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CLASSICAL APPROXIMATION FOR THE CHANGE OF POLARIZATION IN POTENTIAL SCATTERING

by

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I. INTRODUCTION

The problem considered here is that of finding the change of polarization produced by the potential scattering of a particle with charge and magnetic moment. It is difficult to make progress on an exact quantum-mechanical relativistic solution of this problem. However, as shown below, some useful information about the solution can be found through classical considerations. The word classical is used here in the sense of non-quantum but relativistic.

Thomas (1) originally introduced the four-vector description of the polarization of a classical particle and obtained its equation of motion for a particle having normal magnetic moment only subject to external electric and magnetic fields. Later, Bargmann, Michel and Telegdi (2) rederived these equations and pointed out that the classical equation applies exactly for the polarization in the quantum problem when the external fields are static and homogeneous and both the normal and anomalous magnetic moments are included.

Classical relativistic equations of motion for a particle with intrinsic angular momentum also were given by Frenkel (3) and by Kramers (4) using a Lorentz six-vector to describe the polarization. These two approaches are equivalent, as has been shown by Ford and Hirt (5). The four-vector description is used here since it seems to be simpler.

Thomas, Bargmann, Michel and Telegdi limited their work to static homogeneous electric and magnetic fields. Fradkin and Good (6) showed that, even when the fields are nonhomogeneous, the Thomas-Bargmann-Michel-Telegdi equation holds as long as the actions in the problem are large compared to $\hbar$. This means that there is a classical limit to formulae giving the change of polarization produced by electric and magnetic fields. The purpose of this paper is to give this limit for the special case of scattering by a spherically symmetric potential.

It is found that the polarization vector in the classical limit precesses about the angular momentum during the scattering. The problem of finding the total precession angle is reduced to a quadrature. From the derivation it results that for a known potential, the angle of precession of the polarization vector can be divided into two parts: one is that of the contribution due to the normal magnetic moment ($g=2$) of the particle, the other is that due to its anomalous moment. Effects of the anomalous magnetic moment have not been studied before, even for Coulomb scattering.

In Section 2 the derivation of the Thomas-Bargmann-Michel-Telegdi equation is reviewed. Section 3 contains the derivation of the general expression for the polarization precession and the general expression for the differential cross section. The discussion of the validity of the classical approximation is given in Section 4. Detailed results for the special case of a Coulomb field are reported in Section 5. The agreement with the quantum-mechanical results of Fradkin, Weber and Hammer (7) in the appropriate limits is arrived at in Section 6.

II. THE THOMAS-BARGMANN-MICHEL-TELEGDI EQUATION

The Thomas-Bargmann-Michel-Telegdi equation will be derived in this section from first principles. A charged particle with spin moving in an electromagnetic field $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$ is considered. The equations of motion for position and spin are found under the following assumptions;

(a) The charged particle can be treated classically in the sense that it has a position $\vec{X}(t)$ and a polarization or spin in its instantaneous rest system $\vec{\sigma}(t)$.

(b) The effects of field gradients can be neglected and only terms depending on the fields directly need be retained.

Consider a coordinate system fixed in the laboratory and, at any instant, a Lorentz-transformed coordinate system in which the electron is at rest. If the electron has momentum $\vec{p}$ and energy $E = \sqrt{p^2+1}$ in the laboratory (hereafter choose units such that $m = \hbar = c = 1$), the transformation between the two systems is

$$X_\mu^L = a_{\mu\nu} X_\nu^R, \tag{2-1}$$

where

$$a_{ij} = \delta_{ij} + \frac{p_i p_j}{E+1}, \tag{2-2a}$$

$$a_{4i} = -a_{i4} = i p_i , \tag{2-2b}$$

$$a_{44} = E \cdot \tag{2-2c}$$

Greek indices range from 1 to 4, Latin from 1 to 3, and X_4 is it.

Alternatively these formulas can be written as

$$\vec{X}^L = \vec{X}^R + \frac{\vec{V}}{V^2}\left(\frac{1}{\sqrt{1-V^2}} - 1\right)\vec{V}\cdot\vec{X}^R + \frac{\vec{V}t^R}{\sqrt{1-V^2}}, \tag{2-3a}$$

$$t^L = \frac{\vec{V}\cdot\vec{X}^R + t^R}{\sqrt{1-V^2}}, \tag{2-3b}$$

where $\vec{p}$ has been replaced by $\vec{V}/\sqrt{1-V^2}$. In this form they agree with the standard form (8).

Next consider the covariant treatment of velocity and spin. For two neighboring events on the world line of the particle

$$(dX^L)^2-(dt^L)^2=(dX^R)^2-(dt^R)^2=-(d\tau)^2, \tag{2-4}$$

where τ is the proper time. Therefore $d\tau = dt\sqrt{1-v^2}$ is a scalar and the velocity $\frac{dx_\mu}{d\tau}=u_\mu$ is a four vector. The length of this vector must be the same in all Lorentz frames. By evaluating it in the rest system one obtains

$u_\mu u_\mu = -1$.

Let the direction of the spin of the particle in the instantaneous rest system be $\vec{O}$ where $\vec{O}\cdot\vec{O}=1$. Let T_μ be a four vector with components ($\vec{O}$, 0) in the rest system. The Lorentz transformation then gives for the laboratory components

$$T_i = a_{ij}\,O_j = O_i + p_i\frac{\vec{O}\cdot\vec{p}}{E+1}, \tag{2-5a}$$

$$T_4 = a_{4j}\,O_j = i\,\vec{O}\cdot\vec{p}, \tag{2-5b}$$

so

$$O_i = T_i + \frac{i\,p_i\,T_4}{E+1}. \tag{2-6}$$

The values of the scalars that can be formed are found by evaluating them in the rest system:

$$T_\mu T_\mu = \vec{O}\cdot\vec{O} = 1, \qquad (2\text{-}7)$$

$$T_\mu U_\mu = 0. \qquad (2\text{-}8)$$

The equations of motion in the instantaneous system are

$$\frac{d^2\vec{X}}{dt^2} = e\vec{E}, \qquad (2\text{-}9)$$

$$\frac{d\vec{O}}{dt} = \frac{ge}{2}\vec{O}\times\vec{B}. \qquad (2\text{-}10)$$

One can guess the equations of motion in the laboratory system from the following requirements:

(a) They must be covariant, so they will be tensor equations among u_μ , T_ν and the electromagnetic field tensor $F_{\mu\nu}$ where

$$F_{ij} = \varepsilon_{ijk} B_k, \qquad (2\text{-}11)$$

$$F_{i4} = -F_{4i} = -i E_i, \qquad F_{44} = 0. \qquad (2\text{-}12)$$

(b) They must reduce to the correct equations in the rest system.

(c) They must agree with the following conditions:

$$u_\mu u_\mu = -1 , \quad T_\mu T_\mu = 1 , \quad T_\mu u_\mu = 0 . \tag{2-13}$$

Rewrite the rest-system equations in terms of these quantities. In the rest system one has

$$u = (\vec{0}, i) , \quad \tau = t , \quad T = (\vec{\sigma}, 0),$$

$$\frac{du_i}{d\tau} = e\, i\, F_{i4} = e\, F_{i\nu} u_\nu , \tag{2-14}$$

$$\frac{dT_i}{d\tau} = \frac{ge}{2} \epsilon_{ijk} T_j B_k = \frac{ge}{2} F_{i\nu} T_\nu . \tag{2-15}$$

This suggests for the correct equations

$$\frac{du_\mu}{d\tau} = e F_{\mu\nu} u_\nu + \alpha u_\mu u_\nu F_{\nu\rho} T_\rho , \tag{2-16}$$

$$\frac{dT_\mu}{d\tau} = \frac{ge}{2} F_{\mu\nu} T_\nu + \beta u_\mu u_\nu F_{\nu\rho} T_\rho . \tag{2-17}$$

These are covariant, the first terms on the right give the correct limit, and the second terms are zero in the rest system since $u = (\vec{0}, i)$. Choose α , β to make requirement (c) turn out. One finds

$$\frac{1}{2}\frac{d}{d\tau}u_\mu u_\mu = u_\mu \frac{du_\mu}{d\tau}$$

$$= e\,u_\mu F_{\mu\nu} u_\nu + \alpha\, u_\mu u_\mu u_\nu F_{\nu\rho} T_\rho . \qquad (2\text{-}18)$$

Here $F_{\mu\nu}$ is antisymmetric so

$$e\,u_\mu F_{\mu\nu} u_\nu = 0 \qquad (2\text{-}19)$$

and if

$$u_\mu u_\mu = -1 ,$$

it is needed that

$$\alpha = 0 . \qquad (2\text{-}20)$$

Next one finds that

$$\frac{d}{d\tau} T_\mu u_\mu = u_\mu \frac{dT_\mu}{d\tau} + T_\mu \frac{du_\mu}{d\tau}$$

$$= \frac{ge}{2} u_\mu F_{\mu\nu} T_\nu + \beta u_\mu u_\mu u_\gamma F_{\gamma\delta} T_\delta + e T_\mu F_{\mu\nu} u_\nu$$

$$= u_\mu F_{\mu\nu} T_\nu \left(\frac{ge}{2} - \beta - e \right), \tag{2-21}$$

so

$$\beta = (g-2)\frac{e}{2}. \tag{2-22}$$

Finally it is verified that

$$\frac{d}{d\tau} T_\mu T_\mu = 2 T_\mu \frac{dT_\mu}{d\tau} = 0, \tag{2-23}$$

if

$$T_\mu u_\mu = 0. \tag{2-24}$$

The correct equations are then

$$\frac{d}{d\tau} u_\mu = e F_{\mu\nu} u_\nu , \tag{2-25}$$

$$\frac{d}{d\tau} T_\mu = \frac{ge}{2} F_{\mu\nu} T_\nu + (g-2)\frac{e}{2} u_\mu u_\nu F_{\nu\rho} T_\rho , \qquad (2\text{-}26)$$

the last one is the Thomas-Bargmann-Michel-Telegdi equation. These are consistent with $u_\mu u_\mu$, $T_\mu T_\mu$, $T_\mu u_\mu$ being constant. In addition the values

$$u_\mu u_\mu = -1 , \quad T_\mu T_\mu = 1 , \quad T_\mu u_\mu = 0 \qquad (2\text{-}27)$$

are assigned. These eight equations, 2-25 and 2-26, are therefore redundant and one can use

$$u_\mu u_\mu = -1 , \quad T_\mu u_\mu = 0 \qquad (2\text{-}28)$$

to reduce the problem to six equations in $\vec{V}$, $\vec{\sigma}$ consistent with the assignment $\vec{\sigma}\cdot\vec{\sigma} = 1$. One finds, on carrying out this reduction,

$$\frac{d}{dt}(\gamma \vec{V}) = e(\vec{E} + \vec{V}\times\vec{B}) , \qquad (2\text{-}29)$$

$$\gamma \frac{d\vec{\sigma}}{dt} = \frac{ge}{2}\vec{\sigma}\times\left(\vec{B} + \frac{\gamma}{\gamma+1}\vec{E}\times\vec{V}\right) + \qquad (2\text{-}30)$$

$$(g-2)\frac{e}{2}\frac{\gamma^2}{\gamma+1}\vec{\sigma}\times\left[(\vec{E}+\vec{V}\times\vec{B})\times\vec{V}\right]. \qquad (2\text{-}30)$$

The algebra involved in obtaining Equation 2-30 from Equations 2-25 and 2-26 is as follows:

From Equation 2-26, it is seen that

$$\gamma\frac{d\vec{T}}{dt}=\frac{ge}{2}\left[\vec{T}\times\vec{B}-i\,T_4\vec{E}\right]-\frac{e(g-2)}{2}\vec{u}\left[(\vec{u}\times\vec{B})\cdot\vec{T}+\gamma\vec{E}\cdot\vec{T}+i\,\vec{u}\cdot\vec{E}\,T_4\right], \qquad (2\text{-}31)$$

where

$$\gamma=(1-v^2)^{-\frac{1}{2}} \qquad (2\text{-}32)$$

from

$$T_\mu u_\mu=0$$

by solving for T_4 , one finds

$$T_4 = \frac{i\vec{T}\cdot\vec{u}}{\gamma} . \tag{2-33}$$

In Equation 2-31, replace T_4 by $\frac{i\vec{T}\cdot\vec{u}}{\gamma}$ and $\vec{u}$ by $\gamma\vec{V}$. Then it is easy to obtain

$$\gamma\frac{d\vec{T}}{dt} = \frac{ge}{2}\left[\vec{T}\times\vec{B} + \vec{V}\cdot\vec{T}\vec{E}\right]$$

$$-\frac{g-2}{2}e\gamma^2\vec{V}\left[(\vec{V}\times\vec{B})\cdot\vec{T} + \vec{E}\cdot\vec{T}\right.$$

$$\left.-\vec{V}\cdot\vec{E}\,\vec{V}\cdot\vec{T}\right] . \tag{2-34}$$

Furthermore, by using

$$\vec{O} = \vec{T} - \gamma(\gamma+1)^{-1}(\vec{V}\cdot\vec{T})\vec{V} , \tag{2-35}$$

by a straightforward calculation, one obtains,

$$\gamma\frac{d\vec{O}}{dt} = \gamma\frac{d}{dt}\vec{T} - \gamma\frac{d}{dt}\left[\frac{\gamma}{\gamma+1}\vec{V}(\vec{V}\cdot\vec{T})\right] . \tag{2-36}$$

Since $\gamma\frac{d}{dt}\vec{T}$ and $\frac{d}{dt}(\gamma\vec{V}) = e\vec{E} + e\vec{V}\times\vec{B}$ are known, the Equation 2-30 follows from a simple substitution.

III. SOLUTION OF THE CLASSICAL PROBLEM

Let the particle have mass 1, charge e, and magnetic moment $\frac{1}{2} ge$. Suppose it moves in fields $\vec{E} = -\vec{\nabla}\phi$ and $\vec{B} = 0$ where ϕ is spherically symmetric. The equations governing the position $\vec{r}$ and polarization $\vec{\sigma}$ in the classical approximation are

$$\frac{d}{dt}(\gamma \vec{v}) = -e\vec{\nabla}\phi , \tag{3-1}$$

$$\frac{d\vec{\sigma}}{dt} = -e \, \frac{g + g\gamma - 2g}{2(1+\gamma)} \, \vec{\sigma} \times \left(\vec{\nabla}\phi \times \vec{v} \right) , \tag{3-2}$$

where $\vec{v}$ is $\frac{d\vec{r}}{dt}$ and γ is $[1 - v^2]^{-1/2}$. The equation for the polarization is found by specializing Equation 2-30 to the fields considered here. The criterion for validity of these equations will be discussed in the next section.

The orbit equation is independent of the polarization and can be solved separately. Since ϕ depends only on r, $\vec{\nabla}\phi$ can be replaced by $\frac{d\phi}{dr} \vec{r}/r$. Then the angular momentum

$$\vec{L} = \gamma \, \vec{r} \times \vec{v} \tag{3-3}$$

is an integral of motion. The proof that $\vec{L}$ is a constant of motion is given by differentiation with respect to t ;

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\gamma \vec{r} \times \vec{V})$$

$$= \left(\frac{d}{dt}\vec{r}\right) \times \gamma \vec{V} + \vec{r} \times \frac{d}{dt}(\gamma \vec{V})$$

$$= \vec{V} \times \gamma \vec{V} + \vec{r} \times \frac{\vec{r}}{r}\frac{d\phi}{dr} = 0. \qquad (3\text{-}4)$$

It follows that the motion takes place in a plane through the force center. In terms of polar coordinate r , ϑ in this plane, chosen so that ϑ increases for a rotation in the right-handed sense about the angular momentum direction, the magnitude of the angular momentum is

$$L = \gamma r^2 \frac{d\vartheta}{dt}. \qquad (3\text{-}5)$$

The energy

$$E = \gamma + e\phi \qquad (3\text{-}6)$$

is also an integral. The latter can easily be obtained from Equation 3-1 by first taking the dot product with $\vec{V}$ and then integrating.

From Equation 3-3 it follows that

$$\gamma \frac{dr}{dt} = \pm \left[\gamma^2 - 1 - L^2/r^2 \right]^{1/2}, \tag{3-7}$$

where γ is given as a function of r by Equation 3-6. In an ordinary scattering problem the radicand has just one positive root, say r_0. The sign on the right is negative as the particle comes in from infinity to r_0, and positive afterwards. However, in certain pathological cases there is no positive root. For these unphysical cases, the negative sign must be taken, since the particle comes in from infinity and collapses into the force center. These cases will not be considered here, especially since they correspond to $L < 1$ which is outside the validity of the classical description. This will be discussed in Section 5 for a particular potential.

Equations 3-5 and 3-7 combine to give the differential equation of the orbit

$$\frac{dr}{d\vartheta} = \pm r \left[\frac{r^2}{L^2} (\gamma^2 - 1) - 1 \right]^{1/2}. \tag{3-8}$$

The scattering angle, defined to be always positive as shown in Figure 1, is then given by

$$\pi + \epsilon \Theta = 2 \int_{r_0}^{\infty} r^{-1} \left[\frac{r^2}{L^2} (\gamma^2 - 1) - 1 \right]^{-\frac{1}{2}} dr, \tag{3-9}$$

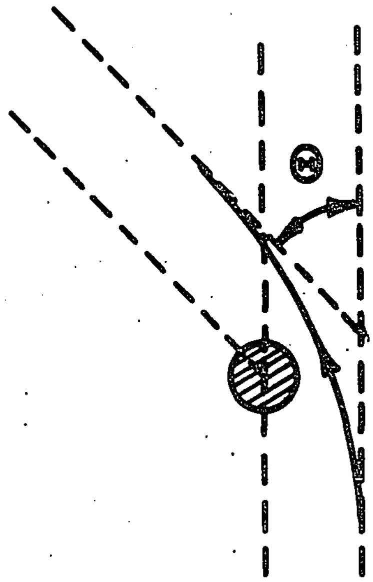


Figure 1a. Attractive scattering

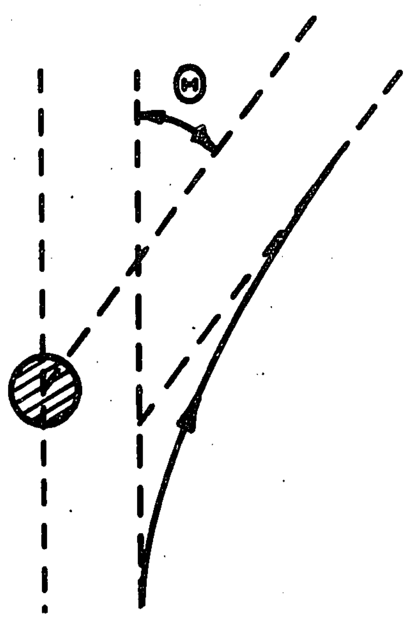


Figure 1b. Repulsive scattering

where ϵ is $+1$ for an attractive potential, -1 for a repulsive potential.

One consequence of Equation 3-2 is that σ is a constant. This is applied in general as argued in Section 2 and also can be seen directly by dotting Equation 3-2 by $\vec{\sigma}$. For the central force field the equation may be rewritten as

$$\frac{d\vec{\sigma}}{dt} = -\epsilon\left[2\gamma r(1+\gamma)\right]^{-1}(g+g\gamma-2\gamma) \cdot \frac{d\phi}{dr}\,\vec{\sigma}\times\vec{L}. \qquad (3\text{-}10)$$

This implies that $\vec{\sigma}\cdot\vec{L}$ is an integral so that, since σ and L are fixed, the angle between $\vec{\sigma}$ and $\vec{L}$ is constant through the motion. The only thing left to determine is the angle through which $\vec{\sigma}$ precesses. Let ω be the precession angle, measured in the right hand sense relative to $\vec{L}$ as shown in Figure 2. Then it follows from Equation 3-10 that

$$\frac{d\omega}{dt} = \epsilon\left[2\gamma r(1+\gamma)\right]^{-1}(g+g\gamma-2\gamma) \cdot L\frac{d\phi}{dr}. \qquad (3\text{-}11)$$

This can be seen as one takes the absolute value of Equation 3-10 and then divides both sides by $\sin\omega$.

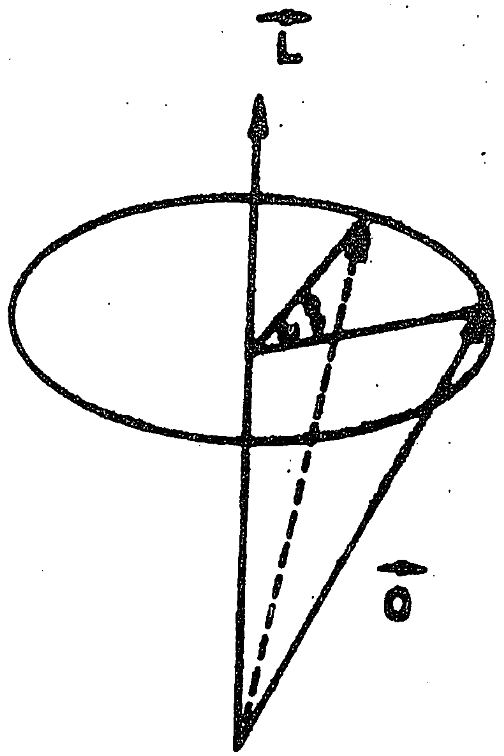


Figure 2. The precession of $\vec{D}$ around $\vec{L}$

Here γ is known as a function of r from Equation 3-6 so the right hand side is a function of r alone. One can use Equation 3-11 to find $\frac{d\omega}{dr}$ and integrate to get the total precession angle Ω,

$$\Omega = e \int_{r_0}^{\infty} (1+\gamma)^{-1} \left[\left(\frac{r}{L}\right)^2 (\gamma^2 - 1) - 1 \right]^{-\frac{1}{2}} (g + g\gamma - 2\gamma) \cdot \frac{d\phi}{dr}\, dr. \tag{3-12}$$

Here

$$\Omega \equiv \Omega_1 + \Omega_2 ,$$

where

$$\Omega_1 = 2e \int_{r_0}^{\infty} (1+\gamma)^{-1} \left[\left(\frac{r}{L}\right)^2 (\gamma^2 - 1) - 1 \right]^{-\frac{1}{2}} \frac{d\phi}{dr}\, dr$$

and

$$\Omega_2 = e(g-2) \int_{r_0}^{\infty} \left[\left(\frac{r}{L}\right)^2 (\gamma^2 - 1) - 1 \right]^{-\frac{1}{2}} \frac{d\phi}{dr}\, dr ,$$

which expresses the contribution to the precession angle due to the normal and the anomalous moments of the particle separately.

The differential cross section $\sigma(\vartheta, \varphi)$ for scattering into the solid angular region around the polar angle ϑ is:

$$\sigma(\Omega)\, d\Omega = \frac{\text{No. of particles scattered into solid angle per unit time}}{\text{Incident intensity}} . \tag{3-13}$$

With respect to Figure 3-a, and for the case of repulsive scattering, the definition of $\sigma(\vartheta)$ is

$$2\pi b\, db = -2\pi \sigma(\vartheta) \sin\vartheta\, d\vartheta , \tag{3-14}$$

where b is the impact parameter as shown. This means that

$$\sigma(\vartheta) = -\frac{1}{2\sin\vartheta}\left[\frac{db^2}{d\Theta}\right]_{\Theta=\vartheta} . \tag{3-15}$$

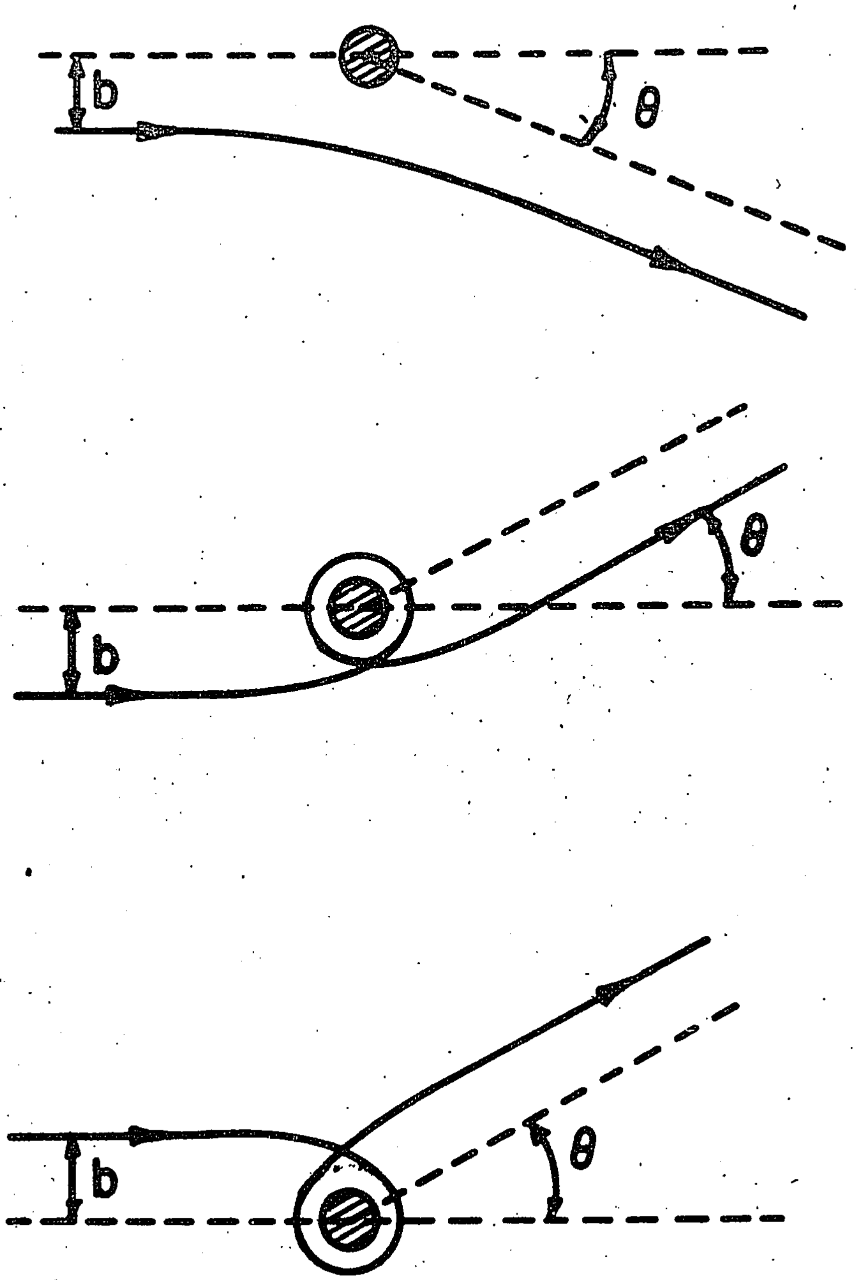
However for the attractive scattering case, the wrap-around effect should be included. The wrap-around effect is divided into the following two cases:

(1) With respect to Figure 3-b, one writes

Figure 3a. Repulsive scattering

Figure 3b. Attractive scattering, direct and wrap around through an angle slightly more than 2π

Figure 3c. Attractive scattering, wrap around through an angle slightly less than 2π


b
θ
θ
b
b
θ

$$2\pi b\,db = -2\pi\,\sigma(\vartheta)\,\sin\vartheta\,d\vartheta$$

$$= -2\pi\,\sigma(\vartheta)\,\sin\vartheta\,d\Theta.$$

(3-16)

Thus, solving for the differential cross section, one obtains

$$\sigma_K(\vartheta) = -\frac{1}{2\sin\vartheta}\left[\frac{db^2}{d\Theta}\right]_{\Theta = 2\pi K + \vartheta},$$

(3-17)

where $K = 0,\ 1,\ 2, \ldots\ldots,$

(2) With respect to Figure 3-c, one writes similarly

$$2\pi b\,db = 2\pi\,\sigma(\vartheta)\,\sin\vartheta\,d\vartheta$$

$$= -2\pi\,\sigma(\vartheta)\,\sin\vartheta\,d\Theta,$$

(3-18)

so that, for the differential cross section,

$$\sigma_j(\vartheta) = \frac{-1}{2\sin\vartheta}\left[\frac{db^2}{d\Theta}\right]_{\Theta=2\pi j-\vartheta,} \qquad (3\text{-}19)$$

where $j = 1, 2, 3, \ldots\ldots,$

Adding all these contributions, one obtains

$$\sigma(\vartheta) = -\frac{1}{2\sin\vartheta}\left\{\sum_{K=0}^{\infty}\left[\frac{db^2}{d\Theta}\right]_{\Theta=2\pi K+\vartheta} + \sum_{j=1}^{\infty}\left[\frac{db^2}{d\Theta}\right]_{\Theta=2\pi j-\vartheta}\right\}. \qquad (3\text{-}20)$$

If the impact parameter b is eliminated through the relation

$$L = pb, \qquad (3\text{-}21)$$

then

$$\sigma(\vartheta) = \frac{-1}{2p^2\sin\vartheta}\left[\sum_{K=0}^{\infty}\left(\frac{\partial L^2}{\partial\Theta}\right)_{\Theta=2\pi K+\vartheta} + \sum_{j=1}^{\infty}\left(\frac{\partial L^2}{\partial\Theta}\right)_{\Theta=2\pi j-\vartheta}\right]. \qquad (3\text{-}22)$$

Notice that the terms in which $K, j = 1, 2, 3, \ldots\ldots$, represent contributions to the cross section from scattering angles greater than 180° which may occur in attractive scattering.

IV. VALIDITY OF THE CLASSICAL DESCRIPTION

If a particle can be described classically, its quantum mechanical description must involve a spatially localized wave packet containing a narrow range in momentum and energy. If a is a characteristic dimension, the spread in momentum is $1/a$, and this must be small compared to the average value so $1/a \ll p$. In order for the packet to remain a packet through the scattering and not be dispersed by the force center, the impact parameter b must be large compared to the size of the packet so $b \gg a \gg 1/p$. Consequently, for a classical description to have meaning, the angular momentum must be restricted by

$$L \gg 1. \tag{4-1}$$

A second criterion for validity can be demonstrated for a spin $\frac{1}{2}$ particle and it will be conjectured to apply in general. Consider, for example, the case of normal moment. The Dirac equation for the particle is

$$\gamma_\mu \pi_\mu \Psi = i \Psi, \tag{4-2}$$

where γ_μ are the Dirac matrices and

$$\Pi_\mu = p_\mu - eA_\mu \tag{4-3}$$

with

$$p_\mu = (\vec{p}, E)$$

and

$$A_\mu = (\vec{A}, \phi),$$

it follows that

$$\gamma_\nu \Pi_\nu \gamma_\mu \Pi_\mu \Psi = -\Psi \tag{4-4}$$

and

$$\gamma_\nu \gamma_\mu \Pi_\nu \Pi_\mu = \Pi_\mu \Pi_\mu - e\vec{\sigma}\cdot\vec{B} + ie\vec{\alpha}\cdot\vec{E}, \tag{4-5}$$

such that

$$\Pi_\mu \Pi_\mu \Psi = \left(e\vec{\sigma}\cdot\vec{B} - ie\vec{\alpha}\cdot\vec{E} - 1\right)\Psi. \tag{4-6}$$

In the present case, $\vec{B}$ is zero. In taking the classical limit, as discussed in detail in Reference 6, one considers functions such that

$$\pi_\mu \Psi = \langle \pi_\mu \rangle \Psi \tag{4-7}$$

is a sufficiently close approximation. In order that Equation 4-6 be consistent with the classical equality

$$\langle \pi_\mu \rangle \langle \pi_\mu \rangle = -1, \tag{4-8}$$

it must be that

$$e \langle \vec{\alpha} \rangle \cdot \vec{E} \ll 1. \tag{4-9}$$

For any Hermitian operator Q and the Hamiltonian H, from Reference 6,

$$\int \Psi^{+} Q (H - e\phi) \Psi \, d\tau + \int [Q (H - e\phi) \Psi]^{+} \Psi \, d\tau$$

$$= 2 \langle H - e\phi \rangle \int \Psi^{+} Q \Psi \, d\tau, \tag{4-10}$$

which may be written in the form

$$\langle [Q, H - e\phi]_{+} \rangle = 2 \gamma \langle Q \rangle. \tag{4-11}$$

An immediate consequence is

$$\langle \vec{\alpha} \rangle = \vec{V}. \tag{4-12}$$

The classical approximation then applies as long as $e\vec{V}\cdot\vec{E} \ll 1$. For the problem under consideration, involving a central potential, this condition reduces to

$$e\left[\left(\frac{d\phi}{dr}\right)\left(\frac{dr}{dt}\right)\right]_{Max.} \ll 1. \tag{4-13}$$

It is conjectured that this condition will also apply for higher spin particles. There are also effects of higher order moments that come in with spins greater than $\frac{1}{2}$. These depend on gradients of the external fields and are not considered in the present work.

V. COULOMB FIELD CASE

Consider the Coulomb potential $\phi = -\frac{\epsilon Z e}{r}$, where Z is positive. Then the integrals in Equations 3-9 and 3-12 namely

$$\pi + \epsilon \Theta = 2 \int_{r_0}^{\infty} r^{-1} \left[\frac{r^2}{L^2} (\gamma^2 - 1) - 1 \right]^{-\frac{1}{2}} dr \qquad (5\text{-}1)$$

and

$$\Omega = e \int_{r_0}^{\infty} (1+\gamma)^{-1} \left[\left(\frac{r}{L}\right)^2 (\gamma^2 - 1) - 1 \right]^{-\frac{1}{2}} \cdot (g + g\gamma - 2\gamma) \frac{\epsilon Z e}{r^2} \, dr \qquad (5\text{-}2)$$

are elementary. One solves

$$E = \gamma + e\phi = \gamma - \frac{\epsilon Z e^2}{r} \qquad (5\text{-}3)$$

for γ , and substitutes into Equation 5-1 and Equation 5-2 to obtain integrals which can be reduced to the form

$$I = \int_{r_0}^{\infty} \frac{1}{r+a} \frac{1}{\left[A r^2 + B r + C \right]^{1/2}} \, dr. \qquad (5\text{-}4)$$

This integral can be evaluated as

$$I = \frac{1}{\sqrt{C - aB + a^2 A}} \cdot$$

$$\arcsin \frac{2(C - aB + a^2 A) + (B - 2aA)(r+a)}{(r+a)\sqrt{4AC - B^2}} \Bigg|_{r_0}^{\infty},$$

(5-5)

where $-\frac{\pi}{2} \leq \arcsin \leq \frac{\pi}{2}$. One sees that the factor

$$\left[\left(\frac{r}{L} \right)^2 (\gamma^2 - 1) - 1 \right]^{-1/2},$$

which is by Equation 5-3

$$\frac{1}{\left[r^2 \left(\frac{E^2 - 1}{L^2} \right) + \frac{2 \epsilon E Z e^2}{L^2} r + \left(\frac{Z^2 e^4}{L^2} - 1 \right) \right]^{1/2}}, \qquad (5\text{-}6)$$

is common to both the integrals. Then the integral in Equation 5-1 reduces to

$$\int_{r_0}^{\infty} \frac{1}{r} \frac{1}{\left[r^2\left(\frac{E^2-1}{L^2}\right) + \frac{2\epsilon E z e^2}{L^2} r + \left(\frac{z^2 e^4}{L^2} - 1\right)\right]^{1/2}} dr \qquad (5\text{-}7)$$

and this is readily evaluated by Equation 5-5. Furthermore,

$$\Omega \equiv \Omega_1 + \Omega_2 , \qquad (5\text{-}8)$$

where Ω_1 is

$$2 z e^2 \epsilon \int_{r_0}^{\infty} \frac{1}{1+\gamma} \cdot \frac{1}{\left[\left(\frac{r}{L}\right)^2 (\gamma^2-1) - 1\right]^{1/2}} \frac{dr}{r^2} \qquad (5\text{-}9)$$

and Ω_2 is

$$(g-2) \epsilon z e^2 \int_{r_0}^{\infty} \frac{1}{\left[\left(\frac{r}{L}\right)^2 (\gamma^2-1) - 1\right]^{1/2}} \frac{dr}{r^2} . \qquad (5\text{-}10)$$

It is seen that

$$\Omega_1 = -2(E+1)\int_{r_0}^{\infty} \frac{dr}{\left(r(E+1)+\epsilon z e^2\right)\left[r^2\left(\frac{E^2-1}{L^2}\right)+\frac{2\epsilon E z e^2}{L^2}r+\left(\frac{z^2e^4}{L^2}-1\right)\right]^{1/2}}$$

$$+2\int_{r_0}^{\infty} \frac{dr}{r\left[r^2\left(\frac{E^2-1}{L^2}\right)+\frac{2\epsilon E z e^2}{L^2}r+\left(\frac{z^2e^4}{L^2}-1\right)\right]^{1/2}} \qquad (5\text{-}11)$$

and both integrals involved have the form illustrated in Equation 5-5. As for the case of

$$\Omega_2 = (g-2)\epsilon z e^2\int_{r_0}^{\infty} \frac{dr}{r^2\left[r^2\left(\frac{E^2-1}{L^2}\right)+\frac{2\epsilon E z e^2}{L^2}r+\left(\frac{z^2e^4}{L^2}-1\right)\right]^{1/2}}, \qquad (5\text{-}12)$$

one sees by means of the transformation

$$\int_{r_0}^{\infty} \frac{dr}{r^2\left[Ar^2+Br+C\right]^{1/2}}$$

$$= \frac{B}{2C}\int_{r_0}^{\infty}\frac{1}{r}\,\frac{dr}{\left[Ar^2+Br+C\right]^{1/2}} - \frac{\left(Ar^2+Br+C\right)^{1/2}}{Cr}\Bigg|_{r_0}^{\infty}, \tag{5-13}$$

that it also can be put into the form of Equation 5-5. Here r_0 is the distance of closest approach and it is found by setting $\frac{dr}{d\theta}=0$ or

$$\frac{r^2}{L^2}\left(\gamma^2-1\right)-1=0, \tag{5-14}$$

with γ given in Equation 5-3. It is convenient to introduce parameters η , λ and β such that

$$\lambda = Ze^2, \qquad \eta = \lambda/L$$

and

$$\beta = \frac{1}{E}\left(E^2-1\right)^{1/2}. \tag{5-15}$$

Then the final results are

$$(1-\eta^2)^{1/2}(\Theta-\pi) = (\epsilon+1)\pi\left[1-(1-\eta^2)^{1/2}\right]$$

$$-2\arctan\left[\frac{\beta}{\eta}(1-\eta^2)^{1/2}\right]$$

(5-16)

and

$$\Omega = \Omega_1 + \Omega_2 , \tag{5-17}$$

where

$$\Omega_1 = \epsilon\Theta - 2\epsilon \arctan\left(\frac{\eta}{E\beta}\right) \tag{5-17a}$$

$$\Omega_2 = (g-2)\,E\eta^2(1-\eta^2)^{-1}\left[\frac{\epsilon\beta}{\eta} + \frac{1}{2}(\pi+\epsilon\Theta)\right]. \tag{5-17b}$$

The Ω_2 term in the last expression corresponds to the part due to $g-2 \neq 0$, or the anomalous moment contribution. These equations apply for $0 \leq \eta < 1$, or $\lambda < L$.

For the Coulomb field, the total energy is

$$E = \sqrt{p_r^2 + \frac{L^2}{r^2} + 1} - \frac{\epsilon\lambda}{r} . \tag{5-18}$$

Here the question arises whether the particle during its motion can approach arbitrarily close to the center. First of all, it is clear that this is never possible if both charges, namely the charge of the incoming particle and the charge of the scatterer, are of the same sign. Futhermore, in the case of attraction, an arbitrarily close approach to the center is *not* possible if $L > |\lambda|$, for in this case the first term in Equation 5-18 is always larger than the second and for $r \to 0$, the right hand side of the equation would approach infinity. On the other hand, if $L < |\lambda|$, than as $r \to 0$, this expression can remain finite although p_r approaches infinity. Thus, if

$$L < |\lambda|, \tag{5-19}$$

the particle during its motion falls into the charge attracting it, in contrast to nonrelativistic mechanics where such a collapse is impossible. However, the above discussion is governed by the validity condition of the classical description as given by Equation 4-1, so that

$$L > |\lambda| \tag{5-20}$$

and the possibility of the particle falling into the scatterer should not be considered here.

The next thing to do is to find the differential cross section for the Coulomb potential scattering.

By defining $\bar{a} \equiv \left(\frac{\lambda}{p}\right)^2$, $b \equiv \frac{L}{p}$, so that $\eta = \frac{\bar{a}^{1/2}}{b}$.
One can rewrite the orbit equation as

$$\Theta = \pi + \frac{(\epsilon+1)\pi\left[1-\left(1-\frac{\bar{a}}{b^2}\right)^{1/2}\right]}{\left(1-\frac{\bar{a}}{b^2}\right)^{1/2}}$$

$$-\frac{2}{\left(1-\frac{\bar{a}}{b^2}\right)^{1/2}} \arctan\left[\beta\left(\frac{b^2}{\bar{a}}-1\right)^{1/2}\right]. \qquad (5\text{-}21)$$

Consider $\bar{a}$ as a fixed parameter, while differentiating with respect to Θ :

$$1 = \frac{db^2}{d\Theta}\left[-\frac{1}{2}\frac{(\epsilon+1)\pi}{\left(1-\frac{\bar{a}}{b^2}\right)^{3/2}}\frac{\bar{a}}{b^4}\right.$$

$$+\frac{1}{\left(1-\frac{\bar{a}}{b^2}\right)^{3/2}}\frac{\bar{a}}{b^4}\arctan\left[\beta\left(\frac{b^2}{\bar{a}}-1\right)^{1/2}\right]$$

$$\left.-\frac{\sqrt{\bar{a}}}{b}\cdot\frac{1}{\left(1-\frac{\bar{a}}{b^2}\right)}\frac{1}{1+\beta^2\left(\frac{b^2}{\bar{a}}-1\right)}\frac{\beta}{\bar{a}}\right]. \qquad (5\text{-}22)$$

By substituting in Equation 5-18, therefore, one finds

$$\frac{db^2}{d\Theta} = - \frac{\bar{a}\left(1 - \frac{\bar{a}}{b^2}\right)}{\frac{1}{2}\left(\frac{\bar{a}}{b^2}\right)^2(\Theta + \epsilon\pi) + \beta \frac{\sqrt{\bar{a}}}{b\left(1 + \beta^2 \frac{b^2}{\bar{a}}\left(1 - \frac{\bar{a}}{b^2}\right)\right)}} . \quad (5\text{-}23)$$

Now by using the result in Equations 3-15, 3-17 and 3-19 and the definition

$$R = \left(\frac{\lambda}{2\beta^2 E \sin^2 \vartheta/2}\right)^2, \quad (5\text{-}24)$$

the cross section for repulsive scattering ($\epsilon = -1$), is found to be

$$\frac{\sigma(\vartheta)}{R} = 2 \tan\frac{\vartheta}{2} \sin\frac{\vartheta}{2}\, \beta^2 \left\{ \left(\frac{1-\eta^2}{\eta^3}\right)\left(\eta(\vartheta - \pi) + \frac{2\beta}{\eta^2 + \beta^2(1-\eta^2)}\right)^{-1} \right\}_{\Theta = \vartheta}.$$

(5-25)

For attractive scattering ($\epsilon = +1$) the cross section is

$$\frac{\sigma(\vartheta)}{R} = 2\tan\frac{\vartheta}{2}\sin^2\frac{\vartheta}{2}\;\beta^2\sum_{k=0}^{\infty}\left\{\frac{1-\eta^2}{\eta^3}\left[\eta\left\{(2k+1)\pi+\vartheta\right\}\right.\right.$$

$$\left.\left.+\frac{2\beta}{\eta^2+\beta^2(1-\eta^2)}\right]^{-1}\right\}_{\Theta=2\pi k+\vartheta}$$

$$+\,2\tan\frac{\vartheta}{2}\sin^2\frac{\vartheta}{2}\;\beta^2\sum_{k=1}^{\infty}\left\{\frac{1-\eta^2}{\eta^3}\left[\eta\left\{(2k+1)\pi-\vartheta\right\}\right.\right.$$

$$\left.\left.+\frac{2\beta}{\eta^2+\beta^2(1-\eta^2)}\right]^{-1}\right\}_{\Theta=2\pi k-\vartheta.} \qquad (5\text{-}26)$$

From the orbit Equation 5-16 , the following relationship can be obtained

$$\cos^2\left[\frac{(1-\eta^2)^{1/2}}{2}(\Theta+\epsilon\pi)\right] = \frac{1}{1+\dfrac{\beta^2}{\eta^2}(1-\eta^2)}. \qquad (5\text{-}27)$$

Consider

$$S \equiv 2\tan\frac{\vartheta}{2}\sin^2\frac{\vartheta}{2}\;\beta^2\,\frac{1-\eta^2}{\eta^3}$$

$$\cdot\left[\eta(\Theta+\epsilon\pi)+\frac{2\beta}{\eta^2\left\{1+\dfrac{\beta^2}{\eta^2}(1-\eta^2)\right\}}\right]^{-1} \qquad (5\text{-}28)$$

$$= \frac{2\tan\frac{\vartheta}{2}\sin^2\frac{\vartheta}{2}\tan^2\left[\frac{(1-\eta^2)^{1/2}(\Theta+\epsilon\pi)}{2}\right]}{\eta^2(\Theta+\epsilon\pi)-\frac{2}{(1-\eta^2)^{1/2}}\tan\left[\frac{(1-\eta^2)^{1/2}(\Theta+\epsilon\pi)}{2}\right]\cos^2\left[\frac{(1-\eta^2)^{1/2}(\Theta+\epsilon\pi)}{2}\right]} . \tag{5-28}$$

Then it is found that

$$S = \frac{\sin\vartheta\left\{\tan\frac{\vartheta}{2}\tan\left[\frac{(1-\eta^2)^{1/2}(\Theta+\epsilon\pi)}{2}\right]\right\}^2}{\eta^2(\Theta+\epsilon\pi)-(1-\eta^2)^{-\frac{1}{2}}\sin\left[(1-\eta^2)^{1/2}(\Theta+\epsilon\pi)\right]} , \tag{5-29}$$

so finally

$$\frac{\sigma(\vartheta)}{R} = \sum_{K,j}\frac{\sin\vartheta\left\{\tan\frac{\vartheta}{2}\tan\left[\frac{(1-\eta^2)^{1/2}(\Theta+\epsilon\pi)}{2}\right]\right\}^2}{\eta^2(\Theta+\epsilon\pi)-(1-\eta^2)^{-\frac{1}{2}}\sin\left[(1-\eta^2)^{1/2}(\Theta+\epsilon\pi)\right]} . \tag{5-30}$$

Here the sum ranges over $K = 0, 1, 2, \ldots\ldots$, where $\Theta = 2\pi k + \vartheta$ and over $j = 1, 2, 3, \ldots\ldots$, where $\Theta = 2\pi j - \vartheta$. This result is the classical relativistic generalization of the Rutherford differential cross section

$$R = \left[\frac{\lambda}{2\beta^2 E \sin^2 \frac{\vartheta}{2}} \right]^2. \qquad (5\text{-}31)$$

For the rest of the discussion only the $g=2$ case is considered in order to make the comparison with quantum mechanical case. Then the parameter η may be eliminated between Equations 5-16 and 5-17 to yield the precession equation

$$\Lambda = \cos\left\{ \frac{1}{2}(\pi + \epsilon\Theta)\left[\frac{1-\Lambda^2}{1-\Lambda^2+\beta^2\Lambda^2} \right]^{\frac{1}{2}} \right\}, \qquad (5\text{-}32)$$

where Λ is defined by

$$\Lambda = E \sin\left[\frac{1}{2}(\Omega - \epsilon\Theta) \right]. \qquad (5\text{-}33)$$

From the precession Equation 5-17 by setting $g=2$ this yields

$$\left[1 + \frac{\beta^2}{\eta^2}(1-\eta^2) \right]^{-\frac{1}{2}} = -\epsilon\Lambda.$$

Then one finds

$$\frac{\beta^2}{\eta^2}(1-\eta^2) = \frac{1-\Lambda^2}{\Lambda^2} = \frac{\beta^2}{\frac{1}{1-\eta^2}-1}, \tag{5-34}$$

so that

$$\frac{1}{1-\eta^2} = 1 + \frac{\beta^2\Lambda^2}{1-\Lambda^2}. \tag{5-35}$$

When this expression is substituted into the orbit equation, it reduces to

$$\Theta = \pi + (\epsilon+1)\pi\left[\left(1+\frac{\beta^2\Lambda^2}{1-\Lambda^2}\right)^{1/2} - 1\right]$$

$$-2\left(1+\frac{\beta^2\Lambda^2}{1-\Lambda^2}\right)^{1/2}\arctan\left(\frac{1-\Lambda^2}{\Lambda^2}\right)^{1/2}. \tag{5-36}$$

However one may write

$$\arcsin x = \arctan\frac{x}{(1-x^2)^{1/2}}$$

$$= \frac{\pi}{2} - \arctan\frac{(1-x^2)^{1/2}}{x}, \tag{5-37}$$

so

$$\Theta = \pi + (\epsilon+1)\pi\left[\left(1+\frac{\beta^2\Lambda^2}{1-\Lambda^2}\right)^{1/2} - 1\right]$$

$$-2\left(1+\frac{\beta^2\Lambda^2}{1-\Lambda^2}\right)^{1/2}\left(\frac{\pi}{2} - \arcsin|\Lambda|\right), \qquad (5\text{-}38)$$

and consequently

$$\pi - \epsilon\Theta = \left(1+\frac{\beta^2\Lambda^2}{1-\Lambda^2}\right)^{1/2}\left(\pi + 2\epsilon \arcsin|\Lambda|\right)$$

$$= \left(1+\frac{\beta^2\Lambda^2}{1-\Lambda^2}\right)^{1/2}\left(\pi - 2\arcsin\Lambda\right). \qquad (5\text{-}39)$$

Then from

$$\arcsin x = \frac{\pi}{2} - \arccos x \qquad (5\text{-}40)$$

it is seen that

$$\Lambda = \cos\left\{\frac{1}{2}(\pi + \epsilon\Theta)\left[\frac{1-\Lambda^2}{1-\Lambda^2+\beta^2\Lambda^2}\right]^{1/2}\right\}. \qquad (5\text{-}41)$$

These classical equations are expected to be valid when the criteria $\eta < \lambda$ and $e\left[\left(\frac{dr}{dt}\right)\left(\frac{d\phi}{dr}\right)\right]_{Max} < 1$ are met. From the relations

$$\gamma = E + \frac{\epsilon\lambda}{r} = E + \epsilon u \tag{5-42}$$

and

$$\gamma \frac{dr}{dt} = \pm \left[(\gamma^2 - 1) - \frac{L^2}{r^2} \right]^{\frac{1}{2}}, \tag{5-43}$$

where

$$u = \frac{\lambda}{r}, \tag{5-44}$$

one finds

$$\frac{dr}{dt} = \pm \frac{1}{E + \epsilon u} \cdot \left[-1 + E^2 + 2\epsilon E u + \frac{\eta^2 - 1}{\eta^2} u^2 \right]^{\frac{1}{2}}. \tag{5-45}$$

Therefore, by defining a quantity

$$Q \equiv \left| \frac{ze^2}{r^2} \frac{dr}{dt} \right| = \frac{1}{\lambda} \frac{u^2}{E + \epsilon u} \cdot \left[-1 + E^2 + 2\epsilon E u + \frac{\eta^2 - 1}{\eta^2} u^2 \right]^{\frac{1}{2}}, \tag{5-46}$$

the condition

$$Q_{max} < 1 \qquad (5\text{-}47)$$

corresponds to the second criterion of validity.

In what follows is the standard procedure to find the maximum value of a quantity like Q. Define $\chi = \frac{\eta^2}{1-\eta^2}$, so that

$$\left.\frac{dQ}{du}\right|_{Q=Q_{max.}} = \frac{-2\epsilon u\left\{u^3 + d_1 u^2 + d_2 u + d_3\right\}}{\lambda\chi(E+\epsilon u)^2\left[(E^2-1)+2\epsilon E u - \frac{1}{\chi}u^2\right]^{1/2}} = 0, \qquad (5\text{-}48)$$

where

$$d_1 = \frac{3}{2}\epsilon E(1-\chi) \qquad (5\text{-}49a)$$

$$d_2 = -\frac{\chi}{2}(6E^2-1) \qquad (5\text{-}49b)$$

$$d_3 = -\epsilon\chi E(E^2-1). \qquad (5\text{-}49c)$$

Consequently the roots of the cubic equation

$$U^3 + d_1 u^2 + d_2 u + d_3 = 0 \tag{5-50}$$

give the extreme value of Q. The answer is given in the following paragraph; a derivation may be found in many books on algebra such as Uspensky's (9).

The solutions of the cubic equation are given by the formula

$$u_j = -\frac{d_1}{3} + 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\delta}{3} + \frac{2\pi j}{3}\right), \qquad j = 0, 1, 2 \tag{5-51}$$

where δ is given by

$$\delta = \cos^{-1}\frac{-\frac{q}{2}}{\left(-\frac{p}{3}\right)^{3/2}}. \tag{5-52}$$

Here δ is limited to the first or the second quadrant, and the expressions for p and q are

$$p = \frac{1}{3}\left[3d_2 - d_1^2\right] \tag{5-53a}$$

$$q = \frac{1}{27}\left[2d_1^3 - 9d_1d_2 + 27d_3\right]. \tag{5-53b}$$

The solutions obtained are tested by substituting each in turn into the Equation 5-46. Then the solution which gives the maximum will be known. By replacing E by $\frac{1}{(1-\beta^2)^{1/2}}$, the answer for $Q_{max.}$ becomes

$$\lambda Q_{max.} = \frac{\xi^2}{(1-\beta^2)(1+\epsilon\xi)} \left(\beta^2 + 2\xi\epsilon - \frac{\xi}{\chi} \right)^{1/2} < \lambda, \tag{5-54}$$

where

$$\chi = \eta^2 / 1-\eta^2 \tag{5-55a}$$

$$\xi = \frac{-\epsilon}{2}(1-\chi) + \rho \cos\left[\frac{1}{3}(\delta + 2\pi - 2\pi\epsilon)\right] \tag{5-55b}$$

$$\rho = \left[(1+\chi)^2 - \frac{2}{3}\chi(1-\beta^2)\right]^{1/2} \tag{5-55c}$$

$$\delta = \arc\cos\left\{ -\epsilon\rho^3 \left[(1+\chi)(1-2\chi-\chi^2) + (1-\beta^2)\chi(\chi+3) \right] \right\}. \tag{5-55d}$$

These criteria are illustrated in Figures 4 and 5. The solid curves on Figures 4 and 5 show λ as a function of β and η for which $Q_{max} = 1$. The inequality 5-54 is satisfied by points to the left of the solid curves. The inequality 4-1, $L \gg 1$, which by the definition of $\eta = \lambda / L$ is equivalent to $\lambda \gg \eta$ is satisfied by points to the left of the dashed line in Figures 4 and 5. Both of these inequalities should be satisfied.

The scattering angle Θ obtained from Equation 5-16 is

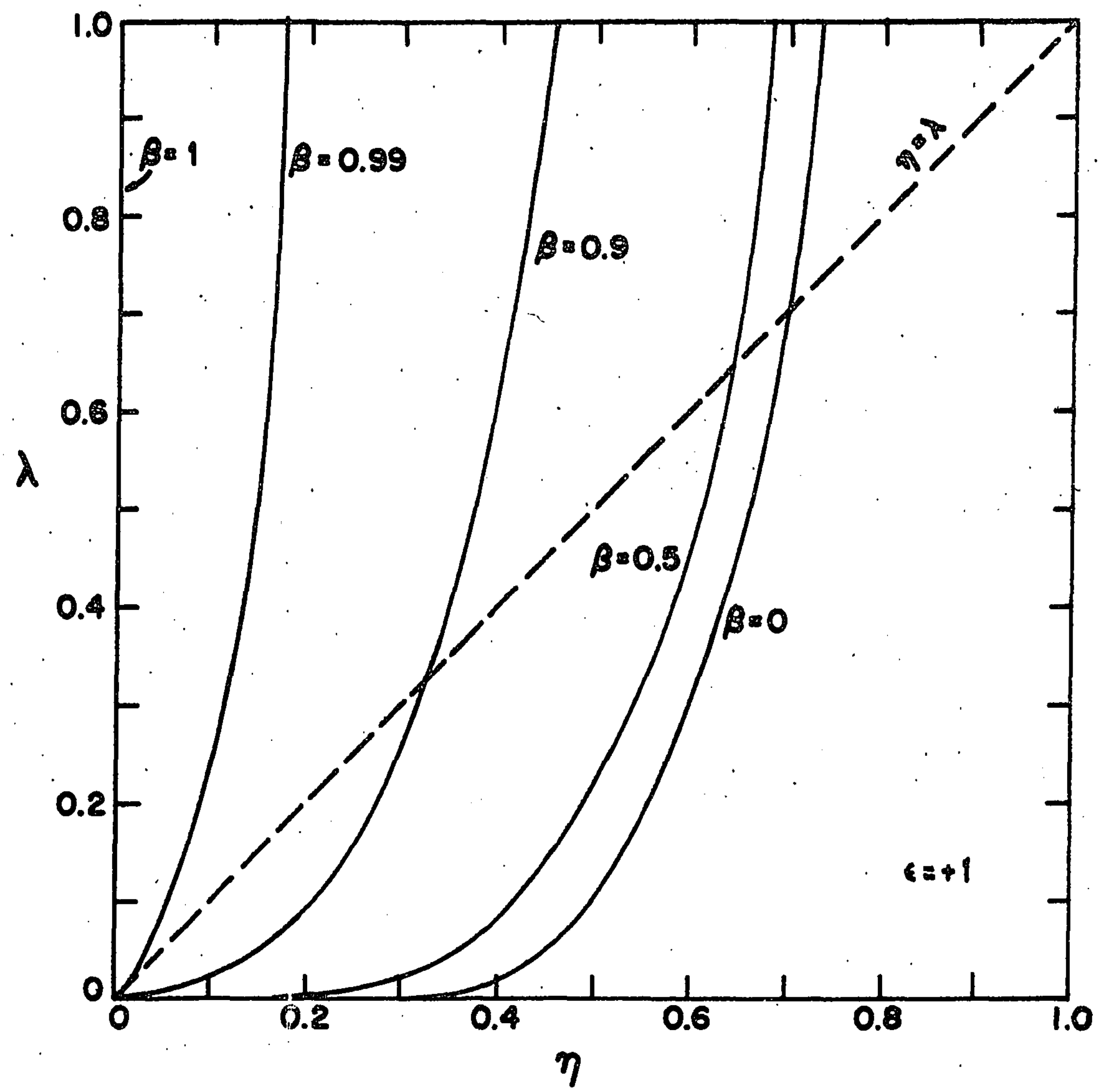


Figure 4. Graph to determine the range of applicability of the classical approximation for attractive Coulomb scattering. The solid lines are plots of λ vs η for various values of β as given by the equation $\lambda Q_{MAX}(\lambda, \beta, \eta) = \lambda$. The dashed line is a plot of $\eta = \lambda$. For given values of λ and β the solid and dashed lines determine two special values of η . The classical approximation applies for values of η less than both these special values.

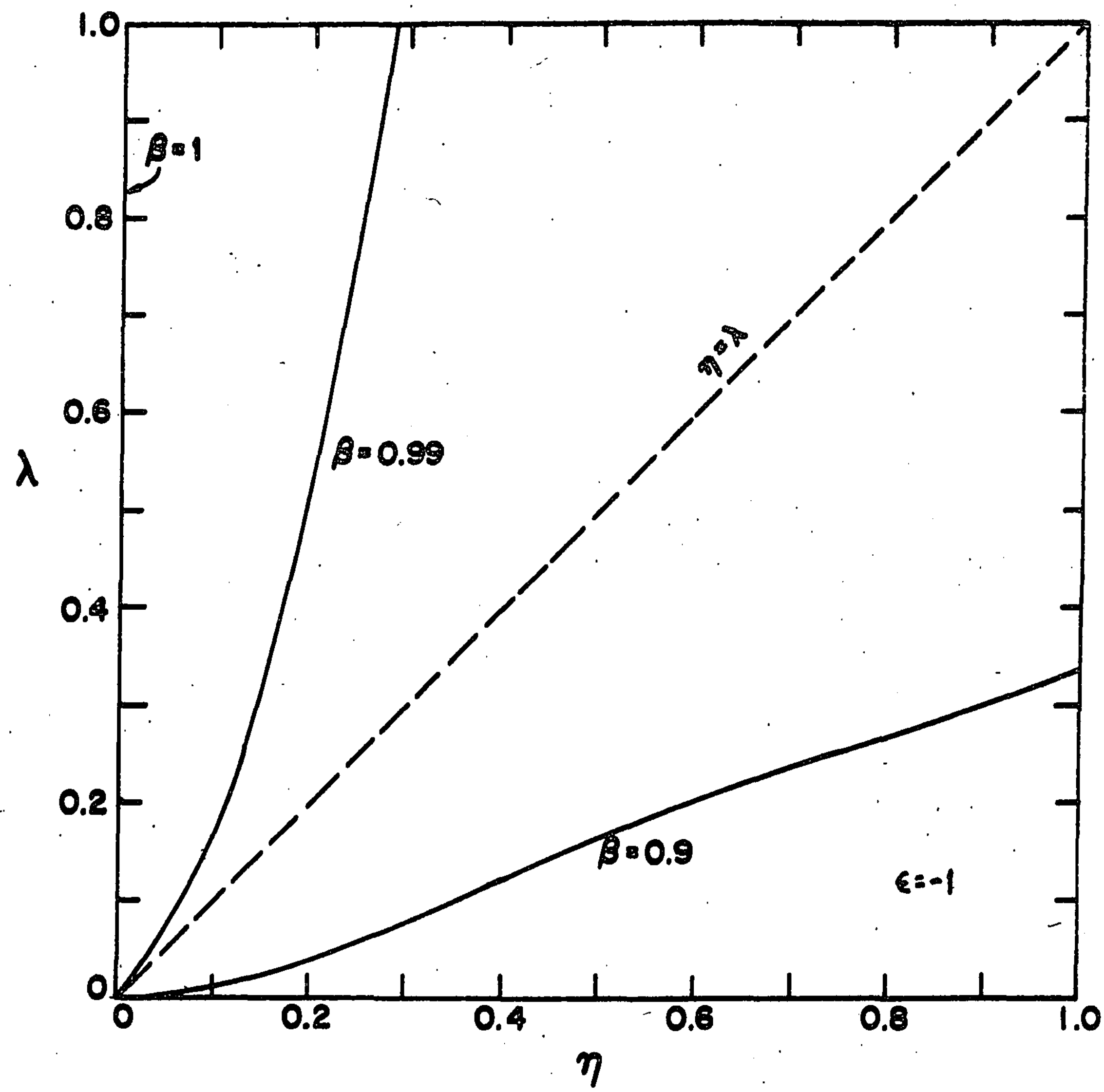


Figure 5. The graph for repulsive Coulomb scattering corresponding to Figure 4. The solid lines are found from $\lambda Q_{MAX}(\lambda, \beta, \eta) = \lambda$.

plotted in Figures 6 and 7. From a comparison of Figures 4 and 6 and again from a comparison of Figures 5 and 7, one sees that the classical results are valid in the extreme relativistic region only for very small Θ. Also, it is seen that the wrap-around effect, or the scattering through angles greater than 2π, occurs only for nonrelativistic attractive scattering.

The preceding equations simplify considerably if one considers only the significant region of small η, or in other words $\lambda/L \ll 1$. In this approximation,

$$\lambda Q_{Max} = 2(3)^{-\frac{3}{2}} \eta^2 \beta^3 (1-\beta^2)^{-1} \tag{5-56}$$

and the equation is simply

$$\eta = \beta \tan \Theta/2 . \tag{5-57}$$

One can expand the differential cross section for small ,
using the facts that

$$\tan \tfrac{1}{2}\left(1-\tfrac{\eta^2}{2}\right)(\vartheta+\epsilon\pi) = \tan \tfrac{1}{2}(\vartheta+\epsilon\pi) - \frac{\eta^2}{4}(\vartheta+\epsilon\pi)\sec^2 \tfrac{1}{2}(\vartheta+\epsilon\pi) \tag{5-58a}$$

and

$$\sin\left(1-\frac{\eta^2}{2}\right)(\vartheta+\epsilon\pi) = \sin(\vartheta+\epsilon\pi)$$

$$-\frac{\eta^2}{2}(\vartheta+\epsilon\pi)\cos(\vartheta+\epsilon\pi) \qquad (5\text{-}58b)$$

for small η, substituting into Equation 5-30 , and taking $K=0$. The result is

$$\frac{\sigma(\vartheta)}{R} = 1 + \frac{\beta^2}{2}\tan^2\frac{\vartheta}{2}\left[(\vartheta+\epsilon\pi)\cot\vartheta - 1\right] + O(\eta^4).$$

(5-59)

The wrap-around effect for nonrelativistic attractive scattering is ignored in this approximation which takes $\Theta=\vartheta$. The contribution to the cross section from the wrap-around effect given by $K, j = 1, 2, 3, \ldots\ldots$, in Equation 3-22 is negligible compared to the direct term $K=0$ except for angles Θ very close to π . For example, it is for $\sigma(\Theta)/R$ at $\beta=0.01$, which is relatively on the nonrelativistic side, that the following table is constructed. The formulae used here are:

Table 1. $\frac{\sigma(\vartheta)}{R}$ at $\beta = 0.01$

Angle ϑ	$\frac{\sigma(\vartheta)}{R}$ direct approx.	$\frac{\sigma(\vartheta)}{R}$ direct exact	$\frac{\sigma(\vartheta)}{R}$ indirect exact	Ratio = $\frac{\text{exact indirect}}{\text{exact direct}}$
60°	$1 + 2.36(10)^{-5}$		$1.355(10)^{-5}$	10^{-5}
90°	$1 - 0.5(10)^{-4}$		0	0
120°	0.9995	1.0009	$1.031(10)^{-4}$	10^{-4}
150°	0.9920	.9240	$4.755(10)^{-3}$	$5(10)^{-3}$
175°	0.3358	.9296	$1.708(10)^{-1}$	.18
178°		1.3995	$6.97\ (10)^{-1}$	.5

Direct term (approximation as η becomes small.)

$$\frac{\sigma}{R} = 1 + \frac{\eta^2}{2}\left\{ (\vartheta + \pi) \cot \vartheta - 1 \right\}. \tag{5-60}$$

Direct term (exact)

$$\frac{\sigma}{R} = \frac{\sin\vartheta \tan^2\frac{\vartheta}{2} \tan^2\left[\frac{(1-\eta^2)^{\frac{1}{2}}}{2}(\vartheta+\pi)\right]}{\eta^2(\vartheta+\pi) - (1-\eta^2)^{-\frac{1}{2}} \sin\left[(1-\eta^2)^{\frac{1}{2}}(\vartheta+\pi)\right]}. \tag{5-61}$$

Single wrap around (exact)

$$\frac{\sigma}{R} = \frac{\sin\vartheta \tan^2\frac{\vartheta}{2} \tan^2\left[\frac{(1-\eta^2)^{\frac{1}{2}}}{2}(3\pi-\vartheta)\right]}{\eta^2(3\pi-\vartheta) - (1-\eta^2)^{-\frac{1}{2}} \sin\left[(1-\eta^2)^{\frac{1}{2}}(3\pi-\vartheta)\right]}. \tag{5-62}$$

In the case of the indirect term, only the $j=1$ case is compared to $K=0$ direct term. From the above table, it is easily seen that the wrap-around effect is negligible except for angles Θ very close to π.

In the approximation of small η, the precession parameter Λ, defined by Equation 5-33, is given by

$$\Lambda = -\epsilon \sin\vartheta/2 \left[1 - \tfrac{1}{4}\epsilon\beta^2 (\pi + \epsilon\vartheta) \tan\vartheta/2 + O(\eta^4) \right].$$

(5-63)

Consequently, in the non-relativistic limit $\beta \to 0$

$$\sin(\Omega - \epsilon\vartheta) = -\frac{\epsilon}{E} \sin\vartheta \left\{ 1 + \tfrac{1}{2}\beta^2 \tan^2\vartheta/2 \right.$$

$$\left. \cdot \left[1 - (\epsilon\pi + \vartheta) \cot\vartheta \right] + O(\beta^4) \right\}, \qquad (5\text{-}64)$$

and in the relativistic limit, $\beta \to 1$, ϑ small

$$\sin(\Omega - \epsilon\vartheta) = -\frac{\epsilon}{E} \sin\vartheta \left[1 + O(1-\beta^2) \right]. \qquad (5\text{-}65)$$

The derivations of these formulas are given next.

For the non-relativistic limit, or for small β , from Equation 5-32, one has

$$\Lambda = \cos\left\{ \frac{\pi+\epsilon\vartheta}{2}\left(1+\frac{\beta^2\Lambda^2}{1-\Lambda^2}\right)^{-\frac{1}{2}} \right\}$$

$$= \cos\left\{ \frac{\pi+\epsilon\vartheta}{2}\left(1-\frac{1}{2}\,\frac{\beta^2\Lambda^2}{1-\Lambda^2}+\cdots\right) \right\}$$

$$= -\epsilon \sin\vartheta/2 + \cos\vartheta/2\ \frac{\pi+\epsilon\vartheta}{2}\cdot\frac{1}{2}\ \frac{\beta^2\Lambda^2}{1-\Lambda^2} + O(\beta^4).$$

(5-66)

Therefore the zero order solution is

$$\Lambda_0 = -\epsilon \sin\vartheta/2\,, \tag{5-67}$$

and by iterating to get the first order correction one finds

$$\Lambda = -\epsilon \sin\vartheta/2\left[1-\frac{\epsilon\beta^2}{4}(\pi+\epsilon\vartheta)\tan\frac{\vartheta}{2}+\cdots\right], \tag{5-68}$$

which by virture of the type of expansion is restricted by the condition

$$\frac{\beta^2}{2}\tan^2\frac{\vartheta}{2} < 1. \tag{5-59}$$

For the relativistic limit, or small $(1-\beta^2)$, $\frac{1}{E^2}$ is small, and

$$\Lambda = \cos\left\{ \frac{\pi+\epsilon\vartheta}{2} (1-\Lambda^2)^{\frac{1}{2}} \left(1 - \frac{\Lambda^2}{E^2}\right)^{-\frac{1}{2}} \right\}$$

$$= \cos\left\{ \frac{\pi+\epsilon\vartheta}{2} (1-\Lambda^2)^{\frac{1}{2}} \left(1 + \frac{\Lambda^2}{2E^2} \cdots\cdots\right)\right\}$$

$$= \cos\left[\frac{\pi+\epsilon\vartheta}{2} (1-\Lambda^2)^{\frac{1}{2}} \right]$$

$$-\left(\frac{\pi+\epsilon\vartheta}{2}\right) \frac{1}{2} (1-\Lambda^2)^{\frac{1}{2}} \frac{\Lambda^2}{E^2} \sin\left[\frac{\pi+\epsilon\vartheta}{2} (1-\Lambda^2)^{\frac{1}{2}} \right]. \qquad (5\text{-}70)$$

The zero order solution is the solution of the transcendental equation

$$\Lambda_0 = \cos\left[\frac{\pi+\epsilon\vartheta}{2} (1-\Lambda_0^2)^{\frac{1}{2}} \right]. \qquad (5\text{-}71)$$

Here ϑ is small, and by iterating to get the 1st order correction one finds

$$\Lambda = \Lambda_0 \left\{ 1 - \frac{1}{E^2} \left(\frac{\pi+\epsilon\vartheta}{4} \right) \Lambda_0 (1-\Lambda_0^2)^{\frac{1}{2}} \right\}, \qquad (5\text{-}72)$$

which by virtue of the type of expansion, is restricted by

$$\frac{\Lambda}{E} < 1. \qquad (5\text{-}73)$$

Therefore,

$$\frac{\arccos \Lambda_0}{(1-\Lambda_0^2)^{\frac{1}{2}}} = \frac{\pi + \epsilon\vartheta}{2}. \qquad (5\text{-}74)$$

By taking the cotangent on both sides of the above equation, one finds

$$\mathrm{Cot}\left\{\frac{\arccos \Lambda_0}{(1-\Lambda_0^2)^{\frac{1}{2}}}\right\} = -\epsilon \tan\frac{\vartheta}{2}. \qquad (5\text{-}75)$$

Expand the left hand side about $\Lambda_0 = 0$,

$$\mathrm{Cot}\left\{\frac{\arccos \Lambda_0}{(1-\Lambda_0^2)^{\frac{1}{2}}}\right\}_{\Lambda_0=0} = 0, \qquad (5\text{-}76)$$

and

$$\left\{\frac{d}{d\Lambda_0}\left[\cot \frac{\arccos \Lambda_0}{(1-\Lambda_0^2)^{\frac{1}{2}}}\right]\right\}_{\Lambda_0=0} = 1. \qquad (5\text{-}77)$$

The result is

$$\Lambda_0 = -\epsilon \tan \vartheta/2. \qquad (5\text{-}78)$$

Now Λ has been defined as

$$\Lambda = E \sin\left(\frac{\Omega - \epsilon\vartheta}{2}\right), \tag{5-79}$$

so that

$$E \sin(\Omega - \epsilon\vartheta) = 2\Lambda\left[1 - (1-\beta^2)\Lambda^2\right]^{\frac{1}{2}}. \tag{5-80}$$

Then, using the expression applicable to the nonrelativistic limit, namely

$$\Lambda = -\epsilon \sin\frac{\vartheta}{2}\left[1 - \frac{\epsilon\beta^2}{4}(\pi + \epsilon\vartheta)\tan\frac{\vartheta}{2} + O(\beta^4)\right], \tag{5-81}$$

and substituting this into the preceding expression, expanding to first order in β^2, one obtains

$$E \sin(\Omega - \epsilon\vartheta) = -\epsilon \sin\vartheta\left\{1 + \frac{\beta^2}{2}\tan^2\frac{\vartheta}{2} \cdot \left[1 - (\epsilon\pi + \vartheta)\cot\vartheta\right] + \cdots\right\}. \tag{5-82}$$

For the relativistic limit, by means of

$$E \sin(\Omega - \epsilon\vartheta) = 2\Lambda\left[1 - \frac{1}{2}\frac{\Lambda^2}{E^2}\right]$$

$$= 2\Lambda_0\left\{1 - \frac{1}{2}\frac{\Lambda_0}{E^2}\left[\Lambda_0 + \frac{\pi + \epsilon\vartheta}{2}(1 - \Lambda_0^2)\right] + O\left(\frac{1}{E^4}\right)\right\}, \qquad (5\text{-}83)$$

and using the expression for Λ_0 appropriate to very small angles, namely $\Lambda_0 = -\epsilon \tan \vartheta/2$ for small ϑ, one obtains, to zero order in $\frac{1}{E^2}$,

$$E \sin(\Omega - \epsilon\vartheta) = -2\epsilon \tan\frac{\vartheta}{2}\left\{1 + O\left(\frac{1}{E^4}\right)\right\}. \qquad (5\text{-}84)$$

VI. COMPARISON WITH THE QUANTUM MECHANICAL RESULTS

It is instructive to compare these results with the quantum mechanical treatment of the Coulomb scattering of a Dirac particle. Fradkin, Weber, and Hammer (7) have studied this problem in terms of the parameters $\lambda = Ze^2$, and $\nu = \epsilon Ze^2/\beta$ and obtained detailed formulas for the wave function expanded in powers of λ through order λ^3 .

Using their formulas, for the relativistic limit (small ν), one obtains the results

$$\frac{\sigma(\vartheta)}{R} = 1 - \left(\frac{\lambda}{\nu}\right)^2 \sin^2\frac{\vartheta}{2} + \pi\left(\frac{\lambda^2}{\nu}\right) \sin\frac{\vartheta}{2}\left(1 - \sin\frac{\vartheta}{2}\right)$$

$$+ O(\lambda^2) + O\left(\frac{\lambda^4}{\nu^2}\right), \qquad (6\text{-}1)$$

$$\sin(\Omega - \epsilon\vartheta) = -\epsilon \sin\vartheta \left[E\left(1-\beta^2 \sin^2\frac{\vartheta}{2}\right)\right]^{-1} \Big\{ 1$$

$$+ \frac{1}{2}\frac{\lambda^2}{\nu}\pi \sec\frac{\vartheta}{2}\left(1 - \sin\frac{\vartheta}{2}\right)\left[1 - (\sin\vartheta)\left(1-\beta^2\sin^2\frac{\vartheta}{2}\right)^{-1}\right]$$

$$+ O(\lambda^2)\Big\}. \qquad (6\text{-}2)$$

The results for the non-relativistic limit (large ν) are

$$\frac{\sigma(\vartheta)}{R} = 1 + \frac{1}{2}\left(\frac{\lambda}{\nu}\right)^2\left(\tan^2\frac{\vartheta}{2}\right)\left[(\vartheta+\epsilon\pi)\cot\vartheta - 1 + O(\lambda^2) + O(\nu^{-2})\right]$$

$$+ (1+\epsilon)\frac{\lambda^2}{\nu}\pi\left(\sin\frac{\vartheta}{2}\right)\left(\tan\frac{\vartheta}{2}\right)\Big[(1-\cos\vartheta/2)\cos\chi$$

$$+ \frac{\nu^{-1}}{4}\left(1+\sec^2\frac{\vartheta}{2}\right)\sin\chi + O(\nu^{-2})\Big], \qquad (6\text{-}3)$$

$$\sin(\Omega-\epsilon\vartheta) = -\epsilon\,\frac{\sin\vartheta}{E}\Big\{1 + \frac{1}{2}\left(\frac{\lambda}{\nu}\right)^2\tan^2\frac{\vartheta}{2}\left[1-(\vartheta+\epsilon\pi)\cot\vartheta\right]$$

$$+ O(\lambda^2) + O(\nu^{-2}) - \frac{1+\epsilon}{2}\left(\frac{\lambda^2}{\nu}\right)\pi\left(1-\sec\frac{\vartheta}{2}\right)\cos\vartheta$$

$$\cdot\left[\cos\chi - \frac{\nu^{-1}}{4}\left(\sec\frac{\vartheta}{2}\right)\left(1+\sec\frac{\vartheta}{2}\right)\sin\chi + O(\nu^{-2})\right]\Big\}, \qquad (6\text{-}4)$$

where

$$\chi = 2\nu\ln\csc\vartheta/2\,. \qquad (6\text{-}5)$$

Equation 6-3 and 6-4 were obtained from an asymptotic expansion for $|\nu| \to \infty$, and in the case of an attractive potential ($\epsilon=+1$) reduce to the form above only for those angles ϑ such that $\chi > 2$.

For the relativistic limit ($\beta \to 1$ or $\nu \to \lambda$), comparing the expressions for the cross section given by Equation 5-59 and 6-1, one sees that within the range where both the classical and quantum equations are valid, here roughly that $\beta^2 \tan^2 \vartheta/2$ is of order $\lambda(1-\beta^2)$ and $\lambda < 1$, the classical and quantum expressions both give

$$\frac{\sigma(\vartheta)}{R} = 1 + O(1-\beta^2) + O(\lambda). \tag{6-6}$$

Also a comparison of Equation 5-65 and Equation 6-2 indicate that within this range the precession angle Ω is given by

$$\sin(\Omega - \epsilon\vartheta) = -\frac{\epsilon}{E} \sin\vartheta \left[1 + O(1-\beta^2) + O(\lambda) \right]. \tag{6-7}$$

For the non-relativistic limit ($\beta \to 0$ or $\nu \to \infty$), a comparison of Equation 5-59 and Equation 6-3 shows that both classical and quantum expressions for the cross section give leading terms of

$$\frac{\sigma(\vartheta)}{R} = 1 + \frac{\beta^2}{2} \tan^2\frac{\vartheta}{2} \left[(\vartheta + \epsilon\pi) \cot\vartheta - 1 \right] + O(\beta^3), \tag{6-8}$$

if for the attractive case one considers an average over a small range of angles centered around ϑ so that the rapidly oscillating terms which are trigonometric function of χ average to zero. Also a comparison of Equations 5-64 and 6-4 indicate that both classical and quantum expressions of the precession of polarization have leading terms of

$$\sin(\Omega-\epsilon\vartheta) = -\frac{\epsilon}{E}\sin\vartheta\left[1+\frac{\beta^2}{2}\tan^2\frac{\vartheta}{2}\left(1-(\epsilon\pi+\vartheta)\cot\vartheta\right)\right]+O(\beta^3), \tag{6-9}$$

where again an average over the rapidly oscillating terms is taken.

There is then agreement between the classical and quantum calculations in the appropriate limits.

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VIII. LITERATURE CITED

1. Thomas, L. H., Phil. Mag. 3, 1 (1927).

2. Bargmann, V., Michel, L. and Telegdi, V. L., Phys. Rev. Letters 2, 435 (1959).

3. Frenkel, J., Z. Physik 37, 243 (1926).

4. Kramers, H. A., Quantum Mechanics, North-Holland Publishing Company, Amsterdam, 1957.

5. Ford, G. W. and Hirt, C. W., Spin Precession of a Classical Particle, Unpublished mimeograph report. Ann Arbor, Michigan, Department of Physics, University of Michigan. 1960.

6. Fradkin, D. M. and Good, R. H., Jr., Rev. Mod. Phys. 33, 343 (1961).

7. Fradkin, D. M., Weber, T. A. and Hammer, C. L., Coulomb Scattering of Dirac Particles, to be published in Annals of Physics, (ca. 1964).

8. Moller, C., The Theory of Relativity, Oxford University Press, London, 1952.

9. Uspensky, J. V., Theory of Equations, McGraw-Hill Book Company, Inc., New York, 1948.