DISPERSION OF GUIDED CIRCUMFERENTIAL WAVES IN A CIRCULAR ANNULUS

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INTRODUCTION

Fatigue cracks have been found to initiate and grow in the radial direction in many of the annulus shaped components in aging helicopters. Those include some of the most critical components such as the rotor hub, connecting links and pitch shaft, etc. At the present time, detection of such radial fatigue cracks relies mostly on visual inspection. A more systematic, automated, and efficient method to detect these cracks must be developed.

Conventional ultrasonic imaging techniques can be used to detect such radial cracks. However, these techniques are impractical for real-time, integrated diagnosis. Recently, it has been proposed [1] that guided ultrasonic waves that propagate in the circumferential direction may be use for the detection of radial fatigue cracks in annulus components.

For this purpose, the propagation of guided circumferential waves must be understood. The objective of this paper is to investigate the dispersion relations for waves that propagate in the circumferential direction of a circular annulus, as shown in Fig 1. Some related work can be found in [2] - [4].

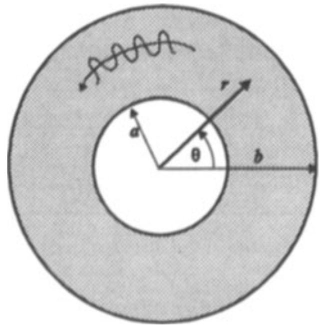

Fig. 1 A circular annulus with inner radius a, outer radius b. The z-direction of the cylindrical coordinate system is perpendicular to the annulus.

PROBLEM FORMULATION

In the cylindrical coordinate system ($r\theta z$) shown in Fig. 1, consider time harmonic waves ($e^{-i\omega t}$) propagating in the θ-direction. Assuming that the particle motion is such that the plane strain deformation in the annulus prevails, namely, the displacement components are

$$u_r = u_r(r,\theta) \quad , \quad u_\theta = u_\theta(r,\theta) \quad , \quad u_z = 0 \quad . \tag{1}$$

The pertinent stress components are

$$\sigma_r = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + 2\mu\frac{\partial u_r}{\partial r} \quad , \tag{2a}$$

$$\sigma_\theta = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + 2\mu\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) \quad , \tag{2b}$$

$$\sigma_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \quad , \tag{2c}$$

where λ and μ are the Lamé constants.

The equations of motion can be solved by introducing the displacement potentials φ and ψ through

$$u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad , \quad u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial r} - \frac{\partial \psi}{\partial r} \quad , \tag{3a,b}$$

where φ and ψ satisfy the reduced wave equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varphi + \frac{\omega^2}{c_L^2}\varphi = 0 \quad , \tag{4a}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi + \frac{\omega^2}{c_T^2}\psi = 0 \quad . \tag{4b}$$

In (4a,b), c_L and c_T are the longitudinal and shear wave velocities, respectively.

For waves propagating in the θ-direction, one may write

$$\varphi = \Phi(r)\exp(ikb\theta) \quad , \qquad \psi = \Psi(r)\exp(ikb\theta) \quad , \tag{5a,b}$$

where, for convenience, the wavenumber k is defined by

$$k = \frac{\omega}{c(b)} \quad . \tag{6}$$

In (6), $c(r)$ is the linear phase velocity for material particles located a distance r from the center. Intuitively, this linear phase velocity should be a function of r. More details on this will be discussed later.

Substitution of (5a,b) into (4a,b), respectively, yields

$$\Phi'' + \frac{1}{r}\Phi' + [(\frac{\omega}{c_L})^2 - (\frac{kb}{r})^2]\Phi = 0 \quad , \quad \Psi'' + \frac{1}{r}\Psi' + [(\frac{\omega}{c_T})^2 - (\frac{kb}{r})^2]\Psi = 0 \quad . \qquad (7a,b)$$

Solutions to (7a,b) are, respectively,

$$\Phi(r) = A_1 J_{kb}(\frac{\omega r}{\kappa c_T}) + B_1 Y_{kb}(\frac{\omega r}{\kappa c_T}) \quad , \quad \Psi(r) = A_2 J_{kb}(\frac{\omega r}{c_T}) + B_2 Y_{kb}(\frac{\omega r}{c_T}) \quad , \qquad (8a,b)$$

where

$$\kappa = \frac{c_L}{c_T} = \sqrt{\frac{2(1-\nu)}{1-2\nu}} \quad , \qquad (9)$$

and ν is the Poisson's ratio, A_1, A_2, B_1, B_2 are constants to be determined by the boundary conditions.

Next, substituting (5) into (3), then into (2a,c) yields

$$\sigma_r(r,\theta) = \frac{\mu \exp(ikb\theta)}{r^2}[\kappa^2 r^2 \Phi'' + (\kappa^2 - 2)r\Phi' - (\kappa^2 - 2)k^2 b^2 \Phi + 2ikbr\Psi' - 2ikb\Psi] \quad ,$$

$$\sigma_{r\theta}(r,\theta) = \frac{\mu \exp(ikb\theta)}{r^2}[-r^2\Psi'' + r\Psi' - k^2 b^2 \Psi + 2ikbr\Phi' - 2ikb\Phi] \quad . \qquad (10a,b)$$

The traction-free boundary conditions at $r = a, b$ are $\sigma_r = \sigma_{r\theta} = 0$. Enforcing these boundary conditions on (10) yields a system of four homogeneous equations for the constants A_1, A_2, B_1, B_2. For non-trivial solutions, the determinant of the coefficient matrix of this system of equations must vanish, i.e.,

$$D(\bar{k},\bar{\omega},\eta) = \begin{Vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{Vmatrix} = 0 \quad , \qquad (11)$$

where the elements d_{ij} are given in the Appendix as functions of non-dimensional wavenumber $\bar{k}$, non-dimensional frequency $\bar{\omega}$ and non-dimensional wall thickness parameter η. These non-dimensional quantities are defined by

$$\bar{k} = kh \quad , \quad h = b - a \quad , \quad \bar{\omega} = \frac{\omega h}{c_T} \quad , \quad \eta = \frac{a}{b} \quad . \qquad (12a,b,c,d)$$

For given values of η, (11) provides the dispersion relationship between the wavenumber and the frequency. Therefore, (11) is called the dispersion equation. Examples of numerical solutions to (11) are presented in the next section.

DISPERSION CURVES

Numerical solutions to the dispersion equation (11) are obtained for $\nu = 0.25$ and $\eta = 0.1$, 0.5 and 0.9, respectively. The corresponding dispersion curves, i.e., the $\overline{\omega} - \overline{k}$ curves, are presented in Fig. 2 - 4. The numerical solution is obtained by giving a value of $\overline{k}$, then finding the root of (11) for the corresponding values of $\overline{\omega}$. An infinite number of real roots exits. Each represents a propagating mode. In the frequency range examined, five propagating modes are found for each of the three cases. The fact that the $\overline{\omega} - \overline{k}$ curves are not straight lines indicates that the waves are dispersive. Note that care must be taken in the root-finding procedure, for the determinant changes its values very rapidly.

For $\overline{k} = 0$, the 4x4 determinant given in (11) can be reduced to a product of two 2x2 subdeterminants. One of the 2x2 subdeterminants corresponds to the dilatational motion represented by the scalar potential φ and the other corresponds to the equivoluminal motion represented by the vector potential ψ. Following the conventions for guided waves in plates [5], the propagating modes corresponding to the dilatational motion are labeled by the letter D, and those corresponding to the equivoluminal motion are labeled by the letter E.

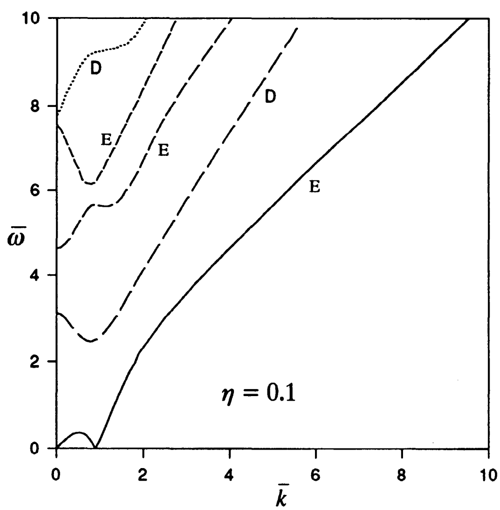

Fig. 2 Dispersion curves of the first five propagating modes in annuli with η = 0.1

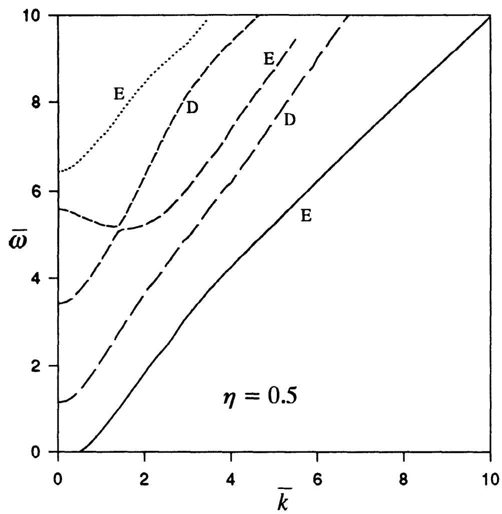

Fig. 3 Dispersion curves of the first five propagating modes in annuli with $\eta = 0.5$

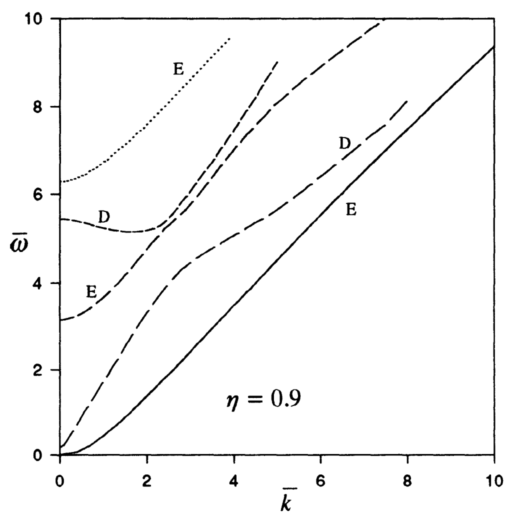

Fig. 4 Dispersion curves of the first five propagating modes in annuli with $\eta = 0.9$

DISCUSSION

The problem studied in this paper is analogous to the problem of guided waves in a plate. The difference is that curvature is involved here. If a plate can be viewed as a cylindrical shell of infinite radius, one obvious effect of the finite radius is that it is no longer possible to define the symmetric and anti-symmetric modes as in the plate case.

Intuitively, the curvature may also affect the dispersion properties significantly. Therefore, in the circular annulus considered here the dispersion relations can be different for material particles located at different distances to the center. However, the dispersion curves shown in Figs. 2-4 are somewhat "curvature independent." This is because the $\overline{\omega} - \overline{k}$ curves obtained from (11) depend on only one geometrical parameter η, which only characterizes the thickness of the wall, not the radius of the annulus. Consequently, the curves in Figs. 2 - 4 can be used for circular annuli of different sizes, as along as they all have the same wall thickness (or the same η). The question is then "how does the curvature get involved?"

To this end, we return to (5). For a time-harmonic wave propagating in the θ-direction, all material particles located on the same radial line should have the same phase factor given by

$$\exp[i(kb\theta - \omega t)] = \exp[i\omega(\frac{kb\theta}{\omega} - t)] = \exp[i\omega(\frac{\theta}{\alpha} - t)] \quad , \tag{13}$$

where

$$\alpha = \frac{\omega}{kb} = \frac{\overline{\omega}}{\overline{k}} \frac{c_T}{b} \tag{14}$$

is called the "angular phase velocity" of the circumferential wave. The second equality in (14) follows from (12).

Now, it is clear from (14) that, although the $\overline{\omega} - \overline{k}$ curves obtained from (11) depend on only the wall thickness (not the radii of the annulus), the angular phase velocity of the circumferential waves does depend on the radii. In other words, the curvature is involved in the angular phase velocity. Interestingly enough, once the $\overline{\omega} - \overline{k}$ relationship is solved from (11) for a given wall thickness, η, the angular phase velocity for annuli of different sizes can be easily obtained from (13), as along as they all have the same wall thickness. In this sense, the dispersion equation can be viewed as universal in that it provides the solutions for annuli of any radii.

In terms of the angular phase velocity, one can compute the linear velocity in the circumferential direction for a material particle at distance r from the center

$$c(r) = r\alpha = \frac{\overline{\omega}}{\overline{k}} \frac{r}{b} c_T \quad . \tag{15}$$

For example, the circumferential wave travels along the outer surface, $r = b$, with the speed of

$$c(b) = \frac{\overline{\omega}}{\overline{k}} c_T = \frac{\omega}{k} \quad . \tag{16}$$

This is consistent with (6).

Now the advantage of defining k through (6) is clear. Note that $\overline{\omega}/\overline{k}$ is the slope of the dispersion curves shown in Figs. 2 - 4. Once the dispersion equation is solved, the propagating (phase) velocity at the outer surface of the annulus, $c(b)$, is obtained from (16). On the other hand, since $c(b)$ can be measured easily on the outer surface of the annulus, (16) provides a tool to determine the dispersion curves experimentally.

It is seen from Figs. 2 - 4 that at higher frequencies, the lowest modes (solid lines) are almost non-dispersive for the three cases presented. However, the linear wave speed $c(b)$ varies with the wall thickness: at higher frequencies, $c(b) > c_T$ for very thick annulus (η = 0.1), and $c(b) < c_T$ for very thin annulus (η = 0.9).

Another point worth mentioning is the limits of the geometrical parameters. Note that in addition to the wavelength $2\pi/k$, there are two other independent length parameters in this problem. Although the choice is somewhat arbitrary, the outer radius b and the wall thickness h are taken to be the two independent parameters in this paper. For convenience, a non-dimensional parameter $\eta = a/b$ was introduced in (12) to characterize the wall thickness. The use of η in (11) makes the dispersion equation independent of the curvature as discussed previously. However, the limits of η must be carefully examined in order to correctly interpret the results.

Obviously, $\eta = 0$ corresponds to a solid disk. Our numerical solutions have shown that when very small values of η are used, (11) predicts the same dispersion curves as those in a solid disk. The limit of $\eta \to 1$, however, needs more attention. Notice that

$$h = b - a = b(1-\eta) \quad . \tag{17}$$

Therefore, two limiting cases may result.

Case (i) $\eta \to 1$ with h being a non-zero value;

Case(ii) $\eta \to 1$ with b being a finite value.

It follows from (16) that Case (i) yields $b \to \infty$. Consequently, this limiting case corresponds to a flat plate of thickness h. On the other hand, Case (ii) yields $h \to 0$, which corresponds to a thin annulus (ring) of radius b.

Note from the preceding discussion, the two limiting cases are differentiated by only different values of b, or different curvature. Now recall that the dispersion equation is curvature independent. Therefore, one can conclude that the dispersion curves for guided waves in a plate and for the circumferential waves in a thin annulus are the same. For example, the dispersion curves in Fig. 4 where η = 0.9 represent an approximation to the dispersion curves of guided waves in a plate of thickness h. In the meantime, if the following relations are used

$$\hat{k} = \frac{\overline{k}}{1-\eta} \quad , \quad \Omega = \frac{\overline{\omega}}{1-\eta} \quad , \tag{18}$$

then, the $\Omega - \hat{k}$ curves calculated from (18) represent the dispersion relationships of the circumferential waves in a thin-wall annulus of radius b.

SUMMARY

The dispersion equation for waves propagating in the circumferential direction of an annulus is derived. It is shown that the resulting $\bar{\omega}-\bar{k}$ curves are universal in that they are curvature independent. Once these dispersion curves are obtained, the curvature dependent angular phase velocity of the circumferential waves can be easily computed. Some limiting cases are also discussed briefly. More detailed studies of the displacement fields will be presented in a separate publication.

ACKNOWLEDGMENTS

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APPENDIX

$$d_{11}=\frac{\Omega^2}{\kappa^2}[J_{\hat{k}-2}(\frac{\Omega}{\kappa})+J_{\hat{k}+2}(\frac{\Omega}{\kappa})-2(\kappa^2-1)J_{\hat{k}}(\frac{\Omega}{\kappa})] \quad , \quad d_{12}=i\Omega^2[J_{\hat{k}-2}(\Omega)-J_{\hat{k}+2}(\Omega)]$$

$$d_{13}=\frac{\Omega^2}{\kappa^2}[Y_{\hat{k}-2}(\frac{\Omega}{\kappa})+Y_{\hat{k}+2}(\frac{\Omega}{\kappa})-2(\kappa^2-1)Y_{\hat{k}}(\frac{\Omega}{\kappa})] \quad , \quad d_{14}=i\Omega^2[Y_{\hat{k}-2}(\Omega)-Y_{\hat{k}+2}(\Omega)]$$

$$d_{21}=\frac{i\Omega^2}{\kappa^2}[J_{\hat{k}-2}(\frac{\Omega}{\kappa})-J_{\hat{k}+2}(\frac{\Omega}{\kappa})] \quad , \quad d_{22}=-\Omega^2[J_{\hat{k}-2}(\Omega)+J_{\hat{k}+2}(\Omega)]$$

$$d_{23}=\frac{i\Omega^2}{\kappa^2}[Y_{\hat{k}-2}(\frac{\Omega}{\kappa})-Y_{\hat{k}+2}(\frac{\Omega}{\kappa})] \quad , \quad d_{24}=-\Omega^2[Y_{\hat{k}-2}(\Omega)+Y_{\hat{k}+2}(\Omega)]$$

$$d_{31}=\frac{\eta^2\Omega^2}{\kappa^2}[J_{\hat{k}-2}(\frac{\eta\Omega}{\kappa})+J_{\hat{k}+2}(\frac{\eta\Omega}{\kappa})-2(\kappa^2-1)J_{\hat{k}}(\frac{\eta\Omega}{\kappa})] \quad ,$$

$$d_{32}=i\eta^2\Omega^2[J_{\hat{k}-2}(\eta\Omega)-J_{\hat{k}+2}(\eta\Omega)]$$

$$d_{33}=\frac{\eta^2\Omega^2}{\kappa^2}[Y_{\hat{k}-2}(\frac{\eta\Omega}{\kappa})+Y_{\hat{k}+2}(\frac{\eta\Omega}{\kappa})-2(\kappa^2-1)Y_{\hat{k}}(\frac{\eta\Omega}{\kappa})] \quad ,$$

$$d_{34}=i\eta^2\Omega^2[Y_{\hat{k}-2}(\eta\Omega)-Y_{\hat{k}+2}(\eta\Omega)]$$

$$d_{41}=\frac{i\eta^2\Omega^2}{\kappa^2}[J_{\hat{k}-2}(\frac{\eta\Omega}{\kappa})-J_{\hat{k}+2}(\frac{\eta\Omega}{\kappa})] \quad , \quad d_{42}=-\eta^2\Omega^2[J_{\hat{k}-2}(\eta\Omega)+J_{\hat{k}+2}(\eta\Omega)]$$

$$d_{43}=\frac{i\eta^2\Omega^2}{\kappa^2}[Y_{\hat{k}-2}(\frac{\eta\Omega}{\kappa})-Y_{\hat{k}+2}(\frac{\eta\Omega}{\kappa})] \quad , \quad d_{44}=-\eta^2\Omega^2[Y_{\hat{k}-2}(\eta\Omega)+Y_{\hat{k}+2}(\eta\Omega)]$$

$$\hat{k}=\frac{\bar{k}}{1-\eta} \quad , \quad \Omega=\frac{\bar{\omega}}{1-\eta}$$