ELASTODYNAMIC SCATTERING CROSS-SECTIONS FOR THREE-DIMENSIONAL WHISKER-LIKE INCLUSIONS

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INTRODUCTION

The problem formulation for scattering of a plane time-harmonic longitudinal wave by a cylindrical inclusion in a homogeneous, isotropic, linearly elastic solid is reduced to the solution of a system of singular integral equations over the inclusion-matrix interface. The inclusion is circular cylindrical with semi-spherical end sections. The singular integral equations are solved by the boundary element method. Once the interface fields have been determined, the scattered far-field is obtained by the use of elastodynamic representation integrals. Of particular interest is the scattering cross-section. The results can be used to determine the attenuation of a longitudinal wave propagating in a solid with a dilute random distribution of whiskers of the shape analyzed in this paper.

The scattering cross sections for a spherical cavity and spherical inclusions in an elastic solid have been discussed in detail by Johnson and Truell[1]. The elastodynamic counterpart of the optical theorem has been derived by Barratt and Collins[2] for a scatterer of arbitrary shape. The formulas to calculate scattering cross-sections have been presented by Gubernatis, Domany and Krumhansl[3]. These authors have also discussed approximations [4] and extensions to introduce multiple scattering effects[5].

SCATTERING AMPLITUDE

We consider the problem of scattering by a volumetric inclusion of surface S, as shown in Fig.1. The host material is assumed to be a homogeneous, isotropic, linearly elastic solid. The scattered wave field u_i^S is expressed as an integral over the surface S of the scatterer:

$$u_i^S(\mathbf{x}) = \int_S U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{y})dS_y - \int_S T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{y})dS_y \tag{1}$$

Review of Progress in Quantitative Nondestructive Evaluation, Vol. 12
Edited by D.O. Thompson and D.E. Chimenti, Plenum Press, New York, 1993

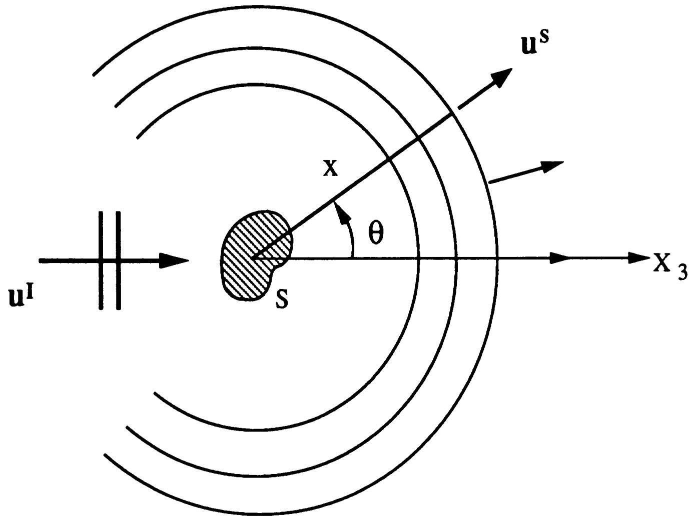

Fig.1 Incident wave and scattered far-field.

where u_i and t_i are the components of the displacement and the traction on the surface of the scatterer. The scattered field u_i^S has been defined as the total field u_i minus the incident field u_i^I

$$u_i^S(\mathbf{x}) = u_i(\mathbf{x}) - u_i^I(\mathbf{x}) \ . \tag{2}$$

For the time-harmonic case, the fundamental solution U_{ij} in Eq.(1) can be found in Ref.[6] and T_{ij} is the traction tensor corresponding to U_{ij}. After the introduction of the far-field approximation, the scattered far-field can be written as[6]:

$$\begin{aligned} u_i^S(\mathbf{x}) &\sim u_i^{SL}(\mathbf{x}) + u_i^{ST}(\mathbf{x}) \\ &= A_i(k_L)\, \frac{e^{ik_L x}}{x} + B_i(k_T)\, \frac{e^{ik_T x}}{x} \end{aligned} \tag{3}$$

where A_i and B_i are longitudinal and transverse scattering amplitudes, respectively. They have the forms

$$A_i(k_L) = \hat{x}_i\hat{x}_j f_j(k_L) \tag{4a}$$

$$B_i(k_T) = (\delta_{ij} - \hat{x}_i\hat{x}_j) f_j(k_T) \tag{4b}$$

where the vector $f_j(k_\alpha)$ is a function of the shape of the scatterer S, the direction of the field point $\hat{x}_i$ from the origin of the coordinate system, the displacement u_i and the traction t_i on the surface of the scatterer, the angular frequency ω ($k_\alpha=\omega/C_\alpha$, α=L,T), and the material properties of the host medium λ, μ and ρ. The vector $f_j(k_\alpha)$ can be expressed as

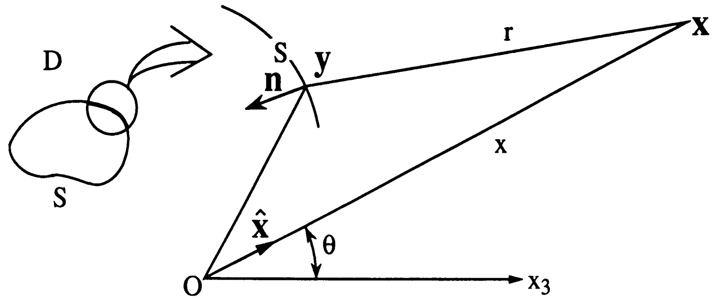

Fig.2 Geometrical relation.

$$f_j(k_\alpha) = \frac{1}{4\pi\rho c_\alpha^2}\int_S [ik_\alpha\lambda\hat{x}_j u_k(\mathbf{y})n_k(\mathbf{y}) + ik_\alpha\mu\{u_j(\mathbf{y})n_k(\mathbf{y}) + u_k(\mathbf{y})n_j(\mathbf{y})\}\hat{x}_k + t_j(\mathbf{y})]e^{-ik_\alpha\hat{\mathbf{x}}\cdot\mathbf{y}}dS_y\,, \qquad (\alpha=L,T)\,. \tag{5}$$

The far-field stress induced by the scattered displacement field given by Eq.(3) can be obtained as

$$\sigma_{ij}^S \sim i\lambda k_L A_k\hat{x}_k\delta_{ij}M(k_L x) + i\mu[k_L(A_i\hat{x}_j + A_j\hat{x}_i)M(k_L x) + k_T(B_i\hat{x}_j + B_j\hat{x}_i)M(k_T x)] \tag{6a}$$

where

$$M(k_\beta x) = \frac{e^{ik_\beta x}}{x}\,, \quad (\beta=L,T) \tag{6b}$$

To evaluate Eqs.(3)-(6), the displacement and the traction on the surface of scatterer must be determined. The geometry of the surface S, field point $\mathbf{x}$, source point $\mathbf{y}$, and unit vector $\hat{\mathbf{x}}$ are shown in Fig.2.

BOUNDARY INTEGRAL EQUATIONS

For the host medium, the boundary integral equation that can be obtained from Eq.(1) is

$$C_{ij}(\mathbf{x})u_j(\mathbf{x}) = \int_S U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{y})dS_y - \int_S T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{y})dS_y + u_i^I(\mathbf{x}), \quad \mathbf{x}\in S \tag{7}$$

where $u_i^I(\mathbf{x})$ is the incident wave and $C_{ij}(\mathbf{x})$ is the free term which is a function of the geometry at the point $\mathbf{x}$. Similarly, the boundary integral equation for the inclusion side of the interface can be written as

$$\bar{C}_{ij}(\mathbf{x})\bar{u}_j(\mathbf{x}) = \int_S \bar{U}_{ij}(\mathbf{x},\mathbf{y})\bar{t}_j(\mathbf{y})dS_y - \int_S \bar{T}_{ij}(\mathbf{x},\mathbf{y})\bar{u}_j(\mathbf{y})dS_y, \quad \mathbf{x}\in S \tag{8}$$

where bars over the physical quantities indicate that the quantities apply to the inclusion. The interface conditions are the conditions of perfect contact,

$$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}), \quad \mathbf{t}(\mathbf{x}) = -\bar{\mathbf{t}}(\mathbf{x}), \quad \mathbf{x}\in S. \tag{9}$$

The use of condition(9) with Eqs.(7) and (8) leads to a system of boundary integral equations for the unknowns **u** and **t**. The details of the numerical solution of these equations by the boundary element method have been discussed in Ref.[7].

SCATTERING CROSS-SECTIONS

For a given angular frequency ω the total cross-section P(ω) is defined as the ratio of the average power flux scattered into all directions to the average intensity of the incident field[3]:

$$P(\omega)=\frac{\langle P^S\rangle}{\langle I^I\rangle} \tag{10}$$

where the scattered power flux takes the form

$$\langle P^S\rangle = \int_A \langle x_i\sigma_{ij}^S\dot{u}_j^S\rangle dA \tag{11}$$

Here A is a surface with unit normal x_i which encloses the scatterer. For a plane wave propagating along the x_3-axis, the incident intensity can be written as

$$\langle I^I\rangle = \langle \sigma_{3j}^I\dot{u}_j^I\rangle \tag{12}$$

where the angular bracket denotes the time-average over the period.

We now consider the following longitudinal wave with unit amplitude

$$\mathbf{u}^I = \mathbf{e}_3\exp\left[i\left(k_L x_3-\omega t\right)\right]. \tag{13}$$

A convenient form to calculate the total cross-section has been given by Gubernatis *et al.*, Ref.[3], in the following form

$$P(\omega) = \frac{4\pi}{k_L}I_m A_3(0) \tag{14a}$$

where A_3 is the average in the azimuthal direction

$$A_3(0) = \frac{1}{2\pi}\int_0^{2\pi} A_3(\theta = 0, \phi)d\phi \tag{14b}$$

In Eq.(14), A_3 is the longitudinal scattering amplitude of Eq.(4a) in the direction of propagation of the longitudinal incident wave.

SCATTERING CROSS-SECTION FOR A WHISKER-LIKE INCLUSION

Figure 3 shows the geometry of a whisker-like inclusion. The inclusion is a circular cylindrical with semi-spherical end sections. The radius of the cylindrical and semi-spherical parts is a, the total length of the inclusion is 2(b+a), and b/a=4 for the calculations of this paper. The material properties were taken as $\bar{E}/E = 5$, $\bar{\nu} = \nu = 0.3$, $\bar{\rho}/\rho = 1$. Two directions have been considered for the propagation direction of the incident plane longitudinal wave. One is along the major axis (x_2-axis) of the whisker and the other is the direction perpendicular to the major axis (x_3-axis). Figure 4 shows the normalized total cross section $P_n = P/G_i$ (i=2,3) as a function of the dimensionless wavenumber ak_L. The total cross-section P was calculated by using the elastodynamic optical theorem given by Eq.(14). For the longitudinal wave propagating along the x_2-axis, the component A_2 defined by Eq.(4a) is used in the expression of Eq.(14). The geometrical cross-sections perpendicular to the x_2-

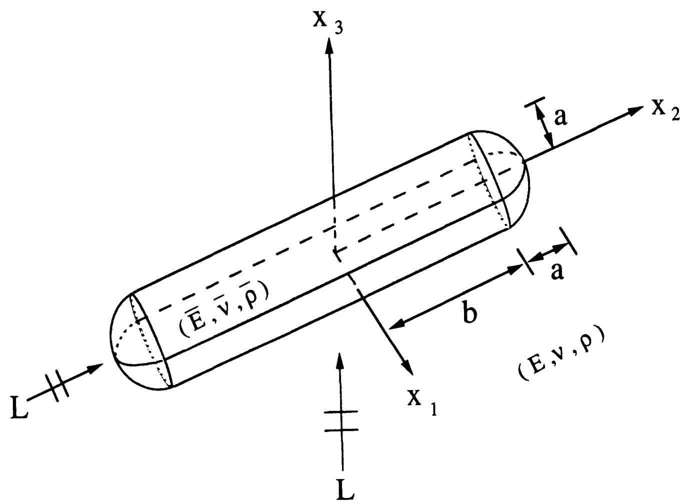

Fig.3 Whisker-like inclusion.

and x_3- axis are, respectively, $G_2 = \pi a^2$ and $G_3 = 4ab + \pi a^2 = (16 + \pi)a^2$ for b/a = 4. In the range of the computed wavenumbers, the normalized cross-section $P_n=P/G_2$ for incidence along the major axis of the cylindrical whisker is greater than P/G_3 for incidence normal to the major axis. It should, however, be noted that the values of the normalization factors G_2 and G_3 are quite different since $G_3/G_2 = (16 + \pi)a^2/\pi a^2$ is about 6. The total cross-section P for longitudinal wave incidence along the major axis is greater than that for incidence normal to the major axis in the small wave number range. At about $ak_L=1.0$, the total cross-section P for longitudinal incidence normal to the major axis begins to exceed that for incidence along the major axis.

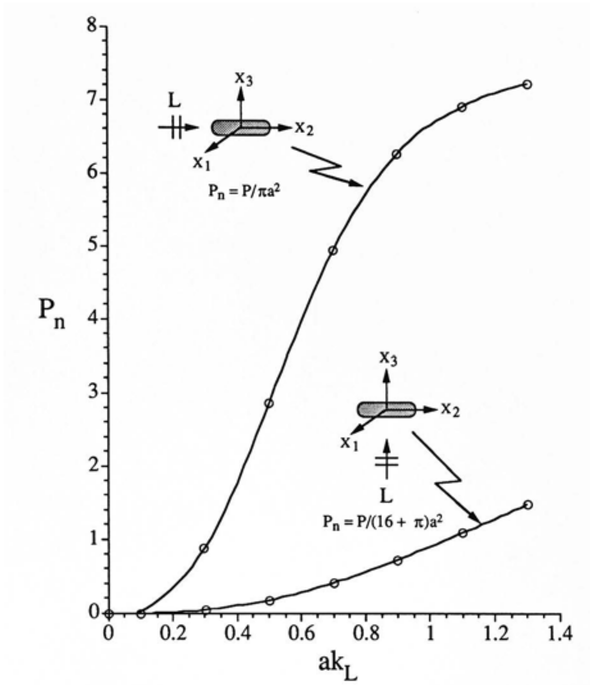

Fig.4 Normalized total cross-section P_n vs. dimensionless wavenumber ak_L.

ATTENUATION

Let us consider a longitudinal wave propagating in a solid with a dilute random distribution of whiskers of the shape analyzed in this paper. We further assume that all whiskers have the same shape, size and orientation, and that all whiskers are of the same material. Under these assumptions, and for a wave propagating along the x_3-axis, the total power loss per unit volume due to scattering is governed by the equation

$$\frac{d\langle I^I\rangle}{dx_3} = - NP(\omega)\langle I^I\rangle \tag{15}$$

where N is the average number of whiskers per unit volume and $P(\omega)$ is the total cross-section for a single whisker. The solution of Eq.(15) is

$$\langle I^I\rangle = \langle I_0^I\rangle e^{-NP(\omega)x_3} \tag{16}$$

where $\langle I_0^I\rangle$ is the intensity of the incident wave at a reference position. Comparison of Eq.(16) with the attenuative wave form $\exp[-\alpha(\omega)x_3]$ leads to the following expression for the attenuation coefficient $\alpha(\omega)$:

$$\alpha(\omega) = \frac{1}{2}NP(\omega) \quad . \tag{17}$$

From Eq.(17), we can estimate the attenuation coefficient $\alpha(\omega)$ from the average number of whiskers per unit volume, N, and the total cross-section, $P(\omega)$, at the specified frequency.

CONCLUSIONS

For incidence of a longitudinal wave on an elastic whisker embedded in an elastic solid, a set of boundary integral equations has been derived for the fields on the surface of the whiskers. These equations have been solved numerically by the boundary element method for values of ak_L up to $ak_L = 1.3$, where a is the radius of the whisker. For wave incidence parallel and normal to the major axis of the whisker, the scattering cross-sections has been calculated as functions of ak_L. The coefficient of attenuation for the propagation of longitudinal waves in a solid with a random distribution of parallel whiskers has been determined in terms of the scattering cross sections.

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