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STABILITY OF SOLUTIONS OF NON-LINEAR DIFFUSION PROBLEMS

by

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$\{x \mid x \text{ satisfies } P\}$, The set of points x which satisfies property P.

(a,b), The open interval $a < x < b$.

$[a,b]$, The closed interval $a \leq x \leq b$.

$[a,b)$, The half open interval $a \leq x < b$.

$(a,b]$, The half open interval $a < x \leq b$.

$x \varepsilon G$, x is a member of G.

$D(f)$, The domain of the function f.

$A \supset B$ or $B \subset A$, A contains B.

$f \varepsilon C'$, f is in class C', i.e. f has continuous first derivatives.

glb, greatest lower bound.

$\overline{R}$, The closure of the set R.

I. INTRODUCTION

This paper discusses sufficient conditions for stability, asymptotic stability and instability of the non-linear diffusion equation with non-linear boundary conditions. The related problem of obtaining bounds for a non-steady state solution is also considered.

The similarity of the problem treated in this paper and the analogous problem in ordinary differential equations leads one to look for a generalization of A. M. Liapunov's direct method (5). Three Russian writers Zubov (13), Volkov (11) and Movchan (7) were able to achieve limited success along this line.

In theory one could apply the methods proposed by Volkov and Movchan to any boundary value problem. To accomplish this we must, however, define Liapunov-like functions. While each author exhibited a function with the necessary characteristics to answer the question of stability for his particular problem this is of no help in finding similar functions for other boundary value problems. This is the same problem that confronted workers in ordinary differential equations when they first began to use Liapunov's direct method; but since that time, they have developed many techniques for finding the necessary Liapunov functions. It may be that similar techniques will be developed for boundary value problems of the type discussed in this paper; but until they are, methods such as Volkov's and Movchan's will have very limited practical value. In fact one can not be sure that satisfactory functions even exist. In a monograph published in 1957, Zubov extended Liapunov's direct method to include differential equations over a metric space. He presented a discussion of an initial value problem involving the partial differential equation

$$u_t = f(x,u,u_x)$$

where x, u, and f are k, n and n-dimensional vectors, respectively. His theory could also be applied to higher order equations but in all cases is limited to initial value problems.

The results we give below depend very heavily on a variation of a lemma originally stated by Westphal (12). The lemma states the following: Suppose the function $F(x,t,u,u_x,u_{xx})$, where F is of class C', is nondecreasing in u_{xx}. Suppose that u(x,t) and v(x,t) satisfy the differential inequalities

$$u_t > F(x,t,u,u_x,u_{xx})$$

$$v_t \leq F(x,t,v,v_x,v_{xx})$$

for all x in (a,b) and t>0. Further suppose the inequality

$$(1.1) \qquad v(x,t) < u(x,t)$$

holds for all $x \epsilon [a,b]$ with t = 0 and for all t > 0 with x=a, x=b. Then we have that (1.1) holds for all $x \epsilon [a,b]$ and t >0.

Prodi (10), Narasimhan (8) and Friedman (4) all made use of variations of Westphal's lemma to obtain extensions to results originally published by Bellman (2). Bellman did not use Westphal's lemma but instead used known results from the theory of multiple Fourier series to convert the partial differential equation into an integral equation. All four authors discussed the stability of solutions of problems which were specializations of the following:

$$u_t = L(u) + F(x,t,u)$$

$$u(a,t) = f_1(t) \text{ and } u(b,t) = f_2(t) \text{ for all } t > 0,$$

$$u(x,0) = \phi(x) \quad \text{for all } x \in [a,b],$$

where $$L(u) = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i}$$

and a_{ij} and b_i are constants. Except for Friedman, they treated only cases in which

$$F(x,t,0) = f_1(t) = f_2(t) = 0$$

and gave sufficient conditions for the stability of the identically zero solution.

The principal technique of the above authors was to devise a function which they could show, by means of Westphal's lemma, bounded the absolute value of the solution of the boundary value problem. McNabb (6) used a similar technique to obtain upper and lower bounding functions in a manner that did not limit him to an explicit form of the partial differential equation

(1.2) $u_t = F(x,u,u_x,u_{xx})$ for all $x \varepsilon G$ and $t > 0$,

where G is an open bounded region in n-space, and F is non-decreasing in u_{xx}. McNabb imposed the boundary conditions

(1.3) $u(x,t) = \emptyset(x)$ for all $x \varepsilon B$,

where B is the boundary of G. Suppose there is a one parameter family $v(x,\lambda)$, with $x \varepsilon G$ and $\lambda \varepsilon [\lambda_1, \lambda_2]$ of solutions of the differential equation $F(x,v,v_x,v_{xx}) = 0$. If there is a $\lambda' \varepsilon [\lambda_1, \lambda_2]$ such that $v(x,\lambda') = \emptyset(x)$ for all $x \varepsilon B$, then $v(x,\lambda')$ is a steady-state solution of Equation (1.2) with boundary conditions (1.3). McNabb's principal result is that $v(x,\lambda')$ is a stable steady-state solution of the above diffusion problem if $v_\lambda(x,\lambda') > 0$, for all $x \varepsilon [a,b]$.

Lakshmikantham took a different approach to the above problem.[1] He developed a theory regarding the stability and boundedness of parabolic equations using Liapunov-like functions. His theory depends on the following lemma which is analogous to Westphal's. Suppose $m(x,t)$ satisfies the differential inequality

(1.4) $m_t \leq G(t,x,m,m_x,m_{xx})$

where $x \varepsilon R$, and R is an open bounded set in n-space. The function m is a non-negative real-valued function and G is nondecreasing in m_{xx}. Suppose there is a real-valued continuous function $W(t,m)$ such that

(1.5) $G(t,x,m,0,0) \leq W(t,m)$ for all $x \varepsilon R$.

[1] V. Lakshmikantham, Mathematics Dept., University of Ontario, Ontario, Canada. Parabolic differential equations and Lyapunov like functions. Private communication. 1964.

Let $r(t)$ be a maximal solution of the differential equation

(1.6) $\quad r'(t) = W(t,r), \qquad r(t_0) = r_0 \geq 0.$

If

(1.7) $\quad r(t) > m(x,t)$ for all $x \varepsilon \overline{R}$, $t = t_o$ and all $x \varepsilon (\overline{R}-R)$, $t > t_o$;

then $\quad r(t) > m(x,t)$ for all $x \varepsilon \overline{R}$, $t \geq t_o$.

Consider the two partial differential equations

(1.8) $\quad u_t = f(t,x,u,u_x,u_{xx})$

(1.9) $\quad v_t = g(t,x,v,v_x,v_{xx})$.

Lakshmikantham defines a function $V(t,x,u,v)$ where u and v are solutions to (1.8) and (1.9), respectively. Suppose $b(r)$ is a non-decreasing continuous real-valued function of a real variable satisfying $b(r) > 0$, for $r > 0$. The function V is required to satisfy the inequality

(1.10) $\quad b(|u-v|) < V(t,x,u,v)$.

We let $\quad \overline{V}(t,x) = V(t,x,u(t,x),v(t,x))$

and form $\overline{V}_t$ making use of (1.8) and (1.9). Suppose V is defined in such a manner that there are functions $G(t,x,\overline{V},\overline{V}_x,\overline{V}_{xx})$ and $W(t,r)$ which satisfy (1.4) and (1.5) where m is replaced by $\overline{V}$. If (1.7) is satisfied, again with m replaced by $\overline{V}$, it then follows from the lemma that

$$\overline{V}(t,x) = V(t,x,u,v) < r(t) \text{ for all } x \varepsilon \overline{R} \text{ and } t > 0.$$

It is possible to reach conclusions regarding the stability and boundedness of the partial differential equations (1.8) and (1.9) from the properties of the solution of the ordinary differential equation, (1.6) and inequality (1.10). We note that condition (1.7) implies certain restrictions on the boundary conditions of the solutions of (1.8) and (1.9). The terms stability and bounded as used here refer to the difference between the solutions of the two diffusion equations. Lakshmikantham's method should be a powerful tool since so much work has been done in this area of ordinary differential equations.

It is possible to extend Lakshmikantham's methods to cover the problems treated in this thesis. It was our experience, however, that it is extremely difficult to find a satisfactory function V. Once the function V has been selected there is little choice in the selection of the functions G and W. For the more obvious choices of V it often turns out that r(t) is unbounded. This tells us nothing regarding the original problem.

The results we present here could be considered an extension of McNabb's (6) work. Chapter 2 gives a variation of Westphal's lemma which applies to the problem we are considering. Chapter 3 gives the principal stability theorems. Chapter 4 discusses how the theorems apply to a particular problem and also shows how the lemma of Chapter 2 may be used to bound the solution of a boundary value problem. For ease of understanding, the results are all given for the case in which the space variable is one dimensional only. An Appendix is added giving a proof of the lemma of Chapter 2 for the n-dimensional case. The proofs of the remaining theorems for the higher dimensional case requires only minor notational changes, hence these proofs are not given.

II. PRELIMINARY RESULTS

The purpose of this chapter is to present a variation of Westphal's lemma which applies to the problem discussed in this thesis. To aid in stating the lemma we introduce the following notation. For a positive real number T let

$$R_T = \{t \mid 0 < t \leq T\} \text{ and } R_{0T} = \{t \mid 0 \leq t \leq T\} .$$

If $T = \infty$ let

$$R_T = \{t \mid t > 0\} \text{ and } R_{0T} = \{t \mid t \geq 0\} .$$

Lemma 1. Suppose u, v, f_1, f_2, g_1, g_2 and F are functions satisfying the following conditions:

i) $u(x,t)$ and $v(x,t)$ are of class C' for all $x \in [a,b]$, and $t \in R_{0T}$, where T is a positive real number or infinity.

ii) u and v are twice continuously differentiable with respect to x for all $x \in (a,b)$ and $t \in R_{0T}$.

iii) $v(x,0) < u(x,0)$ for all $x \in [a,b]$.

iv) $u_x(a,t) = f_1(u(a,t)$ $\quad u_x(b,t) = f_2(u(b,t))$

$v_x(a,t) = g_1(v(a,t))$ $\quad v_x(b,t) = g_2(v(b,t))$

where f_1, f_2, g_1, g_2 are all continuous, with bounded first derivatives.

v) $f_1(u) < g_1(u)$ for all $u \in \overline{[D(f_1) \cap D(g_1)]}$

$f_2(u) > g_2(u)$ for all $u \in \overline{[D(f_2) \cap D(g_2)]}$.

vi) $F(x,t,u,u_x,u_{xx})$ is of class C' and is nondecreasing in u_{xx}.

vii) $F(x,t,v,v_x,v_{xx}) - v_t \geq F(x,t,u,u_x,u_{xx}) - u_t$ for all

$x \varepsilon (a,b)$ and $t \varepsilon R_T$.

Then $v(x,t) < u(x,t)$ for all $x \varepsilon [a,b]$ and $t \varepsilon R_{OT}$.

Proof. We divide the proof into two parts. The first is a proof of the lemma if condition vii is replaced by the condition

vii-a) $F(x,t,v,v_x,v_{xx}) - v_t > F(x,t,u,u_x,u_{xx}) - u_t$

for all $x \varepsilon (a,b)$ and $t \varepsilon R_{OT}$.

Part 1. We assume all the hypotheses of the lemma hold with condition vii replaced by the stronger condition vii-a. Deny the conclusion. Define a function $h(x,t)$ by

(2.1) $h(x,t) = v(x,t) - u(x,t)$.

Let t_1 be the greatest lower bound (glb) of the set

$$S = \{t / h(x,t) \geq 0 \text{ for some } x \varepsilon [a,b]\} .$$

We have from the continuity of h with respect to t and the definition of t_1 that

(2.1a) $\sup_{x \varepsilon [a,b]} h(x,t_1) = 0.$

Hence there is a point $x_1 \varepsilon [a,b]$ such that $h(x_1,t_1) = 0$.

To show that $x_1 \neq a$, suppose $x_1 = a$. Then

$$h_x(a,t_1) = v_x(a,t_1) - u_x(a,t_1)$$
$$= g_1(v(a,t_1)) - f_1(u(a,t_1)) > 0 .$$

Since h is of class C' we have by application of the mean value theorem that $h(x,t_1) > 0$ for some $x > a$ in contradiction to (2.1a). By analogous reasoning $x_1 \neq b$, hence $x_1 \varepsilon (a,b)$. For fixed $t = t_1$, $h(x,t_1)$ is a function of x only; hence it attains an interior maximum at a point $x = x_1$. Therefore we can conclude:

(2.2) $h(x_1,t_1) = 0$ hence $v(x_1,t_1) = u(x_1,t_1)$

(2.3) $h_x(x_1,t_1) = 0$ hence $v_x(x_1,t_1) = u_x(x_1,t_1)$

(2.4) $h_{xx}(x_1,t_1) \leq 0$ hence $v_{xx}(x_1,t_1) \leq u_{xx}(x_1,t_1)$

(2.5) $h_t(x_1,t_1) \geq 0$ hence $v_t(x_1,t_1) - u_t(x_1,t_1) \geq 0$.

Inequality (2.5) follows from the fact that $h(x,t)$ is of class C' and that $h(x_1,t) < 0$ for $t < t_1$. By hypothesis vii-a we have

(2.6) $v_t(x,t)-u_t(x,t) < F(x,t,v,v_x,v_{xx})-F(x,t,u,u_x,u_{xx})$ for all $x \in [a,b]$ and $t \in R_T$.

If we use (2.2), (2.3), (2.4) and the fact that F is a non-decreasing function of its last argument, as is assured by condition vi), we have

(2.7) $F(x_1,t_1,v,v_x,v_{xx}) - F(x_1,t_1,u,u_x,u_{xx}) \leq 0$.

The inequalities (2.6) and (2.7) imply that

(2.8) $v_t(x_1,t_1) - u_t(x_1,t_1) < 0$.

If the set S is non-empty, then it has a glb $t_1 > 0$ and at the point (x_1,t_1) we have both (2.5) and (2.8) holding, a contradiction. So the set S does not have a glb and we conclude that S is empty and the lemma as modified holds.

Part 2. We assume the hypotheses i through vii of Lemma 1 hold. Again we deny the conclusion. Then there exists a $t_1 \in R_T$ and an $x_1 \in (a,b)$ such that

(2.9) $v(x_1,t_1) \geq u(x_1,t_1)$.

We define a function $w(x,t)$ by

$$w(x,t) = v(x,t) + \varepsilon/((n-1)(t+1)^{(n-1)})$$

where $\varepsilon > 0$ and $n \geq 2$. (Both will be specified later.)

We have

$$w_t(x,t) = v_t(x,t) - \varepsilon/(t+1)^n$$

$$w_x(x,t) = v_x(x,t)$$

$$w_{xx}(x,t) = v_{xx}(x,t)$$

$$w(x,0) = v(x,0) + \varepsilon/(n-1)$$

Since F is of class C', it follows that

$$(2.10) \qquad F(x,t,w,w_x,w_{xx}) - w_t = F(x,t,v,v_x,v_{xx}) + \frac{F_v(x,t,v,v_x,v_{xx})(\varepsilon)}{(n-1)(t+1)^{(n-1)}} + O(\varepsilon^2) - v_t + \varepsilon/(t+1)^n$$

$$> F(x,t,v,v_x,v_{xx}) - v_t ,$$

if

$$(2.11) \qquad (\varepsilon/(t+1)^{(n-1)}) \left\{\frac{F_v(x,t,v,v_x,v_{xx})}{n-1} + 1/(t+1)\right\} > 0$$

and $\varepsilon > 0$ is sufficiently small. Let

$$\mu = \min_{\substack{x \in [a,b] \\ t \in [0,t_1]}} F_v(x,t,v(x,t),v_x(x,t),v_{xx}(x,t)).$$

We now choose n so that $[\mu/(n-1)] + [1/(t_1 + 1)] > 0$, or $n > -\mu(t_1+1) + 1$. With this value for n, inequality (2.11) holds for all $x\varepsilon[a,b]$ and $t\varepsilon[0,t_1]$. There is a number $\mu > 0$ such that $g_1(u) - f_1(u) > \mu$ for all $u\varepsilon\overline{[D(f_1)\cap D(g_1)]}$ and $f_2(u) - g_2(u) > \mu$ for all $u\varepsilon\overline{[D(f_2)\cap D(g_2)]}$ since the left sides of both inequalities are positive on a closed set. Hence

$$(2.12) \qquad \begin{aligned} w_x(a,\lambda) &= v_x(a,\lambda) \\ &= g_1(v(a,\lambda)) \\ &> f_1(v(a,\lambda)) + \mu \\ &> f_1(v(a,\lambda)) + \varepsilon M/(n-1)(t+1)^{n-1} \\ &> f_1(v(a,\lambda) + \varepsilon/(n-1)(t+1)^{n-1}) \\ &> f_1(w(a,\lambda)) \end{aligned}$$

if ε is sufficiently small. Here M is an upper bound for $|f'|$. Similarly we can see that the inequality

$$(2.13) \qquad w_x(b,\lambda) < f_2(w(b,\lambda))$$

can also be satisfied for ε sufficiently small. We now select for $\varepsilon > 0$ a value sufficiently small so that inequalities (2.10), (2.12), (2.13) hold and also so that

$$(2.14) \qquad \underset{x\varepsilon[a,b]}{\text{Min}} [u(x,0) - v(x,0)] > \varepsilon \ .$$

We note that there is a positive value of ε satisfying inequality (2.14) since the left side is the minimum of a continuous positive function on a closed interval. Thus we have from (2.10) and hypothesis vii),

$$\begin{aligned} F(x,t,w,w_x,w_{xx}) - w_t &> F(x,t,v,v_x,v_{xx}) - v_t \\ &\geq F(x,t,u,u_x,u_{xx}) - u_t \end{aligned}$$

for all $x \varepsilon [a,b]$ and $t \varepsilon [0,t_1]$. We also have from (2.14) and the fact that $n \geq 2$ that

$$w(x,0) = v(x,0) + \varepsilon/(n-1) < u(x,0) \text{ for all } x\varepsilon[a,b].$$

If we substitute the function $w(x,t)$ for $v(x,t)$ into the statement of our lemma we see that all of the conditions of the lemma are satisfied, with condition vii replaced by vii-a. From Part 1 of this proof it follows that

(2.15) $\quad w(x,t) < u(x,t)$ for all $x\varepsilon[a,b]$ and $t\varepsilon[0,t_1]$.

From inequalities (2.9) and (2.15) and the definition of w we have the contradiction

$$w(x_1,t_1) < u(x_1,t_1) \leq v(x_1,t_1)=w(x_1,t_1)- \varepsilon/(n-1)(t+1)^{n-1}$$

Hence there does not exist a $t_1 \varepsilon R_T$ such that inequality (2.9) holds and the lemma follows.

III. STABILITY THEOREMS

We introduce the following notation. Let Problem D represent the partial differential equation

$$u_t = F(x,u,u_x,u_{xx}) \text{ for all } x \varepsilon [a,b] \text{ and } t > 0$$

with the boundary conditions

$$u_x(a,t) = f_1(u(a,t)) \text{ and } u_x(b,t) = f_2(u(b,t))$$
$$\text{for all } t > 0.$$

We assume that f_1, f_2 are continous with bounded first derivatives, F is of class C', and F is nondecreasing in u_{xx}. We use the notation $u(\emptyset,x,t)$ to represent a solution of Problem D such that

$$u(\emptyset,x,0) = \emptyset(x) \quad \text{for all } x \varepsilon [a,b],$$

where $\emptyset$ is of class C'', u is of class C' for all $x \varepsilon [a,b]$ and $t > 0$ and u is twice continuously differentiable with respect to x for all $x \varepsilon (a,b)$ and $t > 0$.

In this chapter we present theorems concerned with Liapunov-like stability of Problem D. The following definitions are analogous to the corresponding definitions as they are generally used in ordinary differential equations (ODE's) (5). Similar definitions have been used by other writers.

<u>Definition 1.</u> Let $u(\emptyset,x,t)$ be a solution of Problem D. We say that u is a steady-state solution if u is independent of time, i.e. $u(\emptyset,x,t) = \emptyset(x)$ for all $t > 0$.

Definition 2. Let $u(\phi,x,t)$ be a solution of Problem D. Suppose for every $\epsilon > 0$ there is a $\delta > 0$ such that if the function $\psi(x)$ satisfies

$$\max_{x\in[a,b]} |\psi(x) - \phi(x)| < \delta$$

it is true that

$$\max_{\substack{x\in[a,b] \\ t>0}} |u(\phi,x,t) - u(\psi,x,t)| < \epsilon .$$

Then we say $u(\phi,x,t)$ is a stable solution to Problem D.

Definition 3. Let

$$A = \{(x,u) \mid x \in [a,b] \quad \text{and} \quad \psi_1(x) \le u \le \psi_2(x)\} ,$$

where ψ_1 and ψ_2 are arbitrary functions of class C''. Let B be the set of functions defined on the closed interval [a,b] such that $\psi \in B$ implies $\{(x,\psi(x)) \mid x \in [a,b]\} \subset A$. Suppose $u(\phi,x,t)$ is a solution of Problem D such that if $\psi \in B$ we have

$$\lim_{t\to\infty}\left[\max_{x\in[a,b]} |u(\phi,x,t) - u(\psi,x,t)|\right] = 0 .$$

Then we say $u(\phi,x,t)$ is an asymptotically stable solution of Problem D and that A is a region of asymptotic stability.

Definition 4. Let $u(\phi,x,t)$ be a solution of Problem D. Suppose there exists an $\epsilon > 0$ such that for every $\delta > 0$ there is at least one function $\psi(x)$ satisfying both conditions

$$\max_{x\in[a,b]} |\psi(x) - \phi(x)| < \delta$$

and

$$\max_{x\varepsilon[a,b]} |u(\gamma,x,t) - u(\emptyset,x,t)| > \epsilon \quad \text{for some } t > 0.$$

Then we say that $u(\emptyset,x,t)$ is an unstable solution of Problem D.

Our first theorem gives sufficient conditions for a steady-state solution of Problem D to be a stable solution.

Theorem 1. Consider Problem D and assume there exists a one parameter family $v(x,\lambda)$, $\lambda \varepsilon [\lambda_1,\lambda_2]$, of solutions of the ODE.

(3.1) $\quad F(x,v,v_x,v_{xx}) = 0$

satisfying the following four conditions:

i) There is a number $\lambda' \varepsilon (\lambda_1,\lambda_2)$ such that $v_x(a,\lambda') = f_1(v(a,\lambda'))$ and $v_x(b,\lambda') = f_2(v(b,\lambda'))$.

ii) $v_\lambda(x,\lambda) > 0$, for all $x \varepsilon [a,b]$ and $\lambda \varepsilon [\lambda_1,\lambda_2]$.

iii) $v_x(a,\lambda) > f_1(v(a,\lambda))$ and $v_x(b,\lambda) < f_2(v(b,\lambda))$ for $\lambda \varepsilon [\lambda_1,\lambda')$.

iv) $v_x(a,\lambda) < f_1(v(a,\lambda))$ and $v_x(b,\lambda) > f_2(v(b,\lambda))$ for $\lambda \varepsilon (\lambda',\lambda_2]$.

Then if $\emptyset(x) = v(x,\lambda')$, $u(\emptyset,x,t)$ is a steady-state solution of Problem D.

Proof: Assume the hypotheses hold. We must show that given and $\varepsilon > 0$ there is a $\delta > 0$ such that

(3.2) $$\max_{\substack{x\varepsilon[a,b] \\ t > 0}} |u(\emptyset,x,t) - u(\psi,x,t)| < \varepsilon$$

whenever

$$(3.3)\qquad \max_{x\in[a,b]} |\phi(x) - \psi(x)| < \delta .$$

Let $\varepsilon > 0$ be given. Select a number $\bar{\lambda} \in [\lambda_1,\lambda')$ such that

$$(3.4)\qquad \max_{x\in[a,b]} [v(x,\lambda') - v(x,\bar{\lambda})] < \varepsilon$$

and a number $\bar{\bar{\lambda}} \in (\lambda',\lambda_2]$ such that

$$(3.5)\qquad \max_{x\in[a,b]} [v(x,\bar{\bar{\lambda}}) - v(x,\lambda')] < \varepsilon .$$

We define a number δ by

$$(3.6)\qquad \delta = \min\left\{ \min_{x\in[a,b]} (v(x,\lambda') - v(x,\bar{\lambda})) ,\ \min_{x\in[a,b]} (v(x,\bar{\bar{\lambda}}) - v(x,\lambda')) \right\}$$

We note that $\delta > 0$ since $v_\lambda(x,\lambda) > 0$ for all $x\in[a,b]$. Let $\psi(x)$ be an arbitrary function of class C'' satisfying (3.3), then since $\phi(x) = v(x,\lambda')$ we have from (3.6)

$$v(x,\bar{\lambda}) \leq \phi(x) - \delta < \psi(x) < \phi(x) + \delta \leq v(x,\bar{\bar{\lambda}}) \text{ for all } x\in[a,b].$$

Since $v(x,\lambda)$ is a family of solutions of Equation (3.1) and u is a solution to problem D it follows that

$$\begin{aligned} &F(x,v(x,\bar{\lambda}),v_x(x,\bar{\lambda}),v_{xx}(x,\bar{\lambda})) - v_t(x,\bar{\lambda}) \\ &\quad = F(x,u(\psi,x,t),u_x(\psi,x,t),u_{xx}(\psi,x,t)) - u_t(\psi,x,t) \\ &\quad = F(x,v(x,\bar{\bar{\lambda}}),\ v_x(x,\bar{\bar{\lambda}}),v_{xx}(x,\bar{\bar{\lambda}})) - v_t(x,\bar{\bar{\lambda}}) . \end{aligned}$$

If we let $v(x,\bar{\lambda})$ correspond to the function $v(x,t)$ and $u(\psi,x,t)$ correspond to the function $u(x,t)$ of Lemma 1, we see that all the hypotheses of the lemma are satisfied and it follows that $v(x,\bar{\lambda})$ bounds $u(\psi,x,t)$ from below. We may then let $u(\psi,x,t)$ correspond to $v(x,t)$ and $v(x,\bar{\bar{\lambda}})$ correspond to u(x,t) of Lemma 1 and it follows that $v(x,\bar{\bar{\lambda}})$ bounds $u(\psi,x,t)$ from above. Thus it follows that

(3.7) $$v(x,\bar{\lambda}) < u(\psi,x,t) < v(x,\bar{\bar{\lambda}}) \text{ for all } x\varepsilon[a,b] \text{ and } t>0.$$

By (3.4), (3.5) and (3.7) we have

(3.8) $$u(\emptyset,x,t) - \varepsilon = v(x,\lambda') - \varepsilon < v(x,\bar{\lambda}) < u(\psi,x,t)$$

and

(3.9) $$u(\emptyset,x,t) + \varepsilon = v(x,\lambda') + \varepsilon > v(x,\bar{\bar{\lambda}}) > u(\psi,x,t)$$

Combining (3.8) and (3.9) we have (3.2).

In a situation in which it is difficult or impossible to find a one parameter family satisfying the conditions of Theorem 1 it may still be possible to find an upper bound as we show in the following corollary. A similar corollary could be stated establishing a lower bound.

Corollary 1. Suppose there exists a solution v(x), of the ODE (3.1) satisfying the condition

$$v_x(a) < f_1(v(a)) \text{ and } v_x(b) > f_2(v(b)).$$

Then if $u(\psi,x,t)$ is a solution to Problem D where

$$\psi(x) < v(x) \quad \text{for all } x \varepsilon [a,b],$$

we have

$$u(x,t) < v(x) \quad \text{for all } x \varepsilon [a,b] \text{ and } t > 0.$$

Proof: The proof follows immediately from the proof of Theorem 1; v(x) is an upper bound for $u(\psi,x,t)$ for the same reasons that $v(x,\bar{\bar{\lambda}})$ was an upper bound in Theorem 1.

By requiring the function F of Problem D to satisfy one additional condition we can strengthen the conclusion of

Theorem 1 as is shown in the next theorem.

Theorem 2. Let all the hypotheses of Theorem 1 hold. In addition suppose that for all $x \in [a,b]$ and $\lambda \in [\lambda_1,\lambda_2]$

(3.10) $$F_v(x,v(x,\lambda),v_x(x,\lambda),v_{xx}(x,\lambda)) \neq 0.$$

Then $u(\emptyset,x,t)$, where $\emptyset(x) = v(x,\lambda')$ is an asymptotically stable steady-state solution of Problem D and the set

(3.11) $$A = \{(x,u) \mid x \in [a,b] \text{ and } v(x,\lambda_1) < u < v(x,\lambda_2)\}$$

is a region of asymptotic stability.

Proof: The proof consists of several parts but only one will be given in detail since they are all quite similar. We assume the hypotheses hold. Since condition (3.10) requires that F_v has the same sign for all $x \in [a,b]$ and $\lambda \in [\lambda_1,\lambda_2]$, we assume $F_v > 0$ without loss of generality. Let A be the set defined by (3.11) and let B be the set of functions such that $\psi \in B$ implies $\{(x,\psi(x)) \mid x \in [a,b]\} \subset A$. We first show that given any $\varepsilon > 0$ and any $\psi \in B$ there exists a $T' > 0$ such that

(3.12) $$\max_{\substack{x \in [a,b] \\ t > T'}} [u(\psi,x,t) - u(\emptyset,x,t)] < \varepsilon$$

We bound $u(\psi,x,t)$ from above and then show the bound can be decreased as t increases until it is within ε of $v(x,\lambda')$. Let $\varepsilon > 0$ be given and let $\bar{\lambda} \in (\lambda',\lambda_2)$ (See Figure 1) be such that

(3.13) $$v(x,\bar{\lambda}) - v(x,\lambda') < \varepsilon \quad \text{for all } x \in [a,b].$$

We now define three positive numbers μ_1, μ_2, and μ_3 by

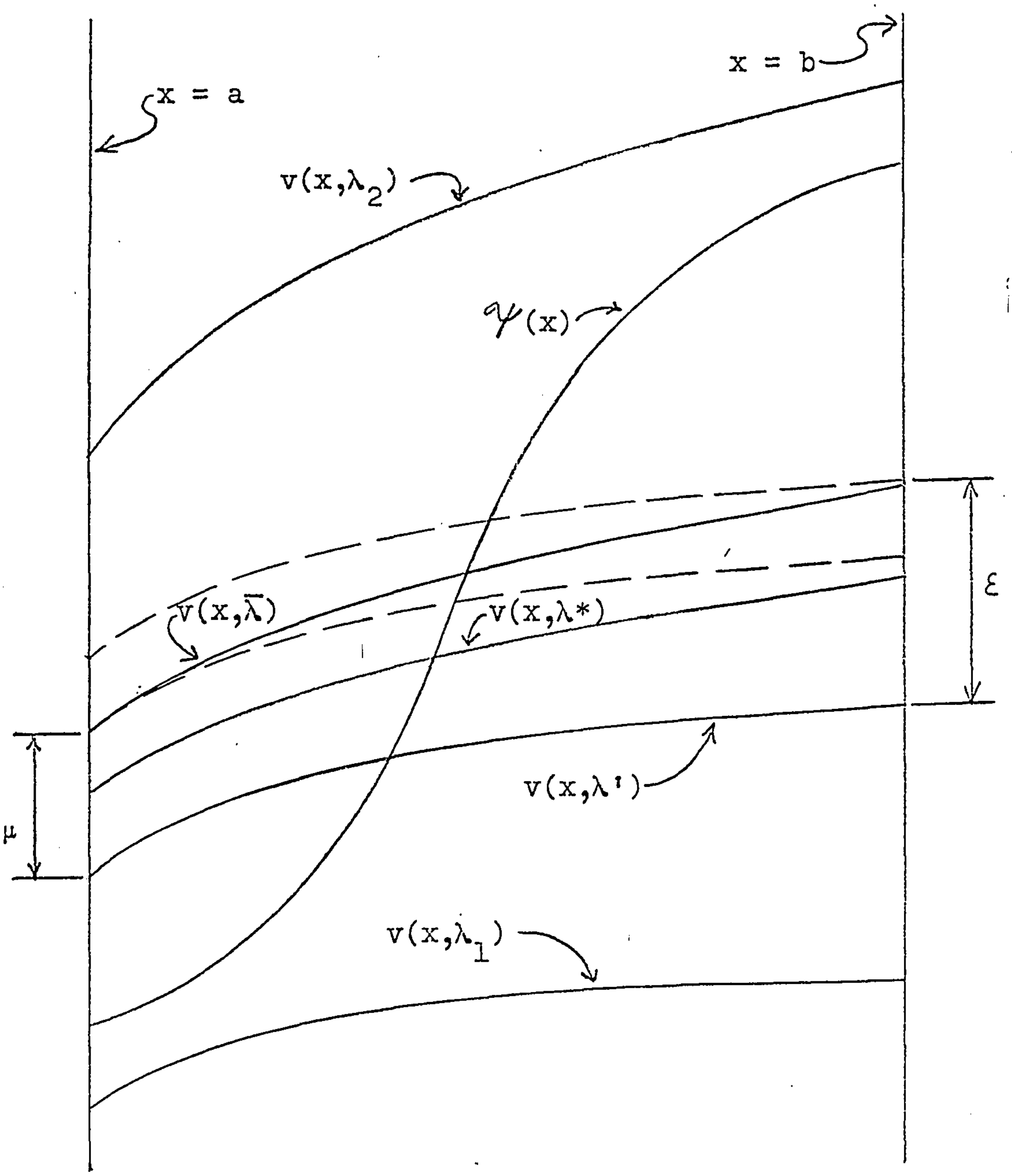

Figure 1. Asmptotic stability

$$\mu_1 = \min_{x \varepsilon [a,b]} [v(x,\bar{\lambda}) - v(x,\lambda')] ,$$

$$\mu_2 = \min_{\substack{x\varepsilon[a,b] \\ \lambda\varepsilon[\lambda',\lambda_2]}} F_v(x,v(x,\lambda),v_x(x,\lambda),v_{xx}(x,\lambda)) ,$$

and

$$\mu_3 = \max_{\substack{x\varepsilon[a,b] \\ \lambda\varepsilon[\lambda_1,\lambda_2]}} v_\lambda(x,\lambda) .$$

Let $\lambda^* \varepsilon (\lambda',\bar{\lambda})$ be such that

$$v(x,\lambda^*) - v(x,\lambda') \leq \mu_1/2 \text{ for } x \,\varepsilon [a,b] .$$

We now define a positive number μ_4 by

(3.14) $$\mu_4 = \min_{\lambda\varepsilon[\lambda^*,\lambda_2]} \{ [f_1(v(a,\lambda)) - v_x(a,\lambda)] , [v_x(b,\lambda) - f_2(v(b,\lambda))] \} .$$

Let $H(\lambda)$ be a function defined for $\lambda \,\varepsilon\, [\bar{\lambda},\lambda_2]$ such that for all $h < H(\lambda)$ we have

(3.15) $$f_1(v(a,\lambda) - h) - f_1(v(a,\lambda)) < 1/2 \, \mu_4$$

and also

(3.16) $$f_2(v(b,\lambda) - h) - f_2(v(b,\lambda)) < 1/2 \, \mu_4 .$$

Let

(3.17) $$\delta_1 = \min_{\lambda\varepsilon[\bar{\lambda},\lambda_2]} [H(\lambda)] .$$

Let $w(x,\lambda)$ be a function defined by

$$w(x,\lambda) = v(x,\lambda) - \delta ,$$

where $\delta > 0$ will be specified later. We have, since F is of class C',

(3.18) $$\begin{aligned} F(x,w,w_x,w_{xx}) &= F(x,v - \delta,v_x,v_{xx}) \\ &= F(x,v,v_x,v_{xx}) - \delta F_v(x,v,v_x,v_{xx}) + O(\delta^2) \end{aligned}$$

$$\leq F(x,v,v_x,v_{xx}) - \delta\mu_2 + 0(\delta^2)$$
$$< F(x,v,v_x,v_{xx})$$
$$= 0,$$

for $\delta > 0$ sufficiently small. We let $\delta_2 > 0$ be such that

$$\delta_2 < \underset{x\varepsilon[a,b]}{\text{Min}} (v(x,\lambda_2) - \Psi(x)) .$$

Let us now assign a positive value to δ, sufficiently small so inequality (3.18) holds and also so that

(3.19) $\qquad \delta \leq \text{Min} \ [(1/3)\ \mu_1, \delta_1, \delta_2] .$

From inequality (3.18) it follows that there exists a positive number μ_5 satisfying

$$F(x,w(x,\lambda),w_x(x,\lambda),w_{xx}(x,\lambda)) < -\mu_5 \text{ for all } x\varepsilon[a,b]$$
$$\text{and } \lambda\varepsilon[\bar{\lambda},\lambda_2].$$

Let $\bar{w}(x,t)$ be a function defined by

$$\bar{w}(x,t) = w(x,\lambda(t)),$$

where

$$\lambda(t) = \lambda' + (\lambda_2 - \lambda')\ e^{-pt} ,$$

$p > 0$ to be specified later. We have

(3.20) $$F(x,\bar{w},\bar{w}_x,\bar{w}_{xx}) - \bar{w}_t = F(x,w,w_x,w_{xx}) + w_\lambda p(\lambda_2-\lambda')\ e^{-pt}$$
$$\leq -\mu_5 + \mu_3 p(\lambda_2-\lambda') .$$

Let $p = \mu_5/\mu_3(\lambda_2-\lambda')$, then the right hand side of inequality (3.20) is zero. Thus the inequality

$$F(x,\bar{w},\bar{w}_x,\bar{w}_{xx}) - \bar{w}_t \leq F(x,u,u_x,u_{xx}) - u_t = 0$$

is satisfied for all $x\varepsilon[a,b]$ and $t\varepsilon R_{T'}$, where T' is the solution of the equation $\lambda(T') = \bar{\lambda}$. From (3.17) and (3.19), inequality (3.15) yields

(3.21) $\qquad f_1(v(a,\lambda) - \delta) - f_1(v(a,\lambda)) > (-1/2)\mu_4$.

The definition of w together with (3.21) and (3.14) gives us

(3.22) $$f_1(w(a,\lambda)) = f_1(v(a,\lambda) - \delta) > f_1(v(a,\lambda)) - \mu_4/2$$
$$> v_x(a,\lambda) = w_x(a,\lambda)$$

Similarly from (3.17), (3.19), inequality (3.15) and (3.14) we obtain

(3.23) $$f_2(w(a,\lambda)) = f_2(v(a,\lambda) - \delta) < f_2(v(b,\lambda)) + \mu_4/2$$
$$< v_x(b,\lambda) = w_x(b,\lambda) \ .$$

Both (3.22) and (3.23) hold for all $\lambda \varepsilon [\bar{\lambda},\lambda_2]$. All conditions of Lemma 1 are now satisfied, and since

$$\bar{w}(x,0) > \psi(x) \quad \text{for all } x \varepsilon [a,b] \ ,$$

it follows that

$$\bar{w}(x,t) > u(\psi,x,t) \quad \text{for all } x\varepsilon[a,b] \text{ and } t \varepsilon R_{T'} .$$

From the definition of $\bar{w}$ and T' it follows that

$$v(x,\bar{\lambda}) > u(\psi,x,T') \text{ for all } x\varepsilon[a,b].$$

Thus by Corollary 1 we have

$$v(x,\bar{\lambda}) > u(\psi,x,t) \text{ for all } x\varepsilon[a,b] \text{ and } t > T',$$

which together with (3.13) gives us (3.12) and completes the first part of the proof.

The next step of the proof is to show that there exists a number T'' such that

(3.24) $$\max_{\substack{x\varepsilon[a,b] \\ t > T''}} [u(\emptyset,x,t) - u(\psi,x,t)] < \varepsilon .$$

The proof of this consists of showing that there is a lower bound for $u(\gamma,x,t)$ which can be increased with time until it is within ε of $u(\emptyset,x,t)$ at some time T''. We do not give the details of this since it differs from the proof of the existence of T' only in minor details.

Let T be the larger of the two numbers T' and T''. Then from (3.12) and (3.24) it follows that

$$\max_{\substack{x\varepsilon[a,b] \\ t > T}} |u(\emptyset,x,t) - u(\gamma,x,t)| < \varepsilon .$$

This completes the proof.

The conclusion of Theorem 2 still holds if (3.10) is replaced by either

(3.25) $F_v = 0$ and $F_{v_x}(x,v(x,\lambda),v_x(x,\lambda),v_{xx}(x,\lambda)) \neq 0$ for all $x\varepsilon[a,b]$ and $\lambda\varepsilon[\lambda_1,\lambda_2]$.

or

(3.26) $F_v = 0$, $F_{v_x} = 0$ and $F_{v_{xx}}(x,v(x,\lambda),v_x(x,\lambda),$ $v_{xx}(x,\lambda)) \neq 0$ for all $x\varepsilon[a,b]$ and $\lambda\varepsilon[\lambda_1,\lambda_2]$.

The proof of the theorem with (3.10) replaced by (3.25) is essentially the same as the one that was given. The difference being that the function $w(x,\lambda)$ is defined by $w(x,\lambda) = v(x,\lambda) - \delta x$. Inequality (3.18) becomes

$$\begin{aligned} F(x,w,w_x,w_{xx}) &= F(x,v - \delta x,v_x - \delta ,v_{xx}) \\ &= F(x,v,v_x,v_{xx}) - F_{v_x}(x;v,v_x,v_{xx}) \\ &\quad + 0(\delta^2) \\ &\leq F(x,v,v_x,v_{xx}) - \delta \bar{\mu}_2 + 0(\delta^2) \end{aligned}$$

$$< F(x,v,v_x,v_{xx})$$
$$= 0$$

for δ sufficiently small. We have made the assumption that F_{v_x} is positive and

$$\bar{\mu}_2 = \max_{\substack{x \in [a,b] \\ \lambda \in [\lambda', \lambda_2]}} F_{v_x}(x,v(x,\lambda),v_x(x,\lambda),v_{xx}(x,\lambda)) .$$

We now select a positive number δ such that inequality (3.27) holds and also such that

$$\max_{\substack{x = a \\ x = b}} |\delta x| \leq \text{Min}[(1/3\ \mu_1), \delta_1, \delta_2]$$

The remainder of the proof follows as before.

If (3.10) is replaced by (3.26) we make the same type of modification of the proof. The function $w(x,\lambda)$ being defined by $w(x,\lambda) = v(x,\lambda) - \delta x^2$.

The next theorem gives sufficient conditions for instability of a solution of Problem D. The conditions for instability amount to reversing certain inequalities in the hypotheses of Theorem 2. It turns out, however, that this gives us more than we need, so we break the theorem into two parts weaking our hypotheses as much as possible. The division is a natural one. Theorem 3a may be thought of as giving sufficient conditions for instability from above, while Theorem 3b does likewise for instability from below.

Theorem 3a. Consider Problem D and assume there exists a one parameter family $v(x,\lambda)$, $\lambda \varepsilon [\lambda', \lambda_2]$, of solutions of

Equation (3.1) satisfying

i) $v_x(a,\lambda') = f_1(v(a,\lambda'))$ and $v_x(b,\lambda') = f_2(v(b,\lambda'))$,

ii) $v_\lambda(x,\lambda) > 0$ for all $x\varepsilon[a,b]$ and $\lambda\varepsilon[\lambda',\lambda_2]$,

iii-a) $v_x(a,\lambda) > f_1(v(a,\lambda))$ and $v_x(b,\lambda) < f_2(v(b,\lambda))$ for all $\lambda\varepsilon(\lambda',\lambda_2]$,

iv) $F_v(x,v(x,\lambda),v_x(x,\lambda),v_{xx}(x,\lambda)) \neq 0$ for all $x\varepsilon[a,b]$ and $\lambda\varepsilon[\lambda',\lambda_2]$.

Then $u(\emptyset,x,t)$, where $\emptyset(x) = v(x,\lambda')$, is an unstable steady-state solution of Problem D.

<u>Theorem 3b</u>. Let all the hypotheses of Theorem 3a hold with the interval $[\lambda',\lambda_2]$ replaced by the interval $[\lambda_1,\lambda']$ and condition iii-a replaced by

ii-b) $v_x(a,\lambda) < f_1(v(a,\lambda))$ and $v_x(b,\lambda) > f_1(v(b,\lambda))$ for all $\lambda\varepsilon(\lambda_1,\lambda']$.

Then $u(\emptyset,x,t)$, where $\emptyset(x) = v(x,\lambda')$, is an unstable steady-state solution of Problem D.

<u>Proof</u>: We sketch the proof of Theorem 3a only since the proof for Theorem 3b follows along the same lines. Assume the hypotheses of Theorem 3a hold and that $F_v > 0$. Let

$$A = \{(x,u) \mid x\varepsilon[a,b] \text{ and } v(x,\lambda') < u < v(x,\lambda_2)\} .$$

We must show there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there is a ψ satisfying the conditions

$$\max_{x\varepsilon[a,b]} |\psi(x) - \emptyset(x)| < \delta$$

and

$$\max_{x\epsilon[a,b]} |u(\psi,x,t) - u(\emptyset,x,t)| > \varepsilon \text{ for some } t > 0.$$

Let

$$= (1/2)\mathrm{Min}[v(x,\lambda_2) - v(x,\lambda')] .$$

We shall show that this ε satisfies the above condition. Let $\delta > 0$ be given. We assume $\delta \leq \varepsilon$ since otherwise there is nothing to prove. Let ψ be any function such that $0 < \psi(x) - v(x,\lambda') < \delta$ for all $x\epsilon[a,b]$. Let $\bar{\lambda}\epsilon(\lambda',\lambda_2)$ be such that

$$\min_{x\epsilon[a,b]}[v(x,\bar{\lambda}) - v(x,\lambda')] \geq \varepsilon , \tag{3.28}$$

and let $\lambda^* > \lambda'$ be such that

$$v(x,\lambda^*) < \psi(x).$$

We define three functions

$$w(x,\lambda) = v(x,\lambda) + h,$$

$$\lambda(t) = \lambda_2 - (\lambda_2 - \lambda^*)\, e^{-pt}$$

and

$$\bar{w}(x,t) = w(x,\lambda(t)) .$$

We must select positive values for h and p sufficiently small so that

$$\bar{w}(x,0) < \psi(x) \text{ for all } x\epsilon[a,b] ,$$

$$w_x(a,\lambda) > f_1(w(a,\lambda)) \text{ and } w_x(b,\lambda) < f_2(w(b,\lambda)) \text{ for all } \lambda\epsilon[\lambda^*,\lambda_2] ,$$

and

$$F(x,\bar{w},\bar{w}_x,\bar{w}_{xx}) - \bar{w}_t \geq F(x,u,u_x,u_{xx}) - u_t \text{ for all } x\epsilon[a,b] \text{ and } t\epsilon R_T ,$$

where T is the solution of the equation $\lambda(T) = \bar{\lambda}$. We omit the details of showing that we can actually find such values for h and p since the procedure is so similar to that followed in the proof of Theorem 2. By application of Lemma 1 we see that

$$\overline{w}(x,t) < u(\psi,x,t) \quad \text{for all } x \varepsilon [a,b] \text{ and } t \varepsilon R_{0T}.$$

But for t = T we have

$$\overline{w}(x,T) = w(x,\overline{\lambda}) < u(\psi,x,T),$$

this together with (3.28) yields the desired result and completes the proof.

We are able to modify the hypotheses of Theorems 3a and 3b in the same manner as we did for Theorem 2. That is, condition iv may be replaced by either (3.25) or (3.26).

IV. EXAMPLES

Example 1

McNabb (6) considered the PDE

(4.1) $u_t = u_{xx} + e^u$

together with the boundary conditions

(4.2) $u(a,t) = A$ and $u(b,t) = B$,

where A and B are constants. The equation

$$u_{xx} + e^u = 0$$

has a one parameter family,

(4.3) $v(x,\lambda) = \lambda - 2(\log \cosh(xe^{\lambda/2}/(2)^{1/2}))$,

of solutions. If for some value λ' of λ we have

$$v(a,\lambda') = A \text{ and } v(b,\lambda') = B,$$

then $v(x,\lambda')$ is a steady-state solution of (4.1) with boundary conditions (4.2). McNabb showed by methods similar to those we used to prove Theorem 1 that if

$$v_\lambda(x,\lambda') > 0 \text{ for } x \varepsilon [a,b],$$

then $v(x,\lambda')$ is a stable steady-state solution. If we eliminate λ between Equation (4.3) and the equation $v_\lambda(x,\lambda)=0$ we obtain the envelope of the family, and in fact the region of stability. McNabb showed by methods somewhat different from those given above that for this problem $v(x,\lambda')$ is asymptotically stable if

(4.4) $a > - ke^{-\lambda'/2}/\sqrt{2}$ and $b < ke^{-\lambda'/2}/ 2$

where k is the real positive solution of the equation

$$\cosh k = k \sinh k.$$

He also showed that the solution is unstable if inequalities (4.4) are reversed.

We consider PDE (4.1) together with the boundary conditions

(4.5) $\qquad u_x(a,t) = f_1(u)$ and $u_x(b,t) = f_2(u)$ for all $t > 0$.

We note that if the family (4.3) satisfies conditions ii, iii and iv of Theorem 1 then $v(x,\lambda')$ is a stable steady-state solution. In fact it is asymptotically stable since Equation (4.1) satisfies condition (3.10). If the family satisfies conditions iii-a or iii-b of Theorems 3a and 3b, respectively, instead of iii and iv of Theorem 1, then $v(x,\lambda')$ is an unstable solution.

Equation (4.1) arises in the study of the self-heating of a slab of combustible material (3). The problem treated by McNabb may be considered as one in which the heat is conducted to the outside slowly enough so that the temperature of the exterior of the slab does not rise above the ambient temperature. A slight increase in the temperature of the interior of the slab will cause an increase in the rate of heat production. If this increased heat production exceeds the accompanying increase in the rate that heat is conducted to the exterior of the slab, then instability results.

In our problem with boundary conditions (4.5), instability could occur for two reasons; the first being the same as for McNabb's problem. The second reason is that the increased heat production caused by a slight increase in internal temperature is not matched by a corresponding heat loss from the sides of the slab. If instability occurs for the second reason the temperature may not increase (or decrease) indefinitely but may approach a second higher (or lower) steady-state solution. What happens depends on the functions f_1 and f_2. There may, in fact, be many steady-state solutions or none at all. In case there are many, some may be asymptotically stable while others are unstable.

Example 2

Consider the PDE

(4.6) $$u_t = (1+u^2)u_{xx} - uu_x^{\,2} \equiv F(x,u,u_x,u_{xx}) \text{ for all } x \in (1,2)$$

together with the boundary condition

(4.7) $$u_x(1,t) = f_1(u) \text{ and } u_x(2,t) = f_2(u).$$

The ODE

$$(1 + u^2)u_{xx} - uu_x^{\,2} = 0$$

has as a one parameter family of solutions

(4.8) $$v(x,\lambda) = \sinh \lambda x.$$

If we differentiate v with respect to x and then eliminate λ between the resulting equation and (4.8) we obtain

$$v_x(1,\lambda) = [1 + v^2(1,\lambda)]^{1/2} \sinh^{-1} v(2,\lambda).$$

Suppose

(4.9) $$f_1(u) > (1+u^2)^{1/2}\sinh^{-1} u \text{ for } u > 0, \ f_1(0) = 0 ,$$

(4.10) $f_1(u) < (1+u^2)^{1/2}\sinh^{-1}u$ for $u < 0$,

(4.11) $f_2(u) < (1/2)(1+u^2)^{1/2}\sinh^{-1}u$ for $u > 0$, $f_2(0) = 0$,

(4.12) $f_2(u) > (1/2)(1+u^2)^{1/2}\sinh^{-1}u$ for $u < 0$,

If (4.9) through (4.12) hold we have, by Theorem 1, that the identically zero solution is stable. In order to apply Theorem 2 and show that we have asymptotic stability we must check to see if $F_v \neq 0$. We have

$$\begin{aligned} F_v(x,v,v_x,v_{xx}) &= 2(v)v_{xx} - v_x^2 \\ &= 2\lambda^2\sinh^2\lambda x - \lambda^2\cosh^2\lambda x \\ &= \lambda^2(\sinh^2\lambda x - 1) \ . \end{aligned}$$

Thus $F_v < 0$ if $\sinh^2 x < 1$ or $u < 1$. Hence we can apply Theorem 2 and we conclude that if the above conditions hold, the identically zero solution is asymptotically stable. If inequalities (4.9) and (4.11) or (4.10) and (4.12) were reversed we would conclude from Theorem 3a or 3b that the trivial solution is unstable.

There is still much information regarding PDE (4.6) with boundary conditions (4.7) that can be obtained by methods similar to those we used in proving Theorems 2 and 3. First note that if conditions (4.9) through (4.12) hold, so that the trivial solution is asymptotically stable, we may be interested in the transient part of the solution. Given a particular function $\psi(x)$, we may proceed as in proof of Theorem 2 to obtain an upper (lower) bound for $u(\psi,x,t)$. Thus for an arbitrary $\varepsilon > 0$ we would find a T such that

$$\max_{\substack{x \varepsilon [1,2] \\ t > T}} u(\gamma, x, t) < \varepsilon \ .$$

In order to apply the theorems on stability or instability it is necessary that certain combinations of inequalities (4.9) through (4.12) all hold or are all reversed. Suppose, for example, inequality (4.9) is reversed while (4.10) through (4.12) hold. Then none of the theorems given in this thesis apply to this problem. It may still be possible to find a bounding function (i.e. a function which bounds the solution $u(\gamma, x, t)$ of the above problem) which will give us the information we seek.

The method of finding the bounding function is as follows. First find a one parameter family of curves that satisfy the necessary boundary conditions. Then make the parameter a function of time in such a way that inequality

$$u_t \geq F(x, u, u_x, u_{xx})$$

or

$$u_t \leq F(x, u, u_x, u_{xx})$$

is satisfied. Which of the inequalities we try to satisfy depends on whether we wish to bound the function from above or below.

As an example of the method, consider the function

(4.13) $\quad S(x,a) = ax$

and the related function

(4.14) $\quad \overline{S}(x,t) = a(t)x.$

We differentiate (4.13) to get

(4.15) $\quad S_x(x,a) = a$

We eliminate a between (4.13) and (4.15) and obtain

$$S_x(1,a) = S(1,a) \text{ and } S_x(2,a) = (1/2)(S(2,a)).$$

The substitution of (4.14) into (4.6) yields

(4.16) $\quad a'(t) = -a^3(t).$

The general solution of (4.16) is

$$a(t) = \pm \frac{1}{[2(t+C)]^{1/2}} \quad ,$$

and therefore

$$\overline{S}(x,t) = \pm \frac{x}{[2(t+C)]^{1/2}}$$

is a solution of (4.11). We suppose the functions f_1 and f_2 are such that

(4.18) $\quad f_1(u) > u$ and $f_2(u) > u/2$ if $u > 0$

and

(4.19) $\quad f_1(u) < u$ and $f_2(u) < u/2$ if $u < 0$.

Given any function $\psi(x)$ we may select values C_1 and C_2 for C in Equation (4.17) so that

$$\overline{S}_1(x,0) = -[1/2C_1]^{1/2}x < \psi(x) < [1/2C_2]^{1/2}x = \overline{S}_2(x,0)$$

Since $\overline{S}_1(x,t)$, $\overline{S}_2(x,t)$ and $u(\psi,x,t)$ are all solutions of (4.6) it follows by Lemma 1 that

$$\overline{S}_1(x,t) < u(\psi,x,t) < \overline{S}_2(x,t) \text{ for all } x \in [a,b] \text{ and } t > 0.$$

Both $\overline{S}_1$ and $\overline{S}_2$ go to zero as t goes to infinity so we see that the identically zero solution is asymptotically stable.

Note that if inequalities (4.18) or (4.19) hold for all u then the region of asymptotic stability is the point set

$$A = \{(x,u) \mid x \epsilon [a,b], \quad -\infty < u < \infty\} .$$

We also see that we are able to obtain upper and lower bounds on the function at any time t. We have no way of knowing how good the bounds are.

V. APPENDIX

We wish to generalize Lemma 1 to n-dimensions. The following notation is used in the statement of Lemma 1a and its proof. Let G be an open bounded region in n-dimensional euclidean space, B the boundary of G. For $x \varepsilon G$ then $x_1, x_2, \ldots, x_n$ represents its coordinates in some fixed cartesian coordinate system. Let

$$D_T = \{(x,t) \mid x \varepsilon G \text{ and } 0 < t \leq T\} \text{ if } T < \infty,$$
$$D_T = \{(x,t) \mid x \varepsilon G \text{ and } t > 0\} \text{ if } T = \infty,$$
$$E_T = \{(x,t) \mid x \varepsilon B \text{ and } 0 < t \leq T\} \text{ if } T < \infty,$$
$$E_T = \{(x,t) \mid x \varepsilon B \text{ and } t > 0\} \text{ if } T = \infty.$$

For functions $u(x,t)$ and $v(x,t)$ defined for all $(x,t) \varepsilon \overline{D}_T$ we let $p_i = u_{x_i}$, $q_i = v_{x_i}$, $r_{ij} = u_{x_i x_j}$, $s_{ij} = v_{x_i x_j}$, $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$, $r = (r_{11}, r_{12}, \ldots, r_{nn})$ and $s = (s_{11}, s_{12}, \ldots, s_{nn})$. Let $u_N(x,t)$ and $v_N(x,t)$ be the derivatives of u and v in the direction of the outward normal to the hypersurface E_T.

Lemma 1a: Suppose u,v,f,g and F are functions satisfying the following conditions:

i) $u(x,t)$ and $v(x,t)$ are of class C' for all $(x,t) \varepsilon \overline{D}_T$ where T is a positive number or infinity.

ii) u and v are twice continuously differentiable with respect to x for all $(x,t) \varepsilon D_T$.

iii) $u_N(x,t) = f(u(x,t))$

$v_N(x,t) = g(v(x,t))$ for all $(x,t)\varepsilon E_T$, where f and g are continuous functions.

iv) $f(u) < g(u)$ for all $u\varepsilon[D(f)\cap D(g)]$.

v) $F(x,t,u,p,r)$ is of class C' and satisfies the condition $\sum_{i,j=1}^{n} F_{p_{ij}} \xi_i \xi_j \geq 0$ for all real numbers ξ_i.

vi) $F(x,t,v,q,s) - v_t \geq F(x,t,u,p,r) - u_t$ for all $(x,t)\varepsilon D_T$.

Then if $v(x,0) < u(x,0)$ for all $x\varepsilon\bar{G}$ we have

$$v(x,t) < u(x,t) \text{ for all } (x,t)\varepsilon\bar{D}_T.$$

Proof. As we did for Lemma 1 we divide the proof into two parts. The first part is a proof of the lemma with condition vi replaced by the condition

vi-a) $F(x,t,v,q,s) - v_t > F(x,t,u,p,r) - u_t$.

Only the proof of the first part is given since the proof of the second part requires no essential change from the proof of the second part of Lemma 1.

Part 1. We assume all the hypotheses hold with condition vi replaced by the stronger condition vi-a. Deny the conclusion. Define a function $h(x,t)$ by

$$h(x,t) = v(x,t) - u(x,t).$$

Let t_1 be the glb of the set

$$S = \{t \mid h(x,t) \geq 0 \text{ for some } x\varepsilon\bar{G}\} .$$

We have from the continuity of h with respect to t and the definition of t_1 that

(A.1) $$\sup_{x\varepsilon\bar{G}}(h(x,t_1)) = 0.$$

Hence there is a point $x_1 \varepsilon \overline{G}$ such that

$$h(x_1,t_1) = 0.$$

To show that $x_1 \not\varepsilon B$, suppose $x_1 \varepsilon B$. Then

$$h_N(x_1,t_1) = v_N(x_1,t_1) - u_N(x_1,t_1)$$
$$g(x_1,t_1) - f(x_1,t_1) > 0 .$$

If $x' \varepsilon G$ is a point on the normal to the hypersurface B at x_1, x' sufficiently close to x_1, we have, from the fact that h is of class C' and by application of the mean value theorem, that $h(x',t) > 0$. But this is a contradiction to (A.1), thus $x_1 \varepsilon G$. For fixed t, $h(x,t_1)$ is a function of x only; hence it attains its interior maximum at the point x_1. Therefore we conclude

(A.2) $\quad h(x_1,t_1) = 0$ hence $v(x_1,t_1) = u(x_1,t_1)$,

(A.3) $\quad h_{x_i}(x_1,t_1) = 0$ hence $p_i(x_1,t_1) = q_i(x_1,t_1)$ for $i = 1,2,\ldots,n$,

$$\sum_{i,j=1}^{n} h_{x_i x_j}(x_1,t_1)\, \xi_i\, \xi_j \leq 0 \text{ hence}$$

$$\sum_{i,j=1}^{n} (s_{ij} - r_{ij})\, \xi_i\, \xi_j \leq 0 \text{ for real } \xi_i,$$

$$h_t(x_1,t_1) \geq 0 \text{ hence } v_t(x_1,t_1) - u_t(x_1,t_1) \geq 0.$$

Suppose that

(A.4) $\quad \sum_{i,j=1}^{n} (F_{p_{ij}})(r_{ij} - s_{ij}) \geq 0$ at the point (x_1,t_1),

then from (A.2), (A.3) and by application of the mean value theorem we have

(A.5) $F(x_1,t_1,u,p,r) \geq F(x_1,t_1,v,q,s)$.

If we transpose terms and evaluate the functions at the point (x_1,t_1), condition vi-a becomes

$$F(x_1,t_1,v,q,s) - F(x_1,t_1,u,p,r) > v_t - u_t \geq 0 \quad \text{or}$$

(A.6) $F(x_1,t_1,u,p,r) < F(x_1,t_1,v,q,s)$.

Since (A.5) and (A.6) cannot both hold we conclude that the set S is empty.

To complete the proof it is necessary to show that (A.4) holds. We define two matrices $A = (F_{p_{ij}})$ and $B = (r_{ij} - s_{ij})$. If $C = AB$ then we see that the left hand side of (A.4) is just the trace of C, so we need to show that the trace of C is non-negative. To accomplish this we first note that the trace of a matrix is invariant under a similarity transformation. This follows from the invariance of the characteristic equation of a matrix under a similarity transformation (9) and the fact that the trace is just the coefficient of the n-1^{st} power of x in the characteristic equation. Let P be a nonsingular matrix such that $P^{-1}AP$ is a diagonal matrix. Both of the matrices $P^{-1}AP$ and $P^{-1}BP$ have nonnegative diagonal elements since both A and B are positive semidefinite. The diagonal elements of $P^{-1}CP = P^{-1}APP^{-1}BP$ are products of the diagonal elements of $P^{-1}AP$ and $P^{-1}BP$ and thus are nonnegative. Hence the trace of $P^{-1}CP$, and therefore the trace of C, is nonnegative.

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