**Crystal structure on Gelfand-Tsetlin-Želobenko patterns**

by

**O'Neill Kingston**

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Jonas Hartwig, Major Professor
Tathagata Basak
Sung-Yell Song
Jason McCullough
David Herzog



Iowa State University

Ames, Iowa

2020

# TABLE OF CONTENTS

## ABSTRACT

We begin by presenting the crystal structure of finite-dimensional irreducible representations of the special linear Lie algebra in terms of Gelfand-Zeitlin patterns. We then define a crystal structure using the set of symplectic Zhelobenko patterns, parametrizing bases for finite-dimensional irreducible representations of sp4. This is obtained by a bijection with Kashiwara-Nakashima tableaux and the symplectic jeu de taquin of Sheats and Lecouvey. We offer some conjectures on the generalization of this structure to rank n as well as a bijection and crystal structure in certain special cases.

# CHAPTER 1. GENERAL INTRODUCTION

The introduction of crystal bases in the 1990's by Kashiwara ([11], [9], [10]) and Lusztig [18] was a breakthrough in the representation theory of Lie algebras and quantum groups, structures that have become ubiquitous in modern physics, algebra and geometry. A crystal basis is a combinatorial object that can essentially be identified with the crystal graph it gives rise to. At the same time, it is a basis for a highest-weight representation of a quantum group (as described, for instance, by Hong and Kang in [8]), and its combinatorial structure is naturally compatible with everything one could hope for: taking tensor products to build larger representations, studying branching rules to understand the behavior of subgroups, and much more. Of particular interest is the fact that crystal bases for all classical Lie groups can be thoroughly described in terms of Kashiwara-Nakashima tableaux (KNT). This is perhaps not surprising, as the standard and semistandard Young tableaux that they generalize have been used extensively in representation theory over the last century. A fact that borders on miraculous is that beyond simply providing a nice basis for a representation of a quantum group, one may define a product on tableaux using a tool called the jeu de taquin (JDT) that coincides exactly with taking tensor products of the associated representations, and which gives rise to a combinatorial structure called the plactic monoid that mirrors the notion of a universal enveloping algebra.

It is well-known that semistandard Young tableaux (SSYT) are in bijection with Gelfand-Tsetlin patterns (ΓЦ) ([7]), arrays of numbers that were introduced specifically to study branching rules for the general linear Lie group but which have found many subsequent applications. Gelfand and Tsetlin also constructed bases for irreducible representations of the orthogonal Lie algebra in [6]. A generalization of these patterns by Želobenko ([22], [23]) (ŽP) shed light on the representation theory of the symplectic Lie algebra, and they were used to provide formulas for the structure constants of algebras related to rational solutions of the Yang-Baxter equation, as Molev describes in [19]. We therefore have two different combinatorial descriptions of the same algebraic objects, one in terms of tableaux and the other in terms of patterns.

If we fix a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ and we let $U(\mathfrak{g})$ be its universal enveloping algebra, then its quantized universal enveloping algebra $U_q(\mathfrak{g})$ is a Hopf algebra whose structure tends toward $U(\mathfrak{g})$ when $q$ approaches 1. This object is also called a quantum group. The category $\mathcal{O}_{\text{int}}$ is the category whose objects are weight modules over $\mathfrak{g}$ with certain interesting properties and whose morphisms are $\mathfrak{g}$-module homomorphisms. In particular, these weight modules admit decompositions into direct sums of irreducible highest weight modules and, as such, not only is the category $\mathcal{O}_{\text{int}}$ closed under finite tensor products, such products are completely reducible. Not too surprisingly, we may analogously define the category $\mathcal{O}^q_{\text{int}}$ of weight modules over $U_q(\mathfrak{g})$ with similarly nice properties.

As Lusztig showed in [17], the $\mathfrak{g}$-modules in $\mathcal{O}_{\text{int}}$ can be deformed into $U_q(\mathfrak{g})$-modules in $\mathcal{O}^q_{\text{int}}$ in such a fashion that the dimensions of their weight spaces are invariant under the deformation. In other words, to understand the representation theory of our quantum group $U_q(\mathfrak{g})$, we need only understand the representation theory of our Lie algebra $\mathfrak{g}$, a well-studied topic indeed for the sort of Lie algebra under discussion. It is especially convenient given that our combinatorial tools are as nice and as powerful as they are: through this winding path of equivalencies, we may summarize the situation by saying that to understand the representation theory of quantum groups, we need only understand the combinatorics of SSYT and KNT, or of ГЦ and ŽP.

Another long-running undercurrent to this area of study comes from a more purely combinatorial perspective. When the utility of applying Young tableaux to problems in representation theory became clear, many more generalizations were made than those discussed above. The era of computers accelerated the growth of interest in this area, and today there are robust communities of mathematicians whose work is focused on coding efficient representations (in the colloquial sense) of these structures, often in Python and Sage, so that these may then be used to attack problems in algebra, combinatorics, geometry and beyond. With a high level of research output surrounding tableaux and tableau-like structures, an active sub-discipline is the effort to identify when two seemingly-different types of structure are in fact equivalent. As described by Sheats in [20], these enumeration problems can frequently be difficult, but they can also illuminate surprising connections between areas of mathematics that appeared to have little in common. In that paper, Sheats gives an algorithm called the symplectic jeu de taquin (SJDT) that he defines on De Concini tableaux ([4]) in order to prove a bijection between those and another formulation known as King tableaux ([12]), which have been shown to be equivalent to KNT ([13]). Moreover, his bijection preserves weights when the tableaux are viewed in the representation theoretic light. In [15] and [16], Lecouvey translates the Sheats SJDT to the KNT setting and describes plactic monoids for types B, C and D, thus greatly clarifying the combinatorial story for quantum groups.

Given how well-understood KNT are and how useful ŽP have proven to be in several active areas of inquiry, some natural questions suggest themselves:

**Question 1.0.1.** *Is there a weight-preserving bijection between KNT and ŽP? If so, can it be formulated in a way that resembles the type A bijection?*

In this thesis, we have partially resolved these questions in the symplectic case:

**Theorem 1.0.2.** *There exists a combinatorial algorithm which provides a weight-preserving bijection between Kashiwara-Nakashima tableaux and Želobenko patterns for type $C_2$. Additionally, a weight-preserving bijection may be given in the special case of what are called hook tableaux in type $C_n$. Moreover, these bijections involve deletion algorithms just as in type A.*

This theorem is proved in chapter 3. Given a KNT, $T$, this process will return a ŽP, $\Gamma$, of the same weight, or vice versa. Determining this algorithm involved the application of the symplectic jeu de taquin of Sheats and Lecouvey, a process wherein a skew tableau (that is, a tableau with holes in it) may be rendered non-skew by repeated "sliding moves" on its boxes.

Since Kashiwara and Nakashima showed that KNT can be endowed with a crystal structure [10], it follows that through this bijection we may give a crystal structure on Želobenko patterns in these restricted cases. It would be desirable for a variety of reasons to be able to give this structure on the collection of patterns in and of itself, not least because this would more clearly illuminate the combinatorial picture on the pattern side. This suggests the following question:

**Question 1.0.3.** *Can a crystal structure be given on ŽP independently of their relationship with* KNT*?*

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & \overline{2} & \overline{1} \\ \hline \overline{2} & \overline{1} \\ \cline{1-2}\end{array} \longleftrightarrow \left\{ \begin{array}{cccc} 4 & & 2 & \\ & 3 & & 0 \\ & & 1 & \\ & & & 0 \end{array} \right\}$$

Figure 1.1: A Kashiwara-Nakashima tableau with associated Želobenko pattern.

Having investigated this fairly extensively, we conjecture the answer to be affirmative. Our initial approach to this problem involved column patterns, a substructure of a ŽP that we defined in parallel to descriptions of the crystal structure on tableaux that involve breaking them into products of columns, a process known as column reading. We also give formulae for the crystal operators in the case of row patterns, a special case of hook patterns, and conjecture that something similar may be achieved for hook patterns in general.

With that question in mind, it also seemed logical to ask the same question in the type A case:

**Question 1.0.4.** *Can a crystal structure be given on Gelfand-Tsetlin patterns independently of the* SSYT-ГЦ *bijection?*

The answer, we discovered, is affirmative:

**Theorem 1.0.5.** *The crystal structure on Gelfand-Tsetlin patterns of type $A_n$ can be explicitly computed based solely on pattern entries.*

This statement is proved by offering formulas for the five functions necessary to form a crystal basis given a Cartan datum: the raising and lowering operators, the string length operators, and the weight function. One advantage of this approach when compared to crystals of tableaux is that, while these formulae are recursive, they may at least be calculated based on the pattern itself without the need to apply a row- or column-reading function, and then further apply the signature rule to the result.

Something particularly interesting about this result is that the formulae for the crystal operators must be given in terms of sums, differences and maxima (or minima) of pattern entries. This is suggestive of some connection to tropical mathematics, which is well-known to have deep connections to crystal basis theory [3]. The formulae in the type A case exhibit this behavior, but as we will discuss in chapter 3, in the type C case especially the complexity of the expressions is very suggestive of the following question:

**Question 1.0.6.** *What is the relationship between the crystal structure on patterns and tropical mathematics?*

With the crystal structure of patterns complete in the general linear case and underway in the symplectic case, future work could also include the investigation of the orthogonal case. This has the potential to be the most difficult setting of the three, but resolving all of them would give crystals of patterns for the Lie algebras of each of the infinite families of classical groups.

# CHAPTER 2. DEFINITIONS

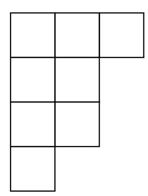
## 2.1 Combinatorial definitions

### 2.1.1 Partitions and Young diagrams

For a fixed integer $N \geq 0$, a *partition* of $N$ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of integers $\lambda_i$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $|\lambda| := \Sigma_{i \geq 1} \lambda_i = N$. The *length* of a partition $\lambda$, $\ell(\lambda)$, is equal to the highest index $i$ for which $\lambda_i > 0$. For $1 \leq i \leq \ell(\lambda)$, the $\lambda_i$ are called the *parts* of $\lambda$. Let $\mathcal{P}(N)$ be the set of partitions of $N$ and put $\mathcal{P} = \bigcup_{N=0}^{\infty} \mathcal{P}(N)$.

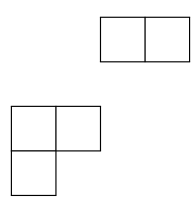
If $\lambda$ is a partition of $N$, the *Young diagram* $\mathrm{YD}(\lambda)$ is a left-justified collection of boxes where the $i$th row has $\lambda_i$ boxes. The *shape* of a Young diagram is its partition $\lambda$. A *tableau* is a Young diagram whose boxes are filled with elements from an alphabet.

$$\mathrm{YD}((3,2,2,1)) =$$



A *subdiagram* is a Young diagram $\mathrm{YD}(\lambda')$ that is contained in Young diagram $\mathrm{YD}(\lambda)$. A *skew diagram* $\mathrm{YD}(\lambda/\lambda')$ is the diagram obtained by subtracting a subdiagram $\lambda'$ from $\lambda$.

$$\mathrm{YD}((4,2,2,1)/(2,2)) =$$



### 2.1.2 Semistandard Young tableaux

A *semistandard Young tableau* (SSYT) of shape $\lambda$ and rank $n-1$ is a tableau of shape $\lambda$ where the boxes are filled with entries from the alphabet is $[n] = \{1, 2, \dots, n\}$ so that each row is weakly increasing from left to right and each column is strictly increasing from top to bottom.

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Let $\mathrm{SSYT}(n, \lambda)$ denote the set of SSYT of rank $n$ and shape $\lambda$.

The *weight* of a SSYT $T$ is given by

$$\mathrm{wt}(T) = k_1\mathbf{e}_1 + \cdots + k_n\mathbf{e}_n,$$

where the $\mathbf{e}_i$ are the standard basis vectors in $\mathbb{R}^n$ and $k_i$ is equal to the number of occurrences of the symbol $i$ in $T$.

### 2.1.3 Gelfand-Tsetlin patterns

Let $n$ be a positive integer and $\lambda$ be a partition with $n$ or fewer parts. A *Gelfand-Tsetlin pattern* [7] with $n$ rows and top row $\lambda$ is a triangular array of integers

$$\Lambda = \left\{ \begin{array}{ccccccccc} \lambda_1^{(n)} & & \lambda_2^{(n)} & & \lambda_3^{(n)} & & \cdots & & \lambda_n^{(n)} \\ & \lambda_1^{(n-1)} & & \lambda_2^{(n-1)} & & \cdots & & \lambda_{n-1}^{(n-1)} & \\ & & \lambda_1^{(n-2)} & & \cdots & & \lambda_{n-2}^{(n-2)} & & \\ & & & \ddots & & \iddots & & & \\ & & & & \lambda_1^{(1)} & & & & \end{array} \right\}$$

where

(i) $\lambda_j^{(i)} \in \mathbb{Z}_{\geq 0}$ for $1 \leq j \leq i \leq n$,

(ii) $\lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_{j+1}^{(i+1)}$ for $1 \leq j \leq i \leq n-1$,

(iii) $\lambda_i^{(n)} = \lambda_i$ for $1 \leq i \leq n$.

Condition (ii) is known as the *interleaving condition*. By convention we set $\lambda_j^{(i)} = 0$ if not $1 \leq j \leq i \leq n$. Let $\Gamma Ц(n, \lambda)$ denote the set of Gelfand-Tsetlin patterns with $n$ rows and top row $\lambda$.

The weight of a ГЦ $\Lambda$ is given by

$$\operatorname{wt}(\Lambda) = \sum_{j=1}^{n} \Big( \sum_{k=1}^{j} \lambda_k^{(j)} - \sum_{k=1}^{j-1} \lambda_k^{(j-1)} \Big) \boldsymbol{e}_j.$$

Given a ГЦ $\Lambda$ we define $\Delta_\ell^{(i)}(\Lambda)$ to be an array of the same shape as $\Lambda$ consisting of a 1 in position $i\ell$ and zeroes everywhere else. Note that this is generally not a valid ГЦ itself.

### 2.1.4 Bijection between tableaux and patterns

There is a well-known (see [3]) and natural bijection between $\mathrm{SSYT}(n, \lambda)$ and $\text{ГЦ}(n, \lambda)$. Given a Gelfand-Tsetlin pattern $\Lambda \in \text{ГЦ}(n, \lambda)$, we obtain a tableau $T = \mathcal{T}(\Lambda) \in \mathrm{SSYT}(n, \lambda)$ by inserting $i$ into the squares of the skew diagram $\mathrm{YD}(\lambda^{(i)}/\lambda^{(i-1)})$, for $i = 1, 2, \ldots, n$, where by convention $\lambda^{(0)}$ is the empty partition. Conversely, given $T \in \mathrm{SSYT}(n, \lambda)$, we obtain a pattern $\Lambda = \mathcal{T}^{-1}(T) \in \text{ГЦ}(n, \lambda)$ as follows. Define the top row $\lambda^{(n)}$ of $\Lambda$ to be the shape of $T$. That is, $\lambda^{(n)} = \lambda$. Then, delete all boxes from $T$ containing the symbol $n$ to obtain tableau $T^{(n-1)}$ and define the next row $\lambda^{(n-1)}$ of $\Lambda$ to be the shape of $T^{(n-1)}$. Continue in this fashion until all the boxes of $T$ have been deleted. Then all the rows of $\Lambda$ have been specified.

**Example 2.1.1.**

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&3&4\\\hline 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array} \;\to\; \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&3\\\cline{1-3} 3\\\cline{1-1}\end{array} \;\to\; \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array} \;\to\; \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}$$

$$\left\{\begin{array}{ccccccc} 4&&4&&2&&1 \end{array}\right\} \longrightarrow \left\{\begin{array}{ccccccc} 4&&4&&2&&1\\ &4&&3&&1& \end{array}\right\} \longrightarrow \left\{\begin{array}{ccccccc} 4&&4&&2&&1\\ &4&&3&&1&\\ &&3&&1&& \end{array}\right\} \longrightarrow \left\{\begin{array}{ccccccc} 4&&4&&2&&1\\ &4&&3&&1&\\ &&3&&1&&\\ &&&2&&& \end{array}\right\}$$

### 2.1.5 Kashiwara-Nakashima tableaux

A *Kashiwara-Nakashima tableau* (KNT) of shape $\lambda$ and rank $n$ is a Young diagram $T \in \mathrm{YD}(\lambda)$ filled with entries from the alphabet

$$\mathcal{A}_{C_n} = \{1 \prec 2 \prec \cdots \prec n-1 \prec n \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{2} \prec \overline{1}\}$$

subject to the following additional constraints:

(i) The entries are strictly increasing from top to bottom and weakly increasing from left to right.

(ii) If the letters $i$ and $\bar{i}$ appear in the same column with $i$ in the $a$-th box from the top and $\bar{i}$ in the $b$-th box from the bottom, then $a + b \leq i$.

(iii) If $T$ has two adjacent columns of either of the forms

$$\begin{array}{|c|c|}\hline \vdots & \vdots \\ \hline i & \\ \hline \vdots & \vdots \\ \hline & j \\ \hline \vdots & \vdots \\ \hline & \bar{j} \\ \hline \vdots & \vdots \\ \hline & \bar{i} \\ \hline \vdots & \vdots \\ \hline \end{array}, \qquad \begin{array}{|c|c|}\hline \vdots & \vdots \\ \hline i & \\ \hline \vdots & \vdots \\ \hline j & \\ \hline \vdots & \vdots \\ \hline \bar{j} & \\ \hline \vdots & \vdots \\ \hline & \bar{i} \\ \hline \vdots & \vdots \\ \hline \end{array},$$

where $i \leq j$, then the vertical distances $d_1$ from the boxes containing $i$ to $j$ and $d_2$ from $\bar{j}$ to $\bar{i}$ are such that $d_1 + d_2 < j - i$.

Note that KNT are technically a generalization of SSYT, and so the latter may be viewed as examples of the former using the alphabet

$$\mathcal{A}_{A_n} = \{1, 2, \ldots, n\}$$

and its associated conditions. For clarity we will continue to refer to SSYT by their original name, as it is typical to do so in the literature.

The *weight* of a KNT $T$ is given by

$$\mathrm{wt}(T) = k_1\mathbf{e}_1 + \cdots + k_n\mathbf{e}_n,$$

where the $\mathbf{e}_i$ are the standard basis vectors in $\mathbb{R}^n$ and $k_i$ is equal to the number of occurrences of the symbol $i$ in $T$ minus the number of occurrences of the symbol $\bar{i}$.

Let $\mathrm{KNT}(n, \lambda)$ denote the set of KNT of rank $n$ and shape $\lambda$, and let $\mathrm{KNT}(n) = \bigcup_{\lambda \in \mathcal{P}} \mathrm{KNT}(n, \lambda)$. Note that if $\lambda$ has more than $n$ parts, then $\mathrm{KNT}(n, \lambda) = \emptyset$ due to conditions (i) and (ii). A tableau satisfying the above conditions is called *admissible* (or, as in Lecouvey in [15], *KN-admissible*).

Let $C$ be a KNT column and let $I = \{z_1 > \cdots > z_r\}$ bet the set of unbarred letters $z$ such that the pair $(z, \overline{z})$ exists in $C$. The column $C$ can be split when there exists a set of $r$ unbarred letters $j = \{t_1 > \cdots > t_r\} \subset \mathcal{A}_{C_n}$ such that:

1. $t_1$ is the greatest letter of $\mathcal{A}_{C_n}$ satisfying $t_1 < z_1, t_1 \notin C$, and $\overline{t}_1 \notin C$,

2. for $i = 2, \ldots, r$, $t_i$ is the greatest letter of $\mathcal{A}_{C_n}$ satisfying $t_i < \min(t_{i-1}, z_i), t_i \notin C$, and $\overline{t}_i \notin C$.

In this case we write:

1. $rC$ for the column obtained by changing $\overline{z}_i$ into $\overline{t}_i$ for each letter $z_i \in I$ in $C$ and reordering if necessary to preserve the ordering of $\mathcal{A}_{C_n}$,

2. $lC$ for the column obtained by changing $z_i$ into $t_i$ for each letter $z_i \in I$ in $C$ and reordering if necessary.

It is a proposition of Sheats in [20] that a column $C$ is admissible if and only if it can be split. A tableau may be put into *split form* by replacing each column $C$ of the tableau with the appropriate $lC$ and $rC$. Lecouvey defines a skew admissible tableau as one in which the columns are admissible and the columns of the split form are weakly increasing from left to right, which is equivalent to the definition of KNT given above. Define $\operatorname{spl}(T)$ to be the split form of $T$.

**Example 2.1.2.** Let $C = 2467\overline{742}$. Then

$$I = \{7, 4, 2\}, \quad J = \{5, 3, 1\}, \quad lC = 1356\overline{742}, \quad rC = 2467\overline{531}.$$

We may now define a *coadmissible* column $C^*$ to be the column obtained from splitting admissible column $C$ and then filling the shape of $C$ with the unbarred letters from $lC$ in increasing order followed by the barred letters of $rC$ in increasing order. Define $\Phi : C \mapsto C^*$ to be this map. Define $\Phi : C \mapsto C^*$ to be the map sending an admissible column $C$ to its coadmissible counterpart $C^*$. A tableau $T$ in which all columns are coadmissible is said to be in *DC2 inadmissible form*, with DC2 referring to the second distance condition.

**Example 2.1.3.** With $C = 2467\overline{742}$, we have $C^* = 1356\overline{531}$. Note that $C^*$ fails to meet the first distance condition on column KNT with respect to 1, 3 and 5.

Given a $T \in \mathrm{KNT}(n)$, we define the quantity $T_i(j)$ to be the number of symbols $j$ occurring in row $i$ of $T$.

### 2.1.6 Želobenko patterns

We define a type C *Želobenko pattern* (ŽP) of rank $n$ associated with partition $\lambda$ with $n$ or fewer parts as an array of non-negative integers of the form

$$\Gamma = \left\{ \begin{array}{cccccccccc}
\lambda_1^{(n)} & & \lambda_2^{(n)} & & \lambda_3^{(n)} & & \cdots & & \lambda_n^{(n)} & \\
 & \lambda_1^{(n)'} & & \lambda_2^{(n)'} & & \lambda_3^{(n)'} & & \cdots & & \lambda_n^{(n)'} \\
 & & \lambda_1^{(n-1)} & & \lambda_2^{(n-1)} & & \cdots & & \lambda_{n-1}^{(n-1)} & \\
 & & & \lambda_1^{(n-1)'} & & \lambda_2^{(n-1)'} & & \cdots & & \lambda_{n-1}^{(n-1)'} \\
 & & & & \lambda_1^{(n-2)} & & \cdots & & \lambda_{n-2}^{(n-2)} & \\
 & & & & & \lambda_1^{(n-2)'} & & \cdots & & \lambda_{n-2}^{(n-2)'} \\
 & & & & & & \cdots & & \cdots & \\
 & & & & & & & \cdots & & \cdots \\
 & & & & & & & & \lambda_1^{(1)} & \\
 & & & & & & & & & \lambda_1^{(1)'}
\end{array} \right\}$$

so that

(i) $\lambda_j^{(i)} \in \mathbb{Z}_{\geq 0}$

(ii) $\lambda_j^{(i)} \geq \lambda_j^{(i)'} \geq \lambda_{j+1}^{(i)}$

(iii) $\lambda_j^{(i)'} \geq \lambda_j^{(i-1)} \geq \lambda_{j+1}^{(i)'}$

(iv) $\lambda_i^{(n)} = \lambda_i$

for all $i, j$. Properties (ii), (iii) are known as the *interleaving conditions*. Let $\check{\mathrm{Z}}\mathrm{P}(n, \lambda)$ denote the set of $\check{\mathrm{Z}}\mathrm{P}$ of rank $n$ associated with a partition $\lambda$.

The weight of a $\check{\mathrm{Z}}\mathrm{P}$ is given by the above expression where we instead define the $k_i$ by

$$k_i = 2(\sum_j \lambda_j^{(i)'}) - (\sum_k \lambda_k^{(i)}) - (\sum_\ell \lambda_\ell^{(i-1)}).$$

As before, given a $\check{\mathrm{Z}}\mathrm{P}$ $\Gamma$ we define $\Delta_\ell^{(i)}(\Gamma)$ to be an array of the same shape as $\Gamma$ consisting of a 1 in position $i\ell$ and zeroes everywhere else, where an apostrophe is applied if appropriate. Note that this is generally not a valid $\check{\mathrm{Z}}\mathrm{P}$ itself.

### 2.1.7 Jeux de taquin

The *jeu de taquin* (the French name for the tile-sliding 15 puzzle) is an algorithm by which skew or punctured semistandard tableau or admissible KNT may have its punctures removed while preserving semistandardness or admissibility. The type A jeu de taquin was developed by Schützenberger, and is equivalent to a row-bumping algorithm devised by Schützenberger and Lascoux [14] for inserting boxes into semistandard tableaux. A good description of the JDT, the bumping algorithm and many of their applications is given by Fulton in [5].

The type A algorithm is as follows. Let $T$ be a semistandard tableau with one box containing a $*$, called the puncture. Begin by comparing the entries of the boxes to the right of and below the box containing $*$.

$$\begin{array}{cc} \vdots & \vdots \\ \hline \multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{a} \\ \hline \multicolumn{1}{|c|}{b} & \\ \cline{1-1} \vdots & \vdots \end{array}$$

If $b \leq a$ or $a$ is not in T, switch $*$ with $b$. Otherwise, since $b > a$ or $b$ is not in the tableau, switch $*$ with $a$. Repeat this process until $*$ occupies an outside corner of $T$, at which point it may be removed from the tableau. Note that this will produce a semistandard skew tableau, or a puncture-free SSYT, with one box fewer than before.

**Example 2.1.4.**

$$
\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 7 \\
\ast & 1 & 2 & 4 & \\
2 & 2 & 4 & 6 & \\
4 & 5 & 5 & & \\
7 & 7 & 8 & &
\end{array}
\to
\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 7 \\
1 & \ast & 2 & 4 & \\
2 & 2 & 4 & 6 & \\
4 & 5 & 5 & & \\
7 & 7 & 8 & &
\end{array}
\to
\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 7 \\
1 & 2 & 2 & 4 & \\
2 & \ast & 4 & 6 & \\
4 & 5 & 5 & & \\
7 & 7 & 8 & &
\end{array}
\to
\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 7 \\
1 & 2 & 2 & 4 & \\
2 & 4 & \ast & 6 & \\
4 & 5 & 5 & & \\
7 & 7 & 8 & &
\end{array}
$$

$$
\to
\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 7 \\
1 & 2 & 2 & 4 & \\
2 & 4 & 5 & 6 & \\
4 & 5 & \ast & & \\
7 & 7 & 8 & &
\end{array}
\to
\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 7 \\
1 & 2 & 2 & 4 & \\
2 & 4 & 5 & 6 & \\
4 & 5 & 8 & & \\
7 & 7 & \ast & &
\end{array}
\to
\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 7 \\
1 & 2 & 2 & 4 & \\
2 & 4 & 5 & 6 & \\
4 & 5 & 8 & & \\
7 & 7 & & &
\end{array}
$$

**Definition 2.1.5.** The *symplectic jeu de taquin*, or SJDT, is an analogous algorithm on skew or punctured KNT, though its description is significantly more involved. This definition is as given by Lecouvey in [15].

Let $T$ be a punctured skew admissible tableau with adjacent columns $C_1$ and $C_2$, with $C_1$ containing the puncture, $\ast$. To perform one step of the SJDT, we must first put $T$ into split form, like so:

$$
\begin{array}{|c|c|c|c|}
\hline
\dots & \dots & \dots & \dots \\
\hline
\ast & \ast & b & b' \\
\hline
a & a' & \dots & \dots \\
\hline
\dots & \dots & & \\
\cline{1-2}
\end{array}
$$

Given $T$ as above, an elementary step of the SJDT is performed as follows:

1. If $a' \leq b$ or the double box $b\ b'$ is empty, then the double boxes $a\ a'$ and $\ast\ \ast$ are permuted. Unsplit the columns to obtain the new tableau.

2. If $a' > b$ or the double box $a\ a'$ is empty, then:

(a) When $b$ is a barred letter, $b$ slides into $rC_1$ to the box containing $*$ and $D_1 = \Phi(C_1) - \{*\} + \{b\}$ is a co-admissible column. Simultaneously the symbol $*$ slides into $lC_2$ to the box containing $b$ and $C_2' = C_2 - \{b\} + \{*\}$ is a punctured admissible column. Then we obtain a new punctured skew admissible tableau $C_1'C_2'$ by setting $C_1' = \Phi^{-1}(D_1)$.

(b) When $b$ is an unbarred letter, $b$ slides into $rC_1$ to the box containing $*$ and gives a new column $C_1' = C_1 - \{*\} + \{b\}$. Simultaneously the symbol $*$ slides into $lC_2$ to the box containing $b$ and $D_2 = \Phi(C_2) - \{b\} + \{*\}$ is a punctured coadmissible column. Then we obtain a new punctured skew tableau $C_1'C_2'$ by setting $C_2' = \Phi^{-1}(D_2)$.

Note that $\Phi$ and $\Phi^{-1}$ are defined on punctured columns by ignoring the puncture.

Repeating this process will eventually result in $*$ occupying an outer, and therefore removable, corner of the resulting tableau. It is a theorem of Lecouvey that iterating this process results in an admissible KNT, with several additional technical steps necessary to show it.

**Example 2.1.6.** For

$$T_1 = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 4 & 5 \\ \hline * & \overline{4} \\ \hline \overline{3} & \overline{1} \\ \hline \overline{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{spl}(T_1) = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 4 \\ \hline 4 & 4 & 5 & 5 \\ \hline * & * & \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{3} & \overline{1} & \overline{1} \\ \hline \overline{1} & \overline{1} & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

we are in case 2 (a) and

$$C_1'C_2' = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 5 & 5 \\ \hline \overline{5} & * \\ \hline \overline{3} & \overline{1} \\ \hline \overline{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

For

$$T_2 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline * & 5 \\ \hline \overline{5} & \overline{5} \\ \hline \overline{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{spl}(T_2) = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline * & * & 4 & 5 \\ \hline \overline{5} & \overline{5} & \overline{5} & \overline{4} \\ \hline \overline{1} & \overline{1} & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

we are in case 2 (b) and

$$C_1'C_2' = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline 4 & * \\ \hline \overline{5} & \overline{4} \\ \hline \overline{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

For

$$T_3 = \begin{array}{|c|c|c|}\hline 4 & * & 4 \\ \hline \overline{5} & \overline{4} & \overline{3} \\ \hline \end{array}$$

we obtain

$$\begin{array}{|c|c|c|}\hline 4 & 4 & * \\ \hline \overline{5} & \overline{4} & \overline{3} \\ \hline \end{array}.$$

In [15], Lecouvey provides an example of a full sequence of applications of the elementary step to a KNT. This requires introducing the notion of $a_1$-admissibility, which is essentially an embedding of the alphabet $\mathcal{A}_{C_n}$ into a larger alphabet with the added symbols $a_1, \overline{a}_1$.

#### 2.1.8 Tropical polynomials

In [3], Bump and Schilling discuss the relationship between crystals and tropical mathematics, a relatively new area of geometric combinatorics and algebraic number theory. A central object of interest in this field is the *tropical semi-ring* $\mathbb{T}$, defined to be the set $\mathbb{R} \cup \{-\infty\}$, where the addition, multiplication and division (denoted by $\oplus, \otimes$ and $\oslash$, respectively) are given by

$$x \oplus y = \max\{x, y\}, \qquad x \otimes y = x + y, \qquad x \oslash y = x - y.$$

Subtraction is not defined. Many authors use $x \oplus y = \min\{x, y\}$ instead, but $x \mapsto -x$ yields an isomorphism between the two versions of the semi-ring. Note that 0 is the multiplicative identity, and $-\infty$ is the additive identity.

The reason the tropical semi-ring is important is that it gives a way to take a polynomial or rational map and "tropicalize" it to a piecewise-linear map, for example $f(x, y, z) = \frac{x+y}{z} \mapsto \max\{x, y\} - z$. Now, starting from a piecewise-linear map $f$, we may attempt to find a polynomial or rational map $f'$ whose tropicalization is $f$. In the event that we can determine (not necessarily uniquely) such a map, we call $f'$ a *geometric lifting* of $f$. The upshot of all of this is that in [1] and [2], Berenstein and Kahzdan proved that crystal bases have geometric liftings which they call *geometric crystals*, algebraic varieties with algebraic maps whose tropicalizations are the weight map, the crystal operators and the Weyl group action.

Speyer and Sturmfels in [21] give an overview of tropical mathematics, from which we take the following definition.

**Definition 2.1.7.** $f(x_1, \ldots, x_n) = \max\{a_1(x_1, \ldots, x_n), \ldots, a_m(x_1, \ldots, x_n)\}$, where $a_i(x_1, \ldots, x_n) = c_i + a_{i1}x_1 + \cdots + a_{in}x_n$ and $c_i, a_{ij} \in \mathbb{Z}$, is called a *tropical (Laurent) polynomial* (see [21]).

### 2.1.9 Example: Combinatorial properties of Kashiwara-Nakashima Tableaux

While initially experimenting with crystals of tableaux in Sage, we observed some interesting sequences occurring in the dimensions of sequences of representations with similar shapes. While many do not currently occur in the Online Encyclopedia of Integer Sequences, one particularly simple one happened to, even though its description is rather obscure. We currently have no guess as to what geometric connection may exist between the structures in question.

**Example 2.1.8.** For $1 \leq n \leq 40$, $|\mathrm{KNT}(2, (n+1, n))|$ is the number of tin cans needed (picture the cans being cut and welded to one another such that four of them point out the vertices of a tetrahedron, and then constructing larger tetrahedra by welding on more cans in the same configuration) to construct a tetrahedron with side length $n$, as seen at `http://oeis.org/A210440`. We conjecture that this is true for all $n$.

## 2.2 Algebraic definitions

### 2.2.1 Quantum groups

These definitions are as in Hong and Kang [8]. A *generalized Cartan matrix* $A = (a_{ij})$ is a $n \times n$ matrix with integral entries such that:

1. For diagonal entries, $a_{ii} = 2$,

2. For non-diagonal entries, $a_{ij} \leq 0$,

3. $a_{ij} = 0$ if and only if $a_{ji} = 0$,

4. $A$ can be written as $DS$, where $D$ is a diagonal matrix and $S$ is a symmetric matrix.

Fix a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ with generalized Cartan matrix $A$ and a finite index set $I$. Let $P^\vee$ be a free abelian group of rank $2|I| - \text{rank } A$ with a $\mathbb{Z}$-basis $\{h_i \mid i \in I\} \cup \{d_s \mid s = 1, \ldots, |I| - \text{rank } A\}$ and let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ be the $\mathbb{C}$-linear space spanned by $P^\vee$. We call $P^\vee$ the *dual weight lattice* and $\mathfrak{h}$ the *Cartan subalgebra* of $\mathfrak{g}$. We also define the *weight lattice* to be

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}.$$

Set $\Pi^\vee = \{h_i \mid i \in I\}$ and choose linearly independent subset $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ satisfying

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_s) = 0 \text{ or } 1 \text{ for } i, j \in I, s = 1, \ldots, |I| - \text{rank } A.$$

The elements of $\Pi$ are called *simple roots*, and the elements of $\Pi^\vee$ are called *simple coroots*. We also define the *fundamental weights* $\Lambda_i \in \mathfrak{h}^*(i \in I)$ to be the linear functionals on $\mathfrak{h}$ given by:

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d_s) = 0 \text{ for } j \in I, s = 1, \ldots, |I| - \text{rank } A.$$

**Definition 2.2.1.** The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ defined as above is said to form a *Cartan datum* associated with the generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$.

**Definition 2.2.2.** Fixing an indeterminate $q$, we define

$$[n]_q = \sum_{0 \leq i < n} q^i = 1 + q + q^2 + \cdots + q^{n-1} = \begin{cases} \frac{1-q^n}{1-q} & \text{for } q \neq 1 \\ n & \text{for } q = 1 \end{cases},$$

$$[n]_q! = [1]_q [2]_q \cdots [n]_q,$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}.$$

These are respectively called the *q-number*, the *q-factorial* and the *q-binomial coefficient*.

**Definition 2.2.3.** Fixing an indeterminate $q$ and a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, the *quantum group* or the quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over $\mathbb{C}(q)$ with 1 generated by the elements $e_i, f_i (i \in I)$ and $q^h (h \in P^\vee)$ with the following defining relations:

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,

2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee$,

3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$,

4. $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,

5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix} 1-a_{ij} \\ k \end{smallmatrix}\right]_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix} 1-a_{ij} \\ k \end{smallmatrix}\right]_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

where $q_i = q^{s_i}$, $K_i = q^{s_i h_i}$.

A $U_q(\mathfrak{g})$-module $V^q$ is called a *weight module* if it admits a *weight space decomposition*

$$V^q = \bigoplus_{\mu \in P} V^q_\mu, \quad \text{where } V^q_\mu = \{v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}.$$

A vector $v \in V^q_\mu$ is called a *weight vector* of weight $\mu$. If $e_i v = 0$ for all $i \in I$, it is called a *maximal vector*. If $V^q_\mu \neq 0$, $\mu$ is called a *weight* of $V^q$ and $V^q_\mu$ is the *weight space* attached to $\mu \in P$. Its dimension $\dim V^q_\mu$ is called the *weight multiplicity* of $\mu$. We will denote by $\mathrm{wt}(V^q)$ the set of weights of the $U_q(\mathfrak{g})$-module $V^q$.

The category $\mathcal{O}^q_{\mathrm{int}}$ consists of $U_q(\mathfrak{g})$-modules $V^q$ satisfying the following conditions:

1. $V^q$ is a weight module and $\dim V^q_\lambda < \infty$ for all $\lambda \in P$,

2. there exist a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\mathrm{wt}(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s),$$

where $D(\lambda) = \{\mu \in P \mid \mu \leq \lambda\}$, and

3. all $e_i$ and $f_i$ ($i \in I$) are locally nilpotent on $V^q$.

The morphisms are taken to be the ordinary $U_q(\mathfrak{g})$-module homomorphisms.

### 2.2.2 Crystals

In this section we recall the definition of crystals, crystals morphisms and their tensor products. Our main reference is [8].

Let $X = (A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum with finite index set $I$.

**Definition 2.2.4.** A *crystal* of type $X$ is a non-empty set $\mathcal{B}$ together with maps

$$\begin{aligned} \mathrm{wt} : \mathcal{B} &\to P, \\ \widetilde{e}_i, \widetilde{f}_i : \mathcal{B} &\to \mathcal{B} \sqcup \{0\}, \quad i \in I, \\ \varepsilon_i, \varphi_i : \mathcal{B} &\to \mathbb{Z} \sqcup \{-\infty\}, \quad i \in I, \end{aligned}$$

satisfying for all $b, b' \in \mathcal{B}$ and $i \in I$:

(i) $\widetilde{f}_i(b) = b'$ if and only if $b = \widetilde{e}_i(b')$, in which case

$$\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i, \qquad \varepsilon_i(b') = \varepsilon_i(b) + 1, \qquad \varphi_i(b') = \varphi_i(b) - 1$$

and we write

$$b \xrightarrow{i} b'.$$

(ii) $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$. In particular, $\varphi_i(b) = -\infty$ if and only if $\varepsilon_i(b) = -\infty$.

(iii) If $\varphi_i(b) = \varepsilon_i(b) = -\infty$, then $\widetilde{e}_i(b) = \widetilde{f}_i(b) = 0$.

The cardinality of $\mathcal{B}$ is the *degree* of the crystal, wt is called the *weight map*, $\widetilde{e}_i$ and $\widetilde{f}_i$ are called *Kashiwara* or *crystal operators*, and $\varphi_i$ and $\varepsilon_i$ are called *string length functions.*

**Definition 2.2.5.** Take $\mathcal{B}$ as the set of vertices and define the $I$-colored arrows on $\mathcal{B}$ by

$$b \xrightarrow{i} b' \text{ if and only if } \widetilde{f}_i b = b' (i \in I).$$

Then $\mathcal{B}$ is given an $I$-colored directed graph structure called the *crystal graph* of

**Definition 2.2.6.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be crystals of type $X$. A *morphism* $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \sqcup \{0\} \to \mathcal{B}_2 \sqcup \{0\}$ such that $\Psi(0) = 0$ and for all $b, b' \in \Psi^{-1}(\mathcal{B}_2)$ and $i \in I$:

(i) $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\Psi(b)) = \varepsilon_i(b), \quad \varphi_i(\Psi(b)) = \varphi_i(b).$

(ii) If $b \xrightarrow{i} b'$ then $\Psi(b) \xrightarrow{i} \Psi(b')$.

If moreover $\Psi$ is bijective as a function $\mathcal{B}_1 \sqcup \{0\} \to \mathcal{B}_2 \sqcup \{0\}$, then $\Psi$ is an *isomorphism*.

**Definition 2.2.7.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be crystals of type $X$. The *tensor product* $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a crystal of type $X$, defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ with crystal structure given by

1. $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,

2. $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), \alpha_i^\vee \rangle\}$,

3. $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle \mathrm{wt}(b_2), \alpha_i^\vee \rangle\}$,

4. $\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

5. $\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$

We denote an element $(b_1, b_2)$ by $b_1 \otimes b_2$ and we set $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

### 2.2.3 Crystal structure on $\mathrm{SSYT}(n, \lambda)$

We recall the crystal structure on semistandard Young tableaux. For details, see e.g. [8], [3].

Let $n$ be a positive integer and $\lambda$ be a partition of length at most $n$. Let $T \in \mathrm{SSYT}(n, \lambda)$. The *far-eastern reading* of $T$, denoted $\mathrm{FarEast}(T)$ is the $|\lambda|$-tuple of letters read off from $T$, reading columns from right to left and each column top to bottom. The map $\mathrm{FarEast} : \mathrm{SSYT}(n, \lambda) \to \{1, 2, \ldots, n\}^{|\lambda|}$ is injective and we denote the inverse map by $\mathrm{FarEast}^{-1}$, defined on the image of FarEast.

For $i \in \{1, 2, \ldots, n-1\}$, the *$i$-bracketing* of a tuple of letters $x = (x_1, x_2, \ldots, x_{|\lambda|})$, denoted in this paper by $[x]_i$, is obtained by crossing out the right-most $i$ having at least one $i+1$ to the right of it, in which case we also cross out the leftmost of those $i+1$'s, and repeating this recursively (ignoring crossed out entries) until $(i, i+1)$ is not a subsequence. The crossed out $i$'s and $(i+1)$'s in $x$ are said to be ($i$-)*bracketed.* Any remaining $i$'s or $(i+1)$'s in $x$ are ($i$-)*unbracketed.*

**Definition 2.2.8** (Crystal structure on $\mathrm{SSYT}(n,\lambda)$, [8],[3])**.** Let $n$ be a positive integer and $\lambda$ a partition with $n$ or fewer parts. Let $P = \mathbb{Z}^n$ with standard basis $\{\boldsymbol{e}_i\}_{i=1}^n$. For $i \in \{1, 2, \dots, n-1\}$ and $T \in \mathrm{SSYT}(n,\lambda)$ define:

$$\mathrm{wt}(T) = N_1(T)\boldsymbol{e}_1 + N_2(T)\boldsymbol{e}_2 + \cdots + N_n(T)\boldsymbol{e}_n, \text{ where } N_i(T) = \#\text{boxes in } T \text{ containing } i,$$

$$\varphi_i(T) = \text{number of } i\text{-unbracketed } i\text{'s in } [\mathrm{FarEast}(T)]_i,$$

$$\varepsilon_i(T) = \text{number of } i\text{-unbracketed } (i+1)\text{'s in } [\mathrm{FarEast}(T)]_i,$$

$$\widetilde{f}_i(T) = \begin{cases} \mathrm{FarEast}^{-1}\big(\text{change leftmost } i \text{ in } [\mathrm{FarEast}(T)]_i \text{ to } i+1\big), & \text{if } \varphi_i(T) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\widetilde{e}_i(T) = \begin{cases} \mathrm{FarEast}^{-1}\big(\text{change rightmost } i+1 \text{ in } [\mathrm{FarEast}(T)]_i \text{ to } i\big), & \text{if } \varepsilon_i(T) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 2.2.9** (see e.g. [8],[3])**.** *Let $n$ be a positive integer and $\lambda$ be a partition with $n$ or fewer parts. The set $\mathrm{SSYT}(n,\lambda)$ equipped with the above maps $\mathrm{wt}, \varphi_i, \varepsilon_i, \widetilde{f}_i, \widetilde{e}_i$ constitutes a crystal of type $A_{n-1}$.*

**Example 2.2.10.** Let $n = 4$ and $\lambda = (5, 2, 2)$ and consider

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} \end{array}$$

Let us compute $\varphi_2(T)$ and $\widetilde{f}_2(T)$. The far-eastern reading of $T$ is

$$\mathrm{FarEast}(T) = (3, 2, 2, 2, 3, 4, 1, 3, 4).$$

To find the 2-bracketing of this, first cross out the rightmost 2 having a 3 somewhere to its right, and also cross out the leftmost of those 3's: $(3, 2, 2, \cancel{2}, \cancel{3}, 4, 1, 3, 4)$. Repeating this step once more, ignoring crossed out entries, we obtain $(3, 2, \cancel{2}, \cancel{2}, \cancel{3}, 4, 1, \cancel{3}, 4)$ at which point the 2-bracketing is finished, as $(2, 3)$ is not a subsequence anymore (ignoring crossed out entries). So

$$[(3, 2, 2, 2, 3, 4, 1, 3, 4)]_2 = (3, 2, \cancel{2}, \cancel{2}, \cancel{3}, 4, 1, \cancel{3}, 4).$$

Now we can compute:

$$\varphi_2(T) = \text{number of 2-unbracketed 2's in } (3, 2, \cancel{2}, \cancel{2}, \cancel{3}, 4, 1, \cancel{3}, 4) = 1$$

$$\begin{aligned} \widetilde{f}_2(T) &= \mathrm{FarEast}^{-1}\big(\text{change rightmost 2 in } (3, 2, \cancel{2}, \cancel{2}, \cancel{3}, 4, 1, \cancel{3}, 4) \text{ to } 3\big) \\ &= \mathrm{FarEast}^{-1}\big((3, 3, \cancel{2}, \cancel{2}, \cancel{3}, 4, 1, \cancel{3}, 4)\big) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} \end{array} \end{aligned}$$

Note that when applying $\mathrm{FarEast}^{-1}$ we ignore the bracketing and preserve the shape of $T$.

**Remark 2.2.11.** The $i$-bracketing can be described directly on the tableaux $T$ as follows. Go through all the columns of $T$ from left to right and do the following. If the column contains an $i$ and there is a thus-far-unbracketed $i+1$ in the same column, or in a column further to the left, then cross out that $i$ along with the rightmost of those $i+1$'s. Then $\varphi_i(T)$ is the number of $i$-unbracketed $i$'s in $T$; $\varepsilon_i(T)$ is the number of $i$-unbracketed $i+1$'s in $T$; $\widetilde{f}_i(T)$ is obtained from $T$ by changing the rightmost $i$-unbracketed $i$ in $T$ to $i+1$; $\widetilde{e}_i(T)$ is obtained from $T$ by changing the leftmost $i$-unbracketed $i+1$ in $T$ to $i$.

### 2.2.4 Crystal structure on KNT$(n, \lambda)$

The *vector representation* of $U_q(\mathfrak{sp}_{2n})$ has crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{\overline{r}} \xrightarrow{r-1} \cdots \xrightarrow{2} \boxed{\overline{2}} \xrightarrow{1} \boxed{\overline{1}}.$$

A tableau $C \in \mathrm{KNT}(r, (1^k))$ is called a *column* of length $k$. Similarly, $R \in \mathrm{KNT}(r, (k))$ is called a *row* of length $k$. As Bump and Schilling describe in [3], for a Young diagram $Y \in \mathrm{YD}(\lambda)$ with

$N$ boxes we wish to define embeddings of $\mathrm{KNT}(r, \lambda)$ into $\mathbb{B}^{\otimes N}$, where $\mathbb{B}$ is the crystal graph of the vector representation. For a column $C$, we define a map $C \mapsto CR(C)$ by

$$\begin{array}{|c|}\hline i_1 \\ \hline \cdots \\ \hline i_k \\ \hline \end{array} \quad \mapsto \quad \boxed{i_1} \otimes \cdots \otimes \boxed{i_k}.$$

Given $T \in \mathrm{KNT}(r, \lambda)$, the *far-eastern reading* of $T$ is given by concatenating the columns of $T$, from right to left, put through the map $CR$. In other words, if we have

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \overline{2} & \overline{1} \\ \hline 2 & \overline{3} & \overline{1} \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \in \mathrm{KNT}(3, (4, 3, 1))$$

then

$$\begin{aligned} CR(T) &= CR(C_4) \otimes CR(C_3) \otimes CR(C_2) \otimes CR(C_1) \\ &= \boxed{\overline{1}} \otimes \boxed{\overline{2}} \otimes \boxed{\overline{1}} \otimes \boxed{2} \otimes \boxed{\overline{3}} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3}. \end{aligned}$$

The *middle-eastern reading* of a tableau $T$, in contrast, moves across the rows from right to left and from top to bottom. Both of these readings can be used for the following rule, but we will choose to apply the far-eastern reading.

The action of $\widetilde{f}_i$ on a tableau $T \in \mathrm{KNT}(2, \lambda)$ may be described by what is called the *signature rule*: if $i = r$, then we proceed analogously to type $A$. That is, symbols $\overline{r}$ in $RR(T)$ are decorated with $-$'s and symbols $r$ are decorated with $+$'s. Then, all instances of a $+$ before a $-$ are bracketed together, starting with the rightmost unbracketed $+$ and the leftmost unbracketed $-$. Then, once no more such inversions remain, $\widetilde{f}_r$ changes the leftmost unbracketed $r$ to a $\overline{r}$, unless none remain in which case $\widetilde{f}_r(T) = 0$.

If $i < r$, the rule is slightly more complicated. As in type $A$, symbols $i + 1$ in $RR(T)$ are given a $-$ while symbols $i$ are given a $+$. However, we also give a $-$ to symbols $\overline{i}$, while symbols $\overline{i+1}$ are marked with $+$. As in the previous case, we now bracket $+$'s to the left of $-$'s with those $-$'s until all such pairs are accounted for. Then $\widetilde{f}_i$ changes the symbol associated with the leftmost unbracketed $+$ to $i + 1$, if it was $i$, or to $\overline{i}$, if it was $\overline{i+1}$. If no unbracketed $+$ remains, then $\widetilde{f}_i(T) = 0$.

The action of $\widetilde{e}_i$ on $T$ is simply the reverse of the above: the symbol bearing the rightmost unbracketed $-$ is changed to its counterpart featuring a $+$ unless none remain. Note that $\varphi_i(T)$ is the number of unbracketed $+$'s, $\varepsilon_i(T)$ is the number of unbracketed $-$'s and $\langle \mathrm{wt}(T), \alpha_i^\vee \rangle = \varphi_i(T) - \varepsilon_i(T)$, as desired.

**Example 2.2.12.** Let $T \in \mathrm{KNT}(3,(3,3))$ be given by

$$T = \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline \overline{3} & \overline{3} & \overline{2} \\ \hline \end{array},$$

and so

$$CR(T) = \boxed{3} \otimes \boxed{\overline{2}} \otimes \boxed{3} \otimes \boxed{\overline{3}} \otimes \boxed{2} \otimes \boxed{\overline{3}}.$$

Then, applying the signature rule, we have

$$\begin{aligned}
\widetilde{f}_3(T) &= \widetilde{f}_3(\boxed{3} \otimes \boxed{\overline{2}} \otimes \boxed{3} \otimes \boxed{\overline{3}} \otimes \boxed{2} \otimes \boxed{\overline{3}}) \\
&= \widetilde{f}_3(\underset{+}{\boxed{3}} \otimes \boxed{\overline{2}} \otimes \underset{+}{\boxed{3}} \otimes \underset{-}{\boxed{\overline{3}}} \otimes \boxed{2} \otimes \underset{-}{\boxed{\overline{3}}}) \\
&= \widetilde{f}_3(\underset{+}{\boxed{3}} \otimes \boxed{\overline{2}} \otimes \underbrace{\underset{+}{\boxed{3}} \otimes \underset{-}{\boxed{\overline{3}}}} \otimes \boxed{2} \otimes \underset{-}{\boxed{\overline{3}}}) \\
&= \widetilde{f}_3(\underbrace{\underset{+}{\boxed{3}} \otimes \boxed{\overline{2}} \otimes \underbrace{\underset{+}{\boxed{3}} \otimes \underset{-}{\boxed{\overline{3}}}} \otimes \boxed{2} \otimes \underset{-}{\boxed{\overline{3}}}}) \\
&= 0.
\end{aligned}$$

Since $T$ has no unbracketed $+$'s and no unbracketed $-$'s with respect to $\widetilde{f}_3$, $\varphi_3(T) = 0$ and $\varepsilon_3(T) = 0$.

**Example 2.2.13.** Let $T \in \mathrm{KNT}(3,(3,3))$ be given by

$$T = \begin{array}{|c|c|c|}\hline 2 & 2 & \overline{3} \\ \hline 3 & \overline{3} & \overline{2} \\ \hline \end{array},$$

and so

$$RR(T) = \boxed{\overline{3}} \otimes \boxed{\overline{2}} \otimes \boxed{2} \otimes \boxed{\overline{3}} \otimes \boxed{2} \otimes \boxed{3}.$$

Then, applying the signature rule, we have

$$\begin{aligned}
\widetilde{f}_2(T) &= \widetilde{f}_2\left(\boxed{\overline{3}} \otimes \boxed{\overline{2}} \otimes \boxed{2} \otimes \boxed{\overline{3}} \otimes \boxed{2} \otimes \boxed{3}\right) \\
&= \widetilde{f}_2\left(\underset{+}{\boxed{\overline{3}}} \otimes \underset{-}{\boxed{\overline{2}}} \otimes \underset{+}{\boxed{2}} \otimes \underset{+}{\boxed{\overline{3}}} \otimes \underset{+}{\boxed{2}} \otimes \underset{-}{\boxed{3}}\right) \\
&= \widetilde{f}_2\left(\underbrace{\underset{+}{\boxed{\overline{3}}} \otimes \underset{-}{\boxed{\overline{2}}}} \otimes \underset{+}{\boxed{2}} \otimes \underset{+}{\boxed{\overline{3}}} \otimes \underset{+}{\boxed{2}} \otimes \underset{-}{\boxed{3}}\right) \\
&= \widetilde{f}_2\left(\underbrace{\underset{+}{\boxed{\overline{3}}} \otimes \underset{-}{\boxed{\overline{2}}}} \otimes \underset{+}{\boxed{2}} \otimes \underset{+}{\boxed{\overline{3}}} \otimes \underbrace{\underset{+}{\boxed{2}} \otimes \underset{-}{\boxed{3}}}\right) \\
&= \boxed{\overline{3}} \otimes \boxed{\overline{2}} \otimes \boxed{3} \otimes \boxed{\overline{3}} \otimes \boxed{2} \otimes \boxed{3}
\end{aligned}$$

and so

$$\widetilde{f}_2(T) = \begin{array}{|c|c|c|}\hline 2 & 3 & \overline{3} \\ \hline 3 & \overline{3} & \overline{2} \\ \hline \end{array}\,.$$

Since $T$ has two unbracketed $+$s and no unbracketed $-$s with respect to $\widetilde{f}_2$, $\varphi_2(T) = 2$ and $\varepsilon_2(T) = 0$.

# CHAPTER 3. CRYSTAL STRUCTURE IN TYPE A

## 3.1 Introduction

In this chapter we give explicit formulae for the crystal operators on $\Gamma Ц(n, \lambda)$ and prove that these equip the set with a crystal basis structure. We then prove that the bijection between the sets $\mathrm{SSYT}(n, \lambda)$ and $\Gamma Ц(n, \lambda)$ is an isomorphism of crystals. The proofs rely heavily on the fact that, by the nature of the bijection, we may obtain various useful combinatorial data by examining certain sums and differences of pattern entries. Of central importance to the crystal isomorphism is that $i$-bracketing is preserved by the map. Clearly it is needed to satisfy all of the appropriate definitions, but it is moreover very much the engine at the heart of the machine that is a crystal basis. The proof, therefore, is heavily focused on demonstrating that the combinatorial quantities we introduce allow us to push $i$-bracketing through the bijection as required.

As we will discuss at the beginning of the next chapter, an unexpected but clear connection with tropical mathematics arises when one attempts to give similar formulae for crystals of symplectic patterns. The same is in fact true in type A, but much as the bijection between SSYT and ΓЦ may be viewed as relying on an almost-trivial application of jeu de taquin whereas the bijection we provide between KNT and ŽP requires a significantly less trivial application of the more complex SJDT, it seems perhaps not coincidental that the type A behavior should resemble a simpler version of the type C behavior. In particular, it is easy to see both the string length operators and the raising and lowering operators as involving relatively simple tropical polynomials of pattern entries. While we do not offer any conjecture as to the significance of this connection, it seems plausible that if one were to shed more light on it, this may alleviate some of the difficulty in attempting to adapt the proof strategy below to type C.

## 3.2 Crystal structure on Gelfand-Tsetlin patterns

In this section we prove the first main result of the paper. We equip the set of Gelfand-Tsetlin patterns with explicit crystal data, and prove that this makes $\Gamma Ц(n, \lambda)$ into a crystal of type $A_{n-1}$. To define the crystal data we will need some notation.

$$\begin{array}{ccccccc} & \lambda_j^{(i+1)} & & \lambda_{j+1}^{(i+1)} & \\ \lambda_{j-1}^{(i)} & & \lambda_j^{(i)} & & \lambda_{j+1}^{(i)} \\ & \lambda_{j-1}^{(i-1)} & & \lambda_j^{(i-1)} & \end{array}$$

Figure 3.1: Part of a Gelfand-Tsetlin pattern

$$\begin{tikzcd} & -\lambda_{j+1}^{(i+1)} \arrow[dl] & \\ +\lambda_j^{(i)} \arrow[dr] & & +\lambda_{j+1}^{(i)} \arrow[ul] \\ & -\lambda_j^{(i-1)} \arrow[ur] & \end{tikzcd}$$

FIGURE 3.2a. $a_j^{(i)}(\Lambda)$

$$\begin{tikzcd} & +\lambda_j^{(i+1)} \arrow[dr] & \\ -\lambda_{j-1}^{(i)} \arrow[ur] & & -\lambda_j^{(i)} \arrow[dl] \\ & +\lambda_{j-1}^{(i-1)} \arrow[ul] & \end{tikzcd}$$

FIGURE 3.2b. $b_j^{(i)}(\Lambda)$

Figure 3.2: Computing diamond numbers in Gelfand-Tsetlin patterns

Let $\Lambda \in \Gamma Ц(n, \lambda)$. We introduce the following *diamond numbers*, which are alternating sums around a diamond shape in $\Lambda$ starting at $\lambda_j^{(i)}$:

$$a_j^{(i)}(\Lambda) := \lambda_j^{(i)} - \lambda_j^{(i-1)} + \lambda_{j+1}^{(i)} - \lambda_{j+1}^{(i+1)}, \quad 0 \le j \le i, \tag{3.2.1a}$$

$$b_j^{(i)}(\Lambda) := -\lambda_j^{(i)} + \lambda_{j-1}^{(i-1)} - \lambda_{j-1}^{(i)} + \lambda_j^{(i+1)}, \quad 1 \le j \le i+1, \tag{3.2.1b}$$

where by convention $\lambda_j^{(i)} = 0$ if not $1 \le j \le i$. Note that,

$$b_j^{(i)}(\Lambda) = -a_{j-1}^{(i)}(\Lambda), \quad 1 \le j \le i+1, \tag{3.2.2}$$

and, by the interleaving conditions,

$$a_0^{(i)}(\Lambda) \le 0, \quad b_{i+1}^{(i)}(\Lambda) \le 0. \tag{3.2.3}$$

For notational convenience we put

$$a_j^{(i)}(\Lambda) = 0 \ \ \forall j > i, \qquad b_j^{(i)}(\Lambda) = 0 \ \ \forall j > i+1. \tag{3.2.4}$$

Next, define these *diamond-sums*:

$$A_j^{(i)}(\Lambda) := \sum_{k=j}^{i} a_k^{(i)}(\Lambda), \quad 0 \le j \le i, \tag{3.2.5}$$

$$B_j^{(i)}(\Lambda) := \sum_{k=1}^{j} b_k^{(i)}(\Lambda), \quad 1 \le j \le i+1. \tag{3.2.6}$$

Note that (3.2.3) imply

$$A_0^{(i)}(\Lambda) \le A_1^{(i)}(\Lambda), \quad B_{i+1}^{(i)}(\Lambda) \le B_i^{(i)}(\Lambda). \tag{3.2.7}$$

The following relation will be useful:

$$A_0^{(i)}(\Lambda) = A_j^{(i)}(\Lambda) - B_j^{(i)}(\Lambda) = -B_{i+1}^{(i)}(\Lambda) \qquad \forall j \in \{0, 1, \ldots, i+1\}. \tag{3.2.8}$$

**Remark 3.2.1.** In [24] the authors give the formula for the $\mathbf{i}_A$-string datum, where $\mathbf{i}_A$ is the reduced long word $(1, 2, 1, 3, 2, 1, \ldots, n-1, n-2, \ldots, 1)$. Converting their notation (their $a_{ij}$ is our $\lambda_i^{(n+i-j)}$) gives the formula

$$d_{i,j}(\Lambda) = \sum_{m=1}^{j-i} (\lambda_m^{(j)} - \lambda_m^{(j-1)}), \qquad 1 \le i < j \le n. \tag{3.2.9}$$

**Definition 3.2.2.** Let $P = \mathbb{Z}^n$ with standard basis $\{\boldsymbol{e}_i\}_{i=1}^n$. Put $\omega_i = \sum_{j=1}^i \boldsymbol{e}_j$. Define for any $\Lambda \in \Gamma Ц(n, \lambda)$ and $i \in \{1, 2, \ldots, n-1\}$:

$$\begin{aligned}
\mathrm{wt}(\Lambda) &= \sum_{j=1}^n \big( \big( \sum_{k=1}^j \lambda_k^{(j)} - \sum_{k=1}^{j-1} \lambda_k^{(j-1)} \big) \big) \boldsymbol{e}_j && (3.2.10)\\
&= A_0^{(1)}(\Lambda)\omega_1 + A_0^{(2)}(\Lambda)\omega_2 + \cdots + A_0^{(n)}(\Lambda)\omega_n;\\
&= -\big(B_2^{(1)}(\Lambda)\omega_1 + B_3^{(2)}(\Lambda)\omega_2 + \cdots + B_{n+1}^{(n)}(\Lambda)\omega_n\big);\\
\varphi_i(\Lambda) &= \max\big\{A_1^{(i)}(\Lambda), A_2^{(i)}(\Lambda), \ldots, A_i^{(i)}(\Lambda)\big\}; && (3.2.11)\\
\varepsilon_i(\Lambda) &= \max\big\{B_1^{(i)}(\Lambda), B_2^{(i)}(\Lambda), \ldots, B_i^{(i)}(\Lambda)\big\}; && (3.2.12)\\
\widetilde{f}_i(\Lambda) &= \begin{cases} \Lambda - \Delta_\ell^{(i)}(\Lambda), & \text{if } \varphi_i(\Lambda) > 0,\\ 0, & \text{if } \varphi_i(\Lambda) = 0, \end{cases} && (3.2.13)\\
&\quad \text{where } \ell = \max\big\{j \in \{1, 2, \ldots, i\} \mid A_j^{(i)}(\Lambda) = \varphi_i(\Lambda)\big\};\\
\widetilde{e}_i(\Lambda) &= \begin{cases} \Lambda + \Delta_\ell^{(i)}(\Lambda), & \text{if } \varepsilon_i(\Lambda) > 0,\\ 0, & \text{if } \varepsilon_i(\Lambda) = 0, \end{cases} && (3.2.14)\\
&\quad \text{where } \ell = \min\big\{j \in \{1, 2, \ldots, i\} \mid B_j^{(i)}(\Lambda) = \varepsilon_i(\Lambda)\big\}.
\end{aligned}$$

**Theorem 3.2.3.** *Let $n$ be any positive integer and $\lambda$ be a partition with $n$ or fewer parts. Then the set $\Gamma Ц(n, \lambda)$ of all Gelfand-Tsetlin patterns with $n$ rows and top row $\lambda$, equipped with the crystal data* $\mathrm{wt}, \widetilde{f}_i, \widetilde{e}_i, \varphi_i, \varepsilon_i$ *as above, is a crystal of type $A_{n-1}$.*

*Proof.* Let $i \in \{1, 2, \ldots, n-1\}$ be arbitrary. First we show that if $\Lambda \in \Gamma Ц(n, \lambda)$ is such that $\varphi_i(\Lambda) > 0$, then $\widetilde{f}_i(\Lambda)$ is a valid Gelfand-Tsetlin pattern. Let $\ell = \max\{j \in \{1, 2, \ldots, i\} \mid A_j^{(i)} = \varphi_i(\Lambda)\}$. Then by definition, $\widetilde{f}_i(\Lambda) = \Lambda - \Delta_\ell^{(i)}(\Lambda)$ which has integer entries, the top row still equals $\lambda$ (since $i < n$), and the interleaving conditions hold everywhere except possibly near the $\lambda_\ell^{(i)}$ entry. More precisely, we must show the following inequalities hold:

$$\begin{array}{ccccc} \lambda_\ell^{(i+1)} & & & & \lambda_{\ell+1}^{(i+1)} \\ & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & \\ & & \lambda_\ell^{(i)} - 1 & & \\ & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & \\ \lambda_{\ell-1}^{(i-1)} & & & & \lambda_\ell^{(i-1)} \end{array} \tag{3.2.15}$$

We have that $\varphi_i(\Lambda) = A_\ell^{(i)}(\Lambda) = a_\ell^{(i)}(\Lambda) + a_{\ell+1}^{(i)}(\Lambda) + \cdots + a_i^{(i)}(\Lambda)$. Note that $a_\ell^{(i)}(\Lambda) > 0$, otherwise $j = \ell + 1$ would satisfy $A_j^{(i)}(\Lambda) = \varphi_i(\Lambda)$ (we can't have $A_{\ell+1}^{(i)}(\Lambda) > \varphi_i(\Lambda)$ by the definitions of $\varphi_i$ and $\ell$), contradicting maximality of $\ell$. Now, $a_\ell^{(i)} > 0$ is equivalent to

$$\lambda_\ell^{(i)} - \lambda_\ell^{(i-1)} + \lambda_{\ell+1}^{(i)} - \lambda_{\ell+1}^{(i+1)} > 0 \tag{3.2.16}$$

by definition of $a_\ell^{(i)}$. Since all entries of $\Lambda$ are integers, (3.2.16) implies that

$$\lambda_\ell^{(i)} - 1 \geq \lambda_\ell^{(i-1)} + \lambda_{\ell+1}^{(i+1)} - \lambda_{\ell+1}^{(i)}. \tag{3.2.17}$$

By the interleaving condition for $\Lambda$,

$$\lambda_{\ell+1}^{(i+1)} \geq \lambda_{\ell+1}^{(i)}, \quad \text{and} \quad \lambda_\ell^{(i-1)} \geq \lambda_{\ell+1}^{(i)}. \tag{3.2.18}$$

Combining (3.2.17) and (3.2.18) we obtain

$$\lambda_\ell^{(i)} - 1 \geq \lambda_\ell^{(i-1)} \quad \text{and} \quad \lambda_\ell^{(i)} - 1 \geq \lambda_{\ell+1}^{(i+1)}, \tag{3.2.19}$$

which are the two rightmost inequalities in (3.2.15). The two leftmost inequalities in (3.2.15) are trivial since $\lambda_\ell^{(i+1)} \geq \lambda^{(i)}$ and $\lambda_{\ell-1}^{(i-1)} \geq \lambda_\ell^{(i)}$ by the interleaving conditions for $\Lambda$. This shows that if $\varphi_i(\Lambda) > 0$ then $\widetilde{f}_i(\Lambda) \in \Gamma Ц(n, \lambda)$.

Next, suppose that $\varepsilon_i(\Lambda) > 0$. We must show that $\widetilde{e}_i(\Lambda) \in \Gamma Ц(n, \lambda)$. We have $\varepsilon_i(\Lambda) = \max\{B_1^{(i)}(\Lambda), \ldots, B_i^{(i)}(\Lambda)\}$. Let $\ell = \min\{j \in \{1, 2, \ldots, i\} \mid B_j^{(i)}(\Lambda) = \varepsilon_i(\Lambda)\}$. Then $\varepsilon_i(\Lambda) = B_\ell^{(i)} = b_1^{(i)}(\Lambda) + b_2^{(i)}(\Lambda) + \cdots + b_\ell^{(i)}(\Lambda)$. As before, $b_\ell^{(i)}(\Lambda) > 0$ by the minimality of $\ell$. So

$$-\lambda_{\ell-1}^{(i)} + \lambda_{\ell-1}^{(i-1)} - \lambda_\ell^{(i)} + \lambda_\ell^{(i+1)} > 0. \tag{3.2.20}$$

We have $\widetilde{e}_i(\Lambda) = \Lambda + \Delta_\ell^{(i)}(\Lambda)$ and hence we must show that

$$\begin{array}{ccccc} \lambda_\ell^{(i+1)} & & & & \lambda_{\ell+1}^{(i+1)} \\ & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & \\ & & \lambda_\ell^{(i)} + 1 & & \\ & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & \\ \lambda_{\ell-1}^{(i-1)} & & & & \lambda_\ell^{(i-1)} \end{array} \tag{3.2.21}$$

Analogously to the previous case, the rightmost two inequalities $\lambda_\ell^{(i)}+1 \geq \lambda_{\ell+1}^{(i+1)}$ and $\lambda_\ell^{(i)}+1 \geq \lambda_{\ell+1}^{(i-1)}$ hold trivially by the interleaving conditions for $\Lambda$. By (3.2.20) we have

$$\lambda_\ell^{(i)} + 1 \leq -\lambda_{\ell-1}^{(i)} + \lambda_{\ell-1}^{(i-1)} + \lambda_\ell^{(i+1)}, \tag{3.2.22}$$

which, together with $\lambda_{\ell-1}^{(i)} \geq \lambda_{\ell-1}^{(i-1)}$ and $\lambda_{\ell-1}^{(i)} \geq \lambda_\ell^{(i+1)}$, hold by the interleaving condition for $\Lambda$, yielding the leftmost two inequalities in (3.2.21). This shows that if $\varepsilon_i(\Lambda) > 0$ then $\widetilde{e}_i(\Lambda) \in \Gamma\text{Ц}(n, \lambda)$.

Next we show that property (i) in the definition of crystal holds. First we show that $\widetilde{f}_i(\Lambda) = \Lambda'$ iff $\widetilde{e}_i(\Lambda') = \Lambda$. Suppose $\widetilde{f}_i(\Lambda) = \Lambda'$. In particular $\varphi_i(\Lambda) > 0$. Then we need to prove $\widetilde{e}_i(\Lambda') = \Lambda$. We have $\Lambda' = \Lambda - \Delta_\ell^{(i)}(\Lambda)$, where $\ell$ is defined by

$$\ell = \max\{j \in \{1, 2, \dots, i\} \mid A_j^{(i)}(\Lambda) = \varphi_i(\Lambda)\}. \tag{3.2.23}$$

First we show that $\varepsilon_i(\Lambda') > 0$. By definition, $\varepsilon_i(\Lambda') = \max\{B_1^{(i)}(\Lambda'), B_2^{(i)}(\Lambda'), \dots, B_i^{(i)}(\Lambda')\}$. So it suffices to show that $B_j^{(i)}(\Lambda') > 0$ for some $j$. For $j = \ell$ we have:

$$B_\ell^{(i)}(\Lambda') = b_1^{(i)}(\Lambda') + b_2^{(i)}(\Lambda') + \cdots + b_\ell^{(i)}(\Lambda') = B_\ell^{(i)}(\Lambda) + 1, \tag{3.2.24}$$

since $b_j^{(i)}(\Lambda') = b_j^{(i)}(\Lambda)$ for $j = 1, \dots, i-1$, while $b_\ell^{(i)}(\Lambda') = b_\ell^{(i)}(\Lambda) + 1$ by definition of $b_j^{(i)}(\Lambda)$. By (3.2.8),

$$B_\ell^{(i)}(\Lambda) + 1 = A_\ell^{(i)}(\Lambda) - A_0^{(i)}(\Lambda) + 1 = \varphi_i(\Lambda) - A_0^{(i)}(\Lambda) + 1. \tag{3.2.25}$$

By definition of $\varphi_i$ we have

$$\varphi_i(\Lambda) - A_0^{(i)}(\Lambda) \geq 0. \tag{3.2.26}$$

Now (3.2.24)-(3.2.26) imply $B_\ell^{(i)}(\Lambda') > 0$, hence $\varepsilon_i(\Lambda') > 0$. It remains to be shown that $\widetilde{e}_i(\Lambda') = \Lambda$. Since $\Lambda = \Lambda' + \Delta_\ell^{(i)}$, we have to show that

$$\ell = \min\{j \in \{1,2,\ldots,i\} \mid B_j^{(i)}(\Lambda') = \varepsilon_i(\Lambda')\}. \tag{3.2.27}$$

For $1 \leq j < \ell$ we saw that $B_j^{(i)}(\Lambda') = B_j^{(i)}(\Lambda)$ and (3.2.8) implies that $B_j^{(i)}(\Lambda) < B_\ell^{(i)}(\Lambda)$, while $B_\ell^{(i)}(\Lambda') = 1 + B_\ell^{(i)}(\Lambda)$. So $\varepsilon_i(\Lambda') \geq B_\ell^{(i)}(\Lambda')$ and we will show equality. For $\ell < j \leq i$ we have, by definition of $b_j^{(i)}(\Lambda)$,

$$B_j^{(i)}(\Lambda') = 2 + B_j^{(i)}(\Lambda), \tag{3.2.28}$$

and by (3.2.8),

$$B_j^{(i)}(\Lambda) = A_j^{(i)}(\Lambda) - A_0^{(i)}(\Lambda), \tag{3.2.29}$$

while by definition of $\ell$, (3.2.23), we have

$$A_j^{(i)}(\Lambda) - A_0^{(i)}(\Lambda) < A_\ell^{(i)}(\Lambda) - A_0^{(i)}(\Lambda). \tag{3.2.30}$$

Thus (3.2.28)-(3.2.30) imply that

$$B_j^{(i)}(\Lambda') \leq 1 + B_\ell^{(i)}(\Lambda) = B_\ell^{(i)}(\Lambda'). \tag{3.2.31}$$

Therefore $\varepsilon_i(\Lambda') = B_\ell^{(i)}(\Lambda')$ and (3.2.27) holds.

The converse is analogous but we provide some details for the sake completeness. Suppose that $\widetilde{e}_i(\Lambda') = \Lambda$. We need to show that $\widetilde{f}_i(\Lambda) = \Lambda'$. We have $\varepsilon_i(\Lambda') > 0$ and $\Lambda = \Lambda' + \Delta_\ell^{(i)}(\Lambda)$ where $\ell = \min\{j \in \{1,2,\ldots,i\} \mid B_j^{(i)}(\Lambda') = \varepsilon_i(\Lambda')\}$. First we show $\varphi_i(\Lambda) > 0$ by showing $A_\ell^{(i)}(\Lambda) > 0$. We have $A_\ell^{(i)}(\Lambda) = A_\ell^{(i)}(\Lambda') + 1$ and $A_\ell^{(i)}(\Lambda') = \varepsilon_i(\Lambda') - B_{i+1}^{(i)}(\Lambda') \geq 0$ by (3.2.8). It remains to show $\widetilde{f}_i(\Lambda) = \Lambda'$. Since $\Lambda' = \Lambda - \Delta_\ell^{(i)}(\Lambda)$, this is equivalent to showing that $\ell = \max\{j \in \{1,2,\ldots,i\} \mid A_j^{(i)}(\Lambda) = \varphi_i(\Lambda)\}$. For $\ell < j \leq i$ we have $A_j^{(i)}(\Lambda) = A_j^{(i)}(\Lambda') = B_j^{(i)}(\Lambda') - B_{i+1}^{(i)}(\Lambda') \leq B_\ell^{(i)}(\Lambda') - B_{i+1}^{(i)}(\Lambda') = A_\ell^{(i)}(\Lambda') = A_\ell^{(i)}(\Lambda) - 1 < A_\ell^{(i)}(\Lambda)$. So $\ell \leq \max\{j \in \{1,2,\ldots,i\} \mid A_j^{(i)}(\Lambda) = \varphi_i(\Lambda)\}$. For $1 \leq j < \ell$ we have $A_j^{(i)} = 2 + A_j^{(i)}(\Lambda') = 2 + B_j^{(i)}(\Lambda') - B_{i+1}^{(i)}(\Lambda') \leq 1 + B_\ell^{(i)} - B_{i+1}^{(i)}(\Lambda') = 1 + A_\ell^{(i)}(\Lambda') = A_\ell^{(i)}(\Lambda)$. This proves the desired equality.

Suppose now that $\widetilde{f}_i(\Lambda) = \Lambda'$ and $\widetilde{e}_i(\Lambda') = \Lambda$ hold. In this case, all the entries of $\Lambda'$ equal those of $\Lambda$, except for one entry $\lambda'^{(i)}_\ell$ in the $i$th row which equals $\lambda^{(i)}_\ell - 1$. Therefore

$$\begin{aligned}
\mathrm{wt}(\Lambda') &= \sum_{j=1}^{n} \Big( \sum_{k=1}^{j} \lambda'^{(j)}_k - \sum_{k=1}^{j-1} \lambda'^{(j-1)}_k \Big) \boldsymbol{e}_j \\
&= -\boldsymbol{e}_i + \boldsymbol{e}_{i+1} + \sum_{j=1}^{n} \Big( \sum_{k=1}^{j} \lambda'^{(j)}_k - \sum_{k=1}^{j-1} \lambda'^{(j-1)}_k ) \Big) \boldsymbol{e}_j \\
&= \mathrm{wt}(\Lambda) - \alpha_i,
\end{aligned}$$

which is equivalent to $\mathrm{wt}(\Lambda') = \mathrm{wt}(\Lambda) + \alpha_i$.

To conclude the proof of (i) we need to show $\varepsilon_i(\Lambda') = \varepsilon_i(\Lambda) + 1$ and $\varphi_i(\Lambda') = \varphi_i(\Lambda) - 1$. For $1 \le j, \ell \le i$ and any $\Lambda \in \Gamma\text{Ц}(n, \lambda)$ we have

$$A^{(i)}_j(\Lambda - \Delta^{(i)}_\ell(\Lambda)) = \begin{cases} A^{(i)}_j(\Lambda), & 0 \le j < \ell, \\ A^{(i)}_j(\Lambda) - 1, & j = \ell, \\ A^{(i)}_j(\Lambda) - 2, & \ell < j \le i. \end{cases}$$

Suppose $\varphi_i(\Lambda) > 0$ and let $\ell = \max\{j \in \{1, 2, \ldots, i\} \mid A^{(i)}_j = \varphi_i(\Lambda)\}$. Then for all $1 \le j \le i$. $A^{(i)}_j(\Lambda - \Delta^{(i)}_\ell(\Lambda) \le \varphi_i(\Lambda) - 1$ with equality for $j = \ell$. Therefore $\varphi_i(\widetilde{f}_i(\Lambda)) = \varphi_i(\Lambda) - 1$.

Let $1 \le j, \ell \le i$. Then for any $\Lambda \in \Gamma\text{Ц}(n, \lambda)$ we have

$$B^{(i)}_j(\Lambda + \Delta^{(i)}_\ell(\Lambda)) = \begin{cases} B^{(i)}_j(\Lambda) & 1 \le j < \ell, \\ B^{(i)}_j(\Lambda) - 1 & j = \ell, \\ B^{(i)}_j(\Lambda) - 2 & \ell < j \le i. \end{cases}$$

Suppose $\varepsilon_i(\Lambda) > 0$ and let $\ell = \min\{j \in \{1, 2, \ldots, i\} \mid B^{(i)}_j(\Lambda) = \varepsilon_i(\Lambda)\}$. Then $B^{(i)}_j(\Lambda + \Delta^{(i)}_\ell(\Lambda)) \le \varepsilon_i(\Lambda) - 1$ with equality for $j = \ell$. Thus $\varepsilon_i(\widetilde{e}_i(\Lambda)) = \varepsilon_i(\Lambda) - 1$.

For property (ii), we verify that for all $\Lambda \in \Gamma\text{Ц}(n, \lambda)$ we have

$$\varphi_i(\Lambda) - \varepsilon_i(\Lambda) = \langle \mathrm{wt}(\Lambda), \alpha_i^\vee \rangle.$$

We have $\varphi_i(\Lambda) = \max\{A_1^{(i)}, A_2^{(i)}, \ldots, A_i^{(i)}\}$ and $\varepsilon_i(\Lambda) = \max\{B_1^{(i)}, B_2^{(i)}, \ldots, B_i^{(i)}\}$. We will use relation (3.2.8). Writing $A_k^{(i)} = A_k^{(i)}(\Lambda)$ for brevity we have

$$\begin{aligned}\varphi_i(\Lambda) - \varepsilon_i(\Lambda) &= \max\{A_1^{(i)}, A_2^{(i)}, \ldots, A_i^{(i)}\} - \max\{A_1^{(i)} - A_0^{(i)}, A_2^{(i)} - A_0^{(i)}, \ldots, A_i^{(i)} - A_0^{(i)}\}\\ &= \max\{A_1^{(i)}, A_2^{(i)}, \ldots, A_i^{(i)}\} - \max\{A_1^{(i)}, A_2^{(i)}, \ldots, A_i^{(i)}\} + A_0^{(i)}\\ &= A_0^{(i)}\\ &= \sum_{k=0}^{i}(\lambda_k^{(i)} + \lambda_{k+1}^{(i)} - \lambda_k^{(i-1)} - \lambda_{k+1}^{(i+1)})\\ &= 2\sum_{k=1}^{i}\lambda_k^{(i)} - \sum_{k=1}^{i-1}\lambda_k^{(i-1)} - \sum_{k=1}^{i+1}\lambda_k^{(i+1)}.\end{aligned}$$

On the other hand, using the first expression for the weight function, we have

$$\mathrm{wt}(\Lambda) = \sum_{j=1}^{n}\big(\sum_{k=1}^{j}\lambda_k^{(j)} - \sum_{k=1}^{j-1}\lambda_k^{(j-1)}\big)\boldsymbol{e}_j$$

so using

$$\langle \boldsymbol{e}_j, \alpha_i^\vee\rangle = \langle \omega_j - \omega_{j-1}, \alpha_i^\vee\rangle = \delta_{ji} - \delta_{j-1,i}$$

we get

$$\begin{aligned}\langle \mathrm{wt}(\Lambda), \alpha_i^\vee\rangle &= \langle\sum_{j=1}^{n}\big(\sum_{k=1}^{j}\lambda_k^{(j)} - \sum_{k=1}^{j-1}\lambda_k^{(j-1)}\big)\boldsymbol{e}_j, \alpha_i^\vee\rangle\\ &= \sum_{j=1}^{n}\big(\sum_{k=1}^{j}\lambda_k^{(j)} - \sum_{k=1}^{j-1}\lambda_k^{(j-1)}\big)(\delta_{ji} - \delta_{j-1,i})\\ &= 2\sum_{k=1}^{i}\lambda_k^{(i)} - \sum_{k=1}^{i-1}\lambda_k^{(i-1)} - \sum_{k=1}^{i+1}\lambda_k^{(i+1)}\end{aligned}$$

This shows that $\varphi_i(\Lambda) - \varepsilon_i(\Lambda) = \langle \mathrm{wt}(\Lambda), \alpha_i^\vee\rangle$, which also equals the coefficient of $\omega_i$ in $\mathrm{wt}(\Lambda)$, proving the second and third equality in (3.2.10).

Lastly, since $\varphi_i(\Lambda)$ and $\epsilon_i(\Lambda)$ are never $-\infty$, condition (iii) in the definition of a crystal is void. □

### 3.2.1 Crystal isomorphism

In this section we prove our second main result which says that the natural bijection $\mathcal{T}$ from $\mathrm{SSYT}(n, \lambda)$ to $\Gamma\text{Ц}(n, \lambda)$ described in Section 2.1.4 is an isomorphism of crystals.

We will let $T_i$ denote the $i$th row of a semistandard Young tableaux $T$, and $T_{\geq \ell}$ the subtableau obtained by deleting the first $\ell - 1$ rows, and similarly for $T_{\leq \ell}$:

$$T = \begin{matrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{matrix} \qquad T_{\geq \ell} = \begin{matrix} T_\ell \\ T_{\ell+1} \\ \vdots \\ T_n \end{matrix} \qquad T_{\leq \ell} = \begin{matrix} T_1 \\ T_2 \\ \vdots \\ T_\ell \end{matrix}$$

The following counting lemma will be useful.

**Lemma 3.2.4.** *Let $\Lambda \in \Gamma Ц(n, \lambda)$ and $T = \mathfrak{T}(\Lambda)$.*

(a) *For all integers $k$ with $1 \leq k \leq n$, the number of letters $i$ in $T_k$ is equal to $\lambda_k^{(i)} - \lambda_k^{(i-1)}$.*

(b) *$a_j^{(i)}(\Lambda)$ counts the number of $i$'s in $T_j$ minus the number of $(i+1)$'s in $T_{j+1}$.*

(c) *$b_j^{(i)}(\Lambda)$ counts the number of $(i+1)$'s in $T_j$ minus the number of $i$'s in $T_{j-1}$.*

(d) *$A_\ell^{(i)}(\Lambda)$ counts the number of $i$'s in $T_{\geq \ell}$ minus the number of $(i+1)$'s in $T_{\geq \ell+1}$.*

(e) *$B_\ell^{(i)}(\Lambda)$ counts the number of $(i+1)$'s in $T_{\leq \ell}$ minus the number of $i$'s in $T_{\leq \ell-1}$.*

*Proof.* (a) The number of boxes in $T_k$ containing a letter from $\{1, 2, \ldots, i\}$ is $\lambda_k^{(i)}$. Then (b) and (c) are immediate by part (a) and the definitions, (3.2.1), of the diamond numbers. Now (d) and (e) follow from parts (b) and (c). $\square$

**Theorem 3.2.5.** *Let $n$ be a positive integer and $\lambda$ a partition with $n$ or fewer parts. The bijection $\mathfrak{T}$ from $\Gamma Ц(n, \lambda)$ to $\mathrm{SSYT}(n, \lambda)$ given in Section 2.1.4 is an isomorphism of crystals.*

*Proof.* Let $\Lambda \in \Gamma Ц(n, \lambda)$, and let $T = \mathfrak{T}(\Lambda)$.

$\underline{\mathrm{wt}(\Lambda) = \mathrm{wt}(T)}$: For each $i \in \{1, 2, \ldots, n-1\}$, by Lemma 3.2.4(a), $\sum_{j=1}^{i} \lambda_j^{(i)} - \sum_{j=1}^{(i-1)} \lambda_j^{(i-1)}$ equals $N_i(T)$, since the letter $i$ cannot occur below the $i$th row in an SSYT.

Let $i \in \{1, 2, \ldots, n-1\}$ be arbitrary. *In the rest of the proof, "bracketing" refers to $i$-bracketing.* Put $A_k^{(i)} = A_k^{(i)}(\Lambda)$ and $B_k^{(i)} = B_k^{(i)}(\Lambda)$ for brevity.

$\underline{\varphi_i(\Lambda) = \varphi_i(T)}$: By definition, $\varphi_i(T)$ is the number of unbracketed $i$'s in $T$. So $\varphi_i(T) \geq \varphi_i(T_{\geq j})$ for any $j \in \{1, 2 \ldots, i\}$. Let $j_1 \geq j_2 \geq \cdots \geq j_k$ be all the rows of $T$ containing at least one unbracketed $i$. Then $\varphi_i(T_{\geq} j_1) = A_{j_1}^{(i)}$ by Lemma 3.2.4(d). Furthermore, $A_{j_1}^{(i)} > A_k^{(i)}$ for $k = i, i-1, \ldots, j_1+1$. Next, $\varphi_i(T_{\geq j_2}) = \varphi_i(T_{\geq j_1}) + \varphi_i(T_{\geq j_2}/T_{\geq j_1}) = A_{j_1}^{(i)} + (A_{j_2}^{(i)} - A_{j_1}^{(i)}) = A_{j_2}^{(i)}$. (Here $T_{\geq j_2}/T_{\geq j_1}$ denotes the subtableau of $T$ consisting of row $j_2$ through row $j_1 - 1$.) And $A_{j_2}^{(i)} > A_k^{(i)}$ for $k = j_1, j_1 - 1, \ldots, j_2 + 1$. Continuing recursively, we eventually obtain that $\varphi_i(T) = \varphi_i(T_{\geq j_k}) = A_{j_k}^{(i)} > A_j^{(i)}$ for $j > j_k$. It remains to be shown that $A_j^{(i)} \leq A_{j_k}^{(i)}$ for $i = j_k - 1, j_k - 2, \ldots, 1$. Since $j_k$ is the top row having unbracketed $i$'s, we have $A_j^{(i)}(\Lambda_{\leq j_k - 1}) \leq 0$ for $j = j_k - 1, j_k - 2, \ldots, 1$, where $\Lambda_{\leq r}$ is defined to be $\mathcal{T}^{-1}(T_{\leq r})$ for all $r$. Since $A_j^{(i)}(\Lambda) - A_{j_k}^{(i)}(\Lambda) = A_j^{(i)}(\Lambda_{\leq j_k - 1})$, this shows the required inequality.

$\underline{\varepsilon_i(\Lambda) = \varepsilon_i(T)}$: This part can be proved completely analogously to the case of $\varphi_i$. But it also follows from the case of $\varphi_i$ and the fact that we already know that $\Gamma Ц(n, \lambda)$ and $\mathrm{SSYT}(n, \lambda)$ are crystals, and hence by property (ii) in the definition of crystal and that $\mathrm{wt}(\Lambda) = \mathrm{wt}(T)$,

$$\varepsilon_i(\Lambda) = \varphi_i(\Lambda) - \langle \mathrm{wt}(\Lambda), \alpha_i^\vee \rangle = \varphi_i(T) - \langle \mathrm{wt}(T), \alpha_i^\vee \rangle = \varepsilon_i(T).$$

$\underline{\mathcal{T}\big(\widetilde{f}_i(\Lambda)\big) = \widetilde{f}_i\big(\mathcal{T}(\Lambda)\big)}$: We have seen already that $\varphi_i(\Lambda) = \varphi_i(T)$. Thus $\widetilde{f}_i(\Lambda) \neq 0$ iff $\widetilde{f}_i(T) \neq 0$. Suppose $\widetilde{f}_i(\Lambda) \neq 0$. Put $\Lambda' = \widetilde{f}_i(\Lambda) = \Lambda - \Delta_\ell^{(i)}(\Lambda)$, where $\ell = \max\{i \in \{1, 2, \ldots, i\} \mid A_j^{(i)}(\Lambda) = \varphi_i(\Lambda)\}$. By definition of the bijection $\mathcal{T}$, the SSYT $\mathcal{T}(\Lambda')$ is obtained from $T$ by changing the rightmost $i$ in row $\ell$ to $i+1$. On the other hand, $\widetilde{f}_i(T)$ is obtained by changing the rightmost unbracketed $i$ in $T$ to $i+1$. So we must show that $\ell$ equals the row index of the rightmost unbracketed $i$ in $T$. First we show that there is an unbracketed $i$ in row $\ell$ of $T$. To do this we derive a series of equivalences. Let $j \in \{1, 2, \dots, i\}$ be arbitrary. Then:

$$
\begin{aligned}
&T \text{ has an unbracketed } i \text{ in row } j \\
\Leftrightarrow\ &\varphi_i(T_{\geq j}) > \varphi_i(T_{\geq j+1}) \\
\Leftrightarrow\ &\varphi_i(\Lambda_{\geq j}) > \varphi_i(\Lambda_{\geq j+1}), \quad \text{where } \Lambda_{\geq k} := \mathcal{T}^{-1}(T_{\geq k}) \\
\Leftrightarrow\ &\max\{A_k^{(i)}(\Lambda_{\geq j}) \mid k = 1, 2, \dots, i\} > \max\{A_k^{(i)}(\Lambda_{\geq j+1}) \mid k = 1, 2, \dots, i\} \\
\Leftrightarrow\ &\max\{A_k^{(i)}(\Lambda) \mid k = j, j+1, \dots, i\} > \max\{A_k^{(i)}(\Lambda) \mid k = j+1, j+2, \dots, i\} \\
\Leftrightarrow\ &A_j^{(i)}(\Lambda) > A_k^{(i)}(\Lambda) \quad \text{for all } k \in \{j+1,\, j+2,\, \dots,\, i\}.
\end{aligned}
$$

The penultimate equivalence holds by the counting lemma, Lemma 3.2.4(d), and that the first row of $T_{\geq j}$ is the $j$th row of $T$ and so on. Now, by definition of $\ell$ we do indeed have

$$A_\ell^{(i)}(\Lambda) > A_k^{(i)}(\Lambda) \quad \text{for all } k \in \{\ell+1,\, \ell+2,\, \dots,\, i\},$$

and therefore by the above series of equivalences there is at least one unbracketed $i$ in row $\ell$ of $T$.

It remains to show that $\ell$ is the row of the rightmost unbracketed $i$ in $T$. Since any $i$ directly to the right of an unbracketed $i$ is itself unbracketed, any unbracketed $i$ further to the right would have to occur among the top $\ell - 1$ rows of $T$. Any unbracketed $i$ among the top $\ell - 1$ rows of $T$ would remain unbracketed when considered as an entry of the truncated tableau $T_{\leq \ell-1}$. So it suffices to show that $T_{\leq \ell-1}$ has no unbracketed $i$'s, or equivalently, that $\varphi_i(T_{\leq \ell-1}) = 0$. Let $\Lambda_{\leq \ell-1} = \mathcal{T}^{-1}(T_{\leq \ell-1})$. As previously shown, $\varphi_i(T_{\leq \ell-1}) = \varphi_i(\Lambda_{\leq \ell-1})$. By Lemma 3.2.4(d), for all $1 \leq j \leq i$:

$$A_j^{(i)}(\Lambda_{\leq \ell-1}) = A_j^{(i)}(\Lambda) - A_\ell^{(i)}(\Lambda),$$

which is less than or equal to zero by definition of $\ell$. Hence $\varphi_i(\Lambda_{\leq \ell-1}) = 0$.

$\underline{\mathcal{T}\big(\widetilde{e}_i(\Lambda)\big) = \widetilde{e}_i\big(\mathcal{T}(\Lambda)\big)}$: We know that $\varepsilon_i(\Lambda) = \varepsilon_i(T)$. Thus $\widetilde{e}_i(\Lambda) = 0$ iff $\widetilde{e}_i(T) = 0$. Suppose that $\widetilde{e}_i(\Lambda) \neq 0$. Put $\Lambda' = \widetilde{e}_i(\Lambda) = \Lambda + \Delta_\ell^{(i)}(\Lambda)$, where

$$\ell = \min\big\{j \in \{1, 2, \dots, i\} \mid B_j^{(i)}(\Lambda) = \varepsilon_i(\Lambda)\big\}.$$

Also recall that

$$\varepsilon_i(\Lambda) = \max\{B_1^{(i)}(\Lambda),\, B_2^{(i)}(\Lambda),\, \dots,\, B_i^{(i)}(\Lambda)\}.$$

By definiton of the bijection $\mathcal{T}$, the SSYT $\mathcal{T}(\Lambda')$ is obtained from $T$ by changing the leftmost $i+1$ in row $\ell$ of $T$ to $i$. On the other hand, $\widetilde{e}_i(T)$ is the SSYT obtained from $T$ by changing the leftmost unbracketed $i+1$ to $i$. So we must show that $\ell$ equals the row index of the row in $T$ which contains the leftmost unbracketed $i+1$.

First we show that row $\ell$ of $T$ contains an unbracketed $i+1$. For this, we derive an equivalent condition. For all $j \in \{1, 2, \ldots, i\}$ we have:

$$\begin{aligned}
&T \text{ contains an unbracketed } i+1 \text{ in row } j+1 \\
\Leftrightarrow\ & \varepsilon_i(T_{\leq j+1}) > \varepsilon_i(T_{\leq j}) \\
\Leftrightarrow\ & \varepsilon_i(\Lambda_{\leq j+1}) > \varepsilon_i(\Lambda_{\leq j}) \quad \text{where } \Lambda_{\leq k} := \mathcal{T}^{-1}(T_{\leq k}) \\
\Leftrightarrow\ & \max\{B_k^{(i)}(\Lambda_{\leq j+1}) \mid k = 1, 2, \ldots, i\} > \max\{B_k^{(i)}(\Lambda_{\leq j}) \mid k = 1, 2, \ldots, i\} \\
\Leftrightarrow\ & B_{j+1}^{(i)} > B_k^{(i)} \quad \text{for all } k \in \{1, 2, \ldots, j\}
\end{aligned}$$

This condition holds for $j+1 = \ell$ by definition of $\ell$. Thus $T$ contains an unbracketed $i+1$ in row $\ell$.

Next we show that no row of $T$ contains an unbracketed $i+1$ further to the left. Such a row $j$ would have to be below $\ell$, i.e. $j \geq \ell + 1$. By the above equivalences we would get

$$B_j^{(i)}(\Lambda) > B_k^{(i)}(\Lambda) \quad \text{for all } k \in \{1, 2, \ldots, j-1\}.$$

In particular, $B_j^{(i)}(\Lambda) > B_\ell^{(i)}(\Lambda)$, which contradicts the definition of $\ell$. This finishes the proof that $\mathcal{T}(\widetilde{e}_i(\Lambda)) = \widetilde{e}_i(\mathcal{T}(\Lambda))$.

<u>Alternative proof that $\mathcal{T}\big(\widetilde{e}_i(\Lambda)\big) = \widetilde{e}_i\big(\mathcal{T}(\Lambda)\big)$</u>: As is well-known, if a function between crystals preserve the string length functions and intertwines the $\widetilde{f}_i$ crystal operators, then it automatically intertwines the $\widetilde{e}_i$ crystal operators. We illustrate this for the convenience of the reader. We know that $\varepsilon_i(\Lambda) = \varepsilon_i(T)$. Thus $\widetilde{e}_i(\Lambda) = 0$ iff $\widetilde{e}_i(T) = 0$. Suppose that $\widetilde{e}_i(\Lambda) \neq 0$. Since $\varphi_i(\mathcal{T}(\widetilde{e}_i(\Lambda)) = \varphi_i(\widetilde{e}_i(\Lambda)) \geq 1$. Thus we have

$$\begin{aligned}
\mathcal{T}\big(\widetilde{e}_i(\Lambda)\big) &= \widetilde{e}_i\widetilde{f}_i\big(\mathcal{T}(\widetilde{e}_i(\Lambda))\big) \\
&= \widetilde{e}_i\mathcal{T}\big(\widetilde{f}_i\widetilde{e}_i(\Lambda)\big) \quad \text{by } \mathcal{T}\widetilde{f}_i = \widetilde{f}_i\mathcal{T} \\
&= \widetilde{e}_i\big(\mathcal{T}(\Lambda)\big).
\end{aligned}$$

□

# CHAPTER 4. BIJECTIONS AND CRYSTAL STRUCTURE IN TYPE C

## 4.1 Introduction

One initial goal for this project was to determine a bijection between Kashiwara-Nakashima tableaux, which are very well-understood in combinatorial terms, and Želobenko patterns. Given the simplicity of the bijection between semistandard Young tableaux and Gelfand-Tsetlin patterns, it seemed reasonable to try to approach the problem by explicitly constructing the bijection in the form of a deletion algorithm. Furthermore, one interesting feature of the type A algorithm is that, after deleting the boxes containing the symbol $i$, for instance, one may view the resulting intermediate tableau as being an element of $\mathrm{SSYT}(i-1, \lambda_{i-1})$ for some partition $\lambda_{i-1}$ contained in the partition $\lambda_i$ from the previous step of the algorithm. Similarly, the associated Gelfand-Tsetlin pattern may be truncated at the $i-1$'st row to obtain a valid pattern in $\Gamma Ц(i-1, \lambda_{i-1})$. There are various applications in the representation theory of $S_n$ and of $\mathfrak{gl}_n$ where, far from being coincidental, this perspective may be used to study induced and restricted representations of the algebraic structures in question. Thus, a useful deletion algorithm in type C would ideally have similar features.

One difference between type A and type C is that when it comes to the latter, everything comes in pairs. In terms of KNT, this is evident from the barred and unbarred varieties of letters in $\mathcal{A}_{C_n}$. For ŽP, it is the rows that come in pairs of equal length. In order for a step of a deletion algorithm to admit an interpretation in terms of the restriction of a representation, it is reasonable to consider removing an associated pair of symbols, or rows, to arrive at the next step. For the process to be as reminiscent as possible of the type A version, it also seemed natural to attempt to remove the pair of symbols or rows associated with the weight $\omega_i$ at step $i$: the boxes containing the symbols $i$ and $\bar{\imath}$ and the rows containing $\lambda_j^{(i)}$ and $\lambda_j^{(i)'}$, to be exact. From the perspective of ŽP this seems exactly as easy to accomplish as it is in type A, but from the perspective of KNT there is a significant complication, in that the ordering of $\mathcal{A}_{C_n}$ places those boxes closer to the "center" of the tableau the closer $i$ is to $n$. So, while deletion in type A involves peeling boxes off of the outside of a tableau, a procedure that obviously preserves semistandardness, deletion in type C involves deleting boxes from the middle of a tableau, an operation that leaves the status of the resulting object very much in question. However, as we defined in the introduction to this thesis, there exist several algorithms, the jeux de taquin, for just such applications. Indeed, it may be useful to imagine, in the course of the type A pattern-tableau bijection, that each step involves the application of the type A jeu de taquin on deleted boxes. It simply happens to be the case that since they are already outer corners due to the structure of type A tableaux, it is a very easy game to play.

The formulation of the symplectic jeu de taquin developed by Sheats and expanded upon by Lecouvey is not the only one available, and this is partly because Kashiwara-Nakashima tableaux are not the only interpretation of type C tableaux. As discussed by Sheats in [20], Lecouvey in [15] and Bump and Schilling in [3], King ([12]) and De Concini ([4]) gave different interpretations of symplectic tableaux, with different applications in representation theory, that may be shown to be equivalent to KNT. However, as the literature on crystal basis theory in particular focuses on KNT, we opted to favor that standard.

Applying the SJDT to a given KNT $T$ is straightforward once one makes the requisite effort to understand the process, but applying it sequentially as the steps of a deletion algorithm raises several new questions. After all, a step of the deletion algorithm may involve deleting all symbols $\bar{i}$ from $T$, followed by recording its shape as a row of the associated Želobenko pattern and then repeating for all symbols $i$. However, since none of the $\bar{i}$'s or $i$'s in $T$ need be outer corners, we must also select an order in which to delete them and then, for each occurrence of each symbol, perform the SJDT until its associated puncture has become one and been removed. This should complete one step of the overall deletion algorithm, and result in a tableau that is valid with respect to $\mathcal{A}_{C_{i-1}}$. Since the application of the JDT on a type A tableau as a part of the algorithm is trivial, that analogy was not helpful in determining a useful order in which to start deleting symbols in our KNT $T$. However, since the SJDT forces punctures to move to the southeast out of $T$, it seemed most natural to begin by deleting the east-most occurrence of the symbol in question so as to remove potential obstructions to its exit. As we will describe, this approach is successful in the case where $n = 2$, so extending it for $n > 2$ seems plausible.

Even for $n = 2$, the bijection is significantly more intricate than that in type A. To summarize the desired algorithm, given $T \in \mathrm{KNT}(2, \lambda)$, first delete all of the $\overline{2}$'s, record the resulting shape, delete all of the 2's, record the shape, delete all of the $\overline{1}$'s, record the shape, and you have now constructed a pattern $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(2, \lambda)$. This would almost be as easy as treating the SJDT as though it were the type A JDT, with the exception of tableaux containing the following subtableau:

$$T_0 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \overline{1} \\ \hline \end{array}.$$

As described in section 1.1.7, if the $\overline{2}$ were deleted and the $\overline{1}$ simply slid to the left, there would be a 1 and a $\overline{1}$ in the same column, which always violates the distance condition on columns for KNT. However, when we apply SJDT we obtain the following sequence of steps:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \overline{1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline * & \overline{1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 2 \\ \hline * & \overline{2} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \overline{2} & * \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \overline{2} \\ \cline{1-1} \end{array},$$

at which point we may delete the newly-created $\overline{2}$, as it is the next-east-most one in the tableau, resulting in tableau

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}.$$

Note that the above algorithm applied to $T_0$ will result in the $\check{\mathrm{Z}}\mathrm{P}$

$$\Gamma_0 = \left\{ \begin{array}{cccc} 2 & & 2 & \\ & 2 & & 0 \\ & & 0 & \\ & & & 0 \end{array} \right\}.$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \overline{1} \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \overline{3} & \overline{1} \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \overline{3} & \overline{2} \\ \hline \end{array}.$$

Figure 4.1: Blocks with interesting deletion algorithm steps in $\mathrm{KNT}(3, \lambda)$ .

If one were to start from $\Gamma_0$ and try to recreate the associated tableau by naively noting that its entries seem to imply the presence of two 2's and two $\overline{2}$'s in the following configuration:

$$T_1 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \overline{2} & \overline{2} \\ \hline \end{array},$$

then one might be surprised to discover that this is not actually an admissible KNT in the sense defined in section 1.1.3. The reason for this is its violation of the distance condition on adjacent columns, with the choice of $i = j = 2$. Observe, however, that it is a tableau in DC2 inadmissible form. The idea behind the algorithm below for the case of $n = 2$, then, is to take the maximal (possibly empty) subtableau exhibiting $T_0$'s behavior from any $T \in \mathrm{KNT}(2, \lambda)$ and begin by swapping it with a block resembling $T_1$ of the same size. This can always be done in a unique way, so this gives a bijection between KNT and what we call KNT$'$. Now, deletion and recording may be performed as desired to generated the ŽP $\Gamma$ associated with $T$. Going from pattern to tableau involves carefully replacing the correct quantity of each symbol to generate an appropriate KNT$'$, before converting it back into a KNT.

So far, our attempts to adapt this to general $n$ have not succeeded, as for $n > 2$ the number of subtableaux requiring non-trivial SJDT application is greater than 1 and how different types of such blocks should interact with one another in larger tableaux is not immediately obvious. In particular, determining how to correctly fill in a KNT$'$ given a general $\Gamma$ will likely require some additional insight.

What is true in general is that applying $\Phi$, the bijection between the sets of admissible and coadmissible columns, column-wise to tableaux gives a bijection between admissible tableaux and tableaux in DC2 inadmissible form. This suggests the possibility of extending the strategy of the rank 2 bijection if a means of constructing a tableau associated with a given pattern were found.

Beyond type $C_2$, in section 3.3 we define hook patterns in type $C_n$ and prove a bijection between hook patterns and hook tableaux. The crystal structure of row patterns, a special case of hook patterns, is then proved in type $C_n$. We conjecture that a similar proof strategy will give the crystal structure of column patterns, similarly defined, and indeed for hook patterns in general.

## 4.2 Weight-preserving bijection in type $C_2$.

### 4.2.1 KNT $\leftrightarrow$ KNT$'$

We define $\mathrm{KNT}'(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$ as follows: let $T \in \mathrm{KNT}(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$. To obtain $T' \in \mathrm{KNT}'(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$, find the maximal occurrence of the subtableau

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & \dots & 1 & 2 & 2 & \dots & 2 \\ \hline \overline{2} & \dots & \overline{2} & \overline{2} & \overline{1} & \dots & \overline{1} \\ \hline \end{array}}_{2k+1} \text{ or } \underbrace{\begin{array}{|c|c|c|c|c|c|}\hline 1 & \dots & 1 & 2 & \dots & 2 \\ \hline \overline{2} & \dots & \overline{2} & \overline{1} & \dots & \overline{1} \\ \hline \end{array}}_{2k}$$

and replace it with

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 2 & \dots & 2 & 2 & 2 & \dots & 2 \\ \hline \overline{2} & \dots & \overline{2} & \overline{2} & \overline{2} & \dots & \overline{2} \\ \hline \end{array}}_{2k+1} \text{ or } \underbrace{\begin{array}{|c|c|c|c|c|c|}\hline 2 & \dots & 2 & 2 & \dots & 2 \\ \hline \overline{2} & \dots & \overline{2} & \overline{2} & \dots & \overline{2} \\ \hline \end{array}}_{2k},$$

respectively, noting that in either case $k$ 1s and $k$ $\overline{1}$s have been exchanged for $k$ 2s and $k$ $\overline{2}$s. Therefore, $\mathrm{wt}(T') = \mathrm{wt}(T)$. It is clear that one can convert a tableau $T' \in \mathrm{KNT}'(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$ back to a tableau $T \in \mathrm{KNT}(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$ by reversing the above process. Tableaux in $\mathrm{KNT}'(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$ violate the distance condition on KNT when $k \geq 1$, but they still have weakly increasing rows and strictly increasing columns.

### 4.2.2 KNT$'$ $\to$ $\check{\mathbf{Z}}\mathbf{P}$

Let $T' \in \mathrm{KNT}'(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$. To obtain $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$, perform the following procedure:

1. Remove $\overline{2}$s starting from the rightmost box available. $\overline{1}$s left with punctures above them then slide up, and $\overline{1}$s with punctures to their left then slide left, in that order of preference.

2. Remove 2s starting from the rightmost box available. $\overline{1}$s with punctures above them then slide up.

3. Remove $\overline{1}$s.

The first row of $\Gamma$ corresponds to the shape $(\lambda_1^{(2)}, \lambda_2^{(2)})$, the second to the shape $(\lambda_1^{(2)'}, \lambda_2^{(2)'})$ of the tableau resulting from step (1), the third to the shape $(\lambda_1^{(1)})$ of the tableau resulting from step (2), and the fourth to the shape $(\lambda_1^{(1)'})$ of the tableau resulting from step (3).

### 4.2.3 $\check{\mathbf{Z}}\mathbf{P} \to \mathrm{KNT}'$

Let $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(2, (\lambda_1^{(2)}, \lambda_2^{(2)}))$:

$$\Gamma = \left\{ \begin{array}{ccccc} \lambda_1^{(2)} & & \lambda_2^{(2)} & & \\ & \lambda_1^{(2)'} & & \lambda_2^{(2)'} & \\ & & \lambda_1^{(1)} & & \\ & & & \lambda_1^{(1)'} & \end{array} \right\}$$

To obtain $T' \in \mathrm{KNT}'(2, (\lambda_1^{(2)}, \lambda_2^{(2)}))$, we reconstruct the tableau as follows:

1. Fill a Young diagram of shape $(\lambda_1^{(1)'})$ with 1s.

2. Extend the shape to $(\lambda_1^{(2)'}, \lambda_2^{(2)'})$ and insert $(\lambda_1^{(2)'} + \lambda_2^{(2)'}) - \lambda_1^{(1)}$ 2s starting from the leftmost box available. If two boxes are tied, choose the one in the first row.

3. Extend the shape to $(\lambda_1^{(2)}, \lambda_2^{(2)})$ and add $\lambda_2^{(2)} + (\lambda_1^{(2)} - \lambda_1^{(2)'}) - ((\lambda_1^{(2)'} + \lambda_2^{(2)'}) - \lambda_1^{(1)})$ $\overline{2}$s to the first row. Fill the rest of the $\overline{2}$s into row 2, and then add $\lambda_1^{(1)} - \lambda_1^{(1)'}$ $\overline{1}$s in the remaining boxes.

**Remark 4.2.1.** In (3), $(\lambda_2^{(2)} + \lambda_1^{(2)} - \lambda_1^{(2)'}) - ((\lambda_1^{(2)'} + \lambda_2^{(2)'}) - \lambda_1^{(1)})$ is the number of things needed to support $\overline{1}$s in row 2 plus the number of $\overline{2}$s in row 1 which are not supporting $\overline{1}$s in row 2 minus the number of 2s, which can do the same job.

### 4.2.4 Proof of theorem

**Theorem 4.2.2.** *For any partition $\lambda$ with $\ell(\lambda) \leq 2$, the algorithms described above provide a bijection between $KNT(2, \lambda)$ and $\check{Z}P(2, \lambda)$.*

*Proof.* Let $\lambda = (\lambda_1^{(2)}, \lambda_2^{(2)})$. Starting from pattern $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(2, \lambda)$, construct a tableau $T \in \mathrm{KNT}(2, \lambda)$ by applying the $\check{\mathrm{Z}}\mathrm{P} \to \mathrm{KNT}' \to \mathrm{KNT}$ procedure. Note that by inspection of $T$ we can recover most of the entries of $\Gamma$: $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are apparent, $\lambda_1^{(1)'}$ is simply the number of 1s in $T$, and $\lambda_1^{(1)} - \lambda_1^{(1)'}$ is the number of $\overline{1}$s. We also know that there are $(\lambda_1^{(2)'} + \lambda_2^{(2)'}) - \lambda_1^{(1)}$ 2s in $T$, so by finding either $\lambda_1^{(2)'}$ or $\lambda_2^{(2)'}$ we can fill in the rest of $\Gamma$. If $\lambda_2^{(2)'} \leq \lambda_1^{(1)'}$ then we're done, as the number of 2s in row 2 is equal to $\lambda_2^{(2)'}$, since $\overline{1}$s can't be below 1s and since $\overline{2}$s can't have occupied any of the first $\lambda_2^{(2)'}$ boxes of row 2.

Suppose that $\lambda_2^{(2)'} > \lambda_1^{(1)'}$. If that is the case, there must be something other than 2s in row 2, so in particular we may consider the skew subtableau $S$ of $T$ of shape $(\lambda_1^{(2)}, \lambda_2^{(2)})/(T_1(1)+T_2(2), T_2(1)+T_2(2))$ whose entries are all $\overline{2}$s and $\overline{1}$s:

$$\begin{array}{cccccc|ccc|cccccc} & & & & & & \overline{2} & \cdots & \overline{2} & \overline{2} & \cdots & \overline{2} & \overline{1} & \cdots & \overline{1} \\ \overline{2} & \cdots & \overline{2} & \overline{1} & \cdots & \overline{1} & \overline{1} & \cdots & \overline{1} & & & & & & \\ & & & & & & & \ell & & & & & & & \end{array}.$$

Note that there may be a rectangular subtableau of $S$ of shape $(\ell, \ell)$ comprised of $\overline{2}$s with $\overline{1}$s beneath them. This can be removed to give a skew tableau $S'$ with no such overlap:

$$\begin{array}{cccccccccccc} & & & & & & \overline{2} & \cdots & \overline{2} & \overline{1} & \cdots & \overline{1} \\ \overline{2} & \cdots & \overline{2} & \overline{1} & \cdots & \overline{1} & & & & & & \end{array}.$$

Now observe that, since $T$ is in $\mathrm{KNT}'(2, \lambda)$, the $\overline{2}$s in $S'$ have 2s above them. Thus we have $T_2(\overline{2}) = \lambda_2^{(2)} - \lambda_2^{(2)'}$, and $\lambda_2^{(2)'} = \lambda_2^{(2)} - T_2(\overline{2})$, both quantities that we can read off of $T$. $\square$

**Proposition 4.2.3.** *The above bijection preserves weights.*

*Proof.* Let $T \in \mathrm{KNT}(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$ with $\mathrm{wt}(T)=(\alpha_1,\alpha_2)$ and let $T'$ be the associated element of $\mathrm{KNT}'(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$. As previously discussed, the KNT $\leftrightarrow$ KNT$'$ procedure is weight-preserving, so $\mathrm{wt}(T')=\mathrm{wt}(T)$. Applying the KNT$'$ $\to$ ŽP procedure to obtain $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(2,(\lambda_1^{(2)},\lambda_2^{(2)}))$, observe that $\mathrm{wt}(\Gamma)=(2\lambda_1^{(1)'}-\lambda_1^{(1)},2(\lambda_1^{(2)'}+\lambda_2^{(2)'})-(\lambda_1^{(2)}+\lambda_2^{(2)})-\lambda_1^{(1)})$ by definition. Since $2\lambda_1^{(1)'}-\lambda_1^{(1)}=\lambda_1^{(1)'}-(\lambda_1^{(1)}-\lambda_1^{(1)'})$ is the number of 1s minus the number of $\overline{1}$s in $T'$ and $2(\lambda_1^{(2)'}+\lambda_2^{(2)'})-(\lambda_1^{(2)}+\lambda_2^{(2)})-\lambda_1^{(1)}=((\lambda_1^{(2)'}+\lambda_2^{(2)'})-\lambda_1^{(1)})-((\lambda_1^{(2)}+\lambda_2^{(2)})-(\lambda_1^{(2)'}+\lambda_2^{(2)'}))$ is the number of 2s minus the number of $\overline{2}$s in $T'$, $\mathrm{wt}(T)=\mathrm{wt}(\Gamma)$. $\square$

### 4.2.5 Discussion of column combinatorics

One powerful advantage of type A crystals of tableaux is that column reading, for example, gives a way to decompose any tableau into simpler parts. As we have seen, this decomposition is central to the definition of bracketing, and it both makes use of and motivates the definition of tensor products of crystals. For this reason, one strategy we used to attempt to give a crystal structure on Želobenko patterns was to decompose them into what we call *column patterns*, which are simply the ŽP associated with their respective column KNT via the bijection. In rank 2, the following is an exhaustive list of possible column tableaux:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline \overline{2}\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline \overline{2}\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline \overline{1}\\\hline\end{array},\ \begin{array}{|c|}\hline \overline{2}\\\hline \overline{1}\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline \overline{2}\\\hline\end{array},\ \begin{array}{|c|}\hline \overline{1}\\\hline\end{array}.$$

Note that a partial order on this set exists in terms of assembling them into larger tableaux, and that each of them clearly gives rise to a Želobenko pattern given our bijection between the two structures in rank 2. For example,

$$\begin{array}{|c|}\hline 2\\\hline \overline{2}\\\hline\end{array} \quad\longleftrightarrow\quad \left\{\begin{array}{ccc} 1 & & 1 \\ & 1 & & 0 \\ & & 0 & \\ & & & 0 \end{array}\right\} \tag{4.2.1}$$

is such a pairing. In much the same way as one may view a tableau as a horizontal stack of its columns, one may think of a pattern as a stack of its column patterns. The upshot of this is potentially quite profound: if we can understand the bracketing of a tableau $T$ in terms of its columns and we can detect which column patterns $\mathcal{T}^{-1}(T)$ possesses based on its entries, then we have a way of giving explicit formulae for the crystal structure of patterns, circuitous though it may be.

Our efforts to provide the missing piece of this puzzle, the count of each type of column pattern making up a given $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(2, \lambda)$, remain incomplete. In short, even in rank 2 the combinatorics in play are daunting. To illustrate this, we give the following example.

**Example 4.2.4.** Let

$$N\left(\boxed{1}\right)$$

be the number of columns consisting just of one box containing a 1 occurring in the tableau $T$ associated with $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(2, \lambda)$. We claim that

$$N\left(\boxed{1}\right) = \max\{0, \lambda_1^{(1)} - \lambda_2^{(2)}\}.$$

The reason is that $\lambda_2^{(2)}$ is the length of the second row of $T$ before the deletion of any symbols, while $\lambda_1^{(1)}$ is the number of 1's in $T$. The only way for the column containing one box with one 1 in it to occur in $T$ is for the latter number to exceed the former, based on our partial order on rank 2 columns.

Continuing in this fashion makes it possible to account for many of the column counts in similarly straightforward ways, but some of them are much more resistant to this approach. For example, we determined that

$$\begin{aligned} N\left(\begin{array}{|c|}\hline 1 \\ \hline x \\ \hline \end{array}\right) &= (\text{number of 1's}) - N\left(\boxed{1}\right) \\ &= \lambda_1^{(1)} - \max\{0, \lambda_1^{(1)} - \lambda_2^{(2)}\} \\ &= \min\{\lambda_1^{(1)}, \lambda_2^{(2)}\}, \end{aligned}$$

where $x \in \{2, \overline{2}\}$. Further,

$$N\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) = \min\{\lambda_1^{(1)}, \lambda_2^{(2)\prime}\},$$

since after deleting symbols $\overline{2}$, the second row contains only 2's and $\overline{1}$'s and only a 2 may appear below a 1. If $\lambda_1^{(1)} > \lambda_2^{(2)'}$, then $T'$, the tableau at the step of the deletion algorithm where all $\overline{2}$'s have been removed and no 2's have been removed, is of the form

$$T' = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & \cdots & 1 & 1 & \cdots \\ \hline 2 & 2 & \cdots & 2 \\ \cline{1-4} \end{array}$$

and so $\lambda_2^{(2)'}$, the length of the second row at that step, counts the columns of interest. However, if $\lambda_1^{(1)} \leq \lambda_2^{(2)'}$, then we have

$$T' = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & \cdots & 1 & x & \cdots \\ \hline 2 & 2 & \cdots & 2 & y & \cdots \\ \hline \end{array}$$

where either $x = 2, y = \overline{1}$ or $x \in \{2, \overline{1}\}$ and $y$ does not occur. Note that in each case, the block of columns containing a 1 and a 2 may be empty.

Using both of these expressions, we may now obtain

$$\begin{aligned} N\left(\begin{array}{|c|} \hline 1 \\ \hline \overline{2} \\ \hline \end{array}\right) &= N\left(\begin{array}{|c|} \hline 1 \\ \hline x \\ \hline \end{array}\right) - N\left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}\right) \\ &= \min\{\lambda_1^{(1)}, \lambda_2^{(2)}\} - \min\{\lambda_1^{(1)}, \lambda_2^{(2)'}\} \\ &= \begin{cases} 0 & \text{for } \lambda_2^{(2)} \geq \lambda_2^{(2)'} \geq \lambda_1^{(1)} \\ \lambda_1^{(1)} - \lambda_2^{(2)'} & \text{for } \lambda_2^{(2)} \geq \lambda_1^{(1)} \geq \lambda_2^{(2)'} \\ \lambda_2^{(2)} - \lambda_2^{(2)'} & \text{for } \lambda_1^{(1)} \geq \lambda_2^{(2)} \geq \lambda_2^{(2)'} \end{cases} . \end{aligned}$$

Using this approach, the counts described above are radically simpler than what one obtains for several of the other rank 2 columns. The difficulty we encountered was that some of the columns may be accounted for combinatorially in more than one way, and the expressions we produced for two different ways did not obviously agree with one another. The complexity of the situation made it relatively easy to decide to pursue other avenues of investigation rather than continuing to try to reach a breakthrough here. Nevertheless, we offer the relatively quaint conjecture that it is possible to conclude this line of thought and end up with explicit formulae.

With that in mind, the form of the column count expressions that we were able to verify is very interesting in that it seems to suggest a connection with tropical geometry. If one had such expressions for all types of rank 2 column, one could then give explicit formulae for bracketing based on pattern entries, something that does not exist for crystals of tableaux. These expressions would inescapably involve maxima or minima of sums and differences of pattern entries, which may be interpreted as tropical polynomials. We offer no conjecture on the potential significance of this interpretation, but as Bump and Schilling discuss in [3], it is well-known that crystal basis theory and tropical geometry are intertwined in various ways. The possibility that Želobenko patterns may be used to illuminate a new facet of this connection is intriguing.

## 4.3 Type C hook patterns

$$\begin{array}{|c|c|c|c|c|c|} \hline 2 & 2 & \overline{3} & \overline{2} & \overline{2} & \overline{1} \\ \hline \overline{3} \\ \cline{1-1} \overline{1} \\ \cline{1-1} \end{array} \quad \longleftrightarrow \quad \left\{ \begin{array}{cccccccc} 6 & & 1 & & 1 & & & \\ & 5 & & 1 & & 0 & & \\ & & 5 & & 1 & & & \\ & & & 3 & & 1 & & \\ & & & & 2 & & & \\ & & & & & 0 & & \end{array} \right\}$$

A *hook tableau* is a tableau of shape $\lambda = (k, 1, \ldots, 1)$ where $0 \le k \le n$ (such a $\lambda$ is called a *hook partition*). Note that row and column tableaux are special cases of hook tableaux. The bijection between hook tableaux and Želobenko patterns is as follows:

**Theorem 4.3.1.** *Let $n$ be a non-negative integer and $\lambda$ a hook partition. There is a weight-preserving bijection between* $\mathrm{KNT}(n, \lambda)$ *and* $\check{Z}P(n, \lambda)$.

*Proof.* Given a hook tableau $T \in \mathrm{KNT}(n, \lambda)$, we obtain Želobenko pattern $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(n, \lambda)$, called a *hook pattern*, by a process similar to that in the rank 2 case. Begin by recording the shape of $T$. Starting from $i = n$ and decrementing by 1 at each step, delete all symbols $\bar{i}$, perform SJDT and record the new shape, then delete all symbols $i$, perform SJDT and record the new shape. To see that this results in a valid Želobenko pattern, observe that there is at most one of each symbol in the first column, so at each step either one of the 1's in the part of the pattern off the first diagonal is switched with a 0 or it isn't, satisfying interleaving in either case. For symbols $i$ or $\bar{i}$ in the first row, removal simply corresponds to decrementing the number in the first diagonal by the number of symbols present, which obviously preserves interleaving.

Given $\Gamma \in \Gamma\text{Ц}(n, \lambda)$, we may construct the associated tableau $T \in \mathrm{KNT}(n, \lambda)$ by reversing this process, similarly to rank 2. Starting from the bottom of the pattern, any increases along the diagonals of $\Gamma$ represent $i$'s and $\bar{i}$'s being added to the tableau at that step. To ensure that the result is a valid KNT, we must insert the symbols being added in such a way that they respect the ordering on rows and columns. In particular, if the next shape has an additional row, which is visible in the pattern in the form of an additional 0 being increased to a 1, then one of the symbols must have been added to the first column, with the rest going into the first row.

To see that this bijection is weight-preserving, note that adding an $i$ or an $\bar{i}$ to the tableau increments or decrements its weight by the same amount as making the stated changes to the entries of $\Gamma$, by the weight formula in section 1.1.6. $\square$

### 4.3.1 Crystal structure on Type C row patterns

Let $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(n, (k))$ where $0 \leq k \leq n$. Since $\mathcal{T}(\Gamma)$ is a row tableau, we call $\Gamma$ a *row pattern.*

**Theorem 4.3.2.** *Let $\Gamma \in \check{Z}P(n, (k))$ where $0 \leq k \leq n$. The crystal structure on row patterns may be given as follows, with the weight as defined in section 1.1.6:*

*For $i \in \{1, \ldots, n-1\}$,*

$$\widetilde{f}_i(\Gamma) = \begin{cases} \Gamma + \Delta_1^{(i+1)'}(\Gamma) + \Delta_1^{(i)}(\Gamma) & \text{if } \lambda_1^{(i+1)} - \lambda_1^{(i+1)'} > 0 \\ \Gamma - \Delta_1^{(i)}(\Gamma) - \Delta_1^{(i)'}(\Gamma) & \text{if } \lambda_1^{(i+1)} - \lambda_1^{(i+1)'} = 0 \text{ and } \lambda_1^{(i)'} - \lambda_1^{(i-1)} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

*For $i = n$, the above formula is simplified to*

$$\widetilde{f}_i(\Gamma) = \begin{cases} \Gamma - \Delta_1^{(i)'}(\Gamma) & \text{if } \lambda_1^{(i)'} - \lambda_1^{(i-1)} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

*For* $i \in \{1, \ldots, n\}$,

$$\varphi_i(\Gamma) = \lambda_1^{(i+1)} - \lambda_1^{(i+1)'} + \lambda_1^{(i)'} - \lambda_1^{(i-1)}.$$

*For* $i \in \{1, \ldots, n-1\}$,

$$\widetilde{e}_i(\Gamma) = \begin{cases} \Gamma - \Delta_1^{(i+1)'}(\Gamma) - \Delta_1^{(i)}(\Gamma) & \text{if } \lambda_1^{(i+1)'} - \lambda_1^{(i)} > 0 \\ \Gamma + \Delta_1^{(i)}(\Gamma) + \Delta_1^{(i)'}(\Gamma) & \text{if } \lambda_1^{(i+1)'} - \lambda_1^{(i)} = 0 \text{ and } \lambda_1^{(i)} - \lambda_1^{(i)'} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

*For* $i = n$, *the above formula is simplified to*

$$\widetilde{e}_i(\Gamma) = \begin{cases} \Gamma + \Delta_1^{(i)'}(\Gamma) & \text{if } \lambda_1^{(i)} - \lambda_1^{(i)'} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

*For* $i \in \{1, \ldots, n\}$,

$$\varepsilon_i(\Gamma) = \lambda_1^{(i+1)'} - \lambda_1^{(i)} + \lambda_1^{(i)} - \lambda_1^{(i)'} = \lambda_1^{(i+1)'} - \lambda_1^{(i)'}.$$

*Proof.* The proof that these provide the crystal structure is combinatorial: note that a row pattern is a special case of a hook pattern. Since a single row is always totally ordered according to the alphabet, no $i$-bracketing may take place. Therefore, if a symbol $i$ or $\overline{i+1}$ occurs in the tableau, $\widetilde{f}_i$ can always change it. Given the order of the symbols in a row and the fact that $\widetilde{f}_i$ will first change the rightmost available symbol, first we check for symbols $\overline{i+1}$ to change into $\overline{i}$'s. These exist in the tableau if and only if $\lambda_1^{(i+1)} - \lambda_1^{(i+1)'} > 0$, by the above bijection for hook patterns. If there are no $\overline{i+1}$'s but there are $i$'s to change into $i+1$'s, then $\lambda_1^{(i+1)} - \lambda_1^{(i+1)'} = 0$ and $\lambda_1^{(i)'} - \lambda_1^{(i-1)} > 0$. The difference in the case of $i = n$ is due to the fact that $\widetilde{f}_n$ only changes $n$'s to $\overline{n}$'s, and there is therefore only one symbol to check, and one pattern element to decrement. The formula for $\widetilde{e}_i$ is analogous. The expressions for $\varphi_i$ and $\varepsilon_i$ count the occurrences of the appropriate symbols in the tableau. Finally, interlacing will be satisfied while modifying $\Gamma$'s entries as long as the conditions are met: $\lambda_1^{(i+1)} - \lambda_1^{(i+1)'} > 0$ means that there is room to increment $\lambda_1^{(i+1)'}$, and it is always the case that $\lambda_1^{(i+1)'} \geq \lambda_1^{(i)}$ so changing both at once is safe, for example. $\square$

### 4.3.2 Crystal structure on Type C column patterns

Let $\Gamma \in \mathrm{KNT}(n, (1, 1, \ldots, 1))$. Since $\mathcal{T}(\Gamma)$ is a column tableau, we call $\Gamma$ a *column pattern*. Note that $\widetilde{f}_i(\mathcal{T}(\Gamma))$ will first change an $i$-unbracketed $i$ to an $i+1$, and then change an $i$-unbracketed $\overline{i+1}$ to an $\overline{i}$ if either is present, since in a column tableau at most one of each symbol occurs. For a column tableau, an $i$ may be $i$-bracketed by an $\overline{i}$ or by an $i+1$, and an $\overline{i+1}$ may be $i$-bracketed is by an $i+1$ or by an $\overline{i}$. So, in order to define the crystal structure on columns, it is necessary to detect all of these possibilities in $\Gamma$. We conjecture that interleaving after the application of $\widetilde{f}_i$ and $\widetilde{e}_i$ coincides with these conditions being met.

Let $\Gamma \in \check{\mathrm{Z}}\mathrm{P}(n, (1, 1, \ldots, 1))$. Define

$$A_i(\Gamma) := \sum_{j=1}^{i} \lambda_j^{(i)'} - \lambda_j^{(i-1)},$$

$$B_i(\Gamma) := \sum_{j=1}^{i+1} \lambda_j^{(i+1)'} - \lambda_j^{(i)},$$

$$C_i(\Gamma) := \sum_{j=1}^{i} \lambda_j^{(i)} - \lambda_j^{(i)'},$$

$$D_i(\Gamma) := \sum_{j=1}^{i+1} \lambda_j^{(i+1)} - \lambda_j^{(i+1)'},$$

and note that $A_i(\Gamma) = 1$ indicates the presence of an $i$ in $\Gamma$, $B_i(\Gamma) = 1$ an $i+1$, $C_i(\Gamma) = 1$ an $\overline{i}$ and, unsurprisingly, $D_i(\Gamma) = 1$ an $\overline{i+1}$. As in type A, it seems reasonable to think that these quantities may represent a good starting point for the development of explicit formulae for the crystal operators for column patterns.

# CHAPTER 5. GENERAL CONCLUSION

Crystal basis theory offers a combinatorial perspective on the representation theory of Lie algebras and other algebraic structures that is both powerful and approachable due to its relative simplicity. In Chapter 1, we provided an overview of the area and some context for the study of Young tableaux and tableaux-like structures.

The type A semistandard Young tableaux are remarkable for the sheer variety of applications they lend themselves to in representation theory and beyond, and not least among these is their ubiquity in the theory of crystal bases. In Chapter 3, we provided a crystal structure on Gelfand-Tsetlin patterns that is independent of that given on semistandard Young tableaux. As patterns have their own diversity of applications, this represents a new way to make connections within and without representation theory. Moreover, the tropical nature of the expressions we obtained presents an intriguing question as to what deeper connection may be at work. Finally, we demonstrated that this crystal structure on patterns is compatible with the existing bijection between Gelfand-Tsetlin patterns and semistandard Young tableaux, rendering translation between the settings as easy as the bijection itself.

The picture in type C remains less clear, but we succeeded in Chapter 4 at giving a bijection between Kashiwara-Nakashima tableaux and Želobenko patterns in rank 2. We also offered some potential strategies for extending this bijection should it be possible to answer some of the combinatorial questions that seem to make type C more challenging to tackle than type A. The special cases of the bijection and of crystal structure in rank $n$ could be helpful along these lines, and when combined with the tensor product rule for crystals they do offer some additional insight in arbitrary rank. Beyond type C, it would be natural to ask the same sort of questions in the orthogonal cases of types B and D. Crystals of tableaux are well-understood here, and patterns exist and have similar applications.

# CHAPTER 6. EXAMPLES

Below we provide several examples of crystals of tableaux with their associated crystals of patterns. First we give crystal graphs for $\mathfrak{sl}_3(\mathbb{C})$, $\lambda = (3, 1)$, followed by some discussion. Then, to illustrate crystals of row patterns and the conjectured crystal of column patterns, we provide crystal graphs for graphs for $\mathfrak{sp}_6(\mathbb{C})$, $\lambda = (2)$ and $\lambda = (1, 1)$.

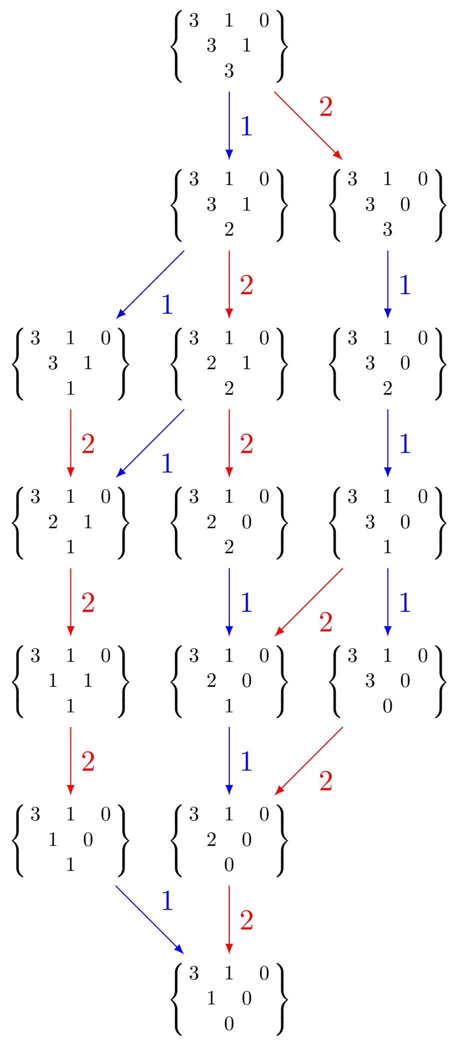


Figure 6.1: The Crystal ΓЦ$(2,(3,1))$

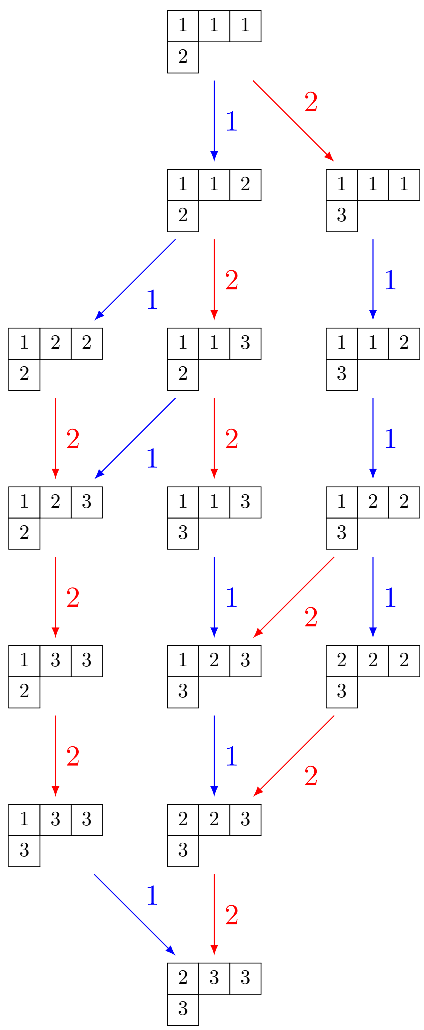


Figure 6.2: The Crystal SSYT$(2,(3,1))$

The above are two isomorphic crystal graphs for $\mathfrak{sl}_3(\mathbb{C})$. If we take the element

$$\Lambda = \left\{ \begin{array}{ccccc} 3 & & 1 & & 0 \\ & 3 & & 1 & \\ & & 2 & & \end{array} \right\}$$

in the crystal of patterns $C$, note that we have

$$A_1^{(1)}(\Lambda) = a_1^{(1)}(\Lambda) = \lambda_1^{(1)} - \lambda_1^{(0)} + \lambda_2^{(1)} - \lambda_2^{(2)} = 2 - 0 + 0 - 1 = 1,$$

which gives

$$\varphi_1(\Lambda) = \max\{A_1^{(1)}(\Lambda)\} = 1.$$

We also have

$$A_1^{(2)}(\Lambda) = a_1^{(2)}(\Lambda) + a_2^{(2)}(\Lambda) = (3 - 2 + 1 - 1) + (1 - 0 + 0 - 0) = 1 + 1 = 2$$

and

$$A_2^{(2)}(\Lambda) = a_2^{(2)}(\Lambda) = 1 - 0 + 0 - 0 = 1,$$

giving

$$\varphi_2(\Lambda) = \max\{A_1^{(2)}(\Lambda), A_2^{(2)}(\Lambda)\} = \max\{2, 1\} = 2.$$

Note also that $\widetilde{f}_1(\Lambda) = \Lambda - \Delta_1^{(1)}(\Lambda)$ so the entry $\lambda_1^{(1)}$ is being decremented, and $\widetilde{f}_2(\Lambda) = \Lambda - \Delta_1^{(2)}(\Lambda)$, giving

$$\widetilde{f}_1(\Lambda) = \left\{ \begin{array}{ccccc} 3 & & 1 & & 0 \\ & 3 & & 1 & \\ & & 1 & & \end{array} \right\} \qquad \widetilde{f}_2(\Lambda) = \left\{ \begin{array}{ccccc} 3 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\}$$

The rest of the crystal operators may be described in a similar way.

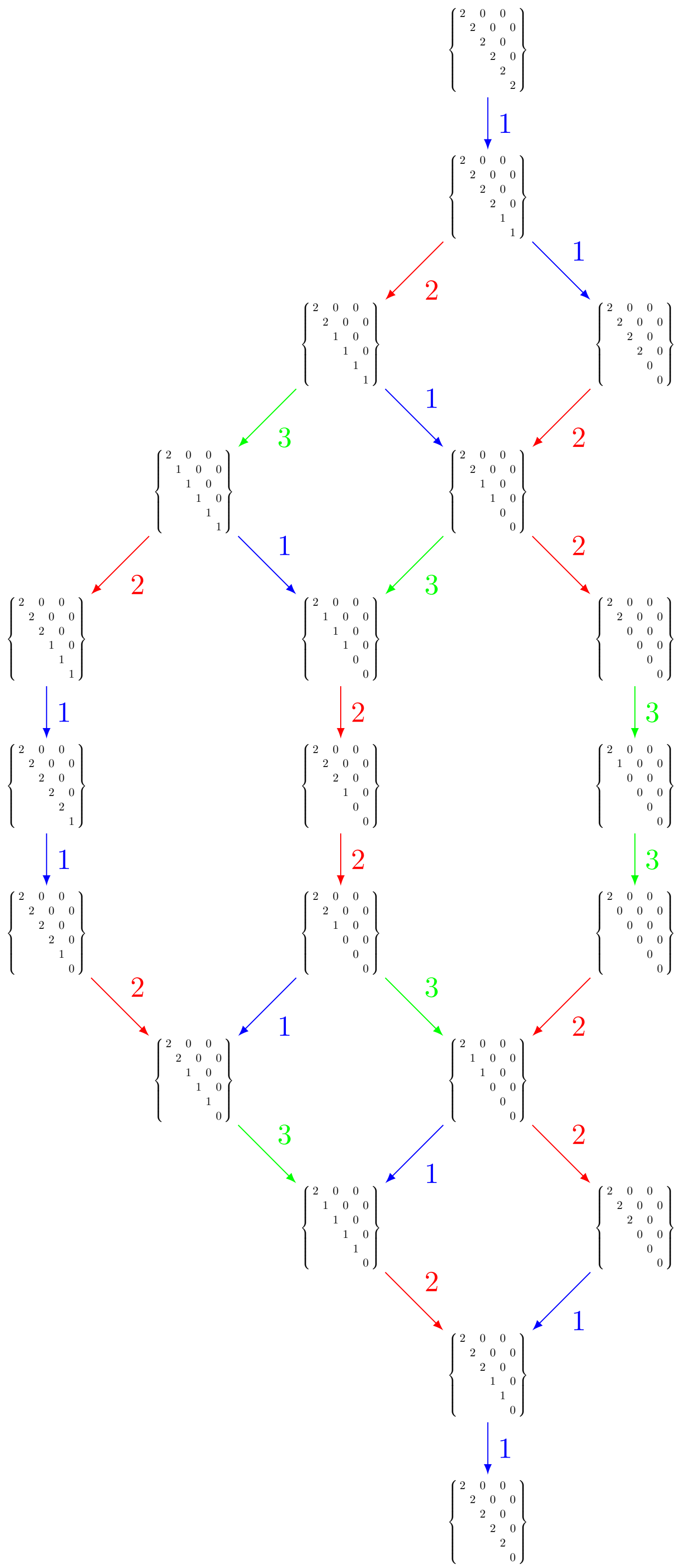


Figure 6.3: The Crystal $\check{\mathrm{Z}}\mathrm{P}(3,(2))$

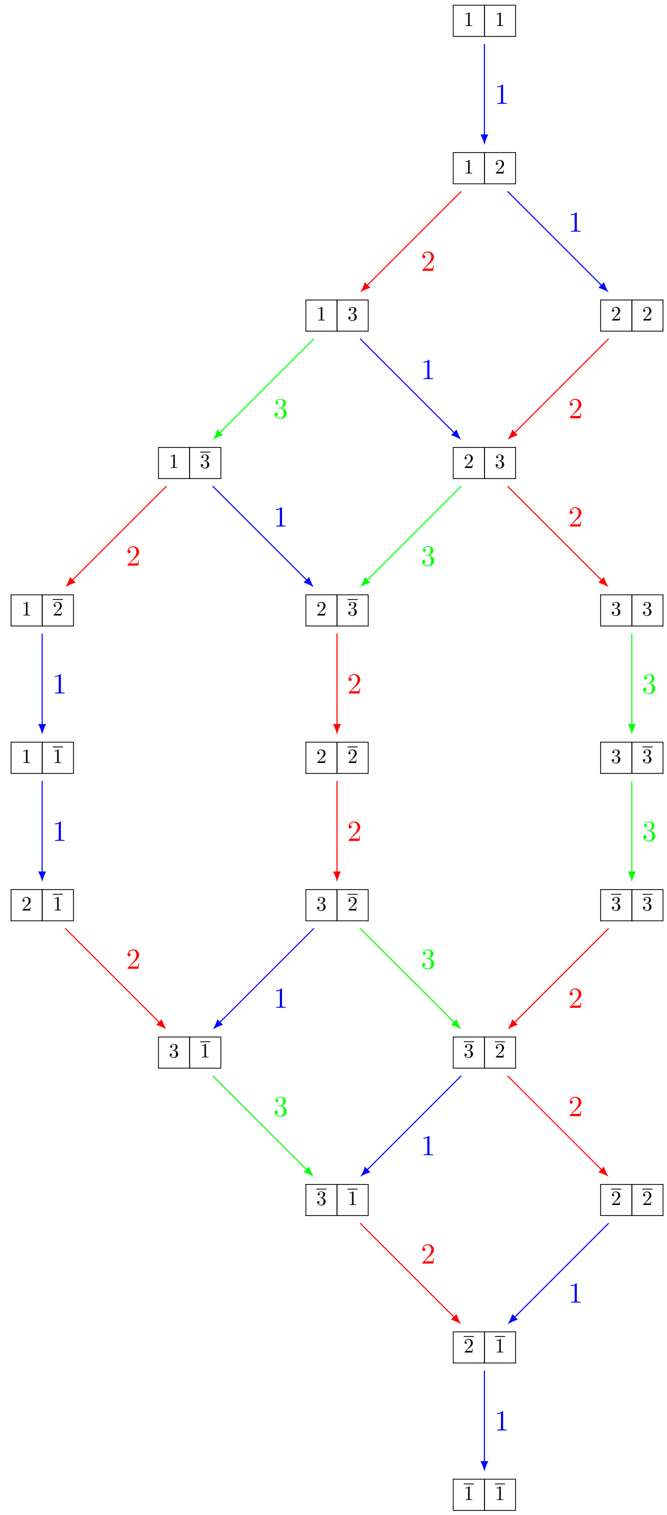



Figure 6.4: The Crystal KNT$(3,(2))$

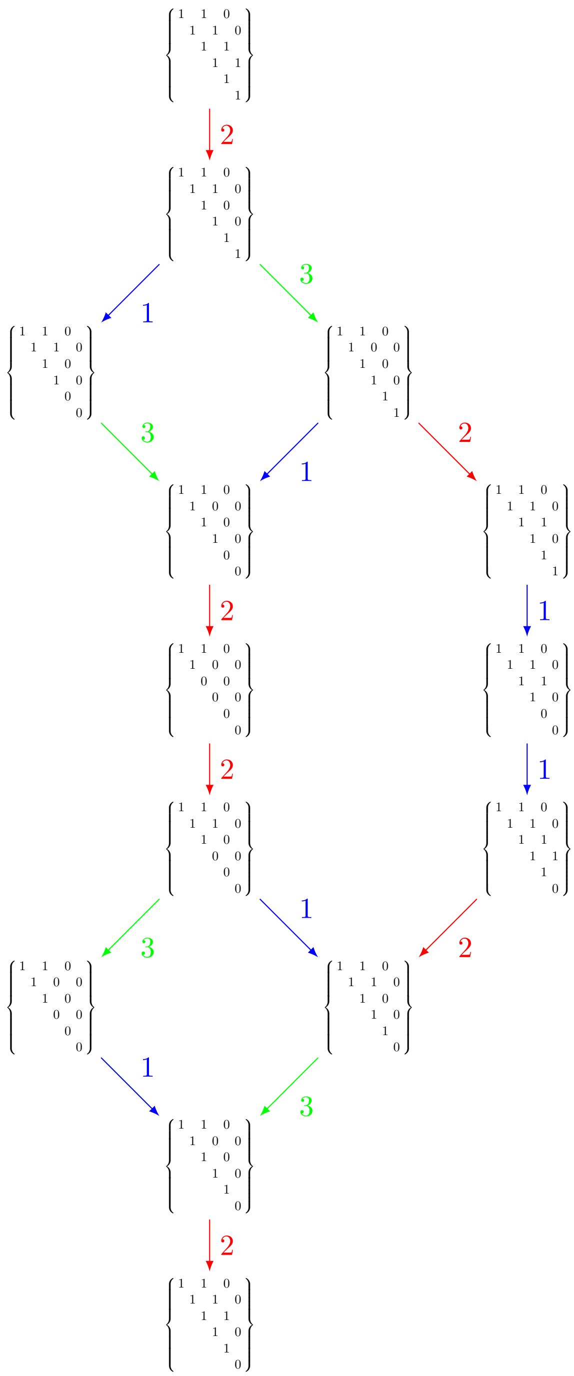


Figure 6.5: The Crystal $\check{\mathrm{Z}}\mathrm{P}(3,(1,1))$

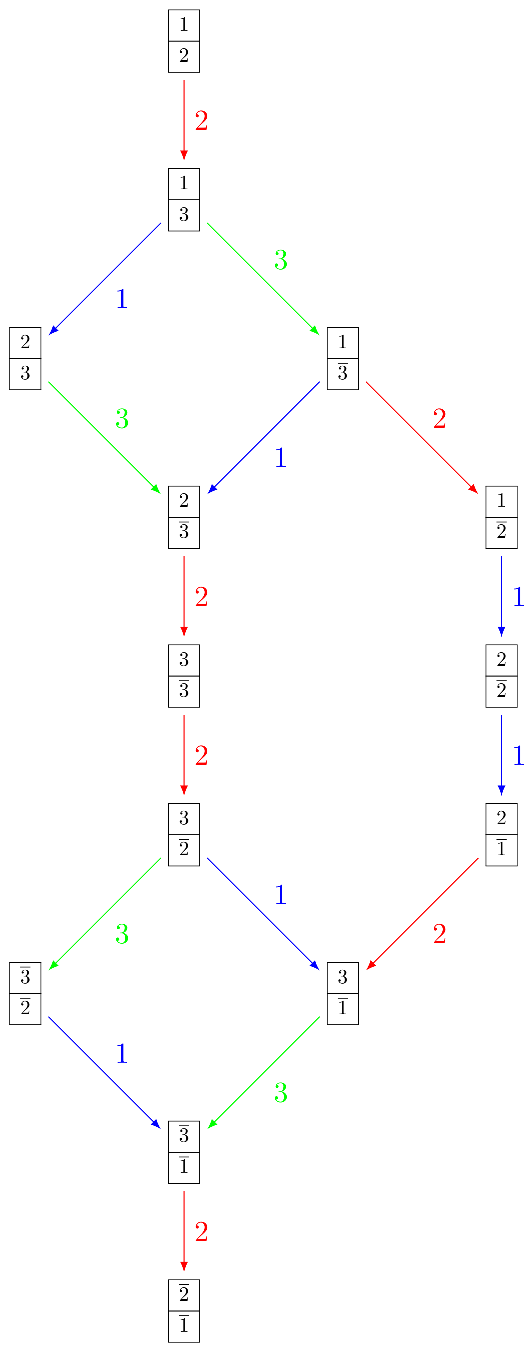



Figure 6.6: The Crystal KNT$(3,(1,1))$

## REFERENCES

[1] Arkady Berenstein and David Kazhdan. Geometric and unipotent crystals. ii. from unipotent bicrystals to crystal bases. *Quantum Groups Contemporary Mathematics*, page 13–88, 2007.

[2] Arkady Berenstein and David Kazhdan. Geometric and unipotent crystals. *Visions in Mathematics*, page 188–236, 2010.

[3] Daniel Bump and Anne Schilling. *Crystal bases: representations and combinatorics.* World Scientific Publishing Co. Pte. Ltd., 2017.

[4] Corrado De Concini. Symplectic standard tableaux. *Advances in Mathematics*, 34(1):1–27, 1979.

[5] William Fulton. *Young tableaux: with applications to representation theory and geometry.* Cambridge Univ. Press, 2003.

[6] I.M. Gelfand and M.L. Tsetlin. Finite-dimensional representations of groups of orthogonal matrices. *Dokl. Akad. Nauk*, 71(5):1017–1020, 1950.

[7] I.M. Gelfand and M.L. Tsetlin. Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk*, 71(5):825–828, 1950.

[8] Jin Hong and Seok-Jin Kang. *Introduction to quantum groups and crystal bases.* American Mathematical Society, 2002.

[9] M. Kashiwara. On crystal bases of the $q$ -analogue of universal enveloping algebras. *Duke Mathematical Journal*, 63(2):465–516, 1991.

[10] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the $q$-analogue of classical lie algebras. *Journal of Algebra*, 165(2):295–345, 1994.

[11] Masaki Kashiwara. Crystalizing the $q$-analogue of universal enveloping algebras. *Communications in Mathematical Physics*, 133(2):249–260, 1990.

[12] R. C. King. Weight multiplicities for the classical groups. *Group Theoretical Methods in Physics Lecture Notes in Physics*, page 490–499, 1975.

[13] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Crystal graphs andq-analogues of weight multiplicities for the root system $a_n$. *Letters in Mathematical Physics*, 35(4):359–374, 1995.

[14] Alain Lascoux and Marcel-P. Schützenberger. Le monoïde plaxique. In *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, volume 109 of *Quad. "Ricerca Sci."*, pages 129–156. CNR, Rome, 1981.

[15] Cédric Lecouvey. Schensted-type correspondence, plactic monoid, and jeu de taquin for type $c_n$. *Journal of Algebra*, 247(2):295–331, 2002.

[16] Cédric Lecouvey. Schensted-type correspondence, plactic monoid, and jeu de taquin for types $b_n$ and $d_n$. *Journal of Algebraic Combinatorics*, 18:99–133, 2003.

[17] G Lusztig. Quantum deformations of certain simple modules over enveloping algebras. *Advances in Mathematics*, 70(2):237–249, 1988.

[18] G. Lusztig. Canonical bases arising from quantized enveloping algebras. ii. *Progress of Theoretical Physics Supplement*, 102:175–201, 1990.

[19] Aleksandr Ivanovič Molev. *Yangians and classical Lie algebras*. American Mathematical Society, 2007.

[20] Jeffrey T. Sheats. A symplectic jeu de taquin bijection between the tableaux of king and of de concini. *Transactions of the American Mathematical Society*, 351(09):3569–3608, Jan 1999.

[21] David Speyer and Bernd Sturmfels. Tropical mathematics. *Mathematics Magazine*, 82(3):163–173, 2009.

[22] Dmitriĭ Petrovich Želobenko. The classical groups. spectral analysis of their finite-dimensional representations. *Russian Mathematical Surveys*, 17(1):1–94, 1962.

[23] Dmitriĭ Petrovich Želobenko. *Compact Lie groups and their representations*, volume 40. American Mathematical Soc., 1973.

[24] H. Watanabe and K. Yamamura. Alcove paths and gelfand-tsetlin patterns. 2019.