Identities relating the Jordan product and the associator in the free nonassociative algebra

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Abstract

We determine the identities of degree ≤ 6 satisfied by the symmetric (Jordan) product $a \circ b = ab + ba$ and the associator $[a, b, c] = (ab)c - a(bc)$ in every nonassociative algebra. In addition to the commutative identity $a \circ b = b \circ a$ we obtain one new identity in degree 4 and another new identity in degree 5. We demonstrate the existence of further new identities in degree 6. These identities define a variety of binary-ternary algebras which generalizes the variety of Jordan algebras in the same way that Akivis algebras generalize Lie algebras.

1 Introduction

1.1 Lie algebras and Akivis algebras

The algebras now called Akivis algebras arose in the study of the geometry of webs and the corresponding analytic loops. They were introduced by Akivis [1] and were given their current name by Hofmann and Strambach [5]. From an algebraic point of view the simplest way to define and motivate Akivis algebras is to consider an arbitrary nonassociative algebra N over some field F with the product denoted by juxtaposition. On N we define two new operations: the binary commutator $[a, b] = ab - ba$ and the ternary associator $[a, b, c] = (ab)c - a(bc)$. We write N^- for the binary-ternary algebra obtained by endowing the underlying vector space of N with these two new operations. It is not difficult to verify that all the identities of degree ≤ 3 satisfied by these two operations in every nonassociative algebra are consequences of the anticommutative identity $[a,a] = 0$ and the Akivis identity (or generalized Jacobi identity)

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = \\ [a,b,c] + [b,c,a] + [c,a,b] - [a,c,b] - [b,a,c] - [c,b,a].$$

By definition an Akivis algebra is a vector space A over a field F with an anticommutative binary operation $[a,b]$ and a ternary operation $[a,b,c]$ which satisfy the Akivis identity. It has been shown by Shestakov [10] (see also Shestakov and Umirbaev [11]) that every Akivis algebra is a subalgebra of N^- for some nonassociative algebra N. Thus the anticommutative and Akivis identities imply all the identities which hold for the commutator and associator in every nonassociative algebra. Akivis algebras are the ultimate generalization of Lie algebras in the context of nonassociative structures.

1.2 Jordan algebras and a new variety of binary-ternary algebras

In this paper we find the corresponding generalization of Jordan algebras. (For the current state of knowledge on Jordan algebras and their identities, see the recent book by McCrimmon [8].) On an arbitrary nonassociative algebra N we introduce two new operations: the symmetric (Jordan) product and the associator:

$$a \circ b = ab + ba, \qquad [a,b,c] = (ab)c - a(bc).$$

We determine the identities of degree ≤ 6 satisfied by these two operations in every nonassociative algebra. We show that:

1. In degree 2 there is only the commutative identity $a \circ b = b \circ a$. (This is obvious.)

2. In degree 3 there are no new identities. (Every identity follows from commutativity.)

3. In degree 4 there are new identities which do not follow from commutativity. They are all consequences of the identity:

$$2[a,a,a] \circ a = [a \circ a, a, a] - [a, a \circ a, a] + [a, a, a \circ a].$$

4. In degree 5 there are new identities which do not follow from commutativity and the identity in degree 4. They are all consequences of the identity displayed in Theorem 3.

5. In degree 6 there exist further new identities which do not follow from the identities in degree 2, 4 and 5.

These identities define a new variety of binary-ternary algebras which is related to the variety of Jordan algebras in the same way that Akivis algebras are related to Lie algebras. At this point two natural questions arise:

1. Are there further identities of degree ≥ 7?
2. Is the T-ideal of all identities finitely generated?

Higher identities, if they exist, would be analogous to the special identities (in particular the Glennie identities) for Jordan algebras.

1.3 Method

We follow an approach to polynomial identities based on combinatorics of nonassociative words, computational linear algebra, and representations of the symmetric group. For convenience we will assume that the coefficient field has characteristic zero. This implies that every identity is equivalent to a set of homogeneous multilinear identities. Since we only consider identities of degree ≤ 6, our results will also hold for fields of characteristic ≥ 7.

2 Degree 3

Theorem 1. *There are no homogeneous multilinear identities in degree 3 involving the symmetric product $a \circ b = ab + ba$ and the associator $[a, b, c] = (ab)c - a(bc)$ apart from those which are consequences of the commutativity of the symmetric product.*

Proof. In degree 3 there are two inequivalent association types for a monomial built from the symmetric product $a \circ b$ and the associator $[a, b, c]$:

$$(a \circ b) \circ c, \qquad [a, b, c].$$

The six permutations of a, b, c give three multilinear monomials of the first type (using the commutativity of the symmetric product):

$$(a \circ b) \circ c, \quad (a \circ c) \circ b, \quad (b \circ c) \circ a,$$

and six multilinear monomials of the second type:

$$[a, b, c], \quad [a, c, b], \quad [b, a, c], \quad [b, c, a], \quad [c, a, b], \quad [c, b, a].$$

Thus we have nine monomials which form an ordered basis for the space A_3 of all multilinear homogeneous polynomials of degree 3 in these two operations.

Using the formulas

$$a \circ b = ab + ba, \qquad [a, b, c] = (ab)c - a(bc),$$

we expand the nine monomials into linear combinations of monomials built from the original nonassociative operation. In degree 3 we have 12 nonassociative monomials: six permutations of the three variables in each of the two association types for a nonassociative binary operation:

$$
\begin{array}{cccccc}
(ab)c, & (ac)b, & (ba)c, & (bc)a, & (ca)b, & (cb)a, \\
a(bc), & a(cb), & b(ac), & b(ca), & c(ab), & c(ba).
\end{array}
$$

These 12 monomials form an ordered basis for the space N_3 of all multilinear homogeneous nonassociative polynomials of degree 3.

We consider the linear map $E_3 \colon A_3 \to N_3$ which sends each basis monomial of A_3 to its expanded form in N_3. The identities we seek are the nonzero elements of the kernel of E_3. Let $[E_3]$ be the 12×9 matrix representing this linear map with respect to the ordered bases given above. The ij-entry of $[E_3]$ is the coefficient of the i-th nonassociative monomial in the expansion of the j-th binary-ternary monomial. The matrix $[E_3]$ is displayed in Table 1. One easily checks that this matrix has full rank, which implies that there are no new identities in degree 3. □

1	0	0	1	0	0	0	0	0
0	1	0	0	1	0	0	0	0
1	0	0	0	0	1	0	0	0
0	0	1	0	0	0	1	0	0
0	1	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	1
0	0	1	−1	0	0	0	0	0
0	0	1	0	−1	0	0	0	0
0	1	0	0	0	−1	0	0	0
0	1	0	0	0	0	−1	0	0
1	0	0	0	0	0	0	−1	0
1	0	0	0	0	0	0	0	−1

Table 1: The expansion matrix in degree 3

3 Degree 4

We write A_n for the vector space with basis consisting of the inequivalent homogeneous multilinear monomials in degree n involving the symmetric product and the associator. (Inequivalent means that we take one representative of each equivalence class with respect to the commutativity of the symmetric product.) We write N_n for the vector space with basis consisting of the homogeneous multilinear monomials in degree n for a nonassociative binary operation. We write $E_n \colon A_n \to N_n$ for the expansion map: the linear map which sends each basis monomial in A_n to its expanded form in N_n. The nonzero elements of $\ker(E_n)$ are the identities in degree n relating the symmetric product and the associator.

We write $[E_n]$ for the matrix representing E_n with respect to a given ordering on the basis monomials of A_n and N_n.

Proposition 1. *The dimension of N_n, which is the number of rows of $[E_n]$, is*

$$\dim(N_n) = \frac{(2n-2)!}{(n-1)!}.$$

Proof. There are $n!$ homogeneous multilinear associative monomials in degree n. The number of association types in degree n for a nonassociative binary operation is the Catalan number

$$K_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

The number of homogeneous multilinear nonassociative monomials is the product $n!K_n$. □

We now give an algorithm for computing a standard ordered basis of A_n. (It seems not to be easy to give a closed form for $\dim(A_n)$, the number of columns of $[E_n]$.) Let T_n be the number of association types for a binary-ternary algebra with a commutative binary operation.

Proposition 2. *We have the following recursive formula for T_n:*

$$T_1 = 1, \qquad T_n = \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} T_{n-i}T_i + e_n \binom{T_{n/2}+1}{2} + \sum_{i+j+k=n} T_i T_j T_k.$$

We define $e_n = 1$ for n even and $e_n = 0$ for n odd. The term involving e_n is included only for even n.

Proof. Consider the last operation performed in each monomial. If this is the symmetric product then the monomial has the form $x \circ y$ where x and y both have degree $< n$. Since the binary operation is commutative we may assume that $\deg(x) \geq \deg(y)$. The first sum in the above expression corresponds to the case $\deg(x) > \deg(y)$. The next term corresponds to the case $\deg(x) = \deg(y)$; this term is included only for even n. If the last operation performed in the monomial is the associator then the monomial has the form $[x, y, z]$ where x, y, z all have degree $< n$. There is no symmetry in this ternary operation. The second sum in the above expression corresponds to this case. □

Here is a short table of the number of association types:

n	1	2	3	4	5	6	7	8	9	10
T_n	1	1	2	6	17	57	191	678	2443	9029

Within each of the T_n association types we apply all permutations of the variables. The commutativity of the symmetric product makes some of the resulting monomials equal. To be precise, for each association type we count the number s of symmetric products which combine two factors of the same association type from a lower degree. The number of multilinear monomials in the current association type is then $n!/2^s$. To find the total number $M_n = \dim A_n$ of multilinear monomials we sum over all association types. Here is a short table of the number of multilinear monomials:

n	1	2	3	4	5	6	7
M_n	1	1	9	75	1095	18585	401625

Theorem 2. *In degree 4, every identity for the symmetric product and the associator follows from commutativity of the symmetric product and the identity*

$$2[a,a,a] \circ a - [a \circ a, a, a] + [a, a \circ a, a] - [a, a, a \circ a].$$

Proof. There are five association types for monomials in a single binary operation:

$$((ab)c)d, \quad (a(bc))d, \quad (ab)(cd), \quad a((bc)d), \quad a(b(cd)).$$

Within each association type we have 24 lexicographically ordered permutations of the variables, giving an ordered basis of 120 monomials for N_4.

There are six association types for monomials built from the symmetric product and the associator:

$$\begin{array}{lll} (((a \circ b) \circ c) \circ d, & [a,b,c] \circ d, & (a \circ b) \circ (c \circ d), \\ [a \circ b, c, d], & [a, b \circ c, d], & [a, b, c \circ d]. \end{array}$$

For clarity we write out the expansions of these monomials:

$$\begin{aligned} (((a \circ b) \circ c) \circ d &= ((ab)c)d + ((ba)c)d + (c(ab)d) + (c(ba))d \\ &\quad + d((ab)c) + d((ba)c) + d(c(ab)) + d(c(ba)), \\ [a,b,c] \circ d &= ((ab)c)d - (a(bc))d + d((ab)c) - d(a(bc)), \\ (a \circ b) \circ (c \circ d) &= (ab)(cd) + (ba)(cd) + (ab)(dc) + (ba)(dc) \\ &\quad + (cd)(ab) + (cd)(ba) + (dc)(ab) + (dc)(ba), \\ [a \circ b, c, d] &= ((ab)c)d - (ab)(cd) + ((ba)c)d - (ba)(cd), \\ [a, b \circ c, d] &= (a(bc))d - a((bc)d) + (a(cb))d - a((cb)d), \\ [a, b, c \circ d] &= (ab)(cd) - a(b(cd)) + (ab)(dc) - a(b(dc)). \end{aligned}$$

Within each of these types we have 24 permutations of the variables. The commutativity of the symmetric product will identify some of the monomials within each association type. To count the monomials correctly we must divide by 2 for each symmetry. The total number of monomials in degree 4 is

$$\frac{4!}{2} + 4! + \frac{4!}{8} + \frac{4!}{2} + \frac{4!}{2} + \frac{4!}{2} = 12 + 24 + 3 + 12 + 12 + 12 = 75.$$

These calculations show that the expansion matrix has size 120×75. We used a computer algebra system to construct this matrix and calculate its row canonical form. We found that $[E_4]$ has rank 74, and so $\ker(E_4)$ is one-dimensional. Therefore, up to a scalar multiple, there is a unique identity in degree 4 relating the symmetric product and the associator. The multilinear form of this identity is

$$I(a,b,c,d) = \\ 2\sum [a,b,c] \circ d - \sum [a \circ b, c, d] + \sum [a, b \circ c, d] - \sum [a, b, c \circ d]. \tag{1}$$

These are sums over all permutations of the variables: the identity I is invariant under all permutations of its arguments. This identity is the linearized form of the identity in the statement of the Theorem. □

The identity I of the last Theorem is the specialization of the Teichmuller identity (see [12], page 136, equation 5) to the one-variable case.

The identity of Theorem 2 does not reduce to the Jordan identity in the associative case. Since every term involves an associator, it collapses to zero in an associative algebra. In contrast, the Akivis identity relating the commutator and associator reduces to the Jacobi identity in the associative case.

4 Degree 5

In the next Theorem we use the associator of the symmetric product:

$$\langle a,b,c \rangle = (a \circ b) \circ c - a \circ (b \circ c).$$

Theorem 3. *In degree 5, every identity for the symmetric product and the associator follows from commutativity of the symmetric product, the identity of degree 4 in Theorem 2, and the identity*

$$\begin{aligned} J = {} & 2\langle a{\circ}a, b, a\rangle \circ b - \langle a{\circ}a, b{\circ}b, a\rangle \\ & + 4\Big([[a,a,a],b,b] - [b,[a,a,a],b] + [b,b,[a,a,a]]\Big) \\ & + \Big(2[a,a{\circ}a,b] \circ b - [a,a{\circ}a,b{\circ}b]\Big) + \Big(2[a,b,a{\circ}a] \circ b - [a,b{\circ}b,a{\circ}a]\Big) \\ & + \Big(2[b,a,a{\circ}a] \circ b - [b{\circ}b,a,a{\circ}a]\Big) - \Big(2[a{\circ}a,a,b] \circ b - [a{\circ}a,a,b{\circ}b]\Big) \\ & - \Big(2[a{\circ}a,b,a] \circ b - [a{\circ}a,b{\circ}b,a]\Big) - \Big(2[b,a{\circ}a,a] \circ b - [b{\circ}b,a{\circ}a,a]\Big) \end{aligned}$$

In this identity, the terms involving only the symmetric product are listed first, then the terms involving only the associator, and finally the terms involving both operations.

Proof. The number of multilinear nonassociative binary monomials is $5!K_5 = 120 \times 14 = 1680$. This is the dimension of N_5 and the number of rows in the matrix $[E_5]$.

There are 17 association types involving the symmetric product and the associator:

$$
\begin{array}{lll}
(((a \circ b) \circ c) \circ d) \circ e, & ([a,b,c] \circ d) \circ e, & ((a \circ b) \circ (c \circ d)) \circ e, \\
[a \circ b, c, d] \circ e, & [a, b \circ c, d] \circ e, & [a, b, c \circ d] \circ e, \\
((a \circ b) \circ c) \circ (d \circ e), & [a,b,c] \circ (d \circ e), & \\
[(a \circ b) \circ c, d, e], & [[a,b,c],d,e], & [a,(b \circ c) \circ d, e], \\
[a,[b,c,d],e], & [a,b,(c \circ d) \circ e], & [a,b,[c,d,e]], \\
[a \circ b, c \circ d, e], & [a \circ b, c, d \circ e], & [a, b \circ c, d \circ e].
\end{array}
$$

Types 1, 3 and 7 use only the symmetric product. Types 10, 12 and 14 use only the associator. The other 11 types use both operations.

Considering the commutativity of the symmetric product in each type we obtain the following number of multilinear monomials:

$$
\begin{aligned}
&\frac{5!}{2} + 5! + \frac{5!}{8} + \frac{5!}{2} + \frac{5!}{2} + \frac{5!}{2} + \frac{5!}{4} + \frac{5!}{2} + \frac{5!}{2} \\
&+ 5! + \frac{5!}{2} + 5! + \frac{5!}{2} + 5! + \frac{5!}{4} + \frac{5!}{4} + \frac{5!}{4} = 1095.
\end{aligned}
$$

This is the dimension of A_5 and the number of columns in the matrix $[E_5]$.

The expansion matrix has size 1680×1095. We used a computer algebra system to construct this matrix and calculate its row canonical form. We found that $[E_5]$ has rank 1070, and so $\ker(E_5)$ has dimension 25. We computed a basis for the nullspace of this matrix: the space of identities in degree 5. The 25 basis identities can be grouped according to the number of terms:

1. twelve identities with 60 terms,
2. three identities with 108 terms,
3. two identities with 110 terms,
4. four identities with 141 terms,
5. three identities with 174 terms,
6. one identity with 210 terms.

We need to separate the new identities in degree 5 from those which follow from the known identity I in degree 4. We can lift identity I to degree 5 in two ways:

$$
I(a \circ b, c, d, e) \quad \text{and} \quad I(a, b, c, d) \circ e.
$$

Since I is a completely symmetric function of its four arguments, we will get 10 liftings of the first type and 5 of the second. These 15 liftings are linearly independent. (This can be verified by putting them into a matrix of size 15×1095 and computing its rank.) Further computations showed that the 15 identities with 60, 110, and 210 terms are linear combinations of the liftings of the identity in degree 4. This shows that they are consequences of the identity I. None of the 10 identities with 108, 141 and 174 terms is a linear combination of the lifted identities. These 10 identities are linearly independent. Therefore they form a basis for a complement of the space of lifted identities in $\ker(E_5)$. We can regard the span of these 10 identities as the space of new identities in degree 5. We chose one of the identities with 108 terms (call it J) and applied every element of the symmetric group S_5 to its terms. This gave us 120 identities which are consequences of J; the span of these 120 identities is the S_5-submodule generated by J. Further computations showed that every one of the new identities is a linear combination of the 120 permutations of the original identity. In other words, the identity J generates (as an S_5-module) the entire space of new identities. This identity is the linearized form of the identity in the Theorem. □

5 Degree 6

The expansion matrix in degree 6 has size 30240×18585: the number of rows is the number of multilinear nonassociative monomials ($6!K_6 = 720 \cdot 42$) and the number of columns is the number of multilinear binary-ternary monomials (see the table preceding Theorem 2). Since it is very difficult to compute with matrices of this size, we need to exploit the structure of the symmetric group algebra to reduce the computations to manageable proportions. The application of the representation theory of the symmetric group to the classification of polynomial identities was introduced by A. I. Malcev [7].

5.1 The symmetric group algebra

We briefly review the structure theory of the group algebra of the symmetric group S_n. By Maschke's theorem the group algebra FS_n is semisimple over a field F of characteristic 0 (or characteristic $p > n$). Therefore, by Wedderburn's theorem the group algebra FS_n is isomorphic to a direct product of full matrix algebras. By the work of Alfred Young we know an explicit description of the matrix units in these full matrix subalgebras of FS_n. For a detailed development of this theory we refer to the books of Rutherford [9] and James and Kerber [6].

To each partition λ of n we associate a frame, and to each frame we associate a set of standard tableaux. Let f_λ be the number of these standard tableaux. We have the decomposition

$$FS_n = M_{f_1}(F) \oplus \cdots \oplus M_{f_t}(F), \tag{2}$$

where t is the number of partitions of n, and we write $f_k = f_{\lambda_k}$ for $1 \leq k \leq t$. The dimensions f_k may be computed from the hook formula, and they satisfy the equation

$$\sum_{k=1}^{t} f_k^2 = n!.$$

Given any permutation π and any partition λ_k there is an algorithm for computing the matrix corresponding to π in M_{f_k} in the natural representation. This shows us how to compute the matrices corresponding to any given element of the group algebra. (See Clifton [4] for a simplification of this algorithm which is very important for the computational implementation of the theory.) Within each full matrix subalgebra $M_{f_k}(F)$ the natural matrix units $E_{ij}^{(k)}$ can be explicitly expressed as linear combinations of permutations using Young's formulas. This shows us how to compute the group algebra element corresponding to any given list of t matrices from the decomposition (2). The matrix units satisfy the familiar relations

$$E_{pq}^{(k)} E_{rs}^{(\ell)} = \delta_{k\ell} \delta_{qr} E_{ps}^{(k)}.$$

Since the full matrix subalgebras are orthogonal, instead of computing with the entire group algebra at once, we may consider one full matrix subalgebra at a time.

5.2 Application of FS_n to polynomial identities

Using these techniques greatly reduces the size of the computations, and permits us to compute with identities of a high degree that would not be attainable otherwise. These methods allow us to tabulate information about identities in the following form. Suppose that we are studying identities of degree n, and that there are $T = T_n$ possible association types. Let λ be a partition of n, and let f_λ be the dimension of the corresponding representation of S_n. The component of a single identity I in representation λ corresponds to a matrix of size $f_\lambda \times T f_\lambda$. If we have a set S of r identities, we stack the corresponding matrices to obtain a matrix of size $r f_\lambda \times T f_\lambda$. The nonzero rows of the row canonical form of this matrix give a set of independent generators for the minimal left ideals generated by these identities in this representation. The number $d(S, \lambda)$ of these nonzero rows is a non-negative integer which cannot be larger than $T f_\lambda$. Let S be a generating set for the identities in degree n satisfied by a given algebra. Let R be a generating set for the identities in degree n which are consequences of identities of degree $< n$ satisfied by the algebra. Then $d(R, \lambda) \leq d(S, \lambda)$; if the inequality is strict we know that the algebra satisfies identities in degree n which do not follow from identities of lower degree. Furthermore, we know that these identities exist in representation λ. For further details of the application of the symmetric group algebra to computation with identities in nonassociative algebras, see previous papers of the authors, especially [2], [3].

5.3 Identities in degree 6

We now present the information described in the last subsection in the case of identities in degree 6 relating the symmetric product and the associator. These results were computed in characteristic $p = 103$. Table 2 contains the following data:

λ	DIM	COM	LIF	ALL	MAX	NEW
6	1	0	12	15	57	3
51	5	82	105	107	285	2
42	9	204	225	228	513	3
411	10	280	292	293	570	1
33	5	132	138	139	285	1
321	16	499	508	509	912	1
3111	10	367	368	368	570	0
222	5	169	171	171	285	0
2211	9	345	345	345	513	0
21111	5	219	219	219	285	0
111111	1	50	50	50	57	0

Table 2: Ranks of identities in degree 6

1. Column 1 (labelled λ) lists the partitions λ of 6 which label the irreducible representations of the symmetric group S_6.

2. Column 2 (labelled DIM) gives the dimensions $f = f_\lambda$ of the irreducible representations corresponding to the partitions in column 1.

3. Column 3 (labelled COM) gives the multiplicities (ranks) of the identities in degree 6 which are implied by the commutativity of the symmetric product; for example $((((a \circ b) \circ c) \circ d) \circ e) \circ f = ((((b \circ a) \circ c) \circ d) \circ e) \circ f$.

4. Column 4 (labelled LIF) gives the multiplicities (ranks) of the identities in degree 6 which are implied by all the identities of lower degree. This includes the commutativity of the symmetric product, together with the lifted forms of the identities in degrees 4 and 5.

5. Column 5 (labelled ALL) gives the multiplicities (ranks) of all identities satisfied by the symmetric product and the associator in degree 6.

6. Column 6 (labelled MAX) gives the dimension of the space of all possible identities for the symmetric product and the associator in degree 6. This is $57f$: the number of association types times the dimension of the irreducible representation.

7. Column 7 (labelled NEW) gives the difference of columns 5 (ALL) and 4 (LIF): this is the number of new identities in degree 6 in the corresponding representation.

With regard to column 4 we make the following remarks. An identity of degree 4 can be lifted to degree 6 either using the binary operation twice or the ternary operation once. To be precise, let $I = I(a, b, c, d)$ be a homogeneous multilinear polynomial of degree 4. Then the S_5-module of consequences of I

in degree 5 is generated by the six identities

$$I(ae,b,c,d),\ I(a,be,c,d),\ I(a,b,ce,d),\ I(a,b,c,de),\ eI(a,b,c,d),\ I(a,b,c,d)e.$$

(Here we denote the binary operation by juxtaposition for simplicity.) Let $I_5 = I_5(a,b,c,d,e)$ denote any one of these identities. Then the S_6-module of consequences of I_5 in degree 6 is generated by the seven identities

$$\begin{gathered} I_5(af,b,c,d,e),\ I_5(a,bf,c,d,e),\ I_5(a,b,cf,d,e),\ I_5(a,b,c,df,e),\\ I_5(a,b,c,d,ef),\ fI_5(a,b,c,d,e),\ I_5(a,b,c,d,e)f. \end{gathered}$$

Altogether this two-step binary lifting process gives a set of 42 identities in degree 6. We may also lift the original identity I directly to degree 6 using the ternary operation, denoted $[a,b,c]$. This gives another seven generators:

$$\begin{gathered} I([a,e,f],b,c,d),\ I(a,[b,e,f],c,d),\ I(a,b,[c,e,f],d),\ I(a,b,c,[d,e,f]),\\ [e,f,I(a,b,c,d)],\ [e,I(a,b,c,d),f],\ [I(a,b,c,d),e,f]. \end{gathered}$$

Thus in total we have a set of 49 identities which generate the S_6-submodule of consequences of I in degree 6. If we have another independent identity J of degree 5, then we must also include the seven (binary) liftings of J to degree 6. This gives a grand total of 56 generators for the consequences in degree 6 of an identity I in degree 4 and an identity J in degree 5.

The data in Table 2 imply the following result. By a minimal identity we mean an identity that generates a minimal left ideal.

Theorem 4. *There are 11 new minimal identities in degree 6 relating the symmetric product and the associator. They occur in the first 6 representations with corresponding multiplicities* $3,2,3,1,1,1$.

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