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HIGH-ENERGY POTENTIAL SCATTERING

by

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I. INTRODUCTION

The study of high-energy scattering is useful because of the information which is furnished about the structure and interactions of the colliding systems. Since the wavelength $\lambda$ of the incident particles decreases as their energy increases, high-energy particles are more sensitive than low-energy ones to the detailed form of the interaction region. When the wavelength is sufficiently short, the shape of the angular distribution of scattered electrons is extremely sensitive to the form of the potential, particularly at large angles. The type of scattering problem considered here will be elastic potential scattering; that is, the interaction between the bombarding particle and target system is represented by a potential independent of the internal coordinates of the target. When it is necessary to be explicit as to the nature of the bombarding particle, we will consider the electron, with charge $\mathcal{e} = -|z|$. The differential cross section $d\sigma$ for scattering of particles into a solid angle $d\Omega$ is

$$d\sigma = |f(\theta, \phi)|^2 d\Omega.$$  

(1.1)

The function $f(\theta, \phi)$ is known as the scattering amplitude, and can be defined by the following equation:

$$f(\theta, \phi) = -\frac{i}{\sqrt{\pi}} \int d\lambda \ e^{-i k_{\perp} \cdot \hat{\lambda}} \ U(\lambda) \ \psi^*(\lambda).$$  

(1.2)
In Eq. (1.2), \( \psi_o^+ \) is the exact solution of the Schrödinger equation with an asymptotic form consisting of the incident plane wave, \( \xi \cdot \mathbf{k}_o \cdot \mathbf{\hat{r}} \), plus outgoing spherical waves. The quantity \( U(\mathbf{r}) \) is a constant times the potential energy, and \( \mathbf{k}_f \) is the propagation vector in the final direction, with magnitude \( k = \frac{\omega}{\mathbf{\hat{r}}} \). If the exact wave function \( \psi_o^+ \) is unknown, one must resort to approximations. A famous approximation is the first Born approximation, wherein \( \psi_o^+ \) is replaced by \( \xi \cdot \mathbf{k}_o \cdot \mathbf{\hat{r}} \). To this approximation, the scattering amplitude is merely the Fourier transform of the potential energy. While this choice is the simplest mathematically, it is often badly in error, especially for large-angle scattering. It is possible to find an improved approximation if the magnitude of the potential is small compared to the incident energy and if the wavelength of the particle is small compared to the range \( R \) of the potential \( (kR \gg 1) \). These conditions insure that the scattering is mainly forward, which is in agreement with our physical picture of high-energy scattering. An approximate non-relativistic high-energy wave function corresponding to forward scattering has been derived by Molière (1) and Glauber (2), where the latter reference is to an excellent set of lecture notes reviewing high-energy collision theory. The approximate wave function may be expressed as

\[
\psi_o^+(\mathbf{r}) = \xi \cdot \mathbf{k}_f \cdot \mathbf{\hat{r}} - \frac{e^{i\mathbf{k}_o \cdot \mathbf{\hat{r}}}}{\mathbf{k}_f} \int_0^{\infty} U(\mathbf{\hat{r}} - \mathbf{k}_o \cdot \mathbf{\hat{r}}) \, d\omega
\]  

(1.3)

The order of magnitude of the potential-dependent phase factor
Thus, the Born approximation is adequate only if \( \frac{V}{R} \ll 1 \). In Part II., a general method is discussed for obtaining wave functions of the type of Eq. (1.3) for both non-relativistic and relativistic charged particles in electric and magnetic fields. This method is an extension of that used by Tolhoek and De Groot (3) in obtaining approximate wave functions for charged particles in constant electric and magnetic fields either parallel or perpendicular to the incident direction of motion.

A high-energy approximation to the scattering amplitude is immediately obtained by substituting Eq. (1.3) into Eq. (1.2). However, the resultant expression is valid only for very small angles, \( \theta \ll \frac{1}{\sqrt{KR}} \). By applying the stationary phase approximation (4) to each term in the infinite Born series, Schiff (5) was able to sum the series in the case of small (\( \theta < \frac{1}{\sqrt{KR}} \)) and large (\( \theta > \frac{1}{\sqrt{KR}} \)) scattering angles. His small-angle result is the same as that mentioned above. Lippmann (6) suggested a two-potential theory as an alternate approach to the Schiff formulas. Essentially, the potential is split into two parts, the first of which produces forward scattering while the second part produces scattering through non-zero angles. The Schiff formulas (5) have been applied to particular potentials by Tiemann (7) and Valk (8).

The failure of the Schiff formulation to cover the entire range of \( \theta \) made it necessary to obtain an exact expression for \( f \) in a form suitable to high-energy calculations. To
obtain this expression for \( f \), and consequently a first approximation valid for all angles, Saxon and Schiff (9) derived a new integral equation for \( \psi_0^+ \) by applying Green's theorem to \( \psi_0^+ \) and a modified Green's function. In Part III, a simplified procedure has been developed by applying Green's theorem to the approximate wave function \( \omega_0^+ \) and the exact Green's function. The result is a more tractable form of \( \psi_0^+ \) from which the desired form of the scattering amplitude follows directly. Following Lippmann's suggestion (6), the resultant \( f(\phi, \theta) \) is rewritten in the form of a two-potential theory. However, his choice of potentials appears to be different than ours. The high-energy scattering amplitude for a spinless Schrödinger particle in a magnetic field is also derived. The problem of vector-wave scattering has been considered by Brown (10), using the Saxon-Schiff form of analysis.

In addition to deriving an exact expression for \( f(\phi, \theta) \), Saxon and Schiff (9) obtained the large- and small-angle approximations from it. However, their derivation of the large-angle formula is open to criticism, as shown in Part III. Also, their method of obtaining validity conditions for the small-angle formula by order-of-magnitude estimates of Fourier transforms does not appear to be reliable.

The derivations of \( f \) described above have been based on non-relativistic scattering theory. As the bombarding particles are high-energy electrons, it is important to obtain the relativistic analogue of \( f \). Schiff (5) used the stationary
high-energy formulation and the phase-shift analysis. This correspondence has been identified for large $\ell$ by Glauber (2) and Verde (11). A different derivation of this relation is obtained in terms of the WKB approximation for the radial wave equation. In addition, a simplified expression for small $\ell$ is obtained in the high-energy limit. If the potential has vanishing slope at the origin, this expression reduces to an asymptotic expansion in $1/k$ first derived by Verde (11).
II. APPROXIMATE WAVE FUNCTIONS

A. Introduction

To determine the theoretical cross section of an electron scattered by a potential \( V(\vec{r}) \), it is sufficient to know the exact wave function for the particle. As this is not always possible, it is desirable to have generalized approximate wave functions. The form of such wave functions is determined by the conditions of the problem. In the specific case of high-energy electron scattering, we expect the scattering to be mostly forward, with only a small probability of the particle being scattered through a large angle. From classical considerations, this expectation is justified when \( V \ll E \), where \( E \) is the incident kinetic energy of the particle. One may see this as follows. Let the \( x \)-axis be the direction of incident motion. After the collision, the particle's transverse momentum \( p_y \), original momentum \( p \), and scattering angle \( \Theta \) are related as

\[
\sin \Theta = \frac{p_y}{p} \quad .
\]  

(2.1)

The momentum \( p_y \) is equal to \( \int F_y \, dt \), where \( F_y \) is the \( y \)-component of the force \( F \), and \( t \) is the time. We now assume that the potential is bounded in space, with range \( R \), and varies appreciably only in a distance of the order of \( R \); that is, \( \frac{\partial V}{\partial y} \) is of order \( \frac{V}{R} \) in magnitude. The time during which the potential acts is of order \( \frac{E}{v} \), where \( v \) is the particle
velocity. Thus, the order of magnitude of $\sin \theta$ is

$$\sin \theta \sim \frac{\psi_R R^y}{F} \sim V_E.$$  \hspace{1cm} (2.2)

When $v \ll E$, $\theta$ will be small.

The wave nature of the electron imposes an additional restriction. If the electrons are to be scattered mainly in the forward direction, reflection effects must be small. Therefore, we require that the potential should change only slowly in a distance of the order of a reduced wave length $\lambda$; that is,

$$\frac{|\hat{v}v|}{v} \lambda < < 1.$$  \hspace{1cm} (2.3)

The order of magnitude of this condition leads to the requirement $\hat{k} R >> 1$. Thus, the two conditions for forward scattering are

$$\begin{cases}
  v << E, \\
  \hat{k} R >> 1
\end{cases}.$$  \hspace{1cm} (2.4)

We have emphasized forward scattering because it is easy to treat mathematically, and because it corresponds to our physical picture of high-energy scattering.

In Section II. B. we will derive generalized approximations for the wave function of a non-relativistic, spinless particle in either a scalar or vector potential, and in Section II. C. we will derive the corresponding formulas for Pauli and Dirac particles. The procedure used will be an
extension of the one-dimensional, constant-potential derivations of Tolhoek and De Groot (3) to variable potentials in three dimensions which satisfy the conditions of Eq. (2.4).

B. Approximate Wave Functions For Scalar and Vector Potentials

1. Scalar Potentials

The potential energy \( V(\vec{r}) \) may be written as

\[
V(\vec{r}) = e \phi(\vec{r})
\]  

(2.5)

where \( e \) is the charge, and \( \phi(\vec{r}) \) is the scalar potential. The charge \( e \) makes a convenient perturbation parameter, for its smallness implies the desired smallness of \( V \). We therefore write the wavefunction as

\[
\psi(\vec{r}) = e^{i \vec{k} \cdot \vec{r}} + i e \phi(\vec{r})
\]  

(2.6)

where

\[
(V^2 + \vec{k}^2 - \frac{2m\epsilon}{\hbar^2} e \phi) \psi(\vec{r}) = 0
\]  

(2.7)

and

\[
(V^2 + \vec{k}^2) e^{i \vec{k} \cdot \vec{r}} = 0
\]  

(2.8)

The symbol \( \vec{k} \) represents the propagation vector, and its magnitude is

\[
|\vec{k}| = \frac{1}{\lambda} = \sqrt{\frac{2m\epsilon}{\hbar^2}}
\]  

(2.9)
The unit vector in the \( \hat{k} \) direction is defined as \( \hat{k} = \frac{\vec{k}}{|\vec{k}|} \). When \( e \) is set equal to zero, \( \psi(\vec{r}) \) reduces to the correct solution of Eq. (2.7). By substituting Eq. (2.6) in Eq. (2.7), and by use of Eq. (2.8), the following exact equation for \( f \) is obtained:

\[
\hat{k} \cdot \vec{\nabla} f(\vec{r}) - \frac{i}{\hbar} \vec{\nabla} \cdot f(\vec{r}) + \frac{2m}{\hbar^2} f(\vec{r}) + \frac{m}{\hbar^2} \varphi(\vec{r}) = 0 .
\]  
(2.10)

If \( f \) is expanded in powers of \( e \), \( f = \Sigma f_n \alpha^n \), the expression for the first approximation \( f_0 \) is

\[
\hat{k} \cdot \vec{\nabla} f_0(\vec{r}) = \frac{i}{\hbar} \vec{\nabla} f_0(\vec{r}) - \frac{m}{\hbar^2} \varphi(\vec{r}) .
\]  
(2.11)

To proceed further, we must require that \( f_0 \) vary only slowly in an electron wavelength; that is,

\[
\frac{|\vec{\nabla} f_0|}{f_0} \ll 1 .
\]  
(2.12)

Then the first term on the right-hand side of Eq. (2.11) can be dropped, and

\[
\hat{k} \cdot \vec{\nabla} f_0^{(n)}(\vec{r}) = -\hbar \frac{\varphi(\vec{r})}{K} ,
\]  
(2.13)

where the superscript is an index of this additional approximation. The solution to Eq. (2.13) is

\[
f_0^{(n)}(\vec{r}) = -\hbar \frac{\varphi(\vec{r})}{\hbar K} \int_0^\infty \varphi(\vec{r} - \hat{k} \cdot \alpha) d\alpha .
\]  
(2.14)

The correctness of the solution is most easily seen if we take \( \hat{k} \) along one of the Cartesian axes, say the \( z \)-axis. Then Eq. (2.14) becomes
\[ f_0^{(\omega)}(\vec{x}) = -\frac{m^2}{\hbar^2 k^2} \int_{-\infty}^{\infty} \varphi(x, y, z') \, dz' \]  

and application to both sides of \( \hat{\mathbf{k}} \cdot \hat{\mathbf{v}} - 2 \frac{\partial}{\partial \mathbf{z}} \) gives Eq. (2.13). The boundary condition is also correct, for \( f_0 \) vanishes for \( -\infty < z < -r \), and Eq. (6) reduces to a plane wave for this range.

A brief comment on the approximations is in order. By the device of dropping higher powers of \( \varepsilon \), we are clearly dropping higher powers of \( V \) than the first. The error introduced may be estimated by comparing the ratio of the last two terms of Eq. (2.10), when \( f \) is replaced by \( f_0^{(\omega)} \). The order of magnitude of this ratio is \( V/\varepsilon \). The second approximation consisted of dropping the \( \mathbf{v} \cdot \mathbf{f}_0 \) term in Eq. (2.11). Comparing the two terms on the right hand side of Eq. (2.11), with \( f_0 \) replaced by \( f_0^{(\omega)} \), we find the ratio to be of order \( 1/k R \). Both of these ratios are small, by Eq. (2.4). Hence, our approximations have neglected only second-order terms.

It is informative and easy to obtain \( f_0^{(1)} \), which contains an amplitude factor. We use the method of successive approximations on Eq. (2.11) by substituting \( \mathbf{v} \cdot \mathbf{f}_0^{(\omega)} \) for \( \mathbf{v} \cdot \mathbf{f}_0 \). The result is

\[ \hat{\mathbf{k}} \cdot \hat{\mathbf{v}} f_0^{(1)}(\vec{x}) = -\frac{i m}{2 \hbar^2 k^2} \mathbf{v} \cdot \mathbf{\nabla}_\omega \varphi(x - \hat{\mathbf{k}} \omega) \, d\omega - \frac{m}{\hbar^2 k^2} \varphi(\vec{x}) \]  

Writing \( f_0^{(1)} = f_0^{(\omega)} + \mathbf{g} \), Eq. (2.16) becomes

\[ \hat{\mathbf{k}} \cdot \hat{\mathbf{v}} \mathbf{g}(\vec{x}) = -\frac{i m}{2 \hbar^2 k^2} \mathbf{g}(\vec{x}) \]  

(2.17)
where

\[ f(x) = \mathcal{A} \int_0^\infty \phi(x - \hat{k} \cdot s) \, ds \quad (2.18) \]

Equation (2.17) has the same form as Eq. (2.13), hence

\[ f(x) = -i \frac{m}{2\hbar^2} \int_0^\infty dt \int_0^\infty \phi[x - \hat{k} (s + t)] \quad (2.19) \]

\[ f(x) = -i \frac{m}{2\hbar^2} \int_0^\infty dt \int_0^\infty \phi[x - \hat{k} (s + t)] \quad (2.20) \]

Therefore, the solution to Eq. (2.16) is

\[ f_0^{(1)} = -i \frac{m}{2\hbar^2} \int_0^\infty \phi(x - \hat{k} \cdot s) \, ds + \]

\[ -i \frac{m}{2\hbar^2} \int_0^\infty dt \int_0^\infty \phi[x - \hat{k} (s + t)] \quad (2.21) \]

Stopping the approximation with \( f_0^{(1)} \), we replace \( f \) in Eq. (2.6) by Eq. (2.21) to yield the approximate wave function,

\[ \psi'(x) = e^{i \hat{k} \cdot x - \frac{m}{2\hbar^2} \int_0^\infty \phi(x - \hat{k} \cdot s) \, ds} \]

\[ \times e^{i \int_0^\infty dt \int_0^\infty \phi[x - \hat{k} (s + t)]} \quad (2.22) \]

The symbols have been simplified in Eq. (2.22) by the abbreviation

\[ \chi(x) = \frac{2m}{\hbar^2} \chi(x) \quad (2.23) \]

The approximation procedure has introduced both a phase shift and an amplitude factor. The phase shift

\[ \delta(x) = -\frac{1}{2} \int_0^\infty \chi(x - \hat{k} \cdot s) \, ds \quad (2.24) \]

has an order of magnitude \( \frac{\chi R}{k} \), which is the product of \( \chi R \) and \( kR \). While we have imposed restrictions on these last two
parameters, their product is, as yet, unrestricted. In Section III. B., it will be shown that the higher-order approximations are small if

\[ \frac{\psi_{R}}{K} \left( \frac{V}{R} \right) \ll 1 \]  

Equation (2.25)

The exponent of the amplitude factor is of order \( \frac{V}{R} \), so that the change in amplitude of the wave as it passes through the potential is small. Equation (2.22) assumes a more familiar form in one dimension. Then \( \psi' \) becomes

\[
\psi'(x) = e^{i k x - i \frac{\psi}{K} \int_{-\infty}^{x} u(y) \, dy} \left[ 1 + \frac{u(x)}{\frac{V}{K}} \right] \]  

Equation (2.26)

Now the WKB approximation, in one dimension, is

\[
\psi(x) \approx \left( 1 - \frac{\psi}{K} \right)^{\frac{1}{2}} e^{i \int_{\infty}^{x} \sqrt{K - u(y)} \, dy} \left[ 1 + \frac{u(x)}{\frac{V}{K}} \right] \]  

Equation (2.27)

Thus, in the limit of large incident energies, the two approximations are equivalent. This equivalence suggests that Eq. (2.25) is the condition of validity for Eq. (2.22), for the next term in the expansion of the WKB exponential is of this order. In three dimensions, the WKB approximation requires

\[ \frac{V}{K} \gg 1 \]  

so that the above equivalence is restricted to one dimension (2).

Because the amplitude factor is close to unity when \( V << K \), \( f_{0}^{(0)} \) is a sufficient approximation. Equation (2.22) then reduces to
\[ \psi(\vec{r}) = e^{i \vec{k} \cdot \vec{r} - i \frac{\hbar}{2m} \int_{-\infty}^{\infty} u(x, y, z') \, dz'} \]  
(2.30)

The phase shift \( \delta(x) \) must be retained for it is negligible only in the case when \( \frac{\hbar}{m} \ll 1 \). Then the wavefunction is, to good approximation, just the plane wave. This limiting case is the well-known Born approximation. To discuss Eq. (2.30), we choose \( \hat{r} = \frac{\hbar}{i} \) :

\[ \psi(\vec{r}) = e^{i \vec{k} \cdot \vec{r} - i \frac{\hbar}{2m} \int_{-\infty}^{\infty} u(x, y, z') \, dz'} \]  
(2.31)

The phase shift is zero until the particle enters the potential (Figure 1). Inside the potential, the particle continues to travel in the \( \vec{z} \)-direction, but suffers a varying phase shift due to its change in wavelength. Beyond the potential, Eq. (2.31) becomes

\[ \psi(\vec{r}) = e^{i \vec{k} \cdot \vec{r} - i \frac{\hbar}{2m} \int_{-\infty}^{\infty} u(x, y, z') \, dz'} \]  
(2.32)

In words, the particle now has a permanent phase shift

\[ \delta(x, y) = -i \frac{\hbar}{2m} \int_{-\infty}^{\infty} u(x, y, z') \, dz' \]  
(2.33)

It is apparent that Eq. (2.30) is an unusual wavefunction. For instance, its asymptotic form does not contain an outgoing spherical wave. Secondly, the permanent phase shift is unphysical. Figure 1 is reminiscent of the geometrical shadow cast by a black sphere. However, it is well known that the geometrical shadow is broken down, in a characteristic distance \( kR^2 \), by small-angle diffraction. One would expect the
Figure 1. Phase surfaces defined by Eq. (2.31) for a repulsive potential.
same thing here, the small-angle diffraction effects destroying the disturbance due to the potential so that far from the potential the wave function of the particle approaches the incident plane wave with no phase shift. In finding cross sections, we need to know either the wave function within the potential boundary or the asymptotic form of the wave function. The objections raised above have dealt only with the asymptotic form of the approximate wave function. Thus, Eq. (2.30) can be used within the potential region. However, Eq. (2.30) is adequate only for the description of small-angle scattering. Schiff (5) has shown that the use of Eq. (2.30) is valid only for the following range of scattering angles:

$$\theta < \frac{K}{\hbar R}$$ (2.34)

2. Vector potentials

In the case of a vector potential \( \mathbf{A}(x) \), the classical Hamiltonian \( H \) reads

$$H = \frac{\hat{p}^2 - \frac{2 A}{\hbar c}}{2m} + E$$ (2.35)

where \( \hat{p} \) is the canonical momentum. By multiplying through by \( \psi \), and by replacing \( \hat{p} \) by \( \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \), Eq. (2.35) becomes

$$\left( \varphi^2 + k^2 - \frac{2 i e A}{\hbar c} \cdot \varphi - \frac{e^2}{\hbar^2 c^2} A^2 \right) \psi(x) = 0$$ (2.36)

where the Lorentz condition, \( \varphi \cdot A = 0 \), has been used. The vector potential is assumed to be bounded in space, with range
As in the case of the scalar potential, we write
\[ \psi(\mathbf{\hat{x}}) = e^{i \mathbf{\hat{x}} \cdot \mathbf{\hat{x}}} + i \mathbf{\hat{x}} f(\mathbf{\hat{x}}) \] (2.6)
and substitute into Eq. (2.36) to obtain an expression for \( f \):
\[ \hat{\mathbf{\hat{R}}} \cdot \nabla f - i \frac{1}{2 \hbar} \nabla^2 f + \frac{2}{\hbar} \mathbf{\hat{R}} \cdot \mathbf{\hat{A}}(\mathbf{\hat{x}}) + \frac{1}{\hbar c} \mathbf{\hat{R}} \cdot \mathbf{\hat{A}}(\mathbf{\hat{x}}) + \frac{2 \hbar^2 c^2}{\hbar^2 c^2 \mathbf{\hat{R}}} f = 0 \] (2.37)
If \( f \) is expanded in powers of \( e \), the differential equation for the first approximation \( f_0 \) becomes
\[ \hat{\mathbf{\hat{R}}} \cdot \nabla f_0 - i \frac{1}{2 \hbar} \nabla^2 f_0 + \frac{1}{\hbar c} \mathbf{\hat{R}} \cdot \mathbf{\hat{A}}(\mathbf{\hat{x}}) \] (2.38)
Again we assume that \( f_0 \) is a slowly varying function in an electron wavelength, and drop the \( \nabla^2 f_0 \) term. Equation (2.38) becomes
\[ \hat{\mathbf{\hat{R}}} \cdot \nabla f_0^{(0)}(\mathbf{\hat{x}}) = \frac{1}{\hbar c} \mathbf{\hat{R}} \cdot \mathbf{\hat{A}}(\mathbf{\hat{x}}) \] (2.39)
with solution
\[ f_0^{(0)}(\mathbf{\hat{x}}) = \frac{1}{\hbar c} \int_0^\infty \mathbf{\hat{R}} \cdot \mathbf{\hat{A}}(\mathbf{\hat{x}} - \hat{\mathbf{\hat{R}}} \cdot \mathbf{\hat{d}}) d\mathbf{\hat{d}} \] (2.40)
An amplitude factor could be obtained in a manner analogous to Eq. (2.16). The error in dropping \( \nabla^2 f_0 \) is of order \( \hbar^2 R \), as before. The error in neglecting higher-order terms in \( e \) is most simply judged by comparison of the last two terms in Eq. (2.37). Their ratio is of order \( \left( \frac{\hbar c^2}{e} \right) \frac{1}{\hbar c} \). Using just Eq. (2.40) as our approximation to \( f \), the approximate wave function for small-angle scattering in a magnetic field is
If both vector and scalar potentials are present, the derivation may be modified in an obvious way to give

\[ \psi'(\vec{x}) = e^{-i\frac{\ell_0}{\hbar_c}} \int_{-\infty}^{\infty} \vec{v}(\vec{x} - \hbar \vec{x}_0) \, d\vec{x} \psi(\vec{x}) \quad (2.42) \]

C. Approximate Wave Functions For Pauli and Dirac Particles

1. Pauli theory

To take account of the electron spin, the starting wave equations must be altered. If we consider a non-relativistic electron in an external magnetic field, the Pauli theory is applicable. The Hamiltonian must be modified by the inclusion of the interaction energy between the electron's magnetic moment and the magnetic field. This additional energy is \(-\vec{\mu} \cdot \vec{B}\), where \(\vec{B}\) is the magnetic field and

\[ \vec{\mu} = \frac{2 \hbar}{2m_e} \vec{\tau} \quad (2.43) \]

is the magnetic moment. The quantity \(\vec{\tau}\) represents the three Pauli spin matrices, with the property

\[ \sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij} \quad (2.43a) \]

With this change, the Hamiltonian becomes

\[ H = (\vec{p} - e \vec{A}) \frac{2}{\hbar m} - \vec{\mu} \cdot \vec{B} = E \quad (2.44) \]

and the corresponding wave equation is
To derive the approximate wave function, the previous procedure must be re-examined. Since the electron has two components of spin, the solution of the free equation,

\[
(\mathbf{p}^2 + \mathbf{k}^2) \psi_0(\mathbf{r}) = 0
\]

is

\[
\psi_0(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{R}} \chi_0(\mathbf{r})
\]

where the ratio of \(a\) to \(b\) determines the polarization. It is evident, therefore, that the form of Eq. (2.6) should be changed to

\[
\psi(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{R}} \psi_0(\mathbf{r})
\]

where \(f\) is a 2 \times 2 matrix. Now, in addition to modifying the phase of the electron wave as it is scattered forward through the potential, the operator \(e^{i\mathbf{k} \cdot \mathbf{R}}\) takes into account the spin change.

An additional problem arises in finding an equation for \(f\). We are now taking derivatives of an exponential operator and must worry about non-commuting factors. The most general form of the function \(f\) is

\[
f(\mathbf{r}) = g(\mathbf{r}) + \frac{1}{\mathbf{r}} \cdot \mathbf{\mathbf{R}}(\mathbf{r})
\]
where \( g \) and \( \{ \hat{f}_x, \hat{f}_y, \hat{f}_z \} \) are scalar functions. Thus, the problem reduces to the differentiation of \( e^{i \vec{\sigma} \cdot \vec{\mathcal{L}}} \). Now
\[
\frac{\partial}{\partial x} e^{i \vec{\sigma} \cdot \vec{\mathcal{L}}(x)}
\] is not equal to \( i [ \vec{\sigma} \cdot \frac{\partial}{\partial x}, \vec{\mathcal{L}}] e^{i \vec{\sigma} \cdot \vec{\mathcal{L}}} \), for the two functions do not commute. However, because of the nature of the Pauli spin matrices, any function of \( \vec{\sigma} \) may be expressed as a linear function of \( \vec{\sigma} \). By expanding \( e^{i \vec{\sigma} \cdot \vec{\mathcal{L}}} \) in a power series, and by use of Eq. (2.43a), we have
\[
\begin{align*}
\xi' \mathcal{L} &= \cos \mathcal{L} + i (\vec{\sigma} \cdot \mathcal{L}) \sin \mathcal{L} \\
(2.50)
\end{align*}
\] where \( \mathcal{L} = \frac{\mathcal{L}_1}{\mathcal{L}_2} \). Differentiation of \( e^{i \vec{\sigma} \cdot \vec{\mathcal{L}}} \) is therefore equivalent to differentiation of the right-hand side of Eq. (2.50), for which the usual rules hold. The result is
\[
\begin{align*}
\frac{\partial}{\partial x} e^{i \vec{\sigma} \cdot \vec{\mathcal{L}}}
&= \xi (\vec{\sigma} \cdot \mathcal{L}) e^{i \vec{\sigma} \cdot \mathcal{L}} + i (\vec{\sigma} \cdot \frac{\partial}{\partial x} e^{i \vec{\sigma} \cdot \mathcal{L}}) \sin \mathcal{L} \\
&= i \vec{\sigma} \cdot \frac{\partial}{\partial x} e^{i \vec{\sigma} \cdot \mathcal{L}} - i (\vec{\sigma} \cdot \frac{\partial}{\partial x} e^{i \vec{\sigma} \cdot \mathcal{L}}) \sin \mathcal{L} \frac{\partial}{\partial x} e^{i \vec{\sigma} \cdot \mathcal{L}} \sin \mathcal{L} (2.51)
\end{align*}
\]
If the magnitude of \( \mathcal{L} \) is small, the second term is of order \( |\mathcal{L}| \) times smaller than the first. The first term, however, is the term we would get if the ordinary rules of differentiation held. Thus, if one is carrying out an approximate analysis in which second order terms in \( \mathcal{L} \) are neglected, one is justified in differentiating \( e^{i \vec{\sigma} \cdot \mathcal{L}} \) by the usual rules. Our procedure will be to derive \( f \) under this assumption that \( \mathcal{L} \) is small.

If Eq. (2.48) is substituted into Eq. (2.45) and the usual rules of differentiation are followed, the following equation is obtained:
The polarization is arbitrary; that is, we want Eq. (2.49) to hold for any \( a \) and \( b \) (zero to one in magnitude). Hence, the bracket must be zero. Keeping only zero order terms in \( e \), and dropping the \( \rho^2 f \) term in the usual way, we have

\[
\hat{\rho} \cdot \nabla f_0^0(\lambda) = \frac{1}{\hbar c} \hat{\rho} \cdot \hat{A}(\lambda) + \frac{1}{2\hbar c} \vec{\sigma} \cdot \vec{B}(\lambda)
\]

and

\[
f_0^0(\lambda) = \frac{1}{\hbar c} \int_{-\infty}^{\infty} \hat{\rho} \cdot \hat{A}(\lambda - \hat{\rho} s) ds + \frac{\vec{\sigma}}{2\hbar c} \int_{-\infty}^{\infty} \vec{B}(\lambda - \hat{\rho} s) ds.
\]

The approximate wave function is

\[
y(\lambda) = \frac{\chi(\lambda)}{\psi(\lambda)} \psi(\lambda),
\]

Comparing Eq. (2.55) with Eq. (2.49), one sees that the first approximation to \( \hat{\rho} \) is \( \hat{\rho}_0 = \frac{\alpha}{2\hbar c} \int_{-\infty}^{\infty} \vec{B}(\lambda - \hat{\rho} s) ds \). The magnitude of \( \hat{\rho}_0 \) is \( \frac{\alpha}{\sqrt{2}} \psi(\lambda) \), which has been assumed small in the derivation of Eq. (2.41). Thus, we have neglected only second-order terms by differentiating \( \lambda^{\alpha} f \) in the usual way, and the derivation of Eq. (2.56) is justified.

2. Dirac Theory

The first-order Dirac equation, with electro-magnetic potentials, is
\[
\begin{align*}
\left[ E - \mathbf{p} \cdot \mathbf{\alpha} + \mathbf{c} \cdot \mathbf{\beta} \cdot (\mathbf{p} - \frac{2}{c} \mathbf{A}) + B \mathbf{m} \cdot \mathbf{c} \right] \psi(x) &= 0 \quad (2.57)
\end{align*}
\]

Here \( \mathbf{\hat{p}} \) stands for the usual momentum operator \( \left( \mathbf{\hat{c}} \mathbf{\hat{p}} \right) \), \( \psi \) is a 4 x 1 spinor, and \( \mathbf{\alpha} \), \( \mathbf{\beta} \) are the 4 x 4 Dirac matrices defined as

\[
\begin{align*}
\mathbf{\alpha} &= \left( \begin{array}{cc} \hat{\sigma}_1 & \hat{\sigma}_2 \\
\hat{\sigma}_3 & \hat{\sigma}_1 \end{array} \right), \\
\mathbf{\beta} &= \left( \begin{array}{cc} 1 & \hat{\sigma}_1 \\
\hat{\sigma}_3 & -1 \end{array} \right).
\end{align*}
\] (2.58)

The components of \( \hat{\sigma} \) are the 2 x 2 Pauli matrices, and each element of \( \hat{\sigma} \), \( \hat{\beta} \) stands for a 2 x 2 matrix. In order to apply conveniently the approximation procedure, we must transform Eq. (2.57) into a second-order equation in the momentum operator. Therefore, we multiply Eq. (2.57) on the left by

\[
\left( E - \mathbf{p} \cdot \mathbf{\alpha} + \mathbf{c} \cdot \mathbf{\beta} \cdot (\mathbf{p} - \frac{2}{c} \mathbf{A}) - B \mathbf{m} \cdot \mathbf{c} \right)
\]

to obtain

\[
\begin{align*}
\left( (E - \mathbf{p} \cdot \mathbf{\alpha}) \mathbf{\hat{c}} \mathbf{\hat{p}} - (\mathbf{p} - \frac{2}{c} \mathbf{A}) \mathbf{\hat{c}} \mathbf{\hat{p}} - \mathbf{m} \cdot \mathbf{c} \mathbf{\hat{c}} \mathbf{\hat{p}} + \frac{2}{c} \mathbf{\hat{c}} \mathbf{\hat{p}} (\mathbf{\hat{c}} \mathbf{\hat{p}}) + \\
+ i \frac{2}{c} \mathbf{\hat{c}} \mathbf{\hat{p}} \mathbf{\hat{c}} \mathbf{\hat{p}} \right) \psi(x) &= 0 \quad (2.59)
\end{align*}
\]

The symbol \( \hat{\sigma} \) stands for

\[
\hat{\sigma} = \left( \begin{array}{cc} \hat{\sigma}_1 & \hat{\sigma}_2 \\
\hat{\sigma}_3 & \hat{\sigma}_1 \end{array} \right)
\] (2.60)

and

\[
\hat{\mathbf{c}} = - \mathbf{\hat{\nabla}} \mathbf{\phi}(x)
\] (2.61)

Equation (2.59) may be rewritten in the following form:

\[
\begin{align*}
\left[ \mathbf{\nabla}^2 + \mathbf{K}^2 - \frac{2 \mathbf{E} \mathbf{\alpha} \mathbf{\beta}}{\hbar \mathbf{c}} + \frac{2 \mathbf{\beta} \mathbf{\alpha}}{\hbar \mathbf{c}} - \frac{2 i \mathbf{E}}{\hbar \mathbf{c}} \mathbf{\nabla} \cdot \mathbf{\phi} - \frac{\mathbf{\beta}}{\hbar \mathbf{c}} \mathbf{\nabla} \right] \psi(x) &= 0
\end{align*}
\] (2.62)
where
\[ k^2 = \frac{E^2 - m^2 c^4}{c^2} . \] (2.62a)

The type of procedure used for the Pauli equation will be used again here. We wish to substitute the wave function
\[ \psi(\vec{r}) = e^{i\vec{r}\cdot\vec{A}} \psi_0(\vec{r}) \] (2.48)
into Eq. (2.62). Now, the wave function \( \psi_0 \) satisfies Eq. (2.46), and is of the form
\[ u_0 = \text{constant} \times \left( -\frac{\vec{p} \cdot \vec{c}}{\varepsilon + mc^2} \alpha \right) - \frac{c(\vec{p}_x - i\vec{p}_y)\beta}{\varepsilon + mc^2}, \]

for positive energies. The function \( \alpha \vec{f}(\vec{r}) \) is now of the form
\[ e^{i\vec{r}\cdot\vec{A}} = u(\vec{r}) + v(\vec{r}, \vec{z}, \vec{r}) \] (2.64)
so that care must again be taken when we differentiate \( e^{i\vec{r}\cdot\vec{A}} \).

However, if the magnitude of \( v \) is small, we have
\[ \frac{\partial}{\partial \vec{x}} e^{i\vec{r}\cdot\vec{A}} = \frac{\partial}{\partial \vec{x}} (1 + i\vec{r}\cdot\vec{A} + \cdots) = i \frac{\partial}{\partial \vec{x}} \psi_0 + \cdots \] (2.65)
\[ \approx i \frac{\partial}{\partial \vec{x}} \psi_0 e^{i\vec{r}\cdot\vec{A}} \] (2.66)
where the last equation is accurate to first order in \( v \).

Therefore, if second-order terms in \( v \) can be neglected, one is justified in differentiating \( e^{i\vec{r}\cdot\vec{A}} \) by the usual rules. Thus
it will be assumed beforehand that the magnitude of \( v \) is small.

By substituting Eq. (2.48) into Eq. (2.62) and by differentiating \( e^{i\tau f} \) in the usual way, one obtains the following equation for \( f \):

\[
\left\{ \hat{\mathbf{r}}, \frac{\partial}{\partial \mathbf{r}} f - \frac{i}{2m c} \mathbf{v} \cdot \mathbf{f} + \frac{e}{2m c} \mathbf{v} \mathbf{f} f^* + \frac{\hbar}{2m c} \mathbf{f} f^* + \frac{e \hbar}{2m c} \mathbf{f} f^* - \frac{e \hbar}{2m c} \mathbf{f} f^* - \frac{e}{2m c} \mathbf{E} \cdot \mathbf{A} \right\} e^{iS} \psi_0 = 0. \tag{2.67}
\]

The polarization of the incoming electron is arbitrary, so the bracket in Eq. (2.62) is set equal to zero. If, as usual, we drop terms in \( c \) and assume the \( v^* f \) term negligible, the equation to be solved is

\[
\hat{\mathbf{r}}, \frac{\partial}{\partial \mathbf{r}} f^{(o)}(\mathbf{r}) = -\frac{e}{\hbar c} \mathbf{v}(\mathbf{r}) + \frac{1}{\hbar c} \hat{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) + \\
+ \frac{1}{2\hbar c} \left[ \mathbf{\nabla} \cdot \mathbf{B}(\mathbf{r}) + i \mathbf{\nabla} \cdot \mathbf{E}(\mathbf{r}) \right]. \tag{2.68}
\]

The solution is

\[
f^{(o)}(\mathbf{r}) = -\frac{e}{\hbar c} \mathbf{v}(\mathbf{r}) \int_0^\infty \mathbf{\nabla} \mathbf{A}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) d\mathbf{s} + \frac{1}{\hbar c} \int_0^\infty \mathbf{\nabla} \mathbf{A}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) d\mathbf{s} \]

\[
+ \frac{1}{2\hbar c} \int_0^\infty \left[ \mathbf{\nabla} \cdot \mathbf{B}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) + i \mathbf{\nabla} \cdot \mathbf{E}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) \right] d\mathbf{s}. \tag{2.69}
\]

The approximate wave function for a Dirac particle is, thus,

\[
\chi(\mathbf{r}) \equiv \psi^{(o)}(\mathbf{r}) \int_0^\infty \left[ \mathbf{\nabla} \cdot \mathbf{B}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) + i \mathbf{\nabla} \cdot \mathbf{E}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) \right] d\mathbf{s} \times \psi_0(\mathbf{r}) 
\]

\[
\times \mathbf{e}^{-\frac{i}{\hbar c} \mathbf{\nabla} \mathbf{A}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) d\mathbf{s} + \frac{i}{\hbar c} \mathbf{\nabla} \mathbf{A}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) d\mathbf{s}}, \tag{2.70}
\]

where, in the relativistic case,

\[
U = \frac{2E \mathbf{v}}{\hbar c^2}. \tag{2.71}
\]

By comparing Eq. (2.69) with Eq. (2.64), one finds that the first approximation to \( v \) is

\[
v_0 = \frac{Z e}{\hbar c} \int_0^\infty \mathbf{\nabla} \mathbf{B}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) d\mathbf{s} + \frac{Z e}{\hbar c} \int_0^\infty \mathbf{\nabla} \mathbf{E}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) d\mathbf{s} + \frac{Z e}{\hbar c} \int_0^\infty \mathbf{\nabla} \mathbf{A}(\mathbf{r} - \hat{\mathbf{r}} \mathbf{s}) d\mathbf{s}.
\]
The term in \( \nu_0 \) containing the magnetic field is of order \( \frac{eB}{mc} \), which is assumed small. The order of magnitude of the \( z \hat{\mathbf{e}} \) term is \( \sqrt{\nu} \). To see this, we note that \( z \hat{\mathbf{e}} = -\mathbf{\phi} \times \mathbf{\hat{r}} \). The force is of order \( \sqrt{\nu} \), the important range of integration is of order \( \nu \), and \( z \) has an expectation value of order \( \nu \), for it mixes small and large components of \( \psi \). Thus, we have as an order of magnitude, \( \frac{eB}{m} \sqrt{\nu} \). But it is well known from relativistic mechanics that \( \nu_\lambda = \frac{eB}{mc} \), which makes the order of magnitude \( \sqrt{\nu} \), as stated. Thus, both terms in \( \nu_0 \) are small, for we have assumed that \( eB \). The derivation of \( \nu_0(\lambda) \) is therefore justified.

In the important case of an electric field alone, we set \( \hat{\mathbf{B}} \) and \( \hat{\mathbf{A}} = 0 \) in Eq. (2.70) to obtain

\[
\nu(\lambda) = \mathbf{e}^{-i\frac{e}{2mc} \int_0^\infty \hat{\mathbf{e}} (\lambda - \hat{\mathbf{r}} \cdot \hat{\mathbf{S}}) \lambda \lambda} u_0 \mathbf{e}^{i\frac{e}{2mc} \int_0^\infty \hat{\mathbf{e}} (\lambda - \hat{\mathbf{r}} \cdot \hat{\mathbf{S}}) \lambda \lambda}
\]  

(2.72)

As the exponent of the exponential operator is of order \( \sqrt{\nu} \), the space-dependent spinor,

\[
u_0'(\lambda) = \mathbf{e}^{-i\frac{e}{2mc} \int_0^\infty \hat{\mathbf{e}} (\lambda - \hat{\mathbf{r}} \cdot \hat{\mathbf{S}}) \lambda \lambda} u_0
\]  

(2.73)

may be approximated by expanding the exponential:

\[
u_0'(\lambda) \approx \mathbf{e}^{[1 + \frac{e}{2mc} \int_0^\infty \hat{\mathbf{e}} (\lambda - \hat{\mathbf{r}} \cdot \hat{\mathbf{S}}) \lambda \lambda]} u_0
\]  

(2.74)

Equation (2.72) is an adequate approximate wave function for the description of relativistic small-angle scattering in an electric field.
III. EXACT NON-RELATIVISTIC TIME-INDEPENDENT SCATTERING

A. Introduction

It is the purpose of this section to derive an exact expression for the non-relativistic cross section of a high-energy particle scattered by a scalar potential \( V(\vec{x}) \). The extension to a relativistic theory will be made in Part IV. In the remainder of Section III. A., we give a brief summary of time-independent scattering theory.

If the particle has incident energy \( E \), its wavefunction is a solution of

\[
\left[ \nabla^2 + k^2 - V(\vec{x}) \right] \psi(\vec{x}) = 0
\]

where

\[
\begin{align*}
k^2 &= \frac{\hbar^2}{\mu} \frac{E}{\hbar} \\
U(\vec{x}) &= \frac{\hbar^2}{\mu} V(\vec{x})
\end{align*}
\]

(3.2)

In order to simplify the derivations, the potential \( U(\vec{x}) \) will be assumed to vanish outside a bounded region. The boundary conditions imposed on \( \psi \) require the asymptotic form of the wavefunction to be the sum of the incident plane wave and an outgoing spherical wave; that is,

\[
\psi^+(\vec{x}) \xrightarrow{|\vec{x}| \to \infty} i \hat{\vec{k}} \cdot \hat{\vec{x}} + \frac{i k r}{r} f(\theta, \varphi)
\]

(3.3)

The plus sign on \( \psi \) refers to the choice of outgoing. (+)
spherical waves rather than incoming (-) spherical waves. The function \( f(\phi, \rho) \) is known as the scattering amplitude. The differential element of cross section, \( d\sigma \), is equal to the flux of particles scattered through solid angle \( d\alpha \), divided by the incident flux. For elastic scattering, one sees from Eq. (3.3) that the differential cross section is

\[
\sigma(\phi, \rho) = \frac{\psi(\phi, \rho)}{\psi(\phi, \rho)} = \left| f(\phi, \rho) \right|^2.
\] (3.4)

Thus, the derivation of the scattering amplitude is the fundamental step in finding cross sections.

A general expression for \( f(\phi, \rho) \) is obtained by writing Eq. (3.1) as an integral equation. If \( G_o(\rho, \rho') \) is the Green's function with outgoing-wave boundary conditions which satisfies the differential equation

\[
(\nabla^2 + \kappa^2) G_o(\rho, \rho') = -\delta(\rho - \rho'),
\] (3.5)

we obtain

\[
\psi_o(\rho) = e^{i\vec{k}_o \cdot \vec{r}} \int d\rho' \ G_o(\rho, \rho') \ \chi(\rho') \ \psi_o(\rho').
\] (3.6)

The subscripts on \( \chi \) and \( \psi_o \) in the preceding equation refer to the direction of motion of the incident particle, whereas the subscript on \( G_o \) denotes the free-space Green's function. The quantity \( \delta(\rho - \rho') \) is the Dirac delta function, and the explicit form of \( G_o(\rho, \rho') \) is

\[
G_o = \frac{1}{4\pi} \frac{\rho - \kappa \rho'}{\rho^2},
\] and

\[
\rho = |\rho - \rho'|.
\] (3.7)
We must look at the asymptotic form of Eq. (3.6) to find $f$. Since $U(\lambda)$ limits the range of integration, the result is

$$
\lim_{|\lambda| \to \infty} i \vec{k}_o \cdot \vec{\lambda} \to - \frac{1}{4\pi} \frac{i k}{\lambda} \int d\lambda' e^{-i \vec{k}_f \cdot \vec{\lambda}'} U(\lambda') \Psi^+_o(\lambda') ,
$$

(3.8)

where the propagation vector in the final direction, $\vec{k}_f$, is defined as

$$
\vec{k}_f = \frac{\lambda}{|\lambda|}
$$

(3.9)

Comparing Eq. (3.8) with Eq. (3.3), we have

$$
f(\theta, \phi) = - \frac{\lambda}{4\pi} \int d\lambda' \phi^+_f(\lambda') U(\lambda') \Psi^+_o(\lambda) ,
$$

$$
\phi^+_f(\lambda) = \int e^{i \vec{k}_f \cdot \vec{\lambda}'}
$$

(3.10)

In Section III. B., Eq. (3.10) will be written in a more convenient form for the study of high-energy scattering. In Section III. C., analogous problem in one dimension is investigated, and in Section III. D., approximate three-dimensional formulas for the limiting cases of small- and large-angle scattering are discussed. Section III. E. contains a brief account of scattering by a magnetic field.

B. High-Energy Scattering In Three Dimensions

The scattering amplitude, Eq. (3.10), is most simply approximated by replacing $\Psi^+_o(\lambda)$ by the incident plane wave, $e^{i \vec{k}_o \cdot \vec{\lambda}}$; that is, by the unscattered part of the wave. Then, one computes only the Fourier transform of the potential to arrive at an approximation to the cross section. This is the
first Born approximation, and is valid when \( \frac{\nu R}{k} << 1, \quad kR >> 1 \). The approximation for \( \psi_0^+ \) may be improved by iterating Eq. (3.6) to obtain the Born series:

\[
\psi_0^+(\vec{x}) = e^{i \vec{k} \cdot \vec{x}} - \int d\vec{x}' G_0^+(\vec{x}, \vec{x}') u(\vec{x}') e^{i \vec{k} \cdot \vec{x}'} + \ldots .
\]  

(3.11)

If \( \frac{\nu R}{k} \) is not small, the first term of Eq. (3.11) is not sufficient, and higher-order Born approximations must be used. These are difficult to compute, and it would seem desirable, therefore, to rewrite Eq. (3.11) in a form leading to a more quickly convergent series. The leading term will be characteristic of the problem. In Section II. B., we obtained an approximate wave function

\[
\omega^+(\vec{x}) = e^{i \vec{k} \cdot \vec{x}} + i \delta(\vec{x}),
\]

(2.30)

\[
\delta(\vec{x}) = -\frac{k}{\nu} \int_0^\infty u(\vec{x} - \hat{k} \vec{x}) d\alpha,
\]

(2.24)

which is accurate provided \( \nu << k \), \( kR >> 1 \). The parameter \( \frac{\nu R}{k} \) is, as yet, unrestricted. For high-energy scattering, \( \omega^+ \) will be the leading term in the rewritten equation for \( \psi^+ \).

1. The exact wave function

Saxon and Schiff (9) obtained an integral equation for \( \psi^+ \), with \( \omega^+ \) as the inhomogeneous term, by applying Green's theorem to \( \psi^+ \) and a suitably modified Green's function. Their derivation can be simplified considerably if one deals with the exact Green's function \( G^+ \), which satisfies
In terms of $G_0^+$, we may write $G^+$ as

$$G^+(\mathbf{x}, \mathbf{x}') = G_0^+(\mathbf{x}, \mathbf{x}') - \int d\mathbf{x}'' G_0^+(\mathbf{x}, \mathbf{x}'') \psi(\mathbf{x}'') G^+(\mathbf{x}'', \mathbf{x}').$$  

(3.12a)

To obtain an equation for $\psi^+$ with $\omega_0^+$ as leading term, we apply Green's theorem to $\omega_0^+$ and $G^+$:

$$\int [ G^+(\mathbf{x}, \mathbf{x}) \nabla^2 \omega_0^+(\mathbf{x}') - \omega_0^+(\mathbf{x}') \nabla^2 G^+(\mathbf{x}', \mathbf{x}) ] \, d\mathbf{x}' = \int [ G^+(\mathbf{x}, \mathbf{x}') \nabla' \omega_0^+(\mathbf{x}') - \omega_0^+(\mathbf{x}') \nabla' G^+(\mathbf{x}', \mathbf{x}) ] \cdot dS'. $$  

(3.13)

The subscript on $\omega_0^+$ again refers to the incident direction of motion. To evaluate the volume term, we must know the differential equation satisfied by $\omega_0^+$. Remembering from Section II. B. that

$$\hat{k} \cdot \nabla \delta(\mathbf{x}) = -\frac{1}{2}k \psi(\mathbf{x}),$$  

(3.13a)

one finds

$$[ \nabla^2 + k^2 - \psi(\mathbf{x}) ] \omega_0^+(\mathbf{x}) = \hat{\phi}_0(\mathbf{x}) \nabla \cdot \delta(\mathbf{x}).$$  

(3.14)

Substituting Eqs. (3.12) and (3.14) into Eq. (3.13), and using reciprocity, $G^+(\mathbf{x}, \mathbf{x}') = G^+(\mathbf{x}', \mathbf{x})$, we obtain

$$\nabla(\mathbf{x}) = \omega_0^+(\mathbf{x}) + \int G^+(\mathbf{x}, \mathbf{x}') \hat{\phi}_0(\mathbf{x}') \nabla' \cdot \delta(\mathbf{x}') \, d\mathbf{x}',$$  

(3.15)

where the surface integral is denoted by
The right-hand side of Eq. (3.15) satisfies the Schrödinger equation, as may be easily demonstrated by application of the operator \( (\mathbf{p}^2 + \mathbf{A}^2 - \mathbf{V}) \) and use of Eqs. (3.12) and 3.14). However, it is not obvious that the asymptotic form of \( \lambda(\mathbf{x}) \) is a plane wave plus outgoing spherical waves. Therefore, it remains to be shown that \( \lambda(\mathbf{x}) \) is the desired solution \( \psi_0^+(\mathbf{x}) \).

It is first necessary to note that, for \( g(\mathbf{x}) \) well-behaved at infinity, the following surface integral vanishes:

\[
\int_{\mathbf{x} \to \infty} \mathcal{S}_x \left[ \varphi_{\lambda} \cdot \mathbf{e}^{i \mathbf{a} \cdot \mathbf{x}} \right] g(\mathbf{x}) = 0 .
\] (3.17)

The quantity \( \mathcal{S}_x \) is non-zero only if \( \mathbf{x} \) lies either in the potential or in a semi-infinite cylinder such that a straight line proceeding from \( \mathbf{x} \) in the direction -\( \mathbf{n} \) pierces the potential (Figure 2). This second region will be called the forward cylinder with axis in the direction \( \mathbf{n} \). The factor \( \mathcal{S}_x \cdot \mathbf{e}^{i \mathbf{a} \cdot \mathbf{x}} \) then limits the surface of integration to the intersection of the sphere \( \mathbf{x} \) with the forward cylinder. In the limit as \( |\mathbf{x}| \to \infty \), the intersection becomes a plane disc with unit normal \( \mathbf{n} \). Then, by use of Eq. (3.13a), Eq. (3.17) becomes

\[
\int_{\mathbf{x} \to \infty} \mathcal{S}_x (\mathbf{n}, \mathbf{e}^{i \mathbf{a} \cdot \mathbf{x}}) g(\mathbf{x}) = \frac{1}{i} \int_{\mathbf{x} \to \infty} \mathcal{S}_x g(\mathbf{x}) \mathbf{e}^{i \mathbf{a} \cdot \mathbf{x}} \lambda(\mathbf{x}) = 0 ,
\] (3.18)

since \( \lambda(\mathbf{x}) \) is bounded in space.
Figure 2. Potential region and semi-infinite forward cylinder.
Equation (3.17) permits us to rewrite Eq. (3.16) as

\[ \mathcal{L}(\lambda) = \int_{\Omega'} \lambda^3 \left[ G^+(\lambda', \lambda) \partial' \phi_0(\lambda') - \phi_0(\lambda') \partial' G^+(\lambda', \lambda) \right] \cdot \end{equation} (3.19) \]

Adding and subtracting unity from \( \lambda \), we have

\[ \mathcal{L}(\lambda) = \int_{\Omega'} \lambda^3 \left[ G^+(\lambda', \lambda) \partial' \phi_0(\lambda') - \phi_0(\lambda') \partial' G^+(\lambda', \lambda) \right] \\
+ \int_{\Omega'} \lambda^3 \left[ G^+(\lambda', \lambda) \partial' \phi_0(\lambda') - \phi_0(\lambda') \partial' G^+(\lambda', \lambda) \right] \cdot \end{equation} (3.20) \]

The second term in Eq. (3.20) vanishes. To see this one notes that because of the outgoing wave boundary condition, \( G^+ \) satisfies

\[ \partial' G^+(\lambda', \lambda) \rightarrow i k \lambda G^+(\lambda', \lambda) . \] (3.21) \]

Then the second term becomes

\[ i k \int_{\Omega'} \lambda^3 \left[ \lambda \partial' \phi_0(\lambda') - 1 \right] \phi_0(\lambda') G^+(\lambda', \lambda) [\lambda - 1] . \] (3.22) \]

Now \( (\lambda \partial' \phi_0 - 1) \) is zero when \( \lambda = \lambda_0 \), while the factor \( \lambda \partial' \phi_0(\lambda') - 1 \) is non-zero only on the intersection of \( \Omega' \) with the forward cylinder, that is, when \( \lambda' = \lambda_0 \). Thus the product of the two factors makes the surface term zero. The first term of Eq. (3.20) is just the surface integral obtained by applying Green's theorem to \( G^+ \) and \( \phi_0 \). Therefore, \( \mathcal{L}(\lambda) \) is equal to the volume integral

\[ \mathcal{L}(\lambda) = \int d\lambda' \left[ G^+(\lambda', \lambda) \partial^2 \phi_0 - \phi_0 \partial\partial' G^+(\lambda', \lambda) \right] \\
= \phi_0(\lambda) - \int d\lambda' G^+(\lambda', \lambda) \cdot \phi(\lambda') \phi_0(\lambda') . \] (3.23) (3.24)
But Eq. (3.24) is just the Chew-Goldberger (12) form of \( \psi_+ \).

It satisfies the correct boundary conditions, for its asymptotic form is that of a plane wave plus outgoing spherical waves. One approximates Eq. (3.24) by approximating \( G^+ \); in this sense, it is not an integral equation for \( \psi_+^\dagger(\vec{x}) \).

Since \( \mathcal{A}(\vec{x}) = \psi_+^\dagger(\vec{x}) \), Eq. (3.15) becomes

\[
\psi_+^\dagger(\vec{x}) = \omega_+^\dagger(\vec{x}) + \int d^3\vec{x}' \ G^+(\vec{x}, \vec{x}') \ \psi_0(\vec{x}') \ \varphi_{+} \ \varphi_{+}^\dagger(\vec{x}')
\]

(3.25)

Equation (3.25) is the exact wavefunction, with \( \omega_+^\dagger \) as leading term. Equation (3.26) is similar in form to Eq. (3.24), with the term in brackets an effective potential.

2. **The exact scattering amplitude**

The presence of the exact Green's function in Eq. (3.25), and the fact that \( \mathcal{A} \) occurs nowhere else in the integrand make it easy to calculate the exact scattering amplitude. The scattering amplitude has been shown to be

\[
f(\vec{k}_f, \vec{k}_o) = \frac{-i}{4\pi} \sum d^3\vec{x} \ \varphi_{+}^\dagger(\vec{x}) \varphi_{+}(\vec{x}) \ \psi_0^+(\vec{x}) \equiv \frac{-i}{4\pi} \ (\varphi_{+} \ \varphi_{+}^\dagger).
\]

(3.10)

Substituting Eq. (3.25) into Eq. (3.10), we obtain

\[
f(\vec{k}_f, \vec{k}_o) = \frac{-i}{4\pi} \ (\varphi_{+} \ \varphi_{+}^\dagger)
\]

(3.27)

By reversing the order of integration in the second term of Eq. (3.27), \( f(\vec{k}_f, \vec{k}_o) \) can be rewritten as
But the solution of the Schrödinger equation having the asymptotic form of the plane wave $\varphi_f$ plus incoming spherical waves is

$$\psi_f^-(x) = \varphi_f(x) - \int d\tilde{x}' G^-(x, \tilde{x}') \varphi_\infty(x) \varphi_f^*(x) .$$

(3.29)

Since

$$[G^-(x, \tilde{x}')]^* = G^+(x, \tilde{x}) ,$$

(3.30)

the star denoting complex conjugation, we have

$$\int d\tilde{x}' \varphi_f^*(x') \varphi_\infty(x') G^+(x', x) = -\psi_\infty^*(x) ,$$

(3.31)

where

$$\psi_\infty(x) = \psi_f^-(x) - \varphi_f(x) .$$

(3.32)

Then, Eq. (3.29) becomes

$$f(R_t, R_s) = -\frac{1}{4\pi} (\varphi_f \cup \varphi_\infty^*) + \frac{1}{4\pi} (\psi_\infty, (\varphi_\infty^* e^{i\phi}) \varphi_\infty) .$$

(3.33)

Equation (3.33) is the desired expression for the scattering amplitude, in which the first term represents the predominant small-angle scattering. The derivation of Eqs. (3.25) and (3.33) is quite general, and, indeed, may be easier in the case of less peculiar approximate wave functions, for it may be evident by inspection that $\varphi_\infty(x)$ is $\varphi_\infty^+(x)$.
procedure is also more flexible than the one used by Saxon and Schiff (9) for, generally, an approximate wave function is more easily found than an approximate Green's function. This follows because the Green's function satisfies an inhomogenous differential equation.

If we define

$$v(x) = v_l(x) + v_r(x),$$

Eq. (3.33) can be put in the following revealing form:

$$f(k_x, k_o) = -$$

This is just the result that would be obtained from two-potential theory (13), where the part of the potential that is to be treated exactly is taken to be $U_f$. The splitting of the potential into two parts is effected by choosing the approximate wave function $w_0^+$, and determining the Schrödinger equation that it satisfies:

$$\left[ -\frac{d^2}{dx^2} + k^2 - U_f(x) \right] w_0^+(x) = 0 \quad (3.37)$$

Because $w_0^+$ describes a particle whose direction of motion is unchanged as it passes through the potential, $U_f$ may be regarded as that part of the potential which generates only forward scattering. Then, $U_r$ produces scattering through non-zero angles. The idea of splitting the potential into a
$U_i$ and a $U_\perp$ has been discussed by Lippmann (6). However, his choice of $U_i$ and $U_\perp$ appears to be different than the one given by Eqs. (3.34) and (3.35). It should be noted that $U_i$ and $U_\perp$ are not real, and that they are non-zero throughout both the potential and the forward cylinder.

Equation (3.33) may be rewritten in a variety of other ways, the two most important being

$$4 \pi f(\vec{k}, \omega) + (\phi_f, \omega \omega') = \int \frac{d^3l}{l} \int d^3z \, \zeta(z') \psi_z(\vec{z}') \frac{i}{\omega} \int d^3z'' \, \phi_0(z'') \psi^*_0(z')$$

$$= \int \frac{d\vec{l}}{l} \int d^3z \, \zeta(\vec{l}) \left( \psi^*_z(\vec{l}) \cdot \nabla \right) \phi_0(z') \psi^*_0(\vec{l}) \left. \right|_{z''}^{\infty} \tag{3.38}$$

Equation (3.38) is the form presented by Saxon and Schiff, and has the advantage that the volume of integration in both terms is limited by the boundaries of the potential. In order to obtain Eq. (3.38) from Eq. (3.33), we choose the $z$-axis parallel to $\vec{z}_0$, and integrate by parts with respect to $z'$. The result is

$$-\int \frac{d\vec{l}}{l} \int d^3z \, \zeta(\vec{l}) \left( \nabla \cdot \frac{i}{\omega} \int d^3z \, U(x, y, z') \psi^*_z(z') \right) \left. \right|_{x''}^{x''} \tag{3.39}$$

Since we can write $\psi_z^*$ as

$$\psi_z^*(\vec{z}) = \phi_z(\vec{z}) - \int G_0^-(\vec{x}, \vec{z}') \psi_+(\vec{z}') d\vec{z}' \tag{3.40}$$

the asymptotic form of $\psi_{ac}$ is clearly an incoming spherical wave. Therefore, $\phi_0(z') \psi^*_0(z')$ varies as $e^{i \frac{k}{\omega} z'}$ at large positive $z'$, and the integral over $z'$ exists and
approaches zero as \( \tau \) approaches +\( \infty \). Thus the first term of Eq. (3.40) clearly vanishes at both limits. Green's theorem applied to the second term now yields Eq. (3.38), the surface integral in Green's theorem having a vanishing integrand because \( U \) vanishes.

One obtains Eq. (3.39) by rewriting the second term on the right-hand side of Eq. (3.33) as

\[
\frac{1}{4\pi}\int d\vec{r} \cdot \vec{v} \cdot (\psi^*_\infty \phi_0 \vec{v} e^{i\phi_0}) - \frac{1}{4\pi}\int d\vec{r} \cdot \vec{v} (\psi^*_s \phi_0) \cdot \vec{v} e^{i\phi_0}.
\]

The first term may be transformed to a surface integral which vanishes by Eq. (3.17).

3. First iteration

To use Eq. (3.33), we must approximate \( \psi_{s\infty} \). The form of Eq. (3.36) suggests setting \( \psi^-_r = \omega^-_f \) as a first approximation, where \( \omega^-_f \) corresponds to \( \psi^-_s \) in the same manner as \( \omega^+_o \) corresponds to \( \psi^+_s \). This will be a valid procedure provided the additional potential \( U_L \) does not seriously distort the modified plane wave; that is, \( U_L \) should be small compared with \( U_j \), and, hence, small compared with \( U \). Again assuming that the distance over which \( U \) changes by an appreciable fraction of itself is of the same order as the range \( R \), we find from Eq. (3.34) that \( U_L \) is of order \( U/k_R \) or \( U^2/k^-_R \), whichever is larger. It is evident that \( U_L \) is small compared to \( U \).

It is desirable to pursue the matter further, for the
semi-quantitative arguments above give no information as to restrictions on the magnitude of $\psi^R_k$. We will have to write $\psi_f^-$ in the form of Eq. (3.25), which requires an expression for $\omega_f^-$. A general argument for the transformation of $\psi^+_o(x)$ to $\psi^-_f(x)$ can be made by brief recourse to time-dependent theory. The result of forward time development of the initial plane wave from time minus infinity to time zero is the total wave function $\psi^+_o(x)$. Because the time-development operator is $\mathcal{L}^{-iH(t_f-t_i)}$, where $H$ is the Hamiltonian operator corresponding to Eq. (3.1), we have

$$\psi^+_o(x) = \lim_{t \to \infty} \mathcal{L}^{-iH t} \phi^+_o(-t)$$

(3.43)

$\psi^-_f$ is generated by developing the final plane wave at time plus infinity backward in time to time zero; that is

$$\psi^-_f(x) = \lim_{t \to \infty} \mathcal{L}^{iH t} \phi^-_f(t)$$

(3.44)

$$= \left( \lim_{t \to \infty} \mathcal{L}^{-iH t} \phi^-_f(t) \right)^*$$

(3.45)

where the star stands for complex conjugation. By use of Eq. (3.43), Eq. (3.45) becomes

$$\psi^-_f(x) = \left[ \psi^+_f(x) \right]^*$$

(3.46)

or

$$\psi^-_f(x) = \left[ \psi^+_f(x) \right]^*$$

(3.47)

Thus, to find $\omega^-_f$, we replace $k_1$ by $-k_f$ in the expression for $\omega^+_o$, and take the complex conjugate. The result is
\[ \omega_f^-(\hat{x}) = e^{i\mathbf{K}_f \cdot \hat{x}} e^{-i\delta_f(\hat{x})} \]  
\[ \delta_f(\hat{x}) = -\frac{i}{\lambda k} \int_0^{\infty} u(\hat{x} + \mathbf{K}_f \cdot \hat{\omega}) d\hat{\omega} \]

The quantity \( \delta_f \) is non-zero only if \( \hat{x} \) lies either in the potential or in a semi-infinite cylinder such that a straight line proceeding from \( \hat{x} \) in the direction \( \mathbf{K}_f \) pierces the potential. This second region will be called the backward cylinder.

The expression for \( \psi_f^- \) can now be written as

\[ \psi_f^-(\hat{x}) = \omega_f^-(\hat{x}) + \int d\hat{x}' G^-(\hat{x}, \hat{x}') \phi_f^-(\hat{x}') \nabla \cdot e^{-i\delta_f(\hat{x}')} \]

so that the complex conjugate of the scattered wave, \( \psi^*_o \), becomes

\[ \psi^*_o(\hat{x}) = \phi^*_o(e^{i\delta_f - 1}) + \int d\hat{x}' G^+(\hat{x}, \hat{x}') \psi^*_o(\hat{x}') \nabla \cdot e^{-i\delta_f(\hat{x}')} \]

To estimate the error of using just the first term of Eq. (3.50a) in Eq. (3.33) for the scattering amplitude, the magnitude of the two terms will be compared, with \( G^+ \) replaced by its first approximation, \( G^+_a \). It will be shown in Section V. C. that, for \( V \ll \varepsilon \),

\[ G^+_a(\hat{x}, \hat{x}') \simeq G^+_o(p) e^{-i\lambda k \int_0^r u(\hat{x} - \mathbf{K}_f \cdot \hat{\omega}) d\hat{\omega}} \]

where
The condition \( kR \gg 1 \), and the factor \( G_0^+ \) enable us to use the stationary phase approximation (4). The stationary phase approximation applied to the integral

\[
I = \int d\zeta' G_0^+(\zeta, \zeta') e^{-i \hat{K}_f \cdot \zeta'} \hat{h}(\zeta')
\]

may be shown to lead to

\[
I \approx -\frac{1}{2ik} \int_0^\infty d\tau \hat{h}(\zeta + \hat{K}_f \tau)
\]

Here \( \hat{h}(\zeta') \) is a function which varies with \( \zeta' \) much more slowly than the plane wave. By use of Eqs. (3.51) and (3.53), the second term of Eq. (3.50a) becomes

\[
\left[ \frac{\partial}{\partial \zeta} e^{-i \hat{K}_F \cdot \zeta} \right]_{\zeta=\zeta'}
\]

In the expression for the scattering amplitude, Eq. (3.33), the effective values of \( \zeta \) are limited to the potential and the forward cylinder. Therefore, except for 180° scattering when the forward and backward cylinders overlap, \( \int_0^\infty d\tau e^{i \hat{K}_F \cdot \zeta'} \) is non-zero only for a range of \( \tau \) of order \( R \). The factor in brackets in Eq. (3.54) thereby limits the \( \tau \) integration to a range of order \( R \). This term in brackets is of order \( \{ v_{KR}, v_{FR} \} \), where the first entry is the order of magnitude when \( v_{KR} \approx 1 \), and the second entry is the order of magnitude for \( v_{FR} > 1 \).
Therefore, Eq. (3.54) is of order \( \{ \frac{V}{K}, \frac{V}{R} (\frac{UR}{R}) \} \). The first term of Eq. (3.50a) is of order \( \{ \frac{UR}{R}, 1 \} \). Therefore, by keeping only the first term in Eq. (3.50a), we introduce an error of

\[ \begin{align*}
\frac{V}{K} & \quad \text{if } \frac{UR}{R} \leq 1, \\
\frac{UR}{R} & \quad \text{if } \frac{UR}{R} > 1.
\end{align*} \]

(3.55)

Thus, the restriction that \( \frac{UR}{R} (\frac{UR}{R}) \leq 1 \) has been obtained. This indicates that a term of order \( \frac{UR}{R} (\frac{UR}{R}) \) has been dropped in the phase of the approximate wave function \( \omega \) (Section II. B.). For \( \phi = 180^\circ \), the above derivation breaks down due to the peculiar nature of \( \omega^\pm \) far from the potential. Difficulties at \( 180^\circ \) are characteristic of the high-energy approximation, and will be encountered again. However, as the difficulties are confined to a single angle, one may still determine cross sections at \( 180^\circ \) by a continuity argument.

C. The Analogous One-Dimensional Case

The simplicity of the one-dimensional scattering problem makes its inclusion worthwhile not only as a comparison with the three-dimensional case, but also to suggest simpler approximate forms of \( f(\vec{k}_f, \vec{k}_r) \). In a manner analogous with the three-dimensional theory, general relations for the transmission coefficient \( |t|^2 \) and the reflection coefficient \( |r|^2 \) are found to be (14).
The plane wave \( \phi_f \) is not the same in the two cases, for the final direction of propagation is in the positive \( x \)-direction for transmission and the negative \( x \)-direction for reflection. Thus, we have \( \phi_f = e^{ikx} \) in Eq. (3.56) and \( \phi_f = e^{-ikx} \) in Eq. (3.57). The modified plane wave, \( \omega_0^+ \), in one dimension is

\[
\omega_0^+(x) = e^{ikx} + i \delta_0(x)
\]

where

\[
\delta_0(x) = -\frac{1}{2ik} \int_{-\infty}^{\infty} \psi_0(y) dy
\]

\[
\frac{\partial}{\partial x} e^{ikx} = \frac{1}{2ik} \psi(x) e^{ikx}
\]

To obtain relations comparable with Eq. (3.33), we substitute the exact high-energy form for \( \psi_0^+(x) \),

\[
\psi_0^+(x) = \omega_0^+(x) + \int dk' G^+(x, k') e^{ikx} \left( \frac{\partial}{\partial k'} + i \delta_0(x') \right)
\]

into Eqs. (3.56) and (3.57) to obtain

\[
\tau = e^{-ikx} \int_{-\infty}^{\infty} \psi_0(y) dy - \frac{1}{2ik} \left( \psi_{ae}, \left( \frac{\partial}{\partial x} e^{ikx} \right) e^{ikx} \right)
\]

\[
\lambda = \frac{1}{2ik} \left( e^{-ikx}, \psi_0^+ \right) - \frac{1}{2ik} \left( \psi_{ae}, \left( \frac{\partial}{\partial x} e^{ikx} \right) e^{ikx} \right)
\]

The scattered wave, \( \psi_{ae} \), stands for \( \psi_f^+ - \phi_f \). Equations (3.62) and (3.63) may be simplified by rewriting them in terms of \( \psi_f^- \), then integrating once by parts and using Eq. (3.60).
The results are

\[ t = e^{-i\frac{\omega}{k} \int_{\infty}^{\infty} V(y) \, dy} - \frac{1}{4k^2} \int_{\infty}^{\infty} V(x) e^{i \delta(x) \frac{\partial}{\partial x} \left[ (\psi^-)^* e^{i k x} \right]} \]  \hspace{1cm} (3.64)

\[ n = -\frac{1}{4k^2} \int_{\infty}^{\infty} V(x) e^{i \delta(x) \frac{\partial}{\partial x} \left[ (\psi^-)^* e^{i k x} \right]} \]  \hspace{1cm} (3.65)

Again, it should be remembered that \( \psi_+^- \) is not the same for both cases, since the final directions of propagation are different.

To obtain the first approximations, one must replace \( \psi_+^- (x) \) by the appropriate \( \omega_+^- (x) \). The three-dimensional form of \( \omega_+^- \), Eq. (3.48), reduces in one dimension to

\[ \omega_+^- (x) = e^{i k x} + i \frac{1}{k} \int_{\infty}^{\infty} V(y) \, dy \]  \hspace{1cm} (3.66)

for transmission, and to

\[ \omega_+^- (x) = e^{-i k x} + i \frac{1}{k} \int_{\infty}^{x} V(y) \, dy \]  \hspace{1cm} (3.67)

for reflection. When Eq. (3.66) is substituted for \( \psi_+^- \) in Eq. (3.64), the first approximation, \( t, \) is given by

\[ t = e^{-i k x} \left\{ e^{i k x} \int_{\infty}^{\infty} V(y) \, dy \left[ 1 - \left(\frac{1}{2ik}\right)^2 \int_{\infty}^{\infty} \frac{\partial^2}{\partial x^2} V^2 (x) \right] \right\} \]  \hspace{1cm} (3.68)

If the second-order term in \( V^2 \) is dropped, \( t \) may be rewritten as

\[ t \approx 1 + \frac{1}{2ik} (\psi_+ ^0 , V \omega_+ ^0 ) \]  \hspace{1cm} (3.69)

a relation which is easily checked by an integration by parts. This result could have been obtained more easily by simply substituting \( \omega_0 ^+ \) for \( \psi_+ ^0 \) in Eq. (3.56). However, the re-
placement of $\psi^+$ by $\omega^+$ in the reflection amplitude, Eq. (3.57), does not give the correct answer, for it fails to take into account a first-order contribution from the second term of Eq. (3.61). Instead we obtain, on substitution of Eq. (3.67) into Eq. (3.65),

$$\rho_1 = \frac{1}{2i \hbar} \int_{-\infty}^{\infty} x \left\{ \psi(x) \frac{d}{dx} \psi(x) \right\}^2 \phi(x) \left( i + \frac{1}{2 \hbar} \right) \psi(x) \right\} \right] \left( 3.70 \right)$$

If the $\psi^+$ term is dropped, $\rho_1$ may be rewritten as

$$\rho_1 \simeq \frac{1}{2i \hbar} \left( \omega^- \cdot U \cdot \omega^+ \right) \left( 3.71 \right)$$

This form of $\rho_1$ may have been expected from the descriptive form of $f(k_f, \tilde{k}_c)$, Eq. (3.36). One dimensional reflection corresponds to 180° scattering, hence one might argue that only $U_L$ will contribute. With $U_f$ set equal to zero, $\psi$ is equal to $U_L$, and the first approximation to the matrix element is just the matrix element of Eq. (3.71).

There is, however, some doubt about the validity of Eq. (3.71). To obtain it, we dropped a term in $U^\pm(x)$ in Eq. (3.70). In general, $U^\pm(x)$ varies more rapidly with $x$ than does $U(x)$, hence one might expect that the rapidly oscillating factor $e^{\pm ikx}$ would annihilate the term in $U$ more completely than the term in $U^\pm$. Again, if the replacement of $U$ by $U_L$ is legitimate for backward scattering, we should expect the Fourier transform of $U_L$ to be large compared with the Fourier transform of $U_f$, for large $k$. The term "Fourier transform" will be used even though $U_f$ and $U_L$ depend on $k$. 
Since, in one dimension,
\[ U_L(x) = - \frac{1}{2ik} \frac{dU}{dx} + \frac{U^2}{4k^2}, \quad (3.72) \]
we have
\[ u_L(2k) = \int_{-\infty}^{\infty} U(x) e^{2ikx} \, dx \]
\[ = \int_{-\infty}^{\infty} U(x) e^{2ikx} + \frac{1}{4k^2} \int_{-\infty}^{\infty} e^{2ikx} U^2(x). \quad (3.73) \]

The Fourier transform of \( u_f = U - U_L \) is then
\[ u_f(2k) = - \frac{1}{2ik} \int_{-\infty}^{\infty} e^{2ikx} U^2(x). \quad (3.74) \]

If we take as an example a simple Gaussian, \( U = U_0 e^{-x^2/R^2} \),
Eqs. (3.74) and (3.75) become
\[ u_L(2k) = U_0 \sqrt{\frac{\pi}{k^2 R}} e^{- (kR)^2} + \frac{U_0^2}{4k^2} \sqrt{\frac{\pi}{R L}} e^{- (kR)^2}, \quad (3.76) \]
\[ u_f(2k) = - \frac{U_0^2}{4k^2} \sqrt{\frac{\pi}{R L}} e^{- (kR)^2}. \quad (3.77) \]

For \( kR >> 1 \), the terms in \( e^{- (kR)^2} \) will dominate unless \( U_0/k^2 \) is extremely small. Thus, for the Gaussian, \( u_f \) and \( u_L \) have Fourier components of equal strength in the case of backscattering. It should be emphasized, however, that this Fourier analysis of \( u_f \) and \( u_L \) is not a conclusive argument against the validity of Eq. (3.71). The integrals which should be compared are the two terms of Eq. (3.70). The evaluation of these integrals is complicated by the presence of the potential-dependent phase factor.

A simple example in which the \( U^2 \) term can be dropped is
the square well; \( U = U_0 \) for \( |x| < R \), \( U = 0 \) for \( |x| > R \). The high-energy reflection coefficient \( |\mathcal{R}_1|^{-1} \) may be compared with the Born approximation \( |\mathcal{R}_B|^{-1} \), and the exact reflection coefficient \( |\mathcal{R}|^{-1} \). From Eq. (3.71) we have

\[
\mathcal{R}_1 = \frac{1}{2iK} \int_R^\infty \alpha \; e^{\pm ikx} U(x) x^{-1/2} \int_R^x U(x') dx'
\]

which leads to

\[
|\mathcal{R}_1|^2 = \left( \frac{U_0 R}{K} \right)^2 j_0^2 \left[ 2kR \left( 1 - \frac{U_0}{2K^2} \right) \right]
\]

where \( j_0 \) is the spherical Bessel function of order zero. The Born approximation is obtained from Eq. (3.71) by setting the potential-dependent phase shifts equal to zero. The result is

\[
|\mathcal{R}_B|^2 = \left( \frac{U_0 R}{K} \right)^2 j_0^2 \left( 2kR \right)
\]

The exact result is found in Schiff's text on quantum mechanics (15) to be

\[
|\mathcal{R}|^2 = \left\{ 1 + \frac{1}{\left[ \frac{U_0}{K^2} j_0(2kR\sqrt{1 - \frac{U_0}{2K^2}}) \right]^2} \right\}^{-1}
\]

The three quantities, \( |\mathcal{R}| \), \( |\mathcal{R}_B| \), and \( |\mathcal{R}_1| \), are plotted against \( kR \) in Figure 3, for the case \( U_0 K^2 = 4 \). It is clear that \( |\mathcal{R}_1| \) is a considerable improvement over the Born approximation. Comparison of their analytical forms demonstrates that this improvement is due to the retention of a term in \( \frac{U_0}{K^2} \).
Figure 3. Reflection amplitudes for a square well.
D. Approximations To $f'$, and Validity Estimates

By analogy with the expressions for $t_1$ and $r_1$ in Section III. C., one might anticipate that, for a suitable range of small $\theta$, 

$$f_+^* = -\frac{1}{4\pi} \left( \phi_f^* + \omega_o^+ \right)$$

and for a suitable range of large $\theta$, 

$$f_-^* = -\frac{1}{4\pi} \left( \omega_f^- + \omega_o^+ \right)$$

Schiff (5) obtained both of these results by summing the infinite Born series, after first approximating each term by the stationary-phase approximation. From this procedure, he was able to argue that $f_+^*$ was valid for scattering angles $\theta < \frac{1}{\sqrt{\nu R}}$, and $f_-^*$ was valid for $\theta > \frac{1}{\sqrt{\nu R}}$. His arguments gave no restrictions on the magnitude of $\nu R$.

If Eqs. (3.82) and (3.83) are valid, one would expect to be able to derive them from the exact expression for $f(\vec{r}_f, \vec{r}_o)$, Eq. (3.33). Replacing $\psi_f^-$ by $\omega_f^-$, we have

$$f'(\vec{r}_f, \vec{r}_o) = -\frac{1}{4\pi} \left( \phi_f^* + \omega_o^+ \right)$$

$$+ \frac{1}{4\pi} \int d\vec{r}_0 \, e^{i \vec{k}_0 \cdot \vec{r}} \left[ e^{i \delta_f(\vec{r})} - 1 \right] \phi_f^* \cdot \delta_o(\vec{r}) \]$$

If the second term in Eq. (3.84) can be dropped for small $\theta$, we obtain Eq. (3.82). Saxon and Schiff (9), hereafter referred to as SS, used order of magnitude estimates to drop their rewritten version of this term [see Eq. (3.38)]. Because
\( a^{i \cdot f - 1} \) is of order \( \{ u_{R}^{\mu}, 0 \} \), and \( e^{i \cdot f \cdot R} \) is of order \( \{ u_{R}^{\mu}, 0 \} \), the order of magnitude of the second term appears to be \( u_{R}^{\mu} R \) for arbitrary \( u_{R}^{\mu} \). The first term, \( f \), has an order of magnitude of at most \( u_{R}^{3} \). The ratio of these two estimates is \( u_{R}^{\mu} \), independent of \( \theta \). However, the second term of Eq. (3.84) can be rewritten in at least two other distinct ways, each leading to different estimates of magnitude. The difficulty is that, in a sense, each integral is a Fourier transform in \( \hat{q} \). When \( a^{i \cdot f \cdot \hat{q}} \) oscillates rapidly, the values of the integrals depend markedly on the exact form of their integrands. Hence, order of magnitude estimates are not reliable for large \( |f| \). However, when \( |f| \) is of order \( u_{R} \), \( a^{i \cdot f \cdot \hat{q}} \) varies slowly in a range \( R \), and order of magnitude estimates can be expected to be accurate. Since \( |f| = k \theta \) for small \( \theta \), we can certainly use Eq. (3.82) when

\[
\theta \approx \frac{1}{u_{R} R}
\]  

(3.85)

For \( \theta > \frac{1}{u_{R} R} \), the first term of Eq. (3.84) will decrease rapidly in magnitude, for the slowly varying components of \( U \) are annihilated by the \( a^{i \cdot f \cdot \hat{q}} \) factor. For some \( \omega \), the effect of \( a^{i \cdot f \cdot \hat{q}} \) will be to reduce \( U \) to the order of magnitude of \( U_{L} \). Only the high frequency components of \( U \) will survive. Since \( |U_{L}| \ll |U| \) (Section III. B.), the two terms of Eq. (3.84) will be roughly the same size. The Schiff (5) estimate of \( \omega \), based on his stationary-phase calcula-
A new difficulty arises on careful examination of the SS (9) derivation of \( f_L' \). Using Eq. (3.38), one may write \( f' \) as

\[
f' ( \vec{k}_f, \vec{k}_e ) = -\frac{i}{4\pi} \left( \varphi^+ ( \mathbf{r}_e, \Omega^+ ) \right)
- \frac{i}{8\pi^2} \int d^2 z' \varphi ( \mathbf{z}_e ) \epsilon_0 \epsilon_i \int_{\mathbf{z}_e}^{\infty} 2 \int_{\mathbf{z}_e}^{\infty} ( \mathbf{z}_e ' - 1 ) \;
\]

where \( \mathbf{z}_e ' = (x, y, z) \). For angles less than 180°, the integral over \( z' \) is well-defined, for the factors \( \varphi ( \mathbf{z}_e ) \) and

\[
[ \epsilon_0 ( \mathbf{z}_e ') - 1 ]
\]

limit the \( z' \) integration to the intersection of the forward and backward cylinders, which excludes infinite values of \( z' \). At \( \theta = 180° \), the two cylinders completely overlap, and \( z' \) may take on all values greater than \( z \).

However, although the integral does not have a definite value, it remains bounded in magnitude. For we can write it as

\[
\int_{z_e}^{\infty} d^2 z' \epsilon_0 \int_{z_e}^{\infty} 2 \int_{z_e}^{\infty} ( \mathbf{z}_e ' - 1 ) \;
\]

where \( z'' \) lies on the "boundary" of the potential. For \( z' > z'' \), the phase \( \delta_0 = -\frac{1}{k} \int_{z_e}^{\infty} \varphi ( \mathbf{z}_e - z') \) does not vary with \( z' \).

The limit as \( a \to \infty \) does not exist, but it is bounded, and, indeed, is of order \( 1/k \).

To obtain \( f' \), SS expand the \( z' \) integral in Eq. (3.86) as an asymptotic series in \( (q_{z_e})^{-1} = (2k \varphi ( \mathbf{z}_e ) \epsilon_i)^{-1} \), an expansion which clearly requires that \( \phi \) should not be too small.

The first two terms in the expansion are (for \( \phi \neq 180° \))
The second term of the expansion is small compared with the first provided

\[
\left\{ \frac{1}{kR \sin^2 \frac{z'}{2}}, \frac{v}{k^2 \sin^2 \frac{z'}{2}} \right\} \ll 1 \quad (3.89)
\]

If the \( z' \) integral is replaced by just the first term in the expansion, aside from various terms of the order of Eq. (3.89), \( f' \) reduces to \( f \). The conditions of validity are assumed to be those of Eq. (3.89). This procedure assumes that the final integration over \( \tilde{z} \) in Eq. (3.86) does not change the relative magnitudes of the first two terms in the asymptotic expansion. To show the danger of this assumption, we consider the following example:

\[
I = \int_{-\infty}^{\infty} \psi(x) \int_{-\infty}^{\infty} e^{i k x} \psi(x) \quad (3.90)
\]

where \( \psi \) is an arbitrary function that vanishes at \( \pm \infty \). Expanding the \( x \)-integral as an asymptotic series in \( 1/k \), we have

\[
\int_{-\infty}^{\infty} e^{i k x} \psi(x) \sim -\frac{i}{kR} e^{i k y} \left[ \psi - \left( \frac{1}{2i k} \right) \frac{\partial \psi}{\partial y} + \ldots \right] \quad (3.91)
\]

The ratio of this second term in brackets to the first is of order \( 1/kR \), which is small. However, on substituting both terms into Eq. (3.90), and performing an integration by parts, we obtain
where the $\frac{1}{2}$ is the contribution of the second term in Eq. (3.91). Thus the contribution of this second term is not small compared with that of the first. In the absence of a proof that no such effect takes place when Eq. (3.88) is substituted in Eq. (3.86), the SS derivation of validity conditions for the large-angle approximation does not appear to be justified.

An equation for $f'$ which resembles the form of Eq. (3.70) for $\Lambda$, may be obtained from Eq. (3.39). Replacing $\varphi_f$ by $\omega_f$ and carrying out the indicated differentiations, we have

$$t'(k_f, k_o) = -\frac{1}{4\pi} (\varphi_f, \varphi_o)$$

$$-\frac{1}{2\pi} \int \text{d}x \int \text{d}y (e^{i\theta \cdot \mathbf{x}} - e^{i\theta \cdot \mathbf{y}}) \frac{1}{(k_o - k_f)^2} \int \text{d}z \int \text{d}z' \text{e}^{i\sigma \cdot \mathbf{z} + i\sigma \cdot \mathbf{z}' + i\sigma \cdot \mathbf{z} - i\sigma \cdot \mathbf{z}'}$$

If desired, this equation can be rewritten in a form in which the ranges of integration are limited to the potential region:

$$t' = -\frac{1}{4\pi} (\varphi_f, \varphi_o) - \frac{1}{2\pi} \int \text{d}x \int \text{d}y \varphi(x, y) \frac{1}{(k_o - k_f)^2} \int \text{d}z \int \text{d}z' \text{e}^{i\sigma \cdot \mathbf{z} + i\sigma \cdot \mathbf{z}' + i\sigma \cdot \mathbf{z} - i\sigma \cdot \mathbf{z}'}$$

For $\theta = 180^\circ$, one has $k_f = -k_o$, $2k_o \cdot \varphi_o \int \text{d}y (x \cdot k_o) = 2 \text{U}(x)$, and Eq. (3.94) becomes

$$t'(k_f, k_o) = -\frac{1}{4\pi} (\omega_f, \omega_o)$$

$$-\frac{1}{4\pi} (\frac{1}{2\pi}) \int \text{d}x \int \text{d}y (\omega_f)^* \omega_o + \int \text{d}z \int \text{d}z' \text{e}^{i\sigma \cdot \mathbf{z} + i\sigma \cdot \mathbf{z}' + i\sigma \cdot \mathbf{z} - i\sigma \cdot \mathbf{z}'}$$

(3.95)
This is the three-dimensional form of Eq. (3.70). The second integral does not have a definite value, but it is bounded in magnitude. Again it is not clear that it is small in magnitude compared to the first term, for, at 180°, only the weak, high-frequency components of $V$ are contributing to the first term.

E. Scattering By a Magnetic Field

We shall consider a high-energy, spinless particle in a magnetic field derivable from a vector potential $\hat{A}$. The vector potential is assumed to be confined to a bounded region of space. The exact solution, $\psi_0^+$, satisfies the equation

$$\left(\nabla^2 + k^2 - 2i\epsilon \hat{A} \cdot \hat{v} - \epsilon^2 \hat{A}^2\right)\psi_0^+(\lambda) = 0$$

(2.36)

with $\epsilon = \frac{\omega}{\epsilon}$, $\hat{v} \cdot \hat{A} = 0$. To obtain a general expression for the scattering amplitude, $f_\lambda(\Delta, \varphi)$, we again express $\psi_0^+(\lambda)$ as the solution of an integral equation,

$$\psi_0^+(\lambda) = \phi_0^-(\lambda) - \int d\lambda' G_0^+(\lambda', \lambda') (2i\epsilon \hat{A} \cdot \hat{v}' + \epsilon^2 \hat{A}^2) \psi_0^+(\lambda')$$

(3.96)

where $G_0^+$ is given by Eq. (3.7). The asymptotic form of Eq. (3.96) is

$$\psi_0^+(\lambda) \xrightarrow{|\lambda| \to \infty} f_\lambda(\Delta) + \frac{1}{4\pi} \frac{e^{ik\lambda}}{\lambda} \int d\lambda' \phi_\lambda^*(\lambda') \left(2i\epsilon \hat{A} \cdot \hat{v}' + \epsilon^2 \hat{A}^2\right) \psi_0^+(\lambda')$$

(3.97)

and the coefficient of $e^{ik\lambda}$ is
Equation (3.98) can be put in a more useful form by noting that

\[ \vec{A} \cdot \vec{\nabla} \psi_0^+ (\vec{r}) = \vec{\nabla} \cdot (\vec{A} \psi_0^+) \]  

(3.99)

and

\[ e^{-i \vec{k}_f \cdot \vec{r}} \psi_0^+ (\vec{r}) = \vec{\nabla} \cdot (e^{-i \vec{k}_f \cdot \vec{r}} \psi_0^+ \vec{A}) \]

(3.100)

When the first term of Eq. (3.100) is substituted into Eq. (3.98), it can be converted into a surface integral which vanishes because \( \vec{A} \) vanishes. Thus, Eq. (3.98) becomes

\[ f_A (\vec{k}_f, \vec{k}_o) = -\frac{i}{4\pi} \int d^2 \vec{r} \ \phi_f^*(\vec{r}) \ [ -2i e \vec{k}_f \cdot \vec{A} (\vec{r}) + e^{i \vec{k}_o \cdot \vec{A}} (\vec{r}) ] \psi_0^*(\vec{r}) \]

(3.101)

The high-energy approximation to the wave function is (Section II. B.)

\[ \psi_0^+ (\vec{r}) = e^{i \vec{k}_o \cdot \vec{r}} + i \eta_0 (\vec{r}) \]

(2.41)

where

\[ \eta_0 (\vec{r}) = e \int_0^\infty \vec{k}_o \cdot \vec{A} (\vec{r} - \hat{k}_o \omega) d\omega \]

(3.102)

As in Section III. B., we wish to write an exact expression for \( \psi_0^+ \) in terms of \( \chi_0^+ (\vec{r}) \) and \( G_A^+ (\vec{r}, \vec{r'}) \), the exact Green's function. The differential equations satisfied by these latter two functions are
\[ (\nabla^2 + k^2 - 2i \varepsilon \mathbf{A} \cdot \mathbf{v} - \varepsilon^2 \mathbf{A}^2) \mathbf{G}_A^+(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \]  
(3.103)

and

\[ (\nabla^2 + k^2 - 2i \varepsilon \mathbf{A} \cdot \mathbf{v} - \varepsilon^2 \mathbf{A}^2) \chi_0^+(\mathbf{r}') = \mathbf{P}_0 \cdot \mathbf{v} \mathbf{e}^{i \eta_0} - 2i \varepsilon \mathbf{A} \cdot \mathbf{v} \mathbf{e}^{i \eta_0} - \varepsilon^2 \mathbf{A}^2 \chi_0^+ \]  
(3.104)

\[ \chi_0^+ \]  
(3.105)

where the identity \[ \mathbf{\hat{r}}_0 \cdot \mathbf{v} \int \mathbf{\hat{r}}_0 \cdot \mathbf{A}(\mathbf{r} - \mathbf{\hat{r}}_0 \cdot \mathbf{A}) \, d\mathbf{\hat{r}}_0 = \mathbf{\hat{r}}_0 \cdot \mathbf{A}(\mathbf{r}) \]  has been used. The differential operator in Eq. (3.105) is just \[ -(\mathbf{\hat{r}} - \varepsilon \mathbf{A})^+ / \mathbf{\hat{r}} \] , where \( \mathbf{\hat{r}} \) is the canonical momentum operator. As \( \chi_0^+ \) has the same functional form as \( \omega_0^+ \), and the boundary conditions remain the same, the equation for \( \psi_o^+(\mathbf{r}) \) may be written in a form analogous to Eq. (3.25):

\[ \psi_o^+(\mathbf{r}) = \chi_o^+(\mathbf{r}) + \int d\mathbf{r}' \mathbf{G}_A^+(\mathbf{r}, \mathbf{r}') \phi_0^-(\mathbf{r}') (\mathbf{\hat{r}} - \varepsilon \mathbf{A})^+ \mathbf{e}^{i \eta_0(\mathbf{r}')}. \]  
(3.106)

Substituting Eq. (3.106) into Eq. (3.101) and interchanging the order of integration in the double integral, we obtain

\[ f_A(\mathbf{k}^+, \mathbf{k}_0^+) = -\frac{1}{4\pi} (\phi_F^+, \wedge \chi_o^+) + \]  
\[ -\frac{1}{4\pi} \int d\mathbf{r}' \phi_0^-(\mathbf{r}') \left( (\mathbf{\hat{r}} - \varepsilon \mathbf{A})^+ \mathbf{e}^{i \eta_0} \right) \int d\mathbf{r}^* \phi_F^* \wedge(\mathbf{r}^*) \mathbf{G}_A^+(\mathbf{r}^*, \mathbf{r}') \]  
(3.107)

where

\[ \wedge(\mathbf{r}) = [ -2 \varepsilon \mathbf{A}^2(\mathbf{r}) + \varepsilon^2 \mathbf{A}^2(\mathbf{r}) ] \]  
(3.108)

The solution of Eq. (2.36) having the asymptotic form of final plane wave plus incoming spherical waves is
\[
\psi_+^- (\mathbf{x}) = \phi_f (\mathbf{x}) - \int d\mathbf{x}' G^- (\mathbf{x}, \mathbf{x}') \left[ 2 \mathbf{e} \cdot \mathbf{A} \cdot \mathbf{v}' + \mathbf{e}^2 \mathbf{A}'^2 \right] \phi_f (\mathbf{x}') \tag{3.109}
\]
\[
= \phi_f (\mathbf{x}) - \int d\mathbf{x}' (G^- (\mathbf{x}, \mathbf{x}') \left[ -2 \mathbf{e} \cdot \mathbf{A} \cdot \mathbf{k}_f + \mathbf{e}^2 \mathbf{A}'^2 \right] \phi_f (\mathbf{x}'). \tag{3.110}
\]

If we use the relation
\[
[ G^- (\mathbf{x}, \mathbf{x}') ]^* = G^+ (\mathbf{x}', \mathbf{x}) , \tag{3.111}
\]
the complex conjugate of Eq. (3.110) may be put in the following form
\[
\int d\mathbf{x}' \phi^*_f (\mathbf{x}') \wedge (\mathbf{x}') G^+ (\mathbf{x}', \mathbf{x}) = - \psi^* (\mathbf{x}) , \tag{3.112}
\]
where, again,
\[
\psi^* (\mathbf{x}) = \psi_+^- (\mathbf{x}) - \phi_f (\mathbf{x}) . \tag{3.113}
\]

Thus Eq. (3.107) reduces to
\[
f_A (\mathbf{r}_f, \mathbf{r}_s) = - \frac{1}{\pi} \left( \phi_f, \wedge \chi^+_c \right) + \frac{1}{\pi} \left( \psi^* , \left[ \mathbf{v} - i \mathbf{e} \cdot \mathbf{A} \mathbf{e}^* \mathbf{e}^* \eta_0 \right] \phi_0 \right) . \tag{3.114}
\]
The structure of Eq. (3.114) is identical with that of Eq. (3.33), for in both cases, the second term contains the square of the kinetic momentum operating on the phase-shift modification.

The first approximation to \( f_A (\mathbf{r}_f, \mathbf{r}_s) \) is obtained by replacing \( \psi_+^- \) by \( \chi^-_f \) in the second term of Eq. (3.114).

Therefore, we must determine an expression for \( \chi^-_f \). The method used in obtaining \( \omega^-_f \), Eq. (3.48), is not immediately applicable for the Hamiltonian operator,
\[ H = -\hbar^2 L_m (\varphi^2 - i \epsilon \mathbf{A} \cdot \mathbf{\nabla} - \epsilon^* \mathbf{A}^* ) \]  

is no longer real. Therefore, Eqs. (3.45) through (3.47) do not hold. From Eq. (3.115), we note that \( H(\mathbf{A}) = \#^*(-\mathbf{A}) \). Thus, Eqs. (3.45) through (3.47) will be correct if we couple the complex conjugation with a sign reversal in the vector potential. Thus, from Eq. (3.47), we have

\[ \psi_f^- (\mathbf{A}, \mathbf{A}) = [ \psi_f^+ (\mathbf{A}, -\mathbf{A}) ]^* \]  

(3.116)

Hence, the expression for \( \chi_f^- \) is

\[ \chi_f^- (\mathbf{A}, \mathbf{A}) = [ \chi_f^+ (\mathbf{A}, -\mathbf{A}) ]^* \]  

(3.117)

\[ = i \mathbf{n} \cdot \mathbf{A} - i \epsilon \int_0^\infty \mathbf{k}_f \cdot \mathbf{A}(\mathbf{A} + \mathbf{n} \cdot \mathbf{A}) \, d\epsilon \]  

(3.118)

F. Summary

A new expression for the exact wave function \( \psi^+(\mathbf{A}) \) is derived by applying Green's theorem to the exact Green's function and \( \omega^+ \). The result is given by Eq. (3.25). By use of this result, the exact high-energy scattering amplitude, Eq. (3.33), is derived. This expression for \( f(\varrho, \phi) \) was first obtained in the form of Eq. (3.38) by Saxon and Schiff (9). Equation (3.33) can also be written in the form of a two potential theory, Eq. (3.36), where the two potentials are given by Eqs. (3.34) and (3.35). Pictorially, one of the potentials generates forward scattering, while the other gives rise to scattering through non-zero angles.
Equation (3.33) can be approximated by Eq. (3.84) provided that $kR \gg 1$ and $\phi_{R}^{\nu}(\nu R/k) \ll 1$. Schiff (5) and Saxon and Schiff (9) have derived additional approximations for small- and large-angle scattering, Eqs. (5.82) and (3.83). However, the Saxon-Schiff derivation of Eq. (3.83) does not appear to be justified. Corresponding expressions for the exact transmission and reflection amplitudes in the high-energy one-dimensional case are given by Eqs. (3.64) and (3.65), while their first approximations are noted in Eqs. (3.68) and (3.70). Finally, the high-energy scattering amplitude for a spinless particle in a magnetic field is given by Eq. (3.114). It is in a form analogous to that of Eq. (3.33).
IV. DIRAC SCATTERING

A. Introduction

The scattering theory presented in Part III. does not satisfy the requirements of special relativity. In order to discuss the scattering of high-energy electrons, one must use the Dirac equation to describe the scattering process. The rest of Section IV. A. is devoted to a review of modifications of the general scattering theory which arise because of this. In Sections IV. B., and IV. C., two derivations of the scattering amplitude are presented; the first is a straight-forward extension of the method used in Part III., while the second is a modified procedure intended to facilitate comparison with the results of Section III. B.

The Dirac equation for a free particle may be expressed in a covariant formalism as

\[ \left( \gamma^\mu i \frac{\partial}{\partial x^\mu} - m \right) \psi(x) = 0 \]

where, for the moment, we set \( \hbar = c = 1 \). The symbol \( \frac{\partial}{\partial x^\mu} \) stands for \( \frac{\partial}{\partial x^\mu} \), where \( \mu \) runs from 0 to 3, and \( x^4 = t \). The \( \gamma^\mu \) are 4 x 4 matrices satisfying the anticommutation rules

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \]

where \( g^{00} = 1 \), \( g^{11} = g^{22} = g^{33} = -1 \), and all the off-diagonal terms are zero. For a plane-wave state of energy \( E \), we must have
\[
\frac{i \partial \phi}{\partial t} = -\epsilon \phi \tag{4.3}
\]

so that Eq. (4.1) may be rewritten as

\[
E \phi = \hbar \phi \tag{4.4}
\]

\[
\hbar = -\gamma^a \gamma^i \partial \omega + \gamma^0 m \quad , \quad a = 1, 2, 3 \tag{4.5}
\]

The Dirac Hamiltonian, h, is Hermitian. By defining

\[
\beta = -\gamma^0 \quad ,
\]

\[
\gamma^a = -\gamma^0 \gamma^a \tag{4.6}
\]

the anticommutation rules become

\[
\gamma^i \gamma^j + \gamma^j \gamma^i = 2 \delta_{ij} \tag{4.7}
\]

\[
\gamma^i \beta + \beta \gamma^i = 0 \quad ,
\]

\[
\beta^2 = 1
\]

Inserting h and c where necessary, and using Eq. (4.6), Eq. (4.4) becomes

\[
[\epsilon - H_0 (\vartheta)] \phi (\vartheta) = 0 \tag{4.8}
\]

where

\[
H_0 (\vartheta) = i \vartheta \cdot \vec{\gamma} - \vec{B} \cdot \vec{m} = -\vec{a} \cdot \vec{p} - \vec{B} \cdot \vec{m} \quad ,
\]

\[
\epsilon = \frac{\mathcal{E}}{\hbar c} \quad ,
\]

\[
\mathcal{M} = \frac{m c}{\hbar} \tag{4.9}
\]

The choice of constants in Eq. (4.8) is guided by the fact that

\[
(\epsilon + H_0) (\epsilon - H_0) = (\epsilon - H_0) (\epsilon + H_0) = \nabla^2 + k^2 \tag{4.10}
\]
where, now,

\[ \mathbf{k}^2 = \mathbf{e}^2 - m^2 \]  

(4.10a)

In the presence of an electromagnetic field, the four-momentum \( \mathbf{p}^\mu \) must be replaced by \( \mathbf{p}^\mu = \mathbf{E}^\mu + \mathbf{A}^\mu \), where \( \mathbf{A}^\mu \) is the four-vector potential. Then Eq. (4.1) becomes

\[ \left( \gamma^\mu i \partial_\mu - \mathbf{e} \gamma^\mu \mathbf{A}_\mu - m \right) \Psi(x) = 0 \]  

(4.11)

which leads, in the time-independent case, to

\[ [ \varepsilon - H_0(\mathbf{\hat{p}}) - V(\mathbf{\hat{r}}) ] \Psi(\mathbf{\hat{r}}) = 0 \]  

(4.12)

where

\[ V = \frac{\gamma^0 \gamma^\mu \mathbf{e} \mathbf{A}_\mu}{\hbar c} \]  

(4.13)

When the vector potential \( \mathbf{A} \) vanishes, \( V \) reduces to the scalar quantity \( V/\hbar c \), where \( V \) is the scalar potential energy. This will be the only case considered.

The positive-energy solutions of Eq. (4.8) are

\[ \phi(\mathbf{\hat{r}}) = \mathbf{u}(\mathbf{\hat{r}}) \mathbf{e}^{i \mathbf{\hat{k}} \cdot \mathbf{\hat{r}}} \]  

(4.14)

where \( \mathbf{u}(\mathbf{\hat{r}}) \) is a 4 x 1 matrix satisfying the equation

\[ (\varepsilon + \mathbf{\hat{r}} \cdot \mathbf{\hat{k}}^\mu + \mathbf{B}_\mu) \mathbf{u} = 0 \]  

(4.15)

There are two linearly-independent positive-energy solutions for \( \mathbf{u} \), corresponding to the two spin states of the electron. When it is necessary to explicitly refer to a spin state \( s \), we
will write the spinor as \( u ( \vec{r} ) \). The \( u ( \vec{r} ) \) satisfy the following covariant orthonormalization property,

\[
\bar{u}^\alpha u^\beta = \delta_{\alpha \beta}, \quad \alpha, \beta = 1, 2, \tag{4.16}
\]

\[
\bar{u} = u^* \beta, \tag{4.16a}
\]

which leads to

\[
u^\alpha u^\beta = \frac{\gamma}{\gamma} = \frac{1}{\sqrt{1 - (\text{velocity})^2}} \tag{4.17}
\]

The latter equation states that an observer measures the moving particle density as one electron per Lorentz-contracted unit volume. The star in Eqs. (4.16) and (4.17) now stands for Hermitian conjugation.

The free-space Green's function corresponding to outgoing spherical waves is a solution of

\[
[\varepsilon - H_0 (\vec{r})] G_0^+ (\vec{r}, \vec{r}') = -\delta (\vec{r} - \vec{r}') \tag{4.18}
\]

and is seen, by inspection, to be

\[
G_0^+ (\vec{r}, \vec{r}') = [\varepsilon + H_0 (\vec{r})] \frac{i}{4\pi} \frac{\vec{k} \cdot \vec{r}'}{r}. \tag{4.19}
\]

The scattering solution of Eq. (4.12) can now be written as

\[
\psi^+_o (\vec{r}) = u_o (\vec{k}_0) \phi^*_o + \int d\vec{r}' G_0^+ (\vec{r}, \vec{r}') \nu (\vec{r}') \psi^+_o (\vec{r}') \tag{4.20}
\]

The asymptotic form of Eq. (4.20) is

\[
\psi^+_o (\vec{r}) \xrightarrow{\| \vec{r} \| \rightarrow \infty} u_o \phi^*_o + \frac{i}{4\pi} \frac{\vec{k} \cdot \vec{r}}{r} \left[\varepsilon + H_0 (\vec{k}_0)\right] \int d\vec{r}' 2 e^{-i\vec{k}_0 \cdot \vec{r}'} \nu (\vec{r}') \psi^+_o (\vec{r}') \tag{4.21}
\]
where

$$H_o(\vec{k}_f) = -\frac{\not{\alpha}}{2} \cdot \vec{k}_f - \beta \mu$$ \hspace{1cm} (4.22)

To obtain a more familiar form, we use the following identity from Dirac theory:

$$\sum_{j=1}^S u^j(\vec{k}) \left[ u^j(\vec{k}) \right]^* = \varepsilon + \frac{H_o(\vec{k})}{2\mu}$$ \hspace{1cm} (4.23)

The sum over s in the preceding equation is a sum over the two spin states. Therefore, Eq. (4.21) becomes

$$\psi_o^+(\vec{r}) \xrightarrow{\lambda \rightarrow \infty} u_o \epsilon i \vec{k}_o \cdot \vec{r}$$

$$-\frac{i \hbar}{4\pi} \frac{\mathcal{S}}{\mathcal{T}} \sum_{j=1}^S u^j(\vec{k}_f) \left[ u^j(\vec{k}_o) \right]^* \int d\vec{r}' e^{-i\vec{k}_f \cdot \vec{r}'} \frac{1}{\varepsilon + \frac{H_o(\vec{k})}{2\mu} + (\lambda')} \psi_o^+(\lambda')$$ \hspace{1cm} (4.24)

The coefficient of $e^{i\vec{k} \cdot \vec{r}}$ is the Dirac scattering amplitude for scattering into a final spin state s. We write this 4 x 1 matrix as

$$F_0^f(\vec{k}_f, \vec{k}_o) = \left[ u_f^j(\vec{k}_f) \right] A^j(\vec{k}_f, \vec{k}_o)$$ \hspace{1cm} (4.25)

where $A^j$ is a scalar and is equal to

$$A^j = -\frac{i \hbar}{4\pi} \int d\vec{r} \left( u_f^j \cdot \not{\alpha} \right) \mathcal{U}(\lambda) \psi_o^+(\lambda) \equiv -\frac{i \hbar}{4\pi} \left( q_f^j \cdot \mathcal{U} \psi_o^+ \right)$$ \hspace{1cm} (4.26)

$$\mathcal{U} = \frac{2\mu}{\hbar^2} \mathcal{V}(\lambda) = 2\mu \mathcal{V}$$ \hspace{1cm} (4.27)

Equation (4.26) has the same form as the non-relativistic scattering amplitude, but now $q_f^j$ and $\psi_o^+$ are Dirac wave functions.

The differential cross section for scattering electrons,
initially in spin state $t$, into final spin state $s$, is

$$
\sigma^s_t = \left| \frac{\langle f^s_t \rangle^* f^s_t}{(u^s_o)^* u^s_t} \right|^2, \quad (4.28)
$$

$$
= \left| A^s_t \right|^2, \quad (4.29)
$$

where Eq. (4.17) has been used in the last step. The cross section corresponding to an unspecified final spin state is

$$
\sigma^s_t = \sum_{s} \left| A^s_t \right|^2. \quad (4.30)
$$

In the first Born approximation, we replace $\psi^s_o(x)$ by $q^s_o(x)$. Then $A^s$ becomes

$$
(A^s_t)_{\text{Born}} = U^s_f U^s_o \rho(\vec{k}_f, \vec{k}_o), \quad (4.31)
$$

where $\rho(\vec{k}_f, \vec{k}_o)$ is the non-relativistic first Born approximation:

$$
\rho(\vec{k}_f, \vec{k}_o) = -\frac{i}{4\pi} \int d\vec{x} \ e^{i \vec{b} \cdot \vec{x}} \ \mathcal{U}(x). \quad (4.32)
$$

Substituting Eq. (4.31) into Eq. (4.30), one obtains

$$
[\sigma^s_t(\theta, \phi)]_{\text{Born}} = \left| \frac{\rho}{3} \right|^2 \left( (u^s_o)^* \left[ \frac{\epsilon + H_o(\vec{k}_o)}{2m} \right] u^s_t \right). \quad (4.33)
$$

This approximate expression for the cross section can be simplified a great deal by averaging over initial spins. Using the fact that the matrices $\rho$ and $\beta$ have zero trace, one obtains the final result (16)

$$
[\sigma(\theta, \phi)]_{\text{Born}} = \frac{1}{2} \sum_{s} \left[ \sigma^s_t(\theta, \phi) \right]_{\text{Born}} = \left( \frac{e}{\hbar} \right)^2 \left( 1 - \frac{\epsilon^2}{\hbar^2} \right) \left| \frac{\rho}{3} \right|^2. \quad (4.34)
$$
Thus, if one knows the non-relativistic Born approximation, the Born cross section for the scattering of a relativistic electron from an unpolarized beam into an unspecified final spin state is also determined via Eq. (4.34).

B. High-Energy Formulation Of the Dirac Scattering Amplitude

In Section II. C. we obtained as a high-energy approximation to the wave function

\[ \Phi(\xi) = 2 \Theta \left( \epsilon - \frac{\xi}{M} \right) + \chi(\xi), \]

where

\[ \chi = \phi(\xi) \]

\[ \psi^+ = \epsilon \psi = \frac{\epsilon}{M} \psi \]

The exponent of the exponential operator is of order \( \sqrt{\epsilon} \), which has been assumed small. It is reasonably clear that the form of the results will be less complicated if this operator is replaced by unity. As our purpose is to derive an exact expression for \( A \), no accuracy is lost for the present, although the simplification will eventually lead to a less accurate first approximation. In the following, we shall suppress the spin index \( s \).

To write an integral equation for \( \psi^+ \) with \( \rho^+ \) as the inhomogeneous term, we make use of the following two differential equations:
The Green's function $G^+$ is a $4 \times 4$ matrix, and a unit $4 \times 4$ matrix is understood on the right-hand side of Eq. (4.37).

The solution of Eq. (4.12) analogous to Eq. (3.25) is, then,

\[
\psi_o^+(\vec{x}) = \pi^+_o(\vec{x}) - \left[ \delta(\vec{x},\vec{x}') \right] e^{i \vec{k}_0 \cdot \vec{x}'} \left[ \vec{\gamma} \cdot \vec{\n} + \nu(\vec{x}') \right] e^{i \delta^o(\vec{x})} \psi_o(\vec{x}). (4.38)
\]

Substituting $\psi_o^+(\vec{x})$ into Eq. (4.26) and inverting the order of integration, we have

\[
A = -\frac{i}{\pi} \frac{\mu}{c} \left( \psi_\phi \cdot \psi_\phi^+ \right) + \frac{i}{\pi} \frac{\mu}{c} \int d\vec{x}' \left[ \int d\vec{x} \psi_\phi^+(\vec{x}) \psi_\phi(\vec{x}') G^+(\vec{x},\vec{x}') \right] \left[ \vec{\gamma} \cdot \vec{\n} + \nu(\vec{x}') \right] e^{i \delta^o(\vec{x})} \psi_o(\vec{x}). (4.39)
\]

A solution of Eq. (4.12) satisfying incoming wave boundary conditions is

\[
\psi_f^-(\vec{x}) = \psi_f(\vec{x}) - \int d\vec{x}' G^-(\vec{x},\vec{x}') \nu(\vec{x}') \psi_f(\vec{x}'). (4.40)
\]

If one takes the Hermitian conjugate of both sides of Eq. (4.40) and uses the relation

\[
\left[ G^-(\vec{x},\vec{x}') \right]^* = G^+(\vec{x}',\vec{x}) \tag{4.41}
\]

the following expression is obtained:

\[
\int d\vec{x} \psi_f^*(\vec{x}) \psi_o(\vec{x}) G^+(\vec{x},\vec{x}') = -2 \varepsilon \psi_o^*(\vec{x}'). \tag{4.42}
\]

where
\[ \psi_{dc}(\lambda) = \psi_f^-(\lambda) - \psi_f^+(\lambda) \quad (4.42a) \]

If Eq. (4.41) is substituted into Eq. (4.39), the expression for \( A \) becomes

\[ A = -\frac{1}{4\pi} \frac{4\pi}{c} \left( \psi_f^-, \ U \right) \zeta^+ \]

\[ -\frac{2\mu}{4\pi} \int d\lambda \, \psi_{dc}(\lambda) \left[ i \hat{r}_0 \cdot \hat{r} \right] [i \hat{\sigma} \cdot \hat{r} \cdot \hat{\sigma}_0(\lambda)] \zeta \]

\[ (4.43) \]

If the initial direction of motion is along the \( \hat{z} \)-axis, Eq. (4.43) can be rewritten with \( U(\lambda) \) as a multiplicative factor in the integrand of the second term:

\[ A = -\frac{1}{4\pi} \frac{4\pi}{c} \left( \psi_f^- \ U \right) \zeta^+ \]

\[ + \frac{M}{4\pi\hbar} \int d\lambda \left[ \hat{\sigma} \int \psi_{dc}(\lambda) \left[ i \hat{r}_0 \cdot \hat{r} \right] [U(\lambda) i \hat{\sigma}_0(\lambda)] \hat{\sigma} \right] \zeta \]

\[ (4.44) \]

The integral over \( \lambda' \) has been shown in Section III. D. to be well-defined. In first approximation, we can replace \( \psi_f^- \) by \( \tilde{z}_f^- \), where

\[ \tilde{z}_f^- (\lambda) = U_f (\hat{r}_f) \hat{z} + i \hat{r}_f \cdot \lambda + \frac{\mu}{2\hbar} \int_{\lambda} U(\lambda + \hat{r}_f \omega) d\omega \]

\[ (4.45) \]

It is possible to express the result for \( A \) in the form of a two-potential theory. One simply rewrites Eq. (4.43) as

\[ A = -\frac{1}{4\pi} \left( \phi_f^-, \left[ -2\mu i \left( \frac{\hat{\sigma} \cdot \hat{r} \cdot \hat{\sigma}_0}{\hat{r} \cdot \hat{\sigma}_0} \right) \right] \zeta^+ \right) \]

\[ -\frac{1}{4\pi} \left( \psi_f^- \left[ U + 2\mu i \left( \frac{\hat{\sigma} \cdot \hat{r} \cdot \hat{\sigma}_0}{\hat{r} \cdot \hat{\sigma}_0} \right) \right] \zeta^+ \right) \]

\[ (4.46) \]

The explicit forms of \( U_f \) and \( U_L \), which generate small- and large-angle scattering, respectively, are therefore
C. A Second High-Energy Formulation Of the Dirac Scattering Amplitude

The expressions for $A$ in Section IV. B. are not in a form which can be compared with the corresponding non-relativistic equations. This is a consequence of using a first-order partial differential equation in the coordinates. A second-order Dirac equation is derived by applying the operator $[\left(\hat{\varepsilon} - \nu\right) + H_0(\hat{\varphi})]$ to the left of Eq. (4.12) to give

\[
\left[ \nabla^2 + k^2 - \nu(\vec{x}) + \nu(\vec{x}) - i (\vec{\omega} \cdot \vec{\varphi}) \right] \psi_0(\vec{x}) = 0 ,
\]

where we now write a subscript $D$ to indicate a Dirac wave function. Transposing the last two terms of Eq. (4.49) to the right-hand side, and using equations of Schrödinger type (however, with relativistic $k$ and $U$),

\[
\left[ \nabla^2 + k^2 - \nu(\vec{x}) \right] G_0^+(\vec{x}, \vec{x}') = -\delta(\vec{x} - \vec{x}') ,
\]

\[
\left[ \nabla^2 + k^2 - \nu(\vec{x}) \right] \psi_0^+(\vec{x}) = 0 ,
\]

we can write the following integral equation for $\psi_0^+$:

\[
\psi_0^+(\vec{x}) = u_0 \psi_0^+(\vec{x})
\]

\[+ \int d\vec{x}' G_0^+(\vec{x}, \vec{x}') \left[ \nu^2(\vec{x}') - i \vec{\omega} \cdot \vec{\varphi}' \nu(\vec{x}') \right] \psi_0^+(\vec{x}') .
\]

It should be noted that both $\psi_0^+$ and $G_0^+$ are scalar functions,
not matrices.

The correctness of the inhomogeneous term can be verified either from Green's theorem, or by the simpler procedure of going to the non-relativistic limit, \( c \to \infty \). Then, since

\[
\nu = \frac{\gamma}{c}, \quad \text{we have} \quad \nu^\top \nu \to \infty \quad \text{and} \quad \vec{z} \cdot \vec{v} \nu \to \infty,
\]

while

\[
U = \frac{\xi}{c^1} U \quad \text{and} \quad \vec{v} . \nu \to \infty. \quad \text{Thus, in the non-relativistic limit, we are solving the problem of a Schrödinger particle with spin, the solution of which is given by the Pauli theory as a 2 \times 1 spinor times the solution of the Schrödinger equation. But this is just the information contained in Eq. (4.52), for as}
\]

\( c \to \infty \), the second term goes to zero, and the spinor \( u \). splits into a non-zero 2 \times 1 matrix, and a zero 2 \times 1 matrix.

Substituting Eq. (4.52) into the general expression for \( A \), Eq. (4.26), we obtain

\[
A = -\frac{i}{c} \frac{D^*}{\ell} u_0 \left( \phi_{f_5}, U \psi_{o_5}^+ \right)
\]

\[
-\frac{i}{c} \int d\vec{x} \left( \frac{\vec{d} \phi_{f_5}}{\ell} \cdot \psi_{o_5}^+ \right) \psi_{o_5}^+ \left( \vec{z}, \vec{x} \right) U(\vec{x}) G_{f_5}^+ \left( \vec{x}, \vec{x}' \right).
\]

(4.53)

Once again we eliminate the Green's function from the expression for \( A \) by noting that the solution of Eq. (4.51), with the asymptotic form of final plane wave plus incoming spherical waves, is

\[
\psi_{f_5}^- \left( \vec{x} \right) = \phi_{f_5}^- \left( \vec{x} \right) - \int d\vec{x}' G_{f_5}^- \left( \vec{x}, \vec{x}' \right) U(\vec{x}') \phi_{f_5}^+ \left( \vec{x}' \right).
\]

(4.54)

We obtain, finally,

\[
A = -\frac{i}{c} \frac{D^*}{\ell} u_0 \left( \phi_{f_5}, U \psi_{o_5}^+ \right)
+ \frac{i}{c} \int d\vec{x} \left( \frac{\vec{d} \phi_{f_5}}{\ell} \cdot \psi_{o_5}^+ \right) \psi_{o_5}^+ \left( \vec{z}, \vec{x} \right).
\]

(4.55)
where, now,

\[ \psi_{sc}(\mathbf{x}) = \psi_{fs}^- (\mathbf{x}) - \psi_{fs}^+ (\mathbf{x}) \]  

(4.56)

In terms of the non-relativistic scattering amplitude \( f(\mathbf{k}_f, \mathbf{k}_o) \) (with, however, \( \mathcal{U} \) replaced by \( \mathcal{V} \) and \( \mathbf{k}^2 = \varepsilon^2 - \frac{m^2}{1} \)), \( A \) can be rewritten as

\[
A = \frac{m}{c} u_f^* u_o f(\mathbf{k}_f, \mathbf{k}_o) + \frac{1}{4\pi} \frac{m}{c} u_f^* \left( \psi_{sc}^+ \left[ \frac{\nu_{sc}}{4e^2} - i \mathbf{A} \cdot \mathbf{\nabla}_{\mathbf{r}} \right] \frac{1}{\nu_{sc}} \psi_{sc}^+ \right) .
\]

(4.57)

This expression is exact and perfectly general in that there has been no mention of the high-energy approximation. The relation for \( A \) has been split up into a term containing the non-relativistic scattering amplitude and a term containing relativistic corrections. As an example, in first Born approximation we set \( \psi_{sc} = \mathcal{O} \) and obtain Eq. (4.31). In the case of high-energy scattering, \( f \) can be written as

\[
f = -\frac{1}{4\pi} \left( \psi_f^- + \omega_o^+ \right) + \frac{1}{4\pi} \left( \psi_{sc}^+ \left( \mathbf{\nu}_{sc} e^{i\mathbf{\sigma} \cdot \mathbf{r}} \right) \frac{1}{\nu_{sc}} \psi_o^+ \right) .
\]

(3.33)

Unlike the second term of Eq. (3.33), the relativistic corrections in the second term of Eq. (4.57) have integrands that are bounded by the potential.

To first approximation in the high-energy formalism, we replace \( \psi_{sc}^+ \) by \( \mathcal{O}^+ \) and \( \psi_{sc} \) by \( (\mathbf{\nu}_f^- - \mathbf{\nu}_{fs}^-) \) in Eq. (4.57). The errors introduced by these replacements are determined in the same manner as in Section III. B. and are found to be of order \{ \nu_{sc}^2, \nu_{sc}^- (\mathbf{\nu}_f^- / \mathbf{R}^2) \} for the approximation
to $\psi_{\sigma, q}^+$, and \( \{ \frac{i}{k_R}, \psi_k^+(u_R^k) \} \) for the approximation to $\psi_{\sigma, e}$. The quantity $f^+ (\vec{r}_f, \vec{r}_t)$ is given by Eq. (3.84).

The second term of Eq. (4.57) becomes

$$
\frac{1}{4\pi} \frac{M}{\varepsilon} u_f^k \int d\vec{r} \ e^{i \vec{p} \cdot \vec{r}} \left( e^{i \sigma \cdot \vec{r}_t - i \sigma \cdot \vec{r}_f} \right) \frac{\psi_f^*}{q \varepsilon^2} - i \vec{\sigma} \cdot \frac{\nabla \psi_f^*}{2\varepsilon} u_0 .
$$

\(4.58\)

The term containing $\tilde{\sigma} \psi$ may be rewritten as a vanishing surface integral plus

$$
\frac{i}{8\pi} \frac{M}{\varepsilon^2} u_f^k \tilde{\sigma} u_0 \cdot \int d\vec{r} u(\vec{r}) \tilde{\sigma} \left[ (\vec{\sigma} \cdot \vec{\nabla}) (e^{i \sigma \cdot \vec{r}_t - i \sigma \cdot \vec{r}_f} \right) e^{i \sigma \cdot \vec{r}_f} .
$$

\(4.59\)

From Eq. (4.15), we can write

$$
(\varepsilon + \vec{\alpha} \cdot \vec{\kappa}_0 + M \beta) u_0 = 0
$$

\(4.60\)

and, also, the Hermitian conjugate equation

$$
u_f^* (\varepsilon + \vec{\alpha} \cdot \vec{\kappa}_f + M \beta) = 0
$$

\(4.61\)

Multiplying Eq. (4.60) on the left by \( u_f^* \), Eq. (4.61) on the right by \( u_0 \), and subtracting, we find

$$
u_f^* (\vec{\alpha} \cdot \frac{\nabla}{\psi_f^*}) u_0 = 0
$$

\(4.62\)

Thus the gradient operator in Eq. (4.59) can be moved through $e^{i \vec{\nabla} \cdot \vec{r}}$ to give

$$
\frac{i}{8\pi} \frac{M}{\varepsilon^2} u_f^k \tilde{\sigma} u_0 \cdot \int d\vec{r} \ e^{i \vec{\sigma} \cdot \vec{r}} \psi_f^* \left[ (\vec{\sigma} \cdot \vec{\nabla}) (e^{i \sigma \cdot \vec{r}_t - i \sigma \cdot \vec{r}_f} \right) e^{i \sigma \cdot \vec{r}_f} .
$$

\(4.63\)
The gradient now brings down only factors of $\psi / \hbar$. For small angles $\theta \ll \sqrt{\eta R}$, order-of-magnitude calculations are meaningful, and the second term of Eq. (4.57) is seen to be of order $\sqrt{\theta} / \epsilon$ times smaller than the first term. For this range of angles, we have

$$f' \simeq -\frac{1}{4\pi} \left( \phi_+^* \cup \omega_0^+ \right),$$

so that the first approximation to $A$, $A_1$, simplifies to

$$\left( A' \right)_{\text{small} \theta} \approx -\frac{1}{4\pi} \omega_+^* \cup_0 \left( \phi_+^* \cup \omega_0^+ \right).$$

Then, in analogy with Eq. (4.34), we have

$$\left( \sigma' \right)_{\text{small} \theta} \approx \left( \frac{2}{3} \right)^4 \left( 1 - \frac{\theta}{\epsilon^2} \sin^2 \frac{\theta}{\epsilon} \right) l \frac{1}{4\pi} \left( \phi_+^* \cup \omega_0^+ \right)^2.$$

By use of his stationary-phase approximation procedure, Schiff (5) obtained both the small-angle form, Eq. (4.65), and an analogous large-angle formula,

$$\left( \sigma' \right)_{\text{large} \theta} \approx \left( \frac{2}{3} \right)^4 \left( 1 - \frac{\theta}{\epsilon^2} \sin^2 \frac{\theta}{\epsilon} \right) l \frac{1}{4\pi} \left( \omega_+^* \cup \omega_0^+ \right)^2.$$

His angular ranges of validity are $\theta \ll \sqrt{\eta R}$ for Eq. (4.65) and $\theta > \sqrt{\eta R}$ for Eq. (4.66). Difficulties in obtaining the non-relativistic large-angle approximation, Eq. (3.83), directly from Eq. (3.33) were discussed in Section III. D. These difficulties prevent us from obtaining Eq. (4.66) directly from $A_1$. 
V. TIME-DEPENDENT SCATTERING THEORY

A. Introduction

The time-dependent Schrödinger equation
\[ i \hbar \frac{\partial \psi(t)}{\partial t} = H \psi(t) \]  \hspace{1cm} (5.1)
\[ H = H_0 + V(x) = \frac{\hbar^2}{2m} + V(x) \]  \hspace{1cm} (5.2)
determines the development in time of the wave function of the interacting particle. It will be assumed, unless otherwise specified, that \( V \) does not depend explicitly on the time. In terms of a state \( \psi(t_o) \) at time \( t_o \), the wave function at time \( t \) may be expressed as
\[ \psi(t) = U(t, t_o) \psi(t_o) \]  \hspace{1cm} (5.3)
where \( U(t, t_o) \) satisfies
\[ i \hbar \frac{\partial U(t, t_o)}{\partial t} = H U(t, t_o) \]  \hspace{1cm} (5.4)
with solution
\[ U(t, t_o) = e^{-\frac{i}{\hbar} H (t-t_o)} \]  \hspace{1cm} (5.5)
The time-development operator, \( U(t, t_o) \), is an unitary operator, and we may thus regard the time development of \( \psi(t) \) as a continuous unitary transformation.

In a collision problem, it is assumed that the initial and final states are plane-wave solutions of Eq. (5.1) with \( V(x) \) set equal to zero, that is, eigenfunctions of the
unperturbed Hamiltonian \( H_0 \). We characterize the initial non-interacting state as \( \phi_{K_0}(-\tau) \) and the final non-interacting state as \( \phi_{K_f}(\tau) \), where \( \tau \) is very large. Because the plane waves are not localized spatially, the use of plane waves as initial states is legitimate only if one introduces an adiabatic decrease in the interaction energy as \( \tau \to \pm \infty \) (17). As the time progresses from \(-\tau\) to \(0\), the time-development operator \( U(0, -\tau) \) rotates the state vector \( \phi_{K_0}(-\tau) \) to a new state, which becomes \( \psi^+(0) \) in the limit \( \tau \to \infty \) (13). The probability amplitude of finding the particle in state \( \phi_{K_f}^\ast \) at time \( t \) is just the scalar product of \( \phi_{K_f}^\ast(t) \) with \( \psi(t) \), or

\[
(\phi_{K_f}^\ast(t), \psi(t)) = \lim_{\tau \to \infty} (\phi_{K_f}^\ast(t), U(t, -\tau) \phi_{K_0}(-\tau)) ,
\]

The transition probability from state \( \phi_{K_0} \) to state \( \phi_{K_f} \) is obtained by taking the limit of Eq. (5.6) as \( t \to \infty \), and then squaring the absolute value of this result. However, as one is interested in transition probabilities per unit time in the determination of cross sections, it is more convenient to determine the rate of transition from state \( \phi_{K_0} \) to state \( \phi_{K_f} \), or

\[
\omega_{K_f, K_0} = \frac{d}{dt} \lim_{\tau \to \infty} |(\phi_{K_f}^\ast(t), U(t, -\tau) \phi_{K_0}(-\tau))|^2 .
\]

This expression is simplified by Lippmann and Schwinger (18) to read

\[
\omega_{K_f, K_0} = \frac{2\pi}{\hbar} \delta(E_f - E_0) |(\phi_f, \psi_0^\ast)|^2 .
\]
where \( \phi_f \) and \( \psi_o^+ \) are now time-independent wave functions. Equation (5.8) is the transition probability per unit time to a single final state. To get rid of the energy-conservation delta function, one integrates over a range of final states,

\[
\frac{\Delta n}{\Delta E_f} \Delta E_f.
\]

The density of states in a range \( E \) to \( E + \Delta E \) is

\[
\frac{\Delta n}{\Delta E} = \left[ \text{spatial volume} \right] \times \frac{\int \frac{\partial \rho}{\partial E} \Delta E}{\hbar^2}\Delta E.
\]

(5.9)

Since the differential cross section is the ratio of the transition probability per unit time to the incident flux, one obtains

\[
\sigma(\theta, \phi) = \left( \frac{1}{4\pi} \right)^2 |\langle \phi_f, \psi_o^+ \rangle|^2 = l + (\theta, \phi) l \quad ,
\]

(5.10)

the same result obtained in the time-independent formulation.

From the above paragraph, it is clear that the time-dependent formulation does not lead to a new result for the expression of the cross section. However, the time-independent wave function with outgoing wave boundary conditions can be written as

\[
\psi_o^+(t) = \lim_{t \to \infty} u(o, t) \phi(-t),
\]

(5.11)

provided the limit exists. In a coordinate representation, Eq. (5.11) becomes

\[
\psi_o^+(\vec{r}) = \lim_{t \to \infty} \int d\vec{r}' \langle \vec{r}' \mid u(o, t) \rangle \phi(-t).
\]

(5.12)

This expression provides an opportunity for obtaining approximate expressions for \( \psi_o^+ \) if approximate expressions for
\( \mathcal{U}(\sigma, -t) \) are available. In addition to its fundamental role in Eq. (5.12), the time-development operator determines the spatial Green's function. If the equation for the exact Green's function [Eq. (3.12)] is written as an operator equation,

\[
\frac{2 \pi}{\hbar} (E - H + i \varepsilon) \mathcal{G}^+ = -1 ,
\]

then one has

\[
\mathcal{G}^+(E) = \frac{\frac{2 \pi}{\hbar}}{E - H + i \varepsilon} \int_0^\infty \frac{d\tau}{2\pi i} e^{-i \mathcal{H}(E + i \varepsilon) \tau} \mathcal{U}(\sigma, -t) \quad \text{(5.14)}
\]

as substitution of Eq. (5.5) for \( \mathcal{U}(\sigma, -t) \) and the subsequent integration will verify. The quantity \( \varepsilon \) is a real, positive increment of energy, which ensures the convergence of the integral at the upper limit. Rewriting Eq. (5.15) in the coordinate representation and suppressing the \( \varepsilon \), one has

\[
\mathcal{G}^+ (\mathbb{R}, \mathbb{R}', E) = \frac{\frac{2 \pi}{\hbar}}{2\pi i} \int_0^\infty \frac{d\tau}{2\pi i} e^{i \mathcal{H} \tau} \langle \mathbb{R}' | \mathcal{U}(\sigma, -t) | \mathbb{R} \rangle \quad \text{(5.16)}
\]

Thus the time-development operator is the fundamental quantity to be determined.

In the following, an approximate propagation function will be deduced, from which the high-energy wave function of Section II. B. as well as an approximate Green's function can be derived. This approximate propagator is then used to construct a physically meaningful stationary-phase analysis, from which the large-and small-angle scattering amplitudes of
Section III. D. are obtained. Finally, exact integral equations for \(\langle \hat{\mathcal{A}}_1 | \mathcal{U}(t, t') | \hat{\mathcal{A}}_2 \rangle\) are obtained in terms of the approximate propagator. These equations lead to two expressions for the wave function. One was presented in Section III. B., while the other is a complicated integral equation.

B. Approximate Propagation Function

It is necessary to have a procedure for obtaining the spatial representation of \(\mathcal{U}(t, t')\) for specific potentials. The form of \(\langle \hat{\mathcal{A}}_1 | \mathcal{U}(t, t') | \hat{\mathcal{A}}_2 \rangle\) most suggestive for our requirements is the Feynman path integral (19)

\[
\langle \hat{\mathcal{A}}_1 | \mathcal{U}(t, t') | \hat{\mathcal{A}}_2 \rangle = A(t_2 - t_1) \int \mathcal{D}(\text{path}) \exp \left\{ \frac{i}{\hbar} S(\text{path}) \right\},
\]

(5.17)

where

\[
\langle \hat{\mathcal{A}}_1 | \mathcal{U}(t, t') | \hat{\mathcal{A}}_2 \rangle \equiv \langle \hat{\mathcal{A}}_1 | \mathcal{U}(t, t, t') | \hat{\mathcal{A}}_2 \rangle.
\]

(5.18)

\[
S = \int_{t_1}^{t_2} L \left[ \dot{\mathcal{X}}(t), \mathcal{X}(t) \right] dt.
\]

(5.19)

The symbol \(S\) stands for the classical action, the time integral of the classical Lagrangian \(L\), taken along the path \(\mathcal{X}(t)\). The action is an extremum for the classical path \(\mathcal{X}_c(t)\). The integration is over all paths connecting the space-time points \((\mathcal{X}_1, t_1)\) and \((\mathcal{X}_2, t_2)\). The quantity \(A(t_2 - t_1)\) is a normalization factor, independent of the potential. It is a measure of the aggregate of all paths in the time interval from \(t_1\) to \(t_2\).

An expression for the propagation function is easily
obtained in the case $V(x) = 0$. We first rewrite Eq. (5.17) as

$$\langle x | 1 \rangle = \xi \frac{i}{\hbar} \int \frac{d\lambda}{(2\pi \hbar)^2} \frac{e^{i \lambda x}}{\sqrt{2\pi \hbar}} \langle x | \lambda \rangle$$

(5.20)

where

$$\langle x | \lambda \rangle = A(t, t_1) \int_{\lambda(t_1)}^{\lambda(t_2)} \rho_{\lambda} \omega \frac{e^{i \lambda \cdot \vec{r} - i \omega t}}{\sqrt{2\pi \hbar}} \langle \lambda | \lambda_c \rangle$$

(5.21)

$S_c$ is the value of Eq. (5.19) when $\lambda(t) = \lambda_c(t)$. For a free particle, the classical path is a straight line, and we have $S_c = S_{f0}$, where

$$S_{f0} = \int_{t_1}^{t_2} T_s \, dt$$

(5.22)

$$T_s = \frac{1}{2} m v_s^2$$

(5.23)

$$v_s = \frac{1}{t_2 - t_1} \frac{\lambda_2 - \lambda_1}{(t_2 - t_1)}$$

(5.24)

The value of $S_c$ is seen to be

$$S_{f0} = \frac{1}{2} m \left( \frac{\lambda_2 - \lambda_1}{(t_2 - t_1)} \right)^2$$

(5.25)

Equation (5.20) becomes

$$\langle x | 1 \rangle = \frac{i}{\hbar} \int \frac{d\lambda}{(2\pi \hbar)^2} \frac{e^{i \lambda \cdot \vec{r} - i \omega t}}{(t_2 - t_1)} \langle x | \lambda \rangle$$

(5.26)

In addition, we must require that $\langle x | 1 \rangle$ propagates a plane wave from time $t_1$ to time $t_2$, that is

$$\frac{i}{\hbar} \int \frac{d\lambda}{(2\pi \hbar)^2} \frac{e^{i \lambda \cdot \vec{r} - i \omega t_2}}{(t_2 - t_1)} \langle x | \lambda \rangle$$

(5.27)

It is sufficient that
\[
\langle 2/1 \rangle_0 = \left( \sqrt{\frac{m}{2\pi i\hbar t_{L}}} \right)^3
\]  
(5.28)

Therefore, if we define \( t_{L1} = (t_L - t_i) \), the expression for the free propagator in three dimensions is

\[
\langle 2/1 \rangle_0 = \left( \sqrt{\frac{m}{2\pi i\hbar t_{L1}}} \right)^3 e^{\frac{i m}{2\pi i\hbar t_{L1}} (\lambda_{L} - \lambda_{1})^2}
\]  
(5.29)

In one dimension, this is modified to

\[
\langle 2/1 \rangle_0 = \sqrt{\frac{m}{2\pi i\hbar t_{L1}}} e^{\frac{i m}{2\pi i\hbar t_{L1}} (\lambda_{L} - \lambda_{1})^2}
\]  
(5.30)

It is possible to approach the derivation of \( \langle 2/1 \rangle \) for a specific potential in a more systematic way by expanding \( \lambda(t) \) in a complete set of orthonormal functions. The integration over all paths then reduces to an integration over the expansion coefficients (20). This method is explained more fully in Appendix A. In addition, a WKB approximation to \( \langle 2/1 \rangle \) is carried out there in one dimension, with the result that

\[
\langle 2/1 \rangle_{\text{WKB}} = e^{\frac{i}{\hbar} \int_{0}^{\infty} (z)} \sqrt{\frac{m}{2\pi i\hbar t_{L1}}} \frac{1}{\sqrt{|\text{det} I|}}
\]  
(A.26)

where "det" stands for determinant, and the elements of the matrix I are (\( m, n \geq 1 \))

\[
I_{mn} = \delta_{mn} + \frac{\tau_{z1}}{m n} \pi r \int_{0}^{\infty} dz \omega(z) \left[ \cos(\frac{m-n}{\pi r} z) - \cos(\frac{m+n}{\pi r} z) \right]
\]  
(A.30)
We wish to infer the form of the 'forward propagator',

\[ \langle 2 | 1 \rangle \sim \text{, which, when substituted in Eq. (5.12), yields the high-energy wave function of Section II. B. The WKB approximation requires that } k R \gg 1 \text{, where } R \text{ is a measure of the range of the potential. To introduce the feature } V \ll E \text{, where } E \text{ is the incident energy, one expands } S_0 \text{ about } S_g, \text{ the action along a straight-line path in space-time. The first two terms in this expansion may be shown to be}

\[ S_w = S_f - \frac{1}{\gamma T_f} \int_{t_1}^{t_2} [V(x_f) - \bar{V}(x_f)] \, dt + \cdots \]  

(5.31)

where

\[ S_f = \int_{t_1}^{t_2} [T_f - V_f] \, dt \]  

(5.32)

\[ \bar{V}(x_f) = \frac{1}{t_1} \int_{t_1}^{t_2} \, dt \text{, } V(x_f(t)) \]  

(5.33)

The second term of Eq. (5.31) introduces a correction of order \( V/E \) relative to the first term for that class of straight-line paths with \( T_g \) of order \( E \). Rewriting Eq. (A.26), we have

\[ \langle 2 | 1 \rangle \sim \frac{m}{2 \pi \hbar t_1} \int_{t_1}^{t_2} \frac{\dot{x}_f}{\hbar} \, dt - \frac{\dot{x}_f}{\hbar} \int_{t_1}^{t_2} \hbar \, dt + \frac{m}{\hbar} \int_{x_1}^{x_2} (S_f - S_{r}) \, \sqrt{dt} \]

(5.34)

But, using Eqs. (5.23) and (5.30), one has

\[ \langle 2 | 1 \rangle = \frac{m}{2 \pi \hbar t_1} \int_{t_1}^{t_2} \frac{\dot{x}_f}{\hbar} \, dt \]

(5.35)
Thus, Eq. (5.34) becomes

$$<\psi|W_{K}|\phi> = \left[<\psi|\phi> e^{-i\frac{\hbar}{\varepsilon} \int_{t',v} \lambda d\tau} \right] \left[ e^{i \frac{\hbar}{\varepsilon} (f_{\psi} - f_{\phi})} (\text{det I})^{-\frac{1}{2}} \right].$$

(5.36)

We will identify the first bracketed factor in Eq. (5.36) as $$<\psi|\phi>_{a}.$$ The factor in $$(S_{\psi} - S_{\phi})$$ is not included, for it will introduce only corrections of order $$\sqrt{v/s}$$ in the phase when the approximate wave function is determined. It will be recalled that in Section II. B., the desired high-energy wave function was obtained from the WKB approximation when only one dimension was involved. There, the amplitude factor was neglected, and analogously, det I will here be replaced by unity. This amounts to dropping the second term in Eq. (A.30). It is to be emphasized that the choice of $$<\psi|\phi>_{a}$$ is dictated solely by the results which are obtained from it. The derivation of the WKB propagation function was merely an attempt at clarifying the origin of the forward propagator.

We may now write

$$<\psi|\phi> = \langle x_{1} | u_{a}(t_{z},t_{1}) | x_{1} \rangle = \frac{1}{\sqrt{\pi \varepsilon^{3}}} e^{i \frac{\hbar}{\varepsilon} S_{a}} = \frac{1}{\sqrt{\pi \varepsilon^{3}}} e^{i \frac{\hbar}{\varepsilon} S_{a}} \langle x_{1} | u_{0}(t_{z},t_{1}) | x_{1} \rangle.$$  (5.37)

It is convenient to rewrite the phase factor in Eq. (5.37a) in terms of a distance

$$\lambda = \frac{x_{1} - x_{1}^{'}}{t_{z} - t_{1}}$$

along the straight-line path. Since

$$x_{1}^{'}(t) = x_{1} + \lambda$$

and

$$d\lambda = \frac{(x_{1} - x_{1}^{'}) d\tau}{t_{z} - t_{1}},$$

Eq. (5.37a) becomes
These equations for \( \langle 2 | 1 \rangle \) may be generalized immediately to three dimensions. Then we have

\[
\langle \vec{r}_2 | V(t, \vec{t}, t_1) | \vec{r}_1 \rangle = e^{-i\frac{\mathbf{E}}{\hbar} t_1} \int_0^{t_1} \mathrm{d}t \mathbf{A}(t) \cdot \mathbf{V}\left(\vec{r}_2(t) + \mathbf{E}t\right) \langle \vec{r}_2 | \mathbf{V}(\vec{r}_2(t_1)) | \vec{r}_1 \rangle,
\]

where, now,

\[
\mathbf{V}(\vec{r}_2(t_1)) = \frac{\mathbf{E}}{\hbar} \int_0^{t_1} \mathbf{V}(\vec{r}_2(t)) \mathrm{d}t.
\]

Equation (5.41) is the main result. However, it is a simple matter to write a forward propagator for the case of a vector potential, \( \mathbf{A}(\vec{r}) \), as well. The Lagrangian for a charged particle in a magnetic field is

\[
\mathcal{L}(\vec{r}, \dot{\vec{r}}) = T + \frac{e}{c} \mathbf{A} \cdot \dot{\mathbf{r}},
\]

where \( \dot{\mathbf{r}} \) is the velocity \( \dot{\vec{r}} \). Using Eqs. (5.35) and (5.37), the expression for the forward propagator is found to be

\[
\langle 2 | 1 \rangle = e^{-i\frac{\mathbf{E}}{\hbar} t_1} \int_0^{t_1} \mathbf{A}[\vec{r}_2(t)] \cdot \dot{\mathbf{r}}_2 \mathrm{d}t \langle 2 | 1 \rangle_0.
\]
Having obtained the approximate propagation function, one has merely to apply Eqs. (5.12) and (5.16) to obtain the corresponding wave function and Green's function. Before proceeding it is convenient to develop a general technique for integrating Eq. (5.12). Let us first consider the one-dimensional integral

\[ I = \int_{-\infty}^{\infty} dx_1 f(x_1) <2\mid 1>. \]

(5.46)

If Eq. (5.30) is substituted for \(<2\mid 1>\), Eq. (5.46) becomes

\[ I = \int_{-\infty}^{\infty} dx_1 f(x_1) \left( \frac{m}{i\hbar t_1} \right)^2 \int_{-\infty}^{\infty} dx_1 f(x_1) e^{i\frac{m}{\hbar t_1}(x_1 - x_2 + v_{\parallel} t_{21})}. \]

(5.47)

The exponential oscillates rapidly when \(x_1\) deviates very far from the straight-line path connecting the space-time points \((x_{10}, t_1)\) and \((x_{20}, t_2)\), where \(x_{10} = x_{20} - v_{\parallel} t_{21}\) and \(v_{\parallel} = \frac{p}{m}\). Analytically, the phase is stationary for the point \(x_{10}\). Thus, if \(f(x_1)\) is a smoothly varying function, the contributions to the integral are restricted to the immediate vicinity of \(x_{10}\). If we expand \(f(x_1)\) about the point \(x_{10}\), Eq. (5.47) becomes
The odd terms in \((x_1 - x_{10})\) all give zero, and

\[
I = \int_{x_1}^{x_{10}} d\chi_i \left\{ f(x_{10}) + \frac{f''(x_{10})}{2!} \chi_i + \ldots \right\} \frac{e^{i k (x_1 - x_{10})^2}}{\sqrt{2\pi i\hbar t_1}}.
\]  

(5.48)

The rest of the series progresses in the same way, the \(n\)th power of \(t_{12}\) is multiplied by the \((2n)\)th derivative of \(f\). We are interested in the limit \(t_{12} \to \infty\). Since \(x_1, v_0, t_{12}\) are fixed, the stationary-phase point approaches \(-\infty\) as \(v_0 t_{12}\). It is clear that there are certain functional forms of \(f\) for which the series in Eq. (5.49) will not only converge in the limit \(t_{12} \to \infty\), but for which only the first term, \(f(x_{10})\), will survive in the same limit. The functional forms of \(f(x_1)\) in which we are interested are

\[
f(x_1) = g[v(x_1)]\quad\text{and}\quad f(x_1) = h\left[\int_{x_1}^{x_{10}} v(y) dy\right],
\]

where \(g\) and \(h\) are differentiable and finite at the point \(x_1 = x_{10}\).

In order that the integral \(\int_{x_1}^{x_{10}} v(x) dx\) be defined, \(v(x_1)\) must drop off faster than \(1/x_1\) at large \(|x_1|\). For the case of \(f(x_1) = g[v(x_1)]\), the even-ordered derivatives evaluated at \(x_{10}\), \(f^{(2n)}(x_{10})\), \(n=1,2,...\), go to zero more rapidly than \(|t_{12}|^{-n}\) in the limit \(t_{12} \to \infty\). For \(f(x_1) = h\left[\int_{x_1}^{x_{10}} v(y) dy\right]\), the even-ordered derivatives evaluated at \(x_{10}\) drop off to zero more rapidly than \(|t_{12}|^{-n}\). Thus, in both cases, Eq. (5.49) reduces to...
Hence, for the specific types of $f$ considered, a comparison of Eq. (5.46) with Eq. (5.51) yields the relation

$$\langle x_1 | x_2 \rangle = \int d^3x \int d^3x' f(x_1) f(x_2) \delta(x_1 - x_2).$$

(5.52)

The generalization of the above procedure to three dimensions complicates the mathematical notation, but does not change the basic form of Eqs. (5.46) to (5.51). This is mainly due to the fact that $\langle \vec{r}_1 : V_0(x, t) \rangle$ is just the product of three one-dimensional free propagators. Qualitatively, we expect Eq. (5.52) to generalize to three dimensions because it contains only factors pertaining to motion in free space. Free motion is independent of the dimensionality of the space. Therefore, in the evaluation of the integral

$$I' = \lim_{\tau_\infty \to -\infty} \int d\vec{r}_1 f(\vec{r}_1) \langle \vec{r}_1 | \vec{r}_1 \rangle \langle \vec{r}_1 | \vec{r}_1 \rangle.$$

(5.53)

where $f(\vec{r}_1) = G[\mathcal{V}(\vec{r}_1)]$ or $H[\int_0^\infty \mathcal{V}(\vec{r}_1 + \vec{r}_2) \mathcal{A}_2]$, we can write

$$\langle \vec{r}_1 | \vec{r}_1 \rangle \langle \vec{r}_1 | \vec{r}_1 \rangle \langle \vec{r}_1 | \vec{r}_1 \rangle.$$

Again one must require that $\mathcal{V}(\vec{r}_1)$ go to zero faster than $|\vec{r}_1|^{-1}$ at large $|\vec{r}_1|$.\]
A somewhat different derivation of Eq. (5.54) can be carried out. The integral, Eq. (5.53), can be written as

\[ I' = \lim_{t_1 \to -\infty} \left( \int \frac{m}{1 + \pi n k t_1} \right)^2 \int d^2 \kappa \left( \frac{\gamma}{1 - \rho \kappa} \right)^2 \]

\[ = \lim_{t_1 \to -\infty} \left( \int \frac{m}{1 + \pi n k t_1} \right)^2 \int d^2 \kappa \left( \frac{\gamma}{1 - \rho \kappa} \right)^2 \]

where the substitution \( \gamma = \lambda - \lambda_{\infty} \) has been made. The phase factor oscillates very rapidly because of the \( \lambda \) dependence and restricts the contributions to the integral from the range \( |\gamma| \approx \frac{1}{2\pi n k t_1} \). Thus, the effective region of integration is limited to a sphere of radius \( \sqrt{2 \pi n k t_1} \) and center at \( \lambda - \frac{\gamma}{1 - \rho \kappa} \). As \( t_1 \) becomes infinite, the radius of the sphere increases only as \( \sqrt{t_1} \) while the center of the sphere recedes to \( -\infty \) as \( t_1 \). We will consider the class of functions \( f \) which assume a constant value throughout the effective sphere of integration in the limit \( t_1 \to -\infty \); that is, \( \max f - \min f \to 0 \) for values of \( \gamma \) within the sphere. Then, \( f \) may be taken outside the integral in Eq. (5.56) and evaluated at any point in the sphere. For convenience, we choose the point of evaluation as \( \gamma = 0 \). Then Eq. (5.56) becomes

\[ I' = \lim_{t_1 \to -\infty} f \left( \lambda - \frac{\gamma}{1 - \rho \kappa} \right) \]

Again, the effect of the integration has been equivalent to the action of a delta function restricting the point \( \lambda \) to
the path of a free particle of velocity \( \vec{v}_0 = \frac{\vec{k}_0}{m} \).

The evaluation of Eq. (5.12) for the forward propagator, \( \langle 2\downarrow \rangle_a \), is now trivial. Substituting Eq. (5.41) into Eq. (5.12), one has

\[
\omega_0^+(\vec{r}_L) = \lim_{t_1 \to \infty} \int d\vec{r}_1 e^{-i\frac{\vec{k}_0}{m} t_1} \vec{v}(\vec{r}_L, \vec{r}_1) \times \langle \vec{r}_L | V_a(0, -t_1) | \vec{r}_1 \rangle \times i \epsilon_{\vec{k}_0 \vec{r}_1} + i \omega t_1.
\]

(5.58)

By use of Eq. (5.54), Eq. (5.58) simplifies to

\[
\omega_0^+(\vec{r}_L) = e^{i\vec{k}_0 \cdot \vec{r}_L} \lim_{t_1 \to \infty} \int d\vec{r}_1 e^{-i\frac{\vec{k}_0}{m} t_1} \vec{v}(\vec{r}_L, \vec{r}_1) f(\vec{r}_1 - \vec{r}_{t_0}).
\]

(5.59)

\[
= e^{i\vec{k}_0 \cdot \vec{r}_L} \lim_{t_1 \to \infty} e^{-i\frac{\vec{k}_0}{m} t_1} \vec{v}(\vec{r}_L, \vec{r}_1 + \vec{v}_0 t_1) \int_0^\infty \vec{v}(\vec{r}_L - \vec{k}_0 x) dx.
\]

(5.60)

since

\[
\vec{v}(\vec{r}_L, \vec{r}_1 + \vec{v}_0 t_1) = \frac{1}{i \vec{v}_0 t_1} \int_0^{1 \vec{v}_0 t_1} \vec{v}(\vec{r}_L - \vec{k}_0 x) dx.
\]

(5.62)

As \( \vec{v}_0 = \frac{\vec{k}_0}{m} \), Eq. (5.61) is just the approximate wave function for forward scattering, Eq. (2.30). The wave function corresponding to the forward propagator for a magnetic field [Eq. (5.45)] is

\[
\chi^+(\vec{r}_L) = \lim_{t_1 \to \infty} \int d\vec{r}_1 e^{i\frac{\vec{k}_0}{m} t_1} \vec{A}(\vec{r}_L - \vec{k}_0 x) \cdot \vec{k} \cdot \vec{r}_1 + i \omega t_1.
\]

(5.63)

\[
= e^{i\vec{k} \cdot \vec{r}_L} \lim_{t_1 \to \infty} \int d\vec{r}_1 e^{i\frac{\vec{k}_0}{m} t_1} \vec{A}(\vec{r}_L - \vec{k}_0 x) \cdot \vec{k} \cdot \vec{r}_1 \int_0^\infty \vec{A}(\vec{r}_L - \vec{k}_0 x) \cdot \vec{k} dx.
\]

(5.64)

\[
= e^{i\vec{k} \cdot \vec{r}_L} \lim_{t_1 \to \infty} \int d\vec{r}_1 e^{i\frac{\vec{k}_0}{m} t_1} \vec{A}(\vec{r}_L - \vec{k}_0 x) \cdot \vec{k} \cdot \vec{r}_1 \int_0^\infty \vec{A}(\vec{r}_L - \vec{k}_0 x) \cdot \vec{k} dx.
\]

(5.65)

\[
= e^{i\vec{k} \cdot \vec{r}_L} \int_0^\infty \vec{A}(\vec{r}_L - \vec{k}_0 x) dx.
\]

(2.41)
We have considered only potentials independent of time. However, the forward propagator is easily modified for a time-dependent potential \( V(\vec{r}, t) \). Equation (5.40) becomes

\[
\langle \vec{r}_1 | e^{-iHt} | \vec{r}_0 \rangle = e^{-iH \int_{t_0}^{t_1} V(\vec{r}_0, \tau) d\tau} \langle \vec{r}_1 | e^{-iH t_1} | \vec{r}_0 \rangle + e^{-iH \int_{t_0}^{t_1} e^{-iH \int_{\tau}^{t_1} V(\vec{r}_0, \tau') d\tau'} d\tau} \langle \vec{r}_1 | e^{-iH t_1} | \vec{r}_0 \rangle
\]

(5.66)

\[
\langle \vec{r}_1 | e^{-iHt} | \vec{r}_0 \rangle = e^{-iH \int_{t_0}^{t_1} V(\vec{r}_0, \tau) d\tau} \langle \vec{r}_1 | e^{-iH t_1} | \vec{r}_0 \rangle + e^{-iH \int_{t_0}^{t_1} e^{-iH \int_{\tau}^{t_1} V(\vec{r}_0, \tau') d\tau'} d\tau} \langle \vec{r}_1 | e^{-iH t_1} | \vec{r}_0 \rangle
\]

(5.67)

since \( \vec{r} = \vec{r}_{t_1}(t_1 - t) \). Then, the approximate wave function is

\[
\omega^+(\vec{r}_1, t_1) = \lim_{t_1 \to \infty} \int d\vec{r}_0 \langle \vec{r}_1 | U_{\omega}(t_1, t_0) | \vec{r}_0 \rangle e^{i\vec{k} \cdot \vec{r}_0} + i\omega t,
\]

(5.68)

\[
\omega^+(\vec{r}_1, t_1) = \lim_{t_1 \to \infty} \int d\vec{r}_0 \langle \vec{r}_1 | U_{\omega}(t_1, t_0) | \vec{r}_0 \rangle e^{i\vec{k} \cdot \vec{r}_0} + i\omega t,
\]

(5.69)

\[
\omega^+(\vec{r}_1, t_1) = \lim_{t_1 \to \infty} \int d\vec{r}_0 \langle \vec{r}_1 | U_{\omega}(t_1, t_0) | \vec{r}_0 \rangle e^{i\vec{k} \cdot \vec{r}_0} + i\omega t,
\]

(5.70)

The Green's functions corresponding to the forward propagator can be obtained from Eq. (5.16):

\[
G^+(\vec{r}_2, \vec{r}_1) = -\frac{i}{2m} \int_0^\infty dt_1 e^{it_1} \langle \vec{r}_2 | U_{\omega}(0, t_1) | \vec{r}_1 \rangle
\]

(5.72)

Substituting Eq. (5.41) for \( \langle \vec{r}_1 | e^{-iHt} | \vec{r}_0 \rangle \), one has

\[
G^+(\vec{r}_2, \vec{r}_1) = -\frac{i}{2m} \int_0^\infty dt_1 e^{it_1} \langle \vec{r}_2 | U_{\omega}(0, t_1) | \vec{r}_1 \rangle
\]

(5.73)

The free-space Green's function, \( G^+_0(\vec{r}) = \frac{\lambda}{4\pi} e^{i\vec{k}\cdot\vec{r}} \), is equal to
Comparison of Eqs. (5.73) and (5.74) shows that the time dependence of the two integrands is the same. The potential-dependent exponential produces only an energy shift of size $\tilde{\nu}$. Thus, $G_\alpha^*$ is just the free-space Green's function with modified propagation number. Defining

$$K(\vec{r}_2, \vec{r}_1) = k \sqrt{|1 - \frac{\tilde{\nu}(\vec{r}_2, \vec{r}_1)}{E}|},$$

one has

$$G_\alpha^*(\vec{r}_2, \vec{r}_1) = G_\alpha^*(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi} \frac{e^{ik\rho}}{\rho}.$$ (5.76)

For high incident energies, $\tilde{\nu} \ll E$, and $K$ simplifies to

$$K \sim k \left[ 1 - \frac{1}{2E} \frac{k}{\rho} \int_0^\infty \frac{d\omega}{\omega} \right].$$ (5.77)

Therefore, in the high-energy limit, the approximate Green's function becomes

$$G_\alpha^*(\vec{r}_2, \vec{r}_1) \approx \frac{1}{4\pi} e^{-ik\rho} - \frac{i}{2E} \int_0^\infty \omega \frac{d\omega}{\omega} v(\vec{r}_2 - \hat{\rho} \omega) d\omega$$

$$= G_0^*(\vec{r}_2, \vec{r}_1) e^{-ik\rho} \int_0^\infty v(\vec{r}_2 - \hat{\rho} \omega) d\omega$$ (5.78)

Equation (5.79) is the approximate Green's function which Saxon and Schiff (9) use in the derivation of their integral equation for $\psi^*$. The case of the vector potential is trivial. If Eq. (5.45) is substituted into Eq. (5.16), the expression for the
approximate Green's function is

\[ G_{\alpha}^{\pm}(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = \sum_{\alpha} \mathcal{E}^{\pm}_{\alpha} \int_{-\infty}^{\infty} \mathcal{A}^{\pm}_{\alpha}(\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1) \cdot \mathbf{\hat{p}} \, d\omega \]

\[ \times (-\frac{\mathbf{E}}{2m_i}) \int_{0}^{\infty} \mathcal{G}_{\alpha}(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) \cdot \mathbf{\hat{p}} \, d\omega \]  

\[ = G_{\alpha}^{\pm}(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) \]  

where Eq. (5.74) has been used for the last step. One should note that Eq. (5.81) satisfies the correct reciprocity condition

\[ G_{\alpha}^{\pm}(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = G_{\alpha}^{\pm}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) \]

There is no simple expression for the approximate Green's function when the potential depends explicitly on the time.

D. Derivation Of Approximate Scattering Amplitudes By Time-Dependent Stationary-Phase Procedure

Schiff's (5) original derivation of \( f' \) for small-and large-angle scattering, Eqs. (3.82) and (3.83), was based on the application of the stationary-phase approximation in a time-independent theory. This type of derivation is also possible in the time-dependent theory. The inclusion of the time coordinate does not complicate the derivation; instead, it emphasizes the basic physical picture behind the mathematics.

We may write an integral equation for \( U(t_2, t_1) \) in terms of the free propagator, \( U_0(t_2, t_1) \), as follows:

\[ U(t_2, t_1) = U_0(t_2, t_1) + \int_{t_1}^{t_2} \frac{1}{\hbar} \int_{t_3}^{t_2} U_0(t_2, t_3) \mathcal{V}(t_3, t_1) \]  

where
The validity of Eq. (5.82) may be readily demonstrated by applying the operator \((i\hbar \frac{\partial}{\partial t} - H_o)\) to both sides of the equation. In addition, the initial condition \(V(t_1, t) = 1\) is clearly fulfilled. Equation (5.82) can be iterated to yield the Born series,

\[
V(t_2, t_1) = V_o(t_2, t_1) + \frac{1}{\hbar} \int_{t_1}^{t_2} dt_3 \, V_o(t_1, t_3) \, V(t_2, t_3) + \cdots \quad (5.85)
\]

With reference to a scattering process, the first term of the series corresponds to no scattering at all. The second term corresponds to a single scattering through an angle of arbitrary size. To introduce the effect of the potential into the first term, we shall attempt to reorganize the Born series into a series where all forward scattering is contained in the first term (when applied to a plane wave), all single large-angle scatterings are grouped in the second term, two large-angle scatterings in the third, etc. Clearly, the inhomogeneous term in the reorganized integral equation must be the forward propagator, \(V_{a}\). We shall express \(V\) as the series

\[
V(t_2, t_1) = V_a(t_2, t_1) + \frac{1}{\hbar} \int_{t_1}^{t_2} dt_3 \, V_a(t_1, t_3) \, V_a(t_2, t_3) + \cdots \quad (5.86)
\]

where the prime denotes omission of forward scattering, in a
sense to be defined more precisely later. The justification of Eq. (5.86) is intuitive rather than mathematical; the equation represents a rearrangement of the infinite Born series in which the nth term describes \( n - 1 \) large-angle scatterings combined with an arbitrary number of forward scatterings. For example, the pictorial description of the second term is forward scattering from time \( t_1 \) to time \( t_3 \) when a single large-angle scattering takes place. This is followed by forward scattering to time \( t_2 \).

Rewriting Eq. (5.86) in the coordinate representation, one has

\[
\langle 211 \rangle = \langle 211 \rangle_\sigma + \frac{1}{\kappa} \int_{t_1}^{t_3} dt_2 \int d\vec{x}_3 \langle 213 \rangle_\sigma \nabla(\vec{x}_2) \langle 311 \rangle_\sigma + \cdots \quad (5.87)
\]

Substituting in Eq. (5.12), and using Eqs. (5.58) and (5.61), we obtain as an approximation to \( \psi^+(x_i) \),

\[
\psi^+_0(x_i) \simeq \omega^+_0(x_i) + \frac{1}{\kappa} \int_{t_1}^{t_3} dt_2 \int d\vec{x}_3 \langle \vec{x}_1 | u_0(x_0, t_2) | \vec{x}_3 \rangle \nabla(\vec{x}_2) \omega^+_0(\vec{x}_2, t_2) \quad (5.88)
\]

If desired, the time integration can be performed by use of Eqs. (5.72) and Eqs. (5.76) and (5.77). The result for high-energies is

\[
\psi^+_0(x_i) \simeq \omega^+_0(x_i) - \int d\vec{x}_3 \; \mathcal{G}^+_0(\rho) \times \frac{-\nabla_0 \int_{t_1}^{t_3} \nabla(\vec{x}_2 - \vec{x}_3) \cdot d\omega}{\nabla(\vec{x}_3) \; \omega^+_0(\vec{x}_3)} \quad (5.89)
\]

However, to obtain expressions for the scattering amplitude,
It is more convenient to substitute Eq. (5.88) into the defining equation for \( f(k_\parallel, k_\perp) \),

\[
f(k_\parallel, k_\perp) = -\frac{i}{4\pi} \left( \varphi_f + \psi_o^+ \right)
\]

(3.10)

The following approximation to \( f \) is obtained:

\[
f' = -\frac{i}{4\pi} \left( \varphi_f + \psi_o^+ \right)
- \frac{1}{4\pi^3} \int d\mathbf{r}_3 \int d\mathbf{r}_1 \int d\mathbf{r}_2 \varphi_f^*(\mathbf{r}_2) V(\mathbf{r}_1) \langle \mathbf{r}_2 | \psi_o^-(0, -t_3) | \mathbf{r}_3 \rangle V(\mathbf{r}_3) \omega_o^+(\mathbf{r}_3, -t_3).
\]

(5.90)

The stationary-phase approximation may be used to simplify the second term of Eq. (5.90). If \( g(\mathbf{r}) \) is a slowly-varying function, integrals of the form

\[
I = \int d\mathbf{r}_1 \quad g(\mathbf{r}_1) \quad \langle \mathbf{r}_1 | \mathbf{r}_2 \rangle \quad e^{i \mathbf{k} \cdot \mathbf{r}_2 - i \omega t_2 + i \mathbf{m} \cdot \mathbf{v}_0 t_2},
\]

(5.91)

may be approximately evaluated by expanding \( g(\mathbf{r}_1) \) in a Taylor series about the stationary-phase point, \( \mathbf{r}_{1o} = \mathbf{r}_2 - \mathbf{k} \mathbf{v}_0 t_2 \).

It should be noted that Eq. (5.54) is not valid here, for we are not taking the limit \( t_1 \to -\infty \). The first two non-zero terms in the expansion are

\[
I = \left[ g(\mathbf{r}_{1o}) + \frac{i}{k} \left( \nabla \cdot g \right)_{\mathbf{r}_{1o}} \right] + \ldots,
\]

(5.93)

\[
p_o = \mathbf{v}_0 t_{21} = |\mathbf{r}_2 - \mathbf{r}_{1o}|.
\]

(5.94)

If we assume that \( g \) varies appreciably only in a distance of order \( R \) and that \( |p_o| \leq R \), the first term of Eq. (5.93) is sufficient provided \( kR \gg 1 \). The condition that \( |p_o| \leq R \)
will be satisfied if the points $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are both restricted to a region with maximum linear dimension $R$.

Returning to Eq. (5.90), one notes that the second term would normally possess two stationary-phase points. One such point arises from the integration over $\hat{\lambda}_2$ and occurs at

$$\hat{\lambda}_3' = \hat{\lambda}_3 - \hat{k}_e v_e t_3,$$

the second comes from the integration over $\hat{\lambda}_1$ and occurs at $\hat{\lambda}_3'' = \hat{\lambda}_3 + \hat{k}_e v_e t_3$. However, the first point is a statement of forward scattering, and must be excluded. The method of stationary phase thus enables us to give a precise mathematical meaning to the term, "omission of forward scattering." The second point will be excluded only if $\hat{k}_e$ differs so little from $\hat{\lambda}_3$ that it is indistinguishable from forward scattering. It will be demonstrated later that

$$\Theta > 1/\sqrt{AR}$$

is the condition for use of the second stationary-phase point. For this case, the second term of Eq. (5.90) becomes

$$-\frac{1}{4\pi i} \int_0^\infty \int d\lambda_3 \left\{ U'(\hat{\lambda}_3 + \hat{k}_e v_e t_3) \right\} e^{-\frac{i}{\hbar} \int_{\omega_0}^{\omega} V(\hat{\lambda}_3 + \hat{k}_e v_e t_3) d\omega} \times e^{i \frac{k}{\hbar} \int_{\omega_0}^{\omega} V(\hat{\lambda}_3 + \hat{k}_e v_e t_3) d\omega}$$

where we have used just the first term of Eq. (5.93). The integral over $t_3$ is easily carried out:

$$\int_0^\infty \left\{ U'(\hat{\lambda}_3 + \hat{k}_e v_e t_3) \right\} e^{-\frac{i}{\hbar} \int_{\omega_0}^{\omega} V(\hat{\lambda}_3 + \hat{k}_e v_e t_3) d\omega} = e^{i \left[ \frac{k}{\hbar} \int_{\omega_0}^{\omega} V(\hat{\lambda} + \hat{k}_e v_e t_3) d\omega \right]}$$

Thus Eq. (5.95) simplifies to

$$-\frac{1}{4\pi} \int d\lambda_3 \left\{ U(\hat{\lambda}_3) e^{i \frac{k}{\hbar} \int_{\omega_0}^{\omega} V(\hat{\lambda}_3) d\omega} \right\} e^{i \frac{\phi}{\hbar} (\hat{\lambda}_3)} \left[ e^{i \phi(\hat{\lambda}_3)} - 1 \right]$$

(5.97)
a term of first order in multiplicative factors of $U$. With this result, Eq. (5.90) becomes

$$f' \sim -\frac{1}{4\pi} (\phi_f, U \omega_o^+) - \frac{1}{4\pi} \int d\lambda_3' \ e^{i \cdot \lambda_3' \cdot \lambda_3^+} U(\lambda_3') \ e^{i \cdot \delta_0} \ e^{i \delta_0}$$

(5.98)

$$= -\frac{1}{4\pi} (\omega_{f}^+, U \omega_o^+)$$

(5.99)

$$= f_i'$$

(3.83)

When $\theta < \sqrt{1/R}$, both stationary-phase points must be excluded. The second term of Eq. (5.90) is then second order in multiplicative factors of the potential and must be dropped relative to the first term. Then we have

$$f \sim -\frac{1}{4\pi} (\phi_f, U \omega_o^+)$$

(5.100)

$$= f_i$$

(3.82)

In the text, the term "forward scattering" has often been used. Since we are describing the particle by wave mechanics, it is reasonable to expect some ambiguity in the interpretation of this term. To determine just what range of angles are included, we shall discuss the free propagation of a plane wave through a distance $R$. The pertinent equations are

$$\int_{+\infty}^{+\infty} \left( \begin{array}{c} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{array} \right) - i \omega t_2 = \int d\lambda_1 <21> \ e^{i \hat{\lambda}_1 \cdot \lambda_1 - i \omega t},$$

$$= \int_{+\infty}^{+\infty} \left( \begin{array}{c} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{array} \right) - i \omega t_2 \left( \frac{m}{\sqrt{2\pi i k t_{21}}} \right) \left( \frac{k |p|}{\sqrt{2\pi i k t_{21}}} \right) \left( \frac{i \omega}{\sqrt{2\pi i k t_{21}}} \right)$$

(5.101)

and $R = v_o t_{21} = \frac{h}{m} t_{21}$. The integral over $\hat{\lambda}_j = \hat{\lambda}_1 - \hat{\lambda}_2$, the stationary-phase point. Quantitatively, the
exponential does not oscillate rapidly in the range
\[ 1/t \approx \sqrt{2\pi\tau_{\text{m}}/m} \]; thus the important contributions to the
integral come from points in a sphere with center at \( \tau_{l} - v_{\text{a}} t_{l} \),
and radius \( \sqrt{2\pi\tau_{\text{m}}/m} = \sqrt{2\pi v_{\text{a}} \tau_{\text{m}}/v_{\text{a}}} = \sqrt{2\pi R} \). If we think
of the amplitude at \( (\tau_{l}, t_{l}) \) as a superposition of disturb­
ances propagated along straight rays from all points \( \tau_{l} \), at
time \( t_{l} \), then only those rays are important that lie in a cone
with vertex at \( \tau_{l} \) and half-angle \( \theta \sim \sqrt{\frac{2\pi}{kR}} \).
Thus, when the stationary-phase approximation is applicable
(\( kR \gg 1 \)), we must assign an angular spread of order \( \sqrt{\frac{2\pi}{kR}} \)
to the term "forward scattering." In the limit \( t_{l} \to -\infty \),
the distance \( R = v_{\text{a}} t_{l} \) becomes infinite, and the half-angle
\( \theta \) goes to zero, the classical result.

E. Exact Integral Equations For the Exact
Time-Development Operator

Having obtained an approximate time-development operator,
one wants to construct an integral equation for the exact \( U \),
with \( U_{\text{a}} \) as the inhomogeneous term. Clearly,
\[
U(t_{l}, t_{l}) = U_{\text{a}}(t_{l}, t_{l})
- \frac{i}{\hbar} \int_{t_{l}}^{t_{l}} d\tau_{3} U(t_{l}, \tau_{3}) \left[ \frac{i\hbar}{\hbar} \frac{\partial}{\partial \tau_{3}} - \mathbf{H} \right] U_{\text{a}}(\tau_{3}, t_{l})
\]
(5.102)
is an acceptable form, as may be verified by applying the op­
erator \( (i\hbar \frac{\partial}{\partial t_{l}} - \mathbf{H}) \) to both sides. The initial condition
\( U(t_{l}, t_{l}) = 1 \) is satisfied, for
\[
\langle \tau_{l} | U_{\text{a}}(t_{l}, t_{l}) | \tau_{l} \rangle = \langle \tau_{l} | U_{\text{a}}(t_{l}, t_{l}) | \tau_{l} \rangle
\]
(5.103)
In the coordinate representation, Eq. (5.102) becomes

\[ \langle 2|1 \rangle = \langle 2|1 \rangle_\alpha + \]

\[ -\frac{i}{\hbar} \int_{t_1}^{t_2} dt_3 \int d\lambda_3^{\ast} \langle 2|3 \rangle \left[ \frac{\hbar}{2m} \frac{\partial}{\partial t_3} + \frac{\hbar^2}{2m} \right] \langle \lambda_3^{\ast} \rangle \langle 3|1 \rangle_\alpha \]  

(5.104)

The operation

\[ \left[ \frac{i}{\hbar} \frac{\partial}{\partial t_3} + \frac{\hbar^2}{2m} \right] \langle 3|1 \rangle_\alpha = 0 \]

(5.104a)

is easily carried out by remembering that

\[ \left[ \frac{i}{\hbar} \frac{\partial}{\partial t_3} + \frac{\hbar^2}{2m} \right] \langle 3|1 \rangle_\alpha = 0 \]

(5.63)

and by using the identity

\[ (\lambda_3^{\ast} - \lambda_1) \cdot \dot{\lambda}_3^{\ast} \langle \lambda_3^{\ast} , \lambda_1 \rangle = \langle \lambda_3^{\ast} \rangle - \langle \lambda_3^{\ast} , \lambda_1 \rangle \]  

(5.105)

Then Eq. (5.104) becomes

\[ \langle 2|1 \rangle = \langle 2|1 \rangle_\alpha + \]

\[ -\frac{i}{\hbar} \int_{t_1}^{t_2} dt_3 \int d\lambda_3^{\ast} \langle 2|3 \rangle \left[ e^{-i\alpha(3,1)} \frac{\partial}{\partial t_3} + e^{i\alpha(3,1)} \right] \langle 3|1 \rangle_\alpha \]  

(5.106)

where

\[ \alpha(3,1) = -\frac{1}{\hbar} \dot{\lambda}_3^{\ast} \langle \lambda_3^{\ast} , \lambda_1 \rangle \]  

(5.107)

Taking the complex conjugate of Eq. (5.106), we have

\[ \langle 1|2 \rangle = \langle 1|2 \rangle_\alpha + \]

\[ +\frac{i}{\hbar} \int_{t_1}^{t_2} dt_3 \int d\lambda_3^{\ast} \langle 1|3 \rangle_\alpha \left[ e^{i\alpha(3,1)} \frac{\partial}{\partial t_3} + e^{-i\alpha(3,1)} \right] \langle 3|2 \rangle \]  

(5.108)

Permuting the space-time coordinates \((\lambda_2, t_2)\) and \((\lambda_1, t_1)\),

and using the fact that \(\alpha(3,1) = -\alpha(1,3)\), we can rewrite
Equation (5.106) will yield a solution for the wave function which involves the unknown Green's function $G^+$, while Eq. (5.109) leads to an integral equation for $\psi^+$. There is, thus, a simplicity in the time-dependent formalism in that the two expressions for $\langle z | 1 \rangle$ are quite similar, whereas the two forms of $\psi^+$ will prove to be very different.

1. Substituting Eq. (5.106) into Eq. (5.12), and using Eq. (5.54), the following expression for $\psi^+$ is obtained:

$$\psi^+_0 (\mathbf{r}_3) = \omega^+_0 (\mathbf{r}_3) + \frac{-i}{ic} \int_0^t dt_3 \int d\mathbf{r}_3 \langle z | 1 \rangle_3 \frac{\mathbf{k} \cdot \mathbf{v}_{3} - i \omega t_3}{\mathbf{k} \cdot \mathbf{v}_{3} + i \omega t_3} \sum_{\mathbf{r}_1} \left[ \mathbf{v} \cdot \mathbf{r}_1 \left( \mathbf{v} \cdot \mathbf{r}_1 \right) \right] \delta (\mathbf{r}_3 - \mathbf{k} + \mathbf{v} t_3) \mathbf{r}_1 \cdot \mathbf{r}_3$$

Using the integral definition of $G^+$, Eq. (5.16), we have the result of Section III. B.,

$$\psi^+_0 (\mathbf{r}_3) = \omega^+_0 (\mathbf{r}_3) + \int d\mathbf{r}_3 G^+ (\mathbf{r}_3, \mathbf{r}_5) \mathbf{v} \cdot \mathbf{r}_3 \left[ \mathbf{v} \cdot \mathbf{r}_3 \mathbf{k} \int_0^t \left. \mathbf{v} \cdot \mathbf{r}_3 - \mathbf{k} t \right| d\mathbf{r}_3 \right]$$

While the time-dependent method is more round-about, it has the advantage of clarifying the approximation scheme. In Eq.
(5.106), the first approximation is clearly the replacement of \( \langle 2 | 3 \rangle \) by \( \langle 2 | 3 \rangle_c \). This corresponds to replacing \( G^+ \) by \( G_{a}^+ \) in the time-independent case, Eq. (3.25).

2. The second integral equation, Eq. (5.109), generates the following form for \( \psi^+ \):

\[
\psi_o^+(\mathbf{x}_3) = \omega_o^+(\mathbf{x}_3) + \nonumber \]

\[
- \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \left[ \varphi_2^+ e^{i \omega (t,3)} \right] \psi_o^+(\mathbf{x}_3) e^{-i \omega t_3} .
\]

The time integration has the form

\[
- \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \left[ \varphi_2^+ \frac{\varphi_3}{\hbar} \nabla (\mathbf{x}_3, \mathbf{x}_3) + \right. \nonumber \]

\[
- \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \left. \nabla (\mathbf{x}_3, \mathbf{x}_3) \right] ,
\]

where

\[
\mathcal{E} (\mathbf{x}_3, \mathbf{x}_3) = E - \nabla (\mathbf{x}_3, \mathbf{x}_3) .
\]

As \( \mathcal{E} \) is independent of time, Eq. (5.113) may be rewritten as

\[
\left[ \left( \frac{\partial^2}{\partial t^2} \right) \mathcal{E} + \frac{1}{4 \pi} \frac{\partial^2}{\partial x^2} \mathcal{E} \right] \times \nonumber \]

\[
\left( - \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \right) .
\]

The integral is just \( G_{a}^+(\mathbf{x}_3, \mathbf{x}_3) \) [Eq. (5.76)], and the derivatives are evaluated through the relation \( \mathcal{E} = \kappa^+ \mathcal{K} \mathcal{I} \mathcal{M} \).

Thus Eq. (5.113) becomes

\[
\left[ \left( \frac{\partial^2}{\partial t^2} \right) \mathcal{E} + \frac{1}{4 \pi} \frac{\partial^2}{\partial x^2} \mathcal{E} \right] \times \nonumber \]

\[
\left( - \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \right) .
\]

Thus Eq. (5.115) becomes

\[
\left[ \left( \frac{\partial^2}{\partial t^2} \right) \mathcal{E} + \frac{1}{4 \pi} \frac{\partial^2}{\partial x^2} \mathcal{E} \right] \times \nonumber \]

\[
\left( - \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \right) .
\]

Thus Eq. (5.115) becomes

\[
\left[ \left( \frac{\partial^2}{\partial t^2} \right) \mathcal{E} + \frac{1}{4 \pi} \frac{\partial^2}{\partial x^2} \mathcal{E} \right] \times \nonumber \]

\[
\left( - \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \right) .
\]

Thus Eq. (5.115) becomes

\[
\left[ \left( \frac{\partial^2}{\partial t^2} \right) \mathcal{E} + \frac{1}{4 \pi} \frac{\partial^2}{\partial x^2} \mathcal{E} \right] \times \nonumber \]

\[
\left( - \frac{\hbar}{4 m_i} \int_{-\infty}^{t_3} dt' \int d\mathbf{x}_3 \langle 2 | 3 \rangle_c \right) .
\]
The integral equation now becomes
\[ \psi_0^+(\lambda_1) = \omega_0^-(\lambda_2) - \int d\lambda_3 \, G_{\alpha}^+(\lambda_1, \lambda_3) \, W_\lambda(\lambda_3, \lambda_2) \, \psi_0^+(\lambda_3) . \]  
(5.120)

The meaning of \( W_\lambda(\lambda_3, \lambda_2) \) is made somewhat more transparent by determining the differential equation satisfied by \( G_{\alpha}^+(\lambda_3, \lambda_2) \). One finds that
\[ \left\{ \frac{\partial^2}{\partial \lambda_3^2} + \omega^2 - [\nu(\lambda_3) - W_\lambda(\lambda_3, \lambda_2)] \right\} G_{\alpha}^+(\lambda_3, \lambda_2) = \]
\[ = -\delta(\lambda_3 - \lambda_2) . \]  
(5.121)

As \( G_{\alpha}^+(\lambda_3, \lambda_2) \) is the Green's function corresponding to the forward propagator, the simplest interpretation of \( \nu(\lambda_3) - W_\lambda(\lambda_3, \lambda_2) \) is as an effective potential which generates only small-angle scattering between the points \( \lambda_3 \) and \( \lambda_2 \).

The effective potential \( W_\lambda \) is a complicated function of the coordinates, but it has two important limiting cases. The first is in the high-energy limit. Then, to first order in \( \nu \), one has
\[ K \sim k \left[ 1 - \frac{1}{2} \nu(\lambda_3, \lambda_2) \right] . \]  
(5.77)

which, when substituted in Eq. (5.119), leads to \( W_\lambda(\lambda_3, \lambda_2) \) etc.
\( \mathcal{W}'(\hat{x}_3', \hat{x}_3) \), where

\[
\mathcal{W}'(\hat{x}_3', \hat{x}_3) = \mathcal{W}(\hat{x}_3', \hat{x}_3) + i \mathcal{W}(\hat{x}_3', \hat{x}_3)
\]

(5.122)

\[
\delta(\hat{x}_3', \hat{x}_3) = -\frac{i}{4\pi} \int_{0}^{\infty} \mathcal{W}(\hat{x}_3', \hat{x}_3) d\omega
\]

(5.123)

Coupled with the high-energy approximation to \( \mathcal{G}_\alpha^\dagger \), Eq. (5.79), Eq. (5.120) becomes

\[
\psi_\alpha^\dagger(\hat{x}_3) = \omega_\alpha^\dagger(\hat{x}_3) + \frac{i}{4\pi} \int_{0}^{\infty} \mathcal{W}(\hat{x}_3', \hat{x}_3) \psi_\alpha^\dagger(\hat{x}_3') d\omega
\]

(5.124)

This equation is the integral equation of Saxon and Schiff (9) and is exact, in spite of the way in which it has been obtained here.

The second limiting case is arrived at in the following way. In Section III. B., we had a potential \( \mathcal{U}(\hat{x}_3) \) which generated only forward scattering along the direction \( \hat{\kappa}_0 \).

Above, we interpreted \( \mathcal{U}(\hat{x}_3) - \mathcal{W}(\hat{x}_3', \hat{x}_3) \) as a potential that generates only forward scattering from \( \hat{x}_3 \) to \( \hat{x}_3' \). This suggests that if, in \( \mathcal{W}_L \), we let \( \hat{x}_3 \) go to minus infinity as

\[
\lim_{|\hat{x}_3| \to \infty} [ -\hat{\kappa}_0, |\hat{x}_3| ]
\]

we should recover \( \mathcal{U}_S(\hat{x}_3) \).

In this limit we have \( \hat{\kappa} \to 0, \hat{\omega} \to 0, \kappa \to \kappa \), and the effective potential \( \mathcal{W}_L(\hat{x}_3', \hat{x}_3) \) reduces to

\[
\mathcal{U}_L(\hat{x}_3) = -\frac{i}{2} \mathcal{W}_L(\hat{x}_3', \hat{x}_3)
\]

(3.34)

The difference \( \mathcal{U}(\hat{x}_3) - \mathcal{U}_L(\hat{x}_3) \) is just \( \mathcal{U}_S(\hat{x}_3) \), by use of Eq. (3.35).
F. Summary

A time-dependent formulation of non-relativistic scattering theory was set up by the introduction of an approximate time-development operator \( \mathcal{U}_\alpha \). The coordinate representation of this operator is given by Eq. (5.41). The approximate high-energy wave function, Eq. (5.61), and an approximate Green's function, Eq. (5.76), were obtained from this propagation function by use of Eqs. (5.12) and (5.16). A time-dependent stationary-phase approximation and an intuitive integral equation relating \( \mathcal{U} \) and \( \mathcal{U}_\alpha \), Eq. (5.86), lead to the Schiff (5) formulas, Eqs. (5.99) and (5.100). Exact integral equations for \( \mathcal{U} \), with \( \mathcal{U}_\alpha \) as inhomogeneous term, are given by Eqs. (5.106) and (5.109), with the corresponding expressions for \( \psi^* \) to be found in Eqs. (3.25) and (5.120).
VI. METHOD OF PARTIAL WAVES

A. Introduction

An alternate approach to scattering theory is the method of partial waves. It is applicable in the important case of a spherically symmetric potential $v(r)$. The incident plane wave is analysed into elementary partial waves of angular momentum $\ell$, and each experiences a phase shift $\delta_\ell$ owing to the effect of the potential. Therefore, after interaction, the sum of these waves is no longer a plane wave but is distorted, corresponding to the scattering which has taken place. The cross section is determined by the $\delta_\ell$.

The reasons for studying this method may be listed as follows:

1. When $kR$ is not large, only the first few $\delta_\ell$ are needed for computing cross sections.

2. Since the potential depends only on $r$, the Schrödinger equation separates, and one has to solve only an ordinary differential equation. If exact solutions cannot be found, approximations for $\delta_\ell$ may be set up which involve only one integration.

3. Approximations of the WKB type are particularly applicable to the radial equation if the standard conditions on the potential, Eq. (2.4), are satisfied. One expects, therefore, a connection between the phase-shift analysis and the three-dimensional high-energy approximation.
Although they will not be studied here, specific scattering problems for which a semi-classical analysis is valid are conveniently dealt with by the phase-shift analysis (21).

The rest of Section VI. A. is devoted to a brief summary of the phase-shift method and the derivation of integral expressions for $J_{l}$. In Section VI. B. the WKB approximation is discussed. Section VI. C. contains alternate approximations which are valuable in determining simplified expressions for $J_{l}$ when $l$ is small and when $l$ is large.

1. Summary of results of partial-wave analysis

The problem is axially symmetric about the direction of motion of the incident particles, say the $z$-axis. The solution of the Schrödinger equation,

$$\left[ \nabla^2 + k^2 - U(r) \right] \psi(r) = 0 \quad (6.1)$$

is

$$\psi(r) = \sum_{\ell=0}^{\infty} a_{\ell} R_{\ell}(\lambda) P_{\ell}(\cos \theta) \quad (6.2)$$

where $r$ and $\theta$ are spherical coordinates. $R_{\ell}$ satisfies the radial equation,

$$\frac{d^2}{dr^2} (\lambda^2 R_{\ell}) + \left[ k^2 - U(r) - \ell(\ell+1) \right] (\lambda^2 R_{\ell}) = 0 \quad (6.3)$$

and $P_{\ell}(\cos \theta)$ is the Legendre polynomial of degree $\ell$.

When $U(\lambda) = 0$, the desired solution to Eq. (6.1) is $e^{ikz}$. 

4. Although they will not be studied here, specific scattering problems for which a semi-classical analysis is valid are conveniently dealt with by the phase-shift analysis (21).

The rest of Section VI. A. is devoted to a brief summary of the phase-shift method and the derivation of integral expressions for $J_{l}$. In Section VI. B. the WKB approximation is discussed. Section VI. C. contains alternate approximations which are valuable in determining simplified expressions for $J_{l}$ when $l$ is small and when $l$ is large.

1. Summary of results of partial-wave analysis

The problem is axially symmetric about the direction of motion of the incident particles, say the $z$-axis. The solution of the Schrödinger equation,

$$\left[ \nabla^2 + k^2 - U(r) \right] \psi(r) = 0 \quad (6.1)$$

is

$$\psi(r) = \sum_{\ell=0}^{\infty} a_{\ell} R_{\ell}(\lambda) P_{\ell}(\cos \theta) \quad (6.2)$$

where $r$ and $\theta$ are spherical coordinates. $R_{\ell}$ satisfies the radial equation,

$$\frac{d^2}{dr^2} (\lambda^2 R_{\ell}) + \left[ k^2 - U(r) - \ell(\ell+1) \right] (\lambda^2 R_{\ell}) = 0 \quad (6.3)$$

and $P_{\ell}(\cos \theta)$ is the Legendre polynomial of degree $\ell$.

When $U(\lambda) = 0$, the desired solution to Eq. (6.1) is $e^{ikz}$. 

4. Although they will not be studied here, specific scattering problems for which a semi-classical analysis is valid are conveniently dealt with by the phase-shift analysis (21).
The solution to the radial equation that vanishes at the origin is \( \frac{\partial}{\partial r} j_x(kr) \), where \( j_x \) is the spherical Bessel function defined as

\[
j_x(kr) = \sqrt{\frac{k}{2\pi r}} J_{\frac{x}{2}}(kr) .
\]

(6.4)

The plane wave can be written as a sum of partial waves,

\[
e^{ikr} = \sum_{l=-\infty}^{\infty} (2l+1) i^l j_x(kr) P_l(\cos \theta) .
\]

(6.5)

The asymptotic form of \( j_x(kr) \) is important:

\[
j_x(kr) \xrightarrow{r \to \infty} \frac{1}{kr} \sin(kr - \frac{\pi}{2} + i\delta) .
\]

(6.6)

In the case of a potential \( V(r) \) that goes to zero faster than \( kr \) at large \( r \), the asymptotic form of \( R_x \) must have the same \( r \) dependence as that of a free particle, but will differ in phase by an amount \( \delta_x \):

\[
R_x(r) \xrightarrow{r \to \infty} \frac{1}{kr} \sin(kr - \frac{\pi}{2} + \delta_x) .
\]

(6.7)

To determine the scattering amplitude \( f(\theta) \), we use the asymptotic form of \( \psi(r) \),

\[
\psi(r) \xrightarrow{r \to \infty} e^{ikr} + f(\theta) \frac{e^{ikr}}{r} .
\]

(6.8)

Substituting Eqs. (6.2) and (6.5) into Eq. (6.8), and using the asymptotic forms of \( R_x \) and \( j_x \), one obtains

\[
f(\theta) = \frac{1}{2ik} \sum_{l=-\infty}^{\infty} (2l+1) e^{il\delta_x} \int_{-\infty}^{\infty} \frac{dx}{x^2} P_l(\cos \theta) ,
\]

(6.9)

\[
da_x = (2l+1) i^l e^{i\delta_x} .
\]

(6.10)
By use of Eq. (6.10), Eq. (6.2) becomes

\[ \psi (x) = \sum_{j=0}^{\infty} i^{x j^2} R_{1/2}(x) \cos \theta . \]  

(6.11)

The total cross section is

\[ \sigma_{\text{total}} = \int |f(\theta)|^2 d\theta = \frac{4\pi}{\hbar^2} \sum_{j=0}^{\infty} (2j+1) \sin^2 \theta . \]  

(6.12)

Therefore, if the \( s_j \) are known, one can find \( f(\theta) \) and \( \sigma_{\text{total}} \).

2. Integral expressions for \( s_j \)

Expressions for \( s_j \) in terms of the exact radial wave function may be obtained in the following way. The radial equation with potential \( U \) is

\[ \frac{d^2}{dx^2}\hat{h}_e + \left[ k^2 - U(x) - \frac{x(x+1)}{r^2} \right] \hat{h}_e = 0 , \]  

(6.13)

where

\[ \hat{h}_e = k r R_e , \]

\[ \hat{h}_e \xrightarrow{\lambda \to 0} 0 , \]

\[ \hat{h}_e \xrightarrow{\lambda \to \infty} \sin (k r - \frac{1}{\lambda} \theta + s_e) . \]  

(6.14)

The radial equation without potential is

\[ \frac{d^2}{dx^2}\hat{g}_e + \left[ k^2 - \frac{x(x+1)}{r^2} \right] \hat{g}_e = 0 , \]  

(6.15)

where

\[ \hat{g}_e = k r \hat{g}_e (k r) , \]

\[ \hat{g}_e \xrightarrow{\lambda \to 0} 0 , \]

\[ \hat{g}_e \xrightarrow{\lambda \to \infty} \sin (k r - \frac{1}{\lambda} \theta) . \]  

(6.16)
In addition, let us define \( \omega_2 \) as a general approximation to \( \hat{\omega}_2 \) which satisfies a differential equation of the form
\[
\frac{d^2 \omega_2}{dx^2} + \left[ \frac{x}{\lambda} - V(x) - \frac{\lambda_2(x)}{\lambda_2} \right] \omega_2 = 0
\]
and the boundary conditions
\[
\omega_2 \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty
\]
\[
\omega_2 \rightarrow \sin \left( \frac{\pi \lambda_2}{2} + \eta \right) \quad \text{as} \quad x \rightarrow \infty
\]
\( \lambda_2(\lambda) \) is a measure of the error in replacing \( \hat{\omega}_2 \) by \( \omega_2 \).

For example, if \( \omega_2 = g_2 \), we have \( \lambda_2 = \lambda = - V(x) \), \( \eta = 0 \).
We will use Eqs. (6.13) to (6.18) to derive two integral expressions for \( \delta_2 \).

a. We substitute \( \hat{\omega}_2 \) and \( g_2 \) into a one-dimensional Green's theorem,
\[
\hat{\omega}_2 \frac{d g_2}{dx} - g_2 \frac{d \hat{\omega}_2}{dx} \int_0^\infty \left[ \hat{\omega}_2 \frac{d g_2}{dx} - g_2 \frac{d \hat{\omega}_2}{dx} \right] dx
\]

The left-hand side vanishes at the lower limit, for \( \hat{\omega}_2 \) and \( g_2 \) are zero. If the upper limit is made very large, the asymptotic forms of \( \hat{\omega}_2 \) and \( g_2 \) may be substituted, yielding \( k \sin \delta_2 \). On the right-hand side, we replace \( R \) by \( \infty \), and substitute Eqs. (6.13) and (6.15). The result is
\[
\sin \delta_2 = -\frac{i}{k} \int_0^\infty V(x) g_2(\lambda x) \hat{\omega}_2(\lambda x) \lambda x d\lambda
\]

From classical considerations, we expect that \( l < \lambda R \) if the \( \lambda R \) partial wave is to be scattered. In the special case \( \lambda R \ll 1 \), the only important phase shift should then be \( \delta_0 \).
Quantum mechanically, the wave functions \( g_k \) and \( \tilde{g}_k \) go rapidly to zero for \( r < r_0 \), the turning point, at which the square bracket in Eq. (6.13) changes sign. If the range of the potential satisfies \( R < r_0 \), the wave function will be very small where \( U(r) \) is non-zero, and \( \delta_k \) will be very small.

The most obvious approximation to Eq. (6.20) is the replacement of \( \tilde{g}_k \) by \( g_k \) to give

\[
\sin \delta_k \approx -k \int_0^\infty \tilde{g}_k(r) U(r) \, d\rho
\]

since \( g_k \) is the free radial function, this approximation resembles the Born approximation. To obtain a validity condition for Eq. (6.21), we assume a monotonically decreasing potential for increasing \( r \), so that \( \delta_k \) is the largest phase shift. Since \( \tilde{g}_k = \sin^2 \frac{k \rho}{r_0} = (1 - \cos \frac{2k \rho}{r_0}) \), Eq. (6.21) becomes

\[
\sin \delta_k \approx -
\frac{k}{r_0} \int_0^\infty \tilde{g}_k(r) U(r) \left( 1 - \cos \frac{2k \rho}{r_0} \right)
\]

If \( k \) is large, and \( U \) varies slowly in a wavelength, the cosine term can be dropped, and the order of magnitude of the integral is \( \frac{U R}{k} \). Therefore, the parameter \( \frac{U R}{k} \) cannot be greater than unity. We have seen that the Born approximation requires \( \frac{U R}{k} \ll 1 \). Hence, Eq. (6.21) is equivalent to the Born approximation only when all the \( \delta_k \) are small.

b. A second drawback to the above approximation is that the asymptotic form of the approximate wave function \( g_k \) gives no estimate of the phase shift. If one uses \( \omega_k \), with
approximate phase shift $\eta_k$, Eq. (6.20) must be modified. Therefore, we substitute $\hbar_k$ and $\omega_k$ in the one-dimensional Green's theorem,

$$\hbar_k \omega_k' - \omega_k' \hbar_k \bigg|_0^R = \int_0^R \left( \hbar_k \omega_k'' - \omega_k \hbar_k'' \right) \, d\lambda \quad (6.23)$$

As before, we take the limit of large $R$ and substitute the asymptotic forms of $\hbar_k$ and $\omega_k$ in the left-hand side of Eq. (6.23). Using Eqs. (6.13) and (6.17) on the right-hand side, we obtain

$$\sin (\delta_k - \eta_k) = \frac{1}{\pi} \int_0^\infty d\lambda \ h_k(\lambda) \ A_k(\lambda) \ W_k(\lambda) \quad (6.24)$$

If $A_k(\lambda)$ is very small, $\eta_k$ will be a good first approximation to $\delta_k$.

**B. WKB Approximation**

We will now apply Eq. (6.24) to the WKB approximation of the radial wave function. As the range of integration in Eq. (6.24) is from zero to infinity, we must find a WKB wave function which is valid for the entire range of $\lambda$.

1. **$l = 0$**

   When $l = 0$, the radial equation reduces to

   $$\frac{\partial^2}{\partial \lambda^2} h_0 + \kappa^2 h_0 = 0 \quad , \quad (6.25)$$

   $$\kappa^2 = k^2 - U(\lambda) \quad , \quad (6.25a)$$

   This equation is identical with the Cartesian case. For potentials $U(\lambda)$ which are slowly varying, finite, and less
than \( k^* \) in magnitude for all \( a \), the WKB approximation is

\[
\overline{w}_o = \frac{c}{\sqrt{\kappa}} \sin \int_0^a \kappa \, \text{d}r \quad (6.26)
\]

The sine function is used rather than an exponential function, for \( \overline{w}_o \) must go to zero at \( r = 0 \). The function \( \overline{w}_o \) satisfies the following differential equation:

\[
\frac{d^2 \overline{w}_o}{dr^2} + \kappa^2 \overline{w}_o = \overline{\lambda}_o \overline{w}_o \quad (6.27)
\]

where

\[
\overline{\lambda}_o = \frac{5}{16} \left( \frac{\nu'}{\kappa^2} \right)^2 + \frac{1}{4} \frac{\nu''}{\kappa^2} \quad (6.28)
\]

We assume a potential that goes to zero at sufficiently large distances. Then the asymptotic form of \( \overline{w}_o \) becomes

\[
\overline{w}_o \xrightarrow{r \to \infty} \frac{c}{\sqrt{\kappa}} \sin \left[ \kappa r + \left( \int_0^a \kappa \, \text{d}r - \kappa r \right) \right] \quad (6.29)
\]

To apply Eq. (6.24), one takes

\[
c = \sqrt{\kappa} \quad (6.29a)
\]

\[
\eta = \lim_{r \to \infty} \left( \int_0^r \kappa \, \text{d}r - \kappa r \right) \quad (6.30)
\]

When \( \nu \ll \kappa \) and \( \nu \kappa (\nu \kappa) \ll 1 \), Eq. (6.30) simplifies to

\[
\kappa \approx -\frac{1}{1} \int_0^\infty \nu (r) \, \text{d}r \quad (6.31)
\]

Equation (6.24) becomes

\[
\sin \left( \delta_0 - \eta \right) = \frac{1}{\kappa} \int_0^\infty \nu_h (r) \, \overline{\lambda}_o (r) \, \overline{w}_o (r) \quad (6.32)
\]

To first approximation, \( h_o \) is replaced by \( \overline{w}_o \). To estimate
the magnitude of the right-hand side of Eq. (6.32), we assume

$$V << k^2, \quad \overline{w}^2 \text{ is always positive and of order unity. } \overline{a}. \text{ is of order } V/k^2. \text{ The magnitude of } \sin (\delta - \eta) \text{ is therefore of order } V/k^2 \approx V/k^2 \cdot \overline{a}. \text{ This is small if } kR >> 1.$$

2. $\lambda \neq 0$

When $\lambda > 0$, one must always deal with a turning point in the radial wave equation. A WKB wave function which holds throughout an interval containing the turning point was derived by Langer (22). It can be expressed as

$$\omega = \sqrt{\frac{k^2}{\mu}} \int_{q_1}^{q_2} \left[ J_{\nu} (\tau) + J_{-\nu} (\tau) \right], \quad (6.33)$$

where

$$J_{\nu} (\tau) = \int_{q_1}^{q_2} q_1 (\tau') d\tau', \quad (6.34)$$

$$\begin{cases} Q_1 (\tau) = \sqrt{k^2 - v(\tau) - (\ell + \lambda)^2} \\ Q_1 (\tau_1) = 0 \end{cases}, \quad (6.35)$$

Equation (6.33) vanishes at the origin and has the asymptotic form of Eq. (6.18), with WKB phase shift

$$\eta_\lambda = \lim_{\lambda \to \infty} \left[ \int_{q_1}^{q_2} \sqrt{k^2 - v(\tau)} - (\ell + \lambda)^2 \right] - \frac{1}{\mu} \int_{q_1}^{q_2} \sqrt{k^2 - (\ell + \lambda)^2}.

(6.36)$$

A detailed discussion of the derivation of Eqs. (6.33) and (6.36) will be found in Appendix B. The material presented there is not new, but it is not found readily in any one text.

In order to apply Eq. (6.24), we must determine $\lambda_\lambda$. \hfill
Equation (B.36) may be rewritten to yield the following differential equation satisfied by $\omega_k$:

$$\frac{d^2 \omega_k}{dx^2} + \left[ k^2 - u(x) - \frac{\lambda (\lambda + 1)}{x^2} \right] \omega_k = 0$$

$$= \left[ \frac{1}{x^2} - \frac{5}{2} \frac{Q_k}{x^2} + \frac{5}{16} \left( \frac{\dot{V}_k}{Q_k} \right)^2 + \frac{1}{4} \frac{V_k''}{Q_k} \right] \omega_k \quad (6.37)$$

$$\lambda_k (x) = \frac{1}{4} x^2 - \frac{5}{3} \frac{Q_k}{x^2} + \frac{5}{16} \left( \frac{\dot{V}_k}{Q_k} \right)^2 + \frac{1}{4} \frac{V_k''}{Q_k} \quad (6.38)$$

Then, substituting Eqs. (6.33), (6.36), and (6.38) in Eq. (6.24), we have

$$\sin (\lambda_k - \eta_k) = \frac{1}{\eta_k} \int_0^\infty dx \lambda_k (x) \lambda_k (x) \omega_k (x) \quad (6.39)$$

$$= \frac{1}{\eta_k} \int_0^\infty dx \lambda_k (x) \omega_k (x) \quad (6.40)$$

The effect on $\lambda_k$ of using $Q$ instead of $Q$ is to add the factor $1/4 \lambda_k$. It is seen from Eq. (B.39) that $\Theta (\lambda_k)$ approaches the value $-\frac{3}{4} \lambda_k$, for $\lambda_k$ close to zero. Since $\lambda_k (x) = \frac{1}{4} x^2 + \Theta (x)$, $\lambda_k (x) \approx -\frac{3}{4} \lambda_k$. Therefore, the use of $Q$ has improved the approximation near the turning point by a factor of about 10.

3. Comments on WKB approximation

Unlike the Born approximation, the WKB approximation does not require the phase shift $\eta_k$ to be small in magnitude. It may be expected to be accurate when the scattering potential
is large, in the sense that \( \frac{V_{R/k}}{kR} \gg 1 \), and kR is large. The particle then suffers many deflections in the scattering process, and approximates the classical behavior. If \( \frac{V_{R/k}}{kR} \) is large, \( \eta_{k} \) will be large for small \( l \). Because the right-hand side of Eq. (6.39) cannot exceed unity, and will usually be much smaller, the percentage error in replacing \( \delta_{\bar{R}} \) by \( \eta_{k} \) will be small when \( \eta_{k} \) is very large. However, it is seldom \( \delta_{\bar{R}} \) alone that is wanted, for example, we must know \( \sin \delta_{R} \) to determine the total cross section. Therefore, \( \sin (\delta_{R} - \eta_{k}) \) must be small if the phase-shift approximation is to be useful.

The expressions for \( \omega_{k} \) and \( \Lambda_{k}(\lambda) \) are complicated, and it is difficult to estimate the \( l \) dependence. However, the right-hand side of Eq. (6.39) will decrease as \( l \) increases, for \( \Theta(\lambda_{i}) \) will decrease and the approximation of \( \omega_{k} \) by \( \omega_{k} \) will be improved. In addition, \( \omega_{k} \) varies as \( \lambda^{l+1} \) near the origin, so that for large \( l \), the region where \( \omega_{k} \) is biggest does not contribute to the integral in Eq. (6.40).

\( \eta_{k} \) also decreases as \( l \) increases, and will generally go to zero at a faster rate than \( \delta_{\bar{R}} \). This behavior follows because \( \eta_{k} \) has been determined solely by the semi-classical features of the problem. The right-hand side of Eq. (6.39) contains the purely quantum mechanical features. For example, if \( U(\lambda) \) is bounded in space, \( \eta_{k} \) will be zero for any \( l \), whose corresponding turning point \( \lambda_{i} \) lies outside the potential. The right-hand side will not be zero, in general, for the wave function describing the particle is not zero in the
classically inaccessible region, and the particle has a non-zero probability of being scattered. Equation (6.21) is the simplest way of obtaining approximate values of \( \delta_\ell \) for \( \ell \geq \ell_1 \).

C. Alternate Approximations

The results of Section VI. B. are sufficiently complicated to obscure the simplifications produced in the high-energy limit. The presence of the centrifugal potential in \( q_\ell \) prevents an expansion of the square root when \( V \ll k^{-1} \). In addition, when \( U(\lambda) \) is set equal to zero, \( \Lambda_\ell (\lambda) \) is not zero, and \( \omega_\ell (\lambda) \) does not reduce to \( \hat{\gamma}_\ell (k \lambda) = k \lambda \hat{\gamma}_\ell (k \lambda) \), the exact solution. This is a fundamental difference between the WKB solutions in \( x \)-space and \( r \)-space. Finally the \( \ell \) dependence is masked. In the following, we shall use the analogy of the three-dimensional high-energy approximation to obtain a high-energy expression for \( \delta_\ell \), useful when \( \lambda \) is small compared with \( kR \). In the range of extremely high incident energies, this expression reduces to a form with a very simple \( \lambda \) dependence. For \( \lambda \) large compared with \( kR \), a simplified expression for \( \delta_\ell \) is found, which is in correspondence with the three-dimensional high-energy formulation.

1. Small \( \ell \) \( \left( \sqrt{\lambda (\ell + 1)} \ll kR \right) \)

   a. We assume that the magnitude of \( U(\lambda) \) is always less than \( k^{-1} \), which requires that \( U(\lambda) \) be non-singular. The case of \( \ell = 0 \) was studied separately in Section VI. B.
There, the wave function,

$$\tilde{\omega}_0(\lambda) = \sqrt{\frac{k}{R}} \sin \int_0^\lambda k \, d\lambda \quad ,$$  

(6.26)

reduced to the correct wavefunction when \( U(\lambda) = 0 \), and

$$K = \sqrt{k^2 - U(\lambda)}$$

was easily simplified when \( U \ll k^2 \).

With \( \rho = \int_0^\lambda k \, d\lambda \), Eq. (6.26) may be rewritten as

$$\tilde{\omega}_0(\lambda) = \sqrt{\frac{k}{R}} \rho \, f_0(\rho) \quad .$$  

(6.41)

An obvious generalization to the case \( \lambda \neq 0 \) is

$$\tilde{\omega}_k(\lambda) = \sqrt{\frac{k}{R}} \rho \, f_k(\rho) \quad .$$  

(6.42)

This expression has the same useful characteristics associated with \( \tilde{\omega}_0 \); in addition, when \( U \ll k^2 \), the argument of \( f_k \) becomes \( k \lambda - \frac{1}{2} k \lambda \). Hence, as in the three-dimensional analysis, the magnitude of \( UR/k \) need not be restricted.

The wave function \( \tilde{\omega}_k \) has the serious drawback that its phase shift is independent of \( \lambda \). For, the asymptotic form of \( \tilde{\omega}_k \) is

$$\tilde{\omega}_k(\lambda) \sim_{\lambda \to \infty} \sin \left( \rho - \frac{\pi}{2} k \lambda \right) = \sin \left( k \lambda - \epsilon \frac{\pi}{2} k + \eta \right) \quad ,$$  

(6.43)

where

$$\eta = \lim_{\lambda \to \infty} \left[ \int_0^\lambda \sqrt{k^2 - U(\lambda')} \, d\lambda' - k \lambda \right] \quad .$$  

(6.30)

It is expected, therefore, that a formulation based on \( \tilde{\omega}_k \) will be useful only when \( \lambda \) is small; that is, when \( \delta_k \) does not differ greatly from \( \eta \).
The differential equation satisfied by \( \overline{\omega}_x \) is
\[
\frac{d^k}{d\lambda^k} \overline{\omega}_x + \left[ k^2 - v(\lambda) - \ell(\ell + 1) \lambda \right] \overline{\omega}_x = \] 
\[
= \left[ \xi_k \left( \frac{v'}{k^2} \right) + \frac{1}{4} \frac{v''}{k^2} + \ell(\ell + 1) \left( \frac{k^2}{\lambda^2} - \frac{1}{\lambda} \right) \right] \overline{\omega}_x \]  \hfill (6.44)

The coefficient of \( \overline{\omega}_x \) on the right-hand side of Eq. (6.44) is defined to be \( \overline{\omega}_x(\lambda) \). As \( v \) is always less than \( k^2 \), and \( K \) is independent of \( \lambda \), the first two terms in \( \overline{\omega}_x(\lambda) \) are always well-behaved. In fact, they are equal to \( \overline{\omega}_x \). The only \( \lambda \) dependence in \( \overline{\omega}_x \) is the multiplicative factor \( \ell(\ell + 1) \).

The quantity \( (k+1)^2 - \frac{1}{\lambda^2} \) must be less singular than \( \frac{1}{\lambda^2} \) at the origin in order that \( \overline{\omega}_x \) will go to zero correctly.

Expanding \( k+1 \) about \( \lambda = 0 \), we obtain
\[
\frac{k+1}{\lambda} \xrightarrow{\lambda \to 0} -\frac{1}{\lambda} \left( \frac{u'}{\lambda^2} \right) \xrightarrow{\lambda \to 0} \frac{1}{\lambda} \]  \hfill (6.45)

Hence, there are no difficulties with this approximation insofar as the differential equation is concerned. Substituting \( \eta \) and \( \overline{\omega}_x \) into Eq. (6.24), we obtain
\[
k \sin (\lambda, -\eta) = \int_0^{\infty} \lambda^2 (\lambda, \eta) \left[ \xi_k \left( \frac{v'}{k^2} \right) + \frac{1}{4} \frac{v''}{k^2} + \ell(\ell + 1) \left( \frac{k^2}{\lambda^2} - \frac{1}{\lambda} \right) \right] \overline{\omega}_x(\lambda) \]  \hfill (6.46)

b. In the high-energy approximation, the following simplifications result:

1. \( \eta = \frac{1}{k} \int_0^{\infty} \sqrt{k^2 - v} \, d\lambda = k \lambda \) \hfill (6.30)

\[
\overline{\omega}_x - \frac{1}{k} \int_0^{\infty} u(\lambda) \, d\lambda - \left( \frac{1}{k} \right)^3 \int_0^{\infty} u^2(\lambda) \, d\lambda + \cdots = \eta \]  \hfill (6.47)
We assume here that \( \psi_\lambda^+ (u/K) \ll 1 \).

5. Equation (6.53) satisfies
\[
\frac{d^2 \vec{\omega}_k}{d\lambda^2} + \left\{ \frac{k^2}{\lambda^2} - U(\lambda) - \frac{1}{4k^2} U'' \right\} + \frac{1}{\lambda^2} \left[ \frac{\lambda^2}{k} - k \lambda \int_0^\lambda u(\lambda) \, d\lambda \right] \vec{\omega}_k = 0 \tag{6.54}
\]
Here, terms in \( u/\lambda^2 \) arising from the expansion of \( K \) have been neglected as indicated by the arrow. The extra terms shown are the errors arising from the form of \( \vec{\omega}_k \) itself.

6. If \( f_k \) is replaced by \( \omega_k \), and the high-energy forms of \( \omega_k, \lambda, \) and \( \lambda_k(\lambda) \) are used, Eq. (6.48) becomes
\[
\sin (x_k - n_0) \propto \frac{1}{k} \int_0^{\hat{k}} \tilde{\omega}_k (k - h_0 \int_0^\lambda u(\lambda) \, d\lambda) \, d\lambda \frac{\lambda^2}{k} - \frac{1}{\lambda^2} \int_0^\lambda \tilde{\omega}_k (k - h_0 \int_0^\lambda u(\lambda) \, d\lambda) \left[ U(\lambda) - \bar{U}(\lambda) \right] \tag{6.55}
\]
The quantity \(- \frac{\lambda}{k} \int_0^\infty \psi \, d\psi\) is retained in the argument of \(\psi_k\) because \(\psi_k\) is an oscillatory function. The parameter \(\psi_k\) is restricted only in that \(\psi_k \ll kR\) must be small. The replacement of \(\psi_k\) by \(\overline{\psi}_k\) will be a good approximation for small \(l\), \(\psi_k \ll 1\), \(kR \gg 1\),

\[
\psi_k \ll 1.
\]

This follows because \(\overline{\psi}_k\) satisfies a differential equation differing from the radial equation for \(\psi_k\) only by second-order terms if \(\psi_k \ll 1\) and \(kR \gg 1\). With respect to the boundary conditions, \(\overline{\psi}_k\) goes to zero at \(\lambda = a\) and, if \(\psi_k \ll 1\), so that \(\eta_e\) is a good approximation to \(\phi_e\), the asymptotic form of \(\overline{\psi}_k\) will be a good approximation for small \(l\) \((l \ll kR)\).

c. Equation (6.55) may be simplified in the limiting case of values of \(k\) so large that \(\psi_k \ll 1\). We can then drop the \(\psi\) term in the argument of \(\phi_k\), and \(\sin (\xi_k - \eta_e)\) may be replaced by \((\phi_k - \eta_e)\). Equation (6.55) becomes

\[
\phi_k - \eta_e = \frac{1}{\lambda k} \int_0^\infty \psi \, d\psi \cos (\xi_k - \eta_e) \psi - \frac{\lambda (\lambda + 1)}{k} \int_0^\infty \frac{\partial^2}{\partial \lambda^2} \phi_k (\kappa \lambda) \psi (\kappa \lambda) d\lambda.
\]

(6.56)

It will now be shown that \(\phi_k\) can be replaced by \(\phi_e\) in Eq. (6.56). The following relation is well known from the theory of Fourier transforms;

\[
\int_0^\infty \psi \, d\psi \, \mathcal{F}(\kappa) \mathcal{G}(\kappa) = \frac{1}{\pi} \int_0^\infty \mathcal{F}(\kappa) \mathcal{G}(\kappa)
\]

(6.57)

where \(\mathcal{F}\) and \(\mathcal{G}\) are the Fourier sine transforms of \(F\) and \(G\) respectively. By use of Eqs. (6.4) and (6.57), Eq. (6.56) may
be rewritten as

\[ S_\lambda - \eta_0 = \frac{1}{8 \pi^2} \int_0^{2k} d\alpha \ P_\lambda \left(1 - \frac{\alpha^2}{2k^2}\right) \ \bar{g}_1(\alpha) \]

\[ - \frac{\ell(l+1)}{2k^2} \int_0^{2k} d\alpha \ P_\lambda \left(1 - \frac{\alpha^2}{2k^2}\right) \ \bar{g}_2(\alpha) \]

where

\[ \int_0^\infty d\alpha \ J_{\lambda + k}(k\alpha) \sin \alpha = \begin{cases} \frac{1}{2k} \ P_\lambda \left(1 - \frac{\alpha^2}{2k^2}\right) & 0 < \alpha < 2k \\ 0 & \alpha > 2k \end{cases} \]  

(6.59)

\[ \bar{g}_1(\alpha) = \int_0^\infty d\alpha \ (\alpha^2 - \frac{\alpha^4}{4k^2}) \sin \alpha \]  

(6.60)

\[ \bar{g}_2(\alpha) = \int_0^\infty d\alpha \ (\alpha - \frac{\alpha^3}{6k^2}) \sin \alpha \]  

(6.61)

\( U \) and its derivatives are smooth functions. Since \( \alpha \) is a reciprocal distance, we expect that for \( \alpha > \frac{1}{\lambda R} \), \( \sin (\alpha \alpha) \) will oscillate rapidly, and \( \bar{g}_1 \) and \( \bar{g}_2 \) will become very small. Therefore, \( \bar{g}_1(\alpha) \) and \( \bar{g}_2(\alpha) \) limit the range of integration in Eq. (6.58) to \( 0 < \alpha < \frac{1}{\lambda R} \), which is a small segment for \( \lambda R \gg \frac{1}{\lambda R} \). The Legendre polynomial may be expanded for small \( \alpha \), with the result

\[ P_\lambda \left(1 - \frac{\alpha^2}{2k^2}\right) = 1 - \frac{\ell(l+1)}{4k^2} \alpha^2 + O\left(\frac{\ell^2 \alpha^4}{k^4}\right) \]  

(6.62)

At \( \alpha = \frac{1}{\lambda R} \), the second term on the right-hand side is \( \frac{\ell(l+1)}{4k^2} \alpha^2 \). For small \( \lambda \), this term is negligible, and we can replace \( P_\lambda \left(1 - \frac{\alpha^2}{2k^2}\right) \) by unity. But this is the same approximation as replacing \( P_\lambda \left(1 - \frac{\alpha^2}{2k^2}\right) \) by

\[ P_0 \left(1 - \frac{\alpha^2}{2k^2}\right) \]  

Thus, for \( \lambda \) satisfying \( \ell(l+1) \ll \lambda R \), one can replace \( P_\lambda \) by \( P_0 \) in Eq. (6.58), and ultimately \( \bar{g}_1 \) by \( \bar{g}_0 \) in Eq. (6.58). Since
we have

\[ \delta_\ell \sim \eta_0 \approx \psi_0 \int_0^{\infty} d\nu \, \frac{d^2 \psi_0}{d\nu^2} \left( 1 - \cos \frac{2\pi \nu}{\lambda^2} \right) \]

In the first term on the right-hand side of the preceding equation, the rapidly oscillating cosine can be dropped, and

\[ \frac{1}{8 \lambda^3} \int_0^{\infty} d\nu \, \frac{d^2 \psi_0}{d\nu^2} = -\left( \frac{1}{2 \lambda} \right)^3 \psi'(0) \]

In general, we cannot drop the cosine in the second term of Eq. (6.64) for \( \psi(0) \) becomes infinite as \( 1/\lambda \) at the lower limit. If we write

\[ a = \eta_0 - \left( \frac{1}{2 \lambda} \right)^3 \psi'(0) \]

\[ b = \frac{1}{2 \lambda^3} \int_0^{\infty} d\nu \, \frac{\psi(0) - \psi'(0)}{\lambda^2} \left( 1 - \cos \frac{2\pi \nu}{\lambda^2} \right) \]

then

\[ \delta_\ell \sim a - b(\ell + 1) b - \left[ \sqrt{2(\ell + 1)} \ll \lambda \right] \]

The \( \lambda \) dependence is now contained solely in the multiplicative factor \( b(\ell + 1) \).

In the special case that \( \psi'(0) = 0 \), \( \psi(0) / \lambda^2 \) is finite at \( \lambda = \xi \), and \( \cos (2\pi \nu / \lambda^2) \) can be dropped in the expression for \( b \). The resultant expression may be integrated by parts to yield
We have now obtained Verde's expansion (11) of $J_z$ in powers of $\eta R$, through order $1/k^3$. Explicitly,

$$J_z \propto -\frac{1}{k R} \int_0^\infty \nu(\lambda) \, d\lambda - \left(\frac{1}{k R}\right)^3 \int_0^\infty \nu^* (\lambda) \, d\lambda - \frac{\lambda (\lambda + 1)}{4 k^3} \int_0^\infty \, d\lambda \left(\frac{d \nu}{d\lambda}\right)$$

(6.69)

where Eq. (6.47) has been used for $\eta_0$.

2. Large $l$ ($l \gg k R$)

When $l$ is large compared to $kR$, the impact parameter $\lambda = (l + \lambda)/k$ lies in the outer regions of the potential. Here $\nu(\lambda)$ is weak, and for a high-speed particle, one expects the scattering to be nearly forward. This suggests that, for large $l$, the phase shift $\delta_z$ corresponds to the phase shift in the high-energy three-dimensional formulation, where a forward-scattering approximation to the wave function was made. In addition, the magnitude of $\nu_R/k$, where $\nu$ is evaluated at the impact parameter, will be small. Thus, the approximation of $J_z$ by $\gamma_z$ in Eq. (6.20) is sufficient. From Eq. (6.21), we can write the following relations for small $\delta_z$:

$$\delta_z \approx -\kappa \int_0^\infty \lambda \, J_z^*(k\lambda) \, \nu(\lambda)$$

(6.21a)

$$= - \nu_z \int_0^\infty \left[ J_z^*(k\lambda) \right] \left[ \lambda \nu(\lambda) \right]$$

(6.70)

As in the case of small $\lambda$, we take the Fourier sine transforms of the two functions in Eq. (6.70) to obtain
\[ \int_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \rho_x(1 - \frac{y}{2\pi}) g(y) \, dy \int_{0}^{\infty} \rho_x \, dy = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \rho_x(1 - \frac{y}{2\pi}) g(y) \, dy \]  \hspace{1cm} (6.71)

where

\[ g(y) = \int_{0}^{\infty} \rho_x \, \sin \omega \, d\omega \]  \hspace{1cm} (6.72)

As argued previously, \( g(y) \) goes to zero rapidly as \( y \) increases, for \( \rho_x \) is a smooth function and does not have large Fourier components. At high energies, only the range of \( y \) for which \( a \ll k \) is important in the evaluation of Eq. (6.71). For small \( \omega \),

\[ p_x(\omega = \omega) \approx \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\sqrt{2\pi}} \omega \right] \]  \hspace{1cm} (6.73)

Therefore, we replace \( \rho_x(1 - \frac{x}{2\pi}) \) by \( \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \omega \, d\omega \) in Eq. (6.71). For large \( \omega \), \( \frac{1}{\sqrt{2\pi}} \omega \omega = (k + \lambda_c) = p \), and

\[ \int_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \rho_x(1 - \frac{y}{2\pi}) g(y) \, dy \int_{0}^{\infty} \rho_x \, d\omega \int_{0}^{\infty} \rho_x \, d\omega = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \rho_x(1 - \frac{y}{2\pi}) g(y) \, dy \]  \hspace{1cm} (6.74)

Because only the range \( a \ll k \) is important, it is legitimate to replace the upper limit of the integral in Eq. (6.74) by \( \infty \). To simplify this expression further, the order of integration must be interchanged. This is not permissible after the change of upper limit, for the double integral is then not absolutely convergent. Nevertheless, in a restricted sense which will be clarified shortly, the correct answer is obtained. Proceeding in this manner, we have
Now the Fourier sine transform of $J_0 \left( \frac{\rho \omega}{w} \right)$ is
\[
\int_0^\infty d\lambda \lambda \sin \lambda s = \begin{cases} 
0 & 0 < \lambda < \frac{K}{w} \\
\frac{1}{\sqrt{\lambda^2 - \frac{K^2}{w^2}}} & \frac{K}{w} < \lambda < \infty
\end{cases},
\] (6.76)
so that
\[
\delta_{KL} \propto -\frac{1}{K} \int_{P_{KL}}^\infty d\lambda \frac{\rho \omega (\lambda)}{\sqrt{\lambda^2 - \rho^2}}.
\] (6.77)
The preceding equation may be put into a more revealing form by choosing the variable $\omega^2 = \lambda^2 - \rho^2$. Then Eq. (6.77) becomes
\[
\delta_{KL} \propto -\frac{1}{K} \int_{\omega_{KL}}^\infty d\omega \ U \left( \sqrt{\omega^2 + \rho^2} \right),
\] (6.78)
an expression previously obtained by both Glauber (2) and Verde (11).

In Eq. (6.78), the potential is integrated along a straight-line trajectory with impact parameter $\rho$. Obviously, we can write
\[
2 \delta_{KL} \propto -\frac{1}{K} \int_{\infty}^{\infty} d\omega \ U \left( \sqrt{\omega^2 + \rho^2} \right),
\] (6.79)
Therefore, let us define a function
\[
2 \delta_{KL} (\omega) = -\frac{1}{K} \int_{\infty}^{\omega} d\omega \ U \left( \sqrt{\omega^2 + \rho^2} \right),
\] (6.80)
where
The approximate wave function for forward scattering in the three-dimensional theory is
\[
\omega = i k z - i k \int_{-\infty}^{\infty} U(x, y, z) = i k z + i \delta(x)
\]  

It is seen that \(2 \delta(z)\) has the same analytical form as \(\delta(x)\), for a central potential. Hence, a particle scattered in the forward direction by a potential \(U(x)\) experiences a phase shift of \(2 \delta(z)\) at a point \((x, y)\), where \(x^2 + y^2 = \frac{z^2}{\lambda^2} = \frac{p^2}{\hbar^2}\) and \(R < \lambda\). It must be remembered, however, that Eq. (6.79) holds only for large \(l\).

b. To obtain Eq. (6.78), we interchanged the order of integration of Eq. (6.74) after first approximating the upper limit \(2k\) by \(\infty\). The interchange is legitimate if the upper limit is left as \(2k\). Then Eq. (6.74) becomes
\[
\delta = -\frac{i}{k} \int_{0}^{\infty} d\nu \nu U(\lambda) \int_{0}^{2k} J_0(\nu s) \sin \nu a
\]  

Now the integral over \(s\) may be written as
\[
\int_{0}^{2k} J_0(\nu s) \sin \nu a = \int_{0}^{\infty} J_0(\nu s) \sin \nu a - \int_{2k}^{\infty} J_0(\nu s) \sin \nu a
\]  

where the complete Fourier integral is given by Eq. (6.76). Since \(2k \ell = 2\rho\) is large, \(J_0(\nu s)\) may be replaced by its asymptotic form in the second term of Eq. (6.84). This second term may be manipulated as follows:
\[ I = \int_{\frac{a}{2k}}^{\infty} ds \, J_0(b s) \sin r s \]  
\[ = \sqrt{\frac{\pi}{2 \hbar}} \int_{\frac{a}{2k}}^{\infty} ds \, \frac{\sin(b s + \frac{\pi}{4}) \sin r s}{\sqrt{s}} \]  
\[ = 2 \sqrt{\frac{\pi}{2 \hbar}} \int_{\frac{a}{2k}}^{\infty} ds \, \sin(b s + \frac{\pi}{4}) \sin(r s) \]  
\[ = \frac{1}{\sqrt{\frac{\pi}{2 \hbar}}} \frac{1}{\sqrt{\lambda - \frac{1}{4}}} \int_{\frac{2k(\lambda - \frac{1}{4})}{2k(\lambda - \frac{1}{4})}}^{\infty} d\omega \cos(\omega^2 \pm \sin \omega^2) \]  
\[ = \frac{1}{\sqrt{\frac{\pi}{2 \hbar}}} \frac{1}{\sqrt{\lambda + \frac{1}{4}}} \int_{\frac{2k(\lambda + \frac{1}{4})}{2k(\lambda + \frac{1}{4})}}^{\infty} d\omega \cos(\omega^2 - \sin \omega^2) \]  
\[ = A \pm B. \]

One uses the + sign in Eq. (6.88) when \( r > \frac{b}{2} \), the - sign when \( r < \frac{b}{2} \). \( B \) is always small regardless of \( r \), for its lower limit is \( \sqrt{2 \pi} \) when \( \lambda = 0 \), and increases for increasing \( \lambda \).

By use of the asymptotic expansions for large \( \omega \),
\[ \int_{0}^{\infty} \cos \omega^2 \, d\omega \approx -\frac{1}{2 \pi} \sin \alpha^2, \]
\[ \int_{0}^{\infty} \sin \omega^2 \, d\omega \approx \frac{1}{2 \pi} \cos \alpha^2, \]  
the following expression for \( B \) is obtained:
\[ B = \frac{1}{2 \sqrt{2 \pi} \sqrt{\lambda + \frac{1}{4}}} \left[ \sin 2k(\lambda + \frac{1}{4}) + \cos 2k(\lambda + \frac{1}{4}) \right]. \]  

If \( r > \frac{3}{2} \frac{b}{2} \) or \( r < \frac{b}{2} \), \( \sqrt{2 \pi} \sqrt{\lambda + \frac{1}{4}} \) is at least \( \sqrt{2 \pi} \), so that we can evaluate \( A \pm \) in this range of \( \lambda \) by making an asymptotic expansion similar to the one for \( B \). The result is
\[ A \pm = \frac{1}{2 \sqrt{2 \pi} \sqrt{\lambda - \frac{1}{4}}} \left[ -\sin 2k(\lambda - \frac{1}{4}) + \cos 2k(\lambda - \frac{1}{4}) \right]. \]  

The sum \( A \pm + B \) may then be written as
\[ I = \cos(2k\lambda) \frac{r}{(\lambda^2 - b^2)} J_0(\lambda r) + \sin(2k\lambda) \frac{r}{(\lambda^2 - b^2)} Y_0(\lambda r) \] (6.93)

a relation which is valid for

\[ \begin{cases} 0 < \lambda < \lambda^* \\ \frac{3}{2} \leq \rho \leq \infty \end{cases} \] (6.94)

As \( p \) is large, \( J_1 \) and \( Y_0 \) are proportional to \( \sqrt{p} \). Also, when \( I \) is integrated over \( r \), the rapidly oscillating sine and cosine terms make the resultant contribution to \( j_0 \) small. Hence, we can drop \( I \) in Eq. (6.84) for the above ranges of \( r \).

To examine the integral \( I \) for \( \frac{\omega}{\omega^*} < \lambda < \frac{3}{2} \lambda^* \), it is convenient to change variables to

\[ \begin{cases} z = \sqrt{\frac{x^2}{\epsilon^2} - 1} \quad \rho > 2 \\ z' = \sqrt{1 - \frac{x^2}{\epsilon^2}} \quad \rho < 2 \end{cases} \] (6.95)

The range of \( z \) and \( z' \) we must investigate is roughly zero to one. With this change of variables in Eqs. (6.83), (6.84), and (6.88), and by dropping \( B \) which is negligible for all \( x \), \( \delta_k \) becomes

\[ \delta_k \approx -\frac{L}{2 \epsilon k} \int_0^\infty dz F(\xi) + \int_0^\infty dz' G(\xi') \] (6.96)

where

\[ F(\xi) = 1 - \sqrt{1 + \xi^2} \left[ 1 - \sqrt{\frac{\xi}{\epsilon}} \cdot x \right] \]

\[ \times \left[ \frac{\omega}{\sqrt{2p(\frac{1}{\omega^2} - 1)}} \right] (6.97) \]
The function $F(z)$ is zero at $z = 0$. For other $z$, it is necessary to investigate the integral

$$
H = \sqrt{\frac{2}{\pi}} \int_{0}^{1} \frac{\text{d}x}{x} \cos x - \sin x.
$$

(6.99)

It rises rapidly from the value zero at $z = 0$ to its maximum value when the upper limit of the integral is of the order of unity; thereafter, for increasing $z$, it oscillates about the value one with a decreasing amplitude (Figure 4). The range of $z$ for which it is rapidly changing is

$$
\sqrt{2 \rho (\sqrt{1 + z^2} - 1)} \sim 1.
$$

Since $p$ is large, $z$ must be small, and therefore $z \sqrt{\rho} \sim 1$, or $z \sim \sqrt{\rho}$. Since $\sqrt{1 + z^2}$ changes so slowly with $z$ in the range $0 < z < 1$, $F(z)$ varies essentially the same way as $H$. It rises rapidly as $z$ varies between zero and $\sqrt{\rho}$, then oscillates with ever decreasing amplitude about the value unity. Because $p$ is large, the contribution to the $z$-integral in Eq. (6.96) from a range of integration of order $\sqrt{\rho}$ is small. Hence, to first approximation, we replace $F(z)$ by unity in Eq. (6.96).

The quantity $G(z)$ is zero at $z = 0$. For increasing $z$, $G$ oscillates about the value zero with decreasing amplitude. To a rough first approximation, we replace $G(z)$ by zero in Eq. (6.96). The result of approximating $F(z)$ by unity, $G(z)$ by zero, is to obtain Eq. (6.78) again.
Figure 4. Sum and difference of Fresnel integrals.
The oscillating nature of \( F(z) \) and \( G(z) \) is the result of diffraction effects associated with the finite wavelength of the incident particle. The distance \( \sqrt{r} \) in which these effects are important is \( \sqrt{\gamma/k} \). This is the same result obtained in Section V. D., where the distance through which the particle is scattered forward is now of the order of the impact parameter \( \lambda \). In the classical limit, \( k \) becomes infinite and, to keep \( \lambda \) non-zero, \( p \) must become infinite also. For \( p \) infinite, we have

\[
F(z) = \begin{cases} 
0 & z = 0 \\
1 & z \neq 0 
\end{cases} 
\]

\[ G(z) = 0 \quad \text{(6.100)} \]

c. The large \( \lambda \) formula, Eq. (6.77), may be compared with the approximation, Eq. (6.70), from which it was derived in the case of the potentials \( U = \omega/\lambda^2 \). Here, \( S \) is an integer greater than unity. From Landau (23), we have

\[
\delta_k \sim -\frac{1}{2k} \int_0^{\infty} d\lambda \frac{\lambda U(\lambda)}{\sqrt{\lambda^{(s-1)}}} = -\frac{\sqrt{\pi}}{4} \sim \frac{k^{s-2}}{\rho^{s-1}} \frac{\Gamma\left(\frac{1}{2} S - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} S\right)} \quad \text{(6.101)}
\]

From Bateman (24), the integral in Eq. (6.70) is given by

\[
\delta_k \sim -\frac{\pi}{2} \int_0^{\infty} d\lambda \lambda U(\lambda) \frac{J_{s+1/2}(k \lambda)}{\lambda^{(s-1)}} = \left(\frac{\pi}{2} \times \frac{k^{s-2}}{\rho^{s-1}}\right) \frac{\Gamma(s-1)}{\Gamma\left(\frac{1}{2} S\right)} \frac{\Gamma(s+3/2-k_1)}{\Gamma\left(\frac{1}{2} S + k_1 + \frac{1}{2}\right)} \quad \text{(6.102)}
\]

The following relation between the gamma functions holds:

\[
\frac{\Gamma(s-1)}{2^{s-1}} = \frac{1}{2} \sqrt{\frac{1}{\pi}} \Gamma\left(\frac{1}{2} S - \frac{1}{2}\right) \frac{\Gamma\left(\frac{1}{2} S\right)}{\Gamma\left(\frac{1}{2} S + \frac{1}{2}\right)} \quad \text{(6.103)}
\]
Hence Eq. (6.102) becomes

\[ \delta_2 \approx -\frac{\sqrt{\pi}}{4} \alpha \frac{r_i^{S-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\Gamma(\frac{1}{2} S - \frac{1}{2})}{\Gamma(\frac{1}{2} S + \frac{1}{2} S - \frac{1}{2})} \]  

(6.104)

Examining the quantity \( \frac{\Gamma(\frac{1}{2} S + \frac{1}{2} S - \frac{1}{2})}{\Gamma(\frac{1}{2} S + \frac{1}{2} S - \frac{1}{2})} \), we note that for each unit by which \( s \) increases, the argument of the numerator decreases by \( \frac{1}{2} \) and the argument of the denominator increases by \( \frac{1}{2} \). Hence, the difference in arguments is 1, and we get a factor of \( \frac{1}{2} \). If \( (\lambda + \frac{1}{2}) \gg (\frac{1}{2} S - \frac{1}{2}) \), the above quantity is approximately \( \frac{1}{2} \) to lowest order in \( \frac{1}{2} \). Therefore, the two forms, Eqs. (6.101) and (6.102), agree when \( \lambda \) is large enough. It is interesting to note that the two forms are identical when \( s = \frac{1}{2} \).

D. Summary

An exact integral expression for \( J_2 \) in terms of the WKB approximation of the radial wave function is given by Eq. (6.39). Because this expression is unwieldy, alternate approximations are considered. A simple relation for \( J_2 \) valid when \( \sqrt{R(\lambda + 1)} \ll k \) and \( \frac{vR}{\lambda} \ll 1 \) is given by Eq. (6.67). When the potential has vanishing slope at the origin, one obtains Eq. (6.69), a relation first obtained by Verde (11). Finally, Eq. (6.96) is an approximate expression for \( m_3 \) valid when \( \lambda \gg k \). In the limit of very large \( \lambda \), Eq. (6.96) simplifies to a well-known expression, Eq. (6.78).


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IX. APPENDIX A

A. Evaluation Of the Feynman Path Integral

From Section (V. B.), we obtain the following equations:

\[
\langle z_{11} \rangle = e^{i\frac{\hbar}{\alpha} L_c} \langle z_{11} \rangle, \tag{5.20}
\]
\[
\langle z_{11} \rangle = A(t_{11}) \int d(\text{paths}) e^{i\frac{\hbar}{\alpha} (L - L_c)} \tag{5.21}
\]

The simplest procedure for evaluating the integral over paths is to Fourier analyze the paths (20). As no new ideas enter in the three-dimensional case, the procedure will be explained for one dimension. A modification in Davison's procedure (20) is introduced by expanding \( f(t) = x(t) - x_c(t) \) in a Fourier series instead of expanding \( x(t) \) alone. Since

\[ f(t_1) = f(t_1) = 0, \]

one can expand \( f(t) \) as

\[ f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi \left( \frac{t-t_1}{t_2-t_1} \right). \tag{A.1} \]

In one dimension, the form of \( \langle z_{11} \rangle \) that propagates a free particle from time \( t_1 \) to time \( t_2 \) is

\[ \langle z_{11} \rangle = \sqrt{\frac{m}{2\pi \hbar t_{11}}} e^{i\frac{\hbar}{\alpha} L_c} \langle z_{11} \rangle. \tag{5.30} \]

We therefore adjust our constants, \( A(t_{21}) \) and \( b_n \), to give this result. They are

\[ A(t_{21}) = \sqrt{\frac{m}{2\pi \hbar t_{21}}} \prod_{n=1}^{\infty} \frac{1}{\sqrt{l_n}}, \tag{A.2} \]
\[ b_n = \frac{a_n}{\hbar^n} \times \sqrt{\frac{\pi \hbar t_{21}}{m}}. \tag{A.3} \]
Thus, Eq. (A.1) becomes
\[ f(t) = \sqrt{\frac{\pi k T L}{m}} \sum_{n=1}^{\infty} \frac{a_n}{\pi} \sin \pi \left( \frac{t-t_1}{t_{21}} \right) \],
\hspace{1cm} (A.4)

and the expression for \( \langle x_l | v(t_1, t_1) | x_i \rangle \) is
\[ = \mathcal{K} \left[ S - S_c \right] \left( \dot{f}, f \right). \]
\hspace{1cm} (A.5)

The expression for \( S - S_c \) in terms of the \( f \) variable is
\[ S - S_c = \int_{t_i}^{t_f} \left[ \frac{1}{2} m \dot{f}^2 + \frac{\partial V(x_c)}{\partial x_c} f - V(f + x_c) + V(x_c) \right] dt. \]
\hspace{1cm} (A.6)

The term, \( \int_{t_i}^{t_f} m \dot{f} x_c dt \), can be integrated by parts to yield
\[ -\int_{t_i}^{t_f} (m \dot{x}_c) f dt = \int_{t_i}^{t_f} v'(x_c) f dt, \]
where the boundary term vanishes because \( f \) vanishes at both end points. Hence, Eq. (A.7) becomes
\[ S - S_c = \int_{t_i}^{t_f} dt \left[ \frac{1}{2} m \dot{f}^2 + \frac{\partial V(x_c)}{\partial x_c} f - V(f + x_c) + V(x_c) \right]. \]
\hspace{1cm} (A.8)

To demonstrate that the constants have been chosen correctly in Eqs. (A.2) and (A.3), we set \( V(x) = 0 \). Then Eq. (A.8) reduces to
\[ S - S_c = \int_{t_i}^{t_f} dt \mathcal{K} m \dot{f}^2 \]
\hspace{1cm} (A.9)

which, on substitution of Eq. (A.4), becomes
\[ S - S_c = \mathcal{K} \pi \sum_{n=1}^{\infty} a_n^2 \]
\hspace{1cm} (A.10)
The orthonormal properties of the trigonometric functions have been used in obtaining the above equation. Thus Eq. (A.5) simplifies to
\[
\langle z | 1 \rangle = \frac{i}{\hbar} \int \mathcal{S}_c \sqrt{\frac{m}{2\pi\hbar t_1}} \prod_{n=1}^N \int_{-\infty}^{\infty} \frac{da_n}{v_{t_n}^n} e^{i \frac{\pi}{\hbar} a_n^2},
\]
(A.11)
which is just Eq. (5.30).

The generalization to the three-dimensional Cartesian case is simple. One now writes
\[
\mathcal{J} = \mathcal{S}_c \sum_{n=1}^N \frac{a_n}{\frac{1}{\hbar} \int \mathcal{S}_c (z_{t_1}) \int \frac{da_n}{v_{t_n}^n} \cdots \int \frac{da_n}{v_{t_n}^n},
\]
(A.12)
where \( \mathcal{S}_c = (a_n, b_n, c_n) \). The integration over paths now reduces to an integration over all the \( a \)'s, \( b \)'s, and \( c \)'s.

Modifying the normalization, Eq. (A.5) becomes
\[
\langle \vec{x}_t | \mathcal{U}(t_1, t_1) | \vec{x}_t \rangle =
\left( \frac{m}{2\pi i \hbar t_1} \right)^{3Nc} \int \frac{da_n}{v_{t_n}^n} \int \frac{db_n}{v_{t_n}^n} \cdots \int \frac{da_n}{v_{t_n}^n} \int \frac{db_n}{v_{t_n}^n} \int \frac{dc_n}{v_{t_n}^n} \mathcal{S}_c [\mathcal{S}_c] (\vec{p}, \vec{p})
(A.13)
The expression for \( S - S_c \) is
\[
S - S_c = \int_{t_1}^{t_2} \left[ \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) - \frac{1}{2} \hbar \frac{m}{\hbar} \dot{\vec{c}}^2 + V(\vec{c}) \right] dt
\]
(A.14)
\[
= \int_{t_1}^{t_2} \left[ \frac{1}{2} m (\vec{p})^2 + \vec{p} \cdot \nabla V(\vec{c}) - V(\vec{c} + \vec{c}) + V(\vec{c}) \right] dt .
\]
(A.15)

B. WKB Approximation In One Dimension

The form of Eq. (5.20) suggests a WKB approximation. The quantity \( \langle \vec{x} | 1 \rangle \) is a measure of the deviation about the
classical path. Since \( j \) goes to zero as \( \gamma \to 0 \), an expansion of \( S-S_0 \) in powers of \( j \) is clearly indicated. For our purposes, it is adequate to derive the approximation in one dimension. Then we can write

\[
V(x) = V(x_c) + \left( V'(x_c) \right)_{x=x_c} f + \int_{x=x_c}^{x} \left( V''(x) \right)_{x=x_c} f^2 + \ldots
\]  

(A.16)

Equation (A.8) becomes

\[
\int_{t_i}^{t_f} \left[ \frac{1}{2} \mu \dot{f}^2 - \frac{1}{2} \frac{\partial^2 V(x_c)}{\partial x_c^2} f^2 \right] dt
\]

where the first variations about the classical path have canceled. On substitution of Eq. (A.4), Eq. (A.17) reduces to

\[
\int_{t_i}^{t_f} \left[ \frac{1}{2} \mu \dot{f}^2 - \frac{1}{2} \frac{\partial^2 V(x_c)}{\partial x_c^2} f^2 \right] dt
\]

(A.18)

where

\[
I_{m,n} = \delta_{m,n} - 2 \pi \int_{t_i}^{t_f} \left( \omega_{m,n} \right) \sin m \pi \tau \sin n \pi \tau d\tau
\]

(A.19)

\[
\left\{ \begin{array}{l}
\omega_{m,n} = \frac{1}{m} \frac{\partial^2 V(x_c)}{\partial x_c^2} \\
\frac{d}{dt} = \frac{t - t_i}{t_f - t_i}
\end{array} \right.
\]

(A.20)

Substituting Eq. (A.19) into Eq. (A.5), the one-dimensional propagation function becomes

\[
\langle X_l | U(t_f, t_i) | X_i \rangle \propto \int_{-\infty}^{\infty} \frac{d \alpha_n}{\sqrt{2 \pi i \alpha_n}} \frac{d \beta_m}{\sqrt{2 \pi i \beta_m}} \ldots \int_{-\infty}^{\infty} \frac{d \alpha_n}{\sqrt{2 \pi i \alpha_n}} \frac{d \beta_m}{\sqrt{2 \pi i \beta_m}} \ldots \int_{-\infty}^{\infty} \frac{d \alpha_n}{\sqrt{2 \pi i \alpha_n}} \frac{d \beta_m}{\sqrt{2 \pi i \beta_m}} 
\]

(A.21)

The integration over the expansion coefficients can be carried out if the axes in "a" space are rotated in order to eliminate cross terms in the exponential. The similarity transformation
to the principal axes gives

\[
\sum_{m,n=1}^{\infty} I_{m,n} a_m a_n = \sum_{m=1}^{\infty} I_m a_m^2 .
\] (A.22)

The Jacobian of this orthogonal transformation is unity, and Eq. (A.21) becomes

\[
\langle \xi \mid 1 \rangle_{\omega_k \xi} = e^{i \pi \int \frac{m}{\lambda} \frac{1}{\sqrt{2 \pi (\pi t_k)}} \left[ \frac{1}{\sqrt{\text{det } I}} \right]}
\] (A.23)

\[
= e^{i \pi \int \frac{m}{\lambda} \frac{1}{\sqrt{2 \pi (\pi t_k)}} \left[ \frac{1}{\sqrt{\text{det } I}} \right]}
\] (A.24)

\[
= e^{i \pi \int \frac{m}{\lambda} \frac{1}{\sqrt{2 \pi (\pi t_k)}} \left[ \frac{1}{\sqrt{\text{det } I}} \right]}
\] (A.25)

\[
= e^{i \pi \int \frac{m}{\lambda} \frac{1}{\sqrt{2 \pi (\pi t_k)}} \left[ \frac{1}{\sqrt{\text{det } I}} \right]}
\] (A.26)

The last step follows because the determinant of a matrix is unchanged by similarity transformation.

In the special case of a harmonic oscillator,

\[\nu = \frac{\hbar}{\lambda} m \omega^2 x^2 ,\]

Eq. (A.26) is exact and

\[
I_{m,n} = \delta_{m,n} \left[ 1 - \frac{\nu^2 \tau_{21}^2}{\hbar^2 \pi^2} \right] .
\] (A.27)

By use of the identity

\[
\prod_{n=1}^{\infty} \sqrt{1 - \frac{\omega^2 t_{21}}{\hbar^2 \pi^2}} = \sqrt{\frac{\sin (\omega t_{21})}{\omega t_{21}}}
\] (A.28)

Eq. (A.26) becomes

\[
\langle z \mid 1 \rangle_{\delta, \text{g.o.}} = e^{i \pi \int \frac{m}{\lambda} \frac{1}{\sqrt{2 \pi i \kappa \sin (\omega t_{21})}}}
\] (A.29)
For a general \( \psi(x) \), further approximations must be made in order to evaluate \( \det I \). Rewriting Eq. (A.19), the expression for \( I_{mn} \) becomes

\[
I_{mn} = \delta_{mn} + \frac{t_{j3}}{m_n \pi^2} \int \frac{d \tilde{z}}{\sqrt{\omega^2 - \omega^2(\tilde{z})}} \left[ \cos (m-n) \pi \tilde{z} - \cos (m+n) \pi \tilde{z} \right] d \tilde{z} \tag{A.30}
\]

Assuming that \( \omega^2 \) is a smoothly varying function of \( \tilde{z} \), one expects that the terms without the cosine factors will be most important. Keeping just those diagonal terms free of oscillatory functions, Eq. (A.30) becomes

\[
I_{mn} \sim \left[ 1 - \frac{\omega^2 - \omega^2(\tilde{z})}{\pi^2} \right] \delta_{mn} \tag{A.31}
\]

\[
\bar{\omega} = \int \omega(\tilde{z}) d \tilde{z} \tag{A.32}
\]

Equation (A.31) has the same form as in the harmonic-oscillator example, and, therefore, we obtain the approximation

\[
\langle z | 1 | \rangle_{WKG} \approx \sqrt{\frac{\hbar}{2 \pi \bar{\omega}}} \int \sqrt{e^{(2j+1) \pi \bar{\omega} t_{11}}} \right) \tag{A.33}
\]
A. WKB Approximation For One-Dimensional Schrödinger Equation

As $\lambda \to 0$, wave optics goes over into geometrical optics. The corresponding limit in wave mechanics is the WKB or semi-classical approximation. Because $p = \frac{\pi}{\lambda}$, $\lambda \to 0$ requires $\frac{\pi}{p} \to 0$ if $p$ is to become the finite momentum of a classical particle. If $\psi = \frac{i}{\lambda} S$ is substituted into the Schrödinger equation,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \left[ E - V(x) \right] \psi = 0,$$

an equation for $S$ is obtained:

$$\frac{S''}{\lambda m} - \frac{i}{\lambda m} S' = E - V(x).$$

Let us expand $S$ in a power series in $\kappa$. The first approximation, $S_0$, is Hamilton's characteristic function,

$$S_0 = \pm \int^x \rho \, dx,$$

$$\rho = \sqrt{i\hbar(E - V)} = \pi \kappa.$$

An estimate of the validity conditions on the approximation $S \approx S_0$ may be obtained by comparing the magnitude of the two terms on the left-hand side of Eq. (B.2):

$$\kappa \left| \frac{S''}{S_0'} \right| \ll 1.$$

Because of $S_0' = \rho$, this condition becomes
The de Broglie wavelength must vary only slightly over distances of order $\lambda$. This condition may be rewritten as

$$\left| \frac{d}{dx} A(x) \right| < \frac{1}{\lambda}$$

If $U$ varies appreciably only in a distance $R$, Eq. (B.6) becomes

$$\frac{U}{k^2} \frac{1}{KR} < \frac{1}{\lambda}$$

This is satisfied if our usual conditions hold.

Carrying out the next approximation, we find

$$\psi(x) = \frac{\xi}{\sqrt{p}} e^{i \phi x} + \frac{\xi}{\sqrt{p}} e^{-i \phi x}$$

If $V > \varepsilon$, $p$ should be replaced by $\sqrt{\rho}$ in the preceding equation.

The WKB approximation clearly fails when $p = 0$, i.e. when $E = V(x)$ at a classical turning point. For example, consider a potential barrier to the left of the turning point $x = a$. The WKB wavefunction valid for $V > \varepsilon$ must be fitted at $x = a$ to the WKB wavefunction valid for $V < \varepsilon$. Since these approximations fail in the vicinity of $x = a$, another method must be used in this range. As we shall always have to deal with a turning point in the radial wave equation for $x > a$, a study of the procedure in the simpler Carte-
sian case is useful.

Let us assume that there is only one turning point, \( x = \alpha \). As an introduction to the formulation, we will assume that the region about \( x = \alpha \) in which the WKB approximation is not valid is small enough so that \( V(x) \) can be represented by a linear function of \( x \) in this range:

\[
V(x) \approx V(\alpha) + \left( \frac{\partial V}{\partial x} \right)_{x=\alpha} (x - \alpha) = E - F_0 q,
\]

where

\[
q = x - \alpha, \quad F_0 = -\left( \frac{\partial V}{\partial q} \right)_{q=0}.
\]

With the change of variables \( y = (\frac{x - \alpha}{F_0})^{\frac{1}{2}} q \), Eq. (B.1) becomes

\[
\frac{d^2 \psi}{dy^2} + y \psi = 0.
\]

The solution is

\[
\psi = \sqrt{y} \left[ A J_{\nu_3} \left( \frac{1}{2} y^{\frac{1}{2}} \right) + B Y_{\nu_3} \left( \frac{1}{2} y^{\frac{1}{2}} \right) \right].
\]

Since \( E - V = F_0 q \) to the above approximation, the momentum becomes

\[
P_0 = \sqrt{2m (E - V)} = \sqrt{2m F_0 q}
\]

and

\[
\int_{\alpha}^{x} P_0 \, dx = \int_{q}^{\frac{\sqrt{2m F_0 q}}{2}} P_0 \, dq = \frac{\nu_3}{2} \sqrt{2m F_0 q^{2\nu}}
\]
where the subscript on $p_0$ is a reminder that $v(x)$ is specified by Eq. (B.9). But
\[
y^{1/\gamma} = \frac{1}{\gamma} \left( \frac{3}{2} \right) \frac{2}{3} y^{3/2},
\]
so
\[
\frac{1}{\gamma} \int_{x}^{\gamma} p_{0} \, dx = \frac{2}{3} y^{3/2}.
\]
Also, because $y$ is proportional to $q$, we obtain
\[
y = \frac{y^{3/2}}{y^{1/2}} \propto \frac{\int_{x}^{\gamma} p_{0} \, dx}{p_0}.
\]
Hence, Eq. (B.12) becomes
\[
\psi(x) = \sqrt{\frac{\int_{x}^{\gamma} p_0 \, dx}{p_0}} \left[ a \, \int_{x}^{\gamma} \left( \frac{1}{\pi} \int_{\alpha}^{\gamma} p_0 \, dx \right) + B' \int_{1/2}^{1/2} \left( \frac{1}{\pi} \int_{\alpha}^{\gamma} p_0 \, dx \right) \right].
\]
If the potential remains greater than $E$ for $-\infty < x < \alpha$, we want our WKB solution to be a dying exponential to the left of $x = \alpha$. The asymptotic form of Eq. (B.18) for $x \ll \alpha$ is
\[
\psi(x) \rightarrow \frac{\alpha'}{2 \sqrt{p_0}} e^{-\sqrt{\frac{1}{\pi p_0}}} \left| \int_{x}^{\gamma} p_0 \, dx \right|, \quad \text{for} \quad x \ll \alpha.
\]
provided that $\alpha' = B'$. For this choice of constants,
\[
\psi(x) \rightarrow \frac{\alpha'}{\sqrt{p_0}} \sin \left( \frac{1}{\pi} \int_{\alpha}^{\gamma} p_0 \, dx + \frac{\pi}{4} \right), \quad \text{for} \quad x > \alpha.
\]
Let us require that $\psi$ take on its asymptotic forms for values of $x$ for which Eq. (B.9) is still valid. Then Eqs. (B.19) and (B.20) without the subscript on the $p$ are the desired WKB approximations for $V > E$ and $V < E$ respectively. They are
called the connection formulas for a barrier to the left.

We will now define \( K = \sqrt{k^2 - U(x)} \), \( f = \int_k^x d\chi \), and discuss an equation of the form

\[
\psi(x) = \sqrt{\frac{2}{k}} \left[ \alpha J_{\nu_3}(f) + \beta J_{-\nu_3}(f) \right]
\]

(B.21)

This function is an exact solution of the Schrödinger equation in the neighborhood of the turning point, where \( f \) reduces to \( \int_x^\infty \frac{U}{k} d\chi \), and it also has an asymptotic form that coincides with the WKB approximation far from the turning point. This suggests that we regard \( \psi(x) \) as a WKB solution that is valid for all \( x \). This approximation was first arrived at by Langer (22). For general \( K \), Eq. (B.21) satisfies

\[
\frac{d^2\psi}{dx^2} + \left[ K^\nu(x) - \Theta(x) \right] \psi = 0
\]

(B.22)

where

\[
\Theta(x) = -\frac{S}{\pi^2} \frac{k^\nu}{\tau_k} + \frac{S}{4} \left[ -\frac{1}{k} \frac{dK^\nu}{dx} \right] - \frac{1}{4} \frac{1}{k^2} \frac{d^2 K^\nu}{dx^2}
\]

(B.23)

The last two terms may be written in terms of \( V \) as

\[
\frac{S}{16} \left( \frac{U'}{k^\nu - U} \right)^2 + \frac{1}{4} \frac{U''}{(k^\nu - U)}
\]

(B.24)

These terms are small if the WKB approximation holds at all.

Assuming that \( U(x) \) approaches a constant at \( x = \pm \infty \), \( f < x \) for large \( |x| \), hence the first term in \( \Theta(x) \) goes to zero at these limits. In the asymptotic regions, \( \Theta \sim K^- \).

Near the turning point \( x = a \), the approximation will be at its worst, for \( K^- \to 0 \). If \( K^- \) is expanded as a power series
about \( x = \alpha \), \( \Theta(\alpha) \) may be found to be (15)

\[
\Theta(\alpha) = \frac{q}{i4\pi} \left[ \left( \frac{\nu''}{\nu'} \right)_{x=\alpha} \right] \frac{1}{i4} \left( \frac{\nu'''}{\nu'} \right)_{x=\alpha}.
\] (B.25)

If \( V(x) \) is a slowly-varying function, \( \Theta(\alpha) \) will be small and Eq. (B.21) will be a good approximation to the wavefunction throughout that interval.

B. WKB Approximation For Radial Wave Equation

When \( \lambda > 0 \), the centrifugal potential \( \lambda(l+\frac{1}{2})/a^2 \) is non-zero, and one must deal with a problem in barrier penetration. The radial equation is

\[
\frac{d}{dr} \phi(r) + \frac{\lambda (l+\frac{1}{2})}{a^2} \phi(r) = 0,
\] (B.26)

\[
\Omega = \kappa - V(\lambda) - \frac{\lambda (l+\frac{1}{2})}{a^2}.
\] (B.27)

\( \Omega \) will be zero at least for one point, say \( \lambda_0 \). Therefore, as the approximate wave function is to be substituted in an integral over all \( \lambda \) to find \( \phi_2 \), we want to use Langer's formula (22), modified to apply to the radial equation.

If one considers \( V(\lambda) + \frac{\lambda (l+\frac{1}{2})}{a^2} \) as an effective potential, then the form of Eq. (B.26) is that of the one-dimensional Schrödinger equation. A first guess at converting the one-dimensional formulas from \( x \) to \( r \) might be to replace \( x \) by \( r \), \( V(x) \) by \( V(\lambda) + \frac{\lambda (l+\frac{1}{2})}{a^2} \). However \( \frac{\lambda (l+\frac{1}{2})}{a^2} \) is singular, and a modification is necessary. This may be illustrated in the following two cases:
1. A WKB wave function converted to \( r \) in the manner described above is

\[
\psi_r = \frac{e^{-i}}{\sqrt{Q}} \sin \left[ \int_{r_0}^{r} Q^\prime(r') \, dr' + \Pi_Q \right]. \tag{B.28}
\]

If we set \( \nu(r) = 0 \), the integration can be carried out. As \( r \to \infty \), \( \psi_r \) becomes

\[
\psi_r \xrightarrow{r \to \infty} D \sin \left[ k r - \frac{\Pi_Q}{\sqrt{2(l+1)}} + \Pi_Q \right]. \tag{B.29}
\]

However, the solution to Eq. (B.26) for \( \nu = 0 \) is Eq. (6.16), which has the asymptotic form

\[
D \sin \left( k r - \frac{\Pi_Q}{\sqrt{2(l+1)}} \right). \tag{B.29a}
\]

Thus, the asymptotic form of Eq. (B.28) is incorrect. Replacing the factor \( \lambda (\lambda + 1) \) in \( Q \) by \( (\lambda + \frac{1}{2})^2 \) removes the difficulty.

2. Equation (B.28) satisfies the following equation:

\[
\frac{d^2 \omega_r}{dr^2} + Q \omega_r = \frac{\lambda}{4} \left( \frac{Q'}{Q} \right) \omega_r - \frac{1}{2} \frac{Q''}{Q} \omega_r . \tag{B.30}
\]

If \( \nu(r) \) is less singular at the origin than \( 1/r^2 \), we have

\[
\frac{Q'}{Q} \xrightarrow{r \to 0} - \frac{1}{r} ,
\]

\[
\frac{Q''}{Q} \xrightarrow{r \to 0} \frac{2}{r^2} . \tag{B.31}
\]

and Eq. (B.30) becomes

\[
\frac{d^2 \omega_r}{dr^2} + \left[ \frac{1}{4r^2} - \frac{\lambda (\lambda + 1)}{r^2} \right] \omega_r \xrightarrow{r \to 0} 0 . \tag{B.32}
\]

Hence, Eq. (B.28) does not satisfy the radial wave equation near the origin. Again, if one uses \( (\lambda + \frac{1}{2})^2 \) instead of \( \lambda (\lambda + 1) \), the difficulty is removed.

The problem is that the two differential equations under
comparison have different singularities; the Schrödinger equation is singular at \( x = \pm \infty \), while the radial equation is singular at \( \lambda = 0 \) and \( \lambda = \infty \). Langer demonstrated that if one changes variables from \( \lambda \) to \( \omega \), where the range of \( \omega \) is \( -\infty \) to \( +\infty \) and \( \lambda = 0 \) corresponds to \( \omega = -\infty \), an equation is obtained which is of the same form as the Schrödinger equation. Comparison of the \( \omega(\lambda) \) and \( \lambda \) equations reveals that the recipe for conversion from \( \lambda \) to \( r \) is to replace \( \lambda \) by \( r \), \( U(\lambda) \) by \( U_r(\lambda) = U(\lambda) + (\ell + \lambda_0)^2/\lambda^2 \). Hence, for the rest of this section, the following propagation number will be used:

\[
Q_r = \lambda^2 - U(\lambda) - \frac{(\ell + \lambda_0)^2}{\lambda^2}
\]  

(B.33)

The modified form of Eq. (B.21) is then

\[
\omega_\lambda(\lambda) = \sqrt{\frac{\rho(\lambda)}{Q_\lambda(\lambda)}} \left[ A \right]_{\lambda_2}^1(f) + B \int_{\lambda_2}^{\lambda_3} (f) \right]
\]  

(B.34)

\[
\left\{
\begin{aligned}
f_\lambda(\lambda) &= \int_{\lambda_2}^{\lambda_3} \rho, \ \mathrm{d}\lambda \\
Q_\lambda(\lambda) &= 0
\end{aligned}
\right.
\]  

(B.35)

The function \( \omega_\lambda \) satisfies

\[
\frac{d^2 \omega_\lambda}{d\lambda^2} + \left[ Q_r(\lambda) - \Theta(\lambda) \right] \omega_\lambda = 0
\]  

(B.36)

where

\[
\Theta(\lambda) = -\frac{5}{36} \frac{Q_r}{f^2} - \frac{1}{4} \frac{d}{d\lambda} \frac{d^2 Q_r}{d\lambda^2} + \frac{5}{16} \left( \frac{d Q_r}{d\lambda} \right)^2
\]  

(B.37)

Replacing \( U \) by \( U_r \) in Eq. (B.25), we have
\[ \Theta (\lambda ) = \frac{q}{140} \left[ \left( \frac{U''}{U'} \right)_{\lambda = \lambda_1} \right]^2 - \frac{1}{14} \left( \frac{U'''}{U'} \right)_{\lambda = \lambda_1}. \]  

(B.38)

The derivatives of \( U \) involve the derivatives of the centrifugal potential. When the turning point \( \lambda_1 \) is close to zero, this potential dominates and

\[ \Theta (\lambda_1) \rightarrow -\frac{3q}{140} \frac{1}{\lambda_1^2}. \]  

(B.39)

If \( U \ll k^2 \), \( r_1 \) is of the order of \( (l+\lambda)/k \) and will be small for small \( l \) and large \( k \). Therefore, \( \Theta (\lambda_1) \) is large for small \( \lambda \), and the WKB approximation may be unreliable.

As \( \lambda \rightarrow \infty \), \( \Theta (\lambda) \rightarrow 0 \). As \( \lambda \rightarrow 0 \), \( Q_{1r} \rightarrow -l(l+1)/\lambda \), for we have guaranteed the correct singularity at \( \lambda = 0 \) by using \( Q_1 \) instead of \( Q \). In the future, we shall assume \( V \) always less than \( k^2 \), so that there will be only one turning point. Then \( \omega_x \) may be used for all \( r \). To obtain the more familiar WKB formulas, we substitute the asymptotic forms for \( \int_{x_0}^{x} \) into Eq. (B.34). As

\[
\int_{x_0}^{x} \sqrt{k^2 - U - (l+\lambda)/\lambda} \, dx, \quad \text{we have}
\]

\[
\int_{x_0}^{x} \xrightarrow[k \to 0]{} i \log \lambda^2 + \lambda^2, \\
\left| \int_{x_0}^{x} \right| \xrightarrow[k \to 0]{} \infty .
\]

(B.40)

For large \( |f_1| \), we have

\[
\omega_x \leftarrow \frac{1}{2} \sqrt{\frac{2}{\pi q_1}} \left[ (B - A) e^{i f_1} + (A e^{i \pi q_2} + B e^{-i \pi q_2}) e^{-i f_1} \right].
\]

(B.41)

Since \( \omega_x \) must go to zero at the origin, \( A \) must equal \( B \).
Therefore, Eq. (B.41) becomes
\[ \omega_1 \rightarrow \frac{1}{n} A \sqrt{\frac{\pi}{\pi_{/}}} \cos \left( \frac{\pi}{\pi_{/}} \right) \rightarrow \omega_{11}. \] (B.42)

Now
\[ \frac{-1}{n} A \rightarrow \frac{1}{n} A \rightarrow \frac{-1}{n} \omega_{11} \rightarrow \omega_{11} \] (B.43)

The factor \((q_1)^{-1}\) yields a factor \(\omega_{11}\), and \(\omega_{11}\) approaches zero as \(\omega_{11}^{\xi+1}\), as it should.

Since \(A = B\), Eq. (B.34) becomes
\[ \omega_2 (\omega) = \frac{1}{n} A \sqrt{\frac{\pi}{\pi_{/}}} \left[ J_{2} (q_1) + J_{-2} (q_1) \right]. \] (B.44)

For \(\omega_{11} >> \omega_{11}\) becomes large and
\[ \omega_2 (\omega) \rightarrow \frac{1}{n} A \sqrt{\frac{\pi}{\pi_{/}}} \cos \left( \frac{\pi}{\pi_{/}} \right) \sin \left( q_1 + \pi_{/} \right). \] (B.45)

Equation (B.45) must have the form \(\sin (k \omega - \omega_{/1} + \eta_{/1})\) for large \(\omega_{/1}\). This requires
\[ A = \sqrt{\frac{k \omega_{/1} \gamma}{\pi_{/1}}}, \]
\[ \eta_{/1} = \lim_{\omega_{/1} \rightarrow \infty} \left[ \frac{\omega_{/1} \sqrt{k \omega_{/1} - l \omega_{/1}}}{\omega_{/1}} - \frac{\omega_{/1} \sqrt{k \omega_{/1} - (l + \gamma) \omega_{/1}}}{\omega_{/1}} \right], \] (B.46)

where we have used
\[ \int \frac{\omega_{/1}}{k \omega_{/1}} \sqrt{\frac{k \omega_{/1} - (l + \gamma) \omega_{/1}}{\omega_{/1}}} \rightarrow \lim_{\omega_{/1} \rightarrow \infty} k \omega_{/1} - (l + \gamma) \omega_{/1}. \] (B.47)

For completeness, the connection between \(\eta_{/1}\) and the classical scattering angle will be derived. The phase shift \(\eta_{/1}\) may be rewritten as
Differentiating with respect to $\lambda$ and rewriting, one has

$$\eta_\lambda = -\kappa \lambda + (l + \lambda \gamma) \eta_\lambda + \lim_{\lambda \to 0} \left( \sqrt{\lambda - v \cdot (l + \lambda \gamma)} \right) \eta_\lambda - \lambda \int_0^\infty d\lambda \quad (B.48)$$

where $\xi = -\kappa (l + \lambda \gamma)$.

Equation (B.49) is often the starting point for applications of semi-classical scattering theory (21).