_{00}}{\partial x^2} = 0 \quad ,$$

$$(57.1)\quad \frac{\partial^2 W_{20}}{\partial y^2} + \rho \frac{\partial^2 W_{20}}{\partial x^2} + \frac{8-3\rho}{10(1-\rho)} h^2 \Delta_4 W_{00} = \frac{8+\rho}{10} h^2 \Delta_2 \left(\frac{\partial^2 W_{00}}{\partial x^2}\right) \quad ,$$

$$(57.2)\quad \frac{\partial^2 W_{40}}{\partial y^2} + \rho \frac{\partial^2 W_{40}}{\partial x^2} = \frac{8+\rho}{10} h^2 \Delta_2 \left(\frac{\partial^2 W_{20}}{\partial x^2}\right) \quad ;$$

$$(58.0)\quad (2-\rho) \frac{\partial^3 W_{00}}{\partial x^2 \partial y} + \frac{\partial^3 W_{00}}{\partial y^3} = 0 \quad ,$$

$$(58.1)\quad (2-\rho) \frac{\partial^3 W_{20}}{\partial x^2 \partial y} + \frac{\partial^3 W_{20}}{\partial y^3} + \frac{8+\rho}{10} h^2 \Delta_2 \left(\frac{\partial^3 W_{00}}{\partial x^2 \partial y}\right)$$

$$+ \frac{8-3\rho}{10(1-\rho)} h^2 \Delta_4 \left(\frac{\partial W_{00}}{\partial y}\right) = 0 \quad ,$$

$$(58.2)\quad (2-\rho) \frac{\partial^3 W_{40}}{\partial x^2 \partial y} + \frac{\partial^3 W_{40}}{\partial y^3} + \frac{8+\rho}{10} h^2 \Delta_2 \left(\frac{\partial^3 W_{20}}{\partial x^2 \partial y}\right) = 0 \quad .$$

Equation (40) is the differential equation defining W_0. After W_0 is determined U_1 and V_1 can be obtained from relations (48) and (49) respectively. When U_0 and V_0 as defined by (45), (46) and (47) are known, W_1 can be obtained from relation (50). It is obvious from an inspection of equations (54) to (58) inclusive that the conditions $G_1 = 0$, $G_2 = 0$ and $N_2 - \frac{\partial H_2}{\partial x} = 0$ impose restrictions on W_0 while the conditions $T_1 = 0$ and $T_2 = 0$ impose restrictions on U_0 and V_0.

The problem will now be completed for three different sets of edge conditions.

B. Solution for the Case of a Plate Pinned on All Four Edges

The boundary (edge) conditions to be imposed on the solutions of the differential equations for this case are $W_o = T_1 = G_1 = 0$ along the edges $x = 0,a$ and $W_o = T_2 = G_2 = 0$ along the edges $y = 0,b$.

The method of solution used in this investigation does not give W_o directly. First W_{oo} is obtained, then W_{2o}, etc., and finally W_o is obtained from equation (16). It has already been pointed out that W_{oo} is the vertical displacement for the corresponding thin plate under the same normal surface load. By observing that conditions (56.0) and (57.0) are precisely the pinned edge conditions from thin plate theory, one may employ the W_{oo} solution of the thin plate problem, in case it has been solved. This solution has already been published by S. Iguchi (7, p.23) for the pinned-pinned case. It is

$$(59)\quad W_{oo} = -\frac{Pa^4}{24D}\left\{\left(1+\frac{\beta y}{b}\right)\left(\frac{x^4}{a^4}-\frac{2x^3}{a^3}+\frac{x}{a}\right)+\alpha\left(\frac{x^5}{5a^5}-\frac{2x^3}{3a^3}+\frac{7x}{15a}\right)\right\}-\frac{P}{aD}\sum_n\frac{\sin\theta x}{\theta^5}\left[i\left\{B_n+(1+\beta)C_n\right\}+j(B_n+C_n)\right]$$

where $B_n = \dfrac{\theta y \text{ ch } \phi \text{ sh } \theta(y-b) + 2 \text{ sh } \phi \text{ sh } \theta(y-b) - \theta(y-b) \text{ sh } \theta y}{\text{sh}^2 \phi}$ *,

$$C_n = \frac{\theta(y-b) \text{ ch } \phi \text{ sh } \theta y - 2 \text{ sh } \phi \text{ sh } \theta y - \theta y \text{ sh } \theta(y-b)}{\text{sh}^2 \phi},$$

$$\theta = \frac{n\pi}{a}, \quad \phi = \theta b, \quad i = 1 - (-1)^n \text{ and } j = -\alpha(-1)^n.$$

From (56.1) and (57.1) it is evident that $\Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial y^2}\right)$ and $\Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial x^2}\right)$ must be evaluated at the edges before a solution for W_{2o} is possible.

At $x = 0,a$: $\Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial x^2}\right) = -\dfrac{P}{D}\left(1 + \dfrac{\alpha x}{a} + \dfrac{\beta y}{b}\right)$ and $\Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial y^2}\right) = 0.$

At $y = 0,b$: $\Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial y^2}\right) = -\dfrac{P}{D}\left(1 + \dfrac{\alpha x}{a} + \dfrac{\beta y}{b}\right)$ and $\Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial x^2}\right) = 0.$

Also at $x = 0,a$: $\dfrac{\partial^2 W_{2o}}{\partial y^2} = 0$; and at $y = 0,b$: $\dfrac{\partial^2 W_{2o}}{\partial x^2} = 0$

because the middle surface maintains a continuous contact with the support in the original plane.

By definition W_{2o} is of the second order of magnitude as compared with W_{oo} and from equation (18.1) $\Delta_4 W_{2o} = 0$. There-

*ch is used for cosh and sh for sinh.

fore one may assume a solution, for W_{20}, of the form

$$(60)\quad W_{20} = -\frac{PA}{2D}\left[\left(1 + \frac{\beta y}{b}\right)(x^2 - ax) + \frac{\alpha}{3a}(x^3 - a^2x)\right]$$

$$-\frac{2P}{aD}\sum_n \frac{\sin\theta x}{\theta^3}\left[F_n \operatorname{ch}\theta y + G_n \operatorname{sh}\theta y + H_n\,\theta y \operatorname{ch}\theta y\right.$$

$$\left.+ I_n\,\theta y \operatorname{sh}\theta y\right]$$

where A, F_n, G_n, H_n and I_n are constants to be determined. Boundary condition (56.1) becomes

$$\frac{\partial^2 W_{20}}{\partial x^2} = \frac{Ph^2(8-3\rho)}{10D(1-\rho)}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right) \quad \text{at } x = 0, a\ .$$

This condition yields $A = -\frac{8-3\rho}{10(1-\rho)}h^2$. The boundary conditions at $y = 0, b$ are $W_{20}=0$ and (57.1) which becomes

$\frac{\partial^2 W_{20}}{\partial y^2} = \frac{Ph^2(8-3\rho)}{10D(1-\rho)}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$. They yield the four relations

$$(a)\quad F_n = -\frac{8-3\rho}{10(1-\rho)}h^2(i+j)\ ,$$

$$(b)\quad F_n \operatorname{ch}\phi + G_n \operatorname{sh}\phi + H_n\phi \operatorname{ch}\phi + I_n\phi \operatorname{sh}\phi$$

$$= -\frac{8-3\rho}{10(1-\rho)}h^2(i+j+\beta i),$$

(c) $F_n + 2I_n = -\frac{8-3\rho}{10(1-\rho)} h^2(i+j)$ and

(d) $(F_n + 2I_n) \operatorname{ch} \phi + (G_n + 2H_n) \operatorname{sh} \phi + H_n\phi \operatorname{ch} \phi + I_n\phi \operatorname{sh} \phi$

$$= -\frac{8-3\rho}{10(1-\rho)} h^2(i+j+\beta i) .$$

The simultaneous solution of this system of equations is

$$I_n = H_n = 0 , \quad G_n = -\frac{8-3\rho}{10(1-\rho)} h^2 \frac{(i+j+\beta i) - (i+j) \operatorname{ch} \phi}{\operatorname{sh} \phi}$$

and $F_n = -\frac{8-3\rho}{10(1-\rho)} h^2(i+j)$. When these values are substituted in equation (60), the solution for W_{20} results. It is easily shown that W_{20} may be written in the form

(61) $W_{20} = -\frac{8-3\rho}{10(1-\rho)} h^2 \Delta_2 W_{00}$.

W_{40} is of the fourth order of magnitude as compared to W_{00} and from equation (18.2) $\Delta_2 W_{40} = 0$. Therefore one assumes a solution of the form

$$W_{40} = \frac{AP}{D}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right) + \frac{Ph^4}{aD}\sum_n \frac{\sin \theta x}{\theta}\left[J_n \operatorname{ch} \theta y + K_n \operatorname{sh} \theta y\right]$$

where A, J_n and K_n are constants to be determined. The boundary conditions at $x = 0,a$ are $W_{40} = 0$ and (56.2) which reduces to $\frac{\partial^2 W_{40}}{\partial x^2} = 0$. These conditions require that $A = 0$. The boundary conditions at $y = 0,b$ are $W_{40} = 0$ and (57.2) which reduces to $\frac{\partial^2 W_{40}}{\partial y^2} = 0$. These conditions require that $J_n = K_n = 0$. Therefore $W_{40} = 0$. By inspection of equations (18) it readily follows that (62) $W_{2n,0} = 0 \quad (n \geqq 2)$.

It remains to determine U_0 and V_0 to complete this case. It has already been shown that W_{01} and $\frac{\partial U_{00}}{\partial x} + \frac{\partial V_{00}}{\partial y}$ are of the same order of magnitude as the load and the load is of the fourth order of magnitude as compared with W_{00}. Therefore

$$\frac{\partial U_{00}}{\partial x} + \frac{\partial V_{00}}{\partial y} = a + bx + cy$$
$$+ \sum_n \frac{\sin \theta x}{a\theta} \left[F_n \text{ ch } \theta y + G_n \text{ sh } \theta y + H_n \theta y \text{ ch } \theta y\right.$$
$$\left. + I_n \theta y \text{ sh } \theta y\right] .$$

However equation (47) requires that this expression be a harmonic function. Therefore $H_n = I_n = 0$. With this background one may consider the following forms of solutions

$$U_{00} = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2$$
$$+ \sum_n \frac{\cos \theta x}{a\theta^2} \left[A_n \text{ ch } \theta y + B_n \text{ sh } \theta y\right] \quad \text{and}$$

$$V_{00} = k_0 + k_1 x + k_3 x^2 + k_4 xy + k_5 y^2 + k_2 y$$
$$+ \sum_n \frac{\sin \theta x}{a\theta^2} \left[C_n \text{ ch } \theta y + D_n \text{ sh } \theta y\right] .$$

In order to obtain a satisfactory solution here it is necessary to observe carefully the method used to obtain W_{00}, W_{20}, etc. In every case a polynomial that satisfied the conditions at $x = 0,a$ was added to a sum that did not conflict with the edge conditions at $x = 0,a$ and that contained sufficient undetermined constants to permit the conditions at $y = 0,b$ to be satisfied. Since the conditions imposed along the edges $x = 0,a$ are not sufficient in number to determine a unique polynomial, this solution must be recognized as a satisfactory solution and not necessarily the unique solution. Garabedian (5) avoided this difficulty by assuming that U_{00} and V_{00} were closed linear functions of x,y. However, this assumption does not permit a satisfactory solution in the pinned-clamped case as will be shown later.

Equation (54.0) is the only condition on U_{00} and V_{00} at $x = 0,a$. It gives

$$c_1 + \rho k_2 = \frac{P\rho}{4G} , \qquad c_4 + 2\rho k_5 = \frac{P\rho\beta}{4Gb} ,$$

$$2c_3 + \rho k_4 = \frac{P\rho\alpha}{4Ga} .$$

Therefore choose $c_1 = k_2 = \frac{P\rho}{2E}$, $c_4 = 2k_5 = \frac{P\rho\beta}{2Eb}$,

$2c_3 = k_4 = \frac{P\rho\alpha}{2Ea}$, $c_0 = c_2 = c_5 = k_0 = k_1 = k_3 = 0$ where the

quantity E is "Young's modulus". Equation (55.0) at $y = 0,b$

yields $D_n = \rho A_n$ and $C_n = \rho B_n$. If the plate be fixed in space by making $V_{oo} = 0$ at $y = 0$ and $U_{oo} = 0$ at $x = 0$, then $B_n = A_n = 0$. Consequently $D_n = C_n = 0$ and the final solutions may be written

$$(63)\quad U_{oo} = \frac{P\rho x}{2E}\left(1 + \frac{\alpha x}{2a} + \frac{\beta y}{b}\right) \quad \text{and}$$

$$(64)\quad V_{oo} = \frac{P\rho y}{2E}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{2b}\right).$$

Since U_{2o} and V_{2o} are of the second order of magnitude as compared to U_{oo} and V_{oo}, they must either be zero or constants. But $V_{2o} = 0$ at $y = 0$ and $U_{2o} = 0$ at $x = 0$ to fix the plate in space. Therefore they are zero everywhere. Consequently $U_{2n,o} = V_{2n,o} = 0 \quad (n \geqq 1)$. This means that (63) and (64) give the complete horizontal displacement of the middle surface. Although not a unique solution it is a rational one in the sense that it yields a displacement at the middle surface which is one-half of what the displacement would be if the plate were resting on a complete foundation.

When U_o and V_o are replaced by their values from solutions (63) and (64), equation (50) yields

$W_1 = -\frac{P}{2E}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$. W_o is given by the sum of W_{oo} from solution (59) and W_{2o} from solution (61). U_1 and V_1 are obtained by substituting the value of W_o in equations (48) and

(49). With all six of these functions known the displacements u, v and w for any point in the plate are easily determined from equations (51), (52) and (53) respectively. Once the displacements are determined it is merely a matter of substitution in the relations $X_x = \frac{2G}{1-2\rho}\left[(1-\rho)\,\frac{\partial u}{\partial x} + \rho\,\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$, etc., to obtain the stresses at any point of the plate.

If $\alpha = \beta = 0$ the solutions given here reduce to those published by Garabedian (5) for a uniform load.

C. Solution for the Case of a Plate Pinned on the Edges x = 0,a and Free on the Edges y = 0,b.

The boundary (edge) conditions to be imposed on the solutions of the differential equations for this case are

$W_o = T_1 = G_1 = 0$ along the edges $x = 0,a$ and

$T_2 = G_2 = N_2 - \frac{\partial H_2}{\partial x} = 0$ along the edges $y = 0,b$.

In this case the thin plate solution for W_{oo} is not available. In order to obtain it a solution satisfying (18.0) or (40.0) is given by

$$(65)\quad W_{oo} = -\frac{Pa^4}{24D}\left[\left(1 + \frac{\beta y}{b}\right)\left(\frac{x^4}{a^4} - \frac{2x^3}{a^3} + \frac{x}{a}\right)\right.$$

$$\left. + \alpha\left(\frac{x^5}{5a^5} - \frac{2x^3}{3a^3} + \frac{7x}{15a}\right)\right] - \frac{P}{aD}\sum_n \frac{\sin\theta x}{\theta^5}\,\cdot$$

$$\left[A_n \operatorname{ch}\theta y + B_n\theta y \operatorname{ch}\theta y + C_n \operatorname{sh}\theta y + D_n\theta y \operatorname{sh}\theta y\right]$$

where the constants A_n, B_n, C_n and D_n are to be determined by the conditions at $y = 0,b$. This solution already satisfies the conditions at $x = 0,a$. The following Fourier expansions which are valid in the interval $0 \leqq x \leqq a$ will be needed.

$$(66)\quad \frac{1}{2}(ax-x^2) = \frac{4}{a}\sum_{n\ \mathrm{odd}} \frac{\sin \theta x}{\theta^3} = \frac{2}{a}\sum_{n} i\ \frac{\sin \theta x}{\theta^3} ,$$

$$(67)\quad \frac{\alpha}{6a}(a^2x-x^3) = \frac{2\alpha}{a}\sum_{n} (-1)^{n+1}\frac{\sin \theta x}{\theta^3} = \frac{2}{a}\sum_{n} j\ \frac{\sin \theta x}{\theta^3} .$$

The substitution of equations (65), (66) and (67) in edge conditions (57.0) and (58.0) at $y = 0,b$ yields the following system of equations.

$$\text{(a)}\quad 2D_n + A_n(1-\rho) = 2\rho(i+j) ,$$

$$\text{(b)}\quad D_n\left[2\ \mathrm{ch}\ \phi + (1-\rho)\phi\ \mathrm{sh}\ \phi\right] + B_n\left[2\ \mathrm{sh}\ \phi + (1-\rho)\phi\ \mathrm{ch}\ \phi\right]$$
$$+ A_n(1-\rho)\ \mathrm{ch}\ \phi + C_n(1-\rho)\ \mathrm{sh}\ \phi = 2\rho(i+j+\beta i) ,$$

$$\text{(c)}\quad B_n\phi(1+\rho) - C_n\phi(1-\rho) = 2\beta i(2-\rho) ,$$

$$\text{(d)}\quad D_n\phi\left[(1+\rho)\ \mathrm{sh}\ \phi - (1-\rho)\phi\ \mathrm{ch}\ \phi\right] - A_n\phi(1-\rho)\ \mathrm{sh}\ \phi$$
$$+ B_n\phi\left[(1+\rho)\ \mathrm{ch}\ \phi - (1-\rho)\phi\ \mathrm{sh}\ \phi\right]$$
$$- C_n\phi(1-\rho)\ \mathrm{ch}\ \phi = 2\beta i(2-\rho) .$$

The simultaneous solution of this system of equations is

$$A_n = \frac{2\rho(i+j)\left[(1+\rho)\ \text{sh}\ \phi - S\right]}{(1-\rho)(R-S)} - \frac{4i\beta\left[\rho\phi S\ \text{sh}\ \phi + (2-\rho)(1-\text{ch}\ \phi)(R-S)\right]}{\phi(1-\rho)(R^2-S^2)},$$

$$B_n = \frac{2\rho(i+j)(1-\text{ch}\ \phi)}{R-S}$$

$$+ \frac{2i\beta\left[(2-\rho)(R-S)\ \text{sh}\ \phi + \rho\phi(R-S\ \text{ch}\ \phi)\right]}{\phi(R^2-S^2)},$$

$$C_n = \frac{2\rho(1+\rho)(i+j)(1-\text{ch}\ \phi)}{(1-\rho)(R-S)}$$

$$+ \frac{2i\beta\left[\rho(1+\rho)\phi(R-S\ \text{ch}\ \phi) - (2-\rho)(R-S)(2\ \text{sh}\ \phi + S)\right]}{\phi(1-\rho)(R^2-S^2)}$$

$$D_n = \frac{2\rho(i+j)\ \text{sh}\ \phi}{R-S} + \frac{2i\beta\left[\rho\phi S\ \text{sh}\ \phi + (2-\rho)(R-S)(1-\text{ch}\ \phi)\right]}{\phi(R^2-S^2)}$$

where $R = (3+\rho)\ \text{sh}\ \phi$ and $S = (1-\rho)\phi$. These values substituted in equation (65) constitute the solution for W_{oo}. If $\alpha = \beta = 0$ this solution reduces to that published by D. L. Holl (8, p.601) for a uniform load.

Again it is evident that $\Delta_2\left(\frac{\partial^2 W_{oo}}{\partial y^2}\right)$, $\Delta_2\left(\frac{\partial^2 W_{oo}}{\partial x^2}\right)$ and $\Delta_2\left(\frac{\partial^3 W_{oo}}{\partial x^2 \partial y}\right)$ must be evaluated at the edges $x = 0, a$, $y = 0, b$ and $y = 0, b$ respectively before a solution for W_{2o} is possible. The Fourier expansion $1 + \frac{\alpha x}{a} = \frac{2}{a}\sum_n (i+j)\ \frac{\sin\ \theta x}{\theta}$ will be needed in evaluating these expressions.

(68)
$$\begin{cases} \text{At } y = 0 : \Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial x^2}\right) = \dfrac{2P}{aD}\displaystyle\sum_n \dfrac{\sin\,\theta x}{\theta}\left[D_n - (i+j)\right] . \\ \text{At } y = b : \Delta_2\left(\dfrac{\partial^2 W_{oo}}{\partial x^2}\right) = \dfrac{2P}{aD}\displaystyle\sum_n \dfrac{\sin\,\theta x}{\theta} \cdot \\ \qquad\qquad \left[B_n \text{ sh } \phi + D_n \text{ ch } \phi - (i+j+\beta i)\right] . \\ \text{At } y = 0 : \Delta_2\left(\dfrac{\partial^3 W_{oo}}{\partial x^2 \partial y}\right) = \dfrac{2P}{abD}\displaystyle\sum_n \dfrac{\sin\,\theta x}{\theta}\left[B_n - \beta i\right] . \\ \text{At } y = b : \Delta_2\left(\dfrac{\partial^3 W_{oo}}{\partial x^2 \partial y}\right) = \dfrac{2P}{abD}\displaystyle\sum_n \dfrac{\sin\,\theta x}{\theta} \cdot \\ \qquad\qquad \left[B_n\phi \text{ ch } \phi + D_n\phi \text{ sh } \phi - \beta i\right] . \end{cases}$$

$$\text{At } x = 0,a : \Delta_2\left(\frac{\partial^2 W_{oo}}{\partial y^2}\right) = 0 .$$

$$\text{Also at } x = 0,a : \frac{\partial^2 W_{2o}}{\partial y^2} = 0 .$$

As in the pinned-pinned case the following biharmonic function which is of the proper order of magnitude is assumed as a solution for W_{2o}.

(69)
$$W_{2o} = -\frac{PA}{2D}\left[\left(1 + \frac{\beta y}{b}\right)(x^2 - ax) + \frac{\alpha}{3a}(x^3 - a^2 x)\right] + \frac{Ph^2}{aD}\sum_n \frac{\sin\,\theta x}{\theta^3}\left[F_n \text{ ch } \theta y + G_n \text{ sh } \theta y\right.$$

$$H_n\theta y \text{ ch } \theta y + I_n\theta y \text{ sh } \theta y\Big] .$$

As before, boundary condition (56.1) yields $A = -\frac{8-3\rho}{10(1-\rho)} h^2$.
When equations (68) and (69) are used in boundary conditions (57.1) and (58.1) at the edges $y = 0,b$, the following system of equations for determining the unknown constants is obtained.

$$\text{(a) } F_n(1-\rho) + 2I_n = \frac{8+\rho}{5} D_n - \frac{4\rho}{5}(i+j) \quad ,$$

$$\text{(b) } F_n(1-\rho) \text{ ch } \phi + G_n(1-\rho) \text{ sh } \phi + H_n \left[2 \text{ sh } \phi + S \text{ ch } \phi\right]$$
$$+ I_n\left[2 \text{ ch } \phi + S \text{ sh } \phi\right] = \frac{8+\rho}{5}(B_n \text{ sh } \phi + D_n \text{ ch } \phi)$$
$$- \frac{4\rho}{5}(i+j+\beta i) \quad ,$$

$$\text{(c) } G_nS - H_n\phi(1+\rho) = \frac{8+\rho}{5} B_n\phi - \frac{4\rho}{5} \beta i \quad ,$$

$$\text{(d) } G_nS \text{ ch } \phi + F_nS \text{ sh } \phi + H_n\phi \left[S \text{ sh } \phi - (1+\rho) \text{ ch } \phi\right]$$
$$+ I_n\phi\left[S \text{ ch } \phi - (1+\rho) \text{ sh } \phi\right] = \frac{8+\rho}{5}(B_n\phi \text{ ch } \phi$$
$$+ D_n\phi \text{ sh } \phi) - \frac{4\rho}{5} \beta i .$$

The simultaneous solution of this system of equations is

$$F_n = \frac{8+\rho}{5(1-\rho)} D_n + \frac{4\rho}{5(1-\rho)}\left[\frac{(i+j)\{S - (1+\rho) \text{ sh } \phi\}}{R-S}\right.$$
$$\left. + \frac{2\beta i\{S\phi \text{ sh } \phi - (R-S)(1 - \text{ch } \phi)\}}{\phi(R^2-S^2)}\right] ,$$

$$G_n = \frac{8+\rho}{5(1-\rho)} B_n - \frac{4\rho}{5(1-\rho)} \left[\frac{(i+j)(1+\rho)(1 - \text{ch}\ \phi)}{R-S} \right.$$

$$\left. + \frac{\beta i\{(1+\rho)\phi(R-S\ \text{ch}\ \phi) + (R-S)(2\ \text{sh}\ \phi + S)\}}{\phi(R^2-S^2)} \right] ,$$

$$H_n = -\frac{4\rho}{5}\left[\frac{(i+j)(1 - \text{ch}\ \phi)}{R-S} - \frac{\beta i\{(R-S)\ \text{sh}\ \phi - \phi(R-S\ \text{ch}\ \phi)\}}{\phi(R^2-S^2)}\right] ,$$

$$I_n = -\frac{4\rho}{5}\left[\frac{(i+j)\ \text{sh}\ \phi}{R-S} + \frac{\beta i\{S\phi\ \text{sh}\ \phi - (R-S)(1 - \text{ch}\ \phi)\}}{\phi(R^2-S^2)}\right] .$$

These values substituted in equation (69) constitute a solution for W_{20}. However it is desirable to express W_{20} in terms of W_{00} if possible. It is easily shown that the above constants may be written in the following form.

$$F_n = \frac{8+\rho}{5(1-\rho)} D_n - \frac{2}{5} A_n - \frac{16i\beta(1 - \text{ch}\ \phi)}{5\phi(1-\rho)(R+S)} ,$$

$$G_n = \frac{8+\rho}{5(1-\rho)} B_n - \frac{2}{5} C_n - \frac{8i\beta(2\ \text{sh}\ \phi + S)}{5\phi(1-\rho)(R+S)} ,$$

$$H_n = -\frac{2}{5} B_n + \frac{8i\beta\ \text{sh}\ \phi}{5\phi(R+S)} ,$$

$$I_n = -\frac{2}{5} D_n + \frac{8i\beta(1 - \text{ch}\ \phi)}{5\phi(R+S)} .$$

Then W_{20} may be written

$$(70)\quad W_{20} = -\frac{8+\rho}{10(1-\rho)}\, h^2 \Delta_2 W_{00} - \frac{2h^2}{5}\frac{\partial^2 W_{00}}{\partial x^2}$$

$$+ \frac{4Ph^2}{5(1-\rho)aD}\sum_n \frac{\sin\theta x}{\theta^3}\Big[i + j + \frac{\beta i y}{b}$$

$$+ \frac{2\beta i}{\phi(R+S)}\Big\{ -2(1-\mathrm{ch}\,\phi)\,\mathrm{ch}\,\theta y - (2\,\mathrm{sh}\,\phi + S)\,\mathrm{sh}\,\theta y$$

$$+ (1-\rho)\,\mathrm{sh}\,\phi\,\theta y\,\mathrm{ch}\,\theta y + (1-\rho)(1-\mathrm{ch}\,\phi)\theta y\,\mathrm{sh}\,\theta y\Big\}\Big].$$

In form (70) it is obvious that with $\alpha = \beta = 0$ this solution reduces to that given by Garabedian (5) for the uniform load.

The following harmonic function of the fourth order of magnitude as compared with W_{00} is assumed for W_{40}.

$$(71)\quad W_{40} = \frac{PA}{D}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$$

$$+ \frac{Ph^4}{aD}\sum_n \frac{\sin\theta x}{\theta}\left[J_n\,\mathrm{ch}\,\theta y + K_n\,\mathrm{sh}\,\theta y\right].$$

The boundary conditions (56.2) and $W_{40} = 0$ at $x = 0, a$ are satisfied when $A = 0$. At $y = 0, b$ boundary conditions (57.2) and (58.2) become

$$\frac{\partial^2 W_{40}}{\partial y^2} + \rho\,\frac{\partial^2 W_{40}}{\partial x^2} = -\frac{(8+\rho)Ph^4}{5aD}\sum_n \theta\,\sin\theta x\left[H_n\,\mathrm{sh}\,\theta y + I_n\,\mathrm{ch}\,\theta y\right],$$

$$(2-\rho)\,\frac{\partial^3 W_{40}}{\partial x^2\partial y} + \frac{\partial^3 W_{40}}{\partial y^3} = \frac{(8+\rho)Ph^4}{5aD}\sum_n \theta^2\,\sin\theta x\left[H_n\,\mathrm{ch}\,\theta y + I_n\,\mathrm{sh}\,\theta y\right].$$

Either one of these equations yields a system of equations from which J_n and K_n are readily determined. The results are

$$J_n = -\frac{8+\rho}{5(1-\rho)} I_n \quad , \qquad K_n = -\frac{8+\rho}{5(1-\rho)} H_n \quad .$$

With these values the solution (71) for W_{40} may be written

$$(72)\quad W_{40} = -\frac{8+\rho}{10(1-\rho)} h^2\left[\Delta_2 W_{20} - \frac{8-3\rho}{10(1-\rho)} Ph^2 \left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)\right] .$$

Again $J_n = \frac{2(8+\rho)}{25(1-\rho)} D_n - \frac{81\beta(8+\rho)(1-\text{ch}\,\phi)}{25\phi(1-\rho)(R+S)}$,

$K_n = \frac{2(8+\rho)}{25(1-\rho)} B_n - \frac{81\beta(8+\rho)\,\text{sh}\,\phi}{25\phi(1-\rho)(R-S)}$ and therefore

$$(73)\quad W_{40} = -\frac{8+\rho}{25(1-\rho)} h^4\Delta_2\left(\frac{\partial^2 W_{00}}{\partial y^2}\right) - \frac{16(8+\rho)Ph^4}{25aD(1-\rho)} \cdot$$

$$\sum_n \frac{1\beta \sin\theta x\ \text{sh}\,\frac{\phi}{2}\ \text{sh}\,\theta\left(y-\frac{b}{2}\right)}{\theta\phi(R+S)} .$$

This solution reduces to that given by Garabedian (5) when $\alpha = \beta = 0$. By a consideration of orders of magnitude or by inspection of equations (18) it is evident that $W_{2n,0} = 0$ $(n>2)$.

Since the boundary conditions which impose restrictions on U_0 and V_0 in this case are identical with those for the pinned-pinned case, the U_o and V_o for this case are given by equations (63) and (64). Likewise equation (50) gives the same value for W_1 ; namely, $W_1 = -\frac{P}{2E}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$. W_o is obtained by adding W_{oo}[solution (65)] , W_{2o}[solution (69)] and W_{4o}[solution (72)] . With W_o evaluated, U_1 and V_1 are determined from equations (48) and (49). The displacements u, v and w for any point in the plate are obtained by substituting the values of W_o, U_o, V_o, W_1, U_1 and V_1 in equations (51), (52) and (53). As mentioned previously, the stresses are readily obtained from the displacements.

D. Solution for the Case of a Plate Pinned on the Edges x = 0,a and Clamped on the Edges y = 0,b.

The boundary conditions to be imposed on the solutions of the differential equations for this case are $W_o = T_1 = G_1 = 0$ along the edges $x = 0,a$ and $W_o = \frac{\partial W_o}{\partial y} = U_o = V_o = 0$ along the edges $y = 0,b$.

Here again the thin plate solution for W_{oo} is not available; therefore assume a solution of $\Delta_4 W_{oo} = -\frac{P}{D}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$ in the form

$$(74)\quad W_{oo} = -\frac{Pa^4}{24D}\left[\left(1 + \frac{\beta y}{b}\right)\left(\frac{x^4}{a^4} - \frac{2x^3}{a^3} + \frac{x}{a}\right)\right.$$

$$+ \alpha \left(\frac{x^5}{5a^5} - \frac{2x^3}{3a^3} + \frac{7x}{15a} \right) \Big] - \frac{P}{aD} \sum_n \frac{\sin \theta x}{\theta^5} \cdot$$

$$\left[A_n \text{ sh } \theta y + B_n \text{ ch } \theta y + C_n \theta y \text{ sh } \theta y + D_n \theta y \text{ ch } \theta y \right] .$$

This solution satisfies the conditions at $x = 0, a$. The conditions at $y = 0, b$ will serve to evaluate the undetermined constants A_n, B_n, C_n and D_n. The following Fourier expansion will be needed.

$$(75) \quad \frac{a^4}{24} \left(\frac{x^4}{a^4} - \frac{2x^3}{a^3} + \frac{x}{a} \right) = \frac{2}{a} \sum_n i \, \frac{\sin \theta x}{\theta^5} \quad ,$$

$$(76) \quad \frac{\alpha a^4}{24} \left(\frac{x^5}{5a^5} - \frac{2x^3}{3a^3} + \frac{7x}{15a} \right) = \frac{2}{a} \sum_n j \, \frac{\sin \theta x}{\theta^5} \quad .$$

When equations (74), (75) and (76) are substituted in the edge conditions there results

$$(77) \quad \begin{cases} B_n = -2(i+j) \quad , \\ B_n \text{ ch } \phi + D_n \phi \text{ ch } \phi + A_n \text{ sh } \phi + C_n \phi \text{ sh } \phi \\ \qquad = -2(i+j+\beta i) \quad , \\ D_n \phi + A_n \phi = -2\beta i \quad , \\ B_n \phi \text{ sh } \phi + D_n \phi (\phi \text{ sh } \phi + \text{ch } \phi) + A_n \phi \text{ ch } \phi \\ \qquad + C_n \phi (\phi \text{ ch } \phi + \text{sh } \phi) = -2\beta i \ . \end{cases}$$

The simultaneous solution of system (77) is

$$A_n = \frac{-2(i+j)(1 - \operatorname{ch}\phi)}{\phi + \operatorname{sh}\phi} - \frac{2\beta i\phi(1 - \operatorname{ch}\phi)}{\phi^2 - \operatorname{sh}^2\phi} ,$$

$$B_n = -2(i+j) \quad ,$$

$$C_n = \frac{2(i+j)\operatorname{sh}\phi}{\phi + \operatorname{sh}\phi} + \frac{2\beta i[\phi^2 \operatorname{sh}\phi + (1 - \operatorname{ch}\phi)(\phi + \operatorname{sh}\phi)]}{\phi(\phi^2 - \operatorname{sh}^2\phi)} ,$$

$$D_n = \frac{2(i+j)(1 - \operatorname{ch}\phi)}{\phi + \operatorname{sh}\phi} + \frac{2\beta i(\operatorname{sh}^2\phi - \phi^2 \operatorname{ch}\phi)}{\phi(\phi^2 - \operatorname{sh}^2\phi)} .$$

These values substituted in equation (74) constitute the solution for W_{00}. If $\alpha = \beta = 0$ this solution reduces to that given by Holl (8, p.606) for a uniform load.

Since W_{20} is of the second order of magnitude as compared with W_{00} and since $\Delta_4 W_{20} = 0$, a solution of the following form is assumed.

$$(78) \quad W_{20} = -\frac{PA}{2D}\left[\left(1 + \frac{\beta y}{b}\right)(x^2 - ax) + \frac{\alpha}{3a}(x^3 - a^2x)\right]$$
$$- \frac{2P}{aD}\sum_n \frac{\sin\theta x}{\theta^3}\left[F_n \operatorname{ch}\theta y + G_n \operatorname{sh}\theta y\right.$$
$$\left. + H_n\,\theta y \operatorname{ch}\theta y + I_n\,\theta y \operatorname{sh}\theta y\right] .$$

As in both previous cases, edge condition (56.1) yields

$A = -\frac{8-3\rho}{10(1-\rho)}h^2$. The constants F_n, G_n, H_n and I_n are to be determined by the conditions $W_{20} = \frac{\partial W_{20}}{\partial y} = 0$ at the edges $y = 0,b$. The resulting system of equations is

$$(a)\ F_n = -(i+j)\frac{8-3\rho}{10(1-\rho)}h^2 \ ,$$

$$(b)\ F_n \operatorname{ch}\phi + G_n \operatorname{sh}\phi + H_n\phi \operatorname{ch}\phi + I_n\phi \operatorname{sh}\phi$$
$$= -\frac{8-3\rho}{10(1-\rho)}h^2(i+j+\beta i) \ ,$$

$$(c)\ G_n\phi + H_n\phi = -\frac{8-3\rho}{10(1-\rho)}h^2\beta i \ ,$$

$$(d)\ F_n\phi \operatorname{sh}\phi + G_n\phi \operatorname{ch}\phi + H_n\phi(\operatorname{ch}\phi + \phi \operatorname{sh}\phi)$$
$$+ I_n\phi(\operatorname{sh}\phi + \phi \operatorname{ch}\phi) = -\frac{8-3\rho}{10(1-\rho)}h^2\beta i \ .$$

If each right member of the system of equations (77) is multiplied by $\frac{8-3\rho}{20(1-\rho)}h^2$, the resulting system of equations will be identical with this system. Therefore the solutions of this system are $\frac{8-3\rho}{20(1-\rho)}h^2$ times the solutions of (77). Hence

$$F_n = \frac{8-3\rho}{20(1-\rho)}h^2B_n \ , \qquad G_n = \frac{8-3\rho}{20(1-\rho)}h^2A_n \ ,$$

$$H_n = \frac{8-3\rho}{20(1-\rho)} h^2 D_n \quad , \qquad\qquad I_n = \frac{8-3\rho}{20(1-\rho)} h^2 C_n \; .$$

By use of these values solution (78) may be written as

$$(79)\quad W_{2o} = -\frac{8-3\rho}{10(1-\rho)} h^2 \frac{\partial^2 W_{oo}}{\partial x^2} \; .$$

For reasons previously stated, W_{4o} is assumed to be

$$(80)\quad W_{4o} = \frac{PA}{D}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$$

$$+ \frac{Ph^4}{aD}\sum_n \frac{\sin \theta x}{\theta}\left[J_n \operatorname{ch} \theta y + K_n \operatorname{sh} \theta y\right] .$$

$W_{4o} = 0$ at $x = 0,a$; therefore $A = 0$.

$W_{4o} = 0$ at $y = 0,b$; therefore $J_n = K_n = 0$.

Hence solution (80) becomes $W_{4o} = 0$. From the concept of orders of magnitude it readily follows that $W_{2n,o} = 0 \quad (n \geqq 2)$.

There remains the task of finding U_o and V_o for this case. The boundary conditions are $T_1 = 0$ along $x = 0,a$ and $U_o = V_o = 0$ along $y = 0,b$. Garabedian (5) assumed, for the case of a uniform load, that U_o and V_o were linear functions of x,y and for this case he obtained the solution $U_o = V_o = 0$. However this solution has $T_1 = 0$ at $x = 0,a$. In order to avoid this difficulty it is necessary to use the boundary conditions at the edges $y = 0,b$ before choosing a definite polynomial part of the solution from the conditions at $x = 0,a$. As before, assume solutions of the form

$$U_{oo} = c_o + c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2$$

$$+\sum_n \frac{\cos\theta x}{a\theta^2}\left[A_n \operatorname{ch}\theta y + B_n \operatorname{sh}\theta y\right],$$

$$V_{oo} = k_o + k_1x + k_2y + k_3x^2 + k_4xy + k_5y^2$$

$$+\sum_n \frac{\sin\theta x}{a\theta^2}\left[C_n \operatorname{ch}\theta y + D_n \operatorname{sh}\theta y\right].$$

The conditions $U_{oo} = V_{oo} = 0$ at $y = 0,b$ give

$$0 = c_o + c_1x + c_3x^2 + \sum_n \frac{\cos\theta x}{a\theta^2} A_n \quad ,$$

$$0 = (c_o + c_2b + c_5b^2) + (c_1 + c_4b)x + c_3x^2$$

$$+\sum_n \frac{\cos\theta x}{a\theta^2}\left[A_n \operatorname{ch}\phi + B_n \operatorname{sh}\phi\right] \quad ,$$

$$0 = k_o + k_1x + k_3x^2 + \sum_n \frac{\sin\theta x}{a\theta^2} C_n \quad ,$$

$$0 = (k_0 + k_2 b + k_5 b^2) + (k_1 + k_4 b)x + k_3 x^2$$

$$+ \sum_n \frac{\sin \theta x}{a\theta^2} \left[C_n \text{ ch } \phi + D_n \text{ sh } \phi \right].$$

By use of the Fourier expansions

$$c_1 x + c_3 x^2 = c_1 \frac{a}{2} + c_3 \frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{2 \cos \theta x}{a\theta^2} \left[2c_3 a(-1)^n - c_1 i \right] \quad \text{and}$$

$$k_0 + k_1 x + k_3 x^2 = \sum_n \frac{2 \sin \theta x}{a\theta^3} \left[k_0 i\theta^2 - \theta^2(-1)^n(k_1 a + k_3 a^2) - 2k_3 i \right]$$

the following results are obtained.

$$c_0 = -\left(c_1 \frac{a}{2} + c_3 \frac{a^2}{3} \right), \qquad c_2 + bc_5 = -\frac{ac_4}{2},$$

$$A_n = 2\left[c_1 i - 2ac_3(-1)^n \right],$$

$$B_n = \frac{2[c_1 i - 2ac_3(-1)^n](1 - \text{ch } \phi) + 2c_4 bi}{\text{sh } \phi},$$

$$C_n = \frac{4k_3 i}{\theta} + 2\theta(-1)^n(k_1 a + k_3 a^2) - 2k_0 i\theta,$$

$$D_n = \frac{4k_3 i(1 - \text{ch } \phi)}{\theta \text{ sh } \phi}$$

$$+ 2\theta \frac{(k_1 a + k_3 a^2)(-1)^n(1 - \text{ch } \phi) - k_0 i(1 - \text{ch } \phi) + k_4(-1)^n ab - i(k_2 b + k_5 b^2)}{\text{sh } \phi}.$$

Boundary condition (54.0) at $x = 0, a$ gives

$$c_1 + \rho k_2 = \frac{P\rho}{4G} , \qquad c_4 + 2\rho k_5 = \frac{P\rho\beta}{4Gb} \qquad \text{and}$$

$$2c_3 + \rho k_4 = \frac{P\rho\alpha}{4Ga} .$$ Therefore choose

$$c_1 = k_2 = \frac{P\rho}{2E} , \qquad c_4 = 2k_5 = \frac{P\rho\beta}{2Eb} ,$$

$$2c_3 = k_4 = \frac{P\rho\alpha}{2Ea} , \qquad c_5 = k_0 = k_1 = k_3 = 0 .$$ Then

$$c_0 = -\frac{P\rho a}{12E}(3+\alpha) , \qquad c_2 = -\frac{P\rho\beta a}{4Eb} ,$$

$$A_n = \frac{P\rho}{E}(i+j) , \qquad B_n = \frac{P\rho\left[(i+j)(1-\operatorname{ch}\phi)+\beta i\right]}{E\ \operatorname{sh}\phi} ,$$

$$C_n = 0 , \qquad D_n = -\frac{P\rho\phi}{2E\ \operatorname{sh}\phi}(2i+2j+\beta i) .$$

The solutions may now be written as

$$(81)\quad U_{00} = \frac{P\rho}{2E}\Bigg[x - \frac{a}{6}(3+\alpha) - \frac{\beta ay}{2b} + \frac{\alpha x^2}{2a} + \frac{\beta xy}{b}$$
$$+\sum_n \frac{2\cos\theta x}{a\theta^2}\left\{(i+j)\operatorname{ch}\theta y + \frac{(i+j)(1-\operatorname{ch}\phi)+\beta i}{\operatorname{sh}\phi}\operatorname{sh}\theta y\right\}\Bigg] ,$$

$$(82)\quad V_{oo} = \frac{P\rho}{2E}\left[y + \frac{\alpha xy}{a} + \frac{\beta y^2}{2b} - \sum_n \frac{\sin\theta x}{a\theta^2}\left\{\frac{2(i+j)+\beta i}{\text{sh}\,\phi}\,\phi\ \text{sh}\,\theta y\right\}\right].$$

When $\alpha = \beta = 0$ these values reduce to

$$(83)\quad U_{oo} = \frac{P\rho}{2E}\left(x-\frac{a}{2}\right) + \frac{P\rho}{aE}\sum_n i\,\frac{\cos\theta x}{\theta^2}\ \frac{\text{ch}\,\theta y\ \text{sh}\,\phi + (1-\text{ch}\,\phi)\ \text{sh}\,\theta y}{\text{sh}\,\phi},$$

$$(84)\quad V_{oo} = \frac{P\rho}{2E}\,y - \frac{P\rho}{aE}\sum_n i\phi\,\frac{\sin\theta x}{\theta^2}\ \frac{\text{sh}\,\theta y}{\text{sh}\,\phi}\ .$$

Since U_{oo} and V_{oo} are quadratic functions, U_{20} and V_{20} are either constants or zero. They are zero at the edges $y = 0, b$; therefore they are zero everywhere. It readily follows that $U_{2n,o} = V_{2n,o} = 0\ (n \geqq 1)$.

W_{oo} from (74) plus W_{20} from (79) gives the value of W_o for this case. U_o and V_o are given by results (81) and (82). After these results are obtained W_1 is given by relation (50), U_1 by (48) and V_1 by (49). When these values are substituted in equations (51), (52) and (53), the displacements u, v and w of any point in the plate, are obtained. When the displacements are known, the stresses can easily be found.

In order to show that the statics of these problems are satisfied, write the resultant vertical shear,

$$N_x = \int_{-h}^{h} X_z dz = G\int_{-h}^{h}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) dz \quad ,$$

as a system of equations by equating terms of like orders of magnitude. Similarly write N_y as a system of equations. The equation of lowest order of magnitude is the corresponding thin plate vertical shear in each case. When this component of the vertical shear is integrated around any closed curve on the surface of the plate, it equals the load on the area bounded by this curve. The components of the resultant vertical shear of the next and higher orders of magnitude vanish when integrated around any closed curve on the plate. Therefore the vertical shear integrated around any closed curve on the plate equals the load on the area bounded by that curve for all three problems solved in this chapter.

Likewise it is interesting to note that the vertical stress $Z_z = -\frac{p}{2}\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)\left(1 + \frac{3z}{2h} - \frac{z^3}{2h^3}\right)$ reduces to one half the load at $z = 0$. This is true for all three problems as the vertical stress does not depend upon the manner in which the plate is supported.

IV. ADDITIONAL RESULTS

A. General Theory for Horizontal Surface Tractions on the Upper and Lower Faces of the Plate

When the problem is restricted to horizontal loads $P_1 = P_2 = 0$. Let $L_1 + L_2 = L$, $L_1 - L_2 = l$, $J_1 + J_2 = J$ and $J_1 - J_2 = j$ where L, l, J and j are functions of x,y. It is possible to prove that L and J are of the same order of magnitude as U_{21} and V_{21}. Likewise l and j are of the same order of magnitude as $\Delta_2 U_{00}$ and $\Delta_2 V_{00}$. Equations (10), (11) and (12) may now be solved for W_0, U_1 and V_1 in essentially the same manner as before. Likewise U_0, V_0 and W_1 may be obtained from equations (13), (14) and (15). The results are

$$(85)\quad \Delta_4 W_0 = -\frac{1}{D}\sum_{n=0}^{\infty} h^{2n+1}\Delta_{2n}Q\left[(2-\rho)\bar{d}_n-\bar{b}_n\right] ,$$

$$(86)\quad \Delta_2\left(\frac{\partial U_1}{\partial x}+\frac{\partial V_1}{\partial y}\right) = -\frac{1}{D}\sum_{n=0}^{\infty} h^{2n+1}\Delta_{2n}Q\left[\rho\bar{d}_n+\bar{b}_n\right] ,$$

$$(87)\quad \Delta_2 W_1 = \frac{1}{4G}\sum_{n=0}^{\infty} h^{2n-1}\Delta_{2n}q\left[(1-\rho)\bar{c}_n+\rho\bar{a}_n\right] ,$$

$$(88)\quad \Delta_2\left(\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right) = -\frac{1}{4G}\sum_{n=0}^{\infty} h^{2n-1}\Delta_{2n}q\left[\rho\bar{c}_n+(1-\rho)\bar{a}_n\right]$$

where $\bar{d}_0 = 0$, $\bar{b}_0 = 1$, $\bar{d}_1 = \frac{1}{6}$, $\bar{c}_0 = 0$, $\bar{a}_0 = 1$,

$$\bar{d}_n = \sum_{i=0}^{n-1} (-1)^i \left[\frac{(i+1)\bar{b}_{n-1-i}}{(1-\rho)} - \frac{i(1-\rho)\bar{d}_{n-1-i}}{(2i+2)!} \right] \qquad (n = 2,3,4,\ldots),$$

$$\bar{b}_n = 6\sum_{i=0}^{n-1} (-1)^i \frac{(i+2)\left[(i+2)\bar{b}_{n-1-i} - (i+1)(1-\rho)\bar{d}_{n-1-i}\right]}{(2i+5)!} \qquad (n = 1,2,3,\ldots),$$

$$\bar{c}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)\bar{a}_{n-1-i} - i\bar{c}_{n-1-i}}{(2i+2)!} \qquad (n = 1,2,3,\ldots),$$

$$\bar{a}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+2)\bar{a}_{n-1-i} - (i+1)\bar{c}_{n-1-i}}{(2i+3)!} \qquad (n = 1,2,3,\ldots),$$

$$Q = \frac{\partial L}{\partial x} + \frac{\partial J}{\partial y} \quad \text{and} \quad q = \frac{\partial l}{\partial x} + \frac{\partial j}{\partial y} .$$

Sibert (12, p.345) has proved that $\bar{a}_n$, $\bar{b}_n$, $\bar{c}_n$ and $\bar{d}_n$ are bounded sequences of constants.

B. Plane and Generalized Plane Stress

When Z_z vanishes everywhere and X_z, Y_z vanish at $z = \pm h$, a state of generalized plane stress exists (10, p.471). Thus the top and bottom surfaces of the plate are free from stress and equations (40) to (53) inclusive reduce to

$$(89)\quad \begin{cases} \Delta_4 W_o = \Delta_4 U_1 = \Delta_4 V_1 = \Delta_2 \left(\dfrac{\partial U_1}{\partial x} + \dfrac{\partial V_1}{\partial y}\right) \\ = \Delta_2 W_1 = \Delta_4 U_o = \Delta_4 V_o = \Delta_2 \left(\dfrac{\partial U_o}{\partial x} + \dfrac{\partial V_o}{\partial y}\right) = 0 \ , \end{cases}$$

$$(90)\quad \begin{cases} U_1 = -\dfrac{\partial W_o}{\partial x} - \dfrac{h^2}{1-\rho}\Delta_2\left(\dfrac{\partial W_o}{\partial x}\right) \ , \\ V_1 = -\dfrac{\partial W_o}{\partial y} - \dfrac{h^2}{1-\rho}\Delta_2\left(\dfrac{\partial W_o}{\partial y}\right) \ , \\ W_1 = -\dfrac{\rho}{1-\rho}\left(\dfrac{\partial U_o}{\partial x} + \dfrac{\partial V_o}{\partial y}\right) \ , \end{cases}$$

$$(91)\quad \begin{cases} u = U_o + U_1 z - \left[\Delta_2 U_o + \dfrac{1}{1-\rho}\dfrac{\partial}{\partial x}\left(\dfrac{\partial U_o}{\partial x} + \dfrac{\partial V_o}{\partial y}\right)\right]\dfrac{z^2}{2!} \\ \qquad + \dfrac{2-\rho}{1-\rho}\Delta_2\left(\dfrac{\partial W_o}{\partial x}\right)\dfrac{z^3}{3!} \ , \\ v = V_o + V_1 z - \left[\Delta_2 V_o + \dfrac{1}{1-\rho}\dfrac{\partial}{\partial y}\left(\dfrac{\partial U_o}{\partial x} + \dfrac{\partial V_o}{\partial y}\right)\right]\dfrac{z^2}{2!} \\ \qquad + \dfrac{2-\rho}{1-\rho}\Delta_2\left(\dfrac{\partial W_o}{\partial y}\right)\dfrac{z^3}{3!} \ , \\ w = W_o + W_1 z + \dfrac{\rho z^2}{2(1-\rho)}\Delta_2 W_o \ . \end{cases}$$

Still further reductions may be made by a consideration of orders of magnitude. W_{o1} and $\dfrac{\partial U_{oo}}{\partial x} + \dfrac{\partial V_{oo}}{\partial y}$ are of the same order of magnitude as the load which in this case is zero. Since these quantities must therefore be zero or constants, they will be assumed zero. The stresses may now be written as

(92.0) $Z_z = 0$,

(92.1) $X_z = \frac{E(z^2 - h^2)}{2(1-\rho^2)} \Delta_2 \left(\frac{\partial W_o}{\partial x}\right)$,

(92.2) $Y_z = \frac{E(z^2 - h^2)}{2(1-\rho^2)} \Delta_2 \left(\frac{\partial W_o}{\partial y}\right)$,

(92.3) $X_x = - \frac{Ez}{1-\rho^2} \Delta_2 W_o + \frac{\partial^2}{\partial y^2}\left[\frac{Ez}{1+\rho} W_o + \frac{E}{1-\rho^2}\left(h^2 z - \frac{2-\rho}{6} z^3\right)\Delta_2 W_o\right]$,

(92.4) $Y_y = - \frac{Ez}{1-\rho^2} \Delta_2 W_o + \frac{\partial^2}{\partial x^2}\left[\frac{Ez}{1+\rho} W_o + \frac{E}{1-\rho^2}\left(h^2 z - \frac{2-\rho}{6} z^3\right)\Delta_2 W_o\right]$,

(92.5) $X_y = - \frac{\partial^2}{\partial x \partial y}\left[\frac{Ez}{1+\rho} W_o + \frac{E}{1-\rho^2}\left(h^2 z - \frac{2-\rho}{6} z^3\right) \Delta_2 W_o\right]$.

When X_z, Y_z and Z_z vanish throughout the plate there is a state of plane stress (10, p.467). Therefore

$\Delta_2 \left(\frac{\partial W_o}{\partial x}\right) = \Delta_2 \left(\frac{\partial W_o}{\partial y}\right) = 0$ and equations (92.3), (92.4) and (92.5) become

(93.1) $X_x = \frac{Ez}{1+\rho}\left[\frac{\partial^2 W_o}{\partial y^2} - \frac{1}{1-\rho} \Delta_2 W_o\right]$,

(93.2) $Y_y = \frac{Ez}{1+\rho}\left[\frac{\partial^2 W_o}{\partial x^2} - \frac{1}{1-\rho} \Delta_2 W_o\right]$ and

(93.3) $X_y = - \frac{Ez}{1+\rho} \frac{\partial^2 W_o}{\partial x \partial y}$ respectively.

These results agree with those given by Love (10, p.473, 470) and also those given by R. V. Southwell (13, p.209, 201).

V. DISCUSSION

Garabedian and Sibert have developed and adequately presented the power series method of approach to plate problems for circular plates. Garabedian has also published some results for uniformily loaded rectangular plates. However, he has never presented his method. The author has adapted the method of Sibert, which he believes is an improvement over that of Garabedian, to rectangular plates. One must recognize, however, that rectangular plate problems are inherently more difficult than circular plate problems. In most circular plate problems the displacements and stresses are assumed independent of the polar coordinate θ. This means that the differential equations defining the displacements are ordinary differential equations. In rectangular plate problems they are partial differential equations. In order to obtain his differential equations, Sibert had to solve systems composed of two differential equations involving one independent and two dependent variables. In this problem it was necessary to solve systems composed of three differential equations which involve three dependent and two independent variables. Again Sibert's ordinary differential equations were solved by the well known process of finding a particular solution and a complementary solution by direct integration. The partial differential equations of this problem are more complicated. Thus the author believes that this is a worthwhile contribution to the literature of plate theory.

The problem is solved for any load (vertical or horizontal applied on the top and bottom faces) which can be represented as a polynomial in x,y continuous over the entire plate. Unfortunately this method is not adapted to discontinuous loads such as a uniform load over a portion of the plate. The difficulty is inherent in the method because of the concept of equating like orders of magnitude. To make this clear, consider the problem of a uniform load over a strip of the plate parallel to the x-axis. It has been proved that $\frac{\partial U_{oo}}{\partial x}$ and $\frac{\partial V_{oo}}{\partial y}$ are of the same order of magnitude as the load; therefore they are constants in this case. Then from equations (54) and (55) it is evident that the T's are constants. Since the T's are zero at the edges, they are zero everywhere. This means that U_{oo} and V_{oo} are zero where the load is zero and constant where the load is constant. Consequently the plate is discontinuous where the load is discontinuous. In other words the method fails for discontinuous loads. Again this discontinuous load cannot be represented by a Fourier series as the series becomes divergent when substituted in the differential equations defining the displacements.

In the special solutions for particular loads the results are very satisfactory in most respects. It is well to point out the two unsatisfactory features of these results. In the case of two edges pinned and two edges free the author failed to find a way to express W_{20} entirely in terms of W_{00}. In regard to the solutions for U_0 and V_0 more information is needed with respect to the precise edge conditions at a pinned edge. It has been mentioned that Love gives $S=0$ as one of the conditions at a pinned edge. However, this condition imposes a restriction on the values of the arbitrary constants α and β. Such a restriction obviously cannot be consistent with physical reality. Although the solutions given for U_0 and V_0 seem fairly satisfactory, they are not presented as the unique solutions.

VI. SUMMARY

1. By the use of recurrence relations the displacements, which satisfy the equilibrium equations, are expressed in terms of U_0, V_0, U_1, V_1, W_0 and W_1.

2. The partial differential equations defining these six functions are obtained for any normal load which can be represented as a polynomial in x,y continuous over the entire plate.

3. These equations are solved for the particular load $P\left(1 + \frac{\alpha x}{a} + \frac{\beta y}{b}\right)$ subject to three different sets of edge conditions: pinned-pinned, pinned-free, pinned-clamped.

4. The results show that the principal part of the vertical displacement of the middle surface is W_{00}, the corresponding thin plate solution. With the one exception noted in the previous chapter, the displacement of the middle surface is given as the thin plate solution plus a correction which is a function of the thin plate solution. Since the results depend upon W_{00}, two thin plate solutions, not hitherto recorded, are given.

5. The partial differential equations for the case of a shearing load are also given.

6. It is shown that this method gives the problems of plane and generalized plane stress very easily.

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