Degrees of freedom of wireless interference network

by

Lei Ke

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Electrical Engineering

Program of Study Committee:
Zhengdao Wang, Major Professor
Aleksandar Dogandžić
Sang W. Kim
Aditya Ramamoorthy
Huaiqing Wu

Iowa State University

Ames, Iowa

2011

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ACRONYMS

AWGN	additive white Gaussian noise
BC	broadcast channel
CSCG	circularly symmetric complex Gaussian
CSI	channel state information
CSIT	channel state information at transmitter
CSIR	channel state information at receiver
DoF	degrees of freedom
FFT	fast Fourier transform
FIC	full interference channel
HK	Han and Kobayashi
IC	interference channel
IFFT	inverse fast Fourier transform
i.i.d.	independent and identically distributed
INR	interference-to-noise ratio
LOS	line-of-sight
MAC	multiple access channel
MIMO	multiple input multiple output
MISO	multiple input single output
RF	radio frequency
SIMO	single input multiple output
SNR	signal-to-noise ratio
ZIC	Z interference channel

ACKNOWLEDGEMENTS

First of all, I appreciate the guidance from my advisor, Dr. Zhengdao Wang. His knowledge and insightful view have been guiding my research all over these years. It is his suggestion that I started working on multi-user communication systems and finally decided to focus on this area for my Ph.D study. I would like to thank him for his patience, kindness and support, especially his confidence in me when I was stuck in research. His attitude in research and personality in everyday life have been and will always be the model for me.

I would like to acknowledge Dr. Aleksandar Dogandžić, Dr. Sang W. Kim, Dr. Aditya Ramamoorthy and Dr. Huaiqing Wu for serving on my committee.

I also want to thank Dr. Huarui Yin, who stimulated my interest in communications when I was still an undergraduate. It was a happy time when he was visiting Iowa State University in 2010. I really enjoyed brainstorming and free talks with him.

During my study in Ames, I was happy to know many friends here: Peng Yu, Dongbo Zhang, Wei Lu, Zhiju Zheng, Kun Qiu, Xi Chen, Neevan Ramalingam, Hakan Topakkaya and many others. The time with them will be a good memory for me.

Words are never enough for expressing my gratitude to my parents. This thesis is dedicated to them.

ABSTRACT

Wireless communication systems are different from the wired systems mainly in three aspects: fading, broadcast, and superposition. Wireless communication networks, and multi-user communication networks in general, have not been well understood from the information-theoretic perspective: the capacity limits of many multi-user networks are not known. For example, the capacity region of a two-user single-antenna interference channel is still not known, though recent result can bound the region up to a constant value. Characterizing the capacity limits of multi-user multiple-input multiple-output (MIMO) interference network is usually even more difficult than the single antenna setup.

To alleviate the difficulty in studying such networks, the concept of degrees of freedom (DoF) has been adopted, which captures the first order behavior of the capacities or capacity regions. One important technique developed recently for quantifying the DoF of multi-user networks is the so-called interference alignment. The purpose of interference alignment is to design the transmit signals structurally so that the interference signals from multiple interferers are aligned to reduce the signal dimensions occupied by interference.

In this thesis, we mainly study two problems related to DoF and interference alignment: 1) DoF region of MIMO full interference channel (FIC) and Z interference channel (ZIC) with reconfigurable antennas, and 2) the DoF region of an interference network with general message demands.

For the first problem, we derive the outer bound on the DoF region and show that it is achievable via time-sharing or beamforming except for one special case. As to this particular special case, we develop a systematic way of constructing the DoF-achieving nulling and beamforming matrices. Our results reveal the potential benefit of using the reconfigurable antenna in

MIMO FIC and ZIC. In addition, the achievability scheme has an interesting space-frequency interpretation.

For the second problem, we derive the DoF region of a single antenna interference network with general message demands, which includes the multiple unicasts and multiple multicasts as special cases. We perform interference alignment using multiple base vectors and align the interference at each receiver to its largest interferer. Furthermore, we show that the DoF region is determined by a subset of receivers, and the DoF region can be achieved by considering a smaller number of interference alignment constraints so as to reduce the number of time expansion.

Finally, as a related research topic, we also include a result on the average throughput of a MIMO interference channel with single-user detector at receivers and without channel state information at transmitters. We present a piecewise linear approximation of the channel throughput under weak, moderate and strong interference regimes. Based on that we determine the optimal number of streams that a transmitter should use for different interference levels.

CHAPTER 1. INTRODUCTION

Fading and interference are two challenges in wireless communication system design. Fading characterizes how the wireless signal gets attenuated through the channel and it has been widely and intensively investigated under point-to-point communication scenario for the past decades. One major advance in wireless communications is the usage of multiple antennas [1, 2], which improves the spectrum efficiency by exploring spatial dimensions. Such improvement is usually termed as multiplexing gain in the wireless literature. Multiple antenna technique can also be used to improve the link quality, i.e., achieving a diversity gain. The goal of diversity is to design the transmit signal such that multiple replicas of the useful signal are available to the receiver, therefore, the error probability due to signal under deep fading can be reduced. The tradeoff between multiplexing and diversity gains is quantified in [3].

There are many multi-user communication systems in practice. One such example is the code division multiple access system, which has been widely used in cellular network or wireless local area network. The rapid growth of throughput demands in practical multi-user system has become the momentum of the research on multi-user communication theory recently. However, in multi-user wireless system, the signal of one user will experience both fading from the channel and suffer from interference of the other transmitters. Therefore, the capacity of multi-user system is in general difficult to solve.

1.1 Capacity of multi-user system with single antenna

There are some multi-user system whose capacity regions are known. For example, the multiple access channel (MAC) and the degraded broadcast channel (BC) [4]. As to the others, we know limited results. In the following, we will briefly review the result on interference channel (IC), whose capacity region, even with only two users, has been a long open problem.

1.1.1 Interference channel

In interference channel, there are multiple transmit-receive pairs, each of them sees the interference from the other pairs. Therefore, if one pair achieves higher rate by increasing the signal-to-noise ratio (SNR), the link quality of the other pairs is sacrificed as more interference is introduced. Hence, unilateral action are in general not overall optimal. For two-user IC, when only one of the two transmit-receive pairs is subject to interference, the interference channel is termed as *Z interference channel (ZIC)*. To avoid confusion, we will call the channel where both pairs are subject to interference the *full interference channel (FIC)*. Although ZIC is simpler than FIC in its structure, the capacity region of ZIC is unsolved.

One major result, discovered in 1975 by Carleial [5], is that interference may not reduce the capacity. That is, if both of the two receivers encounter very strong interference, the receiver can first decode the message of the interference link and then subtract it before decoding its own message. Sato [6] further extended it to strong interference case and Costa, El Gamal [7] presented the achievability and converse proofs. Although Carleial and Sato discussed the problem in Gaussian IC, Costa and El Gamal's proof dealt with the general case, hence the capacity region of general IC under the strong and very strong interference is known.

The weak or mixed interference channel is more difficult to handle, as the receiver can not decode the messages of the other users in general. Various outer bounds have been introduced, e.g., Sato's [8] and Carleial's [9] bounds. However, they are not universally better than each other. As to the achievable rate, the best known result is credited to Han and Kobayashi (HK) [10, 11], where they introduced the concept of private and common messages. A more powerful joint decoder is used at receiver. However, their achievable region needs to be optimized over all the possible input distributions and hence complicated.

One breakthrough in Gaussian IC is from Kramer [12], where he used "genie" method to obtain an outer bound. Etkin, Tse and Wang [13] further used the genie method to developed a close upper bound for the two-user FIC, in the sense that the maximum difference between their upper bounds and a simple HK scheme is at most 1 bit over all the choices of SNR and interference-to-noise ratio (INR). Another important result in Etkin's paper is that based on the first order approximation, i.e., if both SNR and INR are large, the generalized degrees of freedom of Gaussian FIC in some circumstances are the same as that when the interference is completely ignored as noise. This observation suggests that treating interference as noise could be optimal for weak interference scenario and the sum capacity of very weak two-user FIC is settled [14, 15, 16]. The results therein can be extended to interference channel with more than two users in which treating interference as noise is sum capacity optimal.

1.1.2 General Gaussian interference channel

When it comes to interference channel with more than two users, a natural question is whether the HK scheme used in [13] is still near-optimal. Interestingly, it is shown that a structured code is much better than HK scheme for the general multi-user IC. This is due to a recently proposed deterministic channel model, in which the channel strength is quantized and approximated such that there is no background noise and the approximated channel becomes "deterministic". The deterministic channel model has been successfully applied to different wireless channels, including the two-user FIC [17], the ZIC without channel state information at transmitter (CSIT) [18], the many-to-one (generalization of ZIC) and one-to-many interference channels [19], the wireless information flow network (generalization of relay channel) [20], BC without CSIT [21], where the gap between the achievable rate using deterministic channel and capacity region can be bounded up to a constant value.

1.2 Capacity of multi-user system with multiple antennas

The deterministic channel model has been demonstrated to be a powerful tool of bounding the capacity region of different multi-user systems with single antenna. However, when it comes to the multiple antenna case, the approximation it introduced could be arbitrarily large [20, 22]. This is because the deterministic channel only approximate the signal strength and it lacks the ability of modeling the phase information, which is very important in multiple input multiple output (MIMO) wireless communication. Although the capacity regions of some special MIMO multi-user systems are known, the capacity regions of the others, like their single antenna setups, are unknown. In addition, it is difficult to bound the capacity regions within some constant value as deterministic channel model can not be used here. Hence, instead of trying to characterize the capacity region completely, the degrees of freedom (DoF) region is widely used, which characterizes how capacity scales with transmit power as the SNR goes to infinity.

MIMO MAC channel is one of few multi-user systems for which the capacity is known when CSIT is available. As the dual channel of MIMO MAC, MIMO BC has one transmitter and multiple receivers, where the transmitter needs to send different messages to different receivers. The capacity region of MIMO BC is recently found [23] via dirty-paper coding.

As to the MIMO interference channel, its capacity region with very strong and aligned strong interference is also solved [24]. However, the results on other cases are very limited. The sum capacity under the weak interference or noisy interference case can be found in [25, 24]. There are also many results on the capacity regions or sum capacities of certain MIMO interference channels, e.g., MIMO many-to-one interference channel [26], MIMO Z interference channel [27]. In [28], the outer bound of a class of two-user interference channel is obtained, which can be applied to Gaussian MIMO IC. It is shown that capacity region can be bounded up to one bit per complex receive dimension from a general HK achievable region [29]. Such a gap is recently reduced to a constant gap, rather than receive dimension dependent, in [30]. However, the results do not apply to the cases with more than two users.

As the capacity of MIMO interference channel is a very difficult problem, many works are conducted on the achievable rate, e.g., MIMO IC with single user detector [31], the benefit of CSI feedback [32], the connection of Shannon capacity and game theory [33, 34].

1.3 An alternate approach: the DoF studies

The idea of DoF is not a new concept at all. It is well-known as the multiplexing gain in point-to-point communication scenarios. However, it was termed as the spatial DoF in [35], referring to the maximum multiplexing gain, which has slightly different meaning from what it is being used for in recent literature. To clarify, we will call the maximum multiplexing gain as total DoF. The total DoF of MIMO MAC and BC can be directly obtained from their capacity results. And the total DoF of two-user MIMO FIC is obtained in [35]. All these three channels have integer numbers of total DoF, which also corresponds to the number of antennas. Due to this reason, the DoF of MIMO multi-user system were thought to be integer values. Interestingly, it is found that a non-integer multiplexing gain can be achieved over MIMO X channel [36]. The MIMO X channel is a channel with two transmitters and two receivers where each transmitter has messages for both receivers. This simple channel contains MIMO MAC, MIMO BC and MIMO FIC as a portion of itself. And it is shown in that it can achieve more DoF in MIMO X channel than in any of its component.

Some interesting research has been conducted under MIMO X channel setup. The followup work of [37] further investigates this channel. Both the achievability scheme and outer bound are derived and shown to be exactly the same. It is shown that when all the transmitters and receivers have same M number of antennas, the total DoF is $4M/3$. The achievability is based on *interference alignment*, where the terminology was first introduced. The basic idea of interference alignment is to appropriately precode the transmissions so that interference from undesired transmitters is *aligned* at the intended receivers. Such an idea was first used in [36] and clarified in [37]. Although the DoF region is of 4 dimensions, it is completely characterized and it remains the only channel whose DoF region is in a linear space of dimensionality more than three but fully known regardless of the antenna numbers.

Although the capacity region of multi-user MIMO interference channel is not solved, the analysis of DoF is shown to be very useful to reveal the capacity potential of MIMO interference channel. For example, it was conjectured that the channel capacity of interference channel is limited with large number of users [38]. However, it is shown that for K-user M antenna interference channel, total $MK/2$ DoF can be achieved asymptotically via infinite time (frequency) expansion under block fading channel [39]. Indicating the capacity of each user is unbounded regardless of the user number K. Unlike MIMO X channel, the interference alignment scheme for MIMO interference channel requires infinite symbol extension. Although the achievable scheme is not practical, it indeed reveals the potential capacity gain from interference alignment and stimulates the DoF related research. Many exciting results and a number of innovative schemes for multi-user communications have been explored since.

The classic interference alignment scheme proposed in [39] is asymptotic in time. The key idea is to simultaneously satisfy a number of alignment constraints using channel extension, such an alignment scheme was also used in MIMO wireless X network [40], which is the generalization of X channel setup with more users and each transmitter has an independent message for all the receivers. Many alignment schemes has been proposed thereafter, including but not limited to, real interference alignment [41, 42, 43, 44], ergodic interference alignment [45, 46], asymmetric complex signaling [47], blind interference alignment [48, 49] and retrospective interference alignment [50, 51, 52], where the last two schemes are proposed for no CSIT and delayed CSIT cases, respectively.

The CSIT assumption can change the nature of the problem and result in different DoF regions when compared with full CSIT case. For example, It is well-known that the absence of CSIT will not affect the DoF region of MIMO MAC [53]. However, the DoF region of MIMO BC is not changed only when the number of transmit antennas is less than or equal to the minimum number of receive antennas among all the receivers [54, 55] . As to the interference channel, the DoF region of two-user MIMO FIC with CSIT has been obtained in [35], where it is shown that zero forcing is enough to achieve the DoF region. The DoF regions of two-user MIMO BC and FIC without CSIT are considered in [54], where there is an uneven trade-off between the two users. Except for a special case, the DoF region for the FIC is known and achievable. Similar, but more general result of isotropic fading channel can be found in [56]. The DoF regions of the K-user MIMO broadcast, interference and cognitive radio channels are derived in [55] for some cases. However, the special case of two-user in [54] remains unsolved. Recently, it is shown in [48] that if the channel is staggered block fading, we can explore the channel correlation structure to do interference alignment, where the upper bound in the converse can be achieved in some special cases. For example, it is shown that for two-user

MIMO staggered block fading FIC with 1 and 3 antennas at transmitters, 2 and 4 antennas at their corresponding receivers and without CSIT, the DoF pair $(1, 1.5)$ can be achieved. Also recently, it is shown in [57] that the previous outer bound is not tight when the channels are independent and identically distributed (i.i.d.) over time and isotropic over spatial domain. A different solution to the same problem from an *interference localization* point of view can be found in [58]. So by now the DoF region of two-user MIMO FIC is completely known for both the CSIT and the no CSIT cases (channel state information at receiver (CSIR), is always assumed available), provided that the channel is i.i.d. over time and isotropic over spatial domain. However, when the channel is not i.i.d. over coherent block such as in the "staggered" fading channels [48], the DoF could be larger.

The staggered block fading idea in [48] was further clarified in [49], where a blind interference alignment scheme is also proposed for K-user multiple-input single-output (MISO) broadcast channel to achieve DoF outer bound when CSIT is absent. It is shown that although the DoF region is reduced when CSIT is absent, using reconfigurable antenna at receivers can achieve the same total DoF of the CSIT case.

The research of DoF study on delayed CSIT is motivated by [50, 51] for MISO BC. The authors of [52] further explore this problem for different channels. The DoF regions of MIMO two-user FIC and BC with delayed CSIT are reported in [59] and [60], respectively.

Interference alignment is also considered with multicast messages. General message request sets have been considered in [46] where each message is assumed to be requested by an equal number of receivers. Ergodic interference alignment is employed and an achievability scheme is proposed. However, only the achievable sum rate is derived. The precise DoF region has not been identified. The work in [46] has been further extended to the multi-hop equal-length network in [61].

A related effort is the study of the compound MISO BC [62, 43], where the channel between the base station and mobile user is drawn from a finite set. As pointed out in [62], the compound BC can be viewed as a BC with common messages, where each messages is requested by a group of receivers. Therefore, the total DoF of compound BC is also the total DoF of a BC with different multicast groups. It is shown that using real interference alignment scheme [42], the outer bound of compound BC [63] can be achieved regardless of the number of channel states one user can have. In addition, the blind interference alignment scheme [49] is proposed to achieve the total DoF of such a network using reconfigurable antennas. The blind interference alignment scheme is demonstrated to have the ability of multicast transmission as all the receivers within one multicast group using same antenna switching pattern would be able to receive the same intended message.

1.4 Summary of main results

In this section, we summarize the main contributions of the thesis.

Problem 1. *Find out the reason why staggered fading channel can increase the DoF region of two user MIMO full interference channel. Explore the usage of reconfigurable antenna in this channel and quantify the benefit of reconfigurable antennas when there is no CSIT.*

The DoF region of two user MIMO FIC without CSIT was recently solved under the i.i.d. isotropic fading channel assumption. However, it is interesting that using channel correlation structure, the achievable DoF region can be larger. The recent work on reconfigurable antenna shows a way of creating channel variation which can be explored to enhance the DoF region for MISO BC. It is not clear whether such a method can be used for MIMO FIC or not. Our goal is to check the potential benefit of the reconfigurable antenna in MIMO FIC and characterize its DoF region. In addition, we want to find a systematic way of achieving the DoF region for arbitrary antenna number at transmitters and receivers.

Our main contributions for this problem can be summarized as follows:

(i) *We obtain the DoF region as a function of the number of available antenna modes and reveal the incremental gain in DoF that each extra antenna mode can bring.* We successfully use the reconfigurable antenna in MIMO FIC and ZIC. Unlike [49], we use the reconfigurable antenna at one transmitter to create channel fluctuation. We completely characterize the gain offered by configurable antennas. It is shown that in certain cases the reconfigurable antennas can bring extra DoF gains. In these cases, the DoF region is maximized when the number of modes is at least equal to the number of receive antennas at the corresponding receiver.

(ii) *We propose systematic constructions of the beamforming and nulling matrices for achieving the DoF region.* The construction bears an interesting space-frequency interpretation. and it can be used for any number of reconfigurable antennas.

Problem 2. *Characterize the DoF region of interference network with general message demands.*

There are wireless applications where a common message may be demanded by multiple receivers, such as in a wireless video streaming — each broadcast message may be requested by a group of receivers. However, in current work, there are few results on interference alignment with multicast transmission or combination of both unicast and multicast. Only the total DoF for K-user MIMO interference channel have been identified. The DoF region is unknown except for three user case and there is no general way of achieving asymmetric DoF points.

We consider a natural generalization of the multiple unicasts scenario considered in previous works. We consider a setup where there are K transmitters, each with a unique message. There can be J (not necessarily K) receivers, each of which is interested in some arbitrary subset of the K messages, i.e., we consider interference networks with general message demands. This includes as a special case, for instance, multiple unicasts and multiple multicasts. Our main observation is that by appropriately modifying the achievability scheme of [39], we can achieve general points in the DoF region when all the transmitters and receivers have single antenna. We also provide a matching converse bound, that serves to establish the DoF region. To our best knowledge, the DoF region in this scenario has not been considered before. Our main contributions can be summarized as follows:

(i) *Finding the DoF region for single antenna interference networks with general message demands.* We modify the scheme of [39] for the achievability proof. In particular, we achieve general points in the DoF region by using multiple base vectors and aligning the interference at each receiver to its largest interferer.

(ii) *Achieving the DoF region with reduced interference alignment constraints.* We discuss extensions of our approach where the DoF region can be achieved by considering fewer interference alignment constraints, allowing us to work with a smaller level of time-expansion. We demonstrate that essentially the region depends only on a subset of receivers whose demands meet certain characteristics.

1.5 Thesis organization

We briefly give a outline of the thesis in the following.

Chapter 2 introduces the basic concepts and the channel model. In this chapter, we shall discuss the fundamentals of DoF related ideas and notation. We also discuss the different assumptions of channel state information. The system model is presented as well.

Chapter 3 gives an overview of interference alignment techniques. We compare different alignment schemes and summarize the important results on DoF related studies. At the end of this chapter, we review one classic asymptotic interference alignment scheme through a simple example to present the general principle of interference alignment.

Chapter 4 investigates the DoF region of two-user MIMO FIC and ZIC. We quantify how the CSIT affect the DoF region. In particular, we propose to use reconfigurable antenna to change channel coherent time and explore the benefit on DoF region from the channel variation. We propose to perform beamforming jointly on both space and frequency domain to achieve the DoF region.

Chapter 5 focuses on the multi-user interference network. We generalize the K-user M antenna interference channel to an interference network with general message demands where each transmitter has only one message but it can be requested by any receivers. Our results establish the DoF region of single antenna system. Our approach is to use multiple base vectors to construct beamforming columns for all the messages jointly, and aligning the interference signals to the largest one at all the receivers.

Chapter 6 provides one related work on the achievable throughput in multi-user interference channel without CSIT. We consider a simple stream number selection method based on piecewise linear approximation of the channel throughput when there is only single user detector at all the receivers. A solution of mutual information calculation with repeated eigenvalues is presented as well.

Finally, the thesis is concluded in Chapter 7. We also discuss the possible future research directions.

1.6 Notation

We use the following notation: boldface uppercase (lowercase) letters denote matrices (vectors). For a matrix $\boldsymbol{A}$, $\boldsymbol{A}^T$ and $\boldsymbol{A}^H$ denote the transpose and Hermitian of $\boldsymbol{A}$, respectively. $\dim(\boldsymbol{A})$ stands for the dimensionality of a vector space and $\mathrm{span}(\boldsymbol{A})$ is the vector space spanned by the column vectors of matrix $\boldsymbol{A}$. We use $\boldsymbol{I}_m$ to denote a size $m \times m$ identity matrix and $\mathbf{1}_m$ to denote an all-one column vector with length m. We also use notation like $\boldsymbol{A}_{m\times n}$ to emphasize that $\boldsymbol{A}$ is of size $m \times n$. $\mathbf{0}$ and $\mathbf{1}$ denote all one and all zero matrices (vectors), respectively, whose sizes shall be clear based on context. $[\boldsymbol{A}]_{pq}$ is the (p, q)th entry of matrix $\boldsymbol{A}$. And $[\boldsymbol{a}]_p$ is the pth entry of vector $\boldsymbol{a}$. For two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \prec \boldsymbol{B}$ means that the column space of $\boldsymbol{A}$ is a subspace of the column space of $\boldsymbol{B}$. And We use $\boldsymbol{A} \otimes \boldsymbol{B}$ to denote the Kronecker product of $\boldsymbol{A}$ and $\boldsymbol{B}$.

We use $\mathcal{CN}(0, 1)$ to denote the circularly symmetric complex Gaussian (CSCG) distribution with zero mean and unit variance. In addition, we use $\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to denote the distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ for complex Gaussian random vector. Denote $\boldsymbol{g}_n(a) := [1, a, a^2, \ldots, a^{n-1}]^T$. A size $n \times m$ Vandermonde matrix based on a set of element $\{a_1, a_2, \ldots, a_m\}$ is defined as $\boldsymbol{\mathcal{V}}_n(a_1, a_2, \ldots, a_m) = [\boldsymbol{g}_n(a_1), \boldsymbol{g}_n(a_2), \ldots, \boldsymbol{g}_n(a_m)]$. $\mathbb{R}, \mathbb{Z}, \mathbb{C}$ are the real, integer and complex numbers sets. We define $\mathbb{R}_+^K := \{(x_1, x_2, \ldots, x_K) : x_k \in \mathbb{R}, x_k \geq 0, 1 \leq k \leq K\}$ and define $\mathbb{Z}_+^K$ similarly. Finally, we use $\mathcal{I}(\boldsymbol{x}; \boldsymbol{y})$ to denote the mutual information between $\boldsymbol{x}$ and $\boldsymbol{y}$. The differential entropy of a continuous random variable $\boldsymbol{x}$ is denoted as $\mathcal{H}(\boldsymbol{x})$.

CHAPTER 2. DEGREES OF FREEDOM AND WIRELESS INTERFERENCE NETWORK MODEL

One fundamental characterization of communication channel is the channel capacity, which is the maximum transmission rate that can be achieved with some coding scheme such that the information can be decoded with arbitrary small error probability. However, the channel capacities of many channels remain unknown, especially in multi-user communication scenario. The reason is not because of the channel noise but due to the interference from different users. This is especially the case in high SNR region, where the signal and interference is so large that the effect on channel capacity due to noise is negligible. This thesis is primarily focusing on the high SNR region. In this chapter, we will present the preliminary concept of degrees of freedom, wireless channel model, and channel state information assumptions. The system model will also be given in this chapter.

2.1 Degrees of freedom: the basic concept

In this section, we will introduce the basic idea of DoF.

2.1.1 Degrees of freedom: definition

Degrees of freedom is the asymptotic scaling of the capacity with respect to the logarithm of the SNR. More specifically, if the rate of a message as a function of the SNR ρ is $R(\rho)$, then the DoF associated with the message is defined as

$$d = \lim_{\rho \to \infty} \frac{R(\rho)}{\log(\rho)}. \tag{2.1}$$

The DoF is also known as the multiplexing gain [3]. Fig. 2.1 show the capacity of three well-known channels. the DoF of each channel is the slope of capacity curve in high SNR

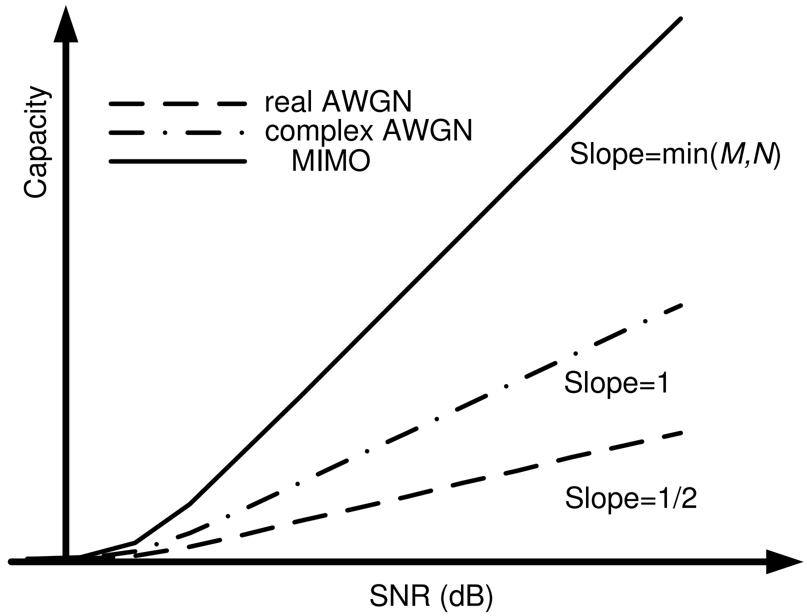

Figure 2.1 DoF of some channels

region. The DoF difference between two channels indicates the difference between channel capacity can be arbitrarily large when SNR goes to infinity. DoF is a simple but important figure of merit for comparing the capacities of different channels. In addition, it is useful to determine whether some transmission scheme is capacity-optimal or not, as any scheme which is capacity-achieving must be DoF optimal as well.

For point-to-point channel, DoF itself is enough to characterize the capacity scaling in high SNR region. However, in multi-user communication scenario, it is necessary to use DoF region to reflect the trade-off among different users.

2.1.2 Degree of freedom region

Consider an interference network with K transmitters and m independent messages $W_i, 1 \leq i \leq m$, where K is not necessarily equal to m. Let $|W_i|$ be the cardinality of message i. Assuming the average power of each transmitter is P and the additive noise has unit variance, there exists some coding scheme such that rate $R_i(P) = \log |W_i|/n$ can be achieved with arbitrary small error probability, where n is the number of channel uses. Let $C(P)$ denotes the capacity region of the interference network, which is the set containing all the achievable

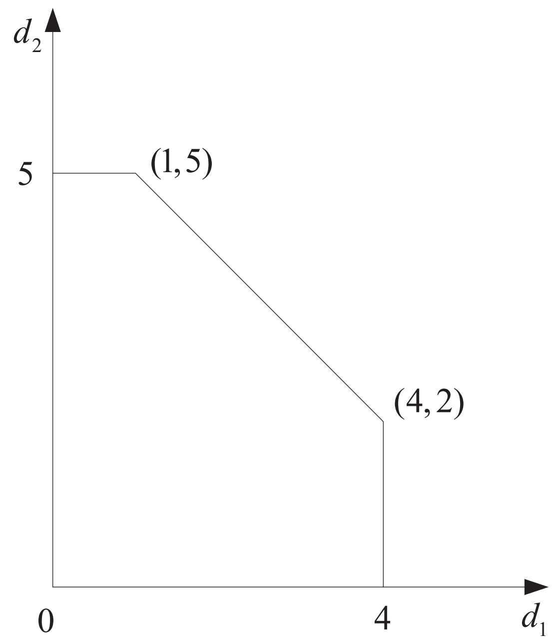

Figure 2.2 DoF region of a two-user MIMO MAC.

rate tuples $\boldsymbol{R}(P) = (R_1(P), R_2(P), \ldots, R_m(P))$. The DoF region is defined as [39, 40]

$$\begin{aligned}\mathcal{D} := \big\{\boldsymbol{d} = (d_1, d_2, \ldots, d_m) \in \mathbb{R}_+^m :\\ \exists (R_1(P), R_2(P), \ldots, R_m(P)) \in C(P),\\ \text{such that } d_i = \lim_{P\to\infty} \frac{R_i(P)}{\log(P)}, \quad 1 \le i \le m\big\}. \quad (2.2)\end{aligned}$$

Instead of trying to characterize the capacity region completely, the DoF region characterizes how capacity region scales with transmit power as the SNR goes to infinity.

Fig. 2.2 shows the DoF region of a two-user MIMO MAC. The receiver has 6 antennas, whereas two transmitters have 4 and 5 antennas, respectively. For other multi-user communication channels, it may be difficult to characterize the DoF regions completely. In that case, *total DoF*, which is the maximum sum DoF, can be a good metric to describe the channel. For example, the total DoF of the MAC channel shown in Fig. 2.2 is 6, which implies that the sum capacity is of the order of $6\log(P)$ at high SNR.

2.2 CSI model

It is well-known that when the transmitter and receiver know the communication channel, the channel capacity may be further enhanced. Such knowledge is usually termed as channel state information (CSI). The availability CSI can be significant in improving system performance. In general, CSI is usually assumed available at the receiver as it can be estimated using training sequence. However, as to transmit side CSI, it can be further divided into many cases. In time division duplex system where the communications in both directions are using the same frequency, the CSIT can be obtained by reciprocity assumption. That is, the CSIR obtained in receiving stage can be converted to CSIT at transmitting stage provided that the channel does not change significantly. Unfortunately, such a reciprocity assumption is not valid in general [64], and the CSIT is usually obtained by the feedback from the receiver [65, 66, 67, 68]. Because of quality and throughput limitations of the feedback link, the CSIT can be imperfect due to decoding error on the feedback link and may also be out of date if the physical channels vary fast. There are many assumptions for the imperfect channel information model, e.g., the transmitters only know the statistics of the channel [69], the quantized channel state information [70], delayed channel state information [51]. Although CSIT is not perfect in general, it is still common to make the perfect CSIT assumption in order to analyze the fundamental system performance as it leads to the performance upper limits of the practical communication systems.

Most CSIR and CSIT are local, which implies that the receivers or transmitters only know the channels that they are associated with. When the transmitters know not only the channels they are using, but also the channels of other transmitters, the CSI is called called *global* CSIT [39].

2.3 Channel model

2.3.1 Single antenna model

One fundamental difference between wired channel and wireless channel is the channel variation. Variation can be either due to moving surroundings and/or moving transmitters (receivers). In addition, due to signal reflection, diffraction and scattering, the signal from the transmitter to receiver will go through different pathes. Therefore, wireless channel is usually modeled as multipath time/frequency varying channel. The channel variation can be further divided into large scale and small scale fadings. The first one characterizes the signal change in macro time/distance, and it is usually due to the natural path loss and the shadowing of large objectives. Whereas the small scale fading characterizes the signal change in micro time/distance, and it is usually due to the reflection and scattering from the local surroundings.

In practice, the wireless channel is modeled statistically. For example, when the receiver has rich statistically independent pathes due to scattering and reflection, it is reasonable to assume that the signal is coming from all directions. In addition, using the central limit theorem, the real and image parts of the channel coefficient can be approximated as i.i.d. Gaussian random variables with zero mean and same variance. Such an assumption will lead to a well-known Rayleigh fading channel model, whose channel magnitude is Rayleigh distributed. Another well-known statistic channel model is the Rician fading channel, where the line-of-sight (LOS) path is also taken into account.

2.3.2 Multiple antenna model

Multiple antenna technique is very useful in wireless communications, which introduces the spatial dimension in addition to the time and frequency dimensions. Assuming there are M antennas at transmitter, N antennas at receiver, the channel can be represented by using matrix representation as

$$\boldsymbol{H} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1M} \\ h_{21} & h_{22} & \cdots & h_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NM} \end{bmatrix}, \tag{2.3}$$

where h_{ji} is the scalar channel from the ith transmitter to the jth receiver. Here, we omit the time index for simplicity. When either transmitter or receiver has signal antenna, the MIMO channel is reduced to single input multiple output (SIMO) or MISO channel, respectively. And the channel is represented by a vector other than matrix.

When all the channel coefficients are Rayleigh distributed, the channel is termed as MIMO Rayleigh fading channel, which can be further characterized as follows [71]

$$\boldsymbol{H} = \boldsymbol{\Phi}_R^{1/2} \boldsymbol{H}_w \boldsymbol{\Phi}_T^{1/2}, \tag{2.4}$$

where $\boldsymbol{H}_w$ is a white Gaussian matrix with all entries i.i.d. Gaussian distributed with zero mean and unit variance. $\boldsymbol{\Phi}_T$ and $\boldsymbol{\Phi}_R$ are the covariance matrices at transmitters and receivers, respectively. This model will be used in Chapter 6.

2.4 System model

In this thesis, we mainly study the MIMO interference channel and its generalization. We first start from the generalized model.

2.4.1 Generalized interference network

We consider a single hop wireless interference network with K transmitters and J receivers. The number of transmit (receive) antennas at the kth transmitter (jth receiver) is denoted as M_k, $1 \leq k \leq K$ (N_j,$1 \leq j \leq J$). We assume that each transmitter has one and only one independent message. For this reason, we do not distinguish the indices for messages and that for transmitters. Each receiver can request an arbitrary set of messages from multiple transmitters. Let $\mathcal{M}_j$ be the set of indices of those messages requested by receiver j.

The channel between transmitter k and receiver j at time instant t is denoted as $\boldsymbol{H}_{jk}(t) \in \mathbb{C}^{N_j \times M_k}, 1 \leq k \leq K, 1 \leq j \leq J$. We make the following assumptions on the channel matrices

1. The channels between transmitters and receivers are frequency flat block fading with coherent time length L. Within each coherent block, the channels between all the transmit antennas and all the receive antennas remain constant.

2. The elements of all the channel matrices at different coherent blocks are independently drawn from some continuous distribution.

3. The channel gains are bounded between a positive minimum value and a finite maximum value to avoid degenerate channel conditions.

4. The probability of $\boldsymbol{H}_{jk}(t)$ belonging to any subset of $\mathbb{C}^{N_j \times M_k}$ that has zero Lebesgue measure is zero.

The received signal at the jth receiver can be expressed as

$$\boldsymbol{y}_j(t) = \sum_{k=1}^{K} \boldsymbol{H}_{jk}(t)\boldsymbol{x}_k(t) + \boldsymbol{z}_j(t), \tag{2.5}$$

where $\boldsymbol{z}_j(t) \in \mathbb{C}^{N_j}$ is an independent CSCG noise with each entry $\mathcal{CN}(0,1)$ distributed, and $\boldsymbol{x}_k(t) \in \mathbb{C}^{M_k}$ is the transmitted signal of the kth transmitter satisfying the following power constraint

$$\mathrm{E}(||\boldsymbol{x}_k(t)||^2) \leq P, \quad 1 \leq k \leq K.$$

Remark 1. *If $J = K$, $M_k = N_j = M, \forall j,k$, $L = 1$ and $\mathcal{M}_j = \{j\}, \forall j$, the general model we considered here reduces to the well-known K-user M antenna interference channel as in [39].*

2.4.2 Time expansion system model

In this thesis, we also introduce a time expansion system model. The model is useful for transmitting a codeword *over τ time slots*, where τ is not necessarily equal to coherent time length L. We use the tilde notation to indicate the time expansion signals, where the number of slots of time expansion signals shall be clear within the context. In general, by default, for a vector $\boldsymbol{x}$, $\tilde{\boldsymbol{x}} = \text{vec}(\boldsymbol{x}(1), \boldsymbol{x}(2), \cdots, \boldsymbol{x}(\tau))$ and for a matrix $\boldsymbol{V}$, $\tilde{\boldsymbol{V}} = \text{diag}(\boldsymbol{V}(1), \boldsymbol{V}(2), \cdots, \boldsymbol{V}(\tau))$.

The time expansion channel of (2.5) over τ time slots can be simply given as

$$\tilde{\boldsymbol{y}}_j = \sum_{k=1}^{K} \tilde{\boldsymbol{H}}_{jk} \tilde{\boldsymbol{x}}_k + \tilde{\boldsymbol{z}}_j. \tag{2.6}$$

2.4.3 Isotropic Fading

Isotropic fading characterizes the fading in spatial domain. We say a complex-valued matrix $\boldsymbol{R}$ is isotropic if its product with any unitary matrix $\boldsymbol{V}$, $\boldsymbol{RV}$, has the same distribution as $\boldsymbol{R}$ [57]. A MIMO channel is isotropic faded if the channel matrix $\boldsymbol{H}$ is isotropic. Similarly, a MISO channel is isotropic if the channel vector $\boldsymbol{h}$ is isotropic. Many channel distributions are isotropic faded, such as the point-to-point MIMO Rayleigh fading channel.

The isotropic fading simply implies that all the channel directions from the transmitter to the receiver are statistically equivalent. Therefore, from the perspective of transmitter, there is no preference on any channel direction when the CSIT is absent. For point-to-point MIMO Rayleigh channel, the capacity achieving scheme when there is no CSIT is to use equal power allocation [2]. When there is CSIT, the capacity achieving scheme is the water-filling solution [2]. However, in terms of DoF, water-filling only provide a power gain, not DoF gain. Hence, isotropic fading is a very good channel assumption in studying the DoF of interference network, especially when there is no CSIT.

2.4.4 Reconfigurable antenna

The model we presented so far assumes the transmitters and receivers are equipped with conventional antennas. In wireless communications, there exists benefit of using reconfigurable antennas [49], which are different from the conventional antennas as they can be switched to different pre-determined modes so that the channel fluctuation can be introduced artificially. We define one antenna *mode* as one possible configuration of a single antenna such that by switching a radio frequency (RF) chain to a different mode, the channel between this antenna and other antennas that form a wireless link is changed. Different antenna modes can be realized via spatially separated physical antennas, or the same physical antenna excited with different polarizations, and so on. The benefit of antenna mode switching lies in the fact that channel variation can be artificially created, without the need to increase the number of RF chains. Although any transmitter or receiver can be equipped with reconfigurable antennas, we only consider using the reconfigurable antennas at transmitters in this thesis.

The reconfigurable transmit antenna can be switched to different modes in different time slots within each coherent block, where the channels between all the transmitter modes and the receive antennas remain constant. Denote S as the total number of antenna modes available at the transmitter. When S is larger than the number of RF chains, the transmitter has the freedom to use different modes at different slots. We assume that the channels between the S modes of the reconfigurable transmit antennas and any receivers is i.i.d. distributed over different time blocks. However, for a given antenna mode usage pattern over the length of a whole coherent block, the effective channel is no longer i.i.d. distributed over the time.

2.5 Summary

In this chapter, we presented the basic idea of DoF and introduced the wireless channel model. DoF indicates the channel capacity scaling at high SNR, which is very useful for analyzing the multi-user wireless system, whose channel capacity remains unknown for many cases.

Wireless channel is different for wired channel mainly in three aspects: fading, broadcasting and superposition. We briefly reviewed the wireless channel for both single antenna and multiple antenna systems. We presented the system model of general interference network at the end of this chapter. The isotropic fading assumption and reconfigurable antenna model are also covered, which will be explored in Chapter 4.

CHAPTER 3. SURVEY OF INTERFERENCE ALIGNMENT

3.1 Interference alignment methods

The basic idea of interference alignment was initially proposed in [36], which was further clarified in [37]. Interference alignment has emerged as an important technique of dealing with interference, which has many different applications in the area of wireless communications [37, 39, 40]. And it is also been applied to other areas, e.g., data storage [72, 73], network coding [74, 75]. Receivers in a wireless setting have to contend with interference from undesired transmitters in addition to ambient noise. Traditional efforts in dealing with interference has focused on reducing the power of interference, whereas in interference alignment the focus is on reducing the dimensionality of the interference subspace. The subspaces of interference from several undesired transmitters are *aligned* so as to minimize the dimensionality of the total interference space. For the K-user M antenna interference channel, it is shown that alignment of interference simultaneously at all the receivers is possible, allowing each user to transmit at approximately half the single-user rate in the high SNR scenario [39]. The idea of interference alignment has been successfully applied to different interference networks [40, 76, 63, 62, 43], demonstrating unlimited capacity potential when the SNR tends to infinity.

There is no universal way of doing interference alignment. Depending on the channel model and CSI assumption, alignment methods can be completely different. In the following part of this section, we will briefly review some existing interference alignment schemes and give comparison of them.

3.1.1 Asymptotic interference alignment

Many interference alignment schemes are based on asymptotic method. The most famous alignment scheme, which was first proposed in [39] for MIMO interference channel is based asymptotic interference alignment. The key assumption is that the channel is time-varying and the transmitters are assumed to have global channel state information. Alignment is achieved by constructing beamforming matrices on all the transmitters jointly over the time expanded channel. The advantage of such a scheme is that it can be applied over many channels as long as the channel coefficients have zero probability to be the same. However, the perfect global CSIT assumption is almost impossible. First, the channel state information must be quantized in practical system therefore the probability that some channel coefficients are the same is no longer zero. The global CSI also introduce large feedback among different users which is impractical. In addition, infinite time expansion is needed to achieve the theoretical DoF upper bound. Nevertheless, Such a scheme is very useful to explore the fundamental limit of communication channels and it has been successfully applied to wireless X network [40], MIMO IC with different number of transmit and receive antennas [77].

3.1.2 Real interference alignment

The real interference alignment[41, 42] is proposed for constant channel model where the asymptotic alignment scheme cannot be used. It requires that all the data are sent using rational numbers. Real interference alignment is based on Khintchine-Groshev theorem in the field of Diophantine approximation. The real interference alignment is an alignment in signal strength domain and it is in fact a lattice alignment, which is studied before under deterministic channel setup [19, 17, 20] and has been applied for K-user interference channel as well [78]. However, one major issue with real interference alignment is that it is impossible to obtain channel gains with infinity resolution. In other words, the channel information must be quantized before it can be used. In such a sense, real interference alignment is not practical as well.

3.1.3 Ergodic interference alignment

Ergodic interference alignment [45] is different from the previously mentioned alignment schemes, which are all used to achieve the DoF in high SNR region. Ergodic interference alignment is proposed to achieve a rate that is proportional to half of a single user rate for a multi-user interference channel, i.e., it is an alignment scheme for any SNR. It uses the fact that for many channel distributions, the channel state can be partitioned into complimentary pairings such that alignment can be performed using these complimentary states. However, the limitation is that it requires the symmetry in the distribution of channel coefficients therefore it cannot been applied for many fading channels. For example, due to the mean components in Rician fading, it is impossible to cancel out the interference via pairing.

3.1.4 Blind interference alignment

The blind interference alignment method is proposed to deal with the scenarios where the transmitters have no CSI. The original concept was proposed in [48], where the authors show that interference alignment is still possible when there is no CSIT but the transmitters know the correlation structure of the channel. Using the staggered fading model [48], the transmitters can jointly construct beamforming matrices such that interference can be aligned. If the channels from a transmitter to its dedicated receiver (the communication link) and the other receivers (interference links) change at different time then this transmitter can send out messages when the communication link changes while interference links remains unchanged. From its own receiver's perspective, the signals are from different directions. However, for those receivers which treat its signal as interference, the interference is coming from the same direction. This property can be used to design alignment schemes such that the interference in the subsequent transmission is aligned to previous transmission. Although channel correlation structure can be used to do interference alignment, the channel coefficients may change slowly over time and it is difficult for transmitters and receivers to know the exact time instant that channel changes. In addition, the interference using staggered fading channel can not be naturally extended to general cases.

The idea was further considered in [49] in MISO broadcast channel, where the receivers are assumed to have reconfigurable antennas, such that the channel can be changed manually. It is shown that with such an ability, the total DoF of a MISO broadcast channel is the same when there is perfect CSIT. In addition, unlike the perfect CSIT case where asymptotic interference alignment is used to achieve the total DoF, only finite number of channel extension is needed with reconfigurable antenna. The reconfigurable antenna is promising as it remove the CSI requirement, and can be used for any SNR, but requires higher hardware complexity.

3.1.5 Retrospective interference alignment

Blind interference alignment was proposed to deal with the case where there is no CSIT. However, the no CSIT assumption is too pessimistic in practical system. Transmitters can still receive some channel state information by using long coding block to combat error events in the feedback link. Therefore the CSI is usually out-dated. For point-to-point memoryless channel, Shannon showed that feedback does not increase channel capacity [79]. However, this is not the case in multi-user communication channels. Recently, the authors in [50, 51] investigated the MISO broadcast channel with delayed CSIT, where the CSI is available to transmitter accurately but with one symbol time delay. They showed that the delayed CSIT can still be very useful to increase the DoF of MISO broadcast channel, which is opposite to the common conjecture that outdated CSIT is not useful at all. The key is to construct precoding matrices at one time instant by only using the channel state in previous time instants. Such a scheme is termed as retrospective interference alignment in [52] and is extended to three-user MIMO interference channel and MIMO X channel as well.

3.2 Interference alignment in three-user SISO IC

In this section, we will briefly review the classic asymptotic interference alignment scheme that was proposed in [39]. To avoid reproduction, we only use a simple three-user SISO interference channel as an example, which is sufficient to reveal the key ideas of interference alignment.

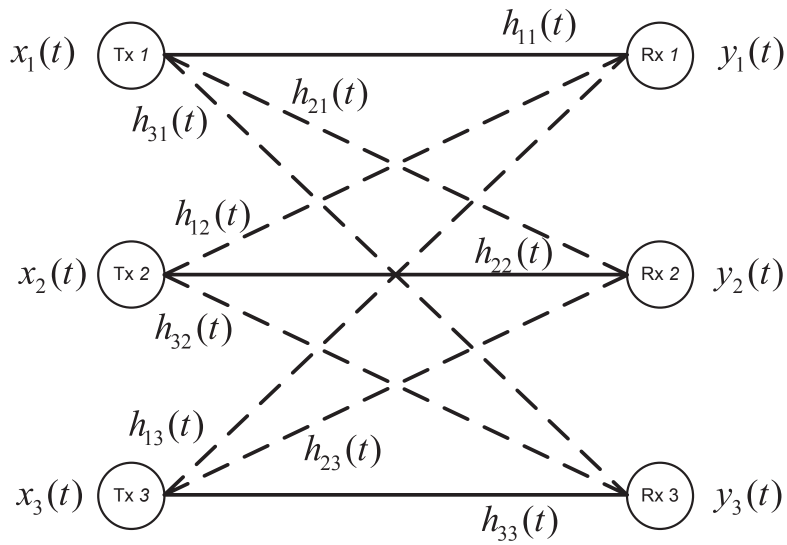

Figure 3.1 System model of three-user SISO interference channel

Consider a three-user interference channel where each transmitter or receiver has only one antenna. The system model can be given as

$$y_1(t) = h_{11}(t)x_1(t) + h_{12}(t)x_2(t) + h_{13}(t)x_3(t) + z_1(t), \tag{3.1}$$

$$y_2(t) = h_{21}(t)x_1(t) + h_{22}(t)x_2(t) + h_{23}(t)x_3(t) + z_2(t), \tag{3.2}$$

$$y_3(t) = h_{31}(t)x_1(t) + h_{32}(t)x_2(t) + h_{33}(t)x_3(t) + z_3(t). \tag{3.3}$$

Here the channel coherent time L is assume to be 1, and all the channel coefficients are assumed to be perfectly known at transmitter, i.e., the transmitters have global channel state information. On the other hand, the receivers only have the channel coefficients that corresponding to the received useful signal and interference. Fig. 3.1 shows the system model.

In [39], the authors proposed to do interference alignment over time expansion channel of τ slots, which can be given as

$$\tilde{\boldsymbol{y}}_1 = \tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{H}}_{13}\tilde{\boldsymbol{x}}_3 + \tilde{\boldsymbol{z}}_1, \tag{3.4}$$

$$\tilde{\boldsymbol{y}}_2 = \tilde{\boldsymbol{H}}_{21}\tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{H}}_{22}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{H}}_{23}\tilde{\boldsymbol{x}}_3 + \tilde{\boldsymbol{z}}_2, \tag{3.5}$$

$$\tilde{\boldsymbol{y}}_3 = \tilde{\boldsymbol{H}}_{31}\tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{H}}_{32}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{H}}_{33}\tilde{\boldsymbol{x}}_3 + \tilde{\boldsymbol{z}}_3. \tag{3.6}$$

Notice that the original channel is scalar channel, therefore the time expanded channel matrix $\tilde{\boldsymbol{H}}_{ji}$ is a diagonal matrix. Apparently, all the receivers have τ signal dimensions over time expanded channel. If one transmitter send out signal over all the τ dimensions, its signal will fill out all the dimensions at the other two receivers, which is not total DoF optimal. To achieve full DoF of this channel we need to do beamforming over time expanded channel and interference alignment.

Let $\tau = 2l+1$, where l is a non-negative integer. Assume that the message from transmitter one is encoded into $l+1$ streams $w_1^{(i)}, 1 \leq i \leq l+1$, where each stream is transmitted by using a length τ beamforming vector $\tilde{\boldsymbol{v}}_1^{(i)}, 1 \leq i \leq l+1$. Therefore, signal $\tilde{\boldsymbol{x}}_1$ can be given as

$$\tilde{\boldsymbol{x}}_1 = \sum_{i=1}^{l+1} w_1^{(i)} \tilde{\boldsymbol{v}}_1^{(i)} \tag{3.7}$$

$$= \underbrace{\begin{bmatrix} \tilde{\boldsymbol{v}}_1^{(1)} & \tilde{\boldsymbol{v}}_1^{(2)} & \dots & \tilde{\boldsymbol{v}}_1^{(l+1)} \end{bmatrix}}_{\tilde{\boldsymbol{V}}_1} \underbrace{\begin{bmatrix} w_1^{(1)} \\ w_1^{(2)} \\ \vdots \\ w_1^{(l+1)} \end{bmatrix}}_{\tilde{\boldsymbol{w}}_1}. \tag{3.8}$$

Similarly, assuming that transmitter 2 and 3 send only l streams, the corresponding beamforming matrices $\tilde{\boldsymbol{V}}_i, i = 2, 3$ and the message vectors $\tilde{\boldsymbol{w}}_i, i = 2, 3$ can be defined accordingly.

We can see that if $\tilde{\boldsymbol{V}}_i$'s are randomly generated, the signal and interference will be overlapping with each other. For example, at receiver 1, the interference space has dimensionality $2l$ as matrix $[\tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{V}}_2, \tilde{\boldsymbol{H}}_{13}\tilde{\boldsymbol{V}}_3]$ has full column rank $2l$. Therefore, it is impossible for transmitter 1 sending $l+1$ interference-free streams.

The key idea of interference alignment is to construct the beamforming matrices in a smart way such that the interference from different transmitters are aligned together to occupy less signal dimension.

To achieve the alignment, the authors of [39] choose the following conditions

$$\tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{V}}_2 = \tilde{\boldsymbol{H}}_{13}\tilde{\boldsymbol{V}}_3, \tag{3.9}$$

$$\tilde{\boldsymbol{H}}_{23}\tilde{\boldsymbol{V}}_3 \prec \tilde{\boldsymbol{H}}_{21}\tilde{\boldsymbol{V}}_1, \tag{3.10}$$

$$\tilde{\boldsymbol{H}}_{32}\tilde{\boldsymbol{V}}_2 \prec \tilde{\boldsymbol{H}}_{31}\tilde{\boldsymbol{V}}_1. \tag{3.11}$$

They further show that the conditions in (3.9)–(3.11) are equivalent to the following

$$\underbrace{\tilde{\boldsymbol{H}}_{21}^{-1}\tilde{\boldsymbol{H}}_{23}\tilde{\boldsymbol{V}}_3}_{\boldsymbol{B}} = \underbrace{\tilde{\boldsymbol{H}}_{21}^{-1}\tilde{\boldsymbol{H}}_{23}\tilde{\boldsymbol{H}}_{12}^{-1}\tilde{\boldsymbol{H}}_{13}\tilde{\boldsymbol{H}}_{32}^{-1}\tilde{\boldsymbol{H}}_{31}}_{\boldsymbol{T}}\underbrace{\tilde{\boldsymbol{H}}_{31}^{-1}\tilde{\boldsymbol{H}}_{32}\tilde{\boldsymbol{V}}_2}_{\boldsymbol{C}}, \tag{3.12}$$

$$\boldsymbol{B} \prec \tilde{\boldsymbol{V}}_1, \tag{3.13}$$

$$\boldsymbol{C} \prec \tilde{\boldsymbol{V}}_1. \tag{3.14}$$

Thanks to the diagonal structure of channel matrices, $\boldsymbol{T}$ is a diagonal matrix as well. And the beamforming matrices can be designed as follows. First, choose

$$\tilde{\boldsymbol{V}}_1 = [\mathbf{1}_\tau, \boldsymbol{T}\mathbf{1}_\tau, \cdots, \boldsymbol{T}^l\mathbf{1}_\tau], \tag{3.15}$$

where $\mathbf{1}_\tau$ is a length τ all one vector. In order to satisfy (3.12)–(3.14) simultaneously, one can choose

$$\boldsymbol{C} = [\mathbf{1}_\tau, \boldsymbol{T}\mathbf{1}_\tau, \cdots, \boldsymbol{T}^{l-1}\mathbf{1}_\tau], \tag{3.16}$$

$$\boldsymbol{B} = [\boldsymbol{T}\mathbf{1}_\tau, \boldsymbol{T}^2\mathbf{1}_\tau, \cdots, \boldsymbol{T}^l\mathbf{1}_\tau]. \tag{3.17}$$

Apparently, any column from $\boldsymbol{C}$ multiplied by $\boldsymbol{T}$ will be a column of $\boldsymbol{B}$ and they are sub matrices of $\tilde{\boldsymbol{V}}_1$. Hence, we have

$$\tilde{\boldsymbol{V}}_2 = \tilde{\boldsymbol{H}}_{32}^{-1}\tilde{\boldsymbol{H}}_{31}[\mathbf{1}_\tau, \boldsymbol{T}\mathbf{1}_\tau, \cdots, \boldsymbol{T}^{l-1}\mathbf{1}_\tau], \tag{3.18}$$

$$\tilde{\boldsymbol{V}}_3 = \tilde{\boldsymbol{H}}_{23}^{-1}\tilde{\boldsymbol{H}}_{21}[\boldsymbol{T}\mathbf{1}_\tau, \boldsymbol{T}^2\mathbf{1}_\tau, \cdots, \boldsymbol{T}^l\mathbf{1}_\tau]. \tag{3.19}$$

To guarantee each receiver can decode its own message, it is necessary to verify that the signal space and interference space are independent at all receivers, this is shown in [39] using two facts: 1) Channel coefficients are randomly drawn, and 2) The beamforming matrices have Vandermonde structure

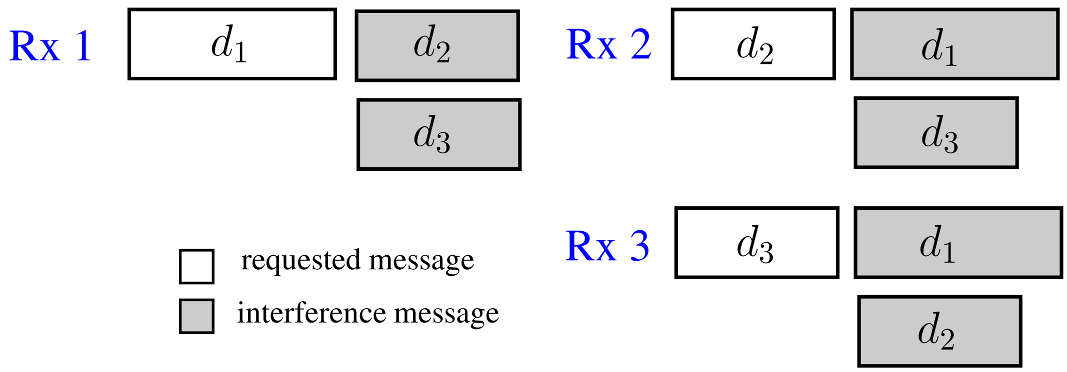

Figure 3.2 Alignment relationship of three user SISO interference channel.

The achievable DoF point is the following

$$(d_1, d_2, d_3) = \left(\frac{l+1}{2l+1}, \frac{l}{2l+1}, \frac{l}{2l+1}\right) \tag{3.20}$$

When $l \to \infty$,

$$(d_1, d_2, d_3) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \tag{3.21}$$

The alignment relationship is shown in Fig. 3.2.

3.3 Summary

In this chapter, we discussed some well-known interference alignment schemes and show their advantages and limitations. In addition, we reviewed the classic asymptotic interference alignment by using a simple example to highlight the key concept. As we pointed before, the interference alignment scheme heavily depends on the channel model and different applications. In the following two chapters, we shall investigate the DoF region of two-user and multi-user systems by exploring different alignment methods.

CHAPTER 4. DOF REGION OF TWO-USER FULL INTERFERENCE CHANNEL AND Z INTERFERENCE CHANNEL WITH RECONFIGURABLE ANTENNAS

4.1 Introduction

Two-user FIC has been studied in [35, 54, 57], where the DoF region with or without CSIT is solved when channel is isotropic block fading. However, it is also shown that there exist potential DoF gain by exploring the channel correlation structure [48]. In this chapter, we investigate the benefit of reconfigurable antennas in two-user FIC and ZIC.

Specifically, we obtain the DoF regions for the cases of:

1. ZIC with CSIT. We show that zero forcing is sufficient for achieving the DoF region in this case (Theorem 1).

2. ZIC and FIC when transmitter one has a number S of antennas modes $S \geq N_1$ (Theorems 2 and 3). Increasing S beyond N_1 does not bring more gains in DoF.

3. ZIC and FIC when $M_1 \leq S < N_1$, in which case each additional antenna mode brings an incremental gain on the DoF region (Theorem 4).

We present joint beamforming and nulling schemes to achieve the DoF region in all cases. When reconfigurable antennas are used, our proposed schemes have an interesting space-frequency coding explanation.

The rest of the chapter is organized as follows. We first present the system model in Section 4.2. Known results on the DoF region of two-user MIMO FIC are also briefly reviewed. The DoF region of ZIC with CSIT is discussed in Section 4.3. The DoF regions of ZIC and

FIC without CSIT when there are enough antenna modes are investigated in Section 4.4. When there are not enough modes, the DoF region is given in Section 4.5. Finally, Section 6.5 concludes this chapter.

4.2 System model and known results

4.2.1 System model

As there are only two transmit-receive pairs in the system and the number of antennas are not the same, the system is termed as an (M_1, N_1, M_2, N_2) system, which can be described as

$$\boldsymbol{y}_1(t) = \boldsymbol{H}_{11}(t)\boldsymbol{x}_1(t) + \boldsymbol{H}_{12}(t)\boldsymbol{x}_2(t) + \boldsymbol{z}_1(t), \tag{4.1}$$

$$\boldsymbol{y}_2(t) = \boldsymbol{H}_{21}(t)\boldsymbol{x}_1(t) + \boldsymbol{H}_{22}(t)\boldsymbol{x}_2(t) + \boldsymbol{z}_2(t). \tag{4.2}$$

Assume the CSIR is always available. We would like to study the DoF regions of MIMO two-user FIC and ZIC with or without CSIT. The DoF region is defined as follows:

$$\mathcal{D} := \left\{(d_1, d_2) \in \mathbb{R}_+^2 : \exists (R_1(P), R_2(P)) \in C(P), \text{such that } d_i = \lim_{P \to \infty} \frac{R_i(P)}{\log(P)}, \quad i = 1, 2\right\}. \tag{4.3}$$

For the no CSIT case, we further assume that one transmitter is equipped with reconfigurable antennas and the channel between different modes to the receivers are isotropic fading. As we stated before, by switching to different modes, the effective channel for the whole block is not isotropic fading and not i.i.d. over the time slots within the block.

One may view our model approximately as a transition from an effective channel where all the links have exactly the same coherent time as in [57] to an effective channel where the links do not have the same coherent time [48]. However, there are two important distinctions between antenna mode switching and variation of channel coherence time: i) Antenna switching can be initiated at will at the transmitter, whereas channel coherence structure is in general not controllable. ii) The resulting equivalent channel from antenna mode switching is not "staggered" [48], so methods therein do not apply here. Similar to [49], we use reconfigurable antennas to explore multiplexing gain other than diversity gain.

4.2.2 Known results on FIC

We first present some known results on DoF region of MIMO full interference channel which will be useful for developing our results.

The total degrees of freedom of two-user MIMO full interference channel with CSIT is developed in [35, Theorem 2], which leads to the following DoF regions:

$$d_i \leq \min(M_i, N_i), \quad i = 1, 2; \tag{4.4}$$

$$d_1 + d_2 \leq \min(\max(N_1, M_2), \max(M_1, N_2), N_1 + N_2, M_1 + M_2). \tag{4.5}$$

An outer bound of degrees of freedom region of two-user MIMO full interference channel without CSIT is as follows [56, Theorem 1]:

$$d_i \leq \min(M_i, N_i), \quad i = 1, 2; \tag{4.6}$$

$$d_1 + \frac{\min(N_1, N_2, M_2)}{\min(N_2, M_2)} d_2 \leq \min(M_1 + M_2, N_1); \tag{4.7}$$

$$\frac{\min(N_1, N_2, M_1)}{\min(N_1, M_1)} d_1 + d_2 \leq \min(M_1 + M_2, N_2). \tag{4.8}$$

Note that the same result is also given in [54], though in a less compact form.

It is shown in [54] that the outer bound given in (4.6)–(4.8) can be achieved by zero forcing or time sharing except for the case $M_1 < N_1 < \min(M_2, N_2)$, for which it was not known how to achieve

$$(d_1, d_2) = \left(M_1, \frac{\min(M_2, N_2)(N_1 - M_1)}{N_1} \right) \tag{4.9}$$

in general. The cases when $N_1 > N_2$ can be converted by switching the user indices. It is shown in [57] that when the channel is isotropic fading and i.i.d. over time, the outer bound given in (4.6)–(4.8) is not tight: if $N_1 \leq N_2$, the DoF region of FIC without CSIT can be given as follows:

$$d_i \leq \min(M_i, N_i), \quad i = 1, 2; \tag{4.10}$$

$$d_1 + \frac{\min(N_1, M_2) - \alpha}{\min(N_2, M_2) - \alpha} (d_2 - \alpha) \leq \min(M_1, N_1). \tag{4.11}$$

where $\alpha = \min(M_1 + M_2, N_1) - \min(M_1, N_1)$. In other words, (4.9) is not achievable when $M_1 < N_1 < \min(M_2, N_2)$, as (4.11) is reduced to

$$d_1 + \frac{M_1}{\min(M_2, N_2) - (N_1 - M_1)} d_2 \leq M_1 + \frac{M_1(N_1 - M_1)}{\min(M_2, N_2) - (N_1 - M_1)} \tag{4.12}$$

and the DoF pair $(d_1, d_2) = (M_1, N_1 - M_1)$ is the corner point of the DoF region.

4.3 Two-user MIMO ZIC with CSIT

In this section, we prove the following theorem.

Theorem 1 (ZIC with CSIT)**.** *The DoF region of a two-user MIMO Z interference channel with CSIT is described by*

$$d_i \leq \min(M_i, N_i), \quad i = 1, 2; \tag{4.13}$$

$$d_1 + d_2 \leq \min(\max(N_1, M_2), N_1 + N_2, M_1 + M_2). \tag{4.14}$$

Proof. We split the proof into the achievability and converse parts, as the following two lemmas. The theorem can be proved by showing the regions given by Lemma 1 and Lemma 2 are the same for all the cases. □

Lemma 1 (Achievability part of Theorem 1)**.** *The following region of two-user MIMO ZIC with CSIT is achievable:*

$$d_i \leq \min(M_i, N_i), \quad i = 1, 2; \tag{4.15}$$

$$\begin{aligned} d_1 + d_2 \leq& \min(N_1, M_1 + \min(N_2, M_2))1(M_2 < N_1) \\ &+ \min(M_2, N_2 + \min(N_1, M_1))1(M_2 \geq N_1) \end{aligned} \tag{4.16}$$

where $1(\cdot)$ *is indicator function.*

Proof. If $M_2 \geq N_1$ and assume transmitter 1 sends d_1 streams, transmitter 2 can send at most $M_2 - N_1$ streams along the null space of $\boldsymbol{H}_{12}$ without interfering receiver 1. Transmitter 2 can also send at most $N_1 - d_1$ streams along the row space of $\boldsymbol{H}_{12}$. Therefore user 2 can decode $\min((M_2 - N_1) + (N_1 - d_1), N_2)$ streams without interfering receiver 1. If $N_1 \geq M_2$

and assume transmitter 2 sends d_2 streams which interfere receiver 1, transmitter 1 can send $\min(N_1 - d_2, M_1)$ decodable streams to receiver 1. Combining these two cases, we have the achievable DoF region shown in this lemma. $\square$

Lemma 2 (Conversepart of Theorem 1)**.** *The region given by* (4.13) *and* (4.14) *is a valid outer bound for the two-user MIMO ZIC with CSIT.*

Proof. It is obvious that adding antennas at the receiver will not shrink the DoF region. Hence, we can add M_1 antennas to receiver 2 resulting an $(M_1, N_1, M_2, M_1 + N_2)$ MIMO FIC, and (4.14) follows from Corollary 1 in [35]. The outer bound of such a MIMO FIC is a valid outer bound of an (M_1, N_1, M_2, N_2) MIMO ZIC. Combining the trivial upper bound on point-to-point system, we have this lemma. $\square$

Based on Lemma 1, zero forcing at receiver is sufficient to achieve the DoF region of ZIC when CSIT is available. The antenna mode switching ability is not needed in this case. However, we shall see later that such an ability is important for the case when CSIT is absent.

4.4 Two-user MIMO ZIC and FIC without CSIT when number of modes $S \geq N_1$

In this section, we describe the DoF regions of two-user ZIC and FIC without CSIT but with transmitter side reconfigurable antennas. We deal with the case that S, the number of antenna modes is at least equal to the N_1. The case $S < N_1$ will be dealt with in Section 4.5.

Based on the antenna number configuration, the achievability scheme of ZIC and FIC without CSIT can be divided into two cases. In the first case, no reconfigurable antenna is needed to achieve an DoF outer bound — reconfigurable antennas are not helpful (Section 4.4.2). In the second case, the outer bound can be achieved with enough transmit side antenna modes (Section 4.4.3): reconfigurable antennas enlarges the DoF region. Our main results in this section are the following two theorems.

Theorem 2 (ZIC with Enough Reconfigurable Antenna Modes)**.** *The DoF region of two-user MIMO Z interference channel without CSIT is described by the following inequalities*

$$d_i \le \min(M_i, N_i), \quad i = 1, 2; \tag{4.17}$$

$$d_1 + \frac{\min(N_1, N_2, M_2)}{\min(N_2, M_2)} d_2 \le \min(M_1 + M_2, N_1). \tag{4.18}$$

if either one of the following is true:

C1) $M_1 < N_1 < \min(M_2, N_2)$ *and transmitter one can switch among* N_1 *antenna modes, or*

C2) (M_1, N_1, M_2, N_2) *do not satisfy the above condition.* □

The DoF region in Theorem 2 is shown in Fig. 4.1.

Theorem 3 (FIC with Enough Reconfigurable Antenna Modes)**.** *The DoF region of two-user MIMO full interference channel without CSIT is described by the inequalities* (4.6)–(4.8) *if any one of the following is true:*

C1) $M_1 < N_1 < \min(M_2, N_2)$ *and transmitter one can switch among* N_1 *antenna modes, or*

C2) $M_2 < N_2 < \min(M_1, N_1)$ *and transmitter two can switch among* N_2 *antenna modes, or*

C3) (M_1, N_1, M_2, N_2) *are not one of the two above cases.* □

4.4.1 Converse part

We first prove the converse part of the two theorems.

Lemma 3 (Transmit antenna mode switching cannot change DoF)**.** *For any channel with non-zero DoF, using reconfigurable antennas with finite number of modes at transmitter would not change the DoF.*

Proof. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the channel input and output, respectively. Let A be the index of transmit antenna modes. We can view the receiver as receiving both $\boldsymbol{x}$ and A. We have

$$\begin{aligned} \mathcal{I}(\boldsymbol{y}; \boldsymbol{x}, A) &= \mathcal{I}(\boldsymbol{y}; A) + \mathcal{I}(\boldsymbol{y}; \boldsymbol{x}|A) && (4.19)\\ &\le \mathcal{H}(A) + \mathcal{I}(\boldsymbol{y}; \boldsymbol{x}|A) && (4.20)\\ &\le o(\log(P)) + \mathcal{I}(\boldsymbol{y}; \boldsymbol{x}|A) && (4.21) \end{aligned}$$

On the other hand,

$$\mathcal{I}(\boldsymbol{y};\boldsymbol{x},A) \geq \mathcal{I}(\boldsymbol{y};\boldsymbol{x}|A) \tag{4.22}$$

Therefore,

$$\mathcal{I}(\boldsymbol{y};\boldsymbol{x},A) = \mathcal{I}(\boldsymbol{y};\boldsymbol{x}|A) + o(\log(P)) \tag{4.23}$$

Hence, transmit mode switching will not affect the DoF of the channel even it is not available to the receiver. □

In the following of this chapter, we assume that the transmit antenna mode switching patten is known to the receiver for simplicity according to Lemma 3.

Lemma 4 (Converse part of Theorem 3)**.** *The outer bound of DoF region of two-user MIMO full interference channel given in* (4.6)–(4.8) *is still valid when either or both transmitters are using antenna mode switching.*

Proof. The outer bound (4.7) has been derived based on the assumption that the rows of $\boldsymbol{H}_{12}$ and those of $\boldsymbol{H}_{22}$ are statistically equivalent [54, 56]. Similarly, the outer bound (4.8) has been derived based on the assumption that the rows of $\boldsymbol{H}_{11}$ and those of $\boldsymbol{H}_{21}$ are statistically equivalent. These assumptions are not affected by antenna mode switching at either or both transmitters. Hence, the DoF outer bound is still valid. □

Lemma 5 (Converse part of Theorem 2)**.** *The outer bound of degrees of freedom region of two-user MIMO Z interference channel without CSIT can be given as follows when transmitter one has the antenna mode switching ability*

$$d_i \leq \min(M_i, N_i), \quad i = 1, 2; \tag{4.24}$$

$$d_1 + \frac{\min(N_1, N_2, M_2)}{\min(N_2, M_2)} d_2 \leq \min(M_1 + M_2, N_1). \tag{4.25}$$

Proof. This is the direct result of [56, Theorem 1] as in (4.6)–(4.8), by noticing that there is no interference from transmitter 1 to receiver 2 hence (4.8) is not longer needed. The antenna switching at transmitter one does not affect the upper bound, for the same reason stated in Lemma 4. □

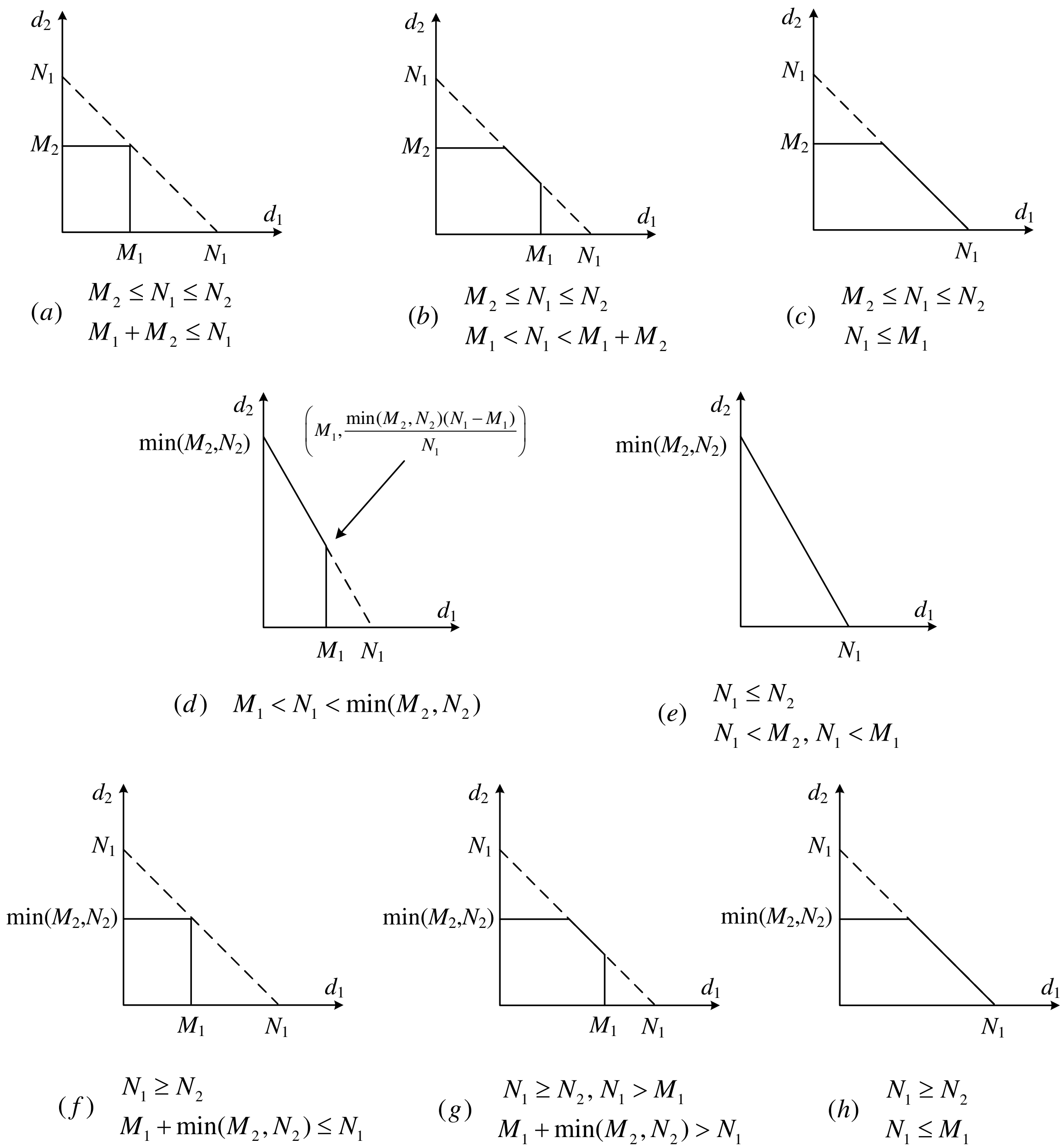

Figure 4.1 DoF region of two-user MIMO ZIC without CSIT when number of antenna modes $S \ge N_1$. Figures (a)–(e) are for the case $N_1 \le N_2$; Figures (f)–(h) are for the case $N_1 \ge N_2$.

4.4.2 Achievability: when antenna mode switching is not needed

In this section, we prove the achievability part for Case C2) of Theorem 2 and Case C3) of Theorem 3. Achievability for the remaining cases are left to Section 4.4.3.

Lemma 6. *For the two-user MIMO Z interference channel without CSIT, when $N_1 \geq N_2$,* (4.25) *is achievable by zero forcing.*

Proof. When $N_1 \geq N_2$, the corresponding outer regions are shown in Fig. 4.1 (f)–(h). Noticing that (4.25) is reduced to $d_1 + d_2 \leq \min(M_1 + M_2, N_1)$, zero forcing is sufficient to achieve the outer bound. □

Lemma 7. *When CSIT is absent, the DoF outer region given by Lemma 5 of a two-user MIMO* (M_1, N_1, M_2, N_2) *ZIC is the same as that of an* $(M_1, N_1, \min(M_2, N_2), \min(M_2, N_2))$ *ZIC.*

Proof. We give the proof case by case. It is trivial that when $M_2 \leq N_2$ reducing the number of antennas at receiver 2 to M_2 will not shrink the DoF region. When $M_2 > N_2$, we can further consider two sub-cases: $N_2 \geq N_1$ and $N_2 < N_1$.

1. When $M_2 > N_2 \geq N_1$, corresponding to Fig. 4.1 (d) and (e), the DoF bound (4.25) becomes $\frac{d_1}{N_1} + \frac{d_2}{N_2} \leq 1$. Hence the DoF outer region is the same as an (M_1, N_1, N_2, N_2) ZIC.

2. When $M_2 > N_2$ and $N_2 < N_1$, the DoF bound (4.25) becomes $d_1 + d_2 \leq \min(M_1 + M_2, N_1)$. Hence, if $M_1 \geq N_1 - N_2$, which implies $M_1 + \min(M_2, N_2) \geq N_1$, the DoF outer region is a pentagon or a tetragon; see Fig. 4.1 (g) and (h). Otherwise, it is a square, see Fig. 4.1 (f). One can show that the region is the same as that of an (M_1, N_1, N_2, N_2) ZIC.

Hence, the lemma holds. □

We also have the following lemma regarding the relationship between DoF regions of ZIC and FIC.

Lemma 8. *When $N_1 \leq N_2$, the MIMO ZIC and FIC have the same DoF regions. Any encoding scheme that is DoF optimal for one channel is also DoF optimal for the other.*

Proof. Any point in the FIC is also trivially achievable in the ZIC because user 2's channel is interference free. Conversely, any point achievable in the ZIC region, is also achievable in FIC. This is based on the fact that the channels are statistically equivalent at both receivers. If receiver 1 can decode user 1's message, then receiver 2, having at least as many antennas, must also be able to decode the same message. Receiver 2 can then subtract the decoded message, which renders the resulting channel the same as in the ZIC. □

Due to Lemma 8, we can translate all achievability schemes from FIC to ZIC and vice versa when $N_1 \leq N_2$. Therefore the achievability schemes in [54] for FIC when $N_1 \leq N_2$ and $M_1 \geq N_1$ can be used for ZIC. Therefore, the achievability part for Case C2) of Theorem 2 is complete.

For the FIC, the achievability for the case $N_1 \leq N_2$, except when $M_1 < N_1 < \min(M_2, N_2)$, is shown in [54]. When $N_1 \geq N_2$, we can swap the indices of the two users, so that except for the Cases C1) and C2) the achievability scheme is known for FIC.

4.4.3 Achievability: with antenna mode switching when $S \geq M_1 N_1$

In this subsection, we prove a weaker version of the achievability for Case C1) of Theorem 2 and Cases C1) and C2) of Theorem 3. Namely, we assume that the number of antenna modes available is $S \geq M_1 N_1$. The scheme is simpler in this case, and the achievability scheme for the case $S = N_1$ will be built upon this case.

Based on Lemma 7 and Lemma 8, we only consider the two-user MIMO ZIC with $M_1 < N_1 < M_2 = N_2$ to prove the Cases C1) for both theorems. Case C2) of Theorem 3 is the Case of C1) with user indices swapped. Therefore, we want to show that the following DoF pair is achievable for ZIC with $S_1 = M_1 N_1$ modes:

$$(d_1, d_2) = \left(M_1, \frac{M_2(N_1 - M_1)}{N_1} \right). \tag{4.26}$$

We first notice that this point cannot be achieved by zero forcing over one time instant. This is because using zero forcing if transmitter 1 sends M_1 streams, transmitter 2 can only send $N_1 - M_1$ streams without interfering receiver 1. If transmitter 2 sends more streams, the desired signal and interference are not separable at receiver 1 as transmitter 2 does not know channel state information so it cannot send streams along the null space of $\boldsymbol{H}_{12}$. A simple example is the $(1, 2, 3, 3)$ case, where the outer bound gives us $(d_1, d_2) = (1, 1.5)$, which is not achievable via zero forcing over one time slot. We make the assumption that the channel coherent time L is at least N_1. Therefore, it is sufficient to show that $(M_1N_1, M_2(N_1 - M_1))$ streams can be achieved in N_1 time slots.

We first develop the beamforming and nulling design by assuming that there are N_1M_1 antenna modes available at transmitter 1 such that it can use different antenna modes in different slots to create channel variation. We will further show that the resultant beamforming and nulling design still work even if there are only N_1 modes available.

The time expansion channel between transmitter 1 and receiver 1 in N_1 time slots is

$$\tilde{\boldsymbol{H}}_{11}=\begin{bmatrix} \boldsymbol{H}_{11}(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H}_{11}(2) & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{H}_{11}(N_1) \end{bmatrix}_{N_1^2 \times N_1 M_1}$$

and the channel between transmitter 2 and receiver 1 is

$$\tilde{\boldsymbol{H}}_{12} = \boldsymbol{I}_{N_1} \otimes \boldsymbol{H}_{12}(1) \tag{4.27}$$

as transmitter 2 does not create channel variation. We will use precoding at transmitter 2 only and nulling at receiver 1 only. Let $\tilde{\boldsymbol{P}}$ be the transmit beamforming matrix at transmitter 2 and $\tilde{\boldsymbol{Q}}$ be the nulling matrix at receiver 1. We propose to use the following structures for them

$$\tilde{\boldsymbol{P}}_{M_2N_1 \times M_2(N_1 - M_1)} = \boldsymbol{P}_{N_1 \times (N_1 - M_1)} \otimes \boldsymbol{I}_{M_2} \tag{4.28}$$

$$\tilde{\boldsymbol{Q}}_{M_1N_1 \times N_1^2} = \boldsymbol{Q}_{M_1 \times N_1} \otimes \boldsymbol{I}_{N_1}. \tag{4.29}$$

The received signal at receiver 1 can be written as

$$\tilde{\boldsymbol{y}}_1 = \tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{P}}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{z}}_1 \tag{4.30}$$

where $\tilde{\boldsymbol{x}}_1$ is a length M_1N_1 vector, and $\tilde{\boldsymbol{x}}_2$ is a length $M_2(N_1 - M_1)$ vector. After applying nulling matrix $\tilde{\boldsymbol{Q}}$, we have

$$\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{y}}_1 = \underbrace{\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{H}}_{11}}_{\tilde{\boldsymbol{A}}}\tilde{\boldsymbol{x}}_1 + \underbrace{\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{P}}}_{\tilde{\boldsymbol{B}}}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{Q}}\tilde{\boldsymbol{z}}_1. \tag{4.31}$$

To achieve the degrees of freedom $(M_1N_1, M_2(N_1 - M_1))$ for both users, it is sufficient to design our $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{Q}}$ to satisfy the following conditions simultaneously

1. $\mathrm{rank}(\tilde{\boldsymbol{A}}) = M_1N_1$,

2. $\mathrm{rank}(\tilde{\boldsymbol{P}}) = M_2(N_1 - M_1)$,

3. $\tilde{\boldsymbol{B}} = \boldsymbol{0}$.

The second condition can be easily satisfied. Because $\mathrm{rank}(\tilde{\boldsymbol{P}}) = \mathrm{rank}(\boldsymbol{P})\mathrm{rank}(\boldsymbol{I}_{M_2})$, we only need to design $\boldsymbol{P}$ such that $\mathrm{rank}(\boldsymbol{P}) = N_1 - M_1$. As to the third condition, notice that

$$\begin{aligned}
\tilde{\boldsymbol{B}} &= (\boldsymbol{Q} \otimes \boldsymbol{I}_{N_1})(\boldsymbol{I}_{N_1} \otimes \boldsymbol{H}_{12}(1))(\boldsymbol{P} \otimes \boldsymbol{I}_{M_2}) \\
&= (\boldsymbol{Q}\boldsymbol{I}_{N_1}\boldsymbol{P}) \otimes (\boldsymbol{I}_{N_1}\boldsymbol{H}_{12}(1)\boldsymbol{I}_{M_2}) \\
&= (\boldsymbol{Q}\boldsymbol{P}) \otimes \boldsymbol{H}_{12}(1).
\end{aligned}$$

It is therefore sufficient (and also necessary) to have $\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{0}$. Then the key is to find a $\boldsymbol{Q}$ such that the equivalent channel of user 1 after nulling

$$\tilde{\boldsymbol{A}} = (\boldsymbol{Q} \otimes \boldsymbol{I}_{N_1})\tilde{\boldsymbol{H}}_{11} \tag{4.32}$$

has full rank M_1N_1 with probability 1. The matrix $\tilde{\boldsymbol{A}}$ is of size $M_1N_1 \times M_1N_1$ and has the following structure

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} q_{11}\boldsymbol{H}_{11}(1) & q_{12}\boldsymbol{H}_{11}(2) & \cdots & q_{1N_1}\boldsymbol{H}_{11}(N_1) \\ q_{21}\boldsymbol{H}_{11}(1) & q_{12}\boldsymbol{H}_{11}(2) & \cdots & q_{2N_1}\boldsymbol{H}_{11}(N_1) \\ \vdots & \vdots & \ddots & \vdots \\ q_{M_11}\boldsymbol{H}_{11}(1) & q_{M_12}\boldsymbol{H}_{11}(2) & \cdots & q_{M_1N_1}\boldsymbol{H}_{11}(N_1) \end{bmatrix}.$$

To show that $\tilde{\boldsymbol{A}}$ has full rank, we need the following lemma, which is known before, and a proof of it can be found in e.g., [80].

Lemma 9. *[80, Lemma 2] Consider an analytic function $h(\boldsymbol{x})$ of several variables $\boldsymbol{x} = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$. If h is nontrivial in the sense that there exists $\boldsymbol{x}_0 \in \mathbb{C}^n$ such that $h(\boldsymbol{x}_0) \neq 0$, then the zero set of $f(\boldsymbol{x})$ $\mathrm{Z} := \{\boldsymbol{x} \in \mathbb{C}^n | h(\boldsymbol{x}) = 0\}$ is of measure (Lebesgue measure in $\mathbb{C}^n$) zero.* □

Because the determinant of $\tilde{\boldsymbol{A}}$ is an analytic polynomial function of elements of $\boldsymbol{H}_{11}(t), t = 1, \ldots, N_1$, we only need to find a specific pair of $\boldsymbol{Q}$ and $\boldsymbol{H}_{11}(t), t = 1, \ldots, N_1$, such that $\tilde{\boldsymbol{A}}$ is full rank. We propose the following:

$$\boldsymbol{Q} = [\boldsymbol{\mathcal{V}}_{N_1}(1, \omega_{N_1}, \ldots, \omega_{N_1}^{M_1-1})]^T, \tag{4.33}$$

where $\omega_{N_1} := \exp(-j2\pi/N_1)$.

Let $\omega := \exp(-j2\pi/N_1^2)$. Take the realizations of $\boldsymbol{H}_{11}(t)$, $t = 1, \ldots, N_1$, as

$$\boldsymbol{H}_{11}(t) = \boldsymbol{\mathcal{V}}_{N_1}(\omega^{t-1}, \omega^{N_1+t-1}, \ldots, \omega^{(M_1-1)N_1+t-1}). \tag{4.34}$$

It can be verified that for such choices of $\boldsymbol{Q}$ and $\boldsymbol{H}_{11}(t)$, $\tilde{\boldsymbol{A}}$ is a Vandermonde matrix:

$$\begin{aligned}\tilde{\boldsymbol{A}} = \boldsymbol{\mathcal{V}}_{M_1N_1}(1, \omega^{N_1}, \ldots, \omega^{(M_1-1)N_1}, \omega^1, \omega^{N_1+1}, \ldots, \omega^{(M_1-1)N_1+1},\\ \ldots, \omega^{N_1-1}, \omega^{2N_1-1}, \ldots, \omega^{M_1N_1-1}),\end{aligned}$$

hence of full rank. We also notice that $\tilde{\boldsymbol{A}}$ is a leading principal minor of a permuted fast Fourier transform (FFT) matrix with size $N_1^2 \times N_1^2$. The permutation is as follows: Index the columns of an FFT matrix $0, 1, \ldots, N_1^2 - 1$, and then permute them in an order shown below:

$$(0, N_1, 2N_1, \ldots, (M_1-1)N_1), (1, N_1+1, 2N_1+1, \ldots, (M_1-1)N_1+1), \ldots$$

Based on Lemma 9, if we choose the nulling matrix using $\boldsymbol{Q}$ as specified in (4.33), $\tilde{\boldsymbol{A}}$ has full rank almost surely. One choice of the corresponding $\boldsymbol{P}$ matrix with respect to (4.33) is the following

$$\boldsymbol{P} = \boldsymbol{\mathcal{V}}_{N_1}(\omega_{N_1}^{-M_1}, \omega_{N_1}^{-(M_1+1)}, \ldots, \omega_{N_1}^{-(N_1-1)}), \tag{4.35}$$

which is orthogonal to $\boldsymbol{Q}$. This completes the achievability part under conditions in Case C1) of Theorem 2 and Cases C1) and C2) of Theorem 3, but with $S \geq M_1N_1$.

4.4.4 Achievability: with antenna mode switching when $S = N_1$

Assuming there are N_1 modes available at transmitter 1 and denote these channel vectors between receive antennas of user 1 and the ith mode as $\boldsymbol{h}_i, 1 \leq i \leq N_1$ and let $\hat{\boldsymbol{H}}_{N_1 \times N_1} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_{N_1}]$. We choose the antenna modes to be switched in the following way:

$$\boldsymbol{H}_{11}(1) = [\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_{M_1}], \tag{4.36}$$

$$\boldsymbol{H}_{11}(2) = [\boldsymbol{h}_2, \boldsymbol{h}_3, \ldots, \boldsymbol{h}_{M_1+1}], \tag{4.37}$$

$$\vdots$$

$$\boldsymbol{H}_{11}(N_1) = [\boldsymbol{h}_{N_1}, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_{M_1-1}]. \tag{4.38}$$

We want to show that under this switching pattern, the equivalent channel $\tilde{\boldsymbol{A}}$ in (4.32) between transmitter one and receiver one after nulling, is still full rank. To show this, indexing the columns of $\tilde{\boldsymbol{A}}$ in (4.32) as $0, 1, \ldots, N_1M_1 - 1$, we then permute and group the columns of $\tilde{\boldsymbol{A}}$ in the following way:

$$(0, M_1, 2M_1, \ldots, (M_1-1)N_1), (1, M_1+1, 2M_1+1, \ldots, (M_1-1)N_1+1), ...$$

Denote the permutation result as $\tilde{\boldsymbol{A}}'$ and it can be expressed as

$$\tilde{\boldsymbol{A}}' = \begin{bmatrix} \hat{\boldsymbol{H}} & \hat{\boldsymbol{H}} & \cdots & \hat{\boldsymbol{H}} \\ \boldsymbol{G}\hat{\boldsymbol{H}} & \omega_{N_1}^{-1}\boldsymbol{G}\hat{\boldsymbol{H}} & \cdots & \omega_{N_1}^{-M_1}\boldsymbol{G}\hat{\boldsymbol{H}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{G}^{M_1}\hat{\boldsymbol{H}} & (\omega_{N_1}^{-1}\boldsymbol{G})^{M_1}\hat{\boldsymbol{H}} & \cdots & (\omega_{N_1}^{-M_1}\boldsymbol{G})^{M_1}\hat{\boldsymbol{H}} \end{bmatrix},$$

where $\boldsymbol{G}$ is a size $N_1 \times N_1$ diagonal matrix and can be expressed as

$$\boldsymbol{G} = \mathrm{diag}(1, \omega_{N_1}, \omega_{N_1}^2, \ldots, \omega_{N_1}^{N_1-1}). \tag{4.39}$$

Notice that $\tilde{\boldsymbol{A}}' = \boldsymbol{R}(\boldsymbol{I}_{M_1} \otimes \hat{\boldsymbol{H}})$, where

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I}_{N_1} & \boldsymbol{I}_{N_1} & \cdots & \boldsymbol{I}_{N_1} \\ \boldsymbol{G} & \omega_{N_1}^{-1}\boldsymbol{G} & \cdots & \omega_{N_1}^{-M_1}\boldsymbol{G} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{G}^{M_1} & (\omega_{N_1}^{-1}\boldsymbol{G})^{M_1} & \cdots & (\omega_{N_1}^{-M_1}\boldsymbol{G})^{M_1} \end{bmatrix}.$$

Recall $\omega_{N_1} = \exp(-j2\pi/N_1)$. To show $\tilde{\boldsymbol{A}}$ is full rank, it is necessary to show $\boldsymbol{R}$ is full rank as $\boldsymbol{I}_{M_1} \otimes \hat{\boldsymbol{H}}$ is full rank with probability 1. It can be verified that via row and column permutations $\boldsymbol{R}$ can be changed to a block diagonal matrix with the ith block being

$$\boldsymbol{\mathcal{V}}_{M_1}(\omega_{N_1}^{i}, \omega_{N_1}^{i-1}, \cdots, \omega_{N_1}^{i-M_1+1}), \tag{4.40}$$

which is full rank due to Vandermonde structure. Hence $\boldsymbol{R}$ is full rank. It follows that $\boldsymbol{A}$ is full rank with probability 1. This completes the achievability part under conditions in Case C1) of Theorem 2 and Cases C1) and C2) of Theorem 3 for $S = N_1$.

4.4.5 Discussion

4.4.5.1 Frequency domain interpretation

We note that the matrix $[\boldsymbol{Q}^{\dagger},\ \boldsymbol{P}]$ is an inverse FFT (IFFT) matrix in our construction (4.28), (4.29), (4.33) and (4.35). This observation yields an interesting frequency domain interpretation of our construction. The signal of user 2 is transmitted over frequencies corresponding to the last $N_1 - M_1$ columns of an IFFT matrix, whereas the first user's signal is transmitted on all frequencies. Due to the antenna mode switching at transmitter 1, the channel between transmitter 1 and receiver 1 is now time-varying and we manually introduce frequency spread. User 1's signal is spread from one frequency bin to all the frequencies while user 2's signal remains in the last $N_1 - M_1$ frequency bins. Therefore the signal in the first M_1 bins is interference free, which can be used to decode user 1's message. The nulling matrix applied at receiver 1 has a projection explanation as well. Left multiplying the left and right hand sides of (4.31) with $\tilde{\boldsymbol{Q}}^{\dagger}$ yields

$$\begin{aligned}\tilde{\boldsymbol{Q}}^{\dagger}\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{y}}_1 &= \tilde{\boldsymbol{Q}}^{\dagger}\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{Q}}^{\dagger}\boldsymbol{Q}\tilde{\boldsymbol{z}}_1 \\ &= ((\boldsymbol{Q}^{\dagger}\boldsymbol{Q}) \otimes \boldsymbol{I}_{N_1})(\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{z}}_1),\end{aligned}$$

where $\boldsymbol{Q}^{\dagger}\boldsymbol{Q}$ is the frequency domain projection matrix. We can see that the signal of user 1 is projected from N_1 frequencies to the first M_1 frequencies.

4.4.5.2 The loss of DoF due to lack of CSIT

In two-user MIMO Z interference channel without CSIT, losing CSIT will not shrink degrees of freedom region if $M_2 \leq N_1$ or $M_2 > N_1 \geq N_2 + M_1$. For all the other cases, the degrees of freedom region is strictly smaller when comparing with the CSIT case.

This observation can be verified case by case. Notice that it is already shown in [54, Theorem 2] that when $M_2 \leq N_1 \leq N_2$ absence of CSIT does not reduce DoF region in two-user MIMO FIC. Because MIMO FIC and ZIC has the same DoF region when $N_1 \leq N_2$. We only need to consider the sub cases when $N_1 > N_2$, corresponding to (f)–(h) in Fig. 4.1.

1. If $M_2 < N_1$ and $N_1 > N_2$, the total DoF of MIMO ZIC is upper bounded by N_1 due to (4.14), so the DoF region remains the same if CSIT is absent.

2. If $M_2 > N_1 > N_2$, the DoF region of MIMO ZIC without CSIT is a square only when $M_1 + N_2 \leq N_1$, same as that of ZIC with CSIT. Otherwise, the maximum total DoF of ZIC with CSIT is $\min(M_2, N_1 + N_2, \min(M_1, N_1) + N_2)$, strictly larger than N_1 which is the maximum total DoF when CSIT is absent, hence loss of CSIT reduces the DoF region.

4.4.5.3 Alternative construction when $N_1/M_1 = \beta \in \mathbb{Z}$

When $N_1/M_1 = \beta \in \mathbb{Z}$, instead of using the $\boldsymbol{Q}$ given in (4.33) we can use the following $\boldsymbol{Q}_{M_1 \times N_1} = \boldsymbol{I}_{M_1} \otimes \mathbf{1}_\beta^T$. We need to show that this $\boldsymbol{Q}$ matrix will lead to a full rank $\tilde{\boldsymbol{A}}$. This can be achieved by choosing $\tilde{\boldsymbol{H}}_{11}$ such that it can be decomposed as $\tilde{\boldsymbol{H}}_{11} = \boldsymbol{I}_{M_1} \otimes \tilde{\boldsymbol{H}}_{11}'$, where

$$\tilde{\boldsymbol{H}}_{11}' = \begin{bmatrix} \boldsymbol{H}_{11}(1) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H}_{11}(2) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{H}_{11}(\beta) \end{bmatrix}_{N_1\beta \times N_1} .$$

For this $\tilde{\boldsymbol{H}}_{11}$

$$\begin{aligned} \tilde{\boldsymbol{A}} &= (\boldsymbol{I}_{M_1} \otimes \mathbf{1}_\beta^T \otimes \boldsymbol{I}_{N_1})(\boldsymbol{I}_{M_1} \otimes \tilde{\boldsymbol{H}}_{11}') \\ &= \boldsymbol{I}_{M_1} \otimes ((\mathbf{1}_\beta^T \otimes \boldsymbol{I}_{N_1})\tilde{\boldsymbol{H}}_{11}'), \end{aligned}$$

which has full rank. For this choice of $\boldsymbol{Q}$, we only use $\beta M_1 = N_1$ antenna modes in N_1 time slots.

Therefore, for the two-user MIMO ZIC and FIC when $M_1 < N_1 < \min(M_2, N_2)$ and $N_1/M_1 = \beta \in \mathbb{Z}$, β fold time expansion is enough to achieve the DoF region. We remark that this can be viewed as the generalization of the case we discussed in Section 4.4.3 for $N_1 = \beta$ and $M_1 = 1$. In fact $\mathbf{1}_\beta^T$ is the nulling matrix $\boldsymbol{Q}$ given in (4.33) when $N_1 = \beta, M_1 = 1$.

4.4.5.4 Successive decoding in ZIC

For the two-user MIMO FIC when $M_1 < N_1 < \min(M_2, N_2)$ and CSIT is absent, we need block decoding at both receivers in general, which introduces decoding delay. Successive interference cancellation decoder can be used at receiver 2 to reduce decoding delay. Taking the case $N_1/M_1 = \beta \in \mathbb{Z}$ as an example, we can use β fold time expansion and choose $\boldsymbol{Q} = \mathbf{1}_\beta^T$. The corresponding $\boldsymbol{P}$ matrix is not necessary to be the last $\beta - 1$ columns of an $\beta \times \beta$ FFT matrix. The following $\boldsymbol{P}$ matrix still satisfies the design constraint

$$\boldsymbol{P}_{\beta\times(\beta-1)} = \begin{bmatrix} \boldsymbol{I}_{\beta-1} \\ \mathbf{1}_{\beta-1}^T \end{bmatrix}. \tag{4.41}$$

Here, $\boldsymbol{P}$ has a nice structure. Every stream of user 2 can be decoded immediately as they are interference free. For other cases where M_1 cannot divide N_1, we can still find a $\boldsymbol{Q}$, $\boldsymbol{P}$ pair through numerical simulation such that the upper diagonal parts of $\boldsymbol{P}$ are all zeros and contain small number of nonzero entries. Such a beamforming matrix can guarantee the immediate decoding of user 2's signal as the interference only comes from the streams decoded already.

4.5 Two-user MIMO ZIC and FIC without CSIT when number of modes $S < N_1$

In this section, we will present our result for the $S < N_1$ case. The main result of this section is the following theorem.

Theorem 4. *When $M_1 < N_1 < \min(M_2, N_2)$ and the antennas of transmitter 1 can be switched among S antenna modes, where $S < N_1$, the DoF region of two-user MIMO ZIC and*

FIC without CSIT is given by the following inequalities

$$d_i \le \min(M_i, N_i), \quad i = 1, 2; \tag{4.42}$$

$$d_1 + \frac{S}{\min(M_2, N_2) - (N_1 - K)} d_2 \le M_1 + \frac{S(N_1 - M_1) + (\min(M_2, N_2) - N_1)(S - M_1)}{\min(M_2, N_2) - (N_1 - S)} \tag{4.43}$$

The DoF region of FIC for $M_2 < N_2 < \min(M_1, N_1)$ *can be obtained by switching the two user indices.* □

The method of proof is heavily based on that in [57] and we list the lemmas therein in Section 4.7 as an appendix. Some notation that is used in this section are the following. Recall that the channel coherent time is L, we simply consider time expanded signal over L time slots and let $t \in [1, L]$ be the index of the slot within one block. In addition, for a time expanded vector $\tilde{\boldsymbol{x}}$, we use $\tilde{\boldsymbol{x}}^n$ or $\{\tilde{\boldsymbol{x}}\}^n$ to denote a sequence of n successive blocks of $\tilde{\boldsymbol{x}}$: $\tilde{\boldsymbol{x}}^n = \text{vec}(\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(nL))$. Furthermore, $\boldsymbol{x}(t)^n$ is the sequence of $\boldsymbol{x}(t)$ which contains all the vector $\boldsymbol{x}$ of the tth slot of all n blocks: $\boldsymbol{x}(t)^n = \text{vec}(\boldsymbol{x}(t), \boldsymbol{x}(t+L), \ldots, \boldsymbol{x}(t+(n-1)L))$. Similar notation is defined for matrices as well. We use $\boldsymbol{H}$ denotes $(\boldsymbol{H}_{11}, \boldsymbol{H}_{12}, \boldsymbol{H}_{21}, \boldsymbol{H}_{22})$, hence $\tilde{\boldsymbol{H}}^n$ denotes all the channel matrices over n blocks. In addition, for a random vector $\boldsymbol{x}$, $\boldsymbol{x}_G$ is a corresponding CSCG vector that has the same covariance matrix as $\boldsymbol{x}$.

4.5.1 The converse part

We prove the converse part of Theorem 4 in the following. Recall that for $M_1 < N_1 < \min(M_2, N_2)$, the proof is equivalent for both FIC and ZIC. We will only show the proof for ZIC. To make the proof self-contained, we will go through some similar steps as in [57], but avoiding details.

The converse is developed based on blocking for every L slots. In each block, the channel $\boldsymbol{H}_{12}, \boldsymbol{H}_{22}$ stay the same with the decomposition $\boldsymbol{H}_{12} = \boldsymbol{W}_{12}\boldsymbol{\Lambda}_{12}\boldsymbol{V}_{12}^{\dagger}$ and $\boldsymbol{H}_{22} = \boldsymbol{W}_{22}\boldsymbol{\Lambda}_{22}\boldsymbol{V}_{22}^{\dagger}$, whereas $\boldsymbol{H}_{11}$ is time-varying among L slots due to antenna mode switching at transmitter 1. Transmitter 1 has S modes with $S < N_1$ and it can adopt arbitrary switching pattern. Let $\underline{\boldsymbol{H}}_{11}$ be an $N_1 \times N_1$ full rank random matrix such that $\underline{\boldsymbol{H}}_{11} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \cdots, \boldsymbol{h}_{N_1}]$ and $\boldsymbol{h}_i, 1 \le i \le S$ is the random vector channel between the ith antenna mode and receive antennas of user 1. We introduce the fictitious vectors $\{\boldsymbol{h}_i, S+1 \leq i \leq N_1\}$ to simplify the proof. We assume $\underline{\boldsymbol{H}}_{11}$ is isotropic fading and i.i.d. over blocks of length L each, where L naturally satisfy $L \geq \lceil S/M_1 \rceil$. We denote the decomposition of $\underline{\boldsymbol{H}}_{11}$ as $\underline{\boldsymbol{W}}_{11}\underline{\boldsymbol{\Lambda}}_{11}\underline{\boldsymbol{V}}_{11}^{\dagger}$.

Furthermore, let $\boldsymbol{E}(t)$ of size $N_1 \times M_1$ denote the antenna mode selection matrix for time t. Let $\boldsymbol{e}_m, 1 \leq m \leq N_1$ be the mth column of $\boldsymbol{I}_{N_1}$. Let $i(t)$ denote the mode index selected by antenna i at time t. Then the ith column of $\boldsymbol{E}(t)$ is $\boldsymbol{e}_{i(t)}$. We have $\boldsymbol{H}_{11}(t) = \underline{\boldsymbol{H}}_{11}\boldsymbol{E}(t)$.

At receiver 1, from Fano's inequality, we have

$$nLR_1 - \delta_{nL} \leq \mathcal{I}(\tilde{\boldsymbol{y}}_1^n; \tilde{\boldsymbol{x}}_1^n | \tilde{\boldsymbol{H}}^n). \tag{4.44}$$

where $\delta_{nL} \to 0$ as $n \to \infty$. Denote

$$\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{H}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{z}}_1 \tag{4.45}$$

$$\tilde{\boldsymbol{r}}_1 = \tilde{\boldsymbol{W}}_{12}^{\dagger}\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_1 \tag{4.46}$$

where $\tilde{\boldsymbol{n}}_1 = \tilde{\boldsymbol{W}}_{12}^{\dagger}\tilde{\boldsymbol{z}}_1$. Using Lemma 13, which says that Gaussian input can reduce the mutual information by at most an $o(\log(P))$ quantity, and two uses of chain rule we have

$$\begin{aligned}
&nLR_1 - n\,o(\log(P)) \\
&\quad \leq \mathcal{I}(\tilde{\boldsymbol{r}}^n; \boldsymbol{x}_{1G}^n | \tilde{\boldsymbol{H}}^n) && (4.47) \\
&\quad = \mathcal{I}\left(\tilde{\boldsymbol{r}}^n; \boldsymbol{x}_{1G}^n | \tilde{\boldsymbol{x}}_2^n, \tilde{\boldsymbol{H}}^n\right) + \mathcal{I}(\tilde{\boldsymbol{r}}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) - \mathcal{I}\left(\{\tilde{\boldsymbol{H}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{z}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right). && (4.48)
\end{aligned}$$

Using Lemma 11, we have

$$\begin{aligned}
\mathcal{I}\left(\{\tilde{\boldsymbol{H}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{z}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) &= \mathcal{I}\left(\{\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{\Lambda}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{z}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) && (4.49) \\
&= \mathcal{I}\left(\{\tilde{\boldsymbol{\Lambda}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) && (4.50) \\
&\geq \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) - n\,o(\log(P)), && (4.51)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}(\tilde{\boldsymbol{r}}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) &= \mathcal{I}\left(\{\tilde{\boldsymbol{W}}_{12}^{\dagger}\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{\Lambda}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) && (4.52) \\
&\leq \mathcal{I}\left(\tilde{\boldsymbol{r}}_1^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) + n\,o(\log(P)). && (4.53)
\end{aligned}$$

Hence R_1 can be further bounded as

$$\begin{aligned} &nLR_1 - n\,o(\log(P)) \\ &\qquad \le \mathcal{I}\left(\tilde{\boldsymbol{r}}^n; \tilde{\boldsymbol{x}}_{1G}^n | \tilde{\boldsymbol{x}}_2^n, \tilde{\boldsymbol{H}}^n\right) + \mathcal{I}(\tilde{\boldsymbol{r}}_1^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) - \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12}^\dagger \tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right). \end{aligned} \tag{4.54}$$

As to receiver 2, using Fano's inequality and Lemma 11, we have

$$nLR_2 - \delta_{nL} \le \mathcal{I}(\tilde{\boldsymbol{y}}_2^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) \tag{4.55}$$

$$= \mathcal{I}\left(\{\tilde{\boldsymbol{W}}_{22}\tilde{\boldsymbol{\Lambda}}_{22}\tilde{\boldsymbol{V}}_{22}^\dagger \tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{z}}_2\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) \tag{4.56}$$

$$\le \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{22}^\dagger \tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_2\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) + n\,o(\log(P)), \tag{4.57}$$

where $\tilde{\boldsymbol{n}}_2 = \tilde{\boldsymbol{W}}_{22}^\dagger \tilde{\boldsymbol{z}}_2$. Hence

$$nLR_2 - n\,o(\log(P)) \le \mathcal{I}\left(\tilde{\boldsymbol{r}}_1^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) - \mathcal{I}\left(\tilde{\boldsymbol{r}}_1^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) + \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{22}^\dagger \tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_2\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{r}}_1^n, \tilde{\boldsymbol{H}}^n\right). \tag{4.58}$$

Notice that by using Gaussian input, the following inequalities hold

$$\mathcal{I}\left(\tilde{\boldsymbol{r}}_1; \tilde{\boldsymbol{x}}_{1G}^n | \tilde{\boldsymbol{x}}_2^n, \tilde{\boldsymbol{H}}^n\right) \le \mathrm{E}\log\left(\det(\boldsymbol{I}_{LN_1} + \frac{P}{M_1}\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{H}}_{11}^\dagger)\right) \tag{4.59}$$

$$= nLM_1\log(P) + nLo(\log(P)), \tag{4.60}$$

$$\mathcal{I}\left(\tilde{\boldsymbol{r}}_1^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) \le n\mathrm{E}\log\left(\frac{\det(\boldsymbol{I}_{LN_1} + \frac{P}{M_2}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{W}}_{12}^\dagger + \frac{P}{M_1}\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{H}}_{11}^\dagger)}{\det(\boldsymbol{I}_{LN_1} + \frac{P}{M_1}\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{H}}_{11}^\dagger)}\right) \tag{4.61}$$

$$= nL(N_1 - M_1)\log(P) + nLo(\log(P)). \tag{4.62}$$

Then let $n \to \infty$, multiply (4.58) with some positive scalar μ, add it with (4.54) and use (4.60), (4.62), we have the following inequality

$$nL[R_1 + uR_2 - o(\log(P))] \le nLM_1\log(P) + \mu nL(N_1 - M_1)\log(P) + \eta, \tag{4.63}$$

where μ is to be determined and

$$\eta = \mu\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{22}^\dagger \tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_2\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) - \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12}^\dagger \tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_1\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) + (1-\mu)\mathcal{I}(\tilde{\boldsymbol{r}}_1^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n). \tag{4.64}$$

Divide (4.63) by $nL\log(P)$ and let $P \to \infty$, we have the following inequality on the DoF of two users

$$d_1 + \mu d_2 \le M_1 + \mu(N_1 - M_1) + \lambda, \tag{4.65}$$

where

$$\lambda = \frac{1}{nL} \lim_{P\to\infty} \frac{\eta}{\log(P)}.$$

Recall that $\tilde{\boldsymbol{r}}_1 = \tilde{\boldsymbol{W}}_{12}^{\dagger}\tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_1$ and $\boldsymbol{H}_{11}(t) = \underline{\boldsymbol{H}}_{11}\boldsymbol{E}(t)$. We define

$$\tilde{\boldsymbol{r}}_2 = \tilde{\underline{\boldsymbol{\Lambda}}}_{11}^{-1}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\underline{\boldsymbol{V}}}_{11}^{\dagger}\tilde{\boldsymbol{E}}\tilde{\boldsymbol{x}}_{1G} + \tilde{\underline{\boldsymbol{\Lambda}}}_{11}^{-1}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{n}}_1 \tag{4.66}$$

$$\tilde{\boldsymbol{r}}_3 = \tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\underline{\boldsymbol{V}}}_{11}^{\dagger}\tilde{\boldsymbol{E}}\tilde{\boldsymbol{x}}_{1G} + \tilde{\underline{\boldsymbol{\Lambda}}}_{11}^{-1}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{n}}_1 \tag{4.67}$$

$$\tilde{\boldsymbol{r}}_4 = \tilde{\underline{\boldsymbol{V}}}_{11}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{E}}\tilde{\boldsymbol{x}}_{1G} + \tilde{\underline{\boldsymbol{V}}}_{11}\tilde{\underline{\boldsymbol{\Lambda}}}_{11}^{-1}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{n}}_1 \tag{4.68}$$

$$\tilde{\boldsymbol{r}}_5 = \tilde{\underline{\boldsymbol{V}}}_{11}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{E}}\tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{n}}_1 \tag{4.69}$$

$$\tilde{\boldsymbol{r}}_6 = \tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{E}}\tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{n}}_1 \tag{4.70}$$

We have

$$\begin{aligned}
\mathcal{I}(\tilde{\boldsymbol{r}}_1^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) &= \mathcal{I}(\tilde{\boldsymbol{r}}_2^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) && (4.71)\\
&= \mathcal{I}(\tilde{\boldsymbol{r}}_3^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) + o(\log(P)) && (4.72)\\
&= \mathcal{I}(\tilde{\boldsymbol{r}}_4^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) + o(\log(P)) && (4.73)\\
&= \mathcal{I}(\tilde{\boldsymbol{r}}_5^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) + o(\log(P)) && (4.74)\\
&= \mathcal{I}(\tilde{\boldsymbol{r}}_6^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) + o(\log(P)), && (4.75)
\end{aligned}$$

where (4.72) due to Lemma 11; (4.71) and (4.73) hold as $\tilde{\underline{\boldsymbol{W}}}_{11}\tilde{\underline{\boldsymbol{\Lambda}}}_{11}$ and $\tilde{\underline{\boldsymbol{V}}}_{11}$ are full rank square matrices. (4.74) holds as changing noise variance will not change the DoF. (4.75) is true because $\tilde{\underline{\boldsymbol{V}}}_{11}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}\tilde{\boldsymbol{V}}_{12}^{\dagger}$ has the same distribution as $\tilde{\boldsymbol{V}}_{12}^{\dagger}$ and $\tilde{\underline{\boldsymbol{V}}}_{11}\tilde{\underline{\boldsymbol{W}}}_{11}^{\dagger}\tilde{\boldsymbol{W}}_{12}$ is independent of $\tilde{\boldsymbol{V}}_{12}^{\dagger}$. To find the DoF order of $\mathcal{I}(\tilde{\boldsymbol{r}}_6^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n)$, we first notice that for each slot t in one block, $\boldsymbol{V}_{12}^{\dagger}$ can be divided into three parts: $\boldsymbol{V}_{12,a}^{\dagger}(t)$, $\boldsymbol{V}_{12,b}^{\dagger}(t)$ and $\boldsymbol{V}_{12,c}^{\dagger}$.

1. $\boldsymbol{V}_{12,a}^{\dagger}(t)$ is of size $M_1 \times M_2$ and consists of M_1 non-zero rows of $\boldsymbol{E}(t)\boldsymbol{V}_{12}^{\dagger}$.

2. $\boldsymbol{V}_{12,c}^{\dagger}$ is of size $(N_1 - S) \times M_2$ and is the same for all $1 \le t \le L$. It consists of $N_1 - S$ rows of $\boldsymbol{V}_{12}^{\dagger}$ that do not appear in any $\boldsymbol{V}_{12,a}(t)^{\dagger}, 1 \le t \le L$.

3. $\boldsymbol{V}_{12,b}^{\dagger}(t)$ is of size $(S - M_1) \times M_2$ and consists of $S - M_1$ rows of $\boldsymbol{V}_{12}^{\dagger}$ that neither in $\boldsymbol{E}(t)\boldsymbol{V}_{12,a}(t)^{\dagger}$ nor in $\boldsymbol{V}_{12,c}^{\dagger}$.

Example 1. *Assume $N_1 = 5$, $M_1 = 2$, $L = 6$, $S = 4$ and $\boldsymbol{V}_{12} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{N_1}]$ where $\boldsymbol{v}_i$'s are $M_2 \times 1$ vectors. Assume $\boldsymbol{E}(t)$ is the following*

$$\boldsymbol{E}(1) = [\boldsymbol{e}_1, \boldsymbol{e}_2], \quad \boldsymbol{E}(2) = [\boldsymbol{e}_1, \boldsymbol{e}_3], \quad \boldsymbol{E}(3) = [\boldsymbol{e}_1, \boldsymbol{e}_4],$$

$$\boldsymbol{E}(4) = [\boldsymbol{e}_1, \boldsymbol{e}_2], \quad \boldsymbol{E}(5) = [\boldsymbol{e}_2, \boldsymbol{e}_4], \quad \boldsymbol{E}(6) = [\boldsymbol{e}_2, \boldsymbol{e}_3].$$

We have

$$\boldsymbol{V}_{12,a}^{\dagger}(1) = [\boldsymbol{v}_1, \boldsymbol{v}_2]^{\dagger}, \quad \boldsymbol{V}_{12,a}^{\dagger}(2) = [\boldsymbol{v}_1, \boldsymbol{v}_3]^{\dagger}, \quad \boldsymbol{V}_{12,a}^{\dagger}(3) = [\boldsymbol{v}_1, \boldsymbol{v}_4]^{\dagger},$$

$$\boldsymbol{V}_{12,a}^{\dagger}(4) = [\boldsymbol{v}_1, \boldsymbol{v}_2]^{\dagger}, \quad \boldsymbol{V}_{12,a}^{\dagger}(5) = [\boldsymbol{v}_2, \boldsymbol{v}_4]^{\dagger}, \quad \boldsymbol{V}_{12,a}^{\dagger}(6) = [\boldsymbol{v}_2, \boldsymbol{v}_3]^{\dagger},$$

$$\boldsymbol{V}_{12,b}^{\dagger}(1) = [\boldsymbol{v}_3, \boldsymbol{v}_4]^{\dagger}, \quad \boldsymbol{V}_{12,b}^{\dagger}(2) = [\boldsymbol{v}_2, \boldsymbol{v}_4]^{\dagger}, \quad \boldsymbol{V}_{12,b}^{\dagger}(3) = [\boldsymbol{v}_2, \boldsymbol{v}_3]^{\dagger},$$

$$\boldsymbol{V}_{12,b}^{\dagger}(4) = [\boldsymbol{v}_3, \boldsymbol{v}_4]^{\dagger}, \quad \boldsymbol{V}_{12,b}^{\dagger}(5) = [\boldsymbol{v}_1, \boldsymbol{v}_3]^{\dagger}, \quad \boldsymbol{V}_{12,b}^{\dagger}(6) = [\boldsymbol{v}_1, \boldsymbol{v}_4]^{\dagger},$$

and $\boldsymbol{V}_{12,c}^{\dagger} = \boldsymbol{v}_5^{\dagger}$. Note that $\boldsymbol{V}_{12,c}^{\dagger}$ remains the same in one block of L slots. □

Suppose receiver 1 receives $\boldsymbol{r}_6$ as in (4.70) and wants to decode the message of $\boldsymbol{x}_2$ that goes through an equivalent channel $\boldsymbol{V}_{12}^{\dagger}$. Then $\boldsymbol{V}_{12,a}^{\dagger}(t)$ are the directions of interference from transmitter at time t, $\boldsymbol{V}_{12,b}^{\dagger}(t)$ are those directions that are temporarily interference-free at time t, and $\boldsymbol{V}_{12,c}^{\dagger}$ are the directions which are interference free for a whole block. The associated noises of the those directions are similarly defined as $\boldsymbol{n}_{2,a}(t), \boldsymbol{n}_{2,b}(t)$ and $\boldsymbol{n}_{2,c}(t)$.

To bound the DoF of $\mathcal{I}(\tilde{\boldsymbol{r}}_6^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n)$ of (4.76), we define

$$\boldsymbol{V}_{12,ab}^{\dagger}(t) = \begin{bmatrix} \boldsymbol{V}_{12,a}^{\dagger}(t) \\ \boldsymbol{V}_{12,b}^{\dagger}(t) \end{bmatrix}, \quad \boldsymbol{n}_{1,ab}(t) = \begin{bmatrix} \boldsymbol{n}_{1,a}(t) \\ \boldsymbol{n}_{1,b}(t) \end{bmatrix}, \tag{4.76}$$

$$\boldsymbol{V}_{12,bc}^{\dagger}(t) = \begin{bmatrix} \boldsymbol{V}_{12,b}^{\dagger}(t) \\ \boldsymbol{V}_{12,c}^{\dagger} \end{bmatrix}, \quad \boldsymbol{n}_{1,bc}(t) = \begin{bmatrix} \boldsymbol{n}_{1,b}(t) \\ \boldsymbol{n}_{1,c}(t) \end{bmatrix}, \tag{4.77}$$

and adopt the following notation for simplicity

$$\boldsymbol{y}_a(t) = \boldsymbol{V}_{12,a}^{\dagger}(t)\boldsymbol{x}_2(t) + \boldsymbol{x}_{1G}(t) + \boldsymbol{n}_{1,a}(t) \tag{4.78}$$

$$\boldsymbol{y}_b(t) = \boldsymbol{V}_{12,b}^{\dagger}(t)\boldsymbol{x}_2(t) + \boldsymbol{n}_{1,b}(t) \tag{4.79}$$

$$\boldsymbol{y}_c(t) = \boldsymbol{V}_{12,c}^{\dagger}\boldsymbol{x}_2(t) + \boldsymbol{n}_{1,c}(t) \tag{4.80}$$

$$\boldsymbol{y}_{bc}(t) = \boldsymbol{V}_{12,bc}^{\dagger}(t)\boldsymbol{x}_2(t) + \boldsymbol{n}_{1,bc}(t) \tag{4.81}$$

In addition, $\boldsymbol{y}_a(t)^n, \boldsymbol{y}_b(t)^n, \boldsymbol{y}_c(t)^n, \boldsymbol{y}_{bc}(t)^n$ are sequences of corresponding vectors of the tth slot over n blocks. The collection of $\boldsymbol{y}_a(1)^n, \boldsymbol{y}_a(2)^n, \ldots \boldsymbol{y}_a(t)^n$ is denoted as $\{\boldsymbol{y}_a^{(1:t)}\}^n$. We also define $\{\boldsymbol{y}_b^{(1:t)}\}^n$, $\{\boldsymbol{y}_c^{(1:t)}\}^n$ and $\{\boldsymbol{y}_{bc}^{(1:t)}\}^n$ similarly. Using the chain rule, we have

$$\begin{aligned}\mathcal{I}(\tilde{\boldsymbol{r}}_6^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n) = {} & \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_{1,c}\}^n; \tilde{\boldsymbol{x}}_2^n | \tilde{\boldsymbol{H}}^n\right) \\ & + \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12,b}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_{1,b}\}^n; \tilde{\boldsymbol{x}}_2^n | \{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_{1,c}\}^n, \tilde{\boldsymbol{H}}^n\right) \\ & + \mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12,a}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{n}}_{1,a}\}^n; \tilde{\boldsymbol{x}}_2^n | \{\tilde{\boldsymbol{V}}_{12,bc}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_{1,bc}\}^n, \tilde{\boldsymbol{H}}^n\right) \end{aligned} \tag{4.82}$$

Now checking the second term in (4.82), we notice that

$$\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12,b}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_{1,b}\}^n; \tilde{\boldsymbol{x}}_2^n | \{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_{1,c}\}^n, \tilde{\boldsymbol{H}}^n\right)$$

$$= \sum_{t=1}^{L} \mathcal{I}\left(\boldsymbol{y}_b(t)^n; \tilde{\boldsymbol{x}}_2^n | \{\boldsymbol{y}_b^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_c^n, \tilde{\boldsymbol{H}}^n\right) \tag{4.83}$$

$$= \sum_{t=1}^{L} \mathcal{H}\left(\boldsymbol{y}_b(t)^n | \{\boldsymbol{y}_b^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_c^n, \tilde{\boldsymbol{H}}^n\right) - \mathcal{H}\left(\boldsymbol{y}_b(t)^n | \tilde{\boldsymbol{x}}_2^n, \{\boldsymbol{y}_b^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_c^n, \tilde{\boldsymbol{H}}^n\right) \tag{4.84}$$

$$= \sum_{t=1}^{L} \mathcal{H}\left(\boldsymbol{y}_b(t)^n | \{\boldsymbol{y}_b^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_c^n, \tilde{\boldsymbol{H}}^n\right) - \mathcal{H}\left(\boldsymbol{y}_b(t)^n | \boldsymbol{x}_2(t)^n, \boldsymbol{y}_b(t)^n, \tilde{\boldsymbol{H}}(t)^n\right) \tag{4.85}$$

$$\leq \sum_{t=1}^{L} \mathcal{H}\left(\boldsymbol{y}_b(t)^n | \boldsymbol{y}_c(t)^n, \boldsymbol{H}(t)^n\right) - \mathcal{H}\left(\boldsymbol{y}_b(t)^n | \boldsymbol{x}_2(t)^n, \boldsymbol{y}_c(t)^n, \boldsymbol{H}(t)^n\right) \tag{4.86}$$

$$= \sum_{t=1}^{L} \mathcal{I}\left(\boldsymbol{y}_b(t)^n; \boldsymbol{x}_2(t)^n | \boldsymbol{y}_c(t)^n, \boldsymbol{H}(t)^n\right) \tag{4.87}$$

$$= \sum_{t=1}^{L} \left[\mathcal{I}\left(\boldsymbol{y}_b(t)^n, \boldsymbol{y}_c(t)^n; \boldsymbol{x}_2(t)^n | \boldsymbol{H}(t)^n\right) - \mathcal{I}\left(\boldsymbol{y}_c(t)^n; \boldsymbol{x}_2(t)^n | \boldsymbol{H}(t)^n\right)\right] \tag{4.88}$$

$$\leq \sum_{t=1}^{L} \left(\frac{N_1 - M_1}{N_1 - S} - 1\right) \mathcal{I}\left(\boldsymbol{y}_c(t)^n; \boldsymbol{x}_2(t)^n | \boldsymbol{H}(t)^n\right) \tag{4.89}$$

$$\leq nL(S - M_1)\log(P) + o(\log(P)) \tag{4.90}$$

where:

- (4.83) and (4.88) follow by chain rule.
- (4.84) and (4.87) are expressing mutual information via entropy.
- (4.85) holds as the second term is the entropy of noise when conditioning on $\boldsymbol{x}_2(t)^n$.

- (4.86) is based on the fact that conditioning reduces entropy.
- (4.89) follows by Lemma 12.
- (4.90) holds due to the fact that the DoF of an $(N_1 - S) \times M_2$ point-to-point MIMO channel is at most $\min(N_1 - S, M_2) = N_1 - S$.

The third term in (4.82) can be bounded in a similar fashion. We have

$$
\begin{aligned}
&\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12,a}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{x}}_{1G} + \tilde{\boldsymbol{n}}_{1,a}\}^n; \tilde{\boldsymbol{x}}_2^n | \{\tilde{\boldsymbol{V}}_{12,bc}^{\dagger}\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{n}}_{1,bc}\}^n, \tilde{\boldsymbol{H}}^n\right) \\
&= \sum_{t=1}^{L} \mathcal{I}\left(\boldsymbol{y}_a(t)^n; \tilde{\boldsymbol{x}}_2^n | \{\boldsymbol{y}_a^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_{bc}^n, \tilde{\boldsymbol{H}}^n\right) && (4.91) \\
&= \sum_{t=1}^{L} \mathcal{H}\left(\boldsymbol{y}_a(t)^n | \{\boldsymbol{y}_a^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_{bc}^n, \tilde{\boldsymbol{H}}^n\right) - \mathcal{H}\left(\boldsymbol{y}_a(t)^n | \tilde{\boldsymbol{x}}_2^n, \{\boldsymbol{y}_a^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_{bc}^n, \tilde{\boldsymbol{H}}^n\right) && (4.92) \\
&= \sum_{t=1}^{L} \mathcal{H}\left(\boldsymbol{y}_a(t)^n | \{\boldsymbol{y}_a^{(1:t-1)}\}^n, \tilde{\boldsymbol{y}}_{bc}^n, \tilde{\boldsymbol{H}}^n\right) - \mathcal{H}\left(\boldsymbol{y}_a(t)^n | \boldsymbol{x}_2(t)^n, \boldsymbol{y}_{bc}(t)^n, \tilde{\boldsymbol{H}}^n\right) && (4.93) \\
&\le \sum_{t=1}^{L} \mathcal{H}\left(\boldsymbol{y}_a(t)^n | \boldsymbol{y}_{bc}(t)^n, \boldsymbol{H}(t)^n\right) - \mathcal{H}\left(\boldsymbol{y}_a(t)^n | \boldsymbol{x}_2(t)^n, \boldsymbol{y}_{bc}(t)^n, \boldsymbol{H}(t)^n\right) && (4.94) \\
&= \sum_{t=1}^{L} \mathcal{I}\left(\boldsymbol{y}_a(t)^n; \boldsymbol{x}_2(t)^n | \boldsymbol{y}_{bc}(t)^n, \boldsymbol{H}(t)^n\right) && (4.95) \\
&\le n \sum_{t=1}^{L} \mathcal{I}\left(\boldsymbol{y}_{aG}(t); \boldsymbol{x}_{2G}(t) | \boldsymbol{y}_{bcG}(t), \boldsymbol{H}(t)\right) && (4.96) \\
&\le nL \log\left(\det\left(\frac{P}{M_1}\boldsymbol{I}_{M_1} + \boldsymbol{I}_{M_1} + \frac{P}{M_2}\boldsymbol{I}_{M_1}\right)\det\left(\frac{P}{M_2}\boldsymbol{I}_{(N_1-M_1)} + \boldsymbol{I}_{(N_1-M_1)}\right)\right) \\
&\qquad - nL \log\left(\det\left(\frac{P}{M_2}\boldsymbol{I}_{(N_1-M_1)} + \boldsymbol{I}_{(N_1-M_1)}\right)\det\left(\frac{P}{M_1}\boldsymbol{I}_{M_1} + \boldsymbol{I}_{M_1}\right)\right) && (4.97) \\
&= o(\log(P)) && (4.98)
\end{aligned}
$$

where:

- (4.91) follows by chain rule.
- (4.92) and (4.95) are expressing mutual information via entropy.
- (4.93) holds as the second term is the entropy of noise when conditioning on $\boldsymbol{x}_2(t)^n$.
- (4.94) is based on the fact that conditioning reduces entropy.

- (4.96) and (4.97) follows by Lemma 14, where the covariance matrix of $\boldsymbol{x}_{1G}(t)+\boldsymbol{n}_{1,a}(t)$ and $\boldsymbol{n}_{1,bc}(t)$ are $\frac{P}{M_1}\boldsymbol{I}_{M_1}+\boldsymbol{I}_{M_1}$ and $\boldsymbol{I}_{(N_1-M_1)}$, respectively. In addition, the optimal input of $\boldsymbol{x}_2(t)$ is CSCG with covariance matrix $\frac{P}{M_2}\boldsymbol{I}_{(N_1-M_1)}$.

Substitute (4.90) and (4.98) in to (4.82), we have

$$\mathcal{I}(\tilde{\boldsymbol{r}}_1^n;\tilde{\boldsymbol{x}}_2^n|\tilde{\boldsymbol{H}}^n)\leq\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_{1,c}\}^n;\tilde{\boldsymbol{x}}_2^n|\tilde{\boldsymbol{H}}^n\right)+nL(S-M_1)\log(P)+o(\log(P)) \tag{4.99}$$

Now we go back to (4.64). Notice that if we choose $\boldsymbol{D}=[\boldsymbol{0}_{N_1\times(\min(M_2,N_2)-N_1)},\boldsymbol{I}_{N_1}]$, $(\boldsymbol{D}\boldsymbol{V}_{22}^{\dagger},\boldsymbol{D}\boldsymbol{n}_2)$ has the same distribution as $(\boldsymbol{V}_{12}^{\dagger},\boldsymbol{n}_1)$ as both $\boldsymbol{V}_{22}$ and $\boldsymbol{V}_{12}$ are uniformly distributed and $\boldsymbol{V}_{22}$ has no fewer columns than $\boldsymbol{V}_{12}$. (Please refer to [57, Sec. IV-C2] for more details). We have the following Markov chain:

$$\tilde{\boldsymbol{x}}_2 - \tilde{\boldsymbol{V}}_{22}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_2 - \tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_1 - \tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_{1,c}. \tag{4.100}$$

Denote $T=\min(M_2,N_2)-(N_1-S)$. Let $\boldsymbol{V}_{22,a}$ contain the first T rows of $\boldsymbol{V}_{22}$, and $\boldsymbol{n}_{2,a}$ contain the first T elements of $\boldsymbol{n}_2$. We can bound η as

$$\begin{aligned}\eta\leq{}&\mu\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{22}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_2\}^n;\tilde{\boldsymbol{x}}_2^n|\{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_{1,c}\}^n,\tilde{\boldsymbol{H}}^n\right)\\&-\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_1\}^n;\tilde{\boldsymbol{x}}_2^n|\{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_{1,c}\}^n,\tilde{\boldsymbol{H}}^n\right)\\&+(1-\mu)nL(S-M_1)\log(P)+o(\log(P)) &(4.101)\\={}&\mu\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{22,a}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_2\}^n;\tilde{\boldsymbol{x}}_2^n|\{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_{1,c}\}^n,\tilde{\boldsymbol{H}}^n\right)\\&-\mathcal{I}\left(\{\tilde{\boldsymbol{V}}_{12,ab}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_1\}^n;\tilde{\boldsymbol{x}}_2^n|\{\tilde{\boldsymbol{V}}_{12,c}^{\dagger}\tilde{\boldsymbol{x}}_2+\tilde{\boldsymbol{n}}_{1,c}\}^n,\tilde{\boldsymbol{H}}^n\right)\\&+(1-\mu)nL(S-M_1)\log(P)+o(\log(P)) &(4.102)\end{aligned}$$

Notice that the size of $\boldsymbol{V}_{12,ab}^{\dagger}$ is $S\times M_2$. Based on Lemma 12, if we choose

$$\mu=\frac{S}{T} \tag{4.103}$$

the difference of the first two mutual information terms of (4.102) is at most in the order of $o(\log(P))$ and we have

$$\lambda\leq\left(1-\frac{S}{T}\right)(S-M_1)=\frac{(\min(M_2,N_2)-N_1)(S-M_1)}{\min(M_2,N_2)-(N_1-S)} \tag{4.104}$$

Recall that $d_1+\mu d_2\leq M_1+\mu(N_1-M_1)+\lambda$. We thus have the outer bound on the sum DoF as shown in (4.43) and the proof of the converse part of Theorem 4 is complete. $\square$

4.5.2 Achievability

In order to show the achievability part of Theorem 4, we only need to construct an achievable scheme for the corner point of the DoF region. Without loss of generality, we assume that $M_2 = \min(M_2, N_2)$; otherwise, transmitter 2 can simply use N_2 transmit antennas. Since $S(N_1 - M_1) + (M_2 - N_1)(S - M_1) = M_2(S - M_1) + M_1(N_1 - S)$, it is sufficient to show that the following DoF pair

$$(d_1, d_2) = (SM_1, M_2(S - M_1) + M_1(N_1 - S)) \tag{4.105}$$

can be achieved over S slots with antenna mode switching at transmitter one among S modes. Similar to Section 4.4.4, we choose the mode switching pattern as follows:

$$\begin{aligned}
\boldsymbol{E}(1) &= [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{M_1}], \\
\boldsymbol{E}(2) &= [\boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_{M_1+1}], \\
&\vdots \\
\boldsymbol{E}(S) &= [\boldsymbol{e}_K, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{M_1-1}].
\end{aligned}$$

We propose to use a generalization of the joint nulling and beamforming design that is investigated in Section 4.4.3. Unlike the frequency nulling that has been used for $S = N_1$, this scheme requires that receiver 1 performs nulling in both frequency and spatial domains. We hereby use two superscripts F and S to indicate the matrices that associated with frequency processing and spatial processing.

The generalized joint nulling and beamforming has the following structure:

$$\tilde{\boldsymbol{Q}} = \boldsymbol{Q}^{\mathsf{F}}_{M_1 \times S} \otimes \boldsymbol{Q}^{\mathsf{S}}_{S \times N_1}, \tag{4.106}$$

$$\tilde{\boldsymbol{P}} = [\tilde{\boldsymbol{P}}_a, \tilde{\boldsymbol{P}}_b], \text{ where} \tag{4.107}$$

$$\tilde{\boldsymbol{P}}_a = [\boldsymbol{P}^{\mathsf{F}}_a]_{S \times (S - M_1)} \otimes [\boldsymbol{P}^{\mathsf{S}}_a]_{M_2 \times M_2}, \tag{4.108}$$

$$\tilde{\boldsymbol{P}}_b = [\boldsymbol{P}^{\mathsf{F}}_b]_{S \times M_1} \otimes [\boldsymbol{P}^{\mathsf{S}}_b]_{M_2 \times (N_1 - S)}. \tag{4.109}$$

The received signal at receiver 1 can be written as

$$\tilde{\boldsymbol{y}}_1 = \tilde{\boldsymbol{H}}_{11}\tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{H}}_{12}[\tilde{\boldsymbol{P}}_a, \tilde{\boldsymbol{P}}_b]\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{z}}_1 \tag{4.110}$$

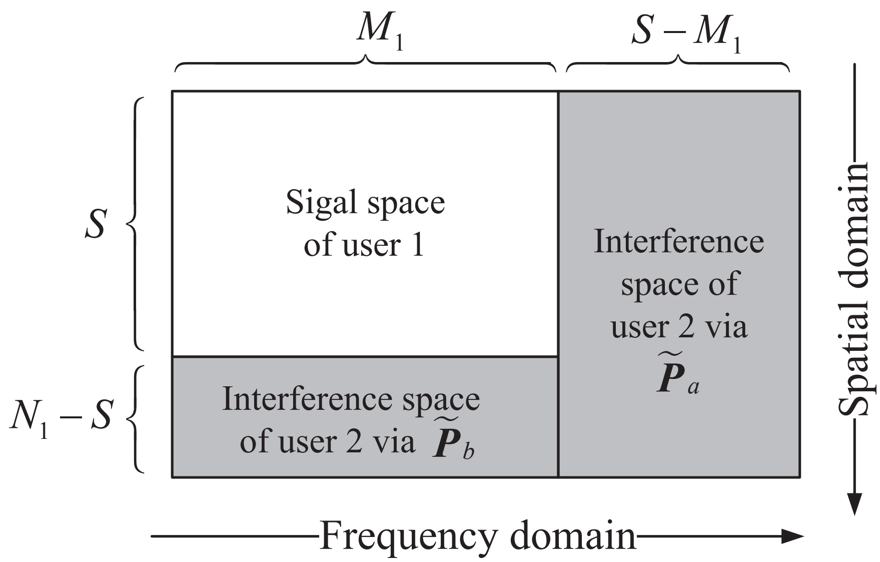

Figure 4.2 Space-Frequency dimension allocation for the two users when $S < N_1$.

where $\tilde{\boldsymbol{x}}_1$ is a length $M_1 S$ vector, and $\tilde{\boldsymbol{x}}_2$ is a length $M_2(S - M_1) + M_1(N_1 - S)$ vector.

After applying nulling matrix $\tilde{\boldsymbol{Q}}$, we have

$$\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{y}}_1 = \underbrace{\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{H}}_{11}}_{\tilde{\boldsymbol{A}}}\tilde{\boldsymbol{x}}_1 + \left[\underbrace{\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{P}}_a}_{\tilde{\boldsymbol{B}}}, \underbrace{\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{P}}_b}_{\tilde{\boldsymbol{C}}}\right]\tilde{\boldsymbol{x}}_2 + \tilde{\boldsymbol{Q}}\tilde{\boldsymbol{z}}_1. \tag{4.111}$$

To achieve the degrees of freedom pair shown in (4.105) for both users, it is sufficient to design our $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{Q}}$ to satisfy the following conditions simultaneously

1. $\text{rank}(\tilde{\boldsymbol{A}}) = M_1 S$,

2. $\text{rank}([\tilde{\boldsymbol{P}}_a, \tilde{\boldsymbol{P}}_b]) = M_2(S - M_1) + M_1(N_1 - S)$,

3. $\tilde{\boldsymbol{B}} = \boldsymbol{0}$,

4. $\tilde{\boldsymbol{C}} = \boldsymbol{0}$.

We propose to use the following realizations:

$$\boldsymbol{Q}^{\mathsf{F}} = [\boldsymbol{\mathcal{V}}_S(1, \omega_S, \ldots, \omega_S^{M_1-1})]^T, \tag{4.112}$$

$$\boldsymbol{P}_a^{\mathsf{F}} = \boldsymbol{\mathcal{V}}_S(\omega_S^{-M_1}, \omega_S^{-(M_1+1)}, \ldots, \omega_S^{-(S-1)}). \tag{4.113}$$

$$\boldsymbol{P}_b^{\mathsf{F}} = (\boldsymbol{Q}^{\mathsf{F}})^{\dagger}, \tag{4.114}$$

$$\boldsymbol{P}_a^{\mathsf{S}} = \boldsymbol{I}_{M_2}, \tag{4.115}$$

$$\boldsymbol{P}_b^{\mathsf{S}} = [\boldsymbol{I}_{N_1-S}; \boldsymbol{0}], \tag{4.116}$$

$$\boldsymbol{Q}^{\mathsf{S}} = \text{null}(\boldsymbol{H}_{12}\boldsymbol{P}_b^{\mathsf{S}})^T, \tag{4.117}$$

where (4.117) means that $\boldsymbol{Q}^{\mathsf{S}}\boldsymbol{H}_{12}\boldsymbol{P}_b^{\mathsf{S}} = \boldsymbol{0}$. Here, we choose $((\boldsymbol{Q}^{\mathsf{F}})^{\dagger}, \boldsymbol{P}_a^{\mathsf{F}})$ to be a size $S \times S$ IFFT matrix, which offers the same frequency domain explanation as discussed in Section 4.4.5; see also Fig. 4.2. It is trivial to see $\tilde{\boldsymbol{B}} = \boldsymbol{0}$. In other words, receiver 1 will simply ignore the signal in the last $S - M_1$ frequencies and only using the signal in the first S frequencies to decode his own message. Therefore, $\tilde{\boldsymbol{P}}_a$ contains the interference directions from all the antennas of transmitter 2 but only in certain frequencies. Now, after applying the frequency nulling, there are N_1S dimensions remaining, which contain both user 1's message and the message of user 2 that is transmitted by $\tilde{\boldsymbol{P}}_b$. Among all the N_1S dimensions, receiver 1 only requires M_1S dimensions to decode his own message, while leaving additional $S(N_1 - M_1)$ dimensions for user 2. Here we choose one possible way of decomposing the remaining dimensions. Transmitter 2 sends some messages in the first M_1 frequencies but only though $N_1 - S$ antennas, as shown in (4.116). Notice that

$$\begin{aligned}
\tilde{\boldsymbol{C}} &= \tilde{\boldsymbol{Q}}\tilde{\boldsymbol{H}}_{12}\tilde{\boldsymbol{P}}_b \\
&= (\boldsymbol{Q}^{\mathsf{F}} \otimes \boldsymbol{Q}^{\mathsf{S}})(\boldsymbol{I}_S \otimes \boldsymbol{H}_{12})(\boldsymbol{P}_b^{\mathsf{F}} \otimes \boldsymbol{P}_b^{\mathsf{S}}) && (4.118) \\
&= (\boldsymbol{Q}^{\mathsf{F}}\boldsymbol{P}_b^{\mathsf{F}}) \otimes (\boldsymbol{Q}^{\mathsf{S}}\boldsymbol{H}_{12}\boldsymbol{P}_b^{\mathsf{S}}) && (4.119)
\end{aligned}$$

which means that the choice of $\boldsymbol{Q}^{\mathsf{S}}$ as given in (4.117) is sufficient to set $\tilde{\boldsymbol{C}} = 0$. It is clear that for the interference signal sent via $\tilde{\boldsymbol{P}}_b$, receiver 1 only need to do spatial zero-forcing in our scheme, which can be seen from the fact $\boldsymbol{Q}^{\mathsf{F}}\boldsymbol{P}_b^{\mathsf{F}} = \boldsymbol{I}_S$ due to (4.114).

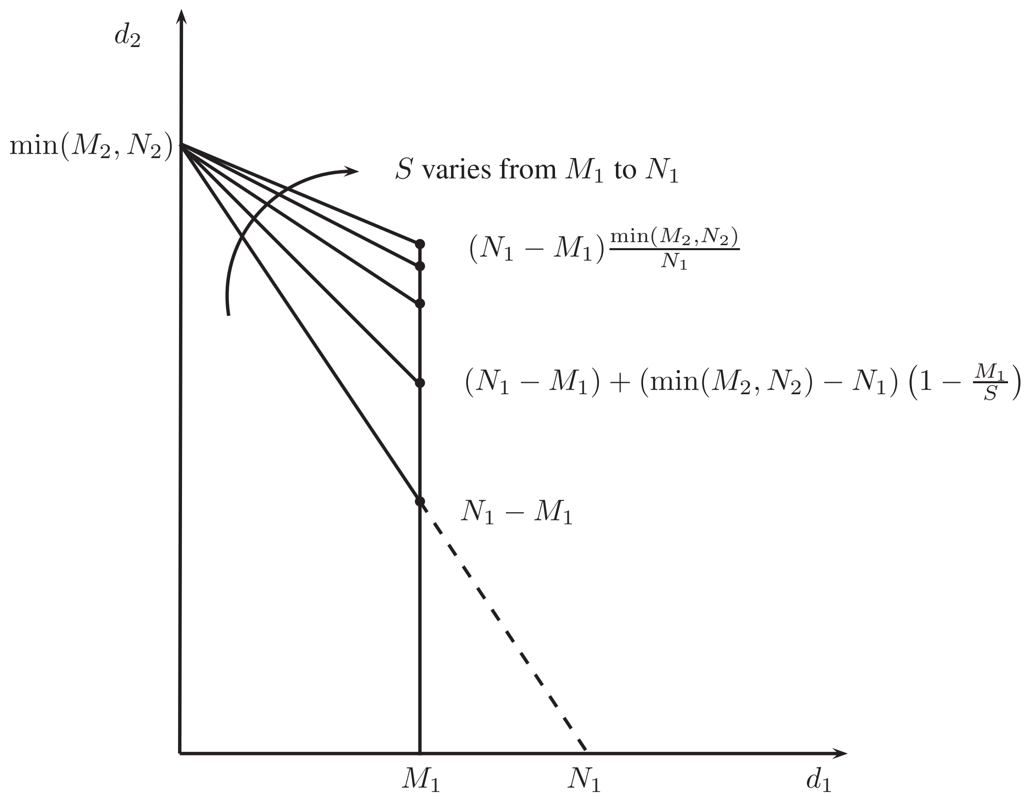

Figure 4.3 The benefit of antenna mode switching on the DoF region, in the case of $M_1 < N_1 < \min(M_2, N_2)$.

To satisfy the second condition, notice that $\mathrm{rank}(\tilde{\boldsymbol{P}}_a) = M_2(S - M_1)$ and $\mathrm{rank}(\tilde{\boldsymbol{P}}_b) = M_1(N_1 - S)$, it is sufficient to show that $\tilde{\boldsymbol{P}}_a \perp \tilde{\boldsymbol{P}}_b$, which is obvious as

$$\tilde{\boldsymbol{P}}_b^{\dagger}\tilde{\boldsymbol{P}}_a = (\boldsymbol{Q}^{\mathsf{F}}\boldsymbol{P}_a^{\mathsf{F}}) \otimes ((\boldsymbol{P}_b^{\mathsf{S}})^{\dagger}\boldsymbol{I}_{M_2}) = \boldsymbol{0} \tag{4.120}$$

because $\boldsymbol{Q}^{\mathsf{F}}\boldsymbol{P}_a^{\mathsf{F}} = \boldsymbol{0}$. This is not surprising as the signal of user 2 transmitted via $\tilde{\boldsymbol{P}}_a$ and $\tilde{\boldsymbol{P}}_b$ are orthogonal in frequency domain. The remaining part is to show the first condition holds, which is true because here $\tilde{\boldsymbol{A}}$ has the same structure as $\tilde{\boldsymbol{A}}'$ of (4.39) with N_1 replaced by S and $\boldsymbol{h}_i$ replaced by $\boldsymbol{Q}^{\mathsf{S}}_{S\times N_1}\boldsymbol{h}_i$.

4.5.3 Discussion

It is not surprising that when $S = N_1$, (4.43) implies

$$d_1 + \frac{N_1}{\min(M_2, N_2)}d_2 \le N_1 \tag{4.121}$$

which is the same as (4.7) and that in [54, Theorem 3] when $M_1 < N_1 < \min(M_2, N_2)$. For the scheme that we discussed above, $\tilde{\boldsymbol{P}}_b$ disappears and it is the DoF achievable scheme that we developed in Section 4.4.3. In addition, when $S = M_1$, (4.43) becomes (4.12) and $\tilde{\boldsymbol{P}}_a$ disappears, the general scheme reduces to the DoF-optimal spatial zero-forcing as shown in [57]. Hence, for one extra mode at transmitter 1, we can further align $\min(M_2, N_2) - N_1$ streams of interference over S slots. The incremental gain per slot is reduced when S increases; see Fig. 4.3. Our result reveals the fundamental benefit that can be obtained from reconfigurable antenna modes when there is no CSIT and $M_1 < N_1 < \min(M_2, N_2)$. In addition, combining with the known results, we know that in order to achieve the DoF region of two-user FIC and ZIC, zero-forcing in frequency and spatial domains suffice regardless of the CSIT assumption.

4.6 Summary

We derived the DoF region for the MIMO Z and full interference channels when perfect channel state information is available at receivers, including i) the Z interference channel with channel state information at the transmitter; ii) the Z and full interference channel without channel state information at the transmitter, but with reconfigurable antennas at the transmitters. For both FIC and ZIC, when the number of antenna modes S at the transmitter with the reconfigurable antennas is not less than the number of receive antennas at the corresponding receiver, the DoF region is maximized and no longer depends on the number of antenna modes. Otherwise, each additional antenna mode can bring extra gain in the DoF region when $M_1 < N_1 < \min(M_2, N_2)$ for both FIC and ZIC, and when $M_2 < N_2 < \min(M_1, N_1)$ for FIC. The incremental gain diminishes as S increases.

The achievability schemes we designed for the reconfigurable antenna cases rely on time expansion and joint beamforming and nulling over the time-expanded channel. Interestingly, they also bear a space-frequency coding interpretation. We completely characterized the DoF regions for both Z and full interference channels when transmitter antenna mode switching is allowed. Our result can specialize to previously known cases when there is no antenna mode switching by simply setting the number of antenna modes equal to the number of transmit antennas. Our work reveals how the channel variation introduced by the extra antenna mode switching brings benefits in the sense of the DoF region.

4.7 Appendix

In this section, we give the lemmas that are used for proving our results. The following lemmas are from [57]. Some symbol notation is slightly changed to be consistent with the notation being used in this thesis.

Lemma 10. *[57, Lemma 1] Let $\boldsymbol{R}(N \times M)$ be an isotropic complex random matrix and define $K = \min(M, N)$. Then there exists a decomposition $\boldsymbol{R} \sim \boldsymbol{W}\boldsymbol{\Lambda}\boldsymbol{V}^\dagger$ such that $\boldsymbol{W}(N \times K)$, $\boldsymbol{\Lambda}(M \times K)$ and $\boldsymbol{V}(M \times K)$ satisfy:*

1. *$\boldsymbol{V}^\dagger\boldsymbol{V} = \boldsymbol{W}^\dagger\boldsymbol{W} = \boldsymbol{I}_K$ and $\boldsymbol{\Lambda}$ is a diagonal random matrix with non-negative elements.*
2. *$\boldsymbol{V}$ is independent of $(\boldsymbol{W}, \boldsymbol{\Lambda})$ and is uniformly distributed on $\mathcal{V} = \{\mathbf{V} \in \mathbb{C}^{M\times K} : \mathbf{V}^\dagger\mathbf{V} = \boldsymbol{I}_K\}$.* $\square$

Lemma 11. *[57, Lemma 2] Let $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ be two $M \times M$ diagonal random matrices with strictly positive diagonal elements almost surely. Let $\boldsymbol{x}$ denote a random vector and $\boldsymbol{u}$ denote a CSCG random vector with arbitrary covariance. Assume that $\boldsymbol{x}$, $\boldsymbol{u}$ and $(\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2)$ are independent. Define random matrix $\boldsymbol{\Lambda}_{\min} = \min(\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2)$, where $\min$ operation is element-wise. Then we have*

$$\begin{aligned}\mathcal{I}(\boldsymbol{\Lambda}_2\boldsymbol{x} + \boldsymbol{u}; \boldsymbol{x}|\boldsymbol{\Lambda}_2) - \mathcal{I}(\boldsymbol{\Lambda}_1\boldsymbol{x} + \boldsymbol{u}; \boldsymbol{x}|\boldsymbol{\Lambda}_1) &\le 2E\log\left(\frac{\det\boldsymbol{\Lambda}_2}{\det\boldsymbol{\Lambda}_{\min}}\right)\\ &\le 2E\log^+\det\boldsymbol{\Lambda}_2 + 2E\left[\log^+\frac{1}{\det\boldsymbol{\Lambda}_{\min}}\right] \qquad (4.122)\end{aligned}$$

where we define $\log^+(x) = \log\max(1; x)$. $\square$

Lemma 12. *[57, Lemma 3] Let $\boldsymbol{x}$ be a random vector in $\mathbb{C}^M$, $\boldsymbol{u}_j \sim \mathcal{CN}(0, \boldsymbol{I}_{K_j})$, $j = 1, 2, 3$, and $K_1 \le K_2 \le M$. In addition, let $\boldsymbol{V}_j$ be a random $M \times K_j$ matrix for $j = 1, 2, 3$. Suppose that conditioned on $\boldsymbol{V}_3 = V_3$, $\boldsymbol{V}_j$ is uniformly distributed on $\mathcal{V}_j = \{\mathbf{V} \in \mathbb{C}^{M\times K_j} | \mathbf{V}^\dagger\mathbf{V} = \boldsymbol{I}_{K_j}$ and $\mathbf{V}^\dagger\mathbf{V}_3 = \mathbf{0}\}$ for $j = 1, 2$. Suppose also that $\boldsymbol{x}, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ and $(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$ are mutually* *independent. Then*

$$\frac{1}{K_1}\mathcal{I}\left(\boldsymbol{V}_1^\dagger \boldsymbol{x} + \boldsymbol{u}_1; \boldsymbol{x} | \boldsymbol{V}_3^\dagger \boldsymbol{x} + \boldsymbol{u}_3, \boldsymbol{V}\right) \geq \frac{1}{K_2}\mathcal{I}\left(\boldsymbol{V}_2^\dagger \boldsymbol{x} + \boldsymbol{u}_2; \boldsymbol{x} | \boldsymbol{V}_3^\dagger \boldsymbol{x} + \boldsymbol{u}_3, \boldsymbol{V}\right). \tag{4.123}$$

Furthermore, suppose $(\boldsymbol{V}_1(i), \boldsymbol{V}_2(i), \boldsymbol{V}_3(i))_{i=1}^n$ *is i.i.d. with identical joint distribution as that of* $(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$ *Then*

$$\frac{1}{K_1}\mathcal{I}\left(\{\boldsymbol{V}_1^\dagger \boldsymbol{x} + \boldsymbol{u}_1\}^n; \boldsymbol{x}^n | \{\boldsymbol{V}_3^\dagger \boldsymbol{x} + \boldsymbol{u}_3\}^n, \boldsymbol{V}^n\right) \geq \frac{1}{K_2}\mathcal{I}\left(\{\boldsymbol{V}_2^\dagger \boldsymbol{x} + \boldsymbol{u}_2\}^n; \boldsymbol{x}^n | \{\boldsymbol{V}_3^\dagger \boldsymbol{x} + \boldsymbol{u}_3\}^n, \boldsymbol{V}^n\right) \tag{4.124}$$

In particular, if $\boldsymbol{V}_3 \equiv \mathbf{0}$, (4.123) *and* (4.124) *become*

$$\frac{1}{K_1}\mathcal{I}\left(\boldsymbol{V}_1^\dagger \boldsymbol{x} + \boldsymbol{u}_1; \boldsymbol{x} | \boldsymbol{V}_1\right) \geq \frac{1}{K_2}\mathcal{I}\left(\boldsymbol{V}_2^\dagger \boldsymbol{x} + \boldsymbol{u}_2; \boldsymbol{x} | \boldsymbol{V}_2\right). \tag{4.125}$$

and

$$\frac{1}{K_1}\mathcal{I}\left(\{\boldsymbol{V}_1^\dagger \boldsymbol{x} + \boldsymbol{u}_1\}^n; \boldsymbol{x}^n | \boldsymbol{V}_1^n\right) \geq \frac{1}{K_2}\mathcal{I}\left(\{\boldsymbol{V}_2^\dagger \boldsymbol{x} + \boldsymbol{u}_2\}^n; \boldsymbol{x}^n | \boldsymbol{V}_2^n\right) \tag{4.126}$$

respectively. □

We restate the result in [57, Theorem 3] as the following lemma using the notation and model we have introduced.

Lemma 13. *Suppose that* $\boldsymbol{w}$ *and* $\boldsymbol{w}_G$ *are two random* M*-vectors,* $\mathbf{H}(N \times M)$ *is a full-rank deterministic matrix, and* $\boldsymbol{v}$ *is a random* N*-vector which is independent of* $\boldsymbol{w}$ *and* $\boldsymbol{w}_G$*. We assume that* $E||\boldsymbol{W}||^2 \leq \gamma$*. Then*

$$\mathcal{I}(\mathbf{H}\boldsymbol{w} + \boldsymbol{v}; \boldsymbol{w}) \leq \mathcal{I}(\mathbf{H}\boldsymbol{w}_G + \boldsymbol{v}; \boldsymbol{w}_G) + \sup_{E||\boldsymbol{a}|| \leq \gamma} \mathcal{I}(\mathbf{H}\boldsymbol{a} + \mathbf{H}\boldsymbol{w}_G; \boldsymbol{a}). \tag{4.127}$$

In particular, if $\boldsymbol{w}_G$ *has distribution* $\mathcal{CN}(\mathbf{0}, \frac{\gamma}{M}\boldsymbol{I})$*, then*

$$\mathcal{I}(\mathbf{H}\boldsymbol{w} + \boldsymbol{v}; \boldsymbol{w}) \leq \mathcal{I}(\mathbf{H}\boldsymbol{w}_G + \boldsymbol{v}; \boldsymbol{w}_G) + \min(M, N). \tag{4.128}$$

Furthermore, for channel model (4.1)–(4.2) *and fixed channel input* $\boldsymbol{x}_1(1), \ldots, \boldsymbol{x}_1(n)$*, we have*

$$\mathcal{I}(\boldsymbol{y}_1^n; \boldsymbol{x}_1^n | \boldsymbol{H}^n) \leq \mathcal{I}(\boldsymbol{y}_1^n; \boldsymbol{x}_{1G}^n | \boldsymbol{H}^n) + n \min(M_1, N_1) \tag{4.129}$$

where

$$\boldsymbol{y}_{1G} = \boldsymbol{H}_{11}\boldsymbol{x}_{1G} + \boldsymbol{H}_{12}\boldsymbol{x}_2 + \boldsymbol{z}_1 \tag{4.130}$$

□

Lemma 14. *[57, Lemma 4] Consider following two channels with M-vector input $\boldsymbol{x}$ and fading matrices $\boldsymbol{A}$ and $\boldsymbol{B}$*

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n}_1 \tag{4.131}$$

$$\boldsymbol{z} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{n}_2 \tag{4.132}$$

where $\boldsymbol{n}_1 \sim \mathcal{CN}(0, \Sigma_1)$ and $\boldsymbol{n}_2 \sim \mathcal{CN}(0, \Sigma_2)$ are mutually independent CSCG noise, and matrix $\left[\begin{smallmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{smallmatrix}\right]$ is isotropic. Furthermore, we assume that $E||\boldsymbol{x}||^2 \leq \gamma$. For notational convenience, let $\boldsymbol{y}_G$ and $\boldsymbol{z}_G$ be the corresponding outputs through model (4.131)–(4.132) with input $\boldsymbol{x}_G \sim \mathcal{CN}(\boldsymbol{0}, \frac{\gamma}{M}\boldsymbol{I}_M)$, respectively. Then

$$\begin{aligned}\mathcal{I}(\boldsymbol{y};\boldsymbol{x}|\boldsymbol{z},\boldsymbol{A},\boldsymbol{B}) &\leq \mathcal{I}(\boldsymbol{y}_G;\boldsymbol{x}_G|\boldsymbol{z}_G,\boldsymbol{A},\boldsymbol{B}) &(4.133)\\ &= E\log\left(\det\left(\begin{bmatrix}\Sigma_1 & \boldsymbol{0}\\ \boldsymbol{0} & \Sigma_2\end{bmatrix} + \frac{\gamma}{M}\begin{bmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{A}^\dagger \boldsymbol{B}^\dagger\end{bmatrix}\right)\right)\\ &\qquad - E\log\left(\det\left(\Sigma_2 + \frac{\gamma}{M}\boldsymbol{B}\boldsymbol{B}^\dagger\right)\det\Sigma_1\right). &(4.134)\end{aligned}$$

Furthermore, we have

$$\mathcal{I}(\boldsymbol{y}^n;\boldsymbol{x}^n|\boldsymbol{z}^n,\boldsymbol{A}^n,\boldsymbol{B}^n) \leq n\mathcal{I}(\boldsymbol{y}_G;\boldsymbol{x}_G|\boldsymbol{z}_G,\boldsymbol{A},\boldsymbol{B}). \tag{4.135}$$

where conditioned on $\boldsymbol{x}^n$, $(\boldsymbol{y}(i),\boldsymbol{z}(i),\boldsymbol{B}(i))_{i=1}^n$ are i.i.d. with identical joint distribution as $(\boldsymbol{y},\boldsymbol{z},\boldsymbol{A},\boldsymbol{B})$ conditioned on $\boldsymbol{x}$. $\square$

CHAPTER 5. DEGREES OF FREEDOM REGION FOR AN INTERFERENCE NETWORK WITH GENERAL MESSAGE DEMANDS

5.1 Introduction

Interference alignment with multicast messages has been studied in compound channel setup for broadcast channel [62, 43]. The compound setting was also explored for X channel and interference channel in [62] as well, where the total number of DoF is shown to be unchanged for these two channels. However, the precise DoF region has not been identified. The interference alignment for a network where each message is assumed to be requested by an equal number of receivers was investigated in [46] using ergodic interference alignment but only the achievable sum rate is reported.

In this chapter, we consider a natural generalization of the multiple unicasts scenario considered in the seminal work of Cadambe and Jafar [39]. The interference networks we investigated allows general message demands, which includes multiple unicasts and multiple multicasts, or combination of them.

The chapter is organized as follows. The system model is given in Section 5.2. We develop an outer bound on the degrees of freedom region of this system, and then show it is achievable in Section 5.3. We discuss approaches for reducing the number of alignment constraints in Section 5.4. Finally, Section 5.5 concludes this chapter.

5.2 System model

We consider a single hop interference network with K transmitters and J receivers, each having M antennas. The received signal at the jth receiver can be expressed as

$$\boldsymbol{y}_j(t) = \sum_{k=1}^{K} \boldsymbol{H}_{jk}(t)\boldsymbol{x}_k(t) + \boldsymbol{z}_j(t), \tag{5.1}$$

Recall that each receiver can request an arbitrary set of messages from multiple transmitters and $\mathcal{M}_j$ is the set of indices of those messages requested by receiver j. Our objective is to study the DoF region of an interference network with general message demands when there is perfect CSI at receivers and global CSI at transmitters. Denote the capacity region of such a system as $C(P)$. The corresponding DoF region is defined as

$$\begin{aligned} \mathcal{D} := \Big\{ \boldsymbol{d} = (d_1, d_2, \cdots, d_K) \in \mathbb{R}_+^K : \exists (R_1(P), R_2(P), \cdots, R_K(P)) \in C(P), \\ \text{such that } d_k = \lim_{P\to\infty} \frac{R_k(P)}{\log(P)}, \quad 1 \le k \le K \Big\}. \end{aligned} \tag{5.2}$$

We assume that the channel coherent time $L = 1$ in this chapter. As we pointed before, if $J = K$ and $\mathcal{M}_j = \{j\}, \forall j$, the general model we considered here will reduce to the well-known K user M antenna interference channel as in [39].

5.3 DoF region of interference network with general message demands

In this section, we prove the DoF region of the interference network with general message demands. Our main result can be summarized as the following theorem.

Theorem 5. *The DoF region of an interference network with general message demands with K transmitters and J receivers, is contained in the following region*

$$\mathcal{D} = \left\{ \boldsymbol{d} \in \mathbb{R}_+^K : \sum_{k\in\mathcal{M}_j} d_k + \max_{i\in\mathcal{M}_j^c}(d_i) \le M, \quad \forall 1 \le j \le J \right\}. \tag{5.3}$$

When $M = 1$, the above outer bound $\mathcal{D}$ is the DoF region. $\square$

5.3.1 Some discussions on the DoF region

5.3.1.1 The converse argument

We now show the region given by (5.3) is an outer bound based on a genie argument [35, 39]. Suppose that for receiver j, there is a genie who provides all the interference messages except for the interference message with the largest DoF. If there are several such messages having the largest DoF, the genie withholds the one with the smallest index. Specifically, the index of this interference message withheld by the genie is denoted as

$$\delta_j = \min\left\{ k \middle| k \in \mathcal{M}_j^c, \text{ and } d_k = \max_{i \in \mathcal{M}_j^c}(d_i) \right\}. \tag{5.4}$$

All other interference messages are revealed by the genie to receiver j. If $\mathcal{M}_j^c$ is an empty set, we define $d_{\delta_j} = 0$.

With the genie's help, all interference messages except for message δ_j can be removed. Since the requested messages $\mathcal{M}_j$ can be decoded, they can be subtracted from the received signal. It is evident that since message δ_j needs to be decoded by some receiver, it holds that $d_{\delta_j} \leq M$ using the point-to-point channel DoF constraint. Therefore, receiver j can also decode message δ_j after subtracting all the other messages as it has M antennas as well. Hence, it follows due to multiple access channel outer bound that any achievable DoF point $(d_1, d_2, \dots, d_K)$ must satisfy

$$\sum_{k \in \mathcal{M}_j} d_k + \max_{i \in \mathcal{M}_j^c}(d_i) \leq M, \quad \forall 1 \leq j \leq J. \tag{5.5}$$

□

5.3.1.2 DoF region representation

The DoF region is a convex polytope. The representation of DoF region in (5.3) is known as the half-space representation in geometry [81]. Another representation of the convex polytope is the vertex representation: as any interior point within a convex polytope can be represented by the linear combination of the vertices. Such a fact is widely used to show the capacity or DoF region with time-sharing argument. That is, it is necessary and sufficient to show the

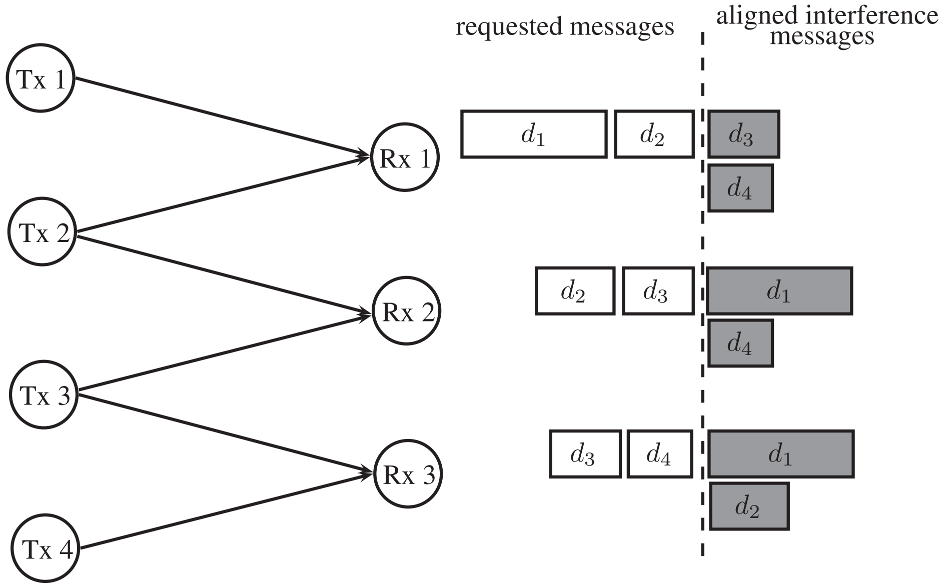

Figure 5.1 Example system setup (left) and alignment for achieving DoF point $(d_1, d_2, d_3) = (1-2d_4, d_4, d_4)$ (right). Message demanding relationship is shown on the left part with arrow lines.

achievability of all the vertices of the capacity or DoF region. Though finding all the vertices of convex polytope is a well-studied problem in geometry [82], the complexity of finding all the vertices of a high dimensional convex polytope is high. In addition, there is no general representation of the vertices based on the half-space representation. Hence, finding the specific achievability scheme for different vertices to validate the DoF region is not efficient at all.

In the following part, we will use a simple example to demonstrate the DoF region and reveal the basic idea of our achievability scheme.

5.3.2 An example of the general message demand and the DoF region

We first show the geometric picture of the DoF region for a specific example. Consider an interference network with 4 transmitters and 3 receivers; see Fig. 5.1. All the transmitters and receivers have single antenna; that is, $M = 1$. Assume $\mathcal{M}_1 = \{1, 2\}$, $\mathcal{M}_2 = \{2, 3\}$ and

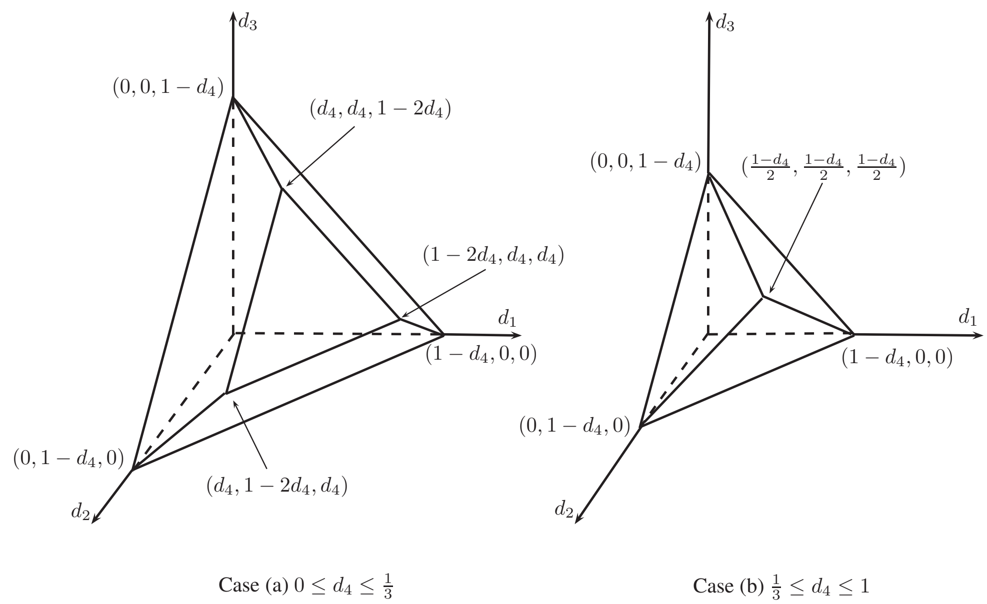

Figure 5.2 DoF region in lower dimensions as a function of d_4.

$\mathcal{M}_3 = \{3,4\}$. The DoF region of the system according to Theorem 5 is as follows

$$\mathcal{D} = \left\{ \boldsymbol{d} \in \mathbb{R}_+^4 \;\middle|\; \begin{array}{l} d_1 + d_2 + d_3 \le 1 \\ d_1 + d_2 + d_4 \le 1 \\ d_2 + d_3 + d_4 \le 1 \\ d_1 + d_3 + d_4 \le 1 \end{array} \right\}. \tag{5.6}$$

The region is 4-dimensional and hence difficult to illustrate. However, if the DoF of one message, say d_4, is fixed, the DoF region of the other messages can be illustrated in lower dimensions as a function of d_4; see Fig. 5.2.

We first investigate the region when $0 \le d_4 \le \frac{1}{3}$, for which the coordinates of the vertices are given in Fig. 5.2, case (a). The achievability of the vertices on the axes is simple as there is no need of interference alignment. Time sharing between the single-user rate vectors $\{\boldsymbol{e}_k, k = 1, 2, \ldots, K\}$ is sufficient. For the remaining three vertices, we only need to show the achievability of one point as the achievability of the others are essentially the same by swapping the message indices.

We now use the scheme based on [39] to do interference alignment and show $(d_1, d_2, d_3, d_4) = (1-2d_4, d_4, d_4, d_4)$ is achievable for any $0 \le d_4 \le \frac{1}{3}$. Recall τ is the duration of the time expansion in number of symbols. Hence, $\tilde{\boldsymbol{H}}_{jk}$ is a size $\tau \times \tau$ diagonal matrix (recall that $M = 1$). Denote the beamforming matrix of transmitter k as $\tilde{\boldsymbol{V}}_k$. First, we want messages 3 and 4 to be aligned at receivers 1. Notice that messages 3 and 4 have the same number of DoF. We choose to design beamforming matrices such that the interference from transmitter 4 is aligned to interference from transmitter 3 at receiver 1. Therefore we have the following constraint

$$\tilde{\boldsymbol{H}}_{14}\tilde{\boldsymbol{V}}_4 \prec \tilde{\boldsymbol{H}}_{13}\tilde{\boldsymbol{V}}_3. \tag{5.7}$$

Note that the interference due to transmitter 1 has a larger DoF at receiver 2; thus, we must align interference from transmitter 4 to interference from transmitter 1 at receiver 2, which leads to

$$\tilde{\boldsymbol{H}}_{24}\tilde{\boldsymbol{V}}_4 \prec \tilde{\boldsymbol{H}}_{21}\tilde{\boldsymbol{V}}_1. \tag{5.8}$$

Similarly at receiver 3, we have

$$\tilde{\boldsymbol{H}}_{42}\tilde{\boldsymbol{V}}_2 \prec \tilde{\boldsymbol{H}}_{41}\tilde{\boldsymbol{V}}_1. \tag{5.9}$$

The alignment relationship is also shown in Fig. 5.1. Notice that d_1 is larger than d_2, d_3 and d_4. Therefore it is possible to design $\tilde{\boldsymbol{V}}_1$ into two parts as $[\tilde{\boldsymbol{V}}_{1a}, \tilde{\boldsymbol{V}}_{1b}]$, where $\tilde{\boldsymbol{V}}_{1a}$ is used for transmitting part of the message 1 with the same DoF as other messages. The second part $\tilde{\boldsymbol{V}}_{1b}$ is used for transmitting the remaining DoF of message 1. In addition, all the columns in $[\tilde{\boldsymbol{V}}_{1a}, \tilde{\boldsymbol{V}}_{1b}]$ are linearly independent.

The design of the first part falls into the framework of classic asymptotic interference alignment scheme in [39]. The beamforming matrices in [39] is chosen from a set of beamforming columns, whose elements are generated from the product of the powers of certain matrices and a vector. We term such a vector as a *base vector* in this chapter. The base vector was chosen to be the all-one vector in [39]. The scheme proposed in [39] was further explored for wireless X network [40] with multiple independent messages at single transmitter, where multiple independent and randomly generated base vectors are used for constructing the beamforming matrices. In our particular example, as no interference is aligned to the second part of message

1, we may choose an independent and randomly generated matrix for $\tilde{\boldsymbol{V}}_{1b}$. However, in general we still need to construct the beamforming matrix structurally using multiple base vectors as we will see in Section 5.3.3. The DoF point can be achieved asymptotically when the number of time expansion τ goes to infinity. We omit further details of beamforming construction for this particular example.

The DoF region of case (b) in Fig. 5.2 can be similarly achieved by showing that the vertex $(d_1, d_2, d_3, d_4) = (\frac{1-d_4}{2}, \frac{1-d_4}{2}, \frac{1-d_4}{2}, d_4)$ is achievable, which will also require the multiple base vector technique.

We remark that the DoF region in this example can also be formulated as the convex hull of the following vertices $\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \frac{1}{3}\mathbf{1}\}$. The achievability of the whole DoF therefore can be alternatively established by showing that $\frac{1}{3}\mathbf{1} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is achievable.

5.3.3 Achievability of DoF region with single antenna transmitters and receivers

We first consider the achievability scheme when all the transmitters and receivers have single antennas. The multiple antenna case can be reduced to the single antenna case and will be discussed later. It is evident that we only need to show any point in $\mathcal{D}$ satisfying

$$d_K \le d_{K-1} \le \cdots \le d_2 \le d_1 \tag{5.10}$$

is achievable, for otherwise the messages can be renumbered appropriately.

5.3.3.1 The set of alignment constraints

The achievability scheme is based on interference alignment over a time expanded channel. Based on (5.10), we impose the following relationship on the cardinality of all the beamforming matrices of the transmitters:

$$|\tilde{\boldsymbol{V}}_K| \le |\tilde{\boldsymbol{V}}_{K-1}| \le \cdots \le |\tilde{\boldsymbol{V}}_2| \le |\tilde{\boldsymbol{V}}_1|. \tag{5.11}$$

At receiver j, we always align the interference messages with larger indices to the interference message with index δ_j, which is the interference message with the largest DoF, given as

$$\delta_j = \min\{k | k \in \mathcal{M}_j^c\} \tag{5.12}$$

Denote $\boldsymbol{T}^{[j]}_{m,n}$ as following

$$\boldsymbol{T}^{[j]}_{m,n} = \tilde{\boldsymbol{H}}^{-1}_{jm}\tilde{\boldsymbol{H}}_{jn}, \tag{5.13}$$

which is the matrix corresponding to the alignment constraint

$$\tilde{\boldsymbol{H}}_{jn}\tilde{\boldsymbol{V}}_n \prec \tilde{\boldsymbol{H}}_{jm}\tilde{\boldsymbol{V}}_m, \tag{5.14}$$

that enforces the interference from message n to be aligned to the interference of message m at receiver j. Based on (5.11), for any $\boldsymbol{T}^{[j]}_{m,n}$ matrix, we always have $n > m$.

For convenience, we define the following set

$$\mathcal{C} := \left\{ (m,n,j) \;\middle|\; \begin{array}{c} j \in \{1,\ldots,J\}, \\ m,n \in \mathcal{M}_j^c, \\ m = \delta_j, n > m \end{array} \right\}. \tag{5.15}$$

In other words, $\mathcal{C}$ is a set of vectors denoting all the alignment constraints. There exists a one-to-one mapping from a vector (m,n,j) in $\mathcal{C}$ to the corresponding matrix $\boldsymbol{T}^{[j]}_{m,n}$.

5.3.3.2 Time expansion and base vectors

It is not difficult to see that the vertices of the DoF region given in (5.3) must be rational as all the coefficients and right hand side bounds are integers (either zero or one). Therefore we only need to consider the achievability of such rational vertices, although the proof below applies to any interior rational points in the DoF region as well.

For any rational DoF point $\boldsymbol{d}$ within $\mathcal{D}$ and satisfying (5.10), we can choose a positive integer κ, such that

$$\kappa\boldsymbol{d} = (\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_K) \in \mathbb{Z}_+^K. \tag{5.16}$$

Towards this end, we propose to use multiple base vectors to construct the beamforming matrices. The total number of base vectors is $\bar{d}_1$. Denote all the base vectors as $\boldsymbol{w}_i, 1 \le i \le \bar{d}_1$. Transmitter k will use base vectors $\boldsymbol{w}_i, 1 \le i \le \bar{d}_k$ to construct its beamforming matrix, and the same base vectors will be used by transmitters $1, 2, \ldots, k-1$ as well, see Fig. 5.3. The

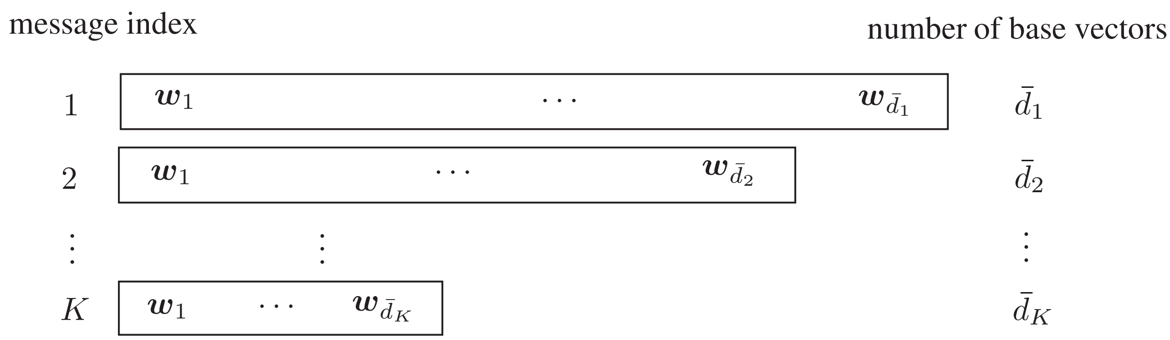

Figure 5.3 Illustration of the base vectors used by different messages.

elements of $\boldsymbol{w}_i$ are independent and identically drawn from some continuous distribution. In addition, we assume that the absolute value of the elements of $\boldsymbol{w}_i$ are bounded between a positive minimum value and a finite maximum value, in the same way that entries of $\boldsymbol{H}_{jk}(t)$ are bounded (see Section 2.4.1). Denote $\Gamma = |\mathcal{C}|$, which is the total number of $\boldsymbol{T}_{m,n}^{[j]}$ matrices as well. Let l be a positive integer, we propose to use $\tau = \kappa(l+1)^{\Gamma}$ fold time expansion.

5.3.3.3 Beamforming matrices design

The beamforming matrices can be generated in the following way

i) Denote Γ_k as the cardinality of the following set

$$\mathcal{C}_k = \{(m,n,j)|(m,n,j) \in \mathcal{C}, n \leq k\} \quad k = 1, 2 \ldots, K, \tag{5.17}$$

which is the number of matrices whose exponents are within $\{0, 1, \ldots, l-1\}$, while the other $\Gamma - \Gamma_k$ matrices can be raised to the power of l. It is evident that $\Gamma_K = \Gamma$, and $\Gamma_1 = 0$.

ii) Transmitter K uses $\bar{d}_K$ base vectors. For base vector $\boldsymbol{w}_i, 1 \leq i \leq \bar{d}_K$, it generates the following l^{Γ} columns

$$\prod_{(m,n,j)\in\mathcal{C}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}} \boldsymbol{w}_i \tag{5.18}$$

where $\alpha_{m,n,j} \in \{0, 1, \ldots, l-1\}$. Hence, the total number of columns of $\tilde{\boldsymbol{V}}_K$ is $\bar{d}_K l^{\Gamma}$.

iii) Similarly, transmitter k uses $\bar{d}_k$ base vectors. For base vector $\boldsymbol{w}_i, 1 \leq i \leq \bar{d}_k$, it generates $l^{\Gamma_k}(l+1)^{\Gamma-\Gamma_k}$ columns

$$\prod_{(m,n,j)\in\mathcal{C}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}} \boldsymbol{w}_i \tag{5.19}$$

where

$$\alpha_{m,n,j} \in \{0,1,\ldots,l\} \qquad n > k \tag{5.20}$$

$$\alpha_{m,n,j} \in \{0,1,\ldots,l-1\} \qquad n \leq k \tag{5.21}$$

In summary, the beamforming design is as follows, for every message, we construct a beamforming column set as

$$\tilde{\mathcal{V}}_k = \left\{ \prod_{(m,n,j)\in\mathcal{C}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}} \boldsymbol{w}_i \,\middle|\, 1 \leq i \leq \bar{d}_k, \alpha_{m,n,j} \in \begin{cases} \{0,1,\ldots,l\} & \text{if } n > k \\ \{0,1,\ldots,l-1\} & \text{otherwise} \end{cases} \right\} \quad 1 \leq k \leq K. \tag{5.22}$$

The beamforming matrix $\tilde{\boldsymbol{V}}_k$ is chosen to be the matrix that contains all the columns of $\tilde{\mathcal{V}}_k$.

5.3.3.4 Alignment at the receivers

Assume $(k,k',j) \in \mathcal{C}$, so that message k' needs to be aligned with message $k < k'$ at receiver j. We now show that this is guaranteed by our design. Let $\boldsymbol{w}_i$, $1 \leq i \leq \bar{d}_{k'}$ be a base vector used by transmitter k', and hence also used by transmitter k. From (5.19), the beamforming vectors generated by $\boldsymbol{w}_i$ at transmitter k can be expressed in the following way

$$\underbrace{\prod_{\substack{(m,n,j)\in\mathcal{C}\\ n\leq k}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}}}_{\alpha_{m,n,j}\in\{0,1,\ldots,l-1\}} \underbrace{\prod_{\substack{(m,n,j)\in\mathcal{C}\\ (m,n)=(k,k')}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}} \prod_{\substack{(m,n,j)\in\mathcal{C}\\ n>k\\ (m,n)\neq(k,k')}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}}}_{\alpha_{m,n,j}\in\{0,1,\ldots,l\}} \boldsymbol{w}_i, \tag{5.23}$$

whereas those at the transmitter k' can be expressed as

$$\underbrace{\prod_{\substack{(m,n,j)\in\mathcal{C}\\ n\leq k'\\ (m,n)\neq(k,k')}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}} \prod_{\substack{(m,n,j)\in\mathcal{C}\\ (m,n)=(k,k')}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}}}_{\alpha_{m,n,j}\in\{0,1,\ldots,l-1\}} \underbrace{\prod_{\substack{(m,n,j)\in\mathcal{C}\\ n>k'}} \left(\boldsymbol{T}_{m,n}^{[j]}\right)^{\alpha_{m,n,j}}}_{\alpha_{m,n,j}\in\{0,1,\ldots,l\}} \boldsymbol{w}_i. \tag{5.24}$$

Comparing the ranges of $\alpha_{k,k',j}$ in (5.23) and (5.24), i.e., the middle terms, it can be verified that the columns in (5.24) multiplied with $\boldsymbol{T}_{k,k'}^{[j]}$, will be a column in (5.23), $\forall(k,k',j) \in \mathcal{C}$. That is, message k' can be aligned to message k for any j such that $(k,k',j) \in \mathcal{C}$.

The alignment scheme works due to the following reasons.

i) Let $\alpha_{m,n,j}(k)$ denote the exponent of the (m,n,j) term for $\tilde{\boldsymbol{V}}_k$. The construction of the beamforming column set guarantees that

$$\max \alpha_{m,n,j}(m) > \max \alpha_{m,n,j}(n), \quad \forall(m,n,j) \in \mathcal{C} \tag{5.25}$$

by setting

$$\max \alpha_{m,n,j}(m) = l, \tag{5.26}$$

$$\max \alpha_{m,n,j}(n) = l-1. \tag{5.27}$$

With (5.25), we are guaranteed all vectors in $\tilde{\boldsymbol{V}}_n$, when left multiplied with $\boldsymbol{T}_{m,n}^{[j]}$ (which has the effect of increasing the exponent of $\boldsymbol{T}_{m,n}^{[j]}$ by one), generates a vector that is within the columns of $\tilde{\boldsymbol{V}}_m$. Hence the alignment is ensured.

For other terms where k is not m or n, $\max \alpha_{m,n,j}(k)$ can be either l or $l-1$.

ii) The base vectors used by transmitter n are also used by transmitter $m < n$. This guarantees that if the interference from transmitter n needs to be aligned with interference from transmitter m, where $m < n$, the alignment is ensured with the condition (5.25).

5.3.3.5 Achievable Rates

It is evident that $\tilde{\boldsymbol{V}}_k$ is a tall matrix of dimension $\kappa(l+1)^{\Gamma} \times \bar{d}_k l^{\Gamma_k}(l+1)^{\Gamma-\Gamma_k}$. We also need to verify it has full column rank by using Lemma 15 (see appendix in Section 5.6) to show that its upper square submatrix is full rank. Notice that all the entries are monomials and the random variables of the monomial are different in different rows. In addition, for a given row $r, 1 \leq r \leq \bar{d}_k l^{\Gamma_k}(l+1)^{\Gamma-\Gamma_k}$, any two entries have different exponents. Therefore, based on Lemma 15, $\tilde{\boldsymbol{V}}_k$ has full column rank and

$$\lim_{l\to\infty} \frac{|\tilde{\boldsymbol{V}}_k|}{\tau} = \lim_{l\to\infty} \frac{\bar{d}_k l^{\Gamma_k}(l+1)^{\Gamma-\Gamma_k}}{\kappa(l+1)^{\Gamma}} = \frac{\bar{d}_k}{\kappa} = d_k. \tag{5.28}$$

5.3.3.6 Separation of the signal and interference spaces

Finally, we need to ensure that the interference space and signal space are linearly independent for all the receivers. Let the set of messages requested by receiver j be $\mathcal{M}_j = \{m_{1,j}, m_{2,j}, \ldots, m_{\beta_j,j}\}$, where $\beta_j = |\mathcal{M}_j|$. For receiver j to be satisfied the following matrix

$$\boldsymbol{\Lambda}_j = \left[\tilde{\boldsymbol{H}}_{jm_{1,j}}\tilde{\boldsymbol{V}}_{m_{1,j}}|\tilde{\boldsymbol{H}}_{jm_{2,j}}\tilde{\boldsymbol{V}}_{m_{2,j}}|\ldots,\tilde{\boldsymbol{H}}_{jm_{\beta_j,j}}\tilde{\boldsymbol{V}}_{m_{\beta_j,j}}|\tilde{\boldsymbol{H}}_{j\delta_j}\tilde{\boldsymbol{V}}_{\delta_j}\right] \tag{5.29}$$

needs to have full rank for all $1 \le j \le J$.

Notice that for any point within $\mathcal{D}$

$$\sum_{m\in\mathcal{M}_j} \bar{d}_m + \bar{d}_{\delta_j} \le \kappa \tag{5.30}$$

always holds (recall $M = 1$). Therefore $\boldsymbol{\Lambda}_j$ is a matrix that is either tall or square. For any row r of its upper square sub-matrix, its elements can be expressed in the following general form:

$$\boldsymbol{H}_{jk}(r) \prod_{(m,n,j)\in\mathcal{C}} \left(\boldsymbol{H}_{jn}^{-1}(r)\boldsymbol{H}_{jm}(r)\right)^{\alpha_{m,n,j}} [\boldsymbol{w}_i]_r. \tag{5.31}$$

The elements from different blocks (that is, different $[\tilde{\boldsymbol{H}}_{jk}\tilde{\boldsymbol{V}}_k]$, $k \in \mathcal{M}_j \cup \{\delta_j\}$) are different due to the fact that $\boldsymbol{H}_{jk}(r)$'s are different, hence the monomials do not have the same random variables. Within one $\tilde{\boldsymbol{H}}_{jk}\tilde{\boldsymbol{V}}_k$, two monomials are different either because they have different $[\boldsymbol{w}_i]_r, 1 \le i \le \bar{d}_k$, or, if they have the same $[\boldsymbol{w}_i]_r$, the associated exponents $\alpha_{m,n,j}$ are different. Thus matrix $\boldsymbol{\Lambda}_j$ has the following properties.

i) Each term is a monomial of a set of random variables.

ii) The random variables associated with different rows are independent.

iii) No two elements in the same row have the same exponents.

Therefore, $\boldsymbol{\Lambda}_j$ has full column rank as its entries satisfy Lemma 15.

Combining the interference alignment and the full-rank arguments, we conclude that any point $\boldsymbol{d}$ satisfying (5.3) is achievable.

5.3.4 Discussion of the achievability of DoF region with multiple antenna transmitters and receivers

In the special case of K-user M antenna interference channel, the achievability of the total DoF is based on antenna splitting argument as in [39]. However, the antenna splitting argument cannot be used to establish the DoF region in general because it relies on the fact that all DoF's are equal when the total DoF is maximized. When multiple antennas are split, the messages transmitted on the individual antennas become additional interference for each other on the receiver side, which may lead to a shrinkage of the DoF region that can be achieved with the split-antenna transmissions. The DoF region of interference network with multiple antennas remains to be quantified.

5.4 Discussion

In this section, we outline some alternative schemes that require a lower level of time-expansion for achieving the same DoF region, and highlight some interesting consequences of the general results developed in Section 5.3.

5.4.1 Group based alignment scheme

The achievability scheme presented in Section 5.3 requires all interference messages at one receiver to be aligned with the largest one. This may introduce more alignment constraints than needed. We give an example here to illustrate this point.

Example 2. *Consider a simple scenario of a single antenna network where there are* 4 *messages and 5 receivers. Without loss of generality, assuming* (5.11) *is true and* $\mathcal{M}_1 = \{1, 2\}$, $\mathcal{M}_2 = \{2\}$, $\mathcal{M}_3 = \{2, 3\}$, $\mathcal{M}_4 = \{2, 3\}$ *and* $\mathcal{M}_5 = \{1, 4\}$*. The alignment constraints associated with the first two receivers will be the following*

$$\tilde{\boldsymbol{H}}_{14}\tilde{\boldsymbol{V}}_4 \prec \tilde{\boldsymbol{H}}_{13}\tilde{\boldsymbol{V}}_3, \tag{5.32}$$

$$\tilde{\boldsymbol{H}}_{23}\tilde{\boldsymbol{V}}_3 \prec \tilde{\boldsymbol{H}}_{21}\tilde{\boldsymbol{V}}_1, \tag{5.33}$$

$$\tilde{\boldsymbol{H}}_{24}\tilde{\boldsymbol{V}}_4 \prec \tilde{\boldsymbol{H}}_{21}\tilde{\boldsymbol{V}}_1. \tag{5.34}$$

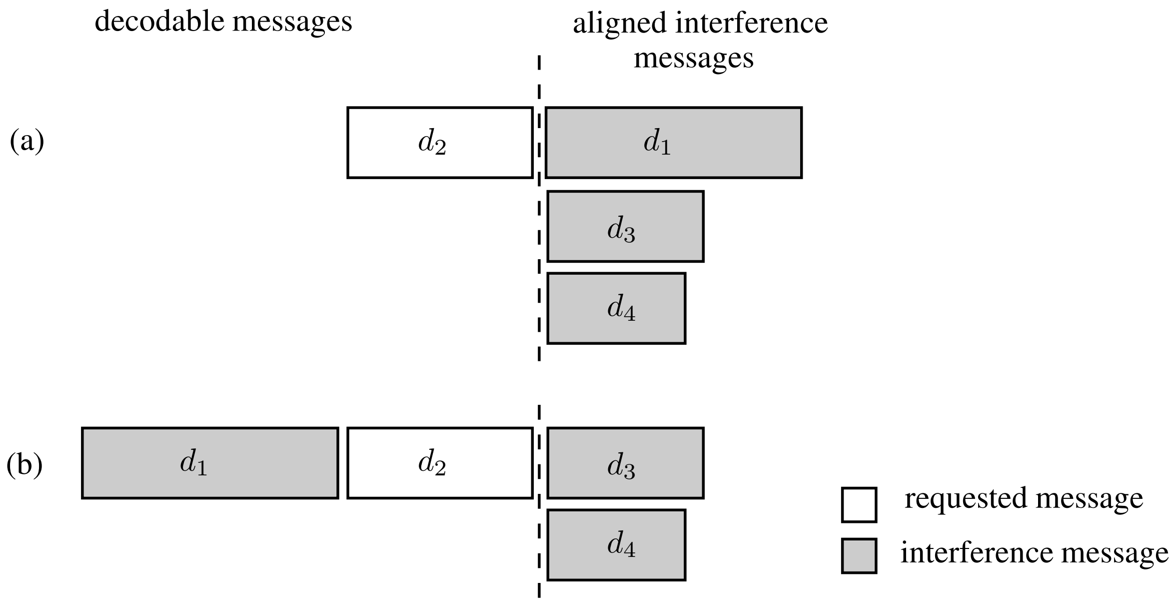

Figure 5.4 Example of alignment: (a) the original scheme, (b) the modified scheme.

However, in this particular case, upon inspection, one can realize that even if receiver 2 also receives message 1, the DoF region will not change. This is because the constraint at receiver 1 dictates that

$$d_1 + d_2 + \max(d_3, d_4) \leq M. \tag{5.35}$$

However, this implies the required constraint at receiver 2,

$$d_2 + \max(d_1, d_3, d_4) \leq M. \tag{5.36}$$

Therefore, receiver 2 can use the same alignment relationship as receiver 1, i.e., it can also decode message 1 without shrinking the DoF region. The difference between the original alignment scheme and the modified scheme of receiver 2 is illustrated in Fig. 5.4. □

The alignment scheme of Section 5.3 can be modified appropriately using the idea of *partially ordered set* (poset)[83].

A poset is a set $\mathcal{P}$ and a binary relation $\leq$ such that for all $a, b, c \in \mathcal{P}$, we have

1. $a \leq a$ (reflexivity).

2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

3. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry).

An element b in $\mathcal{P}$ is the *greatest element* if for every element $a \in P$, we have $a \leq b$. An element $b \in P$ is a *maximal element* if there is no element $a \in \mathcal{P}$ such that $a > b$. If a poset has a greatest element, it must be the unique maximal element, but otherwise there can be more than one maximal element.

For two message request sets $\mathcal{M}_j$ and $\mathcal{M}_{j'}$, we say $\mathcal{M}_j \leq \mathcal{M}_{j'}$ if $\mathcal{M}_j \subseteq \mathcal{M}_{j'}$. With this partial ordering, the collection of message request sets $\{\mathcal{M}_j : 1 \leq j \leq K\}$, with duplicate elements (message sets) removed, forms a poset. Let G denote the number of maximal elements of this poset, and $\bar{\mathcal{M}}_g$ denote the gth maximal element, $1 \leq g \leq G$. We divide the receivers into G group according to the following rule: For receiver j, if there exists a group index g such that $\mathcal{M}_j = \bar{\mathcal{M}}_g$, then receiver j is assigned to group g. Otherwise, $\mathcal{M}_j$ is not a maximal element, we can assign receiver j to any group g such that $\mathcal{M}_j \subset \bar{\mathcal{M}}_g$. In the case where there are multiple maximal elements of the poset that are "larger" than $\mathcal{M}_j$, we can choose the index of any of one of them as the group index of receiver j.

With our grouping scheme, there will be at least one receiver in each group whose message request set is a superset of the message request set of any other user in the same group. There may be multiple such receivers in each group though. In either case, we term one such (or the one in case there is only one) receiver as the *prime* receiver. We choose all the receivers within one group use the same alignment relationship as the prime receiver of that group and the total number of alignment constraints is reduced. In such a way, the receivers in one group can actually decode the same messages requested by the prime receiver of that group, and they can simply discard the messages that they are not interested in.

As to Example 2 given in this section, we can divide 5 receivers into three groups. Receivers 1 and 2 as group 1, receivers 3 and 4 as group 2, receiver 5 as group 3 and prime receivers are 1, 3 and 5. We remark that there are multiple ways of group division as long as one receiver can only belong to one group, e.g., receiver 1 as group 1, receivers 2, 3 and 4 as group 2, receiver 5 as group 3 and prime receivers are 1, 4 and 5.

The group based alignment scheme is still valid as it is the scheme for an interference network whose receiver message requests are from some exclusive sets and multiple receivers having the same request. In other words, it can be viewed as an interference network with multiple multicast groups, where the message request of any multicast group can not be a subset of the requests of the other groups, which of course is a special case of general message requests we considered in Section 5.3.

Corollary 1. *The DoF region of the interference network with general message requests as in Section 6.2 is determined by the prime receivers, adding non-prime receivers to the system will not affect the DoF region.*

Proof. This can be shown as the inequalities (5.3) associated with the non-prime receivers are inactive, therefore the region is dominated by the inequalities of prime receivers. □

5.4.2 DoF region of K user M antenna interference network

Although the DoF region of multiple antenna case is still an open problem for the general case, the DoF region of the K user M antenna interference channel has been obtained based on a simple time sharing argument [84], which we include here for completeness.

Theorem 6. *[84] The DoF region of K user M antenna interference channel is*

$$\mathcal{D} = \{(d_1, d_2, \cdots, d_K) : \quad 0 \leq d_i + d_j \leq M, \forall 1 \leq i, j \leq K, i \neq j\}. \tag{5.37}$$

Proof. Without loss of generality, suppose $d_1^* \geq d_2^* \geq d_k^*, k = 3, \cdots, K$, and $d_i^* + d_j^* \leq d_1^* + d_2^* \leq M$, $\forall i, j \in \{1, 2, \dots K\}$. We would like to show that $(d_1, d_2, \dots, d_K) = (d_1^*, d_2^*, \dots, d_K^*)$ is achievable.

It is obvious that

$$(d_1, d_2, \dots, d_K) = (M, 0, \dots, 0)$$

can be achieved by single user transmission. It is also known from [39] that the point

$$(d_1, d_2, \dots, d_K) = (M/2, M/2, \dots, M/2)$$

is achievable. Trivially, the point

$$(d_1, d_2, \ldots, d_K) = (0, 0, \ldots, 0)$$

is achievable.

By time sharing, with weights $(d_1 - d_2)/M$, $2d_2/M$ and $1 - d_1/M - d_2/M$ among the three points, in that order, it follows that the point

$$(d_1, d_2, \ldots, d_K) = (d_1^*, d_2^*, d_2^*, \ldots, d_2^*)$$

is achievable. This is already at least as large as the DoF we would like to have. □

Remark 2. *The DoF region for a single-antenna interference channel without time-expansion has been independently shown to be the convex hull of* $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_K, \frac{1}{2}\mathbf{1}\}$ *for almost all (in Lebesgue sense) channels [85]. Interestingly, this agrees with DoF region of the* K*-user single antenna interference channel. For, it can be seen from the proof of Corollary 1 that the DoF region given in* (5.37) *can be alternatively formulated as the convex hull of the vectors* $\{\mathbf{0}, M\boldsymbol{e}_1, \ldots, M\boldsymbol{e}_K, \frac{M}{2}\mathbf{1}\}$. *Setting* $M = 1$ *will yield the desired equivalence of the two DoF regions. This equivalence, is non-trivial, however, because it shows that allowing for time-expansion, and time-diversity (channel variation), the DoF region of the interference channel is not increased — the DoF is an inherent spatial (as opposed to temporal) characteristic of the interference channel.*

5.4.3 Length of time expansion

For the K-user M antenna interference channel, the total length of time expansion needed in [39] is smaller than our scheme in order to achieve $KM/2$ total DoF. This is due to the fact that when $J = K$ and $\mathcal{M}_j = \{j\}, \forall j$, it is possible to choose $\tilde{\boldsymbol{V}}_2$ carefully such that the cardinality of $\tilde{\boldsymbol{V}}_2$ is the same as $\tilde{\boldsymbol{V}}_1$ and there is one-to-one mapping between these two. For other asymmetric DoF points, it is in general not possible to choose two messages having the same cardinality of beamforming column sets. The total time expansion needed could be reduced if we use the group based alignment scheme in Section 5.4.1 and/or design the achievable scheme for a specific network with certain DoF.

5.4.4 The total DoF of an interference network with general message demands

As a byproduct of our previous analysis, we can also find the total degrees of freedom for an interference network with general message demands.

Corollary 2. *The total DoF of an interference network with general message demands can be obtained by a linear program shown as follows*

$$\max \sum_{k=1}^{K} d_k$$
$$s.t. \quad \sum_{k \in \mathcal{M}_j} d_k + \max_{i \in \mathcal{M}_j^c}(d_i) \le M, \quad \forall 1 \le j \le J, \boldsymbol{d} \in \mathbb{R}_+^K. \tag{5.38}$$

□

Corollary 3. *If all prime receivers demand β, $1 \le \beta \le K-1$, messages, and each of the K messages are requested by the same number of prime receivers. Then the total DoF is*

$$d_{\mathsf{total}} = \frac{MK}{\beta+1}, \tag{5.39}$$

and is achieved by

$$\boldsymbol{d} = \left(\frac{M}{\beta+1}, \frac{M}{\beta+1}, \ldots, \frac{M}{\beta+1}\right). \tag{5.40}$$

Proof. Based on Corollary 1, we only need to consider G (where G is the number of groups) inequalities that associated with prime receivers. We show that (5.40) achieves the maximum total DoF when all K messages are requested by the same number of prime receivers. Notice that in this case we can expand the inequality of (5.38) into $K-\beta$ inequalities by removing the max() operation. Hence, we will have $G(K-\beta)$ inequalities in total. Since each message is requested by $G\beta/K$ prime receivers, for each d_k it appears $\frac{G\beta}{K}(K-\beta)$ times among the inequalities for prime receivers which request d_k, and it appears $G - \frac{G\beta}{K}$ times otherwise. Summing all the $G(K-\beta))$ inequalities we have

$$\left(\frac{G\beta}{K}(K-\beta) + G - \frac{G\beta}{K}\right) \sum_k d_k \le MG(K-\beta). \tag{5.41}$$

Hence

$$\sum_k d_k \le \frac{MK}{\beta+1}, \tag{5.42}$$

and the corollary is proven. □

Remark 3. *If messages are not requested by the same number of prime receivers it is possible to achieve a higher sum DoF than* (5.39)*. We only need to show an example here. Assuming that there are* 4 *transmitters and* 3 *prime receivers, the message requests are* $\{1,2\},\{1,3\},\{1,4\}$*. If all the transmitters send* $M/3$ *DoF, we could achieve* (5.39)*. However, choosing* $\boldsymbol{d} = (0, \frac{M}{2}, \frac{M}{2}, \frac{M}{2})$ *will lead to sum DoF* $3M/2$ *which is higher.*

5.5 Summary

We derived the DoF region of a single antenna interference network with general message demands. Our achievability scheme for a general point in the DoF region operates by generating beamforming columns with multiple base vectors over time expanded channel, and aligning the interference at each receiver to its largest interferer. Next, we showed that the DoF region is determined by a subset of receivers (called prime receivers), that can be identified by examining the precise demands of all the receivers. In this case, we can divide the receivers into various groups. We provided an alternate interference alignment scheme in this scenario, where the receivers in one group choose the same alignment relationship as the prime receiver of that group; this allows us to reduce the required level of time-expansion.

5.6 Appendix

Lemma 15. *[40, Lemma 1] Consider an* $M \times M$ *square matrix* $\boldsymbol{A}$ *such that* a_{ij}*, the elements in the* i*th row and* j*th column of* $\boldsymbol{A}$*, is of the form*

$$a_{ij} = \prod_{k=1}^{K} \left(x_i^{[k]}\right)^{\alpha_{ij}^{[k]}} \tag{5.43}$$

where $x_i^{[k]}$ *are random variables and all exponents are integers,* $\alpha_{ij}^{[k]} \in \mathbb{Z}$*. Suppose that*

1. $x_i^{[k]} | \{x_{i'}^{[k']}, \forall (i,k) \neq (i',k')\}$ *has a continuous cumulative probability distribution.*

2. $\forall i, j, j' \in \{1,2,\ldots,M\}$ *with* $j \neq j'$

$$\left(\alpha_{ij}^{[1]}, \alpha_{ij}^{[2]}, \ldots, \alpha_{ij}^{[K]}\right) \neq \left(\alpha_{ij'}^{[1]}, \alpha_{ij'}^{[2]}, \ldots, \alpha_{ij'}^{[K]}\right). \tag{5.44}$$

In other words, each random variable has a continuous cdf conditioned on all the remaining variables. Also, any two terms in the same row of the matrix $\boldsymbol{A}$ differ in at least one exponent.

Then, the matrix $\boldsymbol{A}$ has a full rank of M with probability 1. □

Note: When all the $x_i^{[k]}$'s are independently generated from some continuous distribution, the first condition in this lemma still holds.

CHAPTER 6. OTHER WORK: THROUGHPUT ANALYSIS OF MIMO INTERFERENCE CHANNEL WITH STREAM NUMBER SELECTION

6.1 Introduction

The capacity of MIMO system can be improved if there is CSIT. Though it is a power gain, such an improvement can be large, especially at low SNR. While the capacity of a point-to-point MIMO system is relatively well understood, there are fewer results about the capacity of MIMO interference channel. The capacity of MIMO interference channel with transmitter side CSI and single-user detection was investigated in [32]. It was shown that the capacity of an individual link is a convex function for fixed covariance matrix of the other users. However, feeding back of CSI decreases the system payload. Also, accurate transmitter side CSI may not be available if the channel is fast fading.

Multi-user detection is also useful as the receiver can perform interference cancellation to increase the capacity. However, it is only helpful under weak or strong interference, and it does not outperform orthogonal signal when interference level is comparable to signal level [86].

In this chapter, we discuss the achievable throughput by stream number selection in a MIMO interference channel, where there is no CSI at the transmitter. We also assume single-user detection at the receivers. Based on an asymptotic throughput analysis and a piecewise linear approximation of it, we will show how to select optimal number of streams to maximize the individual (and overall) throughput. This chapter is organized as follows. In Section 6.2, we present the system model. The asymptotic analysis and a piecewise linear approximation to the throughput is proposed in Section 6.3. We compare the proposed approximation and the numerical calculations in Section 6.4, and conclude in Section 6.5.

6.2 System model

Consider a K-user MIMO interference channel, assume that the number of transmit and receive antennas are N_t and N_r, respectively. We assume that all the channels are flat fading, and the received signals are corrupted by additive white Gaussian noise (AWGN).

6.2.1 The general model

The received signal of the kth user can be expressed as

$$\boldsymbol{y}_k = \sqrt{\gamma_{l,k}}\boldsymbol{H}_{k,k}\boldsymbol{x}_k + \textstyle\sum_{l=1,l\neq k}^{K} \sqrt{\gamma_{k,l}}\boldsymbol{H}_{k,l}\boldsymbol{x}_l + \boldsymbol{z}_k, \tag{6.1}$$

where $\boldsymbol{y}_k$ and $\boldsymbol{z}_k$ are vectors in $\mathbb{C}^{N_r\times 1}$ denoting the received signal and additive noise at the kth receiver, respectively; $\boldsymbol{x}_k \in \mathbb{C}^{N_t\times 1}$ is the transmitted signal of the kth transmitter, and $\sqrt{\gamma_{k,l}}\boldsymbol{H}_{k,l} \in \mathbb{C}^{N_r\times N_t}$ is the MIMO channel from lth transmitter to kth receiver.

We assume the following: AS1) $\boldsymbol{z}_k \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_{N_r})$. AS2) $\boldsymbol{x}_k \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q}_k)$, where the covariance matrix $\boldsymbol{Q}_k$ has an eigen-value decomposition as $\boldsymbol{Q}_k = \boldsymbol{U}_k\boldsymbol{\Lambda}_k\boldsymbol{U}_k^\dagger$. AS3) All entries of all $\boldsymbol{H}_{k,l}$ matrices are i.i.d. $\mathcal{CN}(0,1)$ distributed. AS4) $\boldsymbol{H}_{k,k}$ is known perfectly at the kth receiver.

The constants $\{\gamma_{k,l} : k,l \in \{1,\ldots,K\}\}$ control the channel magnitudes. We define an $K \times K$ matrix $\boldsymbol{\Gamma}$ whose (k,l)th entry is $\gamma_{k,l}$. The diagonal entries of $\boldsymbol{\Gamma}$ control the SNRs and the off-diagonal entries control INRs.

Define $\bar{\boldsymbol{H}}_k = [\boldsymbol{H}_{k,1},\ldots,\boldsymbol{H}_{k,K}]$, $\bar{\boldsymbol{Q}} = \mathrm{diag}(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_K)$, $\bar{\boldsymbol{\Gamma}} = \mathrm{diag}(\gamma_{k,1}\boldsymbol{I}_{N_t},\ldots,\gamma_{k,K}\boldsymbol{I}_{N_t})$ and

$$\bar{\boldsymbol{H}}_{-k} = [\boldsymbol{H}_{k,1},\ldots,\boldsymbol{H}_{k,k-1},\boldsymbol{H}_{k,k+1},\ldots\boldsymbol{H}_{k,K}] \tag{6.2}$$

$$\bar{\boldsymbol{Q}}_{-k} = \mathrm{diag}(\boldsymbol{Q}_k,\ldots,\boldsymbol{Q}_{k-1},\boldsymbol{Q}_{k+1},\ldots,\boldsymbol{Q}_K) \tag{6.3}$$

$$\bar{\boldsymbol{\Gamma}}_{-k} = \mathrm{diag}(\gamma_{k,1}\boldsymbol{I}_{N_t},\ldots,\gamma_{k,k-1}\boldsymbol{I}_{N_t},\gamma_{k,k+1}\boldsymbol{I}_{N_t},\ldots\gamma_{l,K}\boldsymbol{I}_{N_t}). \tag{6.4}$$

We assume that single user detection is used at each receiver, which means that the interferences are treated as noise. The interference plus noise covariance matrix at receiver k is given by $\boldsymbol{R}_k = \boldsymbol{I}_{N_r} + \bar{\boldsymbol{H}}_{-k}\bar{\boldsymbol{\Gamma}}_{-k}\bar{\boldsymbol{Q}}_{-k}\bar{\boldsymbol{H}}_{-k}^\dagger$. The achievable rate of the kth link with single user detection can be given by [4, 32]

$$C^{(k)} = \mathrm{E}\left\{\log_2\left[\det\left(\boldsymbol{I}_{N_r} + \bar{\boldsymbol{H}}_k\bar{\boldsymbol{\Gamma}}\bar{\boldsymbol{Q}}\bar{\boldsymbol{H}}_k^{\dagger}\boldsymbol{R}_k^{-1}\right)\right]\right\} \tag{6.5}$$

The rank of $\boldsymbol{Q}_k$ determines the *number of streams* sent out simultaneously by the kth transmitter, and is denoted as B_k. Due to the rotational invariance in the statistical distribution of the $\{\boldsymbol{H}_{k,l}\}$ matrices, $\{\boldsymbol{U}_k\}$ do not affect the rates. The value of $C^{(k)}$ in (6.5) does not change if we replace $\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{Q}}_{-k}$ with $\bar{\boldsymbol{\Lambda}}$ and $\bar{\boldsymbol{\Lambda}}_{-k}$, respectively, where $\bar{\boldsymbol{\Lambda}}$ and $\bar{\boldsymbol{\Lambda}}_{-k}$ are similarly defined as $\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{Q}}_{-k}$. Selecting the number of streams will affect the throughput, though.

6.2.2 Simplified model

Having presented the general system model, we specialize the model to a simplified case. In addition to the assumptions we made in the general model, we further assume that AS5) the number of streams per user is the same: $B_k = B, \forall k$. AS6) The SNRs and INRs are the same for all the K users:

$$\gamma_{k,l} = \begin{cases} \gamma, & \text{if } l = k, \\ \eta & \text{if } l \neq k, \end{cases} \qquad k, l = 1, 2, \ldots, K. \tag{6.6}$$

AS7) all the transmitted powers are one: $\mathrm{tr}(\boldsymbol{Q}_k) = 1, \forall k$. AS8) The power is equally allocated among all active streams: $\boldsymbol{Q}_k = \mathrm{diag}[(1/B)\boldsymbol{I}_B, \boldsymbol{0}_{N_t - B}]$. AS9) $\gamma \gg 1$.

Under these simplifying assumptions, we would like to analyze how the number B of streams affects the rate $C^{(k)}$ for given N_t, N_r and K, under various signaling conditions expressed in terms of the SNR γ and INR η.

6.3 Analysis of the throughput

Under the simplifying model considered in Section 6.2.2, all the users have identical throughput, and it is sufficient to analyze the throughput of any user. Define

$$\boldsymbol{D}_1 = \mathrm{diag}[(\gamma/B)\boldsymbol{I}_B, (\eta/B)\boldsymbol{I}_{B(K-1)}], \tag{6.7}$$

$$\boldsymbol{D}_2 = (\eta/B)\boldsymbol{I}_{B(K-1)}. \tag{6.8}$$

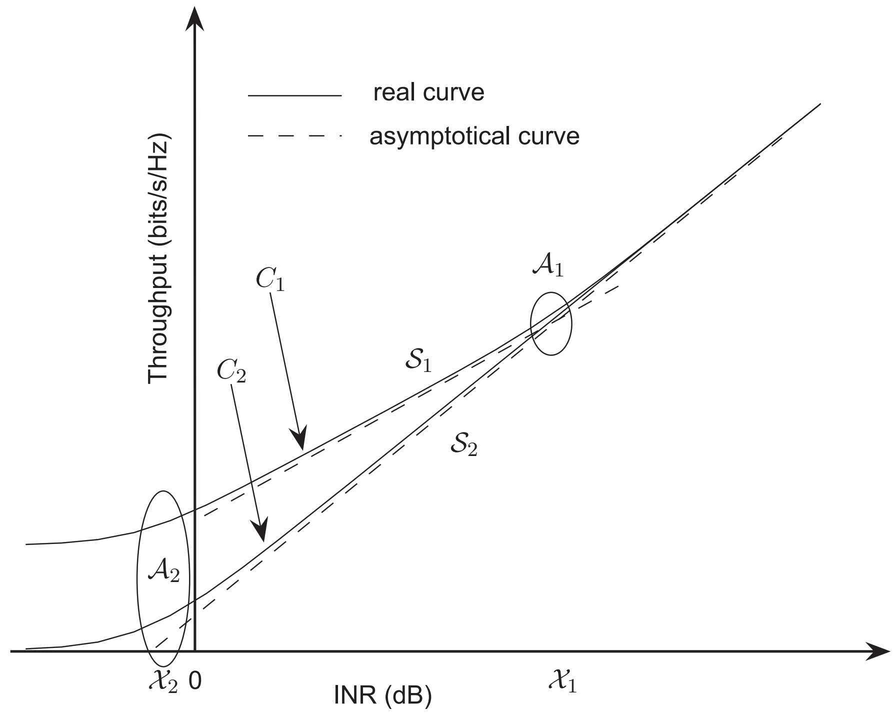

Figure 6.1 Illustration of asymptotic analysis.

Let $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ denote two random matrices of i.i.d. entries $\mathcal{CN}(0,1)$ distributed with sizes $N_r \times BK$ and $N_r \times B(K-1)$, respectively. We can obtain that $\forall k$, $C^{(k)} = C_1 - C_2$, where

$$C_i = \mathrm{E}\left\{\log_2\left[\det\left(\boldsymbol{I}_{N_r} + \boldsymbol{H}_i\boldsymbol{D}_i\boldsymbol{H}_i^{\dagger}\right)\right]\right\}, \quad i = 1, 2. \tag{6.9}$$

Both C_1 and C_2 can be viewed as the capacity of some point-to-point MIMO system. Such single-link MIMO capacity has been relatively well understood, e.g., [2]. Closed-form expressions for computing C_1 and C_2 can be derived based on results in [87, 88]. The results there do not allow for equal diagonal entries in the $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, but modifications can be made to deal with such cases, which is given in Section 6.6 as an appendix.

The numerical expressions, even though providing an exact value of the average rate, do not offer insights about how the system should be optimized. In particular, it does not provide an answer to the question: What should be the optimal number of streams per user? We will discuss this problem with an asymptotic analysis.

6.3.1 General behavior of C_1 and C_2

In this chapter, with a slight abuse of the term DoF, the *number d_r of receive DoF* is the number of receive antennas N_r. Viewing each of C_1 and C_2 as the capacity of a point-to-point MIMO system, we can define the *number of transmit DoF* for C_i, $i = 1, 2$, as the number of diagonal entries that are significantly larger than the remaining ones. And they are denoted as d_t and d_i, respectively, as the total number of transmit DoF and the number of DoF from interference. There are three cases regarding the relationship between γ and η. We will use $\gamma^{(\text{dB})}$ and $\eta^{(\text{dB})}$ to denote the SNR and INR in dB.

1. *Strong interference level*: $\eta \gg \gamma$. The entries of $\boldsymbol{D}_1$ containing γ are negligible, so $d_t = d_i = B(K-1)$. Thus, the slope of C_1 and C_2 as functions of $\eta^{(\text{dB})}$ at large $\eta^{(\text{dB})}$ are the same and can be determined as $\min(N_r, B(K-1))$. If $B(K-1) < N_r$, which means $d_r > d_i$, the receiver still has additional DoF to support communication, which leads to a positive difference between C_1 and C_2. If $B(K-1) \geq N_r$, there will not be enough DoF at the receiver, the difference between C_1 and C_2 goes to zero as η goes to infinity.

2. *Weak interference level:* $\eta \ll \gamma$. In this case, $d_t = B$ and $d_i = 0$. As a result, C_2 is much smaller compared to C_1, and C_1 is the capacity of an $N_r \times B$ MIMO system, which depends on $\gamma^{(\text{dB})}$ linearly for large $\gamma^{(\text{dB})}$ and the slope is $\min(B, N_r)$.

3. *Moderate interference level.* In this case, $d_t = BK$, and $d_i = B(K-1)$. There may be a mismatch between the slope of C_1, which is $\min(N_r, BK)$, and the slope of C_2, which is $\min[N_r, B(K-1)]$, both viewed as functions of $\eta^{(\text{dB})}$. The difference will affect the user throughput $C^{(l)}$.

Fig. 6.1 depicts the general relationship between C_1 and C_2 for fixed SNR γ and varying INR η, as well as their asymptotic straight line approximations. We can see the areas $\mathcal{A}_1$ and $\mathcal{A}_2$ can be viewed as separating the $\eta^{(\text{dB})}$ axis into three regions, corresponding to the three previously discussed cases. The threshold INR levels are denoted as $\mathcal{X}_1$ and $\mathcal{X}_2$. In the middle region, the slope of C_i is denoted as $\mathcal{S}_i$, $i = 1, 2$. The slopes $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same if $B(K-1) \geq N_r$, or different, otherwise. This middle region $\mathcal{X}_2 < \eta^{(\mathrm{dB})} < \mathcal{X}_1$ is more interesting, as the throughput $C^{(l)}$ outside it does not change much.

6.3.2 Approximation of $\mathcal{X}_2$

From Fig. 6.1, we can see that the approximate value of $\mathcal{X}_2$ can be obtained by the intercept of the asymptotic line, which can be calculated from the following lemma.

Lemma 16. *[89, Equation (15)] In an i.i.d. Rayleigh-faded $N \times M$ MIMO system, the intercept or* power offset *of the linear approximation to the capacity for large SNR is*

$$\mathcal{L}(N, M) = \log_2 N + \left(\hat{\gamma} - \sum_{l=1}^{N_{\Uparrow}-N_{\Downarrow}} \frac{1}{l} - \frac{N_{\Uparrow}}{N_{\Downarrow}} \sum_{l=N_{\Uparrow}-N_{\Downarrow}+1}^{N_{\Uparrow}} \frac{1}{l} + 1 \right) \log_2 e \quad (3dB) \tag{6.10}$$

where 3dB=$10\log_{10} 2$, $\hat{\gamma} \approx 0.5772$ is Euler-Mascheroni constant, $N_{\Uparrow} = \max(M, N)$, and $N_{\Downarrow} = \min(M, N)$. The asymptotic slope $\mathcal{S}$ in bits/s/Hz/(3dB) is $N_{\Downarrow}$. □

The equivalent SNR of C_2 equals $(K-1)\eta$, and hence using the lemma we have

$$\mathcal{X}_2 = (10\log_{10} 2)\mathcal{L}(N_r, B(K-1)) - 10\log_{10}(K-1) \ (\mathrm{dB}) \tag{6.11}$$

6.3.3 Approximation of $\mathcal{X}_1$:

The value $\mathcal{X}_1$ indicates an approximate interference level that the capacity becomes DoF limited, which could be investigated from the changes of C_1's slope. It is shown that when the smallest eigenvalue is significantly smaller than 1/SNR, the corresponding virtual channel is negligible [89, 90], and thus the slope changes. We will need the following two lemmas.

Lemma 17. *(Cauchy-Binet formula) Let $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, $\boldsymbol{B} \in \mathbb{C}^{n\times q}$ and $\boldsymbol{C} \in \mathbb{C}^{q\times m}$ where $m \le n$ and $m \le q$. Then*

$$\det(\boldsymbol{ABC}) = \textstyle\sum_{(s)} \sum_{(p)} \det \boldsymbol{A}_{(s)} \cdot \det \boldsymbol{B}_{(p)}^{(s)} \cdot \det \boldsymbol{C}^{(p)}$$

where (s) is an increasingly ordered subset of $\{1, ..., n\}$ of size m; (p) is an increasingly ordered subset of $\{1, \ldots, q\}$; $\boldsymbol{B}_{(p)}^{(s)}$ is a submatrix of $\boldsymbol{B}$ where the rows and columns are those given by (s) and (p), respectively; $\boldsymbol{A}_{(s)}$ is a submatrix of $\boldsymbol{A}$ with columns given by (s); and $\boldsymbol{C}^{(p)}$ is a submatrix of $\boldsymbol{C}$ with rows given by (p). □

Lemma 18. *[91, Lemma II.1] If $\boldsymbol{A}$ is a random matrix of size $m \times n$, with i.i.d. entries $\mathcal{CN}(0,1)$, and $k \leq \min(m,n)$, then*

$$E\left[\det(\boldsymbol{A}_{(s)}^{(p)}(\boldsymbol{A}^{\dagger})_{(u)}^{(v)})\right] = \begin{cases} k!, & \text{if } (p)=(u), (s)=(v) \\ 0, & \text{otherwise} \end{cases} \tag{6.12}$$

where (p) and (u) are increasingly ordered subsets of $\{1,\ldots,m\}$, and (s) and (v) are increasingly ordered subsets of $\{1,\ldots,n\}$, all of size k. □

Using these lemmas, we next evaluate C_1 in two cases:

1. Case 1: $BK \geq N_r$. Define $T = N_r - B(K-1)$. Using Jensen's inequality, C_1 can be approximated as

$$\begin{aligned} C_1 \approx & \mathrm{E}\left\{\log_2\left[\det\left(\boldsymbol{H}_1\boldsymbol{D}_1\boldsymbol{H}_1^{\dagger}\right)\right]\right\} \\ \leq & \log_2\left\{\mathrm{E}\left[\det\left(\boldsymbol{H}_1\boldsymbol{D}_1\boldsymbol{H}_1^{\dagger}\right)\right]\right\} \end{aligned}$$

There are two sub-cases. If $BK - B \geq N_r$, then

$$\begin{aligned} C_1 \approx & \log_2\left\{\sum_{i=0}^{B}\binom{B}{i}\left(\frac{\gamma}{B}\right)^i\binom{BK-B}{N_r-i}\left(\frac{\eta}{B}\right)^{N_r-i}N_r!\right\} \\ = & \log_2\left\{e_0\eta^{N_r}\left[1+\sum_{i=1}^{B}e_i\left(\frac{\gamma}{\eta}\right)^i\right]\right\} \end{aligned} \tag{6.13}$$

If $BK - B < N_r$, then

$$\begin{aligned} C_1 \approx & \log_2\left\{\sum_{j=0}^{BK-Nr}\binom{B}{T+j}\left(\frac{\gamma}{B}\right)^{T+j}\binom{BK-B}{j}\left(\frac{\eta}{B}\right)^{N_r-T-j}N_r!\right\} \\ = & \log_2\left\{f_0\eta^{BK-B}\left[1+\sum_{j=1}^{BK-Nr}f_j\gamma^{N_r}\left(\frac{\gamma}{\eta}\right)^j\right]\right\} \end{aligned} \tag{6.14}$$

where e_i's, f_j's, $i = 0,\ldots,B, j = 0,\ldots,BK-N_r$ are certain coefficients not related to γ and η. When $\gamma \ll \eta$, the terms inside the summations in (6.13) and (6.14) are negligible, so the slope with respect to $\eta^{(\mathrm{dB})}$ will be equal to $\min[B(K-1), N_r]$. When η is close to or smaller than γ, these terms will be comparable to 1 and the corresponding slope of C_1 will be smaller.

2. Case 2: $BK < N_r$. Using the fact that $\det(\boldsymbol{I}+\boldsymbol{AB}) = \det(\boldsymbol{I}+\boldsymbol{BA})$, we have

$$\begin{aligned}C_1 =& \mathrm{E}\left\{\log_2\left[\det\left(\boldsymbol{I}_{BK}+\boldsymbol{D}_1\boldsymbol{H}_1^{\dagger}\boldsymbol{H}_1\right)\right]\right\}\\ \approx& \log_2\left\{\mathrm{E}\left[\det\left(\boldsymbol{D}_1\boldsymbol{H}_1^{\dagger}\boldsymbol{H}_1\right)\right]\right\}\\ \approx& \log_2\left\{\left(\frac{\gamma}{B}\right)^{B}\left(\frac{\eta}{B}\right)^{BK-B}\binom{N_r}{BK}(BK)!\right\}\end{aligned} \tag{6.15}$$

The slope with respect to $\eta^{(\mathrm{dB})}$ is therefore $B(K-1)$.

Based on the discussion on these two cases and the expressions in (6.13) and (6.14), we approximate $\mathcal{X}_1$ as $\gamma^{(\mathrm{dB})}$.

6.3.4 Calculation of $C^{(l)}$ in limit cases

We now calculate the difference between C_1 and C_2 under two extreme cases: $\eta^{(\mathrm{dB})} \to -\infty$ and $\eta^{(\mathrm{dB})} \to \infty$. In the first case, $C_2 \to 0$, and C_1 can be evaluated using Lemma 16 as

$$C_{1,\eta^{(\mathrm{dB})}\to-\infty}(\gamma) \simeq \min(N_r, B)\left(\tfrac{\gamma^{(\mathrm{dB})}}{10\log_2 10} - \mathcal{L}(N_r, B)\right) \tag{6.16}$$

where "$\simeq$" means that the two quantities are asymptotically the same. Similarly, we also have the capacity of C_2 when $\eta \gg \gamma$ as

$$C_{2,\eta\gg\gamma}(\eta) \simeq \min(N_r, BK-B)(\tfrac{\eta^{(\mathrm{dB})}}{10\log_2 10} - \mathcal{L}(N_r, BK-B)). \tag{6.17}$$

The corresponding capacity of C_1 can be approximated as

$$C_{1,\eta\gg\gamma}(\eta) \simeq \min(N_r, BK-B)(\tfrac{\eta^{(\mathrm{dB})}}{10\log_2 10} - \mathcal{L}_{corr}). \tag{6.18}$$

where $\mathcal{L}_{corr}$ is the power offset for correlated Rayleigh fading channel [89, Equation (28)], which can also be approximated through various bounds, e.g. (6.13), (6.14) and (6.15). Subtracting $C_{2,\eta\gg\gamma}(\eta)$ from $C_{1,\eta\gg\gamma}(\eta)$, we have

$$C^{(l)}_{\eta\gg\gamma} \simeq \min(N_r, BK-B)[\mathcal{L}(N_r, BK-B)) - \mathcal{L}_{corr}]. \tag{6.19}$$

Base on the former analysis, the average achievable throughput can be evaluated through piecewise linear approximation, shown in Fig. 6.2, which we summarize as follows:

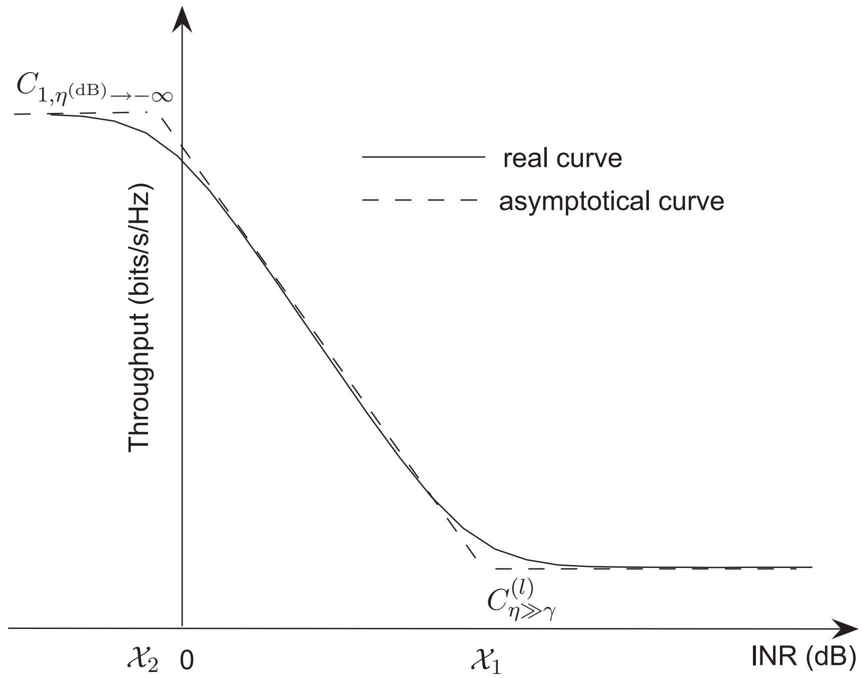

Figure 6.2 Illustration of piecewise linear approximation.

Theorem 7. *The average throughput of MIMO interference channel satisfying the assumption AS1)– AS9) can be evaluated through the following piecewise linear approximation:*

$$C^{(l)} \approx \begin{cases} C_{1,\eta^{(dB)}\to-\infty}, & \text{if } \eta \leq \mathcal{X}_2 \\ C^{(l)}_{\eta\gg\gamma} & \text{if } \eta \geq \mathcal{X}_1 \\ \frac{\eta^{(dB)}-\mathcal{X}_2}{\mathcal{X}_1-\mathcal{X}_2}(C^{(l)}_{\eta\gg\gamma} - C_{1,\eta^{(dB)}\to-\infty}) & \text{otherwise} \end{cases}$$

where $\mathcal{X}_2$ *is given in* (6.11)*,* $\mathcal{X}_1 = \gamma^{(dB)}$*; and* $C_{1,\eta^{(dB)}\to-\infty}$*,* $C_{2,\eta\gg\gamma}(\eta)$ *and* $C_{1,\eta\gg\gamma}(\eta)$ *are as given in* (6.16)*–*(6.18)*.* □

6.4 Simulation and discussion

To evaluate our analysis result, we show the comparison between the piecewise linear approximation and the exact capacity curves in Fig. 6.3. The exact curves were obtained using the modified methods of [87, 88] given in Section 6.6, allowing equal values on the diagonals of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$. The received SNR γ is 30dB. We used $N_t = N_r = K = 4$.

The per user throughput shows different characteristics when the stream number increases, as we discussed. When $\eta > \gamma$, the throughput tends to zero if the number of receive DoF N_r is less than the number of interference streams $B(K-1)$, otherwise it tends to a positive constant. Thus the stream number B in this region should not be larger than $\lfloor N_r/K \rfloor$. Notice that if $B = \lfloor N_r/(K-1) \rfloor$ and it is larger than zero, $C^{(l)}$ is not zero. However, such B is not an optimal choice as the receive DoF is not enough to recover all the data streams from the interference streams. When $\eta < \mathcal{X}_2$, the interference is negligible. We should use as many streams as possible, namely $B = N_t$ streams in this case. As to the middle part, the rate can be approximated as a linear function. In the left part of this region, we still favor large number of streams, but the performance becomes worse and smaller number of streams is preferred as INR increases. The transition INR for stream number change can be obtained either through analytical curve or piecewise linear approximation. And the receiver will feedback the optimal number of streams to the transmitter. In Fig. 6.3, an accurate estimate of critical interference level from exact numerical calculation is η=18dB, whereas the piecewise linear approximation gives 20dB.

Finally, we point out that the analytical analysis will suffer from numerical stability when BK is very large [87], thus the proposed piecewise linear method will be good in analyzing the throughput of MIMO interference channel with a large number of users and can be used to determine the optimal stream number.

6.5 Summary

The achievable throughput of MIMO interference channel using stream number selection is investigated. We assumed that there is no CSI at the transmitters, and single-user detectors. We proposed a piecewise linear approximation to the average throughput in weak, strong, and moderate interference regimes. The rate-transition points can be determined through our piecewise linear approximation. Our results provided rules for determining the optimal number of streams at various interference levels.

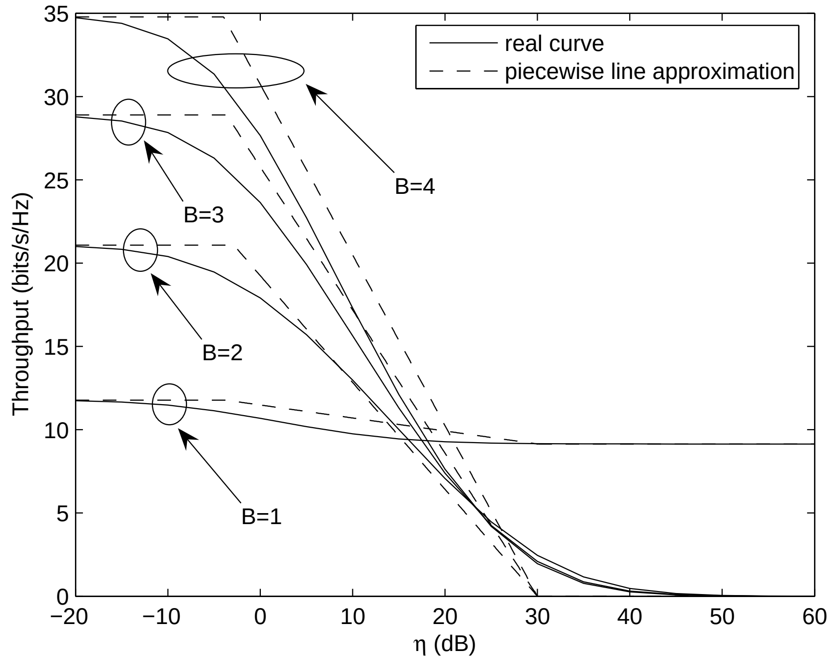

Figure 6.3 Comparison between the analytical curve and piecewise linear approximation. γ=30dB, $N_t = N_r = K = 4$.

6.6 Appendix: Mutual information calculation with repeated eigenvalues

6.6.1 Generic model

We first present the generic model. The quantity we want to calculate has the following mathematical expression

$$\mathcal{I} = \mathrm{E}\left\{\log_2\left[\det\left(\boldsymbol{I}_v + \boldsymbol{W}\boldsymbol{T}\boldsymbol{W}^\dagger\boldsymbol{R}\right)\right]\right\} \tag{6.20}$$

where $\boldsymbol{W}$ is a $v \times u$ matrix with each entry i.i.d. $\mathcal{CN}(0,1)$ distributed. $\boldsymbol{T}$ is an $u \times u$ diagonal matrix with its diagonal entry $[\boldsymbol{T}]_{ii} = 1/t_i$, $\boldsymbol{R}$ is a $v \times v$ diagonal matrix with its diagonal entry $[\boldsymbol{R}]_{jj} = 1/r_j$. We assume that $\boldsymbol{t} = [t_1, \ldots, t_u]$ and $\boldsymbol{r} = [r_1, \ldots, t_v]$ are ordered in non-decreasing (or non-increasing) order. The closed-form solutions of (6.20) have been given in [88]. However, the results therein only consider the case that the eigenvalues of $\boldsymbol{T}$ ($\boldsymbol{R}$) are different. The generic model has been widely used for calculating the mutual information of

$N_r \times N_t$ MIMO point-to-point Rayleigh fading channel which can be given as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}, \tag{6.21}$$

where $\boldsymbol{x} \in \mathbb{C}^{N_t \times 1}$ has zero mean and covariance matrix $\boldsymbol{Q}$. $\boldsymbol{n} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_{N_t})$ is AWGN.

The mutual information of this MIMO system, when $\boldsymbol{H}$ is perfectly known at the receiver and with no CSI at transmitter, is given by [4]

$$\mathcal{I} = \mathrm{E}\left\{\log_2\left[\det\left(\boldsymbol{I}_{N_r} + \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^{\dagger}\right)\right]\right\}. \tag{6.22}$$

Recall (2.4), the full correlated Rayleigh fading case can be represented by

$$\boldsymbol{H} = \boldsymbol{\Phi}_R^{1/2}\boldsymbol{H}_w\boldsymbol{\Phi}_T^{1/2}. \tag{6.23}$$

Denote the eigenvalues of $\boldsymbol{\Phi}_T^{1/2}\boldsymbol{Q}(\boldsymbol{\Phi}_T^{1/2})^{\dagger}$ and $\boldsymbol{\Phi}_R$ as $\boldsymbol{\Lambda}_T$ and $\boldsymbol{\Lambda}_R$, respectively. Using the fact that $\det(\boldsymbol{I} + \boldsymbol{A}\boldsymbol{B}) = \det(\boldsymbol{I} + \boldsymbol{B}\boldsymbol{A})$, $\boldsymbol{\Lambda}_R = \boldsymbol{\Lambda}_R^{\dagger}$ and because the product of $\boldsymbol{H}_w$ with unitary matrix would not change its distribution $\boldsymbol{H}_w$, (6.22) can be modified as follows

$$\mathcal{I} = \mathrm{E}\left\{\log_2\left[\det\left(\boldsymbol{I}_{N_r} + \boldsymbol{H}_w\boldsymbol{\Lambda}_T\boldsymbol{H}_w^{\dagger}\boldsymbol{\Lambda}_R\right)\right]\right\} \tag{6.24}$$

with $\boldsymbol{t} = \mathrm{diag}(\boldsymbol{\Lambda}_T^{-1})$ and $\boldsymbol{r} = \mathrm{diag}(\boldsymbol{\Lambda}_R^{-1})$. Thus, the mutual information of such a full-correlated MIMO channel can be calculated through (6.20), with $v = N_R$, $u = N_t$. When some eigenvalues of $\boldsymbol{\Lambda}_T$ ($\boldsymbol{\Lambda}_R$) are zero, we shall have $u = \mathrm{rank}(\boldsymbol{\Lambda}_T)$ ($v = \mathrm{rank}(\boldsymbol{\Lambda}_R)$) and $\boldsymbol{T}$ ($\boldsymbol{R}$) contains all the nonzero diagonal entries of $\boldsymbol{\Lambda}_T$ ($\boldsymbol{\Lambda}_R$).

6.6.2 Solution of generic model without repeated eigenvalues

We will first re-present the results from [88] in the following before developing the solution when there is repeated eigenvalues. Define $N = \min(u, v)$ and $M = \max(u, v)$. If $v > u$, the value of $\boldsymbol{t}$ and $\boldsymbol{r}$ are interchanged.

The analytical solution of (6.20) is given by

$$\mathcal{I} = \sum_{l=1}^{M-N-1} \frac{l}{N+l} + \sum_{k=1}^{N} \frac{\det \tilde{\boldsymbol{L}}_k}{\det \boldsymbol{L}} - M - 1 \tag{6.25}$$

where $\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}_k, k = 1, \ldots, N$ are $M \times M$ matrices and they can be calculated as follows

1. Calculation of $\boldsymbol{L}$

$$[\boldsymbol{L}]_{ij} = \begin{cases} F(t_i r_j) & j \leq N \\ t_i^{j-1} & j > N \end{cases} \tag{6.26}$$

where $F(x)$ is given as

$$F(x) = e^x \Gamma(M, x) \tag{6.27}$$

and $\Gamma(\alpha, x)$ is the upper incomplete Gamma function. (Note: there are typos in equations (80) and (85) of [88], where $F(t_i r_j, M + z)$ should be $F(t_i r_j, z)$. And because the quantity (6.20) is calculated at $z = 0$, we have a simpler form of $F(x)$ given in (6.27).)

2. Calculation of $\tilde{\boldsymbol{L}}_k$

$\tilde{\boldsymbol{L}}_k$ is given by the matrix $\boldsymbol{L}$, with the kth column replaced by the kth column of an $M \times N$ matrix $\boldsymbol{D}$, whose elements are

$$[\boldsymbol{D}]_{ij} = D(t_i r_j, M) \tag{6.28}$$

$$= \left(e^x x^M (-1)^M h^{(M-1)}(x)\right) \Big|_{x=t_i r_j} \tag{6.29}$$

with the superscript of $h(x)$ indicating the number of derivatives applied and $h(x)$ is

$$h(x) = x^{-1} Ei(-x), \tag{6.30}$$

where $Ei(\cdot)$ is the exponential integral.

6.6.3 Solution of generic model with repeated eigenvalues

When some eigenvalues are the same, multiple rows or columns in $\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}_k$ are the same hence the determinants of these matrices are zero. Define $z_k = \det \tilde{\boldsymbol{L}}_k / \det \boldsymbol{L}$. First, we only consider there are multiple entries of $\boldsymbol{t}$ are the same. Denote the ith row of $\tilde{\boldsymbol{L}}_k$ and $\boldsymbol{L}$ as $\boldsymbol{\alpha}_{k,i}(t_i)$ and $\boldsymbol{\beta}_i(t_i)$, respectively. Then

$$z_k = \frac{\det[\boldsymbol{\alpha}_{k,1}(t_1); \ldots; \boldsymbol{\alpha}_{k,M}(t_M)]}{\det[\boldsymbol{\beta}_1(t_1); \ldots; \boldsymbol{\beta}_M(t_M)]}, \tag{6.31}$$

and we need the following lemma to deal with this problem, which is a modified version of lemma 6 in [88].

Lemma 19. *The ratio of two determinants given in* (6.31) *when some t_i's are the same can be calculated as*

$$\lim_{t_1,\ldots,t_n \to t_1} z_k = \frac{\det[\boldsymbol{\alpha}_{k,1}(t_1); \boldsymbol{\alpha}_{k,1}^{(1)}(t_1); \ldots; \boldsymbol{\alpha}_{k,1}^{(n-1)}(t_1); \boldsymbol{\alpha}_{k,p+1}(t_{p+1}); \ldots; \boldsymbol{\alpha}_{k,M}(t_M)]}{\det[\boldsymbol{\beta}_1(t_1); \boldsymbol{\beta}_1^{(1)}(t_1); \ldots; \boldsymbol{\beta}_1^{(n-1)}(t_1); \boldsymbol{\beta}_{p+1}(t_{p+1}); \ldots; \boldsymbol{\beta}_M(t_M)]}, \tag{6.32}$$

where n is the multiplicity of t_1. □

If there are multiple entries of $\boldsymbol{r}$ are the same, we need to apply Lemma 19 in the column direction once more.

We first define two vectors $\boldsymbol{p} = [p_1, \ldots, p_M]$, $\boldsymbol{q} = [p_1, \ldots, q_N]$, where p_i is the multiplicity of t_i and q_j is the multiplicity of r_j. Using Lemma 19 in both row and column directions, the method presented in Section 6.6.2 can be modified in the following way.

When there are repeated eigenvalues, $\boldsymbol{L}$ can be obtained as follows

$$[\boldsymbol{L}]_{ij} = \begin{cases} F^{(p_i-1,q_j-1)}(t_i r_j) & 1 \le j \le N \\ 0 & N < j < p_i \\ b(i,j) t_i^{a(ij)} & p_i \le j \le M \end{cases} \tag{6.33}$$

where

$$a(i,j) = \begin{cases} j-1, & p_i = 1 \\ \max(a(i-1,j)-1, 0), & p_i \ne 1 \end{cases} \tag{6.34}$$

and

$$b(i,j) = \begin{cases} 1, & \text{if } p_i = 1 \\ \dfrac{a(1,j)!}{a(i,j)!} & \text{otherwise} \end{cases} \tag{6.35}$$

where $F^{(n,m)}(t_i r_j)$ stands for the nth derivative of t_i and the mth derivative of r_j and it can be solved iteratively.

Lemma 20. *The nth derivative of t_i and the mth derivative of r_j can be expressed as*

$$F^{(n,m)}(t_i r_j) = \frac{[x^M P_{n,m}(x) + Q_{n,m}(x) F(x)]|_{x=t_i r_j}}{t_i^n r_j^m}, \tag{6.36}$$

where $P_{n,m}(x)$ and $Q_{n,m}(x)$ are two polynomials of x.

Proof. When $n = 0, m = 0$, we have $P_{0,0}(x) = 0, Q_{0,0}(x) = 1$. Supposing the lemma is true for n and m and applying derivative of t_i to $F^{(n,m)}(t_i r_j)$, we have

$$P_{n+1,m}(x) = (M - n)P_{n,m}(x) + xP'_{n,m}(x) - Q_{n,m}(x), \tag{6.37}$$

$$Q_{n+1,m}(x) = -nQ_{n,m}(x) + xQ'_{n,m}(x) + xQ_{n,m}(x), \tag{6.38}$$

and they both are polynomials of x.

Furthermore, noticing that t_i and r_j are interchangeable, the lemma holds for all $n \geq 0, m \geq 0$. □

We can calculate $\tilde{\boldsymbol{L}}_k$ following the same way in Section 6.6.2, where the entries of matrix $\boldsymbol{D}$ can be calculated using following two lemmas.

Lemma 21. *The mth derivative of $h(x)$ can be expressed as*

$$h^{(m)}(x) = \hat{P}_m(x)e^{-x} + (-1)^m m! x^{-(m+1)} Ei(-x), \tag{6.39}$$

where $\hat{P}_m(x)$ is a polynomial of x.

Proof. If m=0, $\hat{P}_0(x) = 0$. Assuming the lemma holds for m,we have:

$$\hat{P}_{m+1}(x) = \hat{P}'_m(x) - \hat{P}_m(x) + (-1)^m m! x^{-(m+1)-1}. \tag{6.40}$$

Hence the lemma holds for all $m \geq 0$. □

Lemma 22. *If there are repeated values of t_i and r_j, the entries of $\boldsymbol{D}$ matrix can be modified as:*

$$[\boldsymbol{D}]_{ij} = D^{(p_i-1,q_j-1)}(t_i r_j, M), \tag{6.41}$$

where $D^{(n,m)}(t_i r_j, M)$ stands for the nth derivative of t_i and the mth derivative of r_j and it has the general form for $n \geq m$

$$D^{(n,m)}(t_i r_j, M) = \frac{[\check{P}_{n,m}(x) + \check{Q}_{n,m}(x)e^x Ei(-x)]|_{x=t_i r_j}}{t_i^{n-m}}, \tag{6.42}$$

where $\check{P}_{n,m}(x)$ and $\check{Q}_{n,m}(x)$ are two polynomials of x. (To the case that $n < m$, simply switching t_i and r_j as well as the numbers of derivative applied, we can obtain the corresponding value.)

Proof. when $n = 0, m = 0$, $D^{(0,0)}(t_i r_j, M)$ can be given as

$$D^{(0,0)}(t_i r_j, M) = \left[(-1)^M x^M \hat{P}_{M-1}(x) - (M-1)! e^x Ei(-x)\right] \Big|_{x=t_i r_j}. \tag{6.43}$$

Thus, we have $\check{P}_{0,0}(x) = (-1)^M x^M \hat{P}_{M-1}(x)$, $\check{Q}_{0,0}(x) = -(M-1)!$.

Assuming the lemma holds for one n, m pair where $n > m$, then we have

$$\check{P}_{n+1,m}(x) = (m-n)\check{P}_{n,m}(x) + x\check{P}'_{n,m}(x) + \check{Q}_{n,m}(x), \tag{6.44}$$

$$\check{Q}_{n+1,m}(x) = (m-n)\check{Q}_{n,m}(x) + x\check{Q}'_{n,m}(x) + x\check{Q}_{n,m}(x), \tag{6.45}$$

$$\check{P}_{n,m+1}(x) = \check{P}'_{n,m}(x) + x^{-1}\check{Q}_{n,m}(x), \tag{6.46}$$

$$\check{Q}_{n,m+1}(x) = \check{Q}_{n,m}(x) + \check{Q}'_{n,m}(x), \tag{6.47}$$

which are all polynomials of x. Hence the lemma holds for all $n \geq m \geq 0$. □

Then the exact value of (6.20) can be readily calculated using (6.25). However, direct calculation is not stable as $\tilde{\boldsymbol{L}}_k$'s and $\boldsymbol{L}$ are usually ill-conditioned. Thus, row and column scaling should be carried first to improve the numerical stability. Furthermore, noticing that only one column in $\tilde{\boldsymbol{L}}_k$ is different from $\boldsymbol{L}$, we can simplify the calculation by using Gramer's rule [92].

Let $\boldsymbol{V}$ be a $(M-N) \times N$ matrix or an empty matrix (when $M = N$); $\boldsymbol{Z}$ is an $N \times N$ matrix with $[\boldsymbol{Z}]_{k,k} = z_k$. The off-diagonal entries of $\boldsymbol{Z}$ are of no interest. We can view $\boldsymbol{Z}$ as (part of) the solution of the following linear equations

$$\boldsymbol{L}\begin{pmatrix} \boldsymbol{Z} \\ \boldsymbol{V} \end{pmatrix} = \boldsymbol{D} \tag{6.48}$$

The mutual information can then be written using Cramer's rule

$$\mathcal{I} = \left(\sum_{l=1}^{M-N-1} \frac{l}{N+l} + \operatorname{tr}(\boldsymbol{Z}) - M - 1\right) \log_2 e \quad \text{bits} \tag{6.49}$$

and we have the following theorem:

Theorem 8. *The quantity given in* (6.20) *can be calculated through* (6.49) *when* $\boldsymbol{T}$ *and* $\boldsymbol{R}$ *have repeated eigenvalues, where* $\boldsymbol{Z}$ *is the solution of* (6.48)*;* $\boldsymbol{L}$ *is given by* (6.33)*;* $\boldsymbol{D}$ *is given by* (6.41)*.*

CHAPTER 7. CONCLUSIONS AND FUTURE WORK

We investigated the DoF region of wireless interference network in this thesis for both two-user case and multi-user case. We first studied the DoF region of two-user FIC and ZIC. For the ZIC, we derived the DoF region when CSIT is available. As to the no CSIT case, the DoF regions with the usage of reconfigurable transmit antennas are solved for both FIC and ZIC. Although the DoF region of FIC was solved recently under isotropic fading channel assumption, We showed that by using reconfigurable transmit antennas at one transmitter the DoF region can be larger. For this special case, we developed the DoF region outer bound as a function of the number of available transmit antenna modes. Based on that, we investigated the achievability schemes for two cases: 1) when there are sufficient antenna modes, and 2) when there are not so many modes. We proposed a systematic way of constructing beamforming and nulling matrices, which has an interesting space-frequency domain interpretation. Our results closed the gap between the DoF region of two-user FIC with isotropic fading and staggered fading without CSIT, where it was known that more DoF can be achieved in the second case by exploring the channel correlation structures.

We then investigated the DoF region of an interference network with multiple transmitters and receivers, where the number of transmitters is not necessarily equal to the number of receivers. In addition, we considered the case that each receiver can request arbitrary set of messages from different transmitters. The interference network we considered generalizes the multiple unicasts and multiple multicasts scenarios in this area, for which only the total number of DoF is studied. We showed that the DoF region is achievable by generalizing the interference alignment scheme proposed by Cadambe and Jafar. The key is using multiple base vectors to generate beamforming matrices over time-expansion channel and aligning interference signals at each receiver to its largest interferer. The proposed scheme can be used to achieve any rational DoF point in the DoF region, which includes the vertices of the convex DoF region. We also showed that the DoF region depends on a subset of receivers and we developed a modified group based alignment scheme which requires less number of time expansion.

There are still many interesting related problems that have not been considered in this thesis. We give several of them as future research topics in the following

1. *The usage of reconfigurable antenna for other interference networks* The reconfigurable antenna was proposed to be used at receiver side under MIMO BC to achieve the total DoF. In this thesis, we explored its usage at transmitter side. It is interesting to consider the usage of reconfigurable antennas in MIMO X channel as X channel contains both BC and FIC. In [48], it is shown that using reconfigurable antennas for single antenna X channel, the total DoF 4/3 can be achieved without CSIT. However, the generalization to multiple antenna case with different number of antennas is still open.

2. *DoF region of multiple antenna interference networks with general message demands* As we discussed before, the achievability of the single antenna case cannot be directly applied. Interestingly, the achievability scheme of DoF point $(\frac{M}{2}, \frac{M}{2}, \cdots, \frac{M}{2})$ of the K-user M antenna interference channel is based on antenna splitting argument, which is used to achieve the DoF region with the time-sharing argument. The general achievability scheme for the multiple antenna case is still open.

3. *General message demand structures in other interference networks* If transmitters have multiple messages, the receiver demands may result in alignment constraints that cannot be satisfied in a simple manner. One example is the MIMO wireless X network [40]. Its total number of DoF is outer bounded as $\frac{AMN}{M+N-1}$, where M and N are the numbers of transmitters and receivers, respectively. All transmitters and receivers are equipped with A antennas. Each transmitter has different messages for all the receivers. It is shown in [40] that the bound of the total DoF is tight only when the number of antennas A equals 1. The approach for multiple antenna case is based on antenna splitting argument,

which eliminates the possibility of joint coding among different antennas at transmitter side. Apparently, the scheme we proposed in this thesis can only handle the case where transmitters have single message. The DoF region of multiple messages case remains open.

BIBLIOGRAPHY

[1] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, 1998.

[2] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. on Telecomm.*, vol. 10, no. 6, pp. 585–596, 1999.

[3] L. Zheng and D. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inform. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* John Wiley & Sons, Inc., 1991.

[5] A. Carleial, "A case where interference does not reduce capacity (Corresp.)," *IEEE Trans. Inform. Theory*, vol. 21, no. 5, pp. 569–570, May 1975.

[6] H. Sato, "The capacity of the Gaussian interference channel under strong interference (Corresp.)," *IEEE Trans. Inform. Theory*, vol. 27, no. 6, pp. 786–788, June 1981.

[7] M. Costa and A. Gamal, "The capacity region of the discrete memoryless interference channel with strong interference (Corresp.)," *IEEE Trans. Inform. Theory*, vol. 33, no. 5, pp. 710–711, May 1987.

[8] H. Sato, "Two-user communication channels," *IEEE Trans. Inform. Theory*, vol. 23, no. 3, pp. 295–304, Mar. 1977.

[9] A. Carleial, "Outer bounds on the capacity of interference channels (Corresp.)," *IEEE Trans. Inform. Theory*, vol. 29, no. 4, pp. 602–606, Apr. 1983.

[10] T. Han and K. Kobayashi, “A new achievable rate region for the interference channel,” *IEEE Trans. Inform. Theory*, vol. 27, no. 1, pp. 49–60, Jan. 1981.

[11] K. Kobayashi and T. Han, “A further consideration of the hk and cmg regions for the interference channel,” *UCSD-ITA*, 2007.

[12] G. Kramer, “Outer bounds on the capacity of Gaussian interference channels,” *IEEE Trans. Inform. Theory*, vol. 50, no. 3, pp. 581–586, Mar. 2004.

[13] R. H. Etkin, D. N. C. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” *IEEE Trans. Inform. Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.

[14] X. Shang, G. Kramer, and B. Chen, “Outer bound and noisy-interference sum-rate capacity for symmetric Gaussian interference channels,” in *Information Sciences and Systems, 2008. CISS 2008. 42nd Annual Conference on*, 2008, pp. 385–389.

[15] V. Sreekanth Annapureddy and V. Veeravalli, “Sum capacity of the Gaussian interference channel in the low interference regime,” in *Information Theory and Applications Workshop, 2008*, 2008, pp. 422–427.

[16] A. Motahari and A. Khandani, “Capacity bounds for the Gaussian interference channel,” in *Proc. IEEE Intl. Symp. on Info. Theory*, 2008, pp. 250–254.

[17] G. Bresler and D. Tse, “The two-user Gaussian interference channel: a deterministic view,” *European Trans. on Telecomm.*, vol. 19, no. 4, pp. 333–354, Apr. 2008.

[18] Y. Zhu and D. Guo, “Ergodic Fading Z-Interference Channels Without State Information at Transmitters,” *IEEE Trans. Inform. Theory*, vol. 57, no. 5, pp. 2627–2647, May 2011.

[19] G. Bresler, A. Parekh, and D. Tse, “The Approximate Capacity of the Many-to-One and One-to-Many Gaussian Interference Channels,” *IEEE Trans. Inform. Theory*, vol. 56, no. 9, pp. 4566–4592, Sept. 2010.

[20] A. Avestimehr, S. Diggavi, and D. Tse, "Wireless Network Information Flow: A Deterministic Approach," *IEEE Trans. Inform. Theory*, vol. 57, no. 4, pp. 1872–1905, Apr. 2011.

[21] D. Tse and R. Yates, "Fading Broadcast Channels with State Information at the Receivers," 2009. [Online]. Available: http://arxiv.org/abs/0904.3165

[22] M. Anand and P. Kumar, "On approximating Gaussian relay networks with deterministic networks," in *Proc. of IEEE Intl. Inform. Theory Workshop*, 2009, pp. 625–629.

[23] H. Weingarten, Y. Steinberg, and S. Shamai, "The Capacity Region of the Gaussian Multiple-Input Multiple-Output Broadcast Channel," *IEEE Trans. Inform. Theory*, vol. 52, no. 9, pp. 3936–3964, Sept. 2006.

[24] X. Shang, B. Chen, G. Kramer, and H. Poor, "Capacity Regions and Sum-Rate Capacities of Vector Gaussian Interference Channels," *IEEE Trans. Inform. Theory*, vol. 56, no. 10, pp. 5030–5044, Oct. 2010.

[25] V. Annapureddy and V. Veeravalli, "Sum Capacity of MIMO Interference Channels in the Low Interference Regime," *IEEE Trans. Inform. Theory*, vol. 57, no. 5, pp. 2565–2581, May 2011.

[26] V. Cadambe and S. Jafar, "Interference Alignment and a Noisy Interference Regime for Many-to-One Interference Channels," 2009. [Online]. Available: http://arxiv.org/abs/0912.3029

[27] X. Shang, B. Chen, G. Kramer, and H. V. Poor, "MIMO Z-interference channels: capacity under strong and noisy interference," 2009. [Online]. Available: http://arxiv.org/abs/0911.4530

[28] E. Telatar and D. Tse, "Bounds on the capacity region of a class of interference channels," in *Proc. IEEE Intl. Symp. on Info. Theory*, 2007, pp. 2871–2874.

[29] H.-F. Chong, M. Motani, and H. K. Garg, "A comparison of two achievable rate regions for the interference channel," *UCSD-ITA*, 2006.

[30] S. Karmakar and M. K. Varanasi, "The Capacity Region of the MIMO Interference Channel and its Reciprocity to Within a Constant Gap ," 2011. [Online]. Available: http://arxiv.org/abs/1102.0267

[31] R. Blum, "MIMO capacity with interference," *IEEE Journal on Selected Areas in Communications*, vol. 21, no. 5, pp. 793–801, May 2003.

[32] S. Ye and R. Blum, "Optimized signaling for MIMO interference systems with feedback," *IEEE Trans. Signal Processing*, vol. 51, no. 11, pp. 2839–2848, Nov. 2003.

[33] G. Scutari, D. Palomar, and S. Barbarossa, "Optimal Linear Precoding Strategies for Wideband Noncooperative Systems Based on Game Theory – Part I: Nash Equilibria," *Signal Processing, IEEE Transactions on*, vol. 56, no. 3, pp. 1230–1249, Mar. 2008.

[34] ——, "Optimal Linear Precoding Strategies for Wideband Non-Cooperative Systems Based on Game Theory –Part II: Algorithms," *Signal Processing, IEEE Transactions on*, vol. 56, no. 3, pp. 1250–1267, Mar. 2008.

[35] S. Jafar and M. Fakhereddin, "Degrees of freedom for the MIMO interference channel," *IEEE Trans. Inform. Theory*, vol. 53, no. 7, pp. 2637–2642, July 2007.

[36] M. Maddah-Ali, A. Motahari, and A. Khandani, "Signaling over MIMO Multi-Base Systems: Combination of Multi-Access and Broadcast Schemes," in *Proc. IEEE Intl. Symp. on Info. Theory*, 2006, pp. 2104–2108.

[37] S. Jafar and S. Shamai, "Degrees of freedom region of the MIMO X channel," *IEEE Trans. Inform. Theory*, vol. 54, no. 1, pp. 151–170, Jan. 2008.

[38] A. Host-Madsen and A. Nosratinia, "The multiplexing gain of wireless networks," in *Proc. IEEE Intl. Symp. on Info. Theory*, 2005, pp. 2065–2069.

[39] V. Cadambe and S. Jafar, "Interference alignment and degrees of freedom of the K user interference channel," *IEEE Trans. Inform. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[40] ——, "Interference Alignment and the Degrees of Freedom of Wireless X Networks," *IEEE Trans. Inform. Theory*, vol. 55, no. 9, pp. 3893–3908, Sept. 2009.

[41] A. S. Motahari, S. O. Gharan, and A. K. Khandani, "Real Interference Alignment with Real Numbers," 2009. [Online]. Available: http://arxiv.org/abs/0908.1208

[42] A. S. Motahari, S. O. Gharan, M.-A. Maddah-Ali, and A. K. Khandani, "Real Interference Alignment: Exploiting the Potential of Single Antenna Systems," 2009. [Online]. Available: http://arxiv.org/abs/0908.2282

[43] M. A. Maddah-Ali, " On the Degrees of Freedom of the Compound MIMO Broadcast Channels with Finite States," 2009. [Online]. Available: http://arxiv.org/abs/0909.5006

[44] A. Ghasemi, A. S. Motahari, and A. K. Khandani, "Interference Alignment for the K User MIMO Interference Channel," 2009. [Online]. Available: http://arxiv.org/abs/0909.4604

[45] B. Nazer, S. Jafar, M. Gastpar, and S. Vishwanath, "Ergodic interference alignment," in *Proc. IEEE Intl. Symp. on Info. Theory*, 2009, pp. 1769–1773.

[46] B. Nazer, M. Gastpar, S. Jafar, and S. Vishwanath, "Interference alignment at finite SNR: General message sets," in *Proc. of Allerton Conf. Commun., Control, and Computing*, 2009, pp. 843–848.

[47] V. Cadambe, S. Jafar, and C. Wang, "Interference Alignment With Asymmetric Complex Signaling–Settling the Høst–Madsen–Nosratinia Conjecture," *IEEE Trans. Inform. Theory*, vol. 56, no. 9, pp. 4552–4565, Sept. 2010.

[48] S. A. Jafar, "Exploiting channel correlations - simple interference alignment schemes with no CSIT," 2009. [Online]. Available: http://arxiv.org/abs/0910.0555

[49] T. Gou, C. Wang, and S. Jafar, “Aiming Perfectly in the Dark-Blind Interference Alignment Through Staggered Antenna Switching,” *Signal Processing, IEEE Transactions on*, vol. 59, no. 6, pp. 2734–2744, June 2011.

[50] M. Maddah-Ali and D. Tse, “On the Degrees of Freedom of MISO Broadcast Channels with Delayed Feedback,” 2010. [Online]. Available: http://www.eecs.berkeley.edu/Pubs/TechRpts/2010/EECS-2010-122.html

[51] ——, “Completely stale transmitter channel state information is still very useful,” in *Proc. of Allerton Conf. Commun., Control, and Computing*, 2010, pp. 1188–1195.

[52] S. S. Hamed Maleki, Syed A. Jafar, “Retrospective Interference Alignment ,” 2010. [Online]. Available: http://arxiv.org/abs/1009.3593

[53] A. Goldsmith, S. Jafar, N. Jindal, and S. Vishwanath, “Capacity limits of MIMO channels,” *IEEE Journal on Selected Areas in Communications*, vol. 21, no. 5, pp. 684–702, May 2003.

[54] C. Huang, S. Jafar, S. Shamai, and S. Vishwanath, “On degrees of freedom region of MIMO networks without CSIT,” 2009. [Online]. Available: http://arxiv.org/abs/0909.4017

[55] C. S. Vaze and M. K. Varanasi, “The degrees of freedom regions of MIMO broadcast, interference, and cognitive radio channels with no CSIT ,” 2009. [Online]. Available: http://arxiv.org/abs/0909.5424

[56] Y. Zhu and D. Guo, “Isotropic MIMO interference channels without CSIT: The loss of degrees of freedom,” in *Proc. of Allerton Conf. Commun., Control, and Computing*, 2009, pp. 1338–1344.

[57] ——, “The Degrees of Freedom of MIMO Interference Channels without State Information at Transmitters,” 2010. [Online]. Available: http://arxiv.org/abs/1008.5196v1

[58] C. S. Vaze and M. K. Varanasi, "A New Outer-Bound via Interference Localization and the Degrees of Freedom Regions of MIMO Interference Networks with no CSIT," 2011. [Online]. Available: http://arxiv.org/abs/1105.6033

[59] ——, "The Degrees of Freedom Region and Interference Alignment for the MIMO Interference Channel with Delayed CSI," 2011. [Online]. Available: http://arxiv.org/abs/1101.5809

[60] ——, " The Degrees of Freedom Region of the Two-User MIMO Broadcast Channel with Delayed CSI," 2011. [Online]. Available: http://arxiv.org/abs/1101.0306

[61] A. Abdel-Hadi and S. Vishwanath, "On multicast interference alignment in multihop systems," in *Proc. of IEEE Intl. Inform. Theory Workshop*, 2010, pp. 1–4.

[62] T. Gou, S. Jafar, and C. Wang, "On the Degrees of Freedom of Finite State Compound Wireless Networks," *IEEE Trans. Inform. Theory*, vol. 57, no. 6, pp. 3286–3308, June 2011.

[63] H. Weingarten, S. Shamai, and G. Kramer, "On the compound MIMO broadcast channel," in *in Proceedings of Annual Information Theory and Applications Workshop UCSD*, 2007.

[64] A. Paulraj and C. Papadias, "Space-time processing for wireless communications," *Signal Processing Magazine, IEEE*, vol. 14, no. 6, pp. 49–83, June 1997.

[65] S. Simon and A. Moustakas, "Optimizing MIMO antenna systems with channel covariance feedback," vol. 21, no. 3, pp. 406–417, Mar. 2003.

[66] J. Roh and B. Rao, "Transmit beamforming in multiple-antenna systems with finite rate feedback: a VQ-based approach," *IEEE Trans. Inform. Theory*, vol. 52, no. 3, pp. 1101–1112, Mar. 2006.

[67] P. Xia and G. Giannakis, "Design and analysis of transmit-beamforming based on limited-rate feedback," *Signal Processing, IEEE Transactions on*, vol. 54, no. 5, pp. 1853–1863, May 2006.

[68] D. Love, R. Heath, V. Lau, D. Gesbert, B. Rao, and M. Andrews, "An overview of limited feedback in wireless communication systems," vol. 26, no. 8, pp. 1341–1365, Aug. 2008.

[69] V. Veeravalli, Y. Liang, and A. Sayeed, "Correlated MIMO wireless channels: capacity, optimal signaling, and asymptotics," *IEEE Trans. Inform. Theory*, vol. 51, no. 6, pp. 2058–2072, June 2005.

[70] N. Jindal, "MIMO Broadcast Channels With Finite-Rate Feedback," *IEEE Trans. Inform. Theory*, vol. 52, no. 11, pp. 5045–5060, Nov. 2006.

[71] A. Paulraj, R. Nabar, and D. Gore, *Introduction to space-time wireless communications*. Cambridge University Press, 2003.

[72] H. M. Viveck R. Cadambe, Syed A. Jafar, "Distributed Data Storage with Minimum Storage Regenerating Codes - Exact and Functional Repair are Asymptotically Equally Efficient," 2010. [Online]. Available: http://arxiv.org/abs/1004.4299

[73] V. R. Cadambe, C. Huang, S. A. Jafar, and J. Li, "Optimal Repair of MDS Codes in Distributed Storage via Subspace Interference Alignment," 2011. [Online]. Available: http://arxiv.org/abs/1106.1250

[74] A. Das, S. Vishwanath, S. Jafar, and A. Markopoulou, "Network coding for multiple unicasts: An interference alignment approach," in *Information Theory Proceedings (ISIT), 2010 IEEE International Symposium on*, 2010, pp. 1878–1882.

[75] A. Ramakrishnan, A. Das, H. Maleki, A. Markopoulou, S. Jafar, and S. Vishwanath, "Network coding for three unicast sessions: Interference alignment approaches," in *Proc. of Allerton Conf. Commun., Control, and Computing*, 2010, pp. 1054–1061.

[76] C. Suh and D. Tse, "Interference Alignment for Cellular Networks," in *Proc. of Allerton Conf. Commun., Control, and Computing*, 2008, pp. 1037–1044.

[77] T. Gou and S. Jafar, "Degrees of Freedom of the K User $M \times N$ MIMO Interference Channel," *IEEE Trans. Inform. Theory*, vol. 56, no. 12, pp. 6040–6057, Dec. 2010.

[78] V. Cadambe, S. Jafar, and S. Shamai, "Interference Alignment on the Deterministic Channel and Application to Fully Connected Gaussian Interference Networks," *IEEE Trans. Inform. Theory*, vol. 55, no. 1, pp. 269–274, Jan. 2009.

[79] C. Shannon, "The zero error capacity of a noisy channel," *IEEE Trans. Inform. Theory*, vol. 2, no. 3, pp. 8–19, Mar. 1956.

[80] T. Jiang, N. Sidiropoulos, and J. ten Berge, "Almost-sure identifiability of multidimensional harmonic retrieval," *Signal Processing, IEEE Transactions on*, vol. 49, no. 9, pp. 1849–1859, Sept. 2001.

[81] B. Grunbaum, *Convex Polytopes.* Springer, 2003.

[82] T. H. Matheiss and D. S. Rubin, "A survey and comparison of methods for finding all vertices of convex polyhedral sets," *Mathematics of Operations Research*, vol. 5, no. 2, pp. 167–185, May 1980.

[83] B. A. Davey and H. A. Priestley, *Introduction to lattices and order.* Cambridge University Press, 2002.

[84] H. Yin, "Comments on Degrees of freedom region for K-user interference channel with M antennas," 2010. [Online]. Available: http://arxiv.org/abs/1011.3812

[85] Y. Wu, S. Shamai (Shitz), and S. Verdú, "Degrees of freedom of the interference channel: a general formula," in *Proc. IEEE Intl. Symp. on Info. Theory*, 2011, pp. 1344–1348.

[86] X. Shang, B. Chen, and M. Gans, "On the achievable sum rate for MIMO interference channels," *IEEE Trans. Inform. Theory*, vol. 52, no. 9, pp. 4313–4320, Sept. 2006.

[87] P. Smith, S. Roy, and M. Shafi, "Capacity of MIMO systems with semicorrelated flat fading," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2781–2788, Oct. 2003.

[88] S. H. Simon, A. L. Moustakas, and L. Marinelli, "Capacity and Character Expansions: Moment-Generating Function and Other Exact Results for MIMO Correlated Channels," *IEEE Trans. Inform. Theory*, vol. 52, no. 12, pp. 5336–5351, Dec. 2006.

[89] A. Lozano, A. Tulino, and S. Verdu, "High-SNR power offset in multiantenna communication," *IEEE Trans. Inform. Theory*, vol. 51, no. 12, pp. 4134–4151, Dec. 2005.

[90] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication.* Cambridge University Press, 2005.

[91] H. Shin and J. H. Lee, "Capacity of multiple-antenna fading channels: spatial fading correlation, double scattering, and keyhole," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2636–2647, Oct. 2003.

[92] R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge University Press, 1985.