FRADKIN, David Milton, 1931–
SCATTERING WAVE FUNCTION FOR A DIRAC PARTICLE.

Iowa State University of Science and Technology
Ph.D., 1963
Physics, general

University Microfilms, Inc., Ann Arbor, Mich.
SCATTERING WAVE FUNCTION FOR A DIRAC PARTICLE

by

David Milton Fradkin

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

Major Subject: Physics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1963
<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.  INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. THE COULOMB WAVE FUNCTION IN THE JOHNSON-DECK FORM</td>
<td>5</td>
</tr>
<tr>
<td>A. Notation</td>
<td>5</td>
</tr>
<tr>
<td>B. Wave Function and Differential Equations</td>
<td>6</td>
</tr>
<tr>
<td>C. Cross Section for Potential Scattering</td>
<td>12</td>
</tr>
<tr>
<td>D. Polarization of Scattered Wave</td>
<td>20</td>
</tr>
<tr>
<td>III. GREEN'S FUNCTION FORMULATION</td>
<td>23</td>
</tr>
<tr>
<td>A. Sommerfeld-Maue Approximation</td>
<td>23</td>
</tr>
<tr>
<td>B. Green's Function Solution</td>
<td>25</td>
</tr>
<tr>
<td>IV. APPROXIMATION FOR ASYMPTOTIC WAVE FUNCTION</td>
<td>36</td>
</tr>
<tr>
<td>A. Asymptotic Wave Function</td>
<td>36</td>
</tr>
<tr>
<td>B. Potential Scattering Results</td>
<td>41</td>
</tr>
<tr>
<td>V. LIMITING CASES</td>
<td>44</td>
</tr>
<tr>
<td>A. Small Born Parameter</td>
<td>44</td>
</tr>
<tr>
<td>B. Large Born Parameter</td>
<td>45</td>
</tr>
<tr>
<td>VI. REFERENCES</td>
<td>57</td>
</tr>
<tr>
<td>VII. ACKNOWLEDGEMENTS</td>
<td>59</td>
</tr>
<tr>
<td>VIII. APPENDIX A</td>
<td>60</td>
</tr>
<tr>
<td>A. Confluent Hypergeometric Functions</td>
<td>60</td>
</tr>
<tr>
<td>IX. APPENDIX B</td>
<td>64</td>
</tr>
<tr>
<td>A. Coulomb Integral</td>
<td>64</td>
</tr>
</tbody>
</table>
X. APPENDIX C

A. Asymptotic Expansion of $S(\theta, v)$
   1. Integral transformation 73
   2. Born parameter greater than zero 75
   3. Born parameter less than zero 81

B. Asymptotic Expansion of $T(\theta, v)$ 84
I. INTRODUCTION

The traditional way developed by Darwin (1) and Mott (2, 3) to attack the problem of an unbound Dirac particle in a Coulomb field is to obtain an eigenfunction of the Dirac Hamiltonian corresponding to the desired energy which is also a simultaneous eigenfunction of a component $j_z$ of the angular momentum and the Dirac operator $K$ (4, p. 268) whose eigenvalue specifies the total angular momentum and the parity. One may then express the scattering wave function as an expansion in terms of these angular momentum eigenfunctions, where the expansion coefficients are chosen so that for large distances $r$ the wave function behaves like a plane wave plus an outgoing spherical wave. In this way, one obtains the exact scattering wave function in the form of an infinite series. Although the series has been summed numerically for certain ranges of energy and interaction parameters (5-8), in general the series converges slowly, and for analytical manipulations one must resort to an approximation.

Various approximations have been developed for the scattering wave function in a Coulomb field. One of these is the Born approximation which is based on the iteration of a plane wave as the zero order solution in a Green's function formulation of the problem. This approximation gives useful results for high energies where the criterion $\nu = aZe/p < 1$ is satis-
fied. Analytical expressions for various orders in the Born approximation have been developed by Dalitz (9), Gursey (10), and Mitter and Urban (11), among others.

Another type of approximation that has been used is the Sommerfeld-Maue (12, 13) approximation which incorporates the interaction in analogy to the separation of the non-relativistic Schroedinger problem in parabolic coordinates. Modifications of this approximation have been developed by Bethe and Maximon (14), and Johnson et al. (15, 16), who used a Green's function constructed by Meixner (17). The range of validity of this approximation is generally taken to be $\alpha Z/(E \ln E) < 1$.

A third type of approximation used in potential scattering is based on a direct $\alpha Z$ expansion of the exact Mott series. Expansions in increasing order of $\alpha Z$ have been given by Mott (3), McKinley and Feshbach (6), and Johnson et al. (16). The general criteria for these expansions is $\alpha Z < 1, \nu < 1$.

Recently, Johnson and Deck (18) reorganized the infinite Mott series so that it explicitly had the form of an operator acting on a plane wave spinor. The appearance of the plane wave spinor is especially useful in the calculation of matrix elements. However, the operator contains three functions which are expressible by an infinite series, so again an approximation (18, 19) of the series must be made.

In this thesis, a wave function of the Johnson and Deck form is assumed, and the requirement that it is an eigenfunc-
tiation of the Hamiltonian yields a second order partial differential equation for a single function which in turn determines the wave function. The three functions of Johnson and Deck are derivable from this single function. Thus, the problem of finding a scattering wave function is reduced to the non-matrix problem of finding a solution of a single partial differential equation satisfying the asymptotic boundary condition. The cross section for potential scattering and also the polarization of the incident plane wave is exhibited for a wave function in the Johnson and Deck form.

The Sommerfeld-Maue approximation follows naturally as a zero order approximation of the solution of the partial differential equation. An exact Green's function solution is developed, which may then be iterated using the Sommerfeld-Maue approximation as the zero order solution.

As a particular application of this approximation, the asymptotic form of the Coulomb scattering wave function obtained from the first iteration is determined and the relevant potential scattering results are obtained. Here, the wave function is correct to and including \((\alpha Z)^3\) for all values of \(v\), where \(\alpha Z\) and \(v\) are considered as independent parameters. These results are then examined for the limiting cases of small and large Born parameters \(v\). For small \(v\), agreement is obtained with Johnson et al. (16), whereas the large \(v\) expansion yields results not available in the literature. Graphs
of functions relevant to potential scattering are presented for the case of the large Born parameter.
II. THE COULOMB WAVE FUNCTION IN THE JOHNSON-DECK FORM

A. Notation

The Dirac equation for a central potential is \( H\psi = E\psi \)
where
\[
H = -i\vec{\alpha} \cdot \vec{\nabla} + \vec{\beta} + \lambda V(r)
\]
Here, units are used for which \( \hbar = m = c = 1 \). The strength of the interaction is measured by \( \lambda \). For a Coulomb field, \( \lambda = (\alpha Z) \), where \( \alpha \) is the fine structure constant and \( V(r) = -1/r \) so that for \( \lambda > 0 \), the potential is an attractive one. The Born parameter \( \nu \) is defined by \( \nu = \lambda E/p \).

The symbols \( \vec{\alpha} \) and \( \vec{\beta} \) represent the usual Dirac matrices and are given in terms of the Hermitian Dirac matrices \( \gamma_\mu \) by the relations:
\[
\vec{\alpha} = i\gamma_4\vec{\gamma}, \quad \vec{\beta} = \gamma_4.
\]

A set of Dirac matrices that will be referred to is:
\[
\vec{\sigma} = i\gamma_4\gamma_5\vec{\gamma},
\]
\[
\rho_1 = -\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4,
\]
\[
\rho_2 = -i\gamma_1\gamma_2\gamma_3,
\]
\[
\rho_3 = \gamma_4.
\]

These have the convenient algebra:
\[
\sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k ; \\
\rho_i \rho_j = \delta_{ij} + i \varepsilon_{ijk} \rho_k , \\
\sigma_i \rho_j = \rho_j \sigma_i ,
\]

where latin indices range from 1 to 3 and the summation convention is employed. In terms of these, sixteen independent Dirac matrices are 1, \( \overline{P} \), \( \overline{\sigma} \) and \( \rho_i \sigma_j \).

The symbols \( A^\ast \) and \( A^+ \) denote the complex and Hermitian conjugate of any matrix \( A \), respectively. The caret \( \hat{p} \) indicates a unit vector, and the vector cross product is given typically by the notation \( \vec{a} \times \vec{b} \).

B. Wave Function and Differential Equations

A scattering wave function

\[
\psi = D(E, \hat{r}, \hat{p}) U(\hat{p}) , \tag{2.1}
\]

is postulated that has the Johnson-Deck form where the operator \( D \) is defined by

\[
D(E, \hat{r}, \hat{p}) = G + i \lambda M \cdot (\hat{p} - \hat{r}) + i L (\hat{p} \cdot \hat{r}) \cdot \overline{\sigma} . \tag{2.2}
\]

Here, \( U(\hat{p}) \) is a free particle plane wave spinor for positive energy \( E \) and arbitrary polarization, \( \hat{p} \) is a unit vector in the direction of the asymptotic plane wave, and \( G, M, \) and \( L \) are unspecified functions.

Since the wave function is postulated to be an eigenfunction of \( H \), it must satisfy the relation
(H-E)D(E, \hat{r}, \hat{p})U(\hat{p}) = 0 . \tag{2.3}

As a consequence it follows that

\[ \text{trace} \left[ \gamma_A (H-E)D(E, \hat{r}, \hat{p})P_+(E, \hat{p}) \right] = 0 , \tag{2.4} \]

where \( \gamma_A \) is any one of the sixteen independent Dirac matrices, and \( P_+(E, \hat{p}) \) is the free particle positive energy projection operator given by

\[ P_+(E, \hat{p}) = \frac{1}{2E} (\hat{p} \cdot \hat{p} + \beta + E) . \tag{2.5} \]

The usual relativistic connection relates \( p \) and \( E \), namely

\[ p^2 + 1 = E^2 . \]

Letting \( \gamma_A \) be successively the matrices \( 1, \rho, \bar{\sigma}, \rho_1 \bar{\sigma}, \rho_2 \bar{\sigma}, \rho_3 \bar{\sigma} \), Eq. 2.4 yields sixteen equations. Resolving the vector equations in the directions \( \hat{r}, \hat{p}, \) and \( \hat{r} \cdot \hat{p} \), and using the cartesian coordinate system defined by

\[ x = r \sin \theta \cos \phi , \]
\[ y = r \sin \theta \sin \phi , \tag{2.6} \]
\[ z = r \cos \theta , \]

where the angle \( \theta \) is defined by

\[ \cos \theta = \hat{r} \cdot \hat{p} , \tag{2.7} \]

it is found that only seven of the equations are independent. These are:
\[ \frac{\partial G}{\partial \phi} = 0; \]  
\[ \frac{\partial M}{\partial \phi} = 0; \]  
\[ \frac{\partial L}{\partial \phi} = 0; \]  
\[ G = e^{i pr} \left[ \frac{1}{v} \frac{3}{3 r} + \frac{2}{v r} + \frac{\sin \theta}{v r} \frac{3}{3 \theta} \right] e^{-i pr M}; \]  
\[ L = e^{i pr} \left[ \frac{1}{v} \frac{3}{3 r} - \frac{(1 - \cos \theta)}{v r \sin \theta} \frac{3}{3 \theta} \right] e^{-i pr M}; \]  
\[ \left[ \frac{3}{3 r} - \frac{\sin \theta}{r(1 + \cos \theta)} \frac{3}{3 \theta} \right] e^{-i pr G} = \]  
\[ \left[ (1 - \cos \theta) \frac{3}{3 r} + \frac{\sin \theta}{r} \frac{3}{3 \theta} + \frac{(1 + \cos \theta)}{r} \right] e^{-i pr L}; \]  
\[ \left[ \cos \theta \frac{3}{3 r} - \frac{1}{r} \sin \theta \frac{3}{3 \theta} + i \nu \nu - i p \right] G \]  
\[ + \left[ -\sin^2 \theta \frac{3}{3 r} + \frac{1}{r} (\sin^2 \theta - 2) - \frac{1}{r} \sin \theta \cos \theta \frac{3}{3 \theta} \right] L \]  
\[ - \left[ i \nu \left( (1 - \cos \theta) \frac{3}{3 r} + \frac{2}{r} + \frac{1}{r} \sin \theta \frac{3}{3 \theta} \right) \right] M = 0. \]  

Upon substitution of Eqs. 2.11 and 2.12 into Eqs. 2.13 and 2.14, the last two equations may be replaced by

\[ \left[ \frac{3}{3 r} \left( \frac{1}{r \nu} \right) \right] \frac{3}{3 \theta} (1 - \cos \theta) M = 0, \]  
and
\[
\begin{align*}
\frac{r^2 \partial^2 \theta}{\partial r^2} + 3r \frac{\partial \theta}{\partial r} + \left( \frac{1+2 \cos \theta}{\sin \theta} \right) \frac{\partial \theta}{\partial \theta} \\
+ p^2 r^2 - (1 + 2 \nu \nu) \nu \nu + (\lambda \nu \nu)^2 \\
+ r^2 (\frac{\partial}{\partial r} \ln r \nu) \{ - \frac{\partial}{\partial r} + i \nu + \frac{2 \cos \theta}{r (1-\cos \theta)} + \frac{\sin \theta}{r (1-\cos \theta)} \frac{\partial}{\partial \theta} \} \\
- \nu \nu (\nu \nu-1) \nu \{ r \frac{\partial}{\partial \nu} + \frac{\sin \theta}{(1-\cos \theta)} \frac{\partial}{\partial \theta} + \frac{2}{(1-\cos \theta)} \} \\
\end{align*}
\]
\[
M = 0.
\]

For \( \nu \nu \) not equal to a constant, Eq. 2.15 indicates that
\( M = (1 - \cos \theta)^{-1} f(r) \), but by direct substitution, it is
easily seen that this form is incompatible with Eq. 2.16.
Thus the assumption of the Johnson-Deck form is valid only for
\( \nu \nu \) proportional to \( 1/r \). Therefore, the subsequent discussion
will deal only with the Coulomb field.

It is seen that Eq. 2.15 is now identically satisfied.

Defining
\[
a = \cos \theta,
\]
\[
b = \nu \nu \left( \frac{1}{2} + i \nu \right),
\]
the three independent equations specifying the \( r \) and \( a \)
dependence of \( G, L, \) and \( M \) become:

\[
G = e^{ipr} \left[ (a-1) r \frac{\partial}{\partial r} + (1-a^2) \frac{\partial}{\partial a} - 2 \right] e^{-ipr} M;
\]
\[
L = e^{ipr} \left[ -r \frac{\partial}{\partial r} + (a-1) \frac{\partial}{\partial a} \right] e^{-ipr} M;
\]
\[
\begin{bmatrix}
\frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left\{ (1-a^2) \frac{\partial^2}{\partial a^2} - (1+3a) \frac{\partial}{\partial a} \right\} \\
- \frac{2ib}{r} + p^2 + \frac{\lambda^2}{r^2}
\end{bmatrix}
\]

so that the problem has been reduced to a determination of \( M \) alone. This result was not previously known.

The partial differential equation is separable in \( r \) and \( \theta \), and the solution satisfying the asymptotic boundary condition is an expansion in the eigenfunctions having the separation parameter \( k = 1, 2, 3, \ldots \). In agreement with Johnson and Deck (18), this is given by:

\[
M = \sum_{k=1}^{\infty} C(k, \gamma, \nu) \gamma^{\nu-1} e^{-\chi/2} \ {}_1F_1(\gamma-\nu, 2\gamma+1, \chi) \left[ P'_k(a) + P'_k(a) \right],
\]

where

\[
C(k, \gamma, \nu) = (-1)^k \frac{\Gamma(\gamma - \nu)}{\Gamma(2\gamma + 1)} e^{\frac{\gamma \pi i}{2}},
\]

\[
\gamma = \left[ k^2 - \lambda^2 \right]^{1/2},
\]

\[
\chi = -2ipr .
\]

Here, \( {}_1F_1 \) is the confluent hypergeometric function, \( P'_k \) is the derivative of the Legendre polynomial of order \( k \) with respect to its argument.

In terms of a parabolic coordinate system defined by
\[ \zeta_1 = ipr(l + a), \quad \zeta_2 = ipr(l - a), \quad (2.23) \]

Eqs. 2.19 to 2.21 become

\[ G = (\zeta_2 - 2 - 2\zeta_2 \frac{\partial}{\partial \zeta_2})M; \quad (2.24) \]

\[ L = (\zeta_1 + \zeta_2)(\frac{1}{2} - \frac{\partial}{\partial \zeta_1})M; \quad (2.25) \]

\[ \left[ \zeta_1 \frac{\partial^2}{\partial \zeta_1^2} + \zeta_2 \frac{\partial^2}{\partial \zeta_2^2} + 2 \frac{\partial}{\partial \zeta_2} - \frac{1}{4}(\zeta_1 + \zeta_2) \right. \]

\[ - \frac{b}{p} + \frac{\lambda^2}{(\zeta_1 + \zeta_2)} \] \[ \left. \right] M = 0. \quad (2.26) \]

The function \( M \) is not separable in parabolic coordinates due to the term in \( \lambda^2 \). Later it will be shown that neglect of \( \lambda^2 \) gives the Sommerfeld-Maue approximation.

A convenient change of dependent variable that will be used later is

\[ Q = \left[ r(l - a) \right]^{1/2} M. \quad (2.27) \]

The second order partial differential operator now becomes self-adjoint (see Section III-B) and

\[ (Q(\bar{r}) + p^2 + \frac{\lambda^2}{r^2})Q = 0, \quad (2.28) \]

where

\[ Q(\bar{r}) = \frac{1}{r^2} \left[ \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\partial}{\partial a}(1-a^2) \frac{\partial}{\partial a} - \frac{1}{2(l-a)} - 2ibr \right]. \quad (2.29) \]
When acting on functions independent of $\phi$, $0(\vec{r})$ may be written as

$$0(\vec{r}) = \nabla^2 - \left( \frac{1}{2r^2(1-a)} + \frac{2ib}{r} \right), \tag{2.30}$$

where $\nabla^2$ is the Laplacian. Thus, Eq. 2.28 is a Schrödinger equation, but the "potential" is such that it is separable only in spherical coordinates (21).

C. Cross Section for Potential Scattering

For large $r$ the asymptotic form of the Johnson-Deck wave function, that is required to satisfy the asymptotic boundary condition of a plane wave plus an outgoing spherical wave, is given by

$$\psi = e^{i\vec{F} \cdot \vec{r}} U(\hat{\xi}_1, \hat{p}) + \frac{e^{ipr}}{r} D_s(E, \hat{F}, \hat{p}) U(\hat{\xi}_1, \hat{p}), \tag{2.31}$$

where

$$D_s(E, \hat{F}, \hat{p}) = G_s + i\lambda M_s \hat{a} \cdot (\hat{p} - \hat{F}) + iL_s(\hat{p} \cdot \hat{F} \cdot \vec{r}) \tag{2.32}$$

and the incident plane wave has been assigned the polarization direction $\hat{\xi}_1$. This means that the incident plane wave satisfies

$$\{\vec{O}(\hat{p}) \cdot \hat{\xi}_1\} U(\hat{\xi}_1, \hat{p}) = U(\hat{\xi}_1, \hat{p}) \tag{2.33}$$

where the three-vector polarization operator (22) $\vec{O}(\hat{p})$ is given by

$$\vec{O}(\hat{p}) = \vec{\sigma} + \vec{\sigma} \cdot (1 - \vec{\sigma}) \hat{p} \tag{2.34}$$
In the following, the dependence of $D_S$ on $E$ will not be explicitly stated, since the analysis will deal with elastic scattering so only one definite value of energy $E$ will be involved. Also, the convenient notation

$$\vec{P}_1 = \vec{P}, \quad \vec{P}_2 = p\hat{r},$$

is adopted.

Asymptotically, for a given outgoing direction $\hat{\psi}_2$, the scattered wave is itself proportional to a plane wave spinor (of positive energy) traveling in the $\hat{r}$ direction. This means that

$$D_S(\hat{\psi}_2, \hat{\psi}_1) U(\hat{\psi}_2, \hat{\psi}_1) = c(\theta) U(\hat{\psi}_2, \hat{\psi}_1),$$

where $c(\theta)$ is some function of $\theta$, and $\hat{\psi}$ is a definite polarization direction related to $\hat{\psi}_1$ that will be discussed in section II-D. However, if the scattered wave is analyzed not for polarization direction $\hat{\psi}_2$, but is analyzed for arbitrary direction of polarization $\hat{r}$, then the component of the scattered wave is

$$C(\hat{\psi}_2, \hat{\psi}_1; \hat{\psi}_2, \hat{\psi}_1) U(\hat{\psi}_2, \hat{\psi}_1),$$

where

$$C(\hat{\psi}_2, \hat{\psi}_1; \hat{\psi}_2, \hat{\psi}_1) = U^+(\hat{\psi}_2, \hat{\psi}_1) D_S(\hat{\psi}_2, \hat{\psi}_1) U(\hat{\psi}_2, \hat{\psi}_1).$$

Using the fact that

$$U^+(\hat{\psi}, \hat{p}) \hat{a} \cdot \hat{p} U(\hat{\psi}, \hat{p}) = p/E,$$
where $\vec{a} \cdot \vec{p}$ is the current operator in the $\hat{p}$ direction, it is easily shown that the differential cross section for potential scattering from the initial state $U(\hat{\xi}_1, \hat{\rho}_1)$ to the final scattered state $U(\hat{\xi}_2, \hat{\rho}_2)$ is given by

$$d\sigma(\hat{\xi}_2, \hat{\rho}_2; \hat{\xi}_1, \hat{\rho}_1)/d\Omega = |\sigma(\hat{\xi}_2, \hat{\rho}_2; \hat{\xi}_1, \hat{\rho}_1)|^2. \quad (2.40)$$

If the free particle Hamiltonian is defined by

$$H_n = \vec{a} \cdot \vec{p}_n + \beta; \quad n = 1, 2, \quad (2.41)$$

and the relationships

$$\vec{a} \cdot (\hat{p}_1 - \hat{p}_2) = (1/p)(H_1 - H_2), \quad (2.42)$$

$$i\hat{p}_1 \cdot \hat{p}_2 \cdot \vec{a} = (p)^{-2}(\vec{p}_1 \cdot \vec{p}_2 + \beta H_1 + H_2 \beta - 1 - H_2 H_1) \quad (2.43)$$

are used, it is found that for elastic scattering

$$C(\hat{\xi}_2, \hat{\rho}_2; \hat{\xi}_1, \hat{\rho}_1) = U^+(\hat{\xi}_2, \hat{\rho}_2)(A_1 + \beta A_2)U(\hat{\xi}_1, \hat{\rho}_1), \quad (2.44)$$

where $A_1$ and $A_2$ are functions of $\Theta (\hat{p}_1 \cdot \hat{p}_2 = \cos \Theta)$ given by

$$A_1 = G_s - 2(p)^{-2}L_s(E^2 - p^2\cos^2\Theta/2), \quad (2.45)$$

$$A_2 = 2(p)^{-2}L_s E. \quad (2.46)$$

Inserting the appropriate projection operators and finding the differential cross section using standard trace techniques, one obtains the result in the standard form given by Tolhoek (23).
\[ \frac{\partial \sigma(\hat{r}_2, \hat{r}_1; \hat{r}_1 \hat{r}_1)}{\partial \Omega} = I_1(1 + \hat{r}_1 \cdot \hat{r}_2) - I_2(\hat{n} \cdot \hat{r}_1 + \hat{n} \cdot \hat{r}_2) \\
+ I_3(\hat{n} \cdot \hat{r}_2 \cdot \hat{r}_1) - I_4(\hat{n} \cdot \hat{r}_1)(\hat{n} \cdot \hat{r}_2). \]  

(2.47)

Here, \( \hat{n} \) is the normal direction to the scattering plane given by

\[ \hat{n} = (\hat{p}_2 \cdot \hat{p}_1)/|\hat{p}_2 \cdot \hat{p}_1|, \]  

(2.48)

and the four \( I \) functions, which are functions of \( \theta \) but independent of polarization directions, are given by the formulae:

\[ I_1 = A_1^* A_1 \left[ 1 - \frac{(p/E)^2 \sin^2 \theta}{2} \right] + A_2^* A_2 \left[ 1 - \frac{(p/E)^2 \cos^2 \theta}{2} \right] \]
\[ + \frac{2}{E} \Re A_1 A_2^*; \]  

(2.49)

\[ I_2 = \sin \theta \frac{(p/E)^2}{2} \Im A_1 A_2^*; \]  

(2.50)

\[ I_3 = \left( \cot \theta \right) I_4 + \frac{1}{2} \sin \theta \frac{(p/E)^2}{2} (A_1 A_1^* - A_2 A_2^*); \]  

(2.51)

\[ I_4 = \frac{1}{2} \sin^2 \theta \left[ \frac{(E-1)/E}{2} \right]^2 (A_1 A_1^* + A_2 A_2^* - 2 \Re A_1 A_2^*). \]  

(2.52)

These four functions are not independent but are related through the equation

\[ I_4^2 + I_3^2 - 2I_1 I_4 + I_2^2 = 0 \]  

(2.53)

which may be proven by direct substitution.

The functions \( I(\theta) \) given by Eqs. 2.49 to 2.53 may be determined directly from scattering experiments. In a single
scattering experiment with an initially unpolarized beam, the
differential cross section $d\sigma_I/d\Omega$ is obtained from Eq. 2.47 by
summing over final polarization directions and averaging over
the initial polarization directions. This procedure yields

$$d\sigma_I/d\Omega = I_1(\theta)$$  \hspace{1cm} (2.54)

If there is a second scattering (shielded from the direct
incident beam) a long distance $r_{12}$ from the first scatterer,
located at a polar direction $\theta$ with respect to the incident
beam direction $\hat{p}_1$ (where the first scatterer is the coordinate
origin), then the beam incident on the second scatterer is a
plane wave $(r_{12})^{-1} e^{i\mathbf{F}_2 \cdot \mathbf{r}'} D_s(\hat{p}_3, \hat{p}_1) U(\hat{p}_1, \hat{e}_1)$. Here the prime
sign refers to distance measured with respect to the second
scatterer. Substituting this incident wave in Eq. 2.31, and
following the same steps leading to Eq. 2.40, the differential
cross section for scattering from initial state characterized
by $\hat{e}_1, \hat{p}_1$ to a final state, analyzed after the second scat-
tering, characterized by $\hat{e}_3, \hat{p}_3$, is given by

$$d\sigma(\hat{e}_3, \hat{p}_3; \hat{e}_1, \hat{p}_1)/d\Omega' = (r_{12})^{-2} |C(\hat{e}_3, \hat{p}_3; \hat{e}_1, \hat{p}_1)|^2,$$

where

$$C(\hat{e}_3, \hat{p}_3; \hat{e}_1, \hat{p}_1) = U^+(\hat{p}_3, \hat{e}_3) D_s(\hat{p}_3, \hat{p}_2) D_s(\hat{p}_2, \hat{p}_1) U(\hat{p}_1, \hat{e}_1).$$  \hspace{1cm} (2.56)

This matrix element may be written
\[ \sigma(\hat{\xi}_3, \hat{p}_3; \hat{p}_2, \hat{\xi}_1, \hat{p}_1) = \sum_{\varepsilon, \lambda} U^+(\hat{p}_3, \hat{\xi}_3, \hat{p}_2, \hat{\xi}_2) D_{\varepsilon, \lambda}(\hat{p}_3, \hat{p}_2) U_1(\hat{p}_2, \hat{\xi}_2) \]
\[ \times U_1^+(\hat{p}_2, \hat{\xi}_2, \hat{p}_2, \hat{\xi}_1) U(\hat{p}_1, \hat{\xi}_1) \]  

(2.57)

where a complete set of states has been inserted so that \( \varepsilon = \pm 1 \), and the sum over \( \lambda \) refers to both positive and negative energy spinors. For convenience \( \hat{\xi}_2 \) is chosen as the polarization direction related to \( \hat{\xi}_1 \), as indicated by Eq. 2.36. Since \( D_{\varepsilon}(\hat{p}_2, \hat{p}_1) U(\hat{p}_1, \hat{\xi}_1) \) is proportional to \( U(\hat{\xi}_2, \hat{p}_2) \), the sum over \( \lambda \) and \( \varepsilon \) may be trivially performed, and the result is then

\[ \sigma(\hat{\xi}_3, \hat{p}_3; \hat{p}_2, \hat{\xi}_1, \hat{p}_1) = \sigma(\hat{\xi}_3, \hat{p}_3; \hat{\xi}_2, \hat{p}_2) \sigma(\hat{\xi}_2, \hat{p}_2; \hat{\xi}_1, \hat{p}_1), \]  

(2.58)

where the \( \sigma \)'s on the right are defined by Eq. 2.38. Thus, using Eqs. 2.40, 2.55, and 2.58, one finds that

\[ \frac{d\sigma(\hat{\xi}_3, \hat{p}_3; \hat{p}_2, \hat{\xi}_1, \hat{p}_1)}{d\Omega'} = \]
\[ \frac{1}{(r_{12})^2} \frac{d\sigma(\hat{\xi}_3, \hat{p}_3; \hat{\xi}_2, \hat{p}_2)}{d\Omega} \frac{d\sigma(\hat{\xi}_2, \hat{p}_2; \hat{\xi}_1, \hat{p}_1)}{d\Omega} \]  

(2.59)

The differential cross section \( d\sigma_{II}/d\Omega' \) for double scattering from an initially unpolarized beam is obtained by averaging over initial polarization directions (\( \pm \hat{\xi}_1 \)) and summing over final polarization directions (\( \pm \hat{\xi}_3 \)). The calculations are simplified by using the fact (proven in Section
II-D) that if $\mathbf{\hat{n}}_1$ is chosen as $\mathbf{\hat{n}} = (\mathbf{\hat{p}}_2 \cdot \mathbf{\hat{p}}_1) / |\mathbf{\hat{p}}_2 - \mathbf{\hat{p}}_1|$ (or $-\mathbf{\hat{n}}$), then $\mathbf{\hat{n}}_2$ is also $\mathbf{\hat{n}}$ (or $-\mathbf{\hat{n}}$). Substituting Eq. 2.47 into Eq. 2.59 and performing the sums and averages, one obtains the simple result

$$(r_{12})^2 d\sigma / d\Omega' = I_1(\theta)I_1(\theta') + I_2(\theta)I_2(\theta')(\mathbf{\hat{n}} \cdot \mathbf{\hat{n}}')$$

(2.60)

where $\cos \theta = \mathbf{\hat{p}}_1 \cdot \mathbf{\hat{p}}_2$, $\cos \theta' = \mathbf{\hat{p}}_2 \cdot \mathbf{\hat{p}}_3$, $\mathbf{\hat{n}}' = (\mathbf{\hat{p}}_3 \cdot \mathbf{\hat{p}}_1) / |\mathbf{\hat{p}}_3 - \mathbf{\hat{p}}_1|$.

Double scattering may also be analyzed with the formalism of density matrices, as is done, for example by Fano (24) and Rose (25), which yields identical results.

If two measurements of the double scattering are taken at the same angle but $180^\circ$ azimuthally apart with respect to $\mathbf{\hat{n}}_2$, as shown in Fig. 1, then $\mathbf{\hat{n}}'(1) = \mathbf{\hat{n}}'(2)$. Subtracting these two measurements and dividing by the sum of these two measurements, one obtains a measure of the asymmetry

$$I_2(\theta)I_2(\theta') [\mathbf{\hat{n}} \cdot \mathbf{\hat{n}}'(1)] / I_1(\theta)I_1(\theta') .$$

If the whole experiment is coplanar so $\mathbf{\hat{n}}$ is parallel to $\mathbf{\hat{n}}'(1)$, one obtains Mott's (3) definition of the asymmetry $\delta(\theta, \theta')$

$$\delta(\theta, \theta') = I_2(\theta)I_2(\theta') / I_1(\theta)I_1(\theta') .$$

(2.61)

The asymmetry parameter $\mathbf{\hat{G}}(\theta)$, defined by Sherman (8) is the angular function

$$\mathbf{\hat{G}}(\theta) = - I_2(\theta) / I_1(\theta) ,$$

(2.62)
Fig. 1. Double scattering asymmetry arrangement
which is the square root of $\delta(\theta, \theta')$ if the measurement is made so that $\theta$ equals $\theta'$.

With a knowledge of $I_1(\theta)$ from a single scattering experiment, the function $I_2(\theta)$ can be determined by measuring

$$\delta(\theta, \theta') = \left[ \frac{I_2(\theta)}{I_1(\theta)} \right]^2.$$  

Similarly, information regarding $I_3(\theta)$ may be obtained from a triple scattering experiment or from a double scattering experiment if the initial incident beam is prepared in a polarized state. The function $I_4(\theta)$ is related to the other three $I(\theta)$ functions by Eq. 2.53.

D. Polarization of Scattered Wave

For an incident plane wave with definite polarization $\hat{\mathbf{p}}_1$, the scattered wave in the direction $\hat{\mathbf{p}}_2$ has a definite output polarization direction $\hat{\mathbf{g}}_2$ which is a function of $\hat{\mathbf{g}}_1$. This output polarization direction may be determined from the condition that $d\sigma(\hat{\mathbf{g}}_2, \hat{\mathbf{p}}_2; \hat{\mathbf{g}}_1, \hat{\mathbf{p}}_1)/d\Omega$ is a maximum.

Using a set of basis vectors $\hat{\mathbf{p}}_1$, $\hat{\mathbf{n}} \cdot \hat{\mathbf{p}}_1$, and $\hat{\mathbf{n}}$ (where as before $\hat{\mathbf{n}} = (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{p}}_1)/|\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{p}}_1|$), $\hat{\mathbf{g}}_1$ and $\hat{\mathbf{g}}_2$ may be expressed as

$$\hat{\mathbf{g}}_1 = (\sin \phi_1 \cos \theta_1)\hat{\mathbf{p}}_1 + (\sin \phi_1 \sin \theta_1)(\hat{\mathbf{n}} \cdot \hat{\mathbf{p}}_1) + (\cos \phi_1)\hat{\mathbf{n}},$$

$$\hat{\mathbf{g}}_2 = (\sin \phi_2 \cos \theta_2)\hat{\mathbf{p}}_1 + (\sin \phi_2 \sin \theta_2)(\hat{\mathbf{n}} \cdot \hat{\mathbf{p}}_1) + (\cos \phi_2)\hat{\mathbf{n}}.$$  

(2.63)

Eq. 2.47 may then be written as
\[ 2 \frac{d\sigma(\hat{\mathbf{s}}_2, \hat{\mathbf{r}}_2; \hat{\mathbf{s}}_1, \hat{\mathbf{r}}_1)}{d\Omega} = \]

\[
\sin \phi_1 \sin \phi_2 \left[ (I_1 - I_4) \cos (\theta_1 - \theta_2) + I_3 \sin (\theta_1 - \theta_2) \right] \\
+ (1 + \cos \phi_1 \cos \phi_2) I_1 - (\cos \phi_1 + \cos \phi_2) I_2 . \quad (2.64)
\]

In order to find the output polarization direction, the preceding expression is maximized in the usual way with respect to \( \theta_2 \) and \( \phi_2 \). This procedure yields the following relations for the output polarization \( \hat{\mathbf{s}}_2(\theta_2, \phi_2) \):

\[
\sin (\theta_1 - \theta_2) = I_3 (I_1^2 - I_2^2)^{-1/2} , \\
\cos (\theta_1 - \theta_2) = (I_1 - I_4) (I_1^2 - I_2^2)^{-1/2} , \quad (2.65)
\]

and

\[
\sin \phi_2 = (|I_1 - I_2 \cos \phi_1|)^{-1} (I_1^2 - I_2^2)^{1/2} \sin \phi_1 , \\
\cos \phi_2 = (|I_1 - I_2 \cos \phi_1|)^{-1} (I_1 \cos \phi_1 - I_2) , \quad (2.66)
\]

so that the maximum and minimum differential cross sections are given by

\[
\frac{d\sigma(\hat{\mathbf{s}}_2, \hat{\mathbf{r}}_2; \hat{\mathbf{s}}_1, \hat{\mathbf{r}}_1)}{d\Omega} = \frac{1}{2} (I_1 - I_2 \cos \phi_1) , \\
\frac{d\sigma(-\hat{\mathbf{s}}_2, \hat{\mathbf{r}}_2; \hat{\mathbf{s}}_1, \hat{\mathbf{r}}_1)}{d\Omega} = 0 . \quad (2.67)
\]

It is seen that in general, during the scattering, the polarization direction has precessed and nutated about the normal to the plane of scattering from the original direction \( \hat{\mathbf{s}}_1 \) to the final direction \( \hat{\mathbf{s}}_2 \).

The maximum differential cross section cannot be zero.
since this would imply that the expansion coefficients of the scattered wave for the two spinors characterized by polarization directions $\hat{\xi}_2$ and $-\hat{\xi}_2$ are both zero, i.e., there would be no scattered wave. Consequently, choices of $\phi_1 = \pi/2$ and $\phi_1 = 0, \pi$ for Eq. 2.67 lead to the conclusions that $I_1(\theta)$ is always greater than zero, and that $|I_2(\theta)|$ is always less than $I_1(\theta)$.

For the special case of $\phi_1 = 0$ (or $\phi_1 = \pi$), it is seen from Eq. 2.67 that $\phi_2 = 0$ (or $\phi_2 = \pi$). Thus, if the incident polarization direction is parallel (anti-parallel) to the plane of scattering, the output polarization direction of the scattered wave is also parallel (anti-parallel) to the plane of scattering.
III. GREEN'S FUNCTION FORMULATION

A. Sommerfeld-Maue Approximation

The partial differential equation for $M$, Eq. 2.26, suggests a zero order approximation that results from the neglect of the term in $\lambda^2$. This approximation will be designated by the subscript zero. With the neglect of $\lambda^2$, the partial differential equation is separable both in spherical and parabolic coordinates. It will be shown that a consideration of the solution separable in parabolic coordinates leads directly to the well known Sommerfeld-Maue approximation.

In terms of parabolic coordinates, the partial differential equation for $M_0$ can be written as

$$
\begin{bmatrix}
\left[ \zeta_1 \frac{\partial^2}{\partial \zeta_1^2} + (1 + \zeta_1) \frac{\partial}{\partial \zeta_1} + s \right] \\
+ \left[ \zeta_2 \frac{\partial^2}{\partial \zeta_2^2} + (2 - \zeta_2) \frac{\partial}{\partial \zeta_2} - \left( \frac{1}{2} + \frac{b}{p} + s \right) \right]
\end{bmatrix}
\frac{-\zeta_1 - \zeta_2}{2} e^{\frac{\zeta_1 - \zeta_2}{2}} M_0 = 0
$$

(3.1)

The symbol $s$ represents the separation parameter.

The preceding partial differential equation has product solution of the form

$$
M_0(\zeta_1, \zeta_2) = e^{\frac{\zeta_1 - \zeta_2}{2}} {}_1F_1(s,1,-\zeta_1) {}_1F_1(\frac{1}{2} + \frac{b}{p} + s, 2, \zeta_2)
$$

(3.2)

where ${}_1F_1$ is the confluent hypergeometric function whose
properties are summarized in Appendix A. The asymptotic boundary condition of a plane wave plus an outgoing spherical wave is satisfied if the separation parameter $s$ is assigned the value zero. Thus,

$$M_0(\zeta_1, \zeta_2) = c e^{\frac{\zeta_1 - \zeta_2}{2}} P_1(1 + i\nu, 2, \zeta_2)$$

(3.3)

$$= c(i\nu)^{-1} e^{\frac{\zeta_1 - \zeta_2}{2}} \frac{\partial}{\partial \zeta_2} P_1(i\nu, 1, \zeta_2),$$

where $c$ is a normalization constant.

Using Eqs. 2.24 and 2.25 the auxiliary results

$$G_0(\zeta_1, \zeta_2) = -2c e^{\frac{\zeta_1 - \zeta_2}{2}} P_1(i\nu, 1, \zeta_2),$$

(3.4)

$$L_0(\zeta_1, \zeta_2) = 0,$$

(3.5)

are obtained. In order to normalize to unit incident flux, the constant $c$ is chosen as

$$c = -\frac{1}{2} \Gamma(1 - i\nu) e^{\frac{\nu \pi}{2}}.$$  

(3.6)

A substitution of these approximations into the Johnson-Deck form then yields the corresponding approximate wave function:

$$\psi_0 = \Gamma(1 - i\nu) e^{\frac{\nu \pi}{2}} \frac{\zeta_1 - \zeta_2}{2} \left[ 1 + \frac{1}{2\nu} \overline{a} \cdot (\hat{p} - \hat{r}) \frac{\partial}{\partial \zeta_2} \right] P_1(i\nu, 1, \zeta_2) U(\hat{p})$$

(3.7)

$$= \Gamma(1 - i\nu) e^{\frac{\nu \pi}{2}} e^{i\overline{F} \cdot \overline{F}} \left[ 1 - \frac{i}{2\mu} \overline{a} \cdot \overline{\nabla} \right] P_1(i\nu, 1, i(\mu r - \overline{F} \cdot \overline{F})) U(\hat{p}).$$
This is the usual form of the Sommerfeld-Maue wave function.

Through a manipulation of integral representation, Johnson and Deck (18) have also shown that the infinite series for $M$ given in Eq. 2.22 reduces to $M_0$ when $\lambda^2$ is set equal to zero.

B. Green's Function Solution

Since the term proportional to $\lambda^2$ in the partial differential equation for $M$ is also proportional to $r^{-2}$, a Green's function in spherical coordinates can be developed for a corresponding integral equation. In order to do this, the self-adjoint differential operator $O(\overline{r})$, given in Eq. 2.29 for the related function $Q$, is considered and its eigenfunctions are determined.

A differential operator $O(\overline{r})$ is self-adjoint if for any two reasonably differentiable functions of $\overline{r}$, $f_1$ and $f_2$, the operator satisfies the relation

$$f_1 O(\overline{r}) f_2 - f_2 O(\overline{r}) f_1 = \text{divergence } \overline{g}(\overline{r}),$$

where $\overline{g}$ is a function of $f_1$, $f_2$, and their derivatives. If $f_1$ and $f_2$ are bounded eigenfunctions of $O(\overline{r})$ with eigenvalues $\tau_1$ and $\tau_2$, then the integral of the preceding equation over all space gives

$$(\tau_2 - \tau_1) \int f_1 f_2 \, d\overline{r} = 0$$

i.e., the eigenfunctions are orthogonal. For (oscillatory)
continuum eigenfunctions, one considers a finite set which for discrete eigenvalues have their zeros on the surface of a large enclosure. As the boundary of the enclosure is moved away to infinity, the eigenvalue spectrum becomes continuous. Thus, for the continuous eigenfunctions the surface terms arising from \( \text{div} \, \varepsilon(\mathbf{r}) \) become zero, and here also one gets an orthogonality condition between eigenfunctions of the same operator. If the operator were not self-adjoint, orthogonality would exist between the respective eigenfunctions of the operator and its adjoint operator. Thus, the reason for considering the operator \( \mathcal{O}(\mathbf{r}) \) instead of the corresponding operator for \( M \) is that only the eigenfunctions of one operator \( \mathcal{O}(\mathbf{r}) \) are necessary for the following discussion.

For the continuous spectrum, \( s \geq 0 \), the operator \( \mathcal{O}(\mathbf{r}) \) has eigenfunctions \( \psi_k(s, \mathbf{r}) \) that satisfy

\[
0(\mathbf{r}) \psi_k(s, \mathbf{r}) = -s^2 \psi_k(s, \mathbf{r}) .
\] (3.8)

These eigenfunctions have the product form

\[
\psi_k(s, \mathbf{r}) = c_k(s) r^{-1} R(b, k, -2isr) A_k(a) ,
\] (3.9)

where the radial function \( R \) and the angular function \( A_k \) satisfy the separated equations

\[
\left[ \frac{d^2}{dr^2} + \left( \frac{s^2 - 2ib}{r} + \frac{1/4 - k^2}{r^2} \right) \right] R(b, k, -2isr) = 0, \] (3.10)
The regular solutions of these differential equations are, for the angular part,

\[ A_k(a) = \left(\frac{1-a^2}{2k}\right)^{1/2} \left\{ P_k'(a) + P_{k-1}'(a) \right\}, \quad k = 1, 2, \ldots \quad (3.12) \]

which possesses the orthogonality property

\[ \int_{-1}^{1} A_k(a)A_{k'}(a)da = \delta_{k,k'}, \quad (3.13) \]

and for the radial part,

\[ R(b, s, k, -2isr) = (2sr)^h e^{-isr} P_1(h + \frac{b}{s}, 2h, 2isr) , \quad (3.14) \]

where \( h \) is defined by

\[ h = k + 1/2 . \quad (3.15) \]

In the preceding \( s \) is restricted to positive values, since the radial function for negative \( s \) is the same as for positive \( s \) (to a phase factor independent of \( \overline{r} \)), as may be seen by applying Kummer's relation.

The normalization constant \( c_k(s) \) is obtained by requiring the following continuum normalization:

\[ \lim_{\epsilon \to 0^+} \int_{s'=s(1+\epsilon)}^{s'=s(1-\epsilon)} \int \Phi_k(s, \overline{r})\Phi_{k'}(s, \overline{r})d\overline{r} \; ds' = \delta_{k,k'} . \quad (3.16) \]

Using the relation
where $R$ and $R'$ have the same value of $k$ but differ by having parameters $s$ and $s'$ respectively, and also using the asymptotic form of $F_1$, the normalization constant is found to be:

$$c_k(s) = \frac{\left[ \Gamma(h - b/s) \Gamma(h + b/s) \right]^{1/2}}{2\pi \Gamma(2h)} e^{-\frac{i\pi b}{2s}}.$$ (3.18)

The bounded eigenfunctions of the operator $0(\vec{r})$ have the property that as $r \to \infty$, their corresponding radial function goes to zero. Examining the asymptotic form of $F_1$, it can be seen that bounded eigenfunctions occur only for $\text{Im}(b) > 0$, i.e. for $\nu > 0$, and in that case correspond to the replacement $s \to b/(h + n)$, where $n = 0, 1, 2, \ldots$. Consequently, the bounded eigenfunctions are

$$\Phi_k(n, \vec{r}) = c_k(n) r^{-1} R(h + n, k, -\frac{2ibr}{h + n}) \Lambda_k(a).$$ (3.19)

These satisfy the eigenvalue equation

$$0(\vec{r}) \Phi_k(n, \vec{r}) = -\left(\frac{b}{h + n}\right)^2 \Phi_k(n, \vec{r}).$$ (3.20)

The bounded radial function is given by
The symbol \( L_n^{2h-1} \) designates the Laguerre polynomial, as normalized by Morse and Feshbach (26, pp. 784-85).

Using the relation

\[
\int_0^\infty z^{2h} e^{-z} L_n^{2h-1}(z) L_n^{2h-1}(z) dz = \frac{[\Gamma(h + n)]^2}{n!} 2^{(h + n)} ,
\]

it can be shown that if the normalization constant \( c_k(n) \) is chosen as

\[
c_k(n) = \frac{\frac{-i\pi h}{2} e^{-\frac{i\pi}{4}}}{(h + n) \Gamma(2h)} \left[ \frac{b \Gamma(2h + n)}{(2\pi)^n} \right]^{1/2} ,
\]

then the bounded eigenfunctions have the desired normalization,

\[
\int \Phi_k(n, \bar{r}) \Phi_{k'}(n', \bar{r}) d\bar{r} = \delta_{k,k'} \delta_{n,n'} .
\]

In terms of the continuum and bounded eigenfunctions, a Green's function \( G^0(\bar{r}, \bar{r}') \) which behaves asymptotically like an outgoing wave and satisfies

\[
\left[ 0(\bar{r}) + p^2 \right] G^0(\bar{r}, \bar{r}') = \delta(\bar{r} - \bar{r}')
\]

is given by

\[
R(h + n, k, - \frac{2ibr}{h + n}) = \left( \frac{2b}{h + n} \right)^h \frac{1}{e^{\frac{1}{h+n}}} \sum_{l} \left[ \begin{array}{c} \frac{1}{(h+n)} \left( \begin{array}{c} \frac{1}{h+n} \end{array} \right) \right] \frac{2h-1}{-2ibr} .
\]

(3.21)
can be determined, where for the delta function one uses

\[ \delta(\mathbf{r} - \mathbf{r}') \]  

\[ = \sum_{k=1}^{\infty} \int_{0}^{\infty} \psi_k(s, \mathbf{r}) \psi_k(s, \mathbf{r}') ds + \frac{1}{2} \frac{\nu}{|\nu|} + 1 \sum_{n=0}^{\infty} \Phi_k(n, \mathbf{r}) \Phi_k(n, \mathbf{r}') \]  

Consequently,

\[ G^0(\mathbf{r}, \mathbf{r}') \]  

\[ = \sum_{k=1}^{\infty} \int_{0}^{\infty} \frac{\psi_k(s, \mathbf{r}) \psi_k(s, \mathbf{r}') ds}{p^2 - s^2} + \frac{1}{2} \frac{\nu}{|\nu|} + 1 \sum_{n=0}^{\infty} \frac{\Phi_k(n, \mathbf{r}) \Phi_k(n, \mathbf{r}')}{p^2 - \left( \frac{b}{h + n} \right)^2} \]

where the outgoing boundary condition will be invoked in the choice of contour around \( s = p \).

In order to evaluate the integral over \( s \), the relevant integral

\[ I(r, r') = \int_{0}^{\infty} \left[ \frac{c_k(s)}{(p^2 - s^2)} \right]^2 R\left( \frac{b}{s}, k, -2isr \right) R\left( \frac{b}{s}, k, -2isr' \right) ds \]  

is examined. From Eqs. 3.14 and 3.18, this integral can be written

\[ I(r, r') \]

\[ = \frac{(4\pi r')^h}{[2\pi^2(2h)]^2} \int_{0}^{\infty} \frac{e^{-is(r+r')} s^{2h} e^{-s/2} \Gamma(h - \frac{b}{s}) \Gamma(h + \frac{b}{s})}{(p^2 - s^2)} \]  

times \, _1F_1(h + \frac{b}{s}, 2h, 2isr) _1F_1(h + \frac{b}{s}, 2h, 2isr') ds \]  

(3.29)
This integral is evaluated for the case $r > r'$. The corresponding integral for $r < r'$ may then be obtained by the transposition $r \leftrightarrow r'$.

For $r > r'$, the confluent hypergeometric function in $r$ is separated into two parts with different characteristic asymptotic behavior (see Appendix A):

$$\frac{1}{2} F_1\left(h + \frac{b}{s}, 2h, 2isr\right) = \frac{1}{2} F_0\left(h + \frac{b}{s}, 2h, 2isr\right) + \frac{1}{2} F_0\left(h + \frac{b}{s}, 2h, 2isr\right). \quad (3.30)$$

The integral containing $F_0(F_1)$ is then obtained by a consideration of a contour $C_0(C_1)$ in the first (fourth) quadrant of the complex $s$ plane. In order to obtain an outgoing wave for large $r$, the pole at $s = p$ is chosen slightly above the real $s$ axis. The gamma functions contribute other poles in the complex $s$ plane at $s + n = \pm \frac{b}{h + n}$, for $n = 0, 1, 2, \ldots$, since near these values the gamma functions have the behavior

$$\Gamma\left(h + \frac{b}{s}\right) \approx \pm (-1)^n \frac{b}{(h + n)^2} \frac{1}{(s + \frac{b}{h + n})}. \quad (3.31)$$

The contours for the complex integration are given in Figs. 2 and 3. It should be noted that the position of the gamma function poles depend on whether $\nu$ is greater or less than zero. The symbol $L_0(L_1)$ designates that part of the contour $C_0(C_1)$ lying on the positive (negative) imaginary axis.

Using the residue theorem of complex variable theory, the
Fig. 2. Contours for $I(r,r')$ integration, $v > 0$

Fig. 3. Contours for $I(r,r')$ integration, $v < 0$
bracket in Eq. 3.29 is given by

\[
\left\lfloor a \right\rfloor = 2\pi i \text{Residue} \left( C_0 + C_1 \right) - (L_0 + L_1),
\]

since the contribution from the circular path at infinity is zero due to the asymptotic behavior of the F functions.

For \( v \geq 0 \), the residue is given by:

\[
\text{Residue} \left( C_0 + C_1 \right) = \tag{3.33}
\]

\[
\frac{-e^{-ip(r+r')}}{4p} p^{2h} e^{-\frac{1nb}{p}} \frac{\Gamma(h+b)\Gamma(h-b)}{\Gamma(h-n)} F_0^{(h+b/2,2h,2ipr)} F_1^{(h+b/2,2h,2ipr')}
\]

\[
\sum_{n=0}^{\infty} \left[ \frac{-ib(r+r')}{e^{(h+n)}} e^{-in\frac{b}{h+n}} \frac{\Gamma(2h+n)}{\Gamma(2h)} \right]^{(h+n)}
\]

\[
\times \left( \frac{\nu}{|\nu|} \pm 1 \right) \frac{b}{(h+n)^2} F_0^{(h+n,2h,2ipr)} F_1^{(h+n,2h,2ipr')}
\]

It can also be shown that

\[
(L_0 + L_1) = 0, \tag{3.34}
\]

by using Eqs. 8.16 and 8.17, and changing \( s \rightarrow e^{i\pi} s \) on the \( L_0 \) segment.

Defining

\[
R^0_{(b,k,-2ipr)} = (2pr)^{h-1ipr} F_0^{(h+b/2,2h,2ipr)} F_1^{(h+b/2,2h,2ipr')}, \tag{3.35}
\]

which is similar to Eq. 3.14 except that \( F_1 \) has been replaced
by $F^0$, and using Eqs. 3.13 to 3.15, Eqs. 3.21 and 3.23, and Eqs. 3.14 and 3.18, substitution of Eqs. 3.32 and 3.33 into Eq. 3.29 yields the result:

$$I(r,r') = \frac{\pi}{21p} R^0(b,h,-2ipr)R(b,h,-2ipr')$$

$$= \frac{1}{2} \left( \frac{\psi}{|\psi|} + 1 \right) \sum_{n=0}^{\infty} \frac{c_k(n)}{p^2 - \left( \frac{b}{h+n} \right)^2} R(h+n,k,-2ibr)R(h+n,k,-2ibr')$$

for $r > r'$. It is seen from the second term in Eq. 3.36 that a contribution from the integral over the continuum eigenfunctions, properly normalized, precisely cancels the bound state eigenfunctions in Eq. 3.27.

Defining the function $\psi_k^0(p,\theta)$ by

$$\psi_k^0(p,\theta) = c_k(p)r^{-1}R^0(b,p,h,-2ipr)\alpha_k(a),$$

substitution of Eq. 3.36 into Eq. 3.27 finally yields the desired result for the Green's function $G^0$:

$$G^0(\theta,\theta') = \frac{\pi}{21p} \sum_{k=1}^{\infty} \{\psi_k^0(p,\theta)\psi_k(p,\theta')\} \text{ for } r > r'$$

$$= \frac{\pi}{21p} \sum_{k=1}^{\infty} \{\psi_k(p,\theta)\psi_k^0(p,\theta')\} \text{ for } r < r'.$$

With the aid of this Green's function, the partial differential equation, Eq. 2.28, and the required boundary conditions may be replaced by the integral equation
\[ Q(r) = Q_0(r) - \lambda^2 \int d\mathbf{r}' \frac{G^0(r, r') Q(r')}{(r')^2}, \quad (3.39) \]

where

\[ \left[ \phi_0(r) + p^2 \right] Q_0(r) = 0. \quad (3.40) \]

The explicit form of \( Q_0(r) \) corresponding to \( M_0(r) \) is given by

\[
Q_0(r) = \begin{cases} \\
\frac{3\omega}{4p^{3/2}} \sum_{k=1}^{\infty} (-1)^k e^{-\frac{i\pi k}{2}} \sqrt{k} \left[ \frac{\Gamma(n - b/p)}{\Gamma(n + b/p)} \right]^{1/2} \chi_k(p, \Omega) \end{cases} \quad (3.41)
\]

\[
= -\frac{1}{2} \Gamma(1 - i\nu) e^{\frac{\nu\pi}{2}} \left[ r(1 - a) \right]^{1/2} e^{ipra} \frac{1}{\Gamma(1+\nu, 2, ipr(l-a))}.
\]
IV. APPROXIMATION FOR ASYMPTOTIC WAVE FUNCTION

A. Asymptotic Wave Function

The integral equation for $Q(\mathbf{r})$ can be solved by a process of iteration, in which the zero order solution is taken to be $Q_0(\mathbf{r})$. This procedure generates a series in $\lambda^2$ independent of the Born parameter. Thus, the interaction is partially taken into account in the zero order solution (which contains the Born parameter $\nu = aZ\varepsilon/p$) in marked contrast to the entirely interaction independent iteration scheme of the successive Born approximations.

Subsequent discussion will deal with the approximation resulting from one iteration in the limit of large $r$. For this case, Eqs. 3.38 and 3.39 yield

$$Q(\mathbf{r}) = Q_0(\mathbf{r}) - \frac{\lambda^2 \pi}{21p} \sum_{k=1}^{\infty} \psi_k^0(p, r) \int \frac{d\mathbf{r}' \psi_k(p, \mathbf{r}') Q_0(\mathbf{r}')}{(\mathbf{r}')^2}. \quad (4.1)$$

Upon substitution of the infinite series form of $Q_0(\mathbf{r}')$ from Eq. 3.41, the angular integration reduces the resulting double series in Eq. 4.1 to a single series. Using the asymptotic form of $\psi_k^0(p, \mathbf{r})$ and the explicit form of $\psi_k(p, \mathbf{r}')$, Eq. 4.1 becomes:

$$Q(\mathbf{r}) = Q_0(\mathbf{r}) - \lambda^2 \left[ \frac{\mathbf{r}}{1 - a} \right]^{1/2} \frac{e^{\theta\text{sc}}}{21p r} T(\theta, \nu), \quad (4.2)$$

where
The phase factor $\theta_{sc}$ for the outgoing scattered wave is

$$\theta_{sc} = pr + \nu \ln pr(l - a) - 2 \text{ arg } \Gamma(1 + i\nu). \quad (4.4)$$

The symbol $I(k, \nu)$ represents the Coulomb integral defined by

$$I(k, \nu) = \lim_{\epsilon \to 0^+} \int_0^\infty e^{-\epsilon x} x^{2h-2} e^{-ix} \left[ \text{$_1F_1$}(h + \frac{b}{p}, 2h, x e^{\frac{1}{2}}) \right]^2 \, dx. \quad (4.5)$$

In the usual way, a screening parameter $e^{-\epsilon x}$ has been inserted to ensure the existence of the integral. As defined previously, $h = k + 1/2$, $b/p = 1/2 + i\nu$, and $a = \cos \theta$. The Coulomb integral is evaluated in Appendix B, where it is found that

$$I(k, \nu) = \frac{\Gamma(2k)e^{-\nu\pi}}{B(k+1+i\nu, k-i\nu)} \left[ \text{$_1\Psi_1$}(k+1+i\nu) - \text{$_1\Psi_1$}(k-i\nu) \right]. \quad (4.6)$$

Here, $B$ represents the beta function and the symbol $\Psi_1$ designates the logarithmic derivative of the gamma function.

Substituting Eq. 4.6 into Eq. 4.3, the angular function $T(\theta, \nu)$ is expressed as
\[ T(\theta, \nu) = \frac{-(1 - a)}{2i\nu B(-i\nu, 1 + 2i\nu)} e^{-2i\nu \ln(\sin \theta/2)} \text{ times} \]

\[ \sum_{k=1}^{\infty} \frac{1}{k} (P'_k(a) + P'_{k-1}(a)) B(k-i\nu, 1 + 2i\nu) \left[ i\pi + \psi'_1(k+1+i\nu) - \psi'_1(k-i\nu) \right]. \]

Using the relation

\[ (1 - a) [P'_k(a) + P'_{k-1}(a)] = -k [P_k(a) - P_{k-1}(a)], \quad (4.8) \]

the angular function \( T(\theta, \nu) \) assumes the form

\[ T(\theta, \nu) = \frac{e^{-2i\nu \ln(\sin \theta/2)}}{2i\nu B(-i\nu, 1 + 2i\nu)} S(\theta, \nu), \quad (4.9) \]

where \( S(\theta, \nu) \) is defined by

\[ S(\theta, \nu) = \sum_{k=1}^{\infty} (P_k(a) - P_{k-1}(a)) B(k-i\nu, 1 + 2i\nu) \]

\[ \text{times} \left[ i\pi + \psi'_1(k+1+i\nu) - \psi'_1(k-i\nu) \right]. \quad (4.10) \]

In addition to the series representation of the two parameter function \( S(\theta, \nu) \), it is also possible to obtain an integral representation. Using the Eulerian representation for the beta function (27, p. 9)

\[ B(s, t) = \int_0^1 x^{t-1} (1 - x)^{s-1} dx, \quad (4.11) \]

for \( \text{Re } s > 0, \text{Re } t > 0 \), and also using the integral representation for the product of a beta function and the difference of two related logarithmic derivatives of the gamma function (28, p. 314)
$B(s, t) \left[ \psi_1(t) - \psi_1(s + t) \right] = \int_0^1 x^{t-1}(1 - x)^{s-1} \ln x \, dx,$  
(4.12)

where $\Re s > 0$, $\Re t > 0$, the expression for $S(\theta, \nu)$ given in Eq. 4.10 becomes

$$S(\theta, \nu) = \int_0^1 x^{-1\nu-1}(1-x)^{2\nu} \left\{ \sum_{k=1}^{\infty} (P_k(a) - P_{k-1}(a)) x^k \right\} (i\pi - \ln x) \, dx. \quad (4.13)$$

Now, using the generating function expression for Legendre polynomials (27, p. 154)

$$[1 - 2az + z^2]^{-1/2} = \sum_{k=0}^{\infty} z^k P_k(a), \quad (4.14)$$

for $|z| < 1$, the sum in Eq. 4.13 may be performed, yielding the following integral expression for $S(\theta, \nu)$:

$$S(\theta, \nu) = \int_0^1 x^{-1\nu-1}(1-x)^{2\nu} \left\{ \frac{1-x - [1+x^2-2ax]}{[1+x^2-2ax]^{1/2}} \right\} (i\pi - \ln x) \, dx. \quad (4.15)$$

In general, it has not been possible to obtain a closed form expression for $S(\theta, \nu)$. However, in the limit of small $\nu$, a closed form expression may be obtained from the series representation Eq. 4.10, and for large $|\nu|$, a closed form expression may be obtained from an asymptotic expansion derivable from the integral representation Eq. 4.15. These limiting cases are discussed in section V.

The asymptotic wave function may be expressed in terms of the angular function $T(\theta, \nu)$. Using Eq. 4.2, the asymptotic
form of \( Q \) derivable from Eq. 3.41, and Eq. 2.27, the following asymptotic form for \( M \) is obtained:

\[
M(\overline{r}) = \frac{e^{i\Theta_p}}{2ipr(1 - a)} - \frac{e^{i\Theta_{sc}}}{2ipr(1 - a)}(1 + \lambda^2 T(\Theta, \nu)), \tag{4.16}
\]

where the plane wave phase factor \( \Theta_p \) is given by

\[
\Theta_p = pr\alpha - \nu \ln pr(1 - a). \tag{4.17}
\]

From the differential relations given in Eqs. 2.19, 2.20, 2.24, and 2.25, one obtains

\[
G(\overline{r}) = e^{i\Theta_p} - \frac{e^{i\Theta_{sc}}}{2ipr(1-a)}\left[ -2iv(1 + \lambda^2 T(\Theta, \nu)) + \lambda^2 (1-a^2) \frac{\partial T(\Theta, \nu)}{\partial \Theta} \right], \tag{4.18}
\]

\[
L(\overline{r}) = \frac{e^{i\Theta_{sc}}}{2ipr} \lambda^2 \frac{\partial T(\Theta, \nu)}{\partial \Theta}. \tag{4.19}
\]

Using the abbreviations

\[
T(\Theta, \nu) = T, \quad \frac{\partial T(\Theta, \nu)}{\partial \Theta} = T', \tag{4.20}
\]

and remembering that \( a = \cos \Theta \), substitution of Eqs. 4.16, 4.18, and 4.19 into the Johnson-Deck form Eq. 2.1, gives the asymptotic wave function

\[
\psi(\overline{r}) = e^{i\Theta_p} U(\hat{\nu}) + \frac{ie^{i\Theta_{sc}}}{4pr \sin^2 \Theta/2} \left[ -2iv(1 + \lambda^2 T) - \lambda^2 \sin \Theta T' \right]
\]

\[
+ \lambda (1 + \lambda^2 T) \overline{\alpha} (\hat{\nu} \cdot \overline{r}) \quad \left[ U(\hat{\nu}). \right. \tag{4.21}
\]
It is seen that the asymptotic wave function is correct through order $\lambda^3$.

B. Potential Scattering Results

By comparison with the asymptotic forms given in Eqs. 4.21 and 2.32, it is seen that:

$$G_s = \frac{-1}{4p \sin^2 \theta/2} \left[ 2iv(1 + \lambda^2 T) + \lambda^2 \sin \theta T' \right], \quad (4.22)$$

$$M_s = \frac{1}{4p \sin^2 \theta/2} \left[ 1 + \lambda^2 T \right], \quad (4.23)$$

$$L_s = \frac{1}{4p \sin^2 \theta/2} \left[ \lambda^2 \tan \theta/2 T' \right]. \quad (4.24)$$

Consequently, the auxiliary functions $A_1$ and $A_2$ defined by Eqs. 2.45 and 2.46 become

$$A_1 = \frac{-v}{2p \sin^2 \theta/2} \left[ i(1 + \lambda^2 T) + v \tan \theta/2 T' \right], \quad (4.25)$$

$$A_2 = \frac{v}{2p \sin^2 \theta/2} \left[ \frac{\lambda}{p} \tan \theta/2 T' \right]. \quad (4.26)$$

Using the definition of the Rutherford differential cross section

$$R = \left[ \frac{v}{2p \sin^2 \theta/2} \right]^2, \quad (4.27)$$

which is the classical non-relativistic Coulomb differential cross section, and using the abbreviation...
the four I functions of Eqs. 2.49 to 2.52 take the form:

\[
I_1(\theta) = \frac{R}{R} = (1 - \beta^2 \sin^2 \theta/2) \left[ 1 + 2\lambda^2 \text{Re } T + \lambda^4 \text{Im } T^* \right] \\
+ \lambda \beta \sin \theta \left[ \text{Im } T' + \lambda^2 \text{Im } T^* T' \right] \\
+ \lambda^2 \beta^2 \sin^2 \theta/2 \left[ T^* T' \right] ,
\]

\[
I_2(\theta) = -\frac{2\lambda \beta}{E} \sin^2 \theta/2 \left[ \text{Re } T' + \lambda^2 \text{Re } T^* T' \right],
\]

\[
I_3(\theta) = \frac{1}{2} \sin \theta \{ \cos \theta \left( \frac{E-1}{E} \right)^2 + \beta^2 \} \left[ 1 + 2\lambda^2 \text{Re } T + \lambda^4 \text{Im } T^* \right] \\
+ 2\lambda \beta \sin^2 \theta/2 \{ \cos \theta \left( \frac{E-1}{E} \right) + 1 \} \left[ \text{Im } T' + \lambda^2 \text{Im } T^* T' \right] \\
+ \lambda^2 \beta^2 \sin^2 \theta/2 \sin \theta \left[ T^* T' \right],
\]

\[
I_4(\theta) = \frac{1}{2} \sin^2 \theta \left( \frac{E-1}{E} \right)^2 \left[ 1 + 2\lambda^2 \text{Re } T + \lambda^4 \text{Im } T^* \right] \\
+ 2\lambda \beta \left( \frac{E-1}{E} \right) \sin \theta \sin^2 \theta/2 \left[ \text{Im } T' + \lambda^2 \text{Im } T^* T' \right] \\
+ 2\lambda^2 \beta^2 \sin^4 \theta/2 \left[ T^* T' \right].
\]

If the exact solution were used, T would be replaced by T + 0(λ^2). Consequently, in the approximation used, λ^4 TT* and λ^2 Im T*T' occurring in the expressions for the four I(θ) functions, are of the same order of magnitude as the next order neglected correction. However, if one takes the point
of view that the fundamental approximation is in the wave
function, which hopefully is very good, then these contribu­
tions may be retained.

In order to establish a criterion for the validity of
the approximation based on one iteration, one can compare the
first iterated solution $Q(1)(\vec{r})$ with the second iterated
solution $Q(2)(\vec{r})$. Since

$$Q(1)(\vec{r}) = Q_0(\vec{r}) - \lambda^2 \int d\vec{r}'(r')^2 G^0(\vec{r}, \vec{r}') Q_0(\vec{r}') , \quad (4.33)$$

and

$$Q(2)(\vec{r}) = Q(1)(\vec{r}) + \lambda^4 N(\vec{r}) , \quad (4.34)$$

where

$$N(\vec{r}) = \int \int d\vec{r}'d\vec{r}''(r' r'')^2 G^0(\vec{r}, \vec{r}') G^0(\vec{r}', \vec{r}'') Q_0(\vec{r}'') , \quad (4.35)$$

a measure of the validity of the single iteration approxima­
tion is that

$$\frac{\lambda^4 |N(\vec{r})|}{|Q(1)(\vec{r})|} < 1 . \quad (4.36)$$

It is to be emphasized that the formal expansion in $\lambda^2$ is
independent of the Born parameter $\nu = \alpha Z E / p$ and hence is good
for all energies, subject to the condition given in Eq. 4.36.
V. LIMITING CASES

A. Small Born Parameter

It is possible to obtain a closed form expression for the angular function $T(\Theta, \nu)$ in the case of small Born parameter $\nu$. By a Taylor's expansion,

$$i\pi + \psi_1(k+1+iv) - \psi_1(k-iv) = i\pi + \frac{1}{k} - \frac{iv}{k^2} + 2iv\psi_2(k)\ldots, \quad (5.1)$$

where $\psi_2$ is the polygamma function

$$\psi_2(z) = \frac{\partial \psi_1(z)}{\partial z}. \quad (5.2)$$

Thus, through order $\nu$,

$$T(\Theta, \nu) = -\frac{1}{2} \sum_{k=1}^{\infty} (P_k(a) - P_{k-1}(a)) \text{ times} \quad (5.3)$$

$$\left[ \left\{ 1 - 2iv(\gamma + \ln \sin \Theta/2) \right\} \left( \frac{1}{k} + \frac{1}{k^2} \right) + \right.$$

$$\left. \nu \left[ \frac{\pi}{k^2} - \frac{24}{k^3} + \frac{2\pi}{k} \psi_1(z) + \frac{24}{k^2} \psi_2(k) - \frac{24}{k^2} \psi_1(k) \right] \right],$$

where the Euler-Mascheroni constant $\gamma$ is

$$\gamma = -\psi_1(1) = 0.5772157 \ldots \quad (5.4)$$

Also, using the Legendre polynomial identity
\[(1 + \alpha)(P'_k(a) - P'_{k-1}(a)) = \alpha(P_k(a) + P_{k-1}(a)) \quad \text{(5.5)}\]

It is found that through order \(\nu\),

\[\frac{\partial T(\theta, \nu)}{\partial \theta} = \frac{i\nu}{2} \cot \frac{\theta}{2} \sum_{k=1}^{\infty} (P_k(a) - P_{k-1}(a))(\frac{i\pi}{k} + \frac{1}{k^2}) \]

\[+ \frac{\tan \frac{\theta}{2}}{2} \sum_{k=1}^{\infty} (P_k(a) + P_{k-1}(a)) \text{ times} \quad \text{(5.6)}\]

\[\left[ [1 - 2i\nu(\gamma + \ln \sin \frac{\theta}{2})](\frac{i\pi}{k} + \frac{1}{k}) \right] + \nu \left[ \frac{\pi}{k} - \frac{21}{k^2} + 2\pi\psi_1(k) + 21\psi_2(k) - \frac{21}{k}\psi_1(k) \right].\]

For consistent order in \(\lambda\) in the cross section (here \(\nu\) is not considered an independent parameter but equal to \(\lambda/\beta\)), the sums of Eq. 5.3 for \(T(\theta, \nu)\) for zero order in \(\lambda\), and the sums of Eq. 5.6 through first order in \(\lambda\), may be performed using the expressions given by Johnson et al. (16). The scattering results valid to order \(\lambda^4\) are in complete agreement with Johnson et al., and the rather lengthy formulae will not be reproduced here.

**B. Large Born Parameter**

Since the results obtained apply equally well for all Born parameters, it is possible to obtain an asymptotic expansion of the angular function \(T(\theta, \nu)\) for large Born parameters. The method used is to convert the integral representation of
S(θ,ν) given in Eq. 4.15 into a Fourier integral, which can then be evaluated by Lighthill's (29) method of asymptotic evaluation of Fourier transforms. Then, T(θ,ν) is obtained from Eq. 4.9. The details of the asymptotic expansion for ν → ∞ and for ν → -∞ are presented in Appendix C. The results are

\[ \text{Re } T = (4ν^2 \sin \theta)^{-1} \tan^2 \theta/2(\pm \pi + \theta + \sin \theta) + O(ν^{-3}), \quad (5.7) \]

\[ \text{Im } T = -(2ν)^{-1} \tan \theta/2(\theta \pm \pi) + O(ν^{-3}), \quad (5.8) \]

\[ \text{Re } T' = (4ν \cos \theta/2)^{-2} \left( 6 \tan \theta/2 + (\theta \pm \pi)(1 + 3 \tan^2 \theta/2) \right) + O(ν^{-3}), \quad (5.9) \]

\[ \text{Im } T' = -(4ν \cos^2 \theta/2)^{-1} (\pm \pi + \theta + \sin \theta) + O(ν^{-3}), \quad (5.10) \]

where \( \pm \) signifies \( \nu/|\nu| \).

Here, in the asymptotic expansion for ν → ∞, terms containing cos ν(2 ln sin θ/2) and sin ν(2 ln sin θ/2) have been neglected, because for large ν (30, p. 69)

\[ \frac{\sin νx}{x} \to π \delta(x), \]

\[ \frac{\cos νx}{x} \to \frac{1}{x} - \frac{\Theta}{x}. \]

Since the asymptotic expansion is restricted to \( \theta \neq π \), i.e., \( x \neq 0 \), these terms do not contribute.

Upon substitution of these asymptotic expansions into Eqs. 4.20 and 4.30, the following results for the potential scattering I(θ) functions are obtained:
\[ I_1(\theta)/R = 1 + \beta^2 \left[ f_1(\pm, \theta) + \lambda^2 g_1(\pm, \theta) \right] + O(\beta^4) , \quad (5.11) \]
\[ I_2(\theta)/R = - \beta^2 (E\nu)^{-1} \left[ f_2(\pm, \theta) + \lambda^2 g_2(\pm, \theta) \right] + O(\beta^4) , \quad (5.12) \]
\[ I_3(\theta)/R = \beta^2 f_3(\pm, \theta) + O(\beta^4) , \quad (5.13) \]
\[ I_4(\theta)/R = \frac{1}{2} \left[ I_3(\theta)/R \right]^2 , \quad (5.14) \]

where the functions \( f \) and \( g \) are defined by

\[ f_1(\pm, \theta) = - \frac{1}{2} \tan^2 \theta / 2 \left[ 1 + \cos \theta + \cot \theta (\sin \theta + \theta \pm \pi) \right] , (5.15) \]
\[ g_1(\pm, \theta) = \frac{1}{4} \tan^2 \theta / 2 (\theta \pm \pi)^2 , \quad (5.16) \]
\[ f_2(\pm, \theta) = (8)^{-1} \tan^2 \theta / 2 \left[ 6 \tan \theta / 2 \\
+ (\theta \pm \pi)(1 + 3 \tan^2 \theta / 2) \right] , \quad (5.17) \]
\[ g_2(\pm, \theta) = \frac{1}{4} \tan^3 \theta / 2 (\theta \pm \pi)(\sin \theta + \theta \pm \pi) , \quad (5.18) \]
\[ f_3(\pm, \theta) = \tan \theta / 2 \left[ \cos \theta - \frac{1}{2} \tan \theta / 2 (\theta \pm \pi) \right] . \quad (5.19) \]

These functions are plotted in Figs. 4 to 8. The symbol \( \beta \) represents \( \lambda/\nu \), so a term of order \((\lambda \beta)^2\) is actually an expansion in \( \lambda^4 \) and \( \nu^{-2} \).

The Sommerfeld-Maue value of \( f_1 \)

\[ f_{SM} = - \sin^2 \theta / 2 , \quad (5.20) \]

which gives rise to the Mott approximation for the differential cross section \((\lambda^2 = 0)\), is also plotted in Figs. 4 and 5 for comparison.
Mott and Massey (31, p. 80) claim that \( I_1(\theta)/R \) is given by \( 1 + \beta^2 f_{SM} \) for all values of \( \beta \) as long as \( \lambda = \alpha Z \) is small compared with unity. This conclusion was reached on the basis of a small \( \lambda \) expansion of the exact cross section. The results obtained here, as displayed in Figs. 4 and 5, are in sharp disagreement with their assertion. Instead of a common curve for \( +\lambda \) and \(-\lambda \) (attractive and repulsive potential) these results exhibit quite different character for these two cases. From Fig. 4 it is seen that \( f_1(-,\theta) \), pertaining to a repulsive potential, is positive for forward angles, is greater than the Mott value until 90°, thereafter is less than the Mott value. Here, the difference in the \( I_1(\theta)/R \) values,

\[
\frac{I_1(\theta)/R - \{I_1(\theta)/R\}_M}{I_1(\theta)/R} = \beta^2 \{f_1(-,\theta) - f_{SM}\}, \tag{5.21}
\]

for a \( \beta \) (which is less than \( \lambda \) in this large Born parameter approximation) of value 1/10, is of the order of magnitude 0.1 per cent, and for a \( Z \) (unshielded) value of 2 and an appropriate value of \( \beta \), say, 1/300, the difference is of the order of \((10)^{-4}\) per cent. However, the smallness of these differences is a reflection of the \( \beta \) dependence and not the \( \beta \) independent angular factor, i.e., at small \( \beta \) the whole correction itself to the Rutherford value is very small.

In the case of an attractive potential, Fig. 5 shows that \( f_1(+,\theta) \) has a very different character to the Mott \( f_{SM} \). It is
smaller than the Mott value for all forward angles (although the $\lambda^2$ correction proportional to $g_{1}(+,\theta)$ tends in the direction of the Mott value), and becomes positive and much greater than the Mott value for back angles. It is presumed that for $\nu \to \infty$, the apparent divergence of $I_{1}(\theta)/R$ (and $I_{2}(\theta)/R$, $I_{3}(\theta)/R$) as $\theta$ approaches $180^\circ$ is indicative of the unreliability of the approximation in this region, probably due to the inability of the asymptotic expansion of $S(\theta,\nu)$ to adequately represent the function for finite values of $\nu$ (see Appendix C). For the forward angle of $45^\circ$ and a $\beta$ of $1/10$, the difference of $I_{1}(\theta)/R$ is approximately $0.4$ per cent, and for $\beta = 1/300$, the difference is approximately $4(10)^{-4}$ per cent.

The fact that for a repulsive potential ($\nu \to -\infty$), $f_{1}(-,\theta)$ is less than zero, and that for an attractive potential ($\nu \to \infty$), $f_{1}(+,\theta)$ rises steeply at back angles, is in qualitative agreement with the numerical results displayed by Mott and Massey (31, p. 81) for $Z = 80$ and $\nu = 2$, but is in sharp disagreement with the Mott formula given by $f_{SM}$ alone. Exact numerical results are not available in the region of small $\lambda$ and large $\nu$ where the approximation given here applies.

As mentioned earlier, the Mott formula was obtained from a small $\lambda$ expansion, where $\lambda$ and $\beta$ were considered independent parameters. However, one must also consider the relative magnitude of the independent parameter $\beta$. As a general
example, for an arbitrary function \( f(\lambda, \nu) \), the Taylor series expansion for both \( \lambda \) and \( \nu \) small is

\[
f(\lambda, \nu) = f(0, 0) + \left( \frac{\partial f}{\partial \lambda} \right)_0 \lambda + \left( \frac{\partial f}{\partial \nu} \right)_0 \nu + \ldots
\]

whereas a Taylor series expansion for \( \lambda \) small and \( \nu \) large is

\[
f(\lambda, \nu) = f(0, \infty) + \left( \frac{\partial f}{\partial \lambda} \right)_\infty \lambda + \left( \nu^2 \frac{\partial^2 f}{\partial \nu^2} \right)_\infty \nu^{-1} + \ldots
\]

which is quite a different result. Also, for small \( \lambda \) and \( \nu \), it is known (6) that

\[
I_\perp(\theta)/R = 1 + \beta^2 f_{SM} + \pi \lambda \beta \sin \theta/2 \left( 1 - \sin \theta/2 \right) + 0(\lambda^2),
\]

so only in the case that \( \lambda \) is less than \( \beta \) does the conclusion of Mott and Massey hold. Of course, this is true for \( \nu \) less than one, but the point is that for large \( \nu \), the term proportional to \( \lambda \beta \) is no longer present, but essentially is replaced by a term of order \( \beta^2 \) (cf. Eqs. 5.11 and 5.15) when the non-relativistic limit is approached by an expansion in powers of \( 1/\nu \).

From Figs. 6 and 7, it can be seen that for a repulsive potential, the asymmetry parameter \( S(\theta) = -I_2(\theta)/I_1(\theta) \) is always positive, whereas for an attractive potential the asymmetry is generally positive. Only for the case of \( \lambda^2 \) approximately greater than \( 1/6 \) is it possible for the correction term to reverse the sign of the asymmetry in the region of \( \theta = 90^\circ \). The several oscillations in sign of the asymmetry
parameter exhibited by Sherman (8) in the lower energy region of his exact numerical calculations do not appear in these results. However, the largest Born parameter he considered was $|v| = 3$ at $Z = 80$, so this really is not consistent with the approximation $|v| \gg 1, \lambda \ll 1$. Also, the asymmetry parameter is very small and he obtained the value numerically by summing a series of large terms with alternating signs. Thus, the slow oscillations could be more apparent than real in the energy range where they appeared.

Finally, it is to be noted that as expected, in the extreme non-relativistic limit, the differential cross section goes to the Rutherford value, the asymmetry becomes zero, and the polarization direction of the incident wave does not change.
Fig. 4. The $I_1(\theta)/R$ first order correction function $f_1(-,\theta)$ and second order correction function $g_1(-,\theta)$
Fig. 5. The $I_1(\theta)/R$ first order correction function $f_1(+,\theta)$ and second order correction function $g_1(+,\theta)$ (insert depicts logarithm of functions vs. angle)
Fig. 6. The $I_2(\theta)/R$ asymmetry function $f_2(-,\theta)$ and the next order correction function $g_2(-,\theta)$
Fig. 7. The $I_2(\theta)/R$ asymmetry function $f_2(\pm, \theta)$ and the next order correction function $g_2(\pm, \theta)$ (insert depicts logarithm of functions vs. angle)
Fig. 8. The \( I_3(\theta)/R \) functions \( f_3(-, \theta) \) and \( f_3(+, \theta) \) (insert depicts logarithm of function vs. angle)
VI. REFERENCES


VII. ACKNOWLEDGEMENTS

The author wishes to thank Dr. C. L. Hammer for suggesting this problem and directing the subsequent research. His infectious enthusiasm, ideas, and criticisms contributed greatly to the success of this work. The author also wishes to record his gratitude to Dr. T. A. Weber for many valuable discussions on the subject of this thesis, and to Drs. J. M. Keller and R. H. Good, Jr., for their continuing interest in and guidance of the professional development of the author.

The author's wife, Dorothea, has shared with him the uncertainties of research, has provided the necessary stabilizing influence, and her patience and understanding are deeply appreciated.

Finally, the author wishes to thank Dr. G. W. Fox, former Chairman of the Physics Department, for initially encouraging the author to pursue graduate studies at Iowa State University and later extending a personal and friendly welcome.
A. Confluent Hypergeometric Functions

In this appendix some of the properties of the confluent hypergeometric function and associated functions are listed for convenient reference. On the whole, the conventions of Meixner (32) are followed.

The confluent hypergeometric function is a solution of the differential equation

\[ \left[ x \frac{d^2}{dx^2} + (c - x) \frac{d}{dx} - a \right] F = 0 \]  \hspace{1cm} (8.1)

which is regular at the origin. It has the series representation

\[ _1F_1(a, c, x) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(c)}{\Gamma(a)\Gamma(c + n)} \frac{x^n}{n!} \] \hspace{1cm} (8.2)

which indicates its behavior for small \( x \), and for \( |x| \to \infty \), it possesses the asymptotic expansion

\[ _1F_1(a, c, x) \to \frac{\Gamma(c)}{\Gamma(c - a)} e^{i\pi a} x^{-a} \{ 1 - a(1+a-c) + O(\frac{1}{x^2}) \} \]

\[ + \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \{ 1 + \frac{(c-a)(1-a)}{x} + O(\frac{1}{x^2}) \}, \] \hspace{1cm} (8.3)

where \( \epsilon = 1 \) for \( 0 < \arg x < \pi \), and \( \epsilon = -1 \) for \( 0 > \arg x > -\pi \).

The confluent hypergeometric function satisfies the
Kummer relation

\[ {}_1F_1(a, c, x) = e^x {}_1F_1(c-a, c, e^{i\pi x}) \]  \hspace{1cm} (8.4)

and its derivative is related to a contiguous function in the following manner:

\[ \frac{d}{dx} {}_1F_1(a, c, x) = \frac{a}{c} {}_1F_1(a+1, c+1, x) \]  \hspace{1cm} (8.5)

The Eulerian integral representation is given by

\[ {}_1F_1(a, c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt}(1-t)^{c-a-1} t^{a-1} dt \]  \hspace{1cm} (8.6)

where \( \text{Re } c > \text{Re } a > 0 \).

Using a notation adapted from Meixner (32), the confluent hypergeometric function can be separated into two parts with different characteristic asymptotic behavior:

\[ {}_1F_1(a, c, x) = \frac{1}{2} {}_1F^1(a, c, x) + \frac{1}{2} {}_1F^0(a, c, x) \]  \hspace{1cm} (8.7)

These two functions separately satisfy Eq. 8.1 but are not regular at the origin. For \(|x| \to \infty\),

\[ \frac{1}{2} {}_1F^1(a, c, x) \to \frac{\Gamma(c)}{\Gamma(c-a)} e^{ix \alpha} x^{-a} \]  \hspace{1cm} (8.8)

\[ \frac{1}{2} {}_1F^0(a, c, x) \to \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \]  \hspace{1cm} (8.9)

The Mellin-Barnes type integral representations for these functions are:
\[ \frac{1}{2} F^i(a, c, x) = \frac{\Gamma(c)x^{-a}e^{i\pi a\epsilon}}{\Gamma(a)\Gamma(c - a)\Gamma(1 + a - c)2\pi i} \times \frac{1}{i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s)\Gamma(1 + a - c - s)\Gamma(a - s)x^s ds}{\Gamma(s)\Gamma(l + a - c - s)\Gamma(c - a - s)x^{-s}e^{-i\pi s} ds} , \] 

\[ \frac{1}{2} F^0(a, c, x) = \frac{\Gamma(c)x^{a-c}e^{x}}{\Gamma(a)\Gamma(c - a)\Gamma(1 - a)2\pi i} \times \frac{1}{i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s)\Gamma(l - a - s)\Gamma(c - a - s)x^{s}e^{-i\pi s} ds}{\Gamma(s)\Gamma(l + a - s)\Gamma(c - a - s)x^{-s}e^{-i\pi s} ds} . \] 

Here \(|\arg x| < \pi\), and the integration path is such that the poles of \(\Gamma(s)\) are to the left, while the poles of the remaining gamma functions are on the right.

The function \(F^0\) (26, p. 672) is also represented by

\[ \frac{1}{2} F^0(a, c, x) = \frac{\Gamma(c)x^{a-c}e^{x}}{\Gamma(a)\Gamma(c - a)\Gamma(1 - a)2\pi i} \int_0^\infty e^{-t} t^{c-a-1}(1 - \frac{t}{x})^{a-1} \frac{dt}{x} \] 

where specifically in this formula \(0 < \arg x < 2\pi\), and \(\Re c > \Re a > 0\).

For non-negative integer \(n\):

\[ F^0(-n, c, x) = 0, \]  
\[ F^i(c+n, c, x) = 0, \]

from which one may deduce the equivalence

\[ \frac{1}{2} F^0(c+n, c, x) = \frac{1}{2} F^i(c+n, c, x) . \] 

Finally, the following circuit relations are valid:
\[ F^0(a, c, xe^{2i\pi}) = e^{2ni(a-c)} e^{F^1(c-a, c, xe^{i\pi})}, \quad (8.16) \]

\[ F^1(a, c, xe^{2i\pi}) = F^1(a, c, x). \quad (8.17) \]
IX. APPENDIX B

A. Coulomb Integral

In this appendix, the Coulomb integral

$$I(k,v) = \lim_{\epsilon \to 0^+} \int_0^\infty e^{-\epsilon x} x^{2k-1} e^{-ix} \left[ \text{E}_1(k+1+i\nu, 2k+1, e^{2x}) \right] dx$$

is evaluated.

Using the integral representation for the \text{E}_1 functions given in Eq. 8.6, interchanging orders of integration, and doing the Laplace integral in \( x \), the Coulomb integral may be expressed in the form:

$$I(k,v) = c(k) \lim_{\epsilon \to 0^+} \int_0^1 \int_0^1 dt \, ds \left[ \frac{(1-t)(1-s)}{(l-t)(l-s)} \right]^{k-\nu-1} \times (ts)^{k+i\nu} (l-i\epsilon-t-s)^{-2k}$$

where

$$c(k) = (-1)^k \Gamma(2k) \left[ \frac{\Gamma(2k+1)}{\Gamma(k+1+i\nu)\Gamma(k-i\nu)} \right]^2.$$

Upon changing the range of integration by the variable substitutions \( \zeta = t/(1-t) \) and \( \xi = s/(1-s) \), the previous integral assumes the form
I(k,ν) =

\[
\begin{align*}
& c(k) \lim_{\epsilon \to 0^+} \int_0^\infty \frac{d\zeta}{\zeta} \int_0^\infty d\xi \ (\xi \zeta)^{k+i\nu} \left[ (1+\zeta)(1+\xi) \right]^{-1} \times \\
& \left[ 1 - \xi \zeta - i\epsilon(1 + \zeta)(1 + \xi) \right]^{-2k} .
\end{align*}
\] (9.4)

Defining the two auxiliary quantities

\[
A(\zeta) = \frac{1 - i\epsilon - i\epsilon \zeta}{i\epsilon + i\epsilon \zeta + \zeta} ,
\] (9.5)

\[
B(\zeta) = (\zeta + i\epsilon \zeta + i\epsilon) ,
\] (9.6)

the Coulomb integral I(k,ν) may now be written

\[
I(k,\nu) = \lim_{\epsilon \to 0^+} c(k) \int_0^\infty \frac{d\zeta}{\zeta} \left( A(\zeta) \right)^{k+i\nu} G(A(\zeta),\nu) ,
\] (9.7)

where

\[
G(A(\zeta),\nu) = \int_0^\infty \frac{d\xi \xi^{k+i\nu}}{(1+\xi)\left[ \xi - A(\zeta) \right]^{2k}} .
\] (9.8)

For \( \nu = 0 \), the integral \( G(A(\zeta),\nu) \) can be evaluated by partial fractions. For the more general case, \( \nu \neq 0 \), which shall be considered here, the integral \( G(A(\zeta),\nu) \) is evaluated by considering a closed contour in the complex \( \xi \) plane indented around the branch cut taken on the real axis, and completed at infinity, as in Fig. 9. The contribution from the circular arc at infinity is zero, so by the residue theorem of complex variable theory,
\[(7.17)\]

\[8.14\]
\[
G(A(\zeta),\nu) = \frac{\pi i e^{\nu \pi}}{\sinh \nu \pi} \sum \text{Residues} \\
= \frac{\pi i e^{\nu \pi}}{\sinh \nu \pi} \left[ \frac{e^{i \pi (k + i \nu)}}{\left(1 + A(\zeta)\right)^{2k}} + \frac{1}{\Gamma(2k)} \left[ \frac{\delta^{2k-1} \left( \frac{k + i \nu}{1 + \zeta} \right)}{\delta^{2k-1}} \right]_{\zeta = A(\zeta)} \right]
\]

Explicitly evaluating the residue, one obtains
\[
G(A(\zeta),\nu) = \frac{\pi i e^{\nu \pi}}{\sinh \nu \pi} \left[ \frac{e^{i \pi (k + i \nu)}}{\left(1 + A(\zeta)\right)^{2k}} \right. \\
+ \left. \frac{2k-1}{\sum_{s=0}^{2k-1} \Gamma(k+l+i\nu)(-1)^{l+s} \left[ \frac{A(\zeta)}{1 + A(\zeta)} \right]^{k+i\nu-s}} \right] \\
\]

Now, from Eqs. 9.5 and 9.6, \(A(\zeta)\) and \(B(\zeta)\) are in the fourth and first quadrant, respectively, since the \(\xi\) plane has been cut along the real axis. Also, \(A(\zeta)\) and \(B(\zeta)\) may be written:
\[
A(\zeta) = \frac{-c_1(\zeta + \frac{1}{c_1})}{(\zeta + c_1)} , \quad (9.11)
\]
\[
B(\zeta) = (1 + i \epsilon)(\zeta + c_1) = e^{i \epsilon}(\zeta + c_1) \quad (9.12)
\]
where
\[
c_1 = \frac{i \epsilon}{1 + i \epsilon} . \quad (9.13)
\]

If the phase of \(c_1\) is chosen in the first quadrant, then \(A(\zeta)\) and \(B(\zeta)\) have the proper phases. Consequently, to first order
of infinitesimal, $c_1$ is defined by

$$c_1 = e^{i \frac{\pi}{2} - \epsilon},$$

(9.14)

and it follows that $A(\zeta)$ and $B(\zeta)$ are related by

$$[1 + A(\zeta)] = \frac{e^{2\pi i}}{B(\zeta)}.$$

(9.15)

Substituting Eqs. 9.10, 9.11, 9.12, and 9.15 into Eq. 9.7, one obtains

$$I(k, \nu) = \lim_{\epsilon \to 0^+} \frac{c(k) i \pi (-1)^k}{\sinh \pi \nu} \int_0^\infty \frac{d\zeta}{(1 + \zeta)^{2k+1}} \left[ \begin{array}{c} 1 \\ 2k-1 \end{array} \right]$$

$$+ \sum_{s=0}^{2k-1} \frac{\Gamma(k+l+iv)(1+i\epsilon)^{-s}(c_1 \zeta + c_1/c_1^*)^{-s}}{s! \Gamma(k+l+iv-s)(1+\zeta)^{-s}(\zeta+c_1)^{-s}}.$$

(9.16)

Since the integrand is uniformly convergent, and all the phases have been determined unambiguously, the limit may be taken under the integral sign with the result:

$$I(k, \nu) = \frac{c(k) i \pi (-1)^k}{\sinh \pi \nu} \int_0^\infty \frac{d\zeta}{(1 + \zeta)^{2k+1}} \left[ \begin{array}{c} 1 \\ 2k-1 \end{array} \right]$$

$$- \sum_{s=0}^{2k-1} \frac{\Gamma(k+l+iv)e^{i\pi(k+iv-s)}}{s! \Gamma(k+l+iv-s)(1+\zeta)^{2k+1-s}}.$$

(9.17)

The first integral is a standard integral representation (27, p. 9) for the beta function, defined by
\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \]  \hspace{1cm} (9.18) \]

and the remaining integrals are elementary. Using Eqs. 9.3 and 9.18, Eq. 9.17 reduces to

\[ I(k, \nu) = \frac{\Gamma(2k)}{B(k+1+i\nu, k-i\nu)} \frac{1}{2\sinh \pi \nu} \text{ times} \]  \hspace{1cm} (9.19) \]

\[ \left[ B(k+1+i\nu, k-i\nu) - e^{-\pi \nu} \sum_{s=0}^{2k-1} \frac{\Gamma(k+1+i\nu)(-1)^{k-s}}{s! \Gamma(k+1+i\nu-s)(2k-s)} \right] \]

Using the gamma function relation

\[ \Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x} \]  \hspace{1cm} (9.20) \]

and changing the variable of summation by the substitution \( s = 2k - n \), Eq. 9.19 may be written in the more compact form:

\[ I(k, \nu) = \frac{\Gamma(2k)}{B(k+1+i\nu, k-i\nu)} \text{ times} \]  \hspace{1cm} (9.21) \]

\[ \left[ \frac{i\pi}{\sinh \pi \nu} - e^{-\pi \nu} \sum_{n=1}^{2k} \frac{\Gamma(k+1+i\nu, k-n-i\nu)}{n \cdot B(k+1+i\nu, k-i\nu)} \right] \]

In order to simplify the sum in Eq. 9.21, the summand is resolved into partial fractions (33, p. 20) with respect to the parameter \( i\nu \).

\[ \Sigma = \sum_{n=1}^{2k} \frac{B(k+1+i\nu, k-n-i\nu)}{n \cdot B(k+1+i\nu, k-i\nu)} \]  \hspace{1cm} (9.22) \]

\[ = \sum_{n=1}^{2k} \frac{(2k)! \cdot (-1)^n}{n(2k - n)!} \left[ i\nu - (k-1) \right] \frac{1}{[i\nu - (k-2)] \ldots [i\nu - (k-n)]} \]
and the remaining integrals are elementary. Using Eqs. 9.3 and 9.18, Eq. 9.17 reduces to

\[
I(k,\nu) = \frac{\Gamma(2k) \, i\pi}{\left[B(k+1+i\nu, k-i\nu)\right]^2 \sinh \pi \nu} \text{ times}
\]

\[
\left[B(k+1+i\nu, k-i\nu) - e^{-\pi \nu} \sum_{s=0}^{2k-1} \frac{\Gamma(k+1+i\nu)(-1)^{k-s}}{s! \Gamma(k+1+i\nu-s)(2k-s)}\right].
\]

Using the gamma function relation

\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},
\]

and changing the variable of summation by the substitution \(s = 2k - n\), Eq. 9.19 may be written in the more compact form:

\[
I(k,\nu) = \frac{\Gamma(2k)}{B(k+1+i\nu, k-i\nu)} \text{ times}
\]

\[
\left[\frac{i\pi}{\sinh \pi \nu} - e^{-\pi \nu} \sum_{n=1}^{2k} \frac{B(k+1+i\nu, k-n-i\nu)}{n \, B(k+1+i\nu, k-i\nu)}\right].
\]

In order to simplify the sum in Eq. 9.21, the summand is resolved into partial fractions \((33, p. 20)\) with respect to the parameter \(i\nu\).

\[
\Sigma = \sum_{n=1}^{2k} \frac{B(k+1+i\nu, k-n-i\nu)}{n \, B(k+1+i\nu, k-i\nu)}
\]

\[
= \frac{(2k)!(-1)^n}{n(2k-n)!} \frac{1}{[i\nu-(k-1)] \, [i\nu-(k-2)] \ldots \, [i\nu-(k-n)]}
\]
Substitution of this into Eq. 9.24 yields the interesting relation:

\[ \sum_{n=1}^{2k} \frac{B(k+1+i\nu, k-n-i\nu)}{nB(k+1+i\nu, k-i\nu)} = -\sum_{n=1}^{2k} \frac{1}{(i\nu+n-k)} . \tag{9.27} \]

The right hand side of Eq. 9.27 can be expressed in yet another form. Using Eq. 9.20, it is found that in general

\[ \Gamma(k+1+z)\Gamma(k-z) = \frac{\pi(-1)^{k-1}}{\sin(\pi z)} \prod_{n=1}^{2k} (z + n - k) . \tag{9.28} \]

Defining the logarithmic derivative of the gamma function (27, p. 15) \( \psi_1(x) \) by

\[ \frac{3 \ln \Gamma(x)}{3x} = \psi_1(x) , \tag{9.29} \]

and taking the logarithmic derivative of Eq. 9.28, one obtains the relation

\[ \psi_1(k+1+z) - \psi_1(k-z) = -\pi \cot \pi z + \sum_{n=1}^{2k} \frac{1}{z + n - k} . \tag{9.30} \]

Setting \( z = i\nu \), it is found that

\[ \sum_{n=1}^{2k} \frac{1}{i\nu+n-k} = \psi_1(k+1+i\nu) - \psi_1(k-i\nu) - i\pi \coth \pi \nu . \tag{9.31} \]

Finally, substituting Eqs. 9.27 and 9.31 into Eq. 9.21, the Coulomb integral \( I(k, \nu) \) is given by the equivalent expressions:
\[ I(k,\nu) \]

\[ = \frac{\Gamma(2k)}{B(k+1+i\nu, k-i\nu)} \left[ \frac{1}{\sinh \pi \nu} + e^{-\pi \nu} \sum_{n=1}^{2k} \frac{1}{(i\nu + n - k)} \right] \quad (9.32) \]

\[ = \frac{\Gamma(2k) e^{-\nu \pi}}{B(k+1+i\nu, k-i\nu)} \left[ i\pi + \psi_1(k+1+i\nu) - \psi_1(k-1\nu) \right]. \]

To the author's knowledge, the definite integral \( I(k,\nu) \) given by Eq. 9.1, has not been evaluated before. Integrals of this form are only known for those cases where the exponent of \( x \) has integral values greater than \( 2k - 1 \).
A. Asymptotic Expansion of $S(\theta, \nu)$

1. Integral transformation

The two parameter function $S(\theta, \nu)$ is given by the integral expression

$$S(\theta, \nu) = \int_0^1 dx x^{-1+\nu-1}(1-x)^{21\nu} \frac{(1-x-\left[\frac{1+x^2-2x \cos \theta}{1+x^2-2x \cos \theta}\right]^{1/2})}{\left[\frac{1+x^2-2x \cos \theta}{1+x^2-2x \cos \theta}\right]^{1/2}}$$

times $(i\pi - \ln x)$. \hspace{1cm} (10.1)

By a change of integration variable, this integral is transformed to a Fourier integral which can then be evaluated asymptotically using the methods of Lighthill (29).

A one to one mapping of the interior of a unit circle in the $x$ plane into an infinite strip in the $y$ plane is specified by the transformation

$$x = \frac{1}{2}(2 + e^y - \left[e^{2y} + 4e^y\right]^{1/2}), \hspace{1cm} (10.2)$$

where $-\pi < \text{Im}\, y < \pi$, so $|x| < 1$. The inverse transformation is

$$\frac{(1-x)^2}{x} = e^y. \hspace{1cm} (10.3)$$

This mapping is illustrated in Figs. 10 and 11. The branch lines of $x - e^{\pm i\theta}$ which occur in the integrand of Eq. 10.1 are
Fig. 10. Contour in x plane

Fig. 11. Contour in y plane
chosen so that \(1 + x^2 - 2x \cos \theta\) has zero phase along the positive real \(x\) axis, and the branch cut for \(\ln x\) is taken along the negative real axis. These choices are indicated in Fig. 10.

Under this transformation, the integral for \(S(\theta, \nu)\) becomes

\[
S(\theta, \nu) = \int_{-\infty}^{\infty} e^{i\nu y} f(y) \, dy ,
\]

where

\[
f(y) = \frac{(1 - x)(1 - x - [1 + x^2 - 2x \cos \theta]^{1/2})}{(1 + x)[1 + x^2 - 2x \cos \theta]^{1/2}} (i\pi - \ln x),
\]

in which it is understood that \(x\) is the function of \(y\) given explicitly by Eq. 10.2.

For large positive \(y\), \(x = \frac{1}{4} e^{-y}\), and for large negative \(y\), \(x = 1 - \frac{1}{2} e^{-|y|}\), so that

\[
f(y) = -\frac{1}{2} \sin^2 \theta/2 \, y \, e^{-y} \text{ for } y \to \infty ,
\]

and

\[
f(y) = \frac{i\pi}{4} \, e^{-|y|} \text{ for } y \to -\infty.
\]

2. **Born parameter greater than zero**

For \(\nu > 0\), the closed contour just within \(EA'B'C'E\) in the \(x\) plane is considered. The corresponding contour for the \(y\) plane integrand is \(AEF'C'B'D'A\). The integrand is analytic,
and the contributions from $\mathbb{F}'$ and $D'\mathbb{A}$ vanish, so

$$S(\theta, \nu) = - \int_{D'B'C'F'} e^{i\nu y} f(y) \, dy \quad (10.8)$$

Setting $y = \nu + z$, the integral becomes

$$S(\theta, \nu) = - e^{i\nu} \int_{-\infty}^{\infty} e^{i\nu z} f(y) \, dz \quad (10.9)$$

Here

$$2\pi(z) = 2 + e^{i\pi} e^{z} - \left(e^{2i\pi} e^{2z} + 4e^{i\pi} e^{z}\right)^{1/2} \quad (10.10)$$

The singularities of the integrand are at $z = z_1 = 2 \ln 2 \sin \frac{\theta}{2}$ (B'), and at $z = z_0 = 2 \ln 2$ (C').

Expanding about $z - z_0 = \zeta$ with the aid of the binomial theorem, it is found that

$$f(y) = \frac{2i\pi(1 - \cos \theta/2)}{|\zeta|^{1/2} \cos \theta/2} \times \left[ 1 - i e^{\frac{\pi}{4} \frac{\zeta}{|\zeta|} - \frac{1}{2}} \right]^{1/2} \pi^{-1} \quad (10.11)$$

$$+ \frac{1}{4} |\zeta| e^{\frac{i\pi}{2} \frac{\zeta}{|\zeta|} - \frac{1}{2}} (1 - \frac{2(1 + \cos \theta/2)}{\cos^2 \theta/2}) + \ldots \right],$$

where the expansion requires $\theta \neq \pi$. The proper phases for intermediate calculations are given by
\[ x = e^{-1\pi} + 2e^{\frac{1\pi}{2}(\frac{\zeta}{|\zeta|} - 1)} |\zeta|^{1/2} \]

\[ - 2|\zeta|e^{\frac{1\pi}{2}(\frac{\zeta}{|\zeta|} - 1)} + \frac{1}{2}e^{\frac{3\pi}{4}(\frac{\zeta}{|\zeta|} - 1)} |\zeta|^{3/2} + \ldots, \]

\[ x - e^{+i\theta} = 2 \cos \theta/2 e^{\frac{1}{4}(\pi - \theta/2)} \left[ 1 + \frac{e^{+i(\pi - \theta/2)}}{2 \cos \theta/2} (x + 1) \right]. \]

Similarly, expanding about \( z - z_1 = \xi \), it is found that

\[ f(y) = -|\xi|^{-1/2} \tan \theta/2 (\pi + \theta)e^{-\frac{1\pi}{4}(\frac{\xi}{|\xi|} - 1)} \]

\[ \left[ 1 - e^{\frac{1\pi}{4}(\frac{\xi}{|\xi|} - 1)} |\xi|^{1/2} \right] \]

\[ + e^{\frac{1\pi}{2}(\frac{\xi}{|\xi|} - 1)} |\xi| (\frac{2 + \cos^2 \theta/2}{4 \cos^2 \theta/2} + \frac{\tan \theta/2}{\pi + \theta}) + \ldots \],

where \( \theta \neq \pi \). Here, the proper phases for intermediate calculation are given by

\[ x = e^{-1\theta} \left[ 1 - i|\xi| \tan \theta/2 - \frac{i|\xi|^2}{4} \frac{\tan \theta/2}{\cos^2 \theta/2} (2 \cos^2 \theta/2 - e^{i\theta}) \right], \]

\[ x - e^{-i\theta} = |\xi|e^{\frac{1}{2}(\frac{\xi}{|\xi|} - \frac{\xi}{2} - \theta)} e^{\frac{1}{2} \tan \theta/2} \text{ times} \]

\[ \left[ 1 + \left( \frac{1}{2} - \frac{1}{4} \sec^2 \theta/2 e^{i\theta} \right) |\xi| + \ldots \right], \]

\[ x - e^{i\theta} = 2 \sin \theta e^{\frac{1}{2} \frac{1\pi}{4} \sec^2 \theta/2 e^{-i\theta} |\xi| + \ldots} \].
The integral for $S(\Theta, \nu)$ may be written

$$-e^{2\pi \nu} S(\Theta, \nu) = \int_{-\infty}^{\infty} e^{i \nu z} f(y) H(z_1 - z) dz$$

(10.18)

$$+ \int_{-\infty}^{\infty} e^{i \nu z} f(y) H(z_0 - z) H(z - z_1) dz + \int_{-\infty}^{\infty} e^{i \nu z} f(y) H(z - z_0) dz,$$

where the symbol $H(s)$ signifies the step function

$$H(s) = 0 \text{ for } s < 0,$$

$$H(s) = 1 \text{ for } s > 0.$$  

(10.19)

Using Lighthill's theorems (29, p. 46) on asymptotic estimation of Fourier transforms, the integrals in Eq. 10.18 are evaluated by expanding the integrands about their singularities. Thus

$$-e^{2\pi \nu} S(\Theta, \nu) = e^{i \nu z_1} \int_{-\infty}^{\infty} e^{i \nu \xi} f(\xi^-) H(-\xi) d\xi$$

$$+ e^{i \nu z_1} \int_{-\infty}^{\infty} e^{i \nu \xi} f(\xi^-) H(\xi) d\xi + e^{i \nu z_0} \int_{-\infty}^{\infty} e^{i \nu \zeta} f(\zeta^-) H(-\zeta) d\zeta$$

(10.20)

$$+ e^{i \nu z_0} \int_{-\infty}^{\infty} e^{i \nu \zeta} f(\zeta^+) H(\zeta) d\zeta,$$

where symbolically, $f(\xi^-)$ and $f(\xi^-)$ mean $f(y)$ as given in Eq. 10.14 for $\xi > 0$ and $\xi < 0$, respectively. The similar meaning is attached to $f(\zeta^+)$ and $f(\zeta^-)$ in relation to Eq. 10.11.

The integrals are evaluated with the aid of Table 1,
which is modified from Lighthill (29, p. 43).

Table 1. Fourier transforms for \( \nu \neq 0 \)

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( g(\nu) = \int_{-\infty}^{\infty} e^{-i\nu x} f(x) dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>( H(x) )</td>
<td>( -1/\nu )</td>
</tr>
<tr>
<td>( xH(x) )</td>
<td>( -\nu^{-2} )</td>
</tr>
</tbody>
</table>

The result of the indicated operations is the following asymptotic expansion for \( \nu \to \infty \):

\[
S(\theta, \nu) = 2(\nu)^{-1/2} \sec \theta/2 \sqrt{\pi} \exp \left[ -\pi \nu - i\pi/4 + 2i\nu \ln 2 \sin \theta/2 \right]
\times g_1(\theta, \nu) \left[ 1 - \frac{g_2(\theta, \nu)}{8\nu \cos^2 \theta/2 \ g_1(\theta, \nu)} + O(\nu^{-2}) \right]
\]

(10.21)

where

\[
g_1(\theta, \nu) = 1 \sin \theta/2 (\pi + \theta) + 2\pi (1 - \cos \theta/2) \exp(-2i\nu \ln \sin \theta/2),
\]

(10.22)
\[ g_2(\theta, \nu) = \frac{\sin \theta/2 (\pi + \theta)(2 + \cos^2 \theta/2) + 2 \sin \theta \sin \theta/2}{\nu \cos^2 \theta/2} e^{2\nu i \ln \sin \frac{\theta}{2}}. \]  

(10.23)

\[ -2\pi i (1 - \cos \theta/2)(\cos^2 \theta/2 - 2 - 2 \cos \theta/2) \exp(-2i\nu \ln \sin \frac{\theta}{2}) \].

In the derivation, \( \theta \) has been restricted to \( 0 \leq \theta < \pi \). In order to estimate the magnitude of \( \nu \) necessary for this asymptotic expansion to approximate the function \( S(\theta, \nu) \), the condition

\[ 1 > \left| \frac{g_2(\theta, \nu)}{\delta \nu \cos^2 \theta/2 g_1(\theta, \nu)} \right| \]  

(10.24)

is imposed. This restriction may be written in terms of the simple relation between \( \theta = \pi - \Delta \) and \( \nu \),

\[ \Delta^2 > 1/\nu, \]  

(10.25)

so that, approximately, for \( \nu = 100, \theta < 174^\circ \), for \( \nu = 10, \theta < 160^\circ \), and for \( \nu = 4, \theta < 150^\circ \). Thus, at the very best, for finite values of \( \nu \), this asymptotic expansion does not represent the function at extreme back angles. For the singular angle \( \theta = \pi \), one could evaluate \( S(\theta, \nu) \) directly after making this substitution. In that case \( f(y) \) given in Eq. 105 would become singular as \( x \) approaches \( e^{-i\pi} \) in the manner \( 1/|\zeta| \) instead of like \( 1/|\zeta|^{1/2} \). This would lead to the asymptotic expansion of \( S(\theta, \nu) \) having a pre-factor of \( \ln \nu \) rather than \( \nu^{-1/2} \). Since continuous functions of \( \theta \) are of interest, the singular case \( \theta = \pi \) will not be discussed further here.
3. Born parameter less than zero

For \( v < 0 \), the closed contour just within EABCE is considered. The corresponding contour for the \( y \) plane integrand is AEFCBDA. In a similar fashion to the preceding case of \( v > 0 \), it is found that

\[
S(\Theta, v) = -e^{\pi v} \int_{-\infty}^{\infty} e^{ivz} f(y) dz
\]  
(10.26)

except that now \( y = -i\pi + z \), so

\[
2x(z) = 2 + e^{-i\pi} e^z - \left[ e^{-2i\pi} e^{2z} + 4e^{-i\pi} e^z \right]^{1/2}
\]  
(10.27)

The singularities of the integrand are at \( z = z_1 = 2 \ln 2 \sin \theta/2 \) (B), and at \( z = z_0 = 2 \ln 2 \) (C).

Expanding about \( z - z_0 = \zeta \), it is found that

\[
f(y) = 2 \sec \theta/2 (1 - \cos \theta/2) \left[ 1 + |\zeta| (1 - \frac{\tan^2 \theta/2}{2(1 - \cos \theta/2)}) \right] \]
(10.28)

where again the expansion requires \( \Theta \neq \pi \).

The proper phases for intermediate calculations are given by

\[
x = e^{i\pi} + 2|\zeta|^{1/2} \exp \left[ i\pi/4 (1 - \zeta/|\zeta|) \right] \\
- 2|\zeta| \exp \left[ \frac{i\pi}{2} (1 - \frac{\zeta}{|\zeta|}) \right] + \frac{3}{2} |\zeta|^{3/2} \exp \left[ \frac{3i\pi}{4} (1 - \frac{\zeta}{|\zeta|}) \right].
\]  
(10.29)

\[
x - e^{i\Theta} = 2 \cos \theta/2 e^{i(\pi - \theta/2)} \left[ 1 + \frac{1}{2} \sec \theta/2 e^{-i(\pi - \theta/2)} (x+1) \right].
\]  
(10.30)
Expanding about \( z - z_1 = \xi \), it is found that

\[
f(y) = |\xi|^{-1/2}(\pi - \theta) \tan \theta/2 \exp\left[\frac{1}{4}(\frac{\xi}{|\xi|} - 1)\right] \text{times}
\]

\[
\left\{ 1 - |\xi|^{1/2} \exp\left[\frac{1}{4}(1 - \frac{\xi}{|\xi|})\right] (\frac{1}{4} + \frac{1}{2} \sec^2 \theta/2 - \frac{\tan \theta/2}{\pi - \theta}) \right\}.
\]

(10.31)

Here, \( \theta \neq \pi \) also. The proper phases for intermediate calculations are

\[
x = e^{i\theta} \left[ 1 + i\xi \tan \theta/2 \right. \]
\[
\left. + \frac{1}{4} i\xi^2 \sec^2 \theta/2 \tan \theta/2 \left( 2 \cos \theta/2 - e^{-i\theta}\right) \right],
\]

(10.32)

\[
x - e^{i\theta} = \tan \theta/2 |\xi| \exp\left[ -i\pi + i\theta - \frac{i\pi}{2} \frac{\xi}{|\xi|}\right] \text{times} \left[ 1 + (\frac{1}{2} - \frac{1}{4} \sec^2 \theta/2 e^{-i\theta}) \xi \right],
\]

(10.33)

\[
x - e^{-i\theta} = 2 \sin \theta e^{i\pi/2} \left[ 1 + \frac{1}{4} \sec^2 \theta/2 e^{+i\theta} \xi \right].
\]

(10.34)

The integral for \( S(\theta, \nu) \) may be written

\[
\int_{-\infty}^{\infty} e^{i\nu z} f(y) H(z_1 - z) dz
\]

(10.35)

\[
+ \int_{-\infty}^{\infty} e^{i\nu z} f(y) H(z_0 - z) H(z - z_1) dz + \int_{-\infty}^{\infty} e^{i\nu z} f(y) H(z - z_0) dz.
\]

Using the same series of steps following Eq. 10.18, the following asymptotic expansion for \( \nu \to -\infty \) is obtained:
\[ S(\theta, \nu) = -2|\nu|^{-1/2} \tan \theta/2 \sqrt{\pi} \left( \pi - \theta \right) \text{ times} \] (10.36)

\[ \exp \left[ -\pi|\nu| - \frac{\pi}{4} + 2i\nu \ln 2 \sin \theta/2 \right] \left\{ 1 - \frac{1}{2} |\nu|^{-1} g_3(\theta) + o(\nu^{-2}) \right\}, \]

where

\[ g_3(\theta) = \left\lfloor \frac{2 + \cos^2 \theta/2}{4 \cos^2 \theta/2} - \frac{\tan \theta/2}{(\pi - \theta)} \right\rfloor \] (10.37)

The angle \( \theta \) has been restricted by the derivation to \( 0 < \theta < \pi \). As an estimate of the magnitude of \( \nu \) necessary for this asymptotic expansion to approximate the function \( S(\theta, \nu) \), the condition

\[ 1 > \left| \frac{1}{2} \nu^{-1} g_3(\theta) \right| \] (10.38)

is imposed. For \( \theta = \pi - \Delta \), this restriction reduces to

\[ \Delta^2 > \frac{1}{2} |\nu|^{-1} \] (10.39)

so that, approximately, for \( \nu = 100, \theta < 176^\circ \), for \( \nu = 10, \theta < 167^\circ \), and for \( \nu = 4, \theta < 160^\circ \). It is seen that if one uses this criterion, the asymptotic expansion for finite values of \( -|\nu| \) represents the function \( S(\theta, -|\nu|) \) better at back angles then the corresponding expansion for \( S(\theta, |\nu|) \).
B. Asymptotic Expansion of \( T(\theta, \nu) \)

Using the Stirling approximation for the gamma function

\[
\Gamma(z) = e^{-z} z^{z - 1/2} \sqrt{2\pi} \left( 1 + \frac{1}{12z} + \ldots \right),
\]

(10.40)

where \(|\arg z| < \pi\), one finds that

\[
B(-iv, l+2iv) = 2 \sqrt{\pi} |\nu|^{-1/2} \times 
\exp \left\{ -\pi|\nu| + \frac{\nu}{4} + 2iv\ln 2 \right\} \left\{ 1 + 1/(8\nu) + o(\nu^{-2}) \right\}
\]

(10.41)

From the relation of \( S(\theta, \nu) \) to \( T(\theta, \nu) \) given by Eq. 4.9, and from Eq. 10.21, is obtained the asymptotic expansion of \( T(\theta, \nu) \) for \( \nu \to \infty \):

\[
T(\theta, \nu) = \frac{-g_1(\theta, \nu)}{2\nu \cos \theta/2} \left[ 1 - \frac{\sin \theta/2}{4\nu \cos^2 \theta/2} g_4(\theta, \nu) + o(\nu^{-2}) \right],
\]

(10.42)

where \( g_1(\theta, \nu) \) is given by Eq. 10.22, and \( g_4(\theta, \nu) \) is defined by

\[
g_4(\theta, \nu) = \left[ \pi + \theta + \sin \theta + 2\pi i \sin \theta/2 \right] e^{-2i\nu \ln \sin \theta/2}.
\]

(10.43)

Similarly, using Eq. 10.36, the asymptotic expansion of \( T(\theta, \nu) \) for \( \nu \to -\infty \) is found to be:

\[
T(\theta, \nu) = \frac{-i(\pi - \theta)}{2i\nu \cot \theta/2} \left[ 1 - \frac{i}{2\nu} \left( \frac{1}{2 \cos^2 \theta/2} - \tan \theta/2 \right) + \ldots \right].
\]

(10.44)

For convenience, the derivatives of the asymptotic expansion of \( T(\theta, \nu) \) with respect to \( \theta \) are also listed here.
For $\nu \to \infty$, using the abbreviation

$$\chi = 2\nu \ln \sin \theta/2,$$  \hspace{1cm} (10.46)

it is found that

$$\frac{\partial T(\theta, \nu)}{\partial \theta} = \frac{i\pi(1 - \cos \theta/2)}{\sin \theta/2} e^{-i\chi} - \frac{\pi \sin \theta/2}{4\nu \cos^2 \theta/2} e^{-i\chi}$$

$$\frac{1}{4\nu \cos^2 \theta/2} (\pi + \theta + \sin \theta)$$

$$+ \frac{1}{16 \nu^2 \cos^4 \theta/2} \left[ (1 + 2 \sin^2 \theta/2)(\pi + \theta) + 3 \sin \theta \right]$$

$$+ \frac{i\pi e^{-i\chi} \sin \theta/2}{8 \nu^2 \cos^4 \theta/2} \left[ 2 + \sin^2 \theta/2 + \ldots \right] + O(\nu^{-3}),$$  \hspace{1cm} (10.47)

where the dots signify additional terms coming from the next neglected term in the asymptotic expansion of $T(\theta, \nu)$.

Also, for $\nu \to -\infty$,

$$\frac{\partial T(\theta, \nu)}{\partial \theta} = \frac{i}{4|\nu| \cos^2 \theta/2} \left[ \sin \theta - (\pi - \theta) \right]$$

$$+ \frac{1}{8|\nu||2\cos^2 \theta/2} \left[ 3 \tan \theta/2 - \frac{1}{2}(\pi - \theta)(1 + 3 \tan^2 \theta/2) \right]$$

$$+ O(\nu^{-3}).$$

Since the asymptotic expansion of the derivatives of $S(\theta, \nu)$ exist, which could be obtained by differentiating the integral representation with respect to $\theta$ before performing the asymptotic expansion, the derivative of the asymptotic expansion is equal to the asymptotic expansion of the derivative.