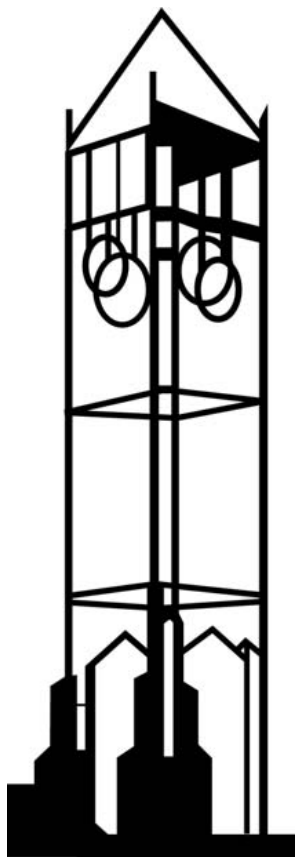


**Sub-Optimality of the Friedman Rule in
Townsend's Turnpike and Stochastic Relocation
Models of Money: Do Finite Lives and Initial Dates
Matter?**

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SUB-OPTIMALITY OF THE FRIEDMAN RULE IN
TOWNSEND'S TURNPIKE AND STOCHASTIC
RELOCATION MODELS OF MONEY: DO FINITE LIVES
AND INITIAL DATES MATTER?*

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March 23, 2005

Abstract

The Friedman rule, a widely studied prescription for monetary policy, is optimal in Townsend's turnpike model of money; it is not so in the overlapping generations version of his stochastic relocation model of money. We investigate these monetary models in the light of this disparity. To that end, we create a modified version of the turnpike model that generates the same stationary monetary equilibria as does the two-period overlapping generations model with random relocation. We exploit this equivalence to explain the aforementioned disparity. We also discuss the importance of whether or not the economy has an initial date.

Keywords: Friedman rule; monetary policy; overlapping generations; turnpike.

JEL classification: E31; E51; E58.

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1 Introduction

In the last two decades, two models of money both due to Robert Townsend, have achieved a high level of popularity in monetary economics. The first is the turnpike model (Townsend, 1980): here infinitely-lived agents with a periodic endowment stream travel along a turnpike and trading restrictions imply that they can trade with each other at trading posts only via the use of money. The second is the random relocation model (Townsend, 1987): typically cast in the two-period overlapping generations framework, this is an environment in which a fraction of two-period lived agents is stochastically relocated to an island (different from the one they were born in) and the only asset they can bring with them is currency.

At first blush, these models share many similarities; they are heterogenous-agent models wherein physical restrictions on the environment preclude the possibility of trade using instruments other than money. Their main apparent difference is the length of the lifespan of agents. Now, it is well-known that the Friedman rule, a famous prescription for monetary policy, is optimal in the turnpike model with infinitely-lived agents¹; it is not so in the random relocation model with two-period lived agents.² This paper investigates this disparity and the role played by the length of agents' lifespans in contributing to it.

This aforementioned divergence is interesting because it mirrors the disparity between theory and practice concerning the optimum quantity of money. Theory has shown the Friedman rule to be optimal in many different environments and under many different assumptions (see, for example, Kimbrough 1986; Chari, Christiano, and Kehoe 1996; Correia and Teles 1996). Yet, in practice, no central bank (CB) has a stated objective of implementing the Friedman rule. If indeed the length of agents' lifespans matters, we would expect most monetary models – in particular, ones in which agents are infinitely lived – to be inadequate in answering practical policy questions about the optimum quantity of

¹See Ljungqvist and Sargent (2001) for a discussion.

²See Smith (2002 a and b) and Bhattacharya, Haslag, and Russell (2004) for elaboration and discussion of this result.

money.

In this paper, we create a modified version of the turnpike model that generates the same stationary monetary equilibria as does the two-period overlapping generations (OG) model with random relocation. The equivalence between the model with infinitely-lived agents and the one with finitely-lived agents is achieved by incorporating the frictions (present in the latter) in the former. We show that the objective function of a benevolent central bank in the two economies is identical, up to a linear transformation. As such, the optimal monetary policy deviates from the Friedman rule in either economy even though agents' lifespans are vastly different (infinity versus two) across them. In addition we show that there is a N -period (N finite) lived "modified" turnpike model that produces the same stationary equilibria as the two-period lived OG model with random relocation. In this sense, we have demonstrated that the equivalence is intact whether agents have finite or infinite lifespans. Therefore, the length of agents' lifespans is not the reason why the Friedman rule is suboptimal.

We extend the analysis further to examine whether it matters if the economy has an initial date or not. We find that the Friedman rule is suboptimal in economies without an initial date while it is the best policy when there is an initial date. When there is an initial date, some agents must be holding the initial money stock. Deviations from the Friedman rule make these agents worse off as money becomes less valuable. In contrast, when there is no initial date, no agent is holding the initial money stock. Thus, deviations from the Friedman rule are less desirable when there is an initial date than when there is not.

Our findings are important on two dimensions. First, we show that the length of life of agents is not important for the suboptimality of the Friedman rule. This finding, therefore, suggests there is nothing special about length of life in OG economies; the same frictions can be introduced in economies populated by infinitely lived agents where they have the same effects. Second, we clarify the role played by an initial period in evaluating optimal monetary policy decisions. When there is no initial date, the CB needs only to be concerned

with equating the social and private costs of using money. In this setting, the Friedman rule is suboptimal. When there is an initial date, the welfare loss to the (initial) holders of money that is created by deviations from the Friedman rule is explicitly taken into account. Deviations from the Friedman rule, therefore, are less desirable.³

Our work is part of a burgeoning literature studying environments with heterogeneity in which the Friedman rule is not optimal (see, for example, the seminal work of Levine (1991), among others). Levine (1991) considers an environment in which there are two types of infinitely-lived agents who randomly become buyers or sellers and information on agents' type is private. If buyers value consumption sufficiently more than sellers do, and if there is some randomness in the economy, then Levine shows that the optimal monetary policy is *expansionary* and not contractionary as the Friedman rule would suggest. We extend the treatment of heterogeneity to include settings in which agents who hold the initial money stock are different from those that do not. Indeed, our results complement Levine's in that we show that it also matters whether society's objective function puts a big enough weight on the welfare of the initial moneyholders.

The remainder of the paper proceeds as follows. Section 2 describes an OG environment in which the Friedman rule is suboptimal. Section 3 describes the turnpike model introduced by Townsend (1980). Section 4 describes a modified turnpike economy with infinitely-lived agents and shows that it solves the same set of equations as the OG environment described in section 2. Section 5 discusses the results and concludes. The appendices contain proofs of the main results as well as an extension to N period lived agents.

³Together, our results add to insights forwarded in Levine (1991). Specifically, he shows that in models with heterogeneous agents, the Friedman rule will be suboptimal. Our results indicate that heterogeneity is not sufficient for the Friedman rule to be suboptimal. Rather, it matters whether society's objective function puts big enough weight on on welfare of the initial moneyholders that also plays an important role.

2 A OG model with random relocation and limited communication

We start by presenting a model economy that is populated by a unit mass of two-period overlapping generations of agents located in two spatially separated locations.⁴ We consider two alternative settings: one with an initial date, in which time is denoted by $t = 1, 2, \dots$, and one with no initial date, in which case time is denoted by $t = \dots, -1, 0, 1, \dots$. Agents receive an endowment of ω units of the single consumption good when young and nothing when old. Only old-age consumption is valued. Let c_t denote old-age consumption of the members of the generation born at date t ; their lifetime utility is given by $u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$, where $\rho \in (0, 1)$. If the economy has an initial date, then there also is an initial old generation. Each member is endowed with an amount of cash M_0 . The consumption of the members of the initial old generation is denoted by c_0^1 .

The timing of events is as follows. Agents receive their endowment at the start of a period. Under standard assumptions discussed in Schreft and Smith (1997), they will find it in their best interest to deposit their entire endowment into a bank before they learn their relocation status. Let α denote the probability (assumed symmetric across locations) that an agent will be relocated. We assume a law of large numbers holds so α is also the measure of agents that is relocated. Goods deposited in the bank can be used to acquire money or they may be put into storage.

Each unit of the consumption good put into storage at date t yields $x > 1$ units of the consumption good at date $t + 1$, where x is a known constant. We assume that currency is the only asset that can be transported between locations and that limited communication prevents the cross-location exchange of privately issued liabilities. Therefore relocated agents will seek to liquidate their holdings of storage to obtain currency. Relocation thus plays the role of a “liquidity preference shock”, and it is natural to assume that banks arise

⁴Our formulation follows Townsend (1987), and Schreft and Smith (1997). Only a concise description of the environment is provided here; the reader is referred to Schreft and Smith (1997) for details.

to insure agents against these shocks. Storage is therefore necessary for banks to exist for without it, agents would use the only store of value available – money balances; then limited communication would have no bite. Upon learning their relocation status, movers redeem their bank deposits in the form of money. In contrast, nonmovers redeem their deposits in the form of goods when old.

The CB can levy lump-sum taxes τ on the endowment of agents by collecting the tax in the form of money balances removed from the economy. In contrast, a lump-sum subsidy is received in the form of a money injection. The money supply evolves according to $M_{t+1} = (1 + z) M_t$ and z is chosen by the CB in a manner that will be explained below. We assume $x \geq 1/(1 + z)$ implying that the return on money is no greater than the return on storage; otherwise, no one would ever want to hold storage and again banks would disappear. Let p_t denote the time t price level; in steady states, $p_{t+1} = (1 + z) p_t$. Since we focus solely on steady-states, we often suppress the time subscript in what follows.

Agents deposit their entire after-tax/transfer endowments with a bank. The bank chooses the gross real return it pays to movers, d^m , and to nonmovers, d^n . In addition, the bank chooses values m (real value of money balances) and s (storage investment) respectively. These choices must satisfy the bank's balance-sheet constraint

$$m + s \leq \omega - \tau. \tag{1}$$

Banks behave competitively, so they take the return on their investments as given. They must hold sufficient liquidity to meet the needs of movers:

$$\alpha d^m (\omega - \tau) \leq \frac{m}{1 + z}. \tag{2}$$

A similar condition for non-movers, who consume all the proceeds from the storage technology, is given by

$$(1 - \alpha) d^n (\omega - \tau) \leq x s. \tag{3}$$

Because of free entry, banks choose their portfolio, in equilibrium, in a way that maximizes

the expected utility of a representative depositor. The bank's problem is written as

$$\max_{d^m, d^n} \frac{(\omega - \tau)^{1-\rho}}{1-\rho} \left\{ \alpha (d^m)^{1-\rho} + (1-\alpha) (d^n)^{1-\rho} \right\} \quad (4)$$

subject to equations (1), (2), and (3).

Let $\gamma \equiv m/(\omega - \tau)$ denote the bank's reserve-to-deposit ratio. Then, since $x \geq (1+z)^{-1}$, equations (1), (2), and (3) hold with equality, the bank's objective function is to choose γ to maximize

$$\frac{(\omega - \tau)^{1-\rho}}{1-\rho} \left\{ \alpha^\rho \left[\frac{\gamma}{1+z} \right]^{1-\rho} + (1-\alpha)^\rho [(1-\gamma)x]^{1-\rho} \right\}. \quad (5)$$

It is easy to check that the reserve to deposit ratio chosen by the bank is given by

$$\gamma(z) = \frac{1}{\left[1 + \frac{1-\alpha}{\alpha} \{(1+z)x\}^{\frac{1-\rho}{\rho}} \right]} \quad (6)$$

and that it increases as $1+z$ decreases. Using (6), it is possible to compute d^m , d^n , τ etc, all as functions of the money growth rate z .

2.1 The economy without an initial date

In this case, there is no initial old to consider and therefore the objective of the CB is to maximize the stationary utility of a representative generation. Formally, the CB chooses the rate of growth of the money supply, z , to maximize (5), where γ is given by equation (6). The solution to this problem is the Golden rule monetary policy [see Freeman, 1993].

Proposition 1 *The steady state welfare maximizing rate of growth of the money supply in this economy is given by $z = 0$.*

The Friedman rule is defined as the rate of growth of the money supply which equates the rate of return of money with the rate of return on storage. Formally, $1 + z^{\text{FR}} = 1/x$. Proposition (1) makes clear that the Friedman rule is different than the stationary welfare maximizing policy in this economy.

The intuition for this result is presented in Bhattacharya, Haslag, and Russell (2004). From the perspective of society, acquiring goods with money is equivalent to storing goods at rate of return of 1. The CB chooses z so that the private marginal cost of using money is equal to its social marginal cost.

2.2 The economy with an initial date

Now consider an economy with an initial date and a mass of initial old who are endowed with money, and for whom consumption is equal to the real value of money balances. Then, $c_1^0 = M_0/p_1$, where it may be checked that $\frac{M_0}{p_1} = \frac{\gamma(z)(\omega - \tau(z))}{(1+z)}$. We posit the following objective function for the CB:

$$W(z) = (1 - \beta) \frac{\left(\frac{M_0}{p_1}\right)^{1-\rho}}{1 - \rho} + \beta \frac{\{\Omega(z)\}^{1-\rho}}{1 - \rho} \Gamma(z) \quad (7)$$

where $\Omega(z) := \omega - \tau(z)$, and $\Gamma(z) := \alpha^\rho \left[\frac{\gamma(z)}{1+z}\right]^{1-\rho} + (1 - \alpha)^\rho [(1 - \gamma(z))x]^{1-\rho}$. It can be verified that this objective function corresponds to the case where the CB maximizes the discounted sum of all generations' expected utility. Under this alternative interpretation, β can be thought of as the CB's discount factor. The CB takes β as given and chooses z to maximize its objective function. We consider different values of the weight for the initial old generation and all other generations. For example, if $\beta = 0$, then the CB only considers the utility of the initial old. Conversely, as $\beta \rightarrow 1$, the weight of the initial old goes to zero and so the CB maximizes the utility of a representative generation (in steady states) and entirely ignores the initial old.

Proposition 2 *The optimal rate of growth of the money supply in this economy is given by*

$$1 + z = 1 - \frac{1 - \beta}{1 - \beta(1 - \alpha^\rho)}, \quad (8)$$

subject to the constraint that $1 + z \geq \frac{1}{x}$.

Given $\alpha > 0$, it is straightforward to see that if $\beta \rightarrow 1$, then $1+z \rightarrow 1$. This corresponds to the optimality of a constant money stock as in Proposition (1). When the CB ignores the initial old, the optimal monetary policy coincides with the optimal policy in an economy with no starting date. As $\beta \rightarrow 0$, in the limit the CB places all weight on the initial old. Since the initial old are made better off by a monetary policy that raises the real value of their money holdings, deflation is best; indeed, the constraint $1+z \geq \frac{1}{x}$ eventually binds and the best feasible monetary policy is the Friedman rule.⁵

In an economy with no initial date, nobody holds the initial stock of money and all agents agree on what the optimal policy is. When there is an initial date, the CB must also take into account the effect of its policy on the agents holding the initial stock of money. The welfare of these agents decreases when the CB deviates from the Friedman rule since such deviations reduce the value of the initial stock of money. Hence, with an initial date, we find that deviations from the FR are less desirable than when there is no initial date.

Note that if $\beta x > 1$, the return on storage is so great, compared to the rate at which the CB weights the expected utility of future generations relative to the initial old, that the CB would like assets in the economy to accumulate. This would allow future generations to consume more than early generations. Of course, the CB cannot force agents to transfer resources to future generations. By increasing z , however, the CB reduces the return on money and can provide incentives for agents to save more in the storage technology.

If we restrict our attention to values of β such that $\beta x \leq 1$, then the CB always chooses the Friedman rule. Indeed, we can rewrite equation (8) to get

$$1+z = \frac{\alpha^\rho}{\frac{1}{\beta} - (1-\alpha^\rho)}.$$

⁵See, also Gahvari (1988) and Ireland (2004). In this proposition, social welfare is lower under the Friedman rule than under a constant-money-stock policy. When there is no initial date, the CB's objective function is a weighted sum of the welfare of identical agents. As the reader will see, when there is an initial date, there is heterogeneity in the form of different money holdings. With an initial date, welfare is non-comparable under different CB policies because one type will realize higher utility and the other type will realize lower utility.

If $\beta x \leq 1$, then

$$\frac{\alpha^\rho}{\frac{1}{\beta} - (1 - \alpha^\rho)} \leq \frac{\alpha^\rho}{x - (1 - \alpha^\rho)} \leq \frac{1}{x}.$$

In many cases the CB would like to set a rate of growth of the money supply lower than the Friedman rule but this is not feasible since in that case nobody would store goods.

3 The Townsend Turnpike Model

We begin this section by reviewing the turnpike model developed in Townsend (1980) and described in detail in Ljungqvist and Sargent (2000).

In the turnpike model, there is a single, perishable consumption good and a countably infinite number of infinitely-lived agents. There are two types of agents, differing in their endowment patterns. Specifically, type- E agents are endowed with ω unit of the consumption good at even dates and nothing at odd dates. Type- O agents are endowed with ω unit of the consumption good at odd dates and nothing at even dates. Each type- O agent is endowed with M_0 units of fiat money at date 0 while the type- E agents are moneyless. Both types of agents wish to consume each period. In addition, there are two restrictions on market participation. First, at each date t , there is a single pairing of one type- E and one type- O populating a market. Second, a type- E will be paired with the specific type- O agents only once. These restrictions, combined with the absence of any common agent or intermediary, eliminates the possibility of private loans and debt issues. It is this friction that opens a social role for fiat money.

Without loss of generality, suppose the recipient is endowed with one unit of the consumption good. Type- O agents are endowed with one unit of the consumption good in odd periods and type- E agents are endowed with one unit of the consumption good in even periods. The optimal consumption allocation for type E in even dates and for type O in odd dates is given by

$$\hat{c} = \frac{(1+z)^{\frac{1}{\rho}} \omega}{(1+z)^{\frac{1}{\rho}} + \beta^{\frac{1}{\rho}}}, \quad (9)$$

with unendowed agents consuming the remainder $1 - \hat{c}$ given by

$$1 - \hat{c} = \frac{\beta^{\frac{1}{\rho}} \omega}{\beta^{\frac{1}{\rho}} + (1+z)^{\frac{1}{\rho}}}.$$

Why is there a gap between \hat{c} and $1 - \hat{c}$? The reason has to do with the odd-even endowment pattern. If the return on money is less than $1/\beta$, agents prefer to consume a unit of good today than to consume tomorrow goods they have bought with the cash equivalent of a unit of goods today. Hence they consume more at dates when they receive their endowments than at other dates. As π increases, the return to money falls thereby increasing the gap between odd and even period consumption. Clearly, the two converge as $(1+z) \rightarrow \beta$. Moreover, the gap increases as $(1+z)$ increases. Notice that the gap introduces fluctuations in agent's consumption sequences which hurts a risk-averse agent in an ex-ante sense. Not surprisingly, the Friedman rule eliminates the fluctuation and implements the first best here.

Proposition 3 *The optimal monetary policy is to set $(1+z) = \beta$, i.e., to follow the Friedman rule.*

In the next section, we present a slight variation to the standard turnpike model that is enough to destroy the optimality of the Friedman rule. As will be evident, the alteration introduced essentially allows us to mimic the OG random relocation with finitely-lived agents in a turnpike model with infinitely lived agents.

4 A modified turnpike economy

In this section, we present a modified version of the model economy introduced in the previous section. The Friedman rule will not be ex-ante optimal for reasons that are similar to those which lead to the suboptimality of the Friedman rule in the random-relocation OG model.

We consider both a version in which there is an initial date and one without. The former is captured by $t = 1, 2, \dots$, and the latter is captured by $t = \dots, -1, 0, 1, \dots$. As in the standard turnpike model, here too there are two types of agents: Type- E (type- O) agents receive an endowment ω at even (odd) dates. There are as many type- E agents as type- O agents. We assume that each agent of type- E is paired with a specific type- O agent. Such a pairing occurs every period and two agents never meet more than once. As before, the single-meeting assumption and the absence of a common agent accounts for why credit is not an acceptable means of payment.⁶

We now introduce the aforementioned modifications to preferences and technology. We modify technology in the sense that we allow agents access to a technology that stores their endowment from date t to date $t + 1$ at a gross return of x . We modify preferences in the following way. Agents no longer like to consume in each period. Specifically, type- E agents do not derive utility from consumption on even dates and type- O agents do not derive utility from consumption on odd dates. Type- E (type- O) agents derive utility from consuming their stored endowment on odd (even) dates. Type- E (type- O) agents also derive utility from consuming type- O 's (type- E 's) endowment good at odd (even) dates. These modifications to the preferences of agents guarantee that, in equilibrium, money is valued and storage is used. In this environment, money is still needed for transactions because agents need a medium of payment for goods they buy from the other type on a date when the other type does not want goods in exchange. Storage is used since it is the only way for agents to consume their own good in periods when they do not receive their endowment.⁷ Also notice that each agent consumes every other period. This allows us

⁶Following the turnpike tradition, we consider infinitely lived agents in this modified economy. In Appendix C, we demonstrate that it is possible to generalize the length of life to any arbitrary number of even periods. We show that the length of life does not matter in the sense that our key findings are retained.

⁷To fix ideas, we suggest the following fable. Suppose milk and cheese are perfect substitutes. All agents receive an endowment of milk (some at odd dates, others at even dates). At the time they receive their endowment, agents are not thirsty but can store the milk to turn it into cheese next period at the rate of

to keep the model in line with the random relocation OG economy presented in section 2 where agents cared only about their old age consumption.

Let c^h denote the consumption of an agent's stored endowment good and c^f denote the consumption of the other type's endowment good. If t_0 is an even date, the preferences of type- E agents at date t_0 can be written as

$$\sum_{t=t_0}^{\infty} \beta^{2(t-t_0)+1} \left[u(c_t^h) + u(c_t^f) \right] \quad (10)$$

and the preferences of type- O agents at date t_0 can be written as

$$\sum_{t=t_0}^{\infty} \beta^{2(t-t_0)} \left[u(c_t^h) + u(c_t^f) \right]. \quad (11)$$

If t_0 is an odd date, the preferences of type- E agents at date t_0 is given by equation (11) while the preferences of type- O agents at date t_0 is given by equation (19). When the economy has an initial date, type- E agents hold money at date zero and get utility only from consuming the endowment good of type- O agents.

The CB chooses z , the rate of growth of the money supply. Money is injected or removed from the economy through lump-sum taxes or subsidies. To facilitate the analogy with the OG setup, we assume the CB levies the tax/subsidy on type- E agents on even dates and on type- O agents at odd dates. Thus, in equilibrium, the CB will levy the tax/subsidy on agents who are holding money.⁸

We assume in this section that β , the agents' discount factor is equal to $1/x$, the inverse of the return on the storage technology. Under this assumption, the Friedman rule equates the return on money with the return on storage and corresponds to a deflation at a rate of β . If we allowed $\beta x > 1$ then agents would want to accumulate assets and the economy

x units of cheese per unit of milk stored. In the following period, agents derive utility from consuming the cheese they have produced as well as the milk that other agents have received as endowment. Money is used to buy fresh milk but not to buy cheese.

⁸Other assumptions about when the taxes/subsidies are being levied do not modify the results. However, they complicate the exposition; for example when lump-sum taxes are raised from each agent in every period, some agents have to hold money for the sole purpose of paying the tax.

would be growing. With $\beta x < 1$, the economy would not grow but we would have to take a stand as to whether the Friedman corresponds to equalizing the return on money and storage or to a deflation at a rate of β . In either case, our main results would hold: Without an initial date the optimal z is greater than the Friedman rule for either definition. With an initial date the optimal z is no greater than the Friedman rule.

4.1 The economy without an initial date

When there is no initial date, the problem faced by both types of agents is identical. Indeed, the problem of a type- E agent on an even date is the same as the problem of a type- O agent on an odd date. The problem of any agent can be written recursively as

$$V = \max \beta \left[u(c^h) + u(c^f) \right] + \beta^2 V, \quad (12)$$

subject to

$$\begin{aligned} m + s &\leq (\omega - \tau), \\ c^f &\leq \frac{m}{1 + z}, \\ c^h &\leq xs. \end{aligned}$$

Under the assumption $u(c) = \frac{c^{1-\rho}}{1-\rho}$, where $\rho \in (0, 1)$, and defining $\gamma \equiv m/(\omega - \tau)$, it is easy to check that (12) reduces to

$$V = \frac{\beta}{1 - \beta^2} \frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1 + z} \right]^{1-\rho} + [(1 - \gamma)x]^{1-\rho} \right\}. \quad (13)$$

Equations (13) and (5) are identical when $\alpha = 1/2$ up to a linear transformation. Thus, the objective function of agents in our modified turnpike economy without an initial date is the same as the objective function of a representative generation in the OG environment. The implication is stated in the following proposition.⁹

⁹It is probably useful to offer an interpretation of the homomorphism that exists. Note that there is an equal number of type-E and type-O agents in the Turnpike model. By this assumption, we guarantee

Proposition 4 *The optimal rate of growth of the money supply in this economy is given by $z = 0$.*

The proof is omitted since this result follows directly from the fact that the CB solves the same problem in the environments in this section and the OG environment without an initial date described in section 2. As before, it follows that the Friedman rule is not optimal in this economy.

To get some intuition about this result it is useful to compare the original turnpike model with our modified version. In the original model, because goods cannot be converted into storage, the choice of z by the central bank only influences how goods are divided between agents in each period. The Friedman rule achieves the efficient allocation because it completely smoothes consumption across types. In the modified model, storage is available, so each agent faces a portfolio-choice problem. The choice is between investing in the storage technology or acquiring money. Note that this is exactly the same choice as that faced by banks in the OG model.

It is possible now to fully elaborate on the role that money plays. Propositions 1 and 4 identify the same optimal monetary policy; the money stock is constant. At $z = 0$, the social objective function is maximized where the social cost and the private opportunity cost of holding money are equal. The private opportunity are easy to conceptualize: for $z > \beta - 1$, storage rate-of-return dominates money. Hence, there is a private opportunity cost to every agent that holds money. In these two model economies, money is the means by which an agent participates in exchanges. Hence, there is a social opportunity costs associated with money because the set of possible trades shrinks if money does not exist.¹⁰ At values

that there is always a match at each reststop along the turnpike. The case with $\alpha = 1/2$ is symmetric to assumption of equal-sized groups in the Turnpike setting. In words, $\alpha = 1/2$ says that the number of movers in the OG random relocation economy who want to acquire units of the consumption good with money are exactly the same size, in terms of the fraction of an island's population, as the number of agents that wish to acquire the consumption by exchanging money for goods in the Turnpike model.

¹⁰For example, in the OG economies, the exchange is an intergenerational transfer. For the modified turnpike economies, the exchange is an inter-reststop transfer.

of $z < 0$, for example, the private opportunity costs are less than the social opportunity costs of the exchanges. In this case, the social opportunity costs of the exchanges exceeds their private opportunity costs of holding money. The "excess" social opportunity cost is borne in the form of lower after-tax resources and, thus, lower welfare. For our purposes, it is important to note that the length of life does not affect these welfare calculations when no initial date is specified.

4.2 The economy with an initial date

At the initial date $t = 1$, the problem for type- O agents is not modified compared to the case without initial date. Thus, we can write

$$V^O = \frac{\beta}{1 - \beta^2} \frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1 + z} \right]^{1-\rho} + [(1 - \gamma)x]^{1-\rho} \right\}.$$

The problem for type- E agents, however, is different. Indeed, at date 1 these agents hold money which they can use to buy the endowment good of type- O agents. From date 2 onward, the problem of type- E agents looks like the problem when there is no initial date. It follows that we can write type- E 's problem

$$u(c_0^f) + \beta V^E,$$

where

$$V^E = \beta \left[u(c^h) + u(c^f) \right] + \beta^2 V^E,$$

subject to

$$\begin{aligned} m + s &\leq (\omega - \tau), \\ c^f &\leq \frac{m}{1 + z}, \\ c^h &\leq xs. \end{aligned}$$

Date-1 consumption for type- E agents is equal to the real value of the money they possess, M_0/p_1 . Equations (16), (17), and (18), in the appendix, hold in this environment.

We can thus write the problem of type- E agents as

$$u(c_0^f) + \beta V^E = u\left(\frac{\gamma\omega}{(1+z) - \gamma z}\right) + \frac{\beta^2}{1 - \beta^2} \frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1+z}\right]^{1-\rho} + [(1-\gamma)x]^{1-\rho} \right\}.$$

Ex-ante welfare in this economy is given by $W = u(c_0^f) + \beta V^E + V^O$. It is the sum of the utility of type- E and type- O agents. Hence, we have

$$W = u\left(\frac{\gamma\omega}{(1+z) - \gamma z}\right) + \frac{\beta}{1 - \beta} \frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1+z}\right]^{1-\rho} + [(1-\gamma)x]^{1-\rho} \right\}. \quad (14)$$

Equations (14) and (7) are the same, up to a linear transformation, when $\alpha = 1/2$. From proposition 2, the CB would like to set $1 + z = \frac{\beta 2^{-\rho}}{1 - \beta(1 - 2^{-\rho})}$. Note that the optimal policy, absent any constraint, would be to set money growth so that the return to money is greater than the return to storage; however this is infeasible. The CB thus sets $1 + z = \beta$, the best feasible policy. The implication is that the Friedman rule is the best policy the CB can achieve.

In the modified turnpike environment, β is the discount factor of agents populating the economy. We have assumed that $\beta = 1/x$ so that the Friedman rule corresponds to a deflation at a rate β and also equate the return on money with the return on storage. Under that assumption, the Friedman rule is the best feasible monetary policy when there is an initial date.

The intuition for the fact that the value of z chosen by the CB is lower in the model with an initial date than in the model without an initial date is the same as with the OG model. With an initial date, some agents are holding the initial stock of money. These agents are made worse off when z increases as the value of the money they hold decreases. Thus deviations from the Friedman rule are less desirable in an economy with an initial date.

Here, the intuition relating marginal social costs to private opportunity costs is modified. At $1 + z = \beta$, the private opportunity costs equal zero. The marginal social cost is also zero. The key difference is that with an initial date, we must take into account the

agents holding the initial stock of money. In this setting, the welfare gain to these initial moneyholders offsets the "excess" social cost borne by those agents that do not have any money at the initial date. With this modified version of social opportunity costs, private opportunity and marginal social costs are equated at Friedman rule.

5 Summary and conclusion

In this paper, we constructed an environment populated by infinitely-lived agents which solves the same set of equations as an OG model. From this we learn that there exists a representation of a model economy populated by infinitely-lived agents that is exactly the same as an OG model economy populated by finitely lived agents. As such, the monetary policy results that are derived in the OG setup carry over to the infinite-horizon setting. More specifically, under conditions in which the Friedman rule is not optimal in the OG model economy, the Friedman rule is not optimal in the model economy populated by agents that live infinitely long. In other words, finite lives do not matter. This allows us to establish that the key element for the suboptimality of the Friedman rule in this type of model is the heterogeneity of agents and not the fact that they have finite lives.

Further, our results demonstrate that the existence of an initial date plays an important role. When there is no initial date, no agent holds the initial stock of money and, thus, all agents agree on what the optimal monetary policy is. When there is an initial date, those agents who hold the initial stock of money are made worse off by an increase in the rate of growth of the money supply. Thus, deviations from the Friedman rule are less desirable when there is an initial date.

Our results address two important issues. First, finite lives are not crucial to an analysis of the *ex-ante* optimal monetary policy. Second, whether or not the initial date is treated explicitly matters. Whether or not agents have finite lives, if there is no initial date, the Friedman rule is sub-optimal. If there is an initial date, the Friedman rule is the best feasible policy.

Our analysis is silent on the effect of changing the length of the decision-making horizon of agents. Notice that in all three models discussed, the standard turnpike, the OG model, and the modified turnpike, agents face two-period problems. The periodic nature of the endowment stream in both the turnpike models converts them into models in which infinitely-lived agents face an infinitely repeated sequence of two-period problems. It would be interesting to consider what effect changing the decision making horizon would have on the optimality of the Friedman rule, a topic we leave to future research.

Appendix

A Proof of Proposition 1

First define

$$\Omega(z) \equiv \omega - \tau(z) = \omega \frac{1+z}{1+z-z\gamma(z)},$$

$$\Gamma(z) \equiv \alpha^\rho \left[\frac{\gamma(z)}{1+z} \right]^{1-\rho} + (1-\alpha)^\rho [(1-\gamma(z))x]^{1-\rho}.$$

The objective function of the planner is thus given by

$$W(z) = \frac{\Omega(z)^{1-\rho}}{1-\rho} \Gamma(z)$$

Taking the derivative of $W(z)$ and setting it equal to zero yields

$$\frac{\Omega(z)^{1-\rho} \Gamma(z)}{1+z} \left[\frac{1+z}{\Omega(z)} \frac{\partial \Omega(z)}{\partial z} + \frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial \Gamma(z)}{\partial z} \right] = 0. \quad (15)$$

It can be verified that

$$\frac{\partial \Gamma(z)}{\partial z} = \alpha^\rho (1-\rho) \frac{1}{1+z} \left(\frac{\gamma(z)}{1+z} \right)^{1-\rho},$$

$$\frac{\partial \Omega(z)}{\partial z} = \Omega(z) \left[\frac{1}{1+z} - \frac{1-\gamma(z) + z \frac{\partial \gamma(z)}{\partial z}}{1+z+z\gamma(z)} \right],$$

and

$$\frac{\partial \gamma(z)}{\partial z} = \frac{1-\rho}{\rho} \frac{\gamma(z)}{1+z} (\gamma(z) - 1).$$

Using these expressions, it is easy to show that

$$\frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial \Gamma(z)}{\partial z} = -\gamma(z),$$

and

$$\frac{1+z}{\Omega(z)} \frac{\partial \Omega(z)}{\partial z} = \frac{\gamma(z) \left[1 + \frac{1-\rho}{\rho} z (\gamma(z) - 1) \right]}{1+z-z\gamma(z)}.$$

Substituting these two expressions into equation (15), it can be established that

$$\frac{\partial W(z)}{\partial z} = 0 \Leftrightarrow z = 0.$$

B Proof of Proposition 2

Note that, in steady states,

$$-\tau = \frac{M_t - M_{t-1}}{p_t} = m \left(\frac{z}{1+z} \right) \quad (16)$$

$$m = \gamma(\omega - \tau) = \frac{\gamma\omega(1+z)}{(1+z) - \gamma z} \quad (17)$$

and hence,

$$\frac{M_0}{p_1} = \frac{M_1}{p_1} \frac{M_0}{M_1} = m \frac{1}{1+z} = \frac{\gamma\omega}{(1+z) - \gamma z}. \quad (18)$$

Substitute the relevant expressions into the CB's objective function and take the derivative with respect to z , to get

$$\begin{aligned} & \left(\frac{\gamma(z)\omega}{1+z-z\gamma(z)} \right)^{-\rho} \left[\frac{\frac{\partial\gamma(z)}{\partial z}\omega(1+z-z\gamma(z)) - \gamma(z)\omega \left(1 - \gamma(z) - z\frac{\partial\gamma(z)}{\partial z} \right)}{(1+z-z\gamma(z))^2} \right] \\ & + \frac{\beta}{1-\beta} \frac{\Omega(z)^{1-\rho}\Gamma(z)}{1+z} \left[\frac{1+z}{\Omega(z)} \frac{\partial\Omega(z)}{\partial z} + \frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial\Gamma(z)}{\partial z} \right] = 0 \end{aligned}$$

It can be verified that

$$\frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial\Gamma(z)}{\partial z} = -\gamma(z)$$

and

$$\frac{1+z}{\Omega(z)} \frac{\partial\Omega(z)}{\partial z} = \frac{\gamma(z) + (1+z)z\frac{\partial\gamma(z)}{\partial z}}{1+z-z\gamma(z)}.$$

After rearranging, we get

$$\begin{aligned} & \left(\frac{\gamma(z)\omega}{1+z-z\gamma(z)} \right)^{-\rho} \omega \left[\frac{\frac{\partial\gamma(z)}{\partial z} (1+z) - \gamma(z) (1-\gamma(z))}{(1+z-z\gamma(z))^2} \right] \\ & + \frac{\beta}{1-\beta} \frac{\Gamma(z)}{1+z} \frac{(\omega(1+z))^{1-\rho}}{(1+z-z\gamma(z))^{1-\rho}} \frac{z \left[\frac{\partial\gamma(z)}{\partial z} (1+z) - \gamma(z) (1-\gamma(z)) \right]}{(1+z-z\gamma(z))} = 0, \end{aligned}$$

which simplifies to

$$\gamma(z)^{-\rho} + \frac{\beta}{1-\beta} \Gamma(z) (\omega(1+z))^{-\rho} z = 0.$$

Note

$$\begin{aligned}\Gamma(z) &= \alpha^\rho \left(\frac{\gamma(z)}{1+z} \right)^{1-\rho} + (1-\alpha)^\rho [(1-\gamma(z))x]^{1-\rho} \\ &= \left(\frac{\alpha}{\gamma(z)} \right)^\rho \frac{\gamma(z)}{(1+z)^{1-\rho}} + \left(\frac{1-\alpha}{1-\gamma(z)} \right)^\rho (1-\gamma(z)) x^{1-\rho}.\end{aligned}$$

From the bank's maximization we have, $\alpha^\rho \frac{1}{1+z} \left(\frac{\gamma(z)}{1+z} \right)^{-\rho} - \left(\frac{1-\alpha}{1-\gamma(z)} \right)^\rho x^{1-\rho} = 0$, so that

$$\Gamma(z) = \left(\frac{\alpha}{\gamma(z)} \right)^\rho \frac{1}{(1+z)^{1-\rho}}.$$

Substitution for Γ in the above expression, we get

$$\gamma^{-\rho} = -\frac{\beta}{1-\beta} \alpha^\rho \gamma^{-\rho} (1+z)^{-1} z.$$

With a little algebra this becomes

$$1+z = 1 - \frac{1-\beta}{1-\beta(1-\alpha^\rho)}.$$

C Does length of life matter?

In this appendix, we show that the length of life does not matter. To achieve this aim, we modify the turnpike economy to let agents live an arbitrary number of even periods. We will consider this modification in two cases: one in which there is an initial date and one without. The former is captured by $t = 1, 2, \dots$, and the latter is captured by $t = \dots, -1, 0, 1, \dots$

All agents live N periods where N is assumed to be even. Members of a new generation are born with an endowment of goods but no money. They receive an endowment in every odd period of their life. Since N is even, agents die after they have used their money to purchase goods during their last period of life. An agent dies after market activities are concluded and is replaced by a newborn agent the following period.

The matching technology is assumed to work as follows. In periods in which they receive an endowment, agents born in period t meet with agents born in period $t - 1$. In periods in which they do not have an endowment, agents born in period t meet with agents born in period $t + 1$.

Preferences have the same structure as in the modified turnpike model. They can be written as

$$\sum_{t=0}^{\frac{N}{2}-1} \beta^{2t+1} \left[u(c_t^h) + u(c_t^f) \right] \quad (19)$$

$$\begin{aligned} m_t + s_t &\leq (\omega_t - \tau_t), \\ c_{t+1}^f &\leq \frac{m_t}{1+z}, \\ c_{t+1}^h &\leq x s_t. \end{aligned}$$

We assume that the CB operates in the same way as it did in the modified turnpike with infinite lives. Again, we assume that β , the agents' discount factor is equal to $1/x$, the inverse of the return on the storage technology.

Since $\beta x = 1$, agents only store goods in periods in which they receive an endowment. Hence, each agent faces a sequence of identical 2 periods problem which can be written

$$\left[u(c^h) + u(c^f) \right] \sum_{t=0}^{\frac{N}{2}-1} \beta^{2t+1}$$

$$\begin{aligned} m + s &\leq (\omega - \tau), \\ c^f &\leq \frac{m}{1+z}, \\ c^h &\leq x s. \end{aligned}$$

We use logic that is similar to the case with infinite lives. Specifically, the central bank chooses the money growth rate to maximize the following program

$$\frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1+z} \right]^{1-\rho} + [(1-\gamma)x]^{1-\rho} \right\} \sum_{t=0}^{\frac{N}{2}-1} \beta^{2t+1}.$$

C.1 The economy without an initial date

We assume that the time rate of preference in the social welfare function is the same as the agent's time rate of preference. Given the agent-replacement assumption, it is equivalent to assess social welfare from the perspective of one agent that lives forever or from the perspective of an infinite sequence of agents who live a finite number of periods. In other words, as long as the size of the population remains constant over time, the welfare in the economy with infinitely lived agents is proportional to the welfare in the economy with finitely lived agents. Since there is no initial date, the planner's objective is proportional to

$$\frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1+z} \right]^{1-\rho} + [(1-\gamma)x]^{1-\rho} \right\}.$$

We can state the following proposition.

Proposition 5 *The optimal rate of growth of the money supply in this economy is given by $z = 0$.*

The proof is omitted since this result follows directly from the fact that the environments in this section, the OG environment without an initial date and the turnpike environment with infinitely lived agents solve the same set of equations.

C.2 The economy with an initial date

The analogy between the modified turnpike model with finite lives and the one with infinitely lived agents also holds in the case in which there is an initial date. At the initial date, the planner is faced with two types of agents: Those that receive their endowment at even dates and those that receive their endowments at odd dates. From the point of view of social welfare, it does not matter how many periods agents that are alive at date $t = 1$ have left to live. Again, as long as the size of the population remains constant (each agent who dies is replaced next period by a new agent) social welfare in an economy with infinitely lived agents is proportional to social welfare in an economy with finitely lived agents. Hence all result that hold for the OG economy and for the modified turnpike economy with infinitely lived agents will also hold for finitely lived agents.

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