

ANALYTICAL SOLUTION TO A VOLTAGE DRIVEN PIEZOELECTRIC ELEMENT

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INTRODUCTION

The performance analysis of piezoelectric crystals or ceramic elements as employed in ultrasonic transducers is typically carried out by electric equivalent circuit models such as KLM [4] or, more recently, by numerical analysis techniques [5]. Piezoelectric finite element formulations, which allow for flexible domain discretization, have emerged as powerful simulation tools for complex sensor configurations. Unfortunately, verification of these numerical programs by analytical theories is difficult, even for the simple one-dimensional single crystal resonator. Indeed, there is a particular need for a transient transducer model since practical NDE requirements often dictate a time domain treatment of the ultrasonic probe response driven by an applied voltage.

This paper discusses an analytical solution for a single piezoelectric element subject to potential excitation applied to the electrodes. Following a separation of variable approach for the mechanical wave equation, we will show that the potential boundary conditions applied to a piezoelectric substrate can be incorporated into the series solution. The resulting transient displacement response is then used as a testbed to examine the numerical displacement predictions of our finite element formulation [7, 8] in the time domain. The importance of the analytical model to verify the numerical field simulations becomes apparent when one studies the influence of different mesh densities and time discretizations.

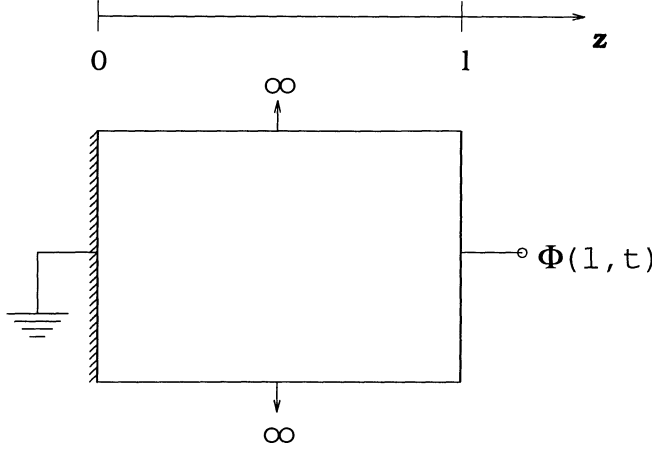


Figure 1: Geometry used to find the one dimensional solution of a piezoelectric element.

PIEZOELECTRIC MODEL FORMULATION

Starting with the one-dimensional equation of motion in the electrostatic limit, we can state the governing relations and constitutive relations in the form [1]:

$$\frac{\partial T_3}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} \quad (1)$$

$$\frac{\partial D_z}{\partial z} = 0 \quad (2)$$

$$D_z = \epsilon_{zz} E_z + e_{z3} \frac{\partial u_z}{\partial z} \quad (3)$$

$$T_3 = -e_3 E_z + c_{33} \frac{\partial u_z}{\partial z} \quad (4)$$

where E_z, D_z , and u_z are the z -components of the electric field, electric displacement, and particle displacement, respectively, and ϵ_{zz} , e_{z3} , c_{33} are the permittivity, piezoelectric and elastic material constants. T_3 is the the z -component of the stress in the xy plane [1]. For the subsequent one-dimensional discussion we will drop the indices.

The electric field in the electrostatic limit is

$$E = -\frac{\partial \Phi}{\partial z} \quad (5)$$

Incorporating (5) into (2) and (3) yields

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{e}{\epsilon} \frac{\partial^2 u}{\partial z^2} \quad (6)$$

which has the generic solution

$$\Phi(z, t) = \frac{e}{\epsilon} u(z, t) + \alpha(t)z + \beta(t) \quad (7)$$

and $\alpha(t)$, $\beta(t)$ are arbitrary integration constants. For the geometry in Figure 1 we

require Φ and u to vanish at $z = 0$, which implies $\beta = 0$. Since the surface at $z = l$ is assumed stress-free, we find from equation (4)

$$-\frac{\partial \Phi}{\partial z} \Big|_{z=l} = \frac{c}{e} \frac{\partial u}{\partial z} \Big|_{z=l} \quad (8)$$

This allows us to determine $\alpha(t)$ such that

$$\alpha(t) = - \left(\frac{e}{\epsilon} + \frac{c}{e} \right) \frac{\partial u}{\partial z} \Big|_{z=l} \quad (9)$$

In light of equation (7), the piezoelectric boundary condition at $z = l$ is

$$\frac{e}{\epsilon} u \Big|_{z=l} - \left(\frac{e}{\epsilon} + \frac{c}{e} \right) \frac{\partial u}{\partial z} \Big|_{z=l} = \Phi \Big|_{z=l} \quad (10)$$

This relation between voltage and displacement on the electrode will be exploited in the mechanical equation of motion, as discussed next.

TRANSIENT WAVE EQUATION IN A FINITE PIEZOELECTRIC DOMAIN

It is well known that the wave equation for piezoelectric media can be cast in the form

$$\frac{\partial^2 u(z, t)}{\partial z^2} - \frac{1}{v_D^2} \frac{\partial^2 u(z, t)}{\partial t^2} = 0 \quad (11)$$

in the interval $t \in [0, \infty)$ and $z \in [0, l]$, where

$$v_D = \sqrt{\frac{c + \frac{e^2}{\epsilon}}{\rho}} \quad (12)$$

is the stiffened wave velocity. Furthermore, we assume the following initial conditions

$$u(z, t) \Big|_{t=0} = 0 \quad (13)$$

$$\frac{\partial u(z, t)}{\partial t} \Big|_{t=0} = 0 \quad (14)$$

and boundary conditions

$$u(z, t) \Big|_{z=0} = 0 \quad (15)$$

$$\frac{\partial u(z, t)}{\partial z} \Big|_{z=l} - \frac{K}{l} u(z, t) \Big|_{z=l} = p \quad (16)$$

where K and p are in general arbitrary constants. We will restrict ourselves to the case where $0 \leq K < 1$, which holds for all practical transducer configurations.

A general solution to (11) can be developed by standard separation of variables [3, 6] techniques

$$u(z, t) = Az + \sum_{n=1}^{\infty} a_n \cos(v_D k_n t) \sin(k_n z) \quad (17)$$

with

$$A = \frac{p}{1-K} \quad (18)$$

and

$$a_n = 4A \frac{k_n \cos(k_n l) - \sin(k_n l)}{2 k_n^2 l - k_n \sin(2k_n l)} \quad (19)$$

The discrete spectrum of k_n is the solution the transcendental equation

$$\tan(k_n l) = \frac{k_n l}{K} \quad (20)$$

in the interval $[\frac{n-1}{l}\pi, \frac{2n-1}{2l}\pi]$. Comparing (10) with (16), we can identify the constant K to be

$$K = \frac{k_e}{1 + k_e}, \quad k_e = \frac{e^2}{c\epsilon} \quad (21)$$

and the constant p is

$$p = -\frac{K\epsilon}{le}\Phi_0 \quad (22)$$

where we assumed that the voltage at $z = l$ is constant

$$\Phi(l, t) = \Phi_0 \quad (23)$$

Inserting (21), (22), and (23) into (17), (18), and (19) we finally obtain the voltage driven displacement response

$$u(z, t) = -\frac{\epsilon\Phi_0 [1 + k_e]}{le} \left[z + \sum_{k=1}^{\infty} b_n \cos(v_D k_n t) \sin(k_n z) \right] \quad (24)$$

where b_n is

$$b_n = \frac{a_n}{A} \quad (25)$$

The analytical series solution is computed by retaining the first 10 terms in the expansion (24). The factors k_1 to k_5 are found by Newton's method while the remaining k_n 's are approximated by $k_n \approx \frac{(n-\frac{1}{2})}{l}\pi$.

MODEL PREDICTIONS AND COMPARISONS WITH FINITE ELEMENT ANALYSIS

We now apply (24) to investigate the displacement response of an $l = 1\text{mm}$ piezoelectric resonator driven by a step voltage $\Phi_0 = 200\text{V}$. The material parameters are those of PZT-5H with $\rho = 7500 \frac{\text{kg}}{\text{m}^3}$, $c = 11.7 \cdot 10^{10} \frac{\text{N}}{\text{m}^2}$, $\epsilon = 13.010 \cdot 10^{-9} \frac{\text{F}}{\text{m}}$, $e = 23.3 \frac{\text{C}}{\text{m}^2}$, and $v_D = 4600 \frac{\text{m}}{\text{s}}$.

Figure 2 depicts the displacement response at three locations $z = l$, $\frac{l}{2}$, and $\frac{l}{4}$. The response is based on a strong electromechanical coupling coefficient k_e of 0.34 for PZT-5H. In contrast, Figure 3 shows the results for a weak coupling coefficient $k_e = 0.03$ for Lithium Niobate with material parameters $\rho = 4700 \frac{\text{kg}}{\text{m}^3}$, $c = 24.510 \cdot 10^{10} \frac{\text{N}}{\text{m}^2}$, $\epsilon = 0.2510 \cdot 10^{-9} \frac{\text{F}}{\text{m}}$, $e = 1.3 \frac{\text{C}}{\text{m}^2}$ and $v_D = 7316 \frac{\text{m}}{\text{s}}$. As can be seen, the displacement response approaches the one dimensional Fourier series solution for an mechanical resonator [2]. Interestingly, the shape of the displacement response is dependent on the coupling coefficient and the time duration. The magnitude of the applied potential only contributes to the magnitude of the responses.

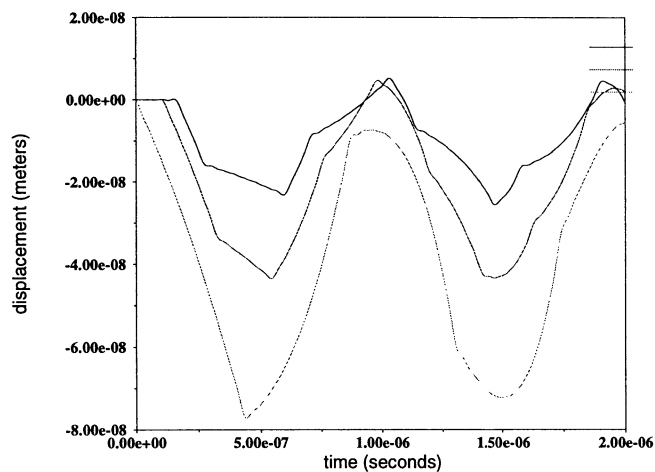


Figure 2: The analytical displacement responses, $u_a(z, t)$, at $z = l$, $z = \frac{l}{2}$, and $z = \frac{l}{4}$ for a PZT-5H resonator.

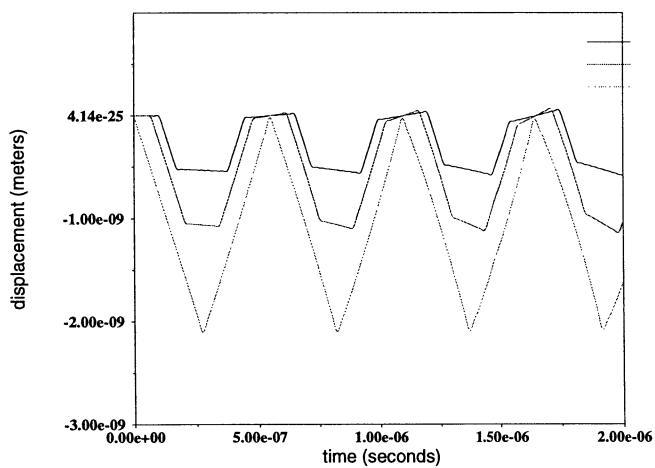


Figure 3: The analytical displacement responses, $u_a(z, t)$, at $z = l$, $z = \frac{l}{2}$, and $z = \frac{l}{4}$ for a Lithium Niobate resonator.

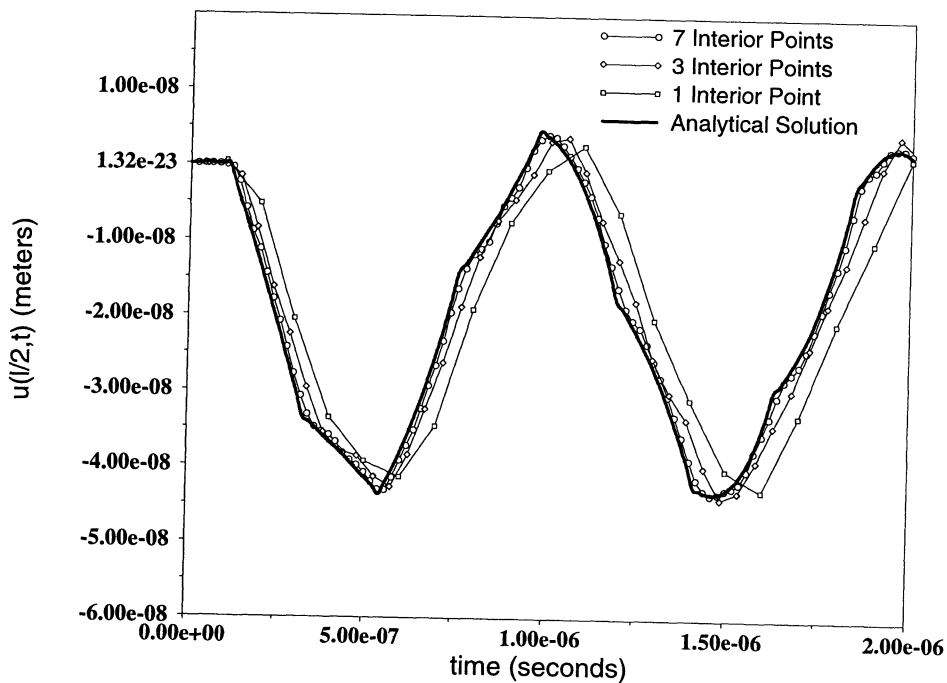


Figure 4: The numerically determined displacement response $u(z, t)$ for PZT-5H at $z = l/2$ for different mesh discretizations. The simulation results are very close to the analytical solution for 7 interior points, which corresponds to a spatial resolution of $\Delta z = 0.125 \text{ mm}$. For higher resolution the simulation and analytical solutions become indistinguishable.

Finally, Figure 4 compares the analytical solution at $z = \frac{l}{2}$ of a PZT-5H crystal with our finite element model [8] for various mesh discretization with Δt chosen such that $\frac{v_D \Delta t}{\Delta x}$ is as close to 1.0 as possible. It can be seen that the numerical predictions approach the analytical solution for a finer mesh size. These one-dimensional results can also be used as a benchmark to determine the suitable node density for two and three dimensional geometries.

CONCLUSIONS

In this paper we develop an explicit analytical solution for piezoelectric media in the time domain. The model is easy to implement and can be directly applied to test the displacement response of numerical methods in the one dimensional limit. This analytical model may be a useful tool in evaluating suitable spatial and temporal discretizations of the piezoelectric finite element method. Indeed, before modeling complex geometries such as ultrasonic array antennas or piezoelectric motors, we always calibrate the numerical code against this analytical solution in the one-dimensional limit.

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