Optimal Buffer Stocks in Neumann--
Economies under Uncertainty

by

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I. Introduction

There exists side-by-side substantial literatures on von Nuemann economies--especially their turnpike properties--[7, 8],--and on optimal growth theory under uncertainty [1, 2, 6]. Typically, the von Neumann models, although deterministic, have a fairly complicated product technology, while the optimal growth models rely on fairly rudimentary product technologies. The present work attempts to bring these two literatures together with the confines of a two-sector von Neumann model under uncertainty. For ease of interpretation, and as the primary application, the model is considered as one of an optimally planned economy [3, 4].

The paper addresses two central questions. The first question concerns the role of inventories in the model. Kornai [5] has shown in the context of a deterministic von Neumann control model that an increase of inventories slows down the growth of the economy. However, Kornai also conjectured inventories might be advantageous. The model developed in this paper gives definite support to Kornai's conjecture. It turns out that for substantial degrees of uncertainty in a sense to be made precise, the optimal plan has a coefficient of resource utilization significantly less than one. The second question concerns the effect of an increase in risk on the model. An early result of the kind was due to Rothschild-Stiglitz [9], in their model of multi-stage planning. They found that for constant returns-to-scale production functions, the median of the probability distribution is a crucial parameter in the optimal plan. The model developed in this paper extends their result. Further, in the case of symmetric probability distributions, an increase in risk not only increases inventories but also shifts the allocation of resources from the starting of new investments to the completion of old investments. In this way, an increase in risk has a double impact on the capital structure of the economy.

The paper is organized as follows. Section II considers the simple case of the one-sector model. Section III investigates the turnpike for the two-sector model under certainty. Sections IV and V consider the two-sector model under uncertainty, including some special cases in Section V.

II. The One-Sector Model

This section reviews a paradigm case of the one-sector model namely linear production with unit period of production. Thus, if F^t is free resources in period t, and $\hat{P}^t$ is projects started in period t, then

(1) $F^{t+1} = F^t - a\hat{P}^t + \alpha_t\hat{P}^t$.

In the deterministic case, $\alpha_t = 1$. The planning objective is terminal value maximization

(2) $\max F^{T+1}$

subject to the transaction equation (1), the initial condition (3),

(3) F^0 given,

and the constraint

(4) $0 \leq \hat{P}^t \leq F^t/a$

It is clear by a standard dynamic programming argument that the optimal policy satisfies

(5) $\hat{P}^t = F^t/a$,

so that the economy grows at the rate of $\frac{F^{t+1}}{F^t} = \frac{1}{a} = \lambda$, the von Neumann rate.

The value of the optimal plan, from (2), (3), and (5) then is

$$(6) \quad \max F^{T+1} = (\lambda)^{T+1} F^0 .$$

Results are not greatly different in the uncertainty case, where α_t is an independently identically distributed random variable with $E\alpha_t = 1$. The planning objective is now expected terminal value maximization

$$(2)' \quad \max EF^{T+1}$$

subject to the same conditions as before. The optimal policy still satisfies (5), and the expected growth rate $\frac{\varepsilon F^{t+1}}{F^t} = \lambda$ as before. Finally, the value of the optimal plan, assuming independence of the various realizations of α, is

$$(6)' \quad \max F^{T+1} = (\lambda)^{T+1} F^0 \prod_{i=0}^{T} \alpha_i = (\lambda)^{T+1} F^0$$

Thus the uncertainty ultimately has no apparent impact on the planning problem. It might be expected that the results of the one-sector model would generalize to the two-sector model. Such is the case for the deterministic model, but rather drastic differences thwart the generalization in the event of uncertainty.

III. The Two-Sector Model under Certainty

The model of the previous section is extended to two-sectors by the addition of an intermediate goods sector. Thus, one now distinguishes between $\hat{P}^t$, projects updated and N^t, new projects started, in period t. The transition equations become

$$(7) \quad F^{t+1} = F^t - a\hat{P}^t - aN^t + \alpha_t \hat{P}^t$$

$$(8) \quad P^{t+1} = P^t - \hat{P}^t + N^t$$

where P^t represents the intermediate good. The choice between starting new projects and updating old ones is reflected in the constraints

(9) $0 \leq \hat{P}^t \leq \min\,(F^t/a,\ P^t)$

since not more projects can be updated than are in process, and

(10) $0 \leq N^t \leq F^t/a - \hat{P}^t$

The objective is to

(11) $\max\ F^{T+1} + aP^{T+1}$,

work in progress being evaluated at its resource cost.

In the deterministic case when $\alpha_t = 1$, the planning problem then is to maximize (11), subject to (7) - (10) and the initial conditions

(12) F^0, P^0 given

Since the objective function exhibits constant returns to scale, it is useful to define new variables that reflect intensities:

(13) $X^t = F^t/aP^t$

(14) $\theta^t = \hat{P}^t/P^t$

(15) $\gamma^t = N^t/P^t$

Rewriting (7) through (10) in the form,

(7)' $X^{t+1} = \dfrac{\alpha_t\theta^t + a(X^t - \theta^t - \gamma^t)}{a(1 - \theta^t + \gamma^t)}$

(8)' $P^{t+1} = P^t(1 - \theta^t + \gamma^t)$

(9)′ $0 \leq \theta^t \leq \min(1, X^t)$

(10)′ $0 \leq \gamma^t \leq X^t - \theta^t$

We seek the maximum of (11) subject to (7)′ - (10)′ and (12). Let us call this problem in the deterministic case ($\alpha_t = 1$) problem I.

Theorem 1. There exists an optimal solution (θ^*_t, γ^*_t) to problem I. It is characterized by

(a) $X^t < X^t_L$ then $\theta^{t*} = X^t$, $\gamma^{t*} = 0$

(b) $X^t \,\varepsilon (X^t_L, X^t_U)$ then $0 < \theta^{t*} < \min(1, X^t)$

$\gamma^{t*} > 0$

such that

$$\theta^{t*} + \gamma^{t*} = X^t$$

and $X^{t+1} = X^{t+1}_U$

(c) $X^t > X^{t+1}_U$ the $\theta^{*t} = 1$

γ^{*t} indeterminate,

but $X^{t+1} \,\varepsilon (X^{t+1}_L, X^{t+1}_U)$

where X^t_L , X^t_U are defined recursively as

(16) $X^t_U = \dfrac{1 + aX^{t+1}_U}{aX^{t+1}_U}$ with the initial conditions $X^T_U = X^T_L = 1.$

$$X^t_L = 1/X^t_U$$

Furthermore, the limit as $T \to \infty$ of $X^t_U = X^*$, the von Neumann growth path, where $X^* = 2 + \lambda$, λ being the von Neumann growth rate ($\lambda = \dfrac{-3 + \sqrt{1 + 4/a}}{2}$).

Sketch of the Proof

There exists an optimal solution by a standard result of linear programming. The characteristic of the optimal solution is proved by induction on i, where $t = T + 1 < i$. Define the optimal stage return function.

$$(17) \quad J^t = \max_{\hat{P}^t,\, N^t} J^{t+1} (F^{t+1}, P^{t+1})$$

In the case $i = 1$, one has

$$J^T = \max_{\hat{P}^T,\, N^T} F^{T+1} + aP^{T+1}$$

subject to (7)′ - (10)′ and (12).

Forming the Lagrangean $\mathcal{L}^T$,

$$(18) \quad \mathcal{L}^T = J^T + \lambda_1(P^T - \hat{P}^T) + \lambda_2(F^T - aN^T - a\hat{P}^T) \ .$$

This is a linear programming problem, whose solution is given by

$$\hat{P}^T = \min(F^T/a,\ P^T),\ N^T \text{ indeterminate.}$$

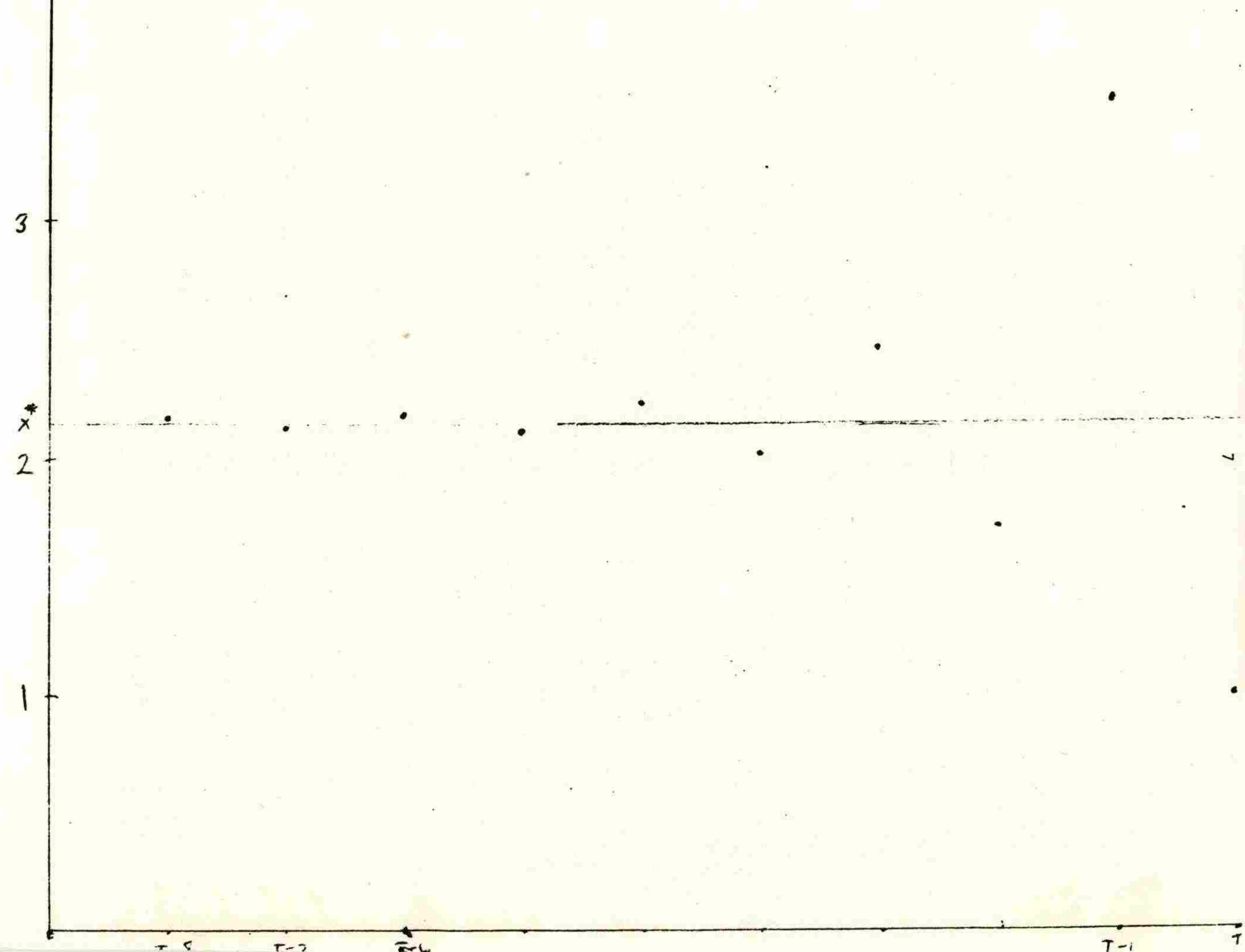
Thus, since $X_U^T = 1$, the solution satisfies condition (a) - (c).

The induction step, which is straightforward but tedious, involves showing that J^t is a concave function of X^t and that the ratio of partial derivatives of J^t satisfies

$$(19) \quad \frac{J_F^t}{J_P^t} \text{ is } \begin{cases} > 1/a & \text{if } X_t < X_t^L \\ = 1/a & \text{if } X_t^L < X_t < X_t^U \\ = 1/1\text{-}a & \text{if } X_t^U < X_t \end{cases}$$

where $J_y^t = \dfrac{\partial J^t}{\partial y^t}$ for any variable y.

FIGURE 1.



Finally, the steady state solution of (13) satisfies the equation for the von Neumann ray,

Intuitively, what the optimal policy says is that the economy at time period t should be aimed as closely as possible to the target X_U^{t+1}, and this target approaches the von Neumann ray for times t sufficiently far removed from T. It is also worth noting that the X_U^t series itself fluctuates according to whether t is odd or even. The first few terms of the sequence are noted in figure one for the case $a = .4$, $\lambda = .158$. Notice that even at times as near to termination as T - 8, X_U^t is quite close to the von Neumann ray $X^* = 2.158$.

IV. The Two-Sector Model under Uncertainty: Solution at T and T-1

In section II it was shown that if planners are risk neutral, then introducing uncertainty into the one sector model has no effect on the expected growth rate of the system or on the optimal allocation of resources. This section considers the impact of uncertainty of the two sector model of the last section. The only formal differences are that α_t, instead of being a constant equal to one in equation (7)′, is now a random variable satisfying

$$(20) \quad \alpha_t = \begin{cases} 1-\varepsilon & \text{with probability} \quad z \\ 1 & \text{with probability} \quad 1-2z \\ 1+\varepsilon & \text{with probability} \quad z \end{cases}$$

$$0 \le \varepsilon \le 1, \quad 0 \le z \le 1/2$$

$$\varepsilon(\alpha_t \, \alpha_{t-i}) = 0 \qquad \text{for } i \neq 0$$

and equation (11) is replaced by its expectation.[1] The case $\varepsilon = 0$, $z = 0$ corresponds to certainty. It turns out that for $t = T$, uncertainty again has no effect on the optimal solution; but at time $t = T - 1$, the situation is complicated considerably by uncertainty.

To this end, introduce the Lagrangean $Ł^t$ (compare [18]),

$$(21) \quad Ł^t = E[J^{t+1}] + \lambda_1(P^t - \hat{P}^t) + \lambda_2(F^t - a\hat{P}^t - aN^t).$$

From (7), (8), and (17), the first order conditions for an optimum are given by

$$(22) \quad Ł^t_{\hat{P}} = E[J^{t+1}_F (\alpha - a) - J^{t+1}_P] - \lambda_1 - \lambda_2 a \leq 0$$

$$(23) \quad Ł^t_N = E[J^{t+1}_P - aJ^{t+1}_F] - \lambda_2 a \leq 0.$$ [2]

In particular at time $t = T$, since J^{T+1}_F, J^{T+1}_P are constant, assuming $(1 - 2a) > 0$ (i.e., the system is productive), it is clear that the optimal strategy at T entails:

$$(24) \quad \hat{P}^T = \mathrm{Min}[P^T, (F^T/a)]$$

Given the terminal value associated with P^{T+1}, it is clear that it is a matter of indifference whether new projects are started at T (if any resources are available).[3] Thus:

$$\begin{aligned}(25) \quad J^T &= F^T + aP^T + (1 - 2a)\ \mathrm{Min}[P^T, F^T/a] \\ &= [(\tfrac{1 - a}{a})F^T + aP^T], \qquad F^T < aP^T \\ &= [F^T + (1 - a)P^T], \qquad F^T > aP^T\end{aligned}$$

Note that the uncertainty has no impact at T, and, as for the case of certainty, J^T is a concave, linear homogeneous function in (F^T, P^T). Further, J^T_F, J^T_P are discontinuous at $F^T = aP^T$.

While the uncertainty has no impact on the optimal decision or the objective function at T, the same need not be true at (T - 1) since $J^T(F^T, P^T)$ is no longer a linear function. The ray ($F^T = aP^T$) represents a vertex of J^T, and at (T - 1) the planner will attempt to steer the economy towards this ray. However, because of the uncertainty, the planner cannot be sure whether he will hit this target, and thus he must decide whether to aim for this target i) on the average ($\alpha = 1$), ii) for an optimistic outcome ($\alpha = 1 + \varepsilon$), or iii) for a pessimistic outcome ($\alpha = 1 - \varepsilon$). As we shall see, the appropriate choice depends on the parameters of the distribution (z, ε), as well as the growth rate of the system (inversely related to a).

From the certainty solution we know that at (T - 1) there exists a range of X^t such that:

(26) $$X^T(1) < 1 \text{ for } X^{T-1} < X^L \equiv \left(\frac{a}{1 + a}\right) \text{ if } \gamma = 0,\ \theta^{T-1} = X^{T-1}\ .$$

In (26), $X^T(1)$ means X^T is evaluated at the outcome $\alpha = 1$, corresponding to the certainty solution. Therefore, for small X^{T-1} the planner will not be able to reach the target ($X^T = 1$) on the average.

Similarly, under certainty, if $X^{T-1} \varepsilon (X^L, X^U) \equiv \left(\frac{a}{1+a}, \frac{1+a}{a}\right)$ then the planner can reach the target without holding any idle reserves, to recapitulate that result:

(27) $$X^T(1) = 1 \text{ for } X^{T-1} \varepsilon \left(\frac{a}{1+a}, \frac{1+a}{a}\right) \text{ if } \theta^{T-1} = \left(\frac{a(1+X)}{1+2a}\right) < 1;\ \gamma^{T-1} = X^{T-1} - \theta^T.$$

Finally, for $X^{T-1} > \left(\frac{1+a}{a}\right)$, reaching the target at T entails holding idle reserves at (T-1), which in turn implies all existing projects must be updated.

Under uncertainty, the first order conditions are:

(28) $L_{\hat{P}}^{T-1} = E[J_F^T(\alpha - a) - J_P^T] - \lambda_1 - d_2 a \leq 0$

(29) $L_N^{T-1} = E[J_P^T - aJ_F^T] - \lambda_2 a \leq 0$

where:

(30) $J_F^T = (\frac{1 - a}{a}),\ J_P^T = a,\quad X^T < 1$

$J_F^T = 1,\ J_P^T = (1 - a),\quad X^T > 1.$

Since $J_F^T\ (1 - a) \geq J_P^T$, one of the two constraints must be binding. If $X^{T-1} < 1$, then $\hat{P}^{T-1} < P^{T-1}$, and $\lambda = 0$; consequently, for $X^{T-1} < 1$:

(31) $N^{T-1} = 0 \Leftrightarrow E[\alpha J_F^T - 2J_P^T] > 0$

From (30), it is apparent $E[\alpha J_F^T - 2J_P^T] > 0$ if $X^T(1) < 1$; thus, for $X^{T-1} < X^L$, the uncertainty has no impact on resource allocation at (T-1). Further, it is apparent that $E[J_P - aJ_F] \gtrless 0$ as $X^T(1) \gtrless 1$. This implies that for $\hat{P}^{T-1} = P^{T-1}$, the planner should aim the economy so that, for $\alpha = 1$, $X^T = 1$. Thus, for large X^{T-1} ($X^{T-1} > X^U = \frac{1 + a}{a}$) the uncertainty again has no impact on resource allocation. The real dilemma for the planner arises for $X^{T-1}\ \epsilon (X^L, X^U) = (\frac{a}{1 + a}, \frac{1 + a}{a})$, where the planner must choose between updating old projects or starting new ones. This is the essential problem introduced by the uncertainty, that of attempting to obtain a balanced allocation for the subsequent period.

Clearly, it would never be desirable to allocate resources such that $X^T > 1$ at $\alpha = 1 - \epsilon$; <u>i.e.</u>, the planner never wants to choose an allocation that will lie above the target with probability one. Thus, uncertainty will affect the resource allocation at (T - 1) only if $E[\alpha J_F - 2J_P] > 0$ for $X^T(1 - \epsilon) < 1$, $X^T(1) > 1$:

$$(32)\quad E[\alpha J_F - 2J_P] = (\frac{1-2a}{a})\ [-a + z(1 + 2a - \varepsilon)]$$

From (32), we see that the impact of the uncertainty cannot be related to the variance of the disturbance alone; it depends on both the magnitude of the disturbance, and the probability of the outcome.[4] Using (32), define z^*:

$$(33)\quad z^* = [a/(1+2a-\varepsilon)] \leq 1/2;\quad \frac{\partial z^*}{\partial \varepsilon} > 0,\ \frac{\partial z^*}{\partial a} > 0.$$

For $z > z^*$, $N = 0$ if $X^T(1 - \varepsilon) < 1$; thus, the optimal allocation at $(T - 1)$ depends upon the magnitude of the disturbance, the probability of outcomes, and the growth rate of the system;[5] these results are collected in

<u>Theorem 2</u>. The optimal allocation of resources in period $T - 1$ is as follows:

(a) If $X^{T-1} < \frac{a}{1+a}$, then for all z, $\hat{P} = F/a$, $N = 0$

(b) If $\frac{a}{1+a} < X^{T-1} < \frac{1+a}{a}$ and $z < z^*$, $\hat{P} = \frac{Pa(1 + X^{T-1})}{1 + 2a}$, $N = F/a - \hat{P}$

(c) If $\frac{1+a}{a} < X^{T-1}$, then for all z, $\hat{P} = P$, $N = P[\frac{(1 + a) + aX^{T-1}}{2a}]$.

(d) If $\frac{a}{1+a} < X^{T-1} < \frac{a}{1+a-\varepsilon}$, then for $z > z^*$, $\hat{P} = F/a$, $N = 0$.

(e) If $\frac{a}{1+a-\varepsilon} < X^{T-1} < \frac{1+a-\varepsilon}{a}$, then for $z > z^*$, $\hat{P} = \frac{Pa(1 + X^{T-1})}{1 + 2a - \varepsilon}$, $N = F/a - \hat{P}$

(f) If $\frac{1+a-\varepsilon}{a} < X^{T-1} < \frac{1+a}{a}$, then for $z > z^*$, $\hat{P} = P$, $N = F/a - \hat{P}$.

Cases (a) - (c) are the same as the certainty solution.

Several things are noteworthy from Theorem 2. First, if the uncertainty has any impact, it is to increase the emphasis on completing existing projects at the expense of new ones; it does not affect the choice between starting new projects or holding inventories. Thus, for $X^{T-1} \geq X^U = (\frac{1+a}{a})$, the uncertainty has no impact on the resource allocation.

Secondly, a mean preserving spread of the distribution (an increase in z that puts more weight in the tails) raises the likelihood that the uncertainty will affect the resource allocation.[6] As noted earlier, the planner attempts to steer the economy towards $X^T = 1$; however, the problem is whether to do this for the worst outcome ($\alpha = 1 - \varepsilon$), or the median one ($\alpha = 1$); the larger the probability of the worst outcome, the more likely it is the planner will steer the economy towards $X^T = 1$ at that outcome. Thus, mean preserving spreads increase the emphasis placed on completing existing projects.

An increase in the magnitude of the disturbance (ε) has ambiguous effects. On the one hand, it increases z^*, and therefore the likelihood the planner will not be affected by the uncertainty. Intuitively, the larger ε means more risk is associated with the completion of existing projects, and hence makes it less desirable to increase the number of these projects. On the other hand, if the planner steers the economy so that $X^T = 1$ for $\alpha = (1 - \varepsilon)$ (i.e., $z > z^*$), then more weight is given to completing these projects to insure the target will be met ($\partial P/\partial \varepsilon \geq 0$ for $z > z^*$).

Finally, note that changes in $\underline{a}$ affect z^*; specifically, $\partial z^*/\partial a > 0$. Since the potential growth rate of the system is inversely related to $\underline{a}$, this means that the larger the potential growth rate of the system, the more likely it is that the uncertainty affects the resource allocation ($z > z^*$). The larger the growth rate of the system, the more desirable it is to insure that projects initiated at $(T - 1)$ can be completed at T--and hence the more weight that is attached to the worst outcome ($\alpha = 1 - a$).

Turning to the effect of the uncertainty on $J^{T-1}(F^{T-1}, P^{T-1})$, it is clear that, due to the concavity of J^T, the uncertainty lowers the expected value of terminal output. This is true even for $z < z^*$, when the uncertainty does not affect the resource allocation pattern. Similarly, increases in the magnitude of the disturbance lower (or leave unaltered) J^{T-1}. Thus, unlike the case of the one sector model, the uncertainty adversely affects the performance of the system. Table I summarizes these results.

V. The Two-Sector Model under Uncertainty: The Solution At T-2

While it was possible to characterize completely the solution at (T - 2) and earlier becomes considerably more complex. The complexity arises from several reasons. First, the objective function at (T - 1) depends on the parameters of the disturbance distribution and, under uncertainty, this function must be evaluated at different points. Thus, the optimal solution will depend on these parameters and a host of potential solutions arise, moreover, there seems to be no simple way to relate the parameters to the optimal solution. Secondly, since the disturbances for each period are assumed uncorrelated, the joint distribution of these disturbances becomes important. Since this joint distribution will be a polynomial, the longer the horizon, the higher the degree of the polynomial, and hence the more complex it becomes to ascertain the properties of an optimal solution. Finally, under certainty we have seen that at any t, there exists a range of the sectoral intensities; $(X_L^t(t), X_U^t(t))$, that completely determine the optimal solution; moreover, at t, the planner aims the economy so that, if feasible, $X^{t+1} = X_U^{t+1}$. Under uncertainty, this rule is greatly complicated; for one thing, the planner must decide whether to allocate resources so that the economy achieves the target "on average," or whether to give more weight to the pessimistic (or optimistic) outcome. Moreover, the numbers of these "targets" proliferate as we move to earlier periods since some will correspond to achieving the "target" for subsequent periods with an optimistic outcome $(\alpha = 1 + \varepsilon)$, others with a pessimistic outcome. Thus, the complexity of the

TABLE I

$J^{T-1}(F^{T-1}, P^{T-1})$

X^{T-1}	$z<z^*$	$z>z^*$
$X \frac{a}{1+a+\varepsilon}$	$J^{T-1} = P^{T-1}[(\frac{1-a-a^2}{a})X^{T-1}+a]$	same as $z<z^*$
$\varepsilon[\frac{a}{1+a+\varepsilon}, \frac{a}{1+a}]$	$J^{T-1} = P^{T-1}[(\frac{1-a-a^2}{a})X^{T-1}+a-z(\frac{1-2a}{a})(X^{T-1}(1+a+\varepsilon)-a)]$	same as $z<z^*$
$(\frac{a}{1+a}, \frac{a}{1+a-\varepsilon})$	$J^{T-1} = \frac{P^{T-1}(1+X^{T-1})}{(1+2a)}[1-\varepsilon z(1-2a)]$	$J^{T-1} = P^{T-1}[(1-a+aX)+z(\frac{1-2a}{a})((1+a-\varepsilon)X-a)$
$(\frac{a}{1+a-\varepsilon}, \frac{1+a-\varepsilon}{a})$	same as above	$J^{T-1} = \frac{P^{T-1}[1+X]}{(1+2a-\varepsilon)}[1-\varepsilon(1-a)]$
$(\frac{1+a-\varepsilon}{a}, \frac{1+a}{a})$	same as above	$J^{T-1} = P^{T-1}[(1-a)X+a-z(\frac{1-2a}{a})(aX-(1+a))$
$\frac{1+a}{a}$	$\frac{P^{T-1}}{2a}[(1-a+aX)-2\varepsilon z(1-2a)]$	same as $z<z^*$

Note: $J_z \leq 0$, $J_\varepsilon \leq 0$ everywhere.

the problem increases enormously as the length of the planning horizon increases.

To discuss the optimal solution at T - 2, it is necessary to distinguish the cases $z < z^*$ from $z > z^*$; the former we consider first. For $z < z^*$, the uncertainty has no impact on decision making at (T - 1); however, this in general, will not be true for (T - 2). In allocating the resources at (T - 2), the planner will, in general, guide the economy so that X^{T-1} equals $(\frac{1 + a}{a})$ - i.e. he will aim towards the VonNeuman growth rate. However, in so directing the economy, the planner must decide whether to focus on the "pessimistic," average or "optimistic" outcome. Moreover, if he chooses the average outcome, for example (so that $X^{T-1}(\alpha = 1) = \frac{1 + a}{a}$), he will need to know whether $X^{T-1}(1 - \varepsilon) \lesseqgtr \frac{a}{1 + a}$; if $X^{T-1}(1 - \varepsilon) < (\frac{a}{1 + a})$, then even at T the target $(X^{T} = 1)$ cannot be achieved. However, if:

$$(34) \quad \varepsilon < \frac{1 + 2a}{(1 + a)^2}$$

then the planner can be sure that $X^{T-1}(1 - \varepsilon) > \frac{a}{1 + a}$, for $X^{T-1}(1) \geq \frac{1 + a}{a}$. We shall assume (34) holds.

As in the case of certainty, if $X^{T-2} < X_L^{T-2} = \frac{1 + a}{2 + a}$, then the economy cannot achieve the target $X^{T-1} = \frac{1 + a}{a}$ even on average. Thus, it is readily verified for this case that $X^{T-2} < (\frac{1 + a}{2 + a})$ entails starting no new projects and using all free resources to finish as many older projects as possible. For $X^{T-2} > (\frac{1 + a}{2 + a})$, the planner must decide whether to start new projects, or continue completing older ones. If the latter course is chosen (N = 0), then $X^{T-1}(1) > (\frac{1 + a}{a})$, and in essence, it means the planner behaves so as to steer the economy towards $X^{T-1} = (\frac{1 + a}{a})$ for the pessimistic outcome. However, if z is small, then less weight will be given to that outcome, and the planner will adopt the certainty solution, aiming the economy so that X^{T-1} $(\alpha = 1) = (\frac{1 + a}{a})$. Formally, the optimal allocation depends on:

(35) sign $E\ [\alpha J_F - 2J_P]$

If $E[\alpha J_F - 2J_P] > 0$ for X^{T-1} $(1 - \varepsilon) < \frac{1 + a}{a} < X^{T-1}$ (1), then more weight is given to the pessimistic outcome, and hence more emphasis is given to completing existing projects. For $E\ [\alpha J_F - 2J_P] < 0$ for X^{T-1} $(1 - \varepsilon) < \frac{1 + a}{a} < X^{T-1}$ (1), the outcome focused upon is that corresponding to $\alpha = 1$.

Table II summarizes the optimal decision rules for the case $z < z^*$. As can be seen from the Table, the optimal solution depends upon the probabilities associated with each outcome. For small z, the uncertainty has no impact for $X^{T-2} \leq (\frac{2 + a}{1 + a})$; however, as z increases, more attention is focused on the pessimistic outcome, and hence on completing existing projects. Note that for $z > \frac{1}{4\varepsilon}$ (which is feasible only for $\varepsilon > 1/2$), all existing projects are completed before new ones are started.

One other feature of the optimal solution is noteworthy. Under certainty if $X^{T-2} > (\frac{2 + a}{1 + a})$, some inventories will be held to insure the economy will achieve its target at (T - 1) - i.e., it is not desirable to use all of the surplus capacity of the economy to start new projects since insufficient resources will exist at (T - 1) to continue along the balanced growth path. However, under uncertainty, the planner is not sure how many resources will be available next period. As long as

TABLE II

Optimal Solution at T-2

$z<z^*, \varepsilon< \frac{1+2a}{(1+a)^2}$

X^{T-2}	$z<z_3$	$z_3<z<\frac{1}{4\varepsilon}$	$z>\frac{1}{4\varepsilon}$
$<\frac{1+a}{2+a}$	$\theta=X,\ \gamma=0;$ $X^{T-1}(1)<\frac{1+a}{a}$	same as $z<z_3$	same as $z<z_3$
$\in(\frac{1+a}{2+a}, \frac{1+a}{2+a-\varepsilon})$	$\theta=\frac{(1+a)(1+X)}{(3+2a)};\ \gamma=X-\theta$ $X^{T-1}(1)=\frac{1+a}{a}$	$\theta=X,\ \gamma=0;$ $X^{T-1}(1-\varepsilon)<\frac{1+a}{a}<X^{T-1}(1)$	same as $z>z_3$
$\in(\frac{1+a}{2+a-\varepsilon}, 1)$	same as above	$\theta=\frac{(1+a)(1+X)}{(1+2a-\varepsilon)}\quad \gamma=\theta-X$ $X^{T-1}(1-\varepsilon)=(\frac{1+a}{a})$	$\theta=X,\ \gamma=0$ $X^{T-1}(1-\varepsilon)>\frac{1+a}{a}$
$\in(1,\frac{2+a-\varepsilon}{1+a})$	same as above	same as above	$\theta=1,\ \gamma=(X-1)$ $X^{T-1}(1-\varepsilon)>(\frac{1+a}{a})$
$\in(\frac{2+a-\varepsilon}{1+a}, \frac{2+a}{1+a})$	same as above	$\theta=1,\ \gamma=X-1$ $X^{T-1}(1-\varepsilon)<\frac{1+a}{a}<X^{T-1}(1)$	same as $z>z_3$
$\in(\frac{2+a}{1+a}, \frac{2+a+\varepsilon}{1+a})$	$\theta=1,\ \gamma=X-1$ $X^{T-1}(1)<\frac{1+a}{a}<X^{T-1}(1+\varepsilon)$	same as $z<z_3$	same as $z<z_3$
$>\frac{2+a+\varepsilon}{1+a}$	$\theta=1,\ \gamma=[\frac{1+\varepsilon+a(X-1)}{(1+2a)}]$	same as $z<z_3$	same as $z<z_3$

Where $Z_3-(1+2a)+z[3+2a+3\varepsilon+8a\varepsilon]-2\varepsilon z^2[3+2a-\varepsilon]=0$

the economy is productive, it will be desirable to start some extra projects (beyond the certainty solution) in order to be able to utilize next period's output, should productivity turn out to be high ($\alpha=(1+\varepsilon)$). Thus, while the uncertainty places less emphasis on new starts at (T-2) if X^{T-2} lies below the VonNeuman ray $(\frac{2+a}{1+a})$, more emphasis is placed on these projects once all existing projects are fully allocated resources.[8]

The optimal solution for z>z* will be qualitatively similar to that sketched above, with the exception that the planner must also decide what target to aim for at (T-1)-ie, should he steer the economy towards $X^{T-1} = (\frac{1+a-\varepsilon}{a})$, or $X^{T-1} = (\frac{1+a}{a})$. As the uncertainty affects the resources allocation at (t + 1) the number of potential targets proliferates. The optimal rule for allocating resources at (T - 2) is still given by equations (22) - (23); further, if $X^{T-2} < 1$, it is clear not all projects can be completed, and hence $\lambda_1 = 0$. As in the previous case, for "small" $X^{T-2}(< \frac{1+a-\varepsilon}{1+2a-\varepsilon})$, $N = 0$, $\hat{P} = F/a$; i.e.,

$$(36) \quad E[\alpha J_P^{T-1} - 2J_P^{T-1}] > 0 \text{ if } X^{T-1}(1) < (\frac{1+a-\varepsilon}{a}).$$

If $X^{T-2} > (\frac{1+a-\varepsilon}{1+2a-\varepsilon})$, the planner can achieve the target $X^{T-1}(1) = (\frac{1+a-\varepsilon}{a})$. The essential question now becomes whether the economy should be "steered" towards $(\frac{1+a-\varepsilon}{a})$, or $(\frac{1+a}{a})$; and also whether the planner should "concentrate" on the average value of X^{T-1}, or the worst outcome. In essence, the optimal rule can be derived by evaluating $E[\alpha J_F - 2J_p]$ in the various regions; (I) $X^{T-1}(1-\varepsilon)<(\frac{1+a-\varepsilon}{a}) < X^{T-1}(1) < (\frac{1+a}{a})$; (II) $X^{T-1}(1-\varepsilon) < (\frac{1+a-\varepsilon}{a}) < \frac{1+a}{a} < X^{T-1}(1)$; (III) $(\frac{1+a+\varepsilon}{a}) < X^{T-1}(1-\varepsilon) < (\frac{1+a}{a}) < X^{T-1}(1)$; and (IV) $X^{T-1}(1-\varepsilon) > (\frac{1+a}{a})$.

To say the least, this is a rather tedious job, and the precise solution depends on the values of (z, ε, and a). From Table I, after some simplification:

(37) $E[\alpha J_F - 2J_P] \gtrless 0$ as $\phi_I \equiv [2(1+a) - z\{8(1+a) - 2\varepsilon + \frac{(1-\varepsilon)(1+6a-\varepsilon)}{1+2a-\varepsilon}\} + 4z^2(3+2a-\varepsilon)] \gtrless 0$

for region I.

(38) $E[\alpha J_F - 2J_P] \gtrless 0$ as $\phi_{II} = [-1 + z(2 + 4\varepsilon + \frac{(1-\varepsilon)(1-2a-\varepsilon)}{1+2a-\varepsilon}) - 4\varepsilon z^2] \gtrless 0$

for region II.

(39) $E[\alpha J_F - 2J_P] \gtrless 0$ as $\phi_{III} = [-1 + z(3+2a+3\varepsilon) - 2z^2(3+2a-\varepsilon)] \gtrless 0$

for region III.

(40) $E[\alpha J_F - 2J_P] \gtrless 0$ as $4\varepsilon z - 1 \gtrless 0$

for region IV.

Once (37)-(40) are evaluated, the solution can be determined. For example, if $\phi_I < 0$, the planner aims the economy so that $X^{T-1}(1) = (\frac{1+a-\varepsilon}{a})$; if $\phi_I > 0$, $\phi_{II} < 0$, then the target is $X^{T-1}(1) = (\frac{1+a}{a})$; if ϕ_I; $\phi_{II} > 0$, $\phi_{III} < 0$, $X^{T-1}(1-\varepsilon) = (\frac{1+a-\varepsilon}{a})$; and so on. Note that $\frac{\partial}{\partial\varepsilon}[E(\alpha J_F - 2J_P)] > 0$--i.e., the larger the disturbance, the greater the incentive to allocate resources to completing existing projects at the expense starting new ones.

To illustrate, consider the following example. We set $a = .4$, $\lambda = .158$ and consider the solution for $\varepsilon = 0$, $\varepsilon = .2$, and $\varepsilon = .6$ for various values of z. The first case corresponds to certainty. In the second case, given the worst outcome, all investment is recovered, though no net output is obtained. In the final case, only current investment, but not prior investment, can be recovered under the worst outcome.

Figure 2 plots the optimal γ^{T-2} as a function of X^{T-2} in the case of certainty. Note that for $X > X_U$, $\theta = 1$ and there is less than full utilization of resources.[9]

Figures 3 and 4 plot the cases $\varepsilon = .2$ and $\varepsilon = .6$ respectively. For all values of z, for $X^{T-2} \leq X_U^{T-2}$, more projects are started than under certainty, the increase being more pronounced as ε increases and as z increases. For all values of z, the zone of full resource utilization extends beyond X_U^{T-2}, except for the case $z > .404$, $\varepsilon = .2$ where the optimal plan aims at $\frac{1+a-\varepsilon}{a}$ under the worst outcomes. Further, except for this case, more new starts are observed past X_U^{T-2} than under certainty. For $\varepsilon = .2$, an increase in z usually decreases γ^{T-2}; but for $\varepsilon = .6$, an increase in z increases γ^{T-2}.

Several inferences can be drawn from the results of this section. First, it seem apparent that the larger the size of the disturbance, the more desirable it is to finish existing projects before newer ones are started. Secondly, in most cases it appears that mean preserving spreads have the same effect. If the probability of deviating from "normal" productivity is small, then it will not affect the resource allocation. However, as z increases, it again becomes more important to finish current projects. Finally, the smaller a (the larger the growth rate of the system) the greater is the importance of completing existing projects. Thus, although uncertainty affects the fine detail of the optimal plan, the features of target seeking and the sequence (update old projects, start new projects, less than fully utilize resources) as a function of relative endowment of the economy persists.

Figure 2. Certainty Solution, t = T - 2.

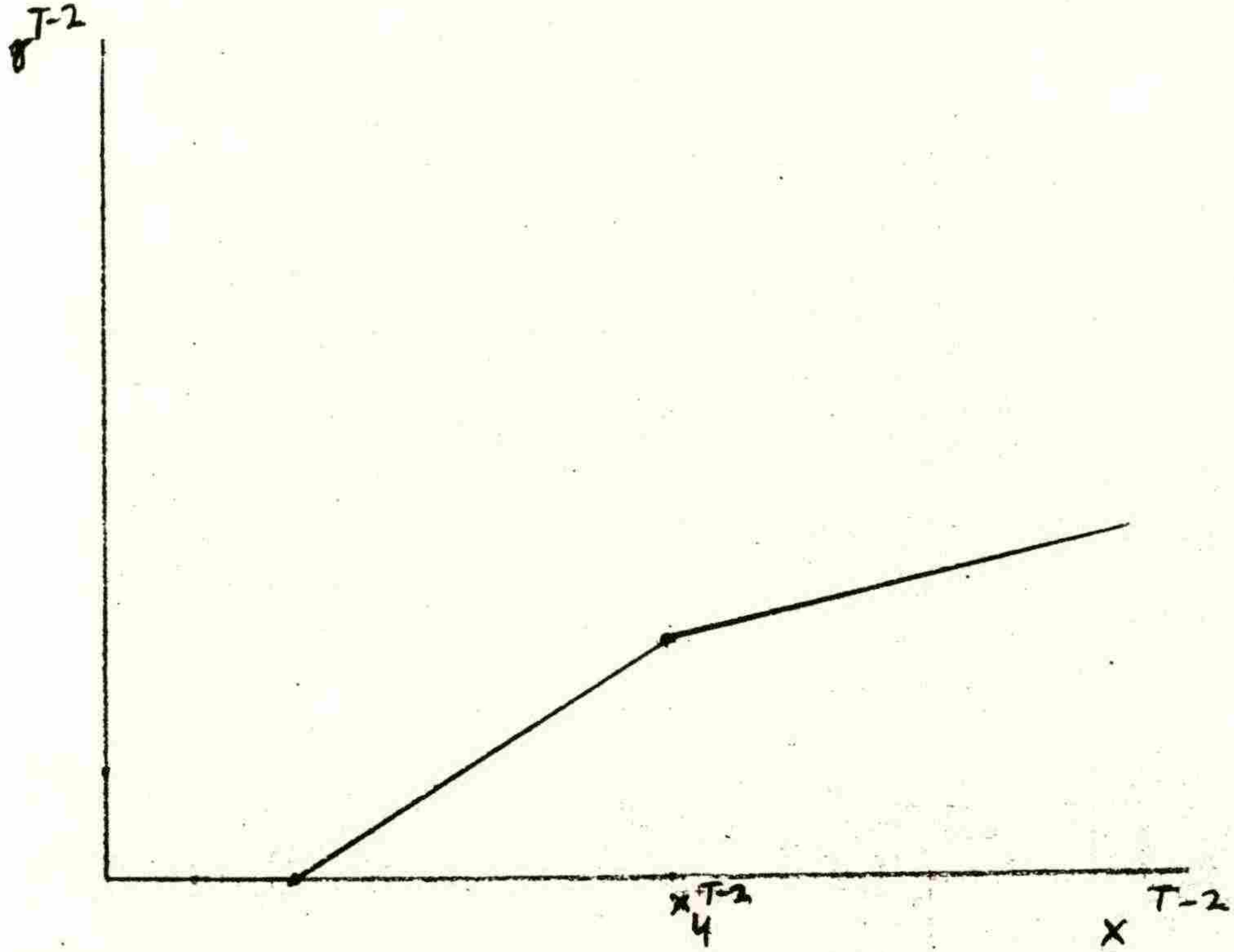
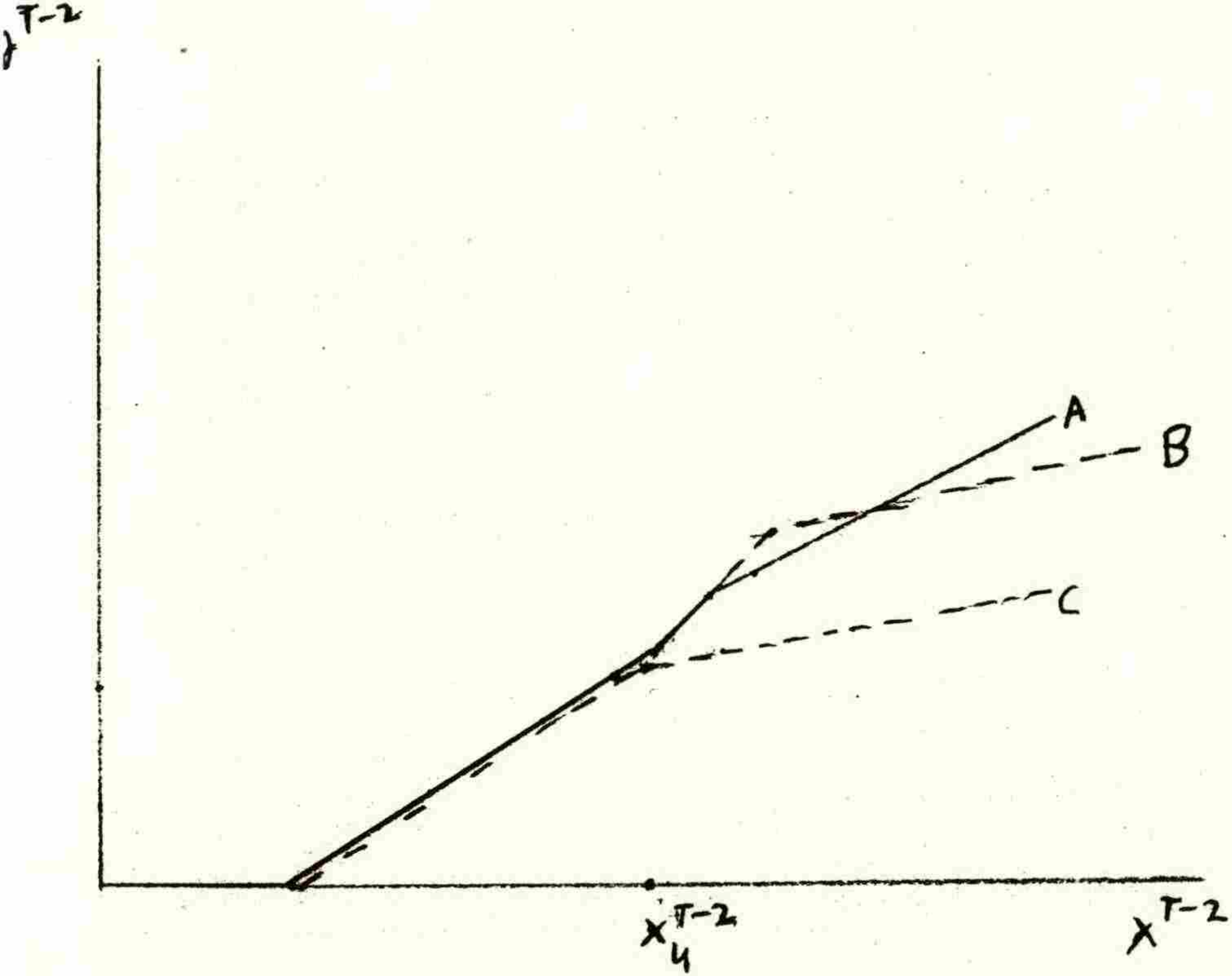
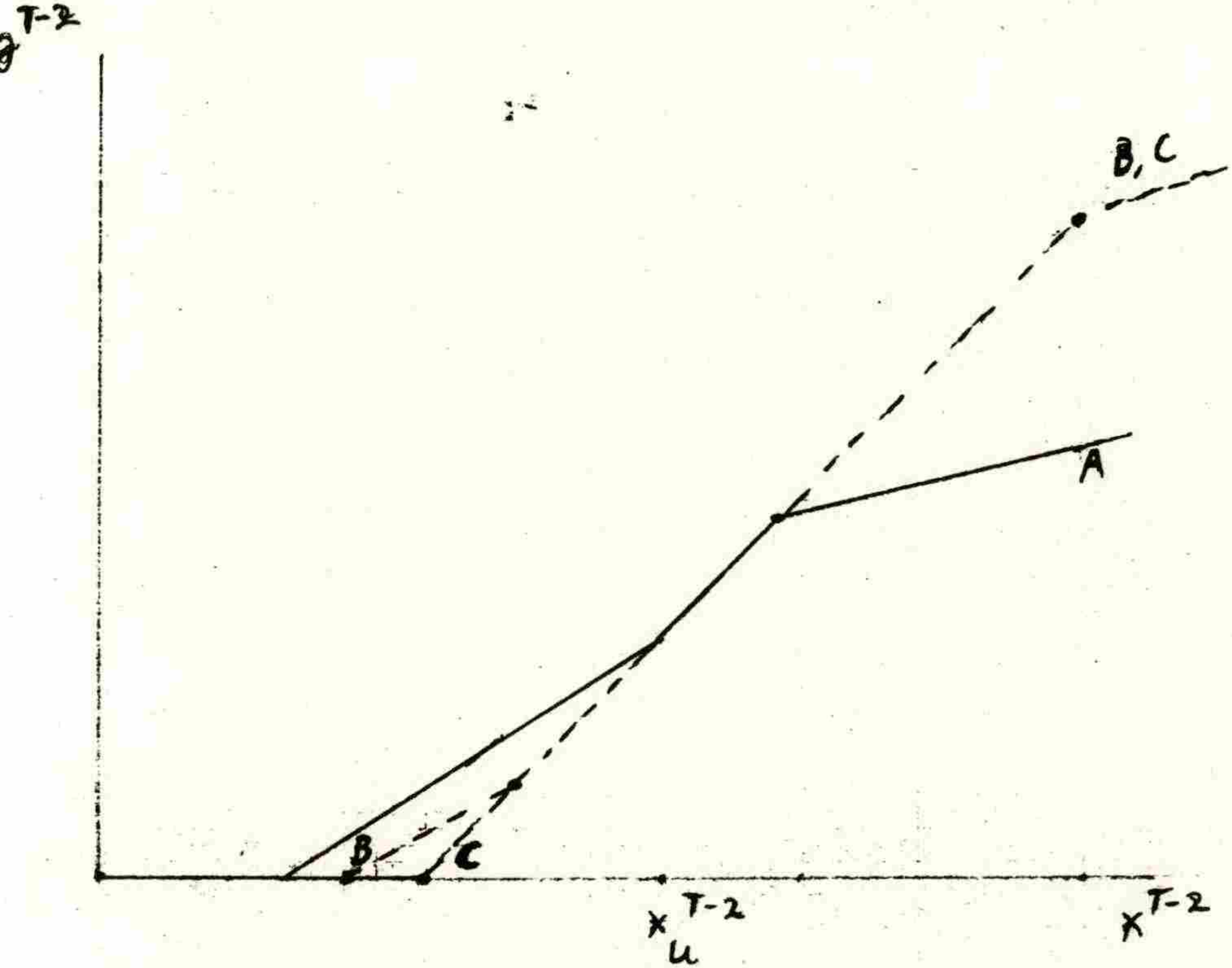


Figure 3. Uncertainty Solution, ϵ = .2



A : $0 < z < .25$
B : $.25 < z < .404$
C : $.404 < z < .5$

Figure 4. Uncertainty Solution, $\epsilon = .6$



A: $0 < z < .333$

B: $.333 < z < .417$

C: $.417 < z < .5$

Footnotes

1. A continuous random variable with mean one leads to qualitatively similar results. The value function, as before, will exhibit constant returns to scale in its arguments.
2. Recall the notation convention of (19).
3. While specifying a different terminal value for P^{T+1} would alter this conclusion, it would not appreciably affect the solution for previous periods, provided $J_F^{T+1} > 2J_P^{T+1}$.
4. If $1 - 2a = 0$, so that the growth rate is zero, it is apparent that nothing can be gained by production, so that the optimal solution degenerates to $\hat{P} = N = 0$.
5. In general, for the continuous distribution, the effect of uncertainty at T-1 is to increase $\hat{P}$ and decrease N.
6. In the discrete case, the increase in z has no effect on resource allocation, provided that $z - z^*$ is one signed. However, for the continuous case one would expect that an increase in z would increase $\hat{P}$.
7. It will always be optimal to complete some existing projects (for a plan of any length) since $\hat{P} = 0$ implies $N = F/a$, $X^{t+1} = 0$, and $E[\alpha J_F - 2J_P] > 0$, a contradiction.
8. The results for $\varepsilon > \frac{1+2a}{(1+a)^2}$ will differ from those cited in the text only to the extent that, for $X^{T-2} = (\frac{1+a}{2+a})$, $N = 0$, $\hat{P} = F/a$, $X^{T-1}(1-\varepsilon) < (\frac{a}{1+a})$, and hence more emphasis will be placed on finishing existing projects before starting new ones.
9. The optimal policy in this case is not unique. The policy depicted aims at X_U^{T-1}. All other optimal policies agree on $0 \leq X \leq X_U^{T-2}$, and aims for the interval (X_L^{T-1}, X_U^{T-1}) for $X^{T-2} > X_U^{T-2}$, thus leading to even less utilization of resources.

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