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LINEAR MEASURE AND OPAQUE SETS

by

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I. INTRODUCTION

The notion of assigning a linear measure to point sets lying in a plane, which is a generalization of arc length, has been considered in many ways. Among these definitions are those of Minkowski (12), Young (16), Janzen (10) and Carathéodory (5). The Minkowski measure is defined similar to Jordan content and thus has many of the same faults. For example, the set of rational numbers in [0, 1] is not Minkowski measurable. Young himself found inconsistencies for his own measure. Janzen measure is not independent of the coordinate system as shown by Gross (9). The definition proposed by Carathéodory is the most widely accepted and the one used in this paper.

Besicovitch (2-4) studied the density properties of plane sets with finite linear measure and discovered many interesting and unexpected relations. Following the definition of Besicovitch, s is a regular point of S if the density of S at s is equal to one. All other points of S are irregular points of S.

For sets of points on a line the linear measure of the set is the same as the Lebesgue measure of the set. Lebesgue's density theorem states that for any measurable set on a line, the set of irregular points of the set has measure zero. However this is not true for linearly measurable plane sets in general. Besicovitch (2) and others have constructed plane sets with positive linear measure for which almost all of the points are irregular. It is shown in Chapter III that if S is linearly measurable then the set of regular points of S is linearly measurable.

The definition of opaque subsets of a square is accredited to Bagemihl (1). In this paper Bagemihl poses the question (1, p. 103): "If S is a linearly measurable opaque set, how small can the linear measure of S be?" Since the projection of S onto a diagonal of the square is that diagonal, the linear measure of S is at least $\sqrt{2}$; a proof is given by Gross (8). Also since the projection of S onto any line has linear measure at least one, the linear measure of S is not less than $\pi/2$. This is a result of Eggleston (6). In Chapter IV it is shown that S has linear measure not less than two, and an example of an opaque set with linear measure $\sqrt{2} + \frac{\sqrt{6}}{2}$ is given.

An opaque set of degree α, for any cardinal number α, is defined in Chapter IV. It is shown that if $2 \leq \alpha \leq c$, there exists an opaque set of degree α and it is clear from the definition that there is no opaque set of degree one or of degree greater than c.

II. DEFINITIONS, NOTATION AND PRELIMINARY THEOREMS

Throughout this paper we will use certain well known results and definitions. Some of these, together with some preliminary theorems, are compiled here for convenience.

All sets of points considered in this paper will be subsets of a fixed Euclidean plane, which will be denoted by E_2. The complement of B with respect to A is defined by

$$A - B = \{x : x \varepsilon A, x \not\varepsilon B\},$$

and the complement of B will mean $E_2 - B$. The null or empty set will be denoted by $\emptyset$. The product or intersection, AB or $A \cap B$, of two sets A and B is defined as

$$AB = A \cap B = \{x : x \varepsilon A \text{ and } x \varepsilon B\}.$$

The product or intersection of a collection of sets $\{S_\alpha\}$ is defined as

$$\bigcap_\alpha S_\alpha = \{x : x \varepsilon S_\alpha \text{ for all } \alpha\}.$$

The sum or union of a collection of sets $\{S_\alpha\}$ is defined as

$$\bigcup_\alpha S_\alpha = \{x : x \varepsilon S_\alpha \text{ for some } \alpha\}.$$

If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are any two points in E_2, the distance between x and y will be defined as

$$\rho(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}.$$

The diameter of a set is defined as $\delta(A) = \sup \rho(x, y)$ where x and y vary over A. The distance between two sets A and B is defined as $d(A, B) = \inf \rho(a, b)$, where a varies over A and b varies over B.

A set whose members are sets will be called a class and classes will be denoted by Greek letters marked with an asterisk, e.g., α^*, β^*, etc. The union of the elements of α^* will be defined by

$$\alpha = \bigcup_{A \in \alpha^*} A.$$

Let S be an arbitrary set in E. Given a positive number ρ, let α^* be a class with elements, A_n, n = 1, 2, ..., where each A_n is an open convex region such that $\delta(A_n) < \rho$ and $\alpha \supset S$. The exterior linear measure of S, denoted by $L^*(S)$, is defined by

$$L^*(S) = \lim_{\rho \to 0} \inf_{\alpha^*} \sum_{n=1}^{\infty} \delta(A_n)$$

where α^* ranges over all classes satisfying the above conditions for a given ρ. Since

$$\inf_{\alpha^*} \sum_{n=1}^{\infty} \delta(A_n)$$

is non-decreasing as $\rho \to 0$,

$$\inf_{\alpha^*} \sum_{n=1}^{\infty} \delta(A_n)$$

approaches a limit, finite or infinite, as $\rho \to 0$. Thus $L^*(S)$ is defined for any set S.

A set S is linearly measurable, hereafter referred to as measurable, if for every set A, $L^*(A) = L^*(AS) + L^*(A - S)$. This criterion for measurability is accredited to Carathéodory.

As defined above, L^* is a regular, metric outer measure (see Munroe, 13) and thus has the following properties:

(1) L^* is non-decreasing

(2) For any class α^* with a countable number of elements, A_n,

$$L^*(\alpha) \leq \sum_{n=1}^{\infty} L^*(A_n)$$

(3) $L^*(\emptyset) = 0$

(4) If $d(A, B) > 0$, then $L^*(A \cup B) = L^*(A) + L^*(B)$

(5) Every Borel set is measurable

These properties will be used implicitly throughout the remainder of this paper. The phrase almost all of S will mean all points of S with the exception of a set of linear measure zero.

<u>Theorem 2.1</u>. If α^* is a class with a countable number

of elements, A_n, such that $L^*(\alpha) < \infty$ and $d(A_n, A_m) > 0$ if $n \neq m$, then

$$L^*(\alpha) = \sum_{n=1}^{\infty} L^*(A_n).$$

Proof. Suppose

$$\sum_{n=1}^{\infty} L^*(A_n) = L^*(\alpha) + c \text{ for some } c > 0.$$

Using an induction argument on Property 4 we have

$$\sum_{n=1}^{K} L^*(A_n) = L^*\left(\bigcup_{n=1}^{K} A_n\right)$$

for any integer K. Since the series converges, there exists an integer N such that

$$\sum_{n=N+1}^{\infty} L^*(A_n) < c/2.$$

Hence

$$\sum_{n=1}^{\infty} L^*(A_n) = \sum_{n=1}^{N} L^*(A_n) + \sum_{n=N+1}^{\infty} L^*(A_n)$$

$$= L^*\left(\bigcup_{n=1}^{N} A_n\right) + \sum_{n=N+1}^{\infty} L^*(A_n) < L^*(\alpha) + \frac{c}{2}.$$

This contradiction shows that

$$\sum_{n=1}^{\infty} L^*(A_n) \leq L^*(\alpha),$$

which together with Property 2 above establishes the theorem.

A circle with center a and radius r will denote the following set,

$$c(a, r) = \{x : \rho(x, a) \leq r\}.$$

Consider the ratio $\frac{L^*(Sc(a, r))}{2r}$. The lower limit, the upper limit and the limit if it exists of this ratio as $r \to 0$, are called respectively the lower exterior density, the upper exterior density and the exterior density of the set S at the point a (a need not belong to S) and are denoted by $\underline{D}^*(a, S)$, $\overline{D}^*(a, S)$ and $D^*(a, S)$, respectively. If S is measurable, the asterisk and the word exterior are omitted.

A point s of S is called a regular point of S if $D^*(s, S)$ exists and $D^*(s, S) = 1$. Any other point of S is called an irregular point of S. A set S is called a regular set if almost all points of S are regular points of S, and if almost all points of S are irregular points of S the set is called an irregular set.

III. LINEAR MEASURE

Throughout this chapter, unless otherwise designated, the term <u>class</u> will denote a countable collection, each of whose members is an open convex region. Also every set denoted by A or B will have finite exterior linear measure.

Results similar to several of those in this chapter have been established by Jeffrey (11) for sets in E_1.

<u>Definition 3.1</u>. Two sets A and B are separated if for every $\epsilon > 0$ there exist two classes α^* and β^* such that $L^*(A - \alpha) < \epsilon$, $L^*(\alpha B) < \epsilon$, $L^*(B - \beta) < \epsilon$ and $L^*(\beta A) < \epsilon$.

Three obvious properties of separated sets are:

I. If A and B are separated and $C \subset A$, then B and C are separated.

II. A and B are separated if $L^*(A) = 0$.

III. A and B are separated in the sense of Definition 3.1 if they are topologically separated, i.e. neither contains a point or limit point of the other.

<u>Lemma 3.1</u>. If A and B are separated then $L^*(AB) = 0$.

<u>Proof</u>. Suppose $L^*(AB) = \epsilon > 0$, and let α^* be any class such that $L^*(A - \alpha) < \epsilon/2$. Since α is an open set it is measurable, hence $L^*(AB) = L^*(\alpha AB) + L^*(AB - \alpha) < L^*(\alpha AB) + \epsilon/2$. Therefore $L^*(\alpha B) \geq L^*(\alpha AB) > \epsilon - \epsilon/2$. This contradiction establishes the lemma.

Definition 3.2. For any set A and B let

$$A_B^0 = \{a : a \varepsilon A, \overline{D}* (a, B) = 0\}$$

and

$$A_B^+ = \{a : a \varepsilon A, \overline{D}* (a, B) > 0\}.$$

A proof of the following lemma may be found in Munroe (13).

Lemma 3.2. If $\{A_n\}$ is a sequence of sets such that

$$A_n \subset A_{n+1}$$

and

$$\lim_{n\to\infty} A_n = A,$$

then

$$\lim_{n\to\infty} L*(A_n) = L*(A).$$

Theorem 3.1. If A and B are sets such that for every $\epsilon > 0$ there exists a class α* such that $L*(A - \alpha) < \epsilon$ and $L*(\alpha B) < \epsilon$, then A and B are separated.

Proof. For any $\epsilon > 0$ there is a class α* such that $L*(A - \alpha) < \epsilon/3$ and $L*(\alpha B) < \epsilon/3$. From Lemma 3.2 there exists an integer N such that

$$L^*(A) - L^*(\alpha' A) < \frac{2\epsilon}{3} \text{ for } \alpha' = \bigcup_{n=1}^{N} \alpha_n, \ \alpha_n \in \alpha^*,$$

therefore $L^*(\alpha' A) > L^*(A) - \frac{2\epsilon}{3}$. Let

$$V_i = \bigcup_{j=1}^{N} v_{ij}$$

where v_{ij} is closed,

$$v_{ij} \subset \alpha_j, \ v_{(i-1)j} \subset v_{ij} \text{ and } \bigcup_{i=1}^{\infty} v_{ij} = \alpha_j.$$

For M sufficiently large, from Lemma 3.2 we have

$$L^*(\alpha' A) - L^*(AV_M) < \epsilon/3,$$

therefore

$$L^*(AV_M) > L^*(\alpha' A) - \epsilon/3 > L^*(A) - \epsilon .$$

Since V_M and the complement of α, $E_2 - \alpha$, are disjoint closed sets and V_M is bounded there exists a $\delta > 0$ such that $d(V_M, E_2 - \alpha) = \delta$. Let β^* be a class such that $\beta \supset B - \alpha$ and $d(V_M, \beta) \geq \delta/2$. Then $L^*(B - \beta) \leq L^*(\alpha B) < \epsilon$, therefore $L^*(V_M A) + L^*(\beta A) = L^*((V_M A) \cup (\beta A)) \leq L^*(A)$. Thus $L^*(\beta A) \leq L^*(A) - L^*(V_M A) < \epsilon$ so A and B are separated by Definition 3.1.

<u>Lemma 3.3</u>. If A_1 and A_2 are separated sets and A_1 and A_3 are separated sets then A_1 and $A_2 \cup A_3$ are separated sets.

Proof. For any $\epsilon > 0$ there are two classes α^* and β^* such that

$$L^*(A_2 - \alpha) < \epsilon/2, \; L^*(\alpha A_1) < \epsilon/2, \; L^*(A_3 - \beta) < \epsilon/2$$

and

$$L^*(\beta A_1) < \epsilon/2.$$

Let $\gamma^* = \alpha^* \cup \beta^*$, then

$$L^*(\gamma A_1) \leq L^*(\alpha A_1) + L^*(\beta A_1) < \epsilon/2 + \epsilon/2 = \epsilon$$

and

$$L^*((A_2 \cup A_3) - \gamma) = L^*((A_2 - \gamma) \cup (A_3 - \gamma))$$

$$\leq L^*(A_2 - \gamma) + L^*(A_3 - \gamma) \leq L^*(A_2 - \alpha) + L^*(A_3 - \beta)$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Since a class γ^* satisfying the above inequalities can be found for any $\epsilon > 0$, A_1 and $A_2 \cup A_3$ are separated by Theorem 3.1.

Theorem 3.2. If $A_0 = \{a : a \in A, \overline{D}^*(a, A) = 0\}$ then $L^*(A_0) = 0$.

Proof. Given any $\epsilon > 0$, for each $a \in A_0$ there exists an $R > 0$ such that for all $r < R$

$$(1) \qquad \frac{L^*(Ac(a, r))}{2r} < \epsilon.$$

For a fixed ϵ, let E_n denote the set of points of A_0 such that Inequality 1 holds for all $r < 1/n$. Since

$$\lim_{n\to\infty} E_n = A_0 \quad \text{and} \quad E_m \subset E_n$$

if m < n, Lemma 3.2 guarantees the existence of a positive integer N such that for all n > N

$$(2) \qquad L^*(E_n) > L^*(A_0) - \epsilon$$

For some K > N, let α^* be any class such that

$$\alpha \supset E_K, \ \delta(\alpha_n) < 1/K \quad \text{and} \quad \sum_{n=1}^{\infty} \delta(A_n) < L^*(A_0) + \epsilon .$$

About each set α_n circumscribe a circle c_n of radius $\delta(\alpha_n)$ with center at some point of E_K. Then

$$\frac{L^*(Ac_n)}{2\delta(\alpha_n)} < \epsilon$$

from Inequality 1. But

$$\bigcup_{n=1}^{\infty} c_n \supset \alpha \supset E_K$$

so

$$L^*(A_0) - \epsilon < L^*(E_K) = L^*(E_K(\bigcup_{n=1}^{\infty} c_n)) \leq \sum_{n=1}^{\infty} L^*(E_K c_n)$$

$$\leq \sum_{n=1}^{\infty} L^*(Ac_n) < 2\,\epsilon \sum_{n=1}^{\infty} \delta(\alpha_n) < 2\,\epsilon(L^*(A_0) + \epsilon).$$

Since ϵ is arbitrary, this is true only if $L^*(A_0) = 0$.

A family of F_A of circles covers A in the sense of Vitali if every point of A is the center of a sequence of circles of F_A with radii approaching zero.

The proofs of Lemma 3.4 and Lemma 3.5 may be found in W. Sierpiński (15). Lemma 3.4 applies to any plane set.

Lemma 3.4. If F_A covers A in the sense of Vitali, then there exists a sequence of mutually exclusive circles $\{c_n\}$, $c_n \in F_A$, such that if C_n is a circle (not necessarily belonging to F_A) with the same center, and with radius three times that of c_n, then

$$\bigcup_{n=1}^{\infty} C_n \supset A.$$

Lemma 3.5. If $A_1 = \{a : a \in A, \overline{D}^*(a, A) > 1\}$ then $L^*(A_1) = 0$.

Randolph (14) has shown that if $\underline{D}^*(a, A) \geq k > 0$ for almost all a in A and F_A is a family of circles covering A in the sense of Vitali, then there exists a sequence of mutually disjoint circles, $\{c_n\}$, of F_A such that

$$L^*(A - \bigcup_{n=1}^{\infty} c_n) = 0.$$

The following theorem is a generalization of Randolph's result in the sense that it is only required that $\bar{D}^*(a, A) \geq k > 0$ for almost all a in A, however the circles c_n may not be members of a given F_A.

Theorem 3.3. If there exists a $k > 0$ such that $\bar{D}^*(a, A) \geq k$ for almost all $a \varepsilon A$, then for any $\epsilon > 0$ there exists a mutually exclusive sequence of circles $\{c_n\}$ such that each c_n has a point of A as its center with radius less than ϵ and

$$L^*(A - \bigcup_{n=1}^{\infty} c_n) = 0.$$

Proof. From the conditions of the theorem and Lemma 3.5, the set $A' = \{a : a \varepsilon A, k \leq \bar{D}^*(a, A) \leq 1\}$ has the property that $L^*(A) = L^*(A')$. At each point $a \varepsilon A'$ there exists a sequence of circles $\{c(a, r_n)\}$, $r_n < \epsilon$, $r_n \to 0$ such that

(1) $$k/2 < \frac{L^*(Ac(a, r_n))}{2r_n}$$

and

(2) $$\frac{L^*(Ac(a, 3r_n))}{6r_n} < 2$$

Let C denote this family of circles. Hence from Lemma 3.4 there exists a sequence of disjoint circles $\{c_n\}$, $c_n = c(a_n, r_n)$ such that

$$\bigcup_{n=1}^{\infty} c(a_n, \quad 3r_n) \supset A'.$$

Since Inequality 1 is true for each c_n and they are disjoint

$$k \sum_{n=1}^{N} r_n < \sum_{n=1}^{N} L^*(Ac_n) = L^*(A(\bigcup_{n=1}^{N} c_n))$$

$$\leq L^*(A(\bigcup_{n=1}^{\infty} c_n)) \leq L^*(A).$$

Thus the series

$$\sum_{n=1}^{\infty} r_n$$

converges and

$$(3) \qquad k \sum_{n=1}^{\infty} r_n \leq L^*(A(\bigcup_{n=1}^{\infty} c_n)).$$

Since

$$\bigcup_{n=1}^{\infty} c(a_n, 3r_n) \supset A'$$

and Inequality 2 holds for each $c(a_n, 3r_n)$ we have

$$(4)\qquad L^*(A) \leq \sum_{n=1}^{\infty} L^*(Ac(a_n, 3r_n)) < 12 \sum_{n=1}^{\infty} r_n$$

Therefore from Inequalities 3 and 4,

$$kL^*(A) < 12k \sum_{n=1}^{\infty} r_n \leq 12L^*(A(\bigcup_{n=1}^{\infty} c_n)) .$$

Let

$$S_1 = \bigcup_{n=1}^{N} c(a_n, r_n)$$

then there exists an integer N such that $kL^*(A) < 12L^*(AS_1)$. Thus $(12 - k)L^*(A) > 12(L^*(A) - L^*(AS_1))$. Since S_1 is closed it is linearly measurable so that $L^*(A) = L^*(AS_1) + L^*(A - S_1)$ and hence

$$(5)\qquad L^*(A - S_1) < \frac{12 - k}{12} L^*(A)$$

Now let C_1 denote the family of circles which are in C that have no point in common with S_1. Since S_1 is closed, for every $a \varepsilon A' - S_1$ there exists a circle $c(a, r)$ such that $S_1 c(a, r) = \emptyset$, so that $C_1 \supset A' - S_1$. Proceeding exactly as above, using C_1 in place of C and $A' - S_1$ in place of A', we obtain a set S_2 which is the union of mutually exclusive circles of C_1 and the following relation, analogous to

Inequality 5

(6) $$L^*((A - S_1) - S_2) < \frac{12 - k}{12} L^*(A - S_1).$$

From Inequalities 5 and 6 we have

$$L^*((A - S_1) - S_2) < (\frac{12 - k}{12}) L^*(A - S_1)$$
$$< (\frac{12 - k}{12})^2 L^*(A)$$

Continuing this process we have mutually exclusive sets $\{S_n\}$ each consisting of mutually exclusive circles of C such that for every positive integer M,

$$L^*(A - \bigcup_{n=1}^{\infty} S_n) \leq L^*(A - \bigcup_{n=1}^{M} S_n) < (\frac{12 - k}{12})^M L^*(A)$$

But $k > 0$ and $L^*(A) < \infty$ so that

$$L^*(A - \bigcup_{n=1}^{\infty} S_n) = 0.$$

<u>Theorem 3.4</u>. Given any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists a mutually exclusive sequence of circles $\{c_n\}$ such that each c_n has a point of A as its center, radius less than ϵ_1 and

$$L^*(A - \bigcup_{n=1}^{\infty} c_n) < \epsilon_2.$$

<u>Proof</u>. Let $A_0 = \{a : a \in A, \overline{D}^*(a, A) = 0\}$ and

$A_n = \{a : a \varepsilon A, 1/n \leq \bar{D}^*(a, A) \leq 1\}$ and

$A' = \{a : a \varepsilon A' \bar{D}^*(a, A) > 1\}$.

$$A_1 \subset A_2 \subset \ldots \subset A_n \subset A_{n+1} \subset \ldots \subset A - (A_0 \cup A')$$

and

$$\lim_{n\to\infty} A_n = A - (A_0 \cup A').$$

By Theorem 3.2 and Lemma 3.5 $L^*(A_0 \cup A') = 0$ so by Lemma 3.2

$$\lim_{n\to\infty} L^*(A_n) = L^*(A - (A_0 \cup A')) = L^*(A).$$

Hence given any $\epsilon_2 > 0$ there is an integer N such that $L^*(A_n) > L^*(A) - \epsilon_2$ if $n > N$. By Theorem 3.3 for each positive integer n there exists a mutually exclusive sequence of circles $\{c_{n(K)}\}$ such that each $c_{n(K)}$ has a point of A_n as its center with radius less than ϵ_1 and

$$L^*(A_n - \bigcup_{K=1}^{\infty} c_{n(K)}) = 0.$$

Therefore

$$L^*(A - \bigcup_{K=1}^{\infty} c_{n(K)}) = L^*(A) - L^*(A(\bigcup_{K=1}^{\infty} c_{n(K)}))$$

$$\leq L^*(A) - L^*(A_n(\bigcup_{K=1}^{\infty} c_{n(K)})) = L^*(A) - L^*(A_n) < \epsilon_2.$$

Lemma 3.6. If the sets $\{A_n\}$ are such that

$$A_n \subset A_{n+1}, \quad \lim_{n\to\infty} A_n = A$$

and the sets A_n and B are separated for each n, then the sets A and B are separated.

Proof. Since A_n and B are separated, for every $\epsilon > 0$ there exists a class α_n^* such that $L^*(A_n - \alpha_n) < \epsilon/2^n$ and $L^*(\alpha_n B) < \epsilon/2^n$. Let

$$\alpha^* = \bigcup_{n=1}^{\infty} \alpha_n^*,$$

then

$$L^*(A - \alpha) = L^*(A - \bigcup_{n=1}^{\infty} \alpha_n) = L^*(\bigcup_{K=1}^{\infty} (A_K - \bigcup_{n=1}^{\infty} \alpha_n))$$

$$\leq \sum_{K=1}^{\infty} L^*(A_K - \bigcup_{n=1}^{\infty} \alpha_n) \leq \sum_{K=1}^{\infty} L^*(A_K - \alpha_K)$$

$$< \sum_{K=1}^{\infty} \epsilon/2^K = \epsilon$$

and

$$L^*(\alpha B) = L^*(B(\bigcup_{n=1}^{\infty} \alpha_n)) \leq \sum_{n=1}^{\infty} L^*(\alpha_n B) < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

Therefore the sets A and B are separated by Theorem 3.1.

Theorem 3.5. The sets A_B^0 and B are separated.

Proof. For any pair of positive numbers ϵ, δ, we define two sets as follows:

$$A(\epsilon, \delta) = \left\{a : a \in A_B^0, \frac{L^*(Bc(a, r))}{2r} < \epsilon/2, \right.$$

$$\left. \text{for every } r < \delta \right\}$$

and

$$B(\epsilon) = \left\{b : b \in B, \epsilon < \frac{L^*(Bc(b, r_n))}{2r_n}, \right.$$

$$\left. \text{for some } \{r_n\} \to 0 \right\}.$$

Suppose for some pair of numbers ϵ_1, δ_1 that the sets $A(\epsilon_1, \delta_1)$ and $B(\epsilon_1)$ are not separated. From III following Definition 3.1 there is a sequence $\{b_n\}$, $b_n \in B(\epsilon_1)$ such that

$$\lim_{n\to\infty} b_n = a_0 \text{ for some } a_0 \in A(\epsilon_1, \delta_1).$$

For every δ_2, $0 < 2\delta_2 < r_N < \delta_1$, there exists a b_n such that $c(b_n, r_N - \delta_2) \subset c(a_0, r_N)$. From the definition of $A(\epsilon_1, \delta_1)$ and $B(\epsilon_1)$ we have

$$2\epsilon_1(r_N - \delta_2) < L^*(Bc(b_n, r_N - \delta_2))$$

$$\leq L^*(Bc(a_0, r_N)) < \epsilon_1 r_N.$$

This inequality is contradictory, hence $A(\epsilon, \delta)$ and $B(\epsilon)$ are separated for every pair of numbers ϵ, δ.

For any given $\epsilon > 0$, $A(\epsilon, 1/m) \subset A(\epsilon, 1/n)$ if $m < n$ and

$$\lim_{n\to\infty} A(\epsilon, 1/n) = A_B^0;$$

also $B(1/m) \subset B(1/n)$ if $m < n$ and

$$\lim_{n\to\infty} B(1/n) = B - B_0$$

where $B_0 = \{b : b \varepsilon B, \overline{D}*(b, B) = 0\}$. Since $A(\epsilon, 1/n)$ and $B(1/m)$ are separated, A_B^0 and $B(1/m)$ are separated by Lemma 3.6. Similarly A_B^0 and $B - B_0$ are separated. Therefore A_B^0 and B are separated since $L*(B_0) = 0$ from Theorem 3.2.

<u>Theorem 3.6</u>. There is no $E \subset A_B^+$ such that $L*(E) > 0$ and E and B are separated.

<u>Proof</u>. Suppose there exists a set $E \subset A_B^+$ such that $L*(E) = \lambda > 0$ and E and B are separated. Then for each $e \varepsilon E$ there exists a $d > 0$ such that

$$(1) \qquad \frac{L*(Bc(e, r_n))}{2r_n} > d$$

for a properly chosen sequence $\{r_n\} \to 0$ as $n \to \infty$. Let $E(d)$ be the part of E for which Inequality 1 holds for a given d. Then $E(1/m) \subset E(1/n)$ if $m < n$ and

$$\lim_{n\to\infty} E(1/n) = E.$$

Thus by Lemma 3.2 there is an N such that $L^*(E') = \lambda' > \lambda/2$ where $E' = E(1/N)$. By hypothesis, E and B are separated, hence E' and B are separated. Therefore for any $\epsilon > 0$ there exists a class α^* such that

(2) $$L^*(E' - \alpha) < \epsilon \quad \text{and} \quad L^*(\alpha B) < \epsilon .$$

Let $E'' = \{e : \varepsilon\ E', \bar{D}^*(e, E') \leq 1\}$. By Lemma 3.5, $L^*(E'') = L^*(E')$. Let C be the family of circles $\{c(e, r)\}$ such that $e \varepsilon E''$, $c(e, r) \subset \alpha$, $\frac{L^*(E'c(e, 3r))}{6r} < 2$ and Inequality 1 is satisfied. From Lemma 3.4 there exists a sequence of disjoint circles $\{c(e_n, r_n)\}$ such that

$$c(e_n, r_n) \varepsilon C$$

and

$$\bigcup_{n=1}^{\infty} c(e_n, 3r_n) \supset \alpha E''.$$

Let

$$C' = \bigcup_{n=1}^{\infty} c(e_n, r_n).$$

From Inequality 1,

$$L^*(\alpha B) \geq L^*(C'B) = \sum_{n=1}^{\infty} L^*(Bc(e_n, r_n)) > 2/N \sum_{n=1}^{\infty} r_n .$$

Also

$$L^*(\alpha E') = L^*(\alpha E'') \leq L^*(E''(\bigcup_{n=1}^{\infty} c(e_n, 3r_n)))$$

$$= L^*(E'(\bigcup_{n=1}^{\infty} c(e_n, 3r_n))) \leq \sum_{n=1}^{\infty} L^*(E'c(e_n, 3r_n))$$

$$< 12 \sum_{n=1}^{\infty} r_n.$$

From Inequality 2, $L^*(\alpha E') > L^*(E') - \epsilon$. Therefore

$$L^*(\alpha B) > 2/N \sum_{n=1}^{\infty} r_n > 2/N \frac{L^*(\alpha E')}{12}$$

$$> \frac{1}{6N} (L^*(E') - \epsilon) > \frac{1}{6N} (\lambda/2 - \epsilon).$$

Since ϵ is arbitrary and N is fixed, this contradicts Inequality 2 and establishes the theorem.

<u>Theorem 3.7</u>. There is no $E \subset A_B^+$ such that $L^*(E) > 0$ and E and B_A^+ are separated.

<u>Proof</u>. By Theorem 3.5 B_A^0 and A are separated, hence B_A^0 and A_B^+ are separated. If $E \subset A_B^+$ is such that $L^*(E) > 0$ then by Theorem 3.6 E and B are not separated, but E and B_A^0 are separated. Therefore E and B_A^+ are not separated by Lemma 3.3.

<u>Theorem 3.8</u>. The sets A_B^0 and A_B^+ are separated.

<u>Proof</u>. For any pair of positive numbers ϵ, δ, we define two sets as follows:

$$A_0(\epsilon, \delta) = \{a: a \in A_B^0, \frac{L^*(Bc(a, r))}{2r} < \epsilon/2,$$
$$\text{for every } r < \delta\}$$

and

$$A_+(\epsilon) = \{a: a \in A_B^+, \epsilon < \frac{L^*(Bc(a, r_n))}{2r_n},$$
$$\text{for some } \{r_n\} \to 0\}.$$

Suppose for some pair of numbers ϵ_1, δ_1, that the sets $A_0(\epsilon_1, \delta_1)$ and $A_+(\epsilon_1)$ are not separated. From III following Definition 3.1 there is a sequence $\{a_n\}$, $a_n \in A_+(\epsilon_1)$ such that

$$\lim_{n \to \infty} a_n = a_0 \text{ for some } a_0 \in A_0(\epsilon_1, \delta_1).$$

For every δ_2, $0 < 2\delta_2 < r_N < \delta_1$, there exists an a_n such that $c(a_n, r_N - \delta_2) \subset c(a_0, r_N)$. From the definition of $A_0(\epsilon_1, \delta_1)$ and $A_+(\epsilon_1)$ we have,

$$2\epsilon_1(r_N - \delta_2) < L^*(Bc(a_n, r_N - \delta_2)$$
$$\leq L^*(Bc(a_0, r_N)) < 2r_N\epsilon_{1/2} = r_N\epsilon_1.$$

This inequality is contradictory, hence $A_0(\epsilon, \delta)$ and $A_+(\epsilon)$ are separated for every pair of numbers ϵ, δ.

For any given $\epsilon > 0$,

$$A_0(\epsilon, 1/m) \subset A_0(\epsilon, 1/n) \text{ if } n > m$$

and

$$\lim_{n\to\infty} A_0(\epsilon,\ 1/n) = A_B^0;$$

also

$$A_+(1/m) \subset A_+(1/n) \text{ if } m < n$$

and

$$\lim_{n\to\infty} A_+(1/n) = A_B^+.$$

Therefore by Lemma 3.2, for any $\lambda_1 > 0$ there exists a pair of integers n, m such that $L^*(A_B^+) - L^*(A_+(1/m)) < \lambda_1$ and $L^*(A_B^0) - L^*(A_0(1/m,\ 1/n)) < \lambda_1$.

Now suppose A_B^0 and A_B^+ are not separated. Then there exists a $\lambda > 0$ such that if α^* is any class with the property that $L^*(A_B^0 - \alpha) < \lambda$, then $L^*(\alpha A_B^+) \geq \lambda$. From above, there exists a pair of integers N, M, such that

$$L^*(A_B^0) - L^*(A_0(1/N,\ 1/M)) < \lambda/2$$

and

$$L^*(A_B^+) - L^*(A_+(1/N)) < \lambda/2.$$

If α^* is any class such that $L^*(A_0(1/N,\ 1/M) - \alpha) < \lambda/2$, then

$$L^*(A_B^0 - \alpha) = L^*(A_B^0) - L^*(\alpha A_B^0)$$

$$\leq L^*(A_B^0) - L^*(\alpha A_0(1/N,\ 1/M))$$

$$= L^*(A_B^0) - (L^*(A_0(1/N,\ 1/M)) - L^*(A_0(1/N,\ 1/M) - \alpha))$$

$$< \lambda/2 + \lambda/2 = \lambda.$$

therefore

$$\lambda \leq L^*(\alpha A_B^+) = L^*(A_B^+) - L^*(A_+(1/N))$$

$$+ L^*(\alpha A_+(1/N)) - (L^*(A_B^+ - \alpha) - L^*(A_+(1/N) - \alpha))$$

$$\leq L^*(A_B^+) - L^*(A_+(1/N)) + L^*(\alpha A_+(1/N))$$

$$< \lambda/2 + L^*(\alpha A_+(1/N)),$$

so that $L^*(\alpha A_+(1/N)) > \lambda/2$. Hence $A_+(1/N)$ and $A_0(1/N,\ 1/M)$ are not separated. This contradiction proves the theorem.

Theorem 3.9. If A and B are separated sets then $L^*(A) + L^*(B) = L^*(A \cup B)$.

Proof. Since A and B are separated, for any given $\epsilon > 0$ there exists a class α^* such that $L^*(A - \alpha) < \epsilon$ and $L^*(\alpha B) < \epsilon$. Let $E = A \cup B$ then $L^*(\alpha E) + L^*(E - \alpha) = L^*(E)$ since α is open and hence measurable. Therefore $L^*(\alpha A) + L^*(B - \alpha) \leq L^*(E)$. However

$$L^*(\alpha A) + L^*(A - \alpha) = L^*(A)$$

and

$$L^*(\alpha B) + L^*(B - \alpha) = L^*(B),$$

therefore

$$L^*(A) + L^*(B) = L^*(\alpha A) + L^*(B - \alpha) + L^*(A - \alpha)$$

$$+ L^*(\alpha B) < L^*(\alpha A) + L^*(B - \alpha) + 2\epsilon \leq L^*(E) + 2\epsilon.$$

Since ϵ is arbitrary we have $L^*(A) + L^*(B) \leq L^*(E) = L^*(A \cup B)$. But $L^*(A \cup B) \leq L^*(A) + L^*(B)$, therefore $L^*(A) + L^*(B) = L^*(A \cup B)$.

Besicovitch (3) has proven the following lemma.

<u>Lemma 3.7</u>. Let α^*_ρ denote any class such that $\delta(\alpha_n) < \rho$ for each $\alpha_n \varepsilon \alpha^*_\rho$. Then for any set A and any $\epsilon > 0$ there exists a $\rho_1 > 0$ such that

$$L^*(A\alpha_\rho) < \sum_{n=1}^{\infty} \delta(\alpha_n) + \epsilon$$

for all $\rho < \rho_1$.

<u>Theorem 3.10</u>. $L^*(A^+_B) = L^*(B^+_A) = L^*(A^+_B \cup B^+_A)$.

<u>Proof</u>. Suppose $L^*(A^+_B) = L^*(B^+_A) + c$ for some $c > 0$. It follows from Lemma 3.7 that a number $\rho > 0$ may be fixed such that for any class α^* with $\delta(\alpha_n) < \rho$ we have

$$L^*(\alpha A^+_B) < \sum_{n=1}^{\infty} \delta(\alpha_n) + c/4.$$

Now choose α^* such that $\alpha \supset B^+_A$ with $\delta(\alpha_n) < \rho$ and

$$\sum_{n=1}^{\infty} \delta(\alpha_n) < L^*(B^+_A) + c/4.$$

Thus $L^*(\alpha A^+_B) < L^*(B^+_A) + c/2$. Let $E = A^+_B - \alpha$; then since α is measurable,

$$L^*(E) = L^*(A_B^+ - \alpha) = L^*(A_B^+) - L^*(\alpha A_B^+)$$

$$= L^*(B_A^+) + c - L^*(\alpha A_B^+)$$

$$> L^*(B_A^+) + c - (L^*(B_A^+) + c/2) = c/2.$$

Since $\alpha E = \emptyset$ for any $\epsilon > 0$ we have $L^*(B_A^+ - \alpha) = 0 < \epsilon$ and $L^*(\alpha E) = 0 < \epsilon$. Hence B_A^+ and E are separated, contradicting Theorem 3.7, so that $L^*(A_B^+) \leq L^*(B_A^+)$. The same argument shows that $L^*(B_A^+) \leq L^*(A_B^+)$. Therefore $L^*(B_A^+) = L^*(A_B^+)$.

Now suppose $L^*(A_B^+ \cup B_A^+) = L^*(A_B^+) + c$ for some $c > 0$. By Lemma 3.8 there is a class α^* such that

$$\alpha \supset A_B^+, \quad \sum_{n=1}^{\infty} \delta(\alpha_n) < L^*(A_B^+) + c/4$$

and

$$L^*(\alpha(A_B^+ \cup B_A^+)) < \sum_{n=1}^{\infty} \delta(\alpha_n) + c/4.$$

Therefore $L^*(\alpha(A_B^+ \cup B_A^+)) < L^*(A_B^+) + c/2$. Let $E = (A_B^+ \cup B_A^+) - \alpha$ then

$$L^*(E) = L^*((A_B^+ \cup B_A^+) - \alpha)$$

$$= L^*(A_B^+ \cup B_A^+) - L^*(\alpha(A_B^+ \cup B_A^+))$$

$$= L^*(A_B^+) + c - L^*(\alpha(A_B^+ \cup B_A^+))$$

$$> L^*(A_B^+) + c - (L^*(A_B^+) + c/2) = c/2.$$

Since $\alpha \supset A_B^+$ and $\alpha E = \emptyset$, $E \subset B_A^+$. Then for any $\epsilon > 0$ $L^*(A_B^+ - \alpha) = 0 < \epsilon$ and $L^*(\alpha E) = 0 < \epsilon$, hence E and A_B^+ are separated, contradicting Theorem 3.7. Hence $L^*(A_B^+ \cup B_A^+) \leq L^*(A_B^+)$ so $L^*(A_B^+ \cup B_A^+) = L^*(A_B^+) = L^*(B_A^+)$.

<u>Theorem 3.11</u>. If C is any one of the sets A_B^+, B_A^+ or $A_B^+ \cup B_A^+$ then:

(i) $L^*(A) = L^*(A_B^0) + L^*(A_B^+)$

(ii) $L^*(A \cup B) = L^*(A_B^0) + L^*(B_A^0) + L^*(C)$

(iii) $L^*(A) + L^*(B) = L^*(A \cup B) + L^*(C)$

<u>Proof</u>. (i) The sets A_B^0 and A_B^+ are separated by Theorem 3.8 so that Equality i follows from Theorem 3.9.

(ii) From Definition 3.2, $A \cup B = A_B^0 \cup B_A^0 \cup (A_B^+ \cup B_A^+)$. By Theorem 3.5 A_B^0 and B are separated also B_A^0 and A are separated. By Theorem 3.8 A_B^0 and A_B^+ are separated also B_A^0 and B_A^+ are separated. Hence, A_B^0, B_A^0 and $A_B^+ \cup B_A^+$ are mutually separated sets. Therefore $L^*(A \cup B) = L^*(A_B^0) + L^*(B_A^0) + L^*(A_B^+ \cup B_A^+)$ by Theorem 3.9. Thus Equality ii follows from Theorem 3.10.

(iii) From Equalities i and ii and Theorem 3.10,

$$L^*(A) + L^*(B) = L^*(A_B^0) + L^*(A_B^+) + L^*(B_A^0) + L^*(B_A^+)$$

$$= L^*(A_B^0) + L^*(B_A^0) + 2L^*(C) = L^*(A \cup B) + L^*(C).$$

Theorem 3.12. A necessary and sufficient condition for A to be measurable is that $L^*(C_A^+) = 0$ where $C = E_2 - A$.

Proof. Let W be any set such that $L^*(W) < \infty$. We first show that if $L^*(C_A^+) = 0$ then $L^*(W) = L^*(AW) + L^*(W - A)$. Let $E = W - A$. Then $E \subset C$ so $E_A^+ \subset C_A^+$ and hence $L^*(E_A^+) = 0$. Also since $AW \subset A$, $E_{AW}^+ \subset E_A^+$ so that $L^*(E_{AW}^+) = 0$. Therefore by Theorem 3.11, Equality iii, we have

$$L^*(AW) + L^*(E) = L^*(E \cup AW) + L^*(E_{AW}^+)$$

$$= L^*(E \cup AW) = L^*(W).$$

Now suppose $L^*(C_A^+) > 0$. Let $C_n = \{c : c \in C_A^+, \overline{D}^*(c, A) > 1/n\}$. It follows from Lemma 3.4 that there is an integer n_0 such that $0 < L^*(C_m) \leq 6mL^*(A) < \infty$ for $m \geq n_0$. Let B be one of the sets C_n for $n \geq n_0$, then $B_A^+ = B$. Also let $W = A_B^+ \cup B_A^+$. Then by Theorem 3.10, $L^*(W) = L^*(A_B^+)$ $= L^*(B_A^+) = L^*(B) > 0$. Thus $L^*(AW) + L^*(W - A) = L^*(A_B^+)$ $+ L^*(B_A^+) = 2L^*(W) \neq L^*(W)$. Hence A is not measurable.

Theorem 3.13. If A is a measurable set and $A = B \cup C$, then B and C are measurable if B and C are separated.

Proof. Let

$$B' = \{b : b \notin C, \overline{D}*(b, C) > 0\},$$

$$C' = \{c : c \notin B, \overline{D}*(c, B) > 0\}$$

and

$$E = \{e : e \notin A, \overline{D}*(e, A) > 0\}.$$

Then $C_B^+ = (CC') \cup F$ where

$$F = \{c : c \varepsilon BC, \overline{D}*(c, B) > 0\}.$$

Therefore $L*(F) = 0$ by Lemma 3.1. From Theorem 3.9 and Theorem 3.11, Equality iii, we have

$$L*(A) = L*(B) + L*(C) = L*(B) + L*(C) - L*(C_B^+)$$

$$= L*(B) + L*(C) - L*(B_C^+).$$

Thus $L*(C_B^+) = L*(B_C^+) = L*(CC') = L*(BB') = 0$. Since A is measurable $L*(E) = 0$ by Theorem 3.12. Since $B' \subset E \cup (BB')$ and $C' \subset E \cup (CC')$ we have $L*(B') = L*(C') = 0$. Hence B and C are measurable by Theorem 3.12.

<u>Theorem 3.14</u> Let A be a set such that $\overline{D}*(a, A) \leq 1$ for all $a \varepsilon A$, then the set R of regular points of A and the set B of irregular points are separated.

<u>Proof</u>. For any pair of positive numbers ϵ, δ, we define two sets as follows:

$$R(\epsilon, \delta) = \{a : a \varepsilon R,\ 1 - \epsilon/2 < \frac{L*(Ac(a, r))}{2r}$$

$$\text{for every } r < \delta\}$$

and

$$B(\epsilon) = \left\{a : a \in B, \frac{L^*(Ac(a, r_n))}{2r_n} < 1 - \epsilon, \right.$$
$$\left. \text{for some } \{r_n\} \to 0\right\}.$$

Suppose for some pair of numbers ϵ_1, δ_1, that the sets $R(\epsilon_1, \delta_1)$ and $B(\epsilon_1)$ are not separated. Then there is a sequence $\{a_n\}$, $a_n \in R(\epsilon_1, \delta_1)$ such that

$$\lim_{n \to \infty} a_n = b_1$$

for some $b_1 \in B(\epsilon_1)$ by III following Definition 3.1. For every δ_2, $0 < \delta_2 < r_N < \delta_1$ there exists an a_n such that $c(a_n, r_N - \delta_2) \subset c(b_1, r_N)$. From the definition of $R(\epsilon_1, \delta_1)$ and $B(\epsilon_1)$ we have,

$$(1 - \epsilon_1/2)\ 2(r_N - \delta_2) < L^*(Ac(a_n, r_N - \delta_2))$$
$$\leq L^*(Ac(b_1, r_N)) < (1 - \epsilon_1)\ 2r_N.$$

Since δ_2 can be chosen arbitrarily small, independent of ϵ_1 and δ_1 (N depends on δ_1) the above inequality is contradictory so that $R(\epsilon, \delta)$ and $B(\epsilon)$ are separated for every pair of numbers ϵ, δ.

For any given $\epsilon > 0$, $R(\epsilon, 1/m) \subset R(\epsilon, 1/n)$ if $m < n$ and

$$\lim_{n \to \infty} R(\epsilon, 1/n) = R;$$

also $B(1/m) \subset B(1/n)$ if $m < n$ and

$$\lim_{n\to\infty} B(1/n) = B.$$

Therefore by Lemma 3.2, for any $\lambda_1 > 0$ there exists a pair of numbers n, m such that $L^*(B) - L^*(B(1/n)) < \lambda_1$ and $L^*(R) - L^*(R(1/n, 1/m)) < \lambda_1$.

Now suppose R and B are not separated. Then there exists a $\lambda > 0$ such that if α^* is any class with the property that $L^*(R - \alpha) < \lambda$, then $L^*(\alpha B) \geq \lambda$. From above, there exists a pair of integers N, M such that $L^*(R) - L^*(R(1/N, 1/M)) < \lambda/2$ and $L^*(B) - L^*(B(1/N)) < \lambda/2$. If α^* is any class such that $L^*(R(1/N, 1/M) - \alpha) < \lambda/2$ then

$$L^*(R - \alpha) = L^*(R) - L^*(\alpha R) \leq L^*(R) - L^*(\alpha R(1/N, 1/M))$$

$$= L^*(R) - (L^*(R(1/N, 1/M)) - L^*(R(1/N, 1/M) - \alpha))$$

$$< \lambda/2 + \lambda/2 = \lambda.$$

Therefore

$$\lambda \leq L^*(\alpha B) = L^*(\alpha B(1/N)) + L^*(B) - L^*(B(1/N))$$

$$- (L^*(B - \alpha) - L^*(B(1/N) - \alpha)) \leq L^*(\alpha B(1/N))$$

$$+ L^*(B) - L^*(B(1/N)) < L^*(\alpha B(1/N)) + \lambda/2,$$

so that $L^*(\alpha B(1/N)) > \lambda/2$. Hence $R(1/N, 1/M)$ and $B(1/N)$ are not separated. This contradiction establishes the theorem.

Corollary 3.14.1. Let A be any set, then the set R of regular points and the set B of irregular points are separated.

Proof. Let $A_1 = \{a : a \varepsilon A, \bar{D}^*(a, A) > 1\}$ then from Lemma 3.5, $L^*(A_1) = 0$. Therefore

$$L^*(Ac(a, r)) = L^*((A - A_1)\ c(a, r))$$

so $R - A_1$ is the set of regular points of $A - A_1$. Also $B = B_1 \cup A_1B$ where $B_1 = (A - A_1) - R$. Since $L^*(A_1) = 0$, R and A_1 are separated and by Theorem 3.14 $R - A_1$ and B_1 are separated, hence R and B are separated by Lemma 3.3.

Theorem 3.15. If the set A is measurable, then R and B are measurable where R is the set of regular points of A and B is the set of irregular points of A.

Proof. By Corollary 3.14.1, B and R are separated and $A = B \cup R$. The theorem thus follows from Theorem 3.13.

IV. OPAQUE SUBSETS OF A SQUARE

In this section L will denote a straight line and the sets Q and Q° will be defined to be:

$$Q = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

and

$$Q^{\circ} = \{(x, y) : 0 < x < 1, 0 < y < 1\}.$$

All sets will be subsets of Q unless otherwise stated. If B is any set or ordinal number $|B|$ will denote the cardinal number of B. The cardinal numbers of the set of all real numbers and the set of all positive integers will be denoted by c and d respectively.

<u>Definition 4.1</u>. A set $A \subset Q$ is an opaque set if $L \cap Q \neq \emptyset$ implies $L \cap A \neq \emptyset$.

<u>Theorem 4.1</u>. If A is an opaque set, then $A' = \{p : p \text{ is a limit point of } A\}$ is an opaque set.

<u>Proof</u>. Let L be any line such that $L \cap Q \neq \emptyset$ and L_1 be a line parallel to L such that $L_1 \cap Q \neq \emptyset$. Let d_1 denote the distance between L and L_1. We define a sequence of lines $\{L_n\}$ as follows. For $n \geq 2$ the line L_n is parallel to L and between L and L_1 at a distance d_1/n from L. Since A is opaque, for each n there is a point $p_n \in A$ such that p_n is a point of L_n. The points p_n are in Q, therefore there exists a subsequence $\{p_{n(K)}\}$ such that

$$\lim_{K\to\infty} p_{n(K)} = p$$

where p is a point of L and $p \varepsilon A'$. Since L is arbitrary, A' is opaque.

Theorem 4.2. If A is a closed opaque set then there is a set $B \subset A$ such that B is a perfect opaque set.

Proof. Since A is closed there is a perfect set $B \subset A$ such that $|A - B| \leq d$. A proof of this statement may be found in Goffman (7).

Let L be any line such that $L \cap Q \neq \emptyset$, and choose L_n as in the proof of Theorem 4.1. Let b_n be a point of B between L_n and L. There is such a point since there is a nondenumerable set of points of A between L_n and L and $|A - B| \leq d$. Since the points b_n are in Q there is a convergent subsequence $\{b_{n(K)}\}$ such that

$$\lim_{K\to\infty} b_{n(K)} = b$$

and b is a point of L. Since $b_{n(K)} \varepsilon B$ and B is perfect, $b \varepsilon L \cap B$. Thus B is opaque.

Theorem 4.3. If A_n are nested, closed opaque sets, then

$$\bigcap_n A_n = A$$

is an opaque set.

Proof. Let L be any line such that $L \cap Q \neq \emptyset$. Then

$L \cap A_n = B_n \neq \emptyset$ and B_n is closed since $B_n = (L \cap Q)A_n$ is the intersection of closed sets. Also $B_{n+1} \subset B_n$ since $A_{n+1} \subset A_n$ and $B_{n+1} = (L \cap Q)\ A_{n+1} \subset (L \cap Q)A_n = B_n$. By the Cantor Product Theorem,

$$\bigcap_n B_n = B \neq \emptyset$$

and since $B_n \subset A_n$ we have

$$\bigcap_n B_n \subset \bigcap_n A_n .$$

Therefore $L \cap A \supset L \cap B = B$ is not empty, hence A is opaque.

The following examples show that Theorems 4.2 and 4.3 are not true if the set A in Theorem 4.2 is not closed and the sets A_n of Theorem 4.3 are not closed.

<u>Example 4.1</u>. Let $A = (Q - (Q^\circ \cup \{p_1\} \cup \{p_2\})) \cup \{p_3\}$ where $p_1 = (1/2,\ 0)$, $p_2 = (1/2,\ 1)$ and $p_3 = (1/2,\ 1/2)$. Then A is opaque however it does not contain a perfect opaque subset.

<u>Example 4.2</u>. Let $A = \{p : p \varepsilon Q,\ p = (x,\ y),\ x \neq 1/2\}$ and $A_n = A \cup l_n$ where $l_n = \{p : p = (1/2,\ y),\ 0 < y < 1/n\}$. Then A_n is opaque, $A_{n+1} \subset A_n$ for each n, but

$$\bigcap_n A_n = A$$

which is not opaque.

A set B is well-ordered by the ordering $<$ if, given any non-empty subset A of B, there exists an element $a \in A$ such that $a < a'$ for any $a' \in A - a$. A set B is best-well-ordered by the ordering $<$ if B is well-ordered by $<$ and for every $b \in B$, $|B_b| < |B|$ where $B_b = \{p : p \in B, p < b\}$. The axiom of choice implies that every set can be well-ordered.

Theorem 4.4. Every set B can be best-well-ordered.

Proof. Let $<$ denote an ordering which well-orders B and $B_1 = \{b : b \in B, |B_b| = |B|\}$. If B_1 is empty then B is best-well-ordered by $<$. If B_1 is not empty then there is a first element of b' of B_1. Let $B_2 = \{b : b \in B, b < b'\}$, then $|B_2| = |B|$ so there exists a one-to-one correspondence between B_2 and B. Since B_2 is well-ordered by $<$ a well-ordering $<'$ is induced on B by this one-to-one correspondence between B and B_2. From the definition of B_2 and b' it follows that the ordering $<'$ is a best-well-ordering.

Definition 4.2. A set A is an opaque set of degree α if $L \cap Q^\circ \neq \emptyset$ implies $|L \cap A| = \alpha$.

Theorem 4.5. For any cardinal number α, such that $2 \leq \alpha \leq c$, there exists an opaque set, A, of degree α.

Proof. The set A will be defined by transfinite induction. There are c lines which intersect Q°; best-well-order the set of these lines to form a transfinite sequence, $L_1, L_2, \ldots, L_\gamma, \ldots$. Suppose $A_\beta \subset L_\beta \cap Q$ has been defined for all $\beta < \gamma$ such that L_β contains exactly α points of A_β and at most α points of

$$\bigcup_{\beta<\gamma} A_\beta$$

are collinear. If L_γ contains exactly α points of

$$\bigcup_{\beta<\gamma} A_\beta,$$

let $A_\gamma = \emptyset$. If L_γ contains η points of

$$\bigcup_{\beta<\gamma} A_\beta,\ \eta < \alpha,$$

then A_γ must be a subset of $L_\gamma \cap Q$ such that

$$L_\gamma \cap (\bigcup_{\beta\leq\gamma} A_\beta)$$

contains α points and such that at most α points of

$$\bigcup_{\beta\leq\gamma} A_\beta$$

are collinear.

To complete the proof we need only to show that for any γ, $|\gamma| < c$, there exist points in $L_\gamma \cap Q$ which satisfy the conditions of the construction.

First suppose $\alpha = n$ is a positive integer greater than or equal to two. Let ν be the cardinality of all possible n-tuples, whose entries are elements of

$$\bigcup_{\beta<\gamma} A_\beta .$$

Then ν is greater than or equal to the cardinality of the set of distinct lines, each containing n points of

$$\bigcup_{\beta<\gamma} A_\beta .$$

Since $L_1, L_2, \ldots, L_\gamma, \ldots$ is best well-ordered

$$\left|\bigcup_{\beta<\gamma} A_\beta\right| \leq |\gamma|\, n < c.$$

If

$$\left|\bigcup_{\beta<\gamma} A_\beta\right|$$

is not finite then

$$\nu = \left|\bigcup_{\beta<\gamma} A_\beta\right|^n = \left|\bigcup_{\beta<\gamma} A_\beta\right| < c,$$

while if

$$\left|\bigcup_{\beta<\gamma} A_\beta\right|$$

is finite so is ν. Hence at most ν points of $L_\gamma \cap Q$ will not satisfy the conditions of the construction; therefore the construction is valid for any finite α.

If α is an infinite cardinal number, $\alpha < c$, we may choose any points of $L_\gamma \cap Q$ for A_γ. Since if

$$L \cap \left(\bigcup_{\beta < \gamma} A_\beta \right)$$

contains no more than α points, then

$$L \cap \left(\bigcup_{\beta \leq \gamma} A_\beta \right)$$

contains no more than α points.

There is no opaque set of degree one, since the set must contain at least two points and there is a line containing these points. The sets $Q - Q^\circ$ and Q are opaque sets of degrees two and c respectively.

The linear measure of the projection of a measurable set A onto a line perpendicular to the direction θ is measurable (14) and will be denoted by $L(P(A, \theta))$. A set will be referred to as a measurable set if it is linearly measurable and of finite measure.

The following theorems will be used in the remainder of this section. The proofs of Theorems 4.6 and 4.7 may be found in Besicovitch (3, 4), respectively, and the proofs of Theorems 4.8 and 4.9 may be found in Eggleston (6).

Theorem 4.6. The set of all regular points of a measurable set is a regular set and the set of irregular points is an irregular set.

Corollary 4.6.1. Any measurable set is the sum of a measurable regular set and a measurable irregular set.

Proof. This corollary follows immediately from Theorem 4.6 and Theorem 3.15.

Theorem 4.7. The projection of a measurable irregular set on almost all directions is of measure zero.

Corollary 4.7.1. If A is a measurable irregular set then A is not opaque.

Proof. The measure of the projection of an opaque set in any direction is greater than or equal to one.

Theorem 4.8. If A is a measurable regular set then $L(P(A, \Theta))$ is a continuous function of Θ.

Theorem 4.9. If A is a measurable regular set then

$$\int_0^{2\pi} L(P(A, \Theta))\, d\Theta \leq 4L(A).$$

Theorem 4.10. If A is a measurable opaque set and $A = A_1 \cup A_2$ where A_1 is a measurable regular set and A_2 is a measurable irregular set then $L(P(A, \Theta)) = L(P(A_1, \Theta))$.

Proof. Such decompositions exist by Corollary 4.6.1. Since A is opaque, $L(P(A, \Theta)) = L(P(Q, \Theta))$, $L(P(Q, \Theta))$ is a continuous function of Θ and by Theorem 4.8 $L(P(A_1, \Theta))$ is a continuous function of Θ. Since $A_1 \subset A$, $L(P(A_1, \Theta)) \leq L(P(A, \Theta))$. By Theorem 4.7, $L(P(A_2, \Theta)) = 0$ for almost all values of Θ. Hence

$$L(P(A, \Theta)) \leq L(P(A_1, \Theta)) + L(P(A_2, \Theta)) = L(P(A_1, \Theta))$$

for almost all values of Θ. Since $L(P(A, \Theta))$ and $L(P(A_1, \Theta))$ are continuous functions of Θ, $L(P(A, \Theta)) = L(P(A_1, \Theta))$.

The following theorem improves the previously known bound for measurable opaque sets. (See Introduction.)

Theorem 4.11. If A is a measurable opaque set, then $L(A) \geq 2$.

Proof. Since A is opaque, $L(P(A, \Theta)) = L(P(Q, \Theta))$ and hence

$$\int_0^{2\pi} L(P(A, \Theta))\, d\Theta = 4\int_0^{\pi/2} \sqrt{2} \cos (\pi/4 - \Theta)\, d\Theta = 8.$$

Let A_1 be the set of regular points of A and $A_2 = A - A_1$. Then by Theorems 3.15, 4.6 and 4.10, $L(P(A_1, \Theta)) = L(P(A,\Theta))$ so by Theorem 4.9,

$$\int_0^{2\pi} L(P(A_1, \Theta))\, d\Theta = \int_0^{2\pi} L(P(A, \Theta))\, d\Theta = 8 \leq 4L(A).$$

Hence $L(A) \geq 2$.

Example 4.3. Let $f(p)$ be the sum of the distances from p to each of the points (0, 1), (1, 1) and (1, 0). The minimum value of this function is

$$f(\overline{p}) = \frac{\sqrt{2} + \sqrt{6}}{2}.$$

The set consisting of the three closed line segments connecting $\bar{p}$ and (0, 1), $\bar{p}$ and (1, 1), $\bar{p}$ and (1, 0) and the closed line segment connecting (0, 0) and (1/2, 1/2) is opaque and has measure

$$\sqrt{2} + \frac{\sqrt{6}}{2}.$$

It is not known whether or not there exists a measurable opaque set, A, such that

$$2 \leq L(A) < \sqrt{2} + \frac{\sqrt{6}}{2}.$$

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