FLAW CHARACTERIZATION AND SIZING USING SENSITIVITY ANALYSIS AND THE BOUNDARY ELEMENT METHOD

Daniel A. Tortorelli

Department of Mechanical and Industrial Engineering
University of Illinois
Urbana, Il 61801

Zi-Xian Wang

Department of Theoretical and Applied Mechanics
University of Illinois
Urbana, Il 61801

Guna Krishnasamy

PDA Engineering
2975 Redhill Ave.
Costa Mesa, CA 92626

INTRODUCTION

The scattered field from an arbitrary shaped flaw due to a known incident field can be obtained numerically using the boundary element method [1]. In this so-called forward problem the flaw shape, it's location, the incident field and the properties of the material are always known apriori. However, in nondestructive evaluation all information regarding the flaw shape is not known apriori. Instead, a finite number of scattered field measurements are available for a known incident field from which the flaw shape is to be determined. Problems of this type are referred to as inverse problems. Here we propose a means of solving the inverse problem which combines numerical optimization, the boundary element method and shape sensitivity analysis. In this approach the forward problem for an assumed flaw shape is initially solved. Then for the assumed shape the sensitivities of the scattered field with respect the different shape parameters which describe the flaw are computed. The solution to the forward problem, the sensitivities and the experimental measurement of the scattered field are then used as the driving mechanism for the optimization (cf. [2],[3],[4],[5],[6], and [7]). The optimization problem minimizes the error between the computed and the experimentally measured scattered field by appropriately redefining the shape parameters.

In this paper the solution strategy for the inverse problem is presented for identifying the shape and size of a single void. Here the forward problem and the integral equations for evaluating the sensitivities are given as integral equations and solved using the boundary element method. This solution strategy may be extended to identifying the shape and size of a crack wherein the integral equations are hypersingular in nature [8].

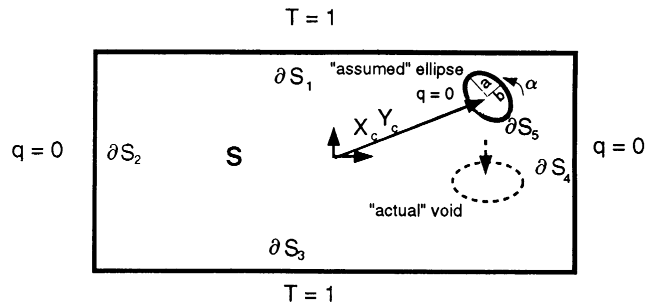

Figure 1. Heat conducting body s and the location of the ellipse

The sensitivity of the scattered field with respect to the shape parameters can be computed using different approaches such as finite difference, direct differentiation or adjoint method. For inverse problems, the adjoint variable method offers a significant numerical advantage. However, the adjoint method introduces volume integrals which are not easily computed with the boundary element method. To remedy this situation we present an adjoint variable formulation where the unfavorable volume integral is converted into a surface integral which may be computed via the boundary element method.

Here we study the Laplace equation. It is believed that the same method may be used to solve the inverse problem for the elastodynamic case.

THE INVERSE PROBLEM

The objective of this inverse problem is to identify a small ellipsoidal void in a large rectangular domain. The ellipse is defined by its center point component X_c, Y_c; its axis lengths a,b; and it orientation α.(see fig.1)

To identify the void, we compare the results between numerical and experimental response measurements. Specifically, we have experimentally obtained the flux over surface for the test specimens with the rectangular domain which contains "actual" void. Next we perform a BEM analysis for an "assumed" void. If the computed flux matches the experimental flux, then we presume that the "actual" and "assumed" voids coalesce, and hence, we have determined the void position and size in the test specimens.

The temperature T, and flux $q = \frac{\partial T}{\partial \mathbf{n}}$ for a homogeneous isotropic heat conducting body s in fig.1 are governed by a BVP. The boundary condition over the ellipse boundary ∂s_5 is given by $q = 0$. Over the rectangular surface, the temperature is held constant, $T = 1$, on the boundaries ∂s_1 and ∂s_3, while a constant flux $q = 0$ is enforced over the surface ∂s_2 and ∂s_4. The mixed boundary value problem is given by

$$\begin{aligned} \Delta_1 T(x, \Phi) &= 0 \qquad \text{for}(x, \Phi) \in s \\ \frac{\partial T}{\partial \mathbf{n}} &= 0 \qquad \text{for}(x, \Phi) \in \partial s_2 \ , \ \partial s_4 \text{ and } \partial s_5 \\ T &= 1 \qquad \text{for}(x, \Phi) \in \partial s_1 \text{ and } \partial s_3 \end{aligned} \tag{1}$$

To evaluate the difference between the "assumed" and "actual" voids, an error functional is defined:

$$G(\Phi) = \frac{1}{2}\int_{\partial s_1}(q - q_{exp})^2 dax \tag{2}$$

where q_{exp} is the flux over $\partial \mathsf{s}_1$ which is determined from experiment measurements and q is the flux calculated from a BEM analysis. To locate the "actual" position of the ellipse, we minimize G with respect to $\Phi = \{\mathsf{X}_c, \mathsf{Y}_c, \mathsf{a}, \mathsf{b}, \alpha\}^T$. To perform the minimization we combine sensitivity analysis, numerical optimization and the boundary element method.

ADJOINT SHAPE SENSITIVITY ANALYSIS

To derive shape sensitivity method, the domain parameterization and an adjoint variable approach are used. In the domain parameterization method, a reference domain is introduced. Then, the body configuration is expressed as a function of the referential configuration, i.e.

$$\mathsf{s} = \mathsf{f}(\mathsf{S}, \Phi) \tag{3}$$

where f is an invertable mapping. Equations 1 must be transformed to the reference configuration (see [9]).

$$\begin{aligned} \mathrm{div}_1(J(\mathsf{X},\Phi)\mathbf{J}^{-1}(\mathsf{X},\Phi)\mathbf{J}^{-T}(\mathsf{X},\Phi)\nabla_1\hat{\mathsf{T}}(\mathsf{X},\Phi)) = 0 \quad & \mathrm{for}(\mathsf{X},\Phi) \in S \\ \hat{\mathsf{T}}(\mathsf{X},\Phi) = 1 \quad & \mathrm{for}(\mathsf{X},\Phi) \in \partial S_1 \text{ and } \partial S_3 \\ \mathbf{J}^{-T}(\mathsf{X},\Phi)\nabla_1\hat{\mathsf{T}}(\mathsf{X},\Phi)\cdot J(\mathsf{X},\Phi)\mathbf{J}^{-T}(\mathsf{X},\Phi)\mathrm{N}(\mathsf{X})/K = 0 \quad & \mathrm{for}(\mathsf{X},\Phi) \in \partial S_2,\ \partial S_4 \text{ and } \partial S_5 \end{aligned} \tag{4}$$

where $\hat{\mathsf{T}}(\mathsf{X},\Phi) = \mathsf{T}(f(\mathsf{X},\Phi),\Phi)$, $J(\mathsf{X},\Phi) = \det\mathbf{J}(\mathsf{X},\Phi)$, $\mathbf{J}(\mathsf{X},\Phi) = \nabla_1 f(\mathsf{X},\Phi)$, and $K = J \parallel \mathbf{J}^{-T}\mathrm{N} \parallel$.

The above equations may be solved for T and q by using the isoparametric boundary element methods.

In the reference domain equation 2 becomes

$$\hat{G}(\Phi) = \frac{1}{2}\int_{\partial S_1}(q(f(\mathsf{X},\Phi),\Phi) - q_{exp})^2 K da_X \tag{5}$$

In the adjoint variable approach, we define the augmented functional $\hat{G}^+$ by multiplying equation 4.1 by an arbitrary function $\hat{\lambda}$ and integrating over domain S and adding to $\hat{G}$, i.e.

$$\begin{aligned} \hat{G}^+(\Phi) = \hat{G}(\Phi) &= \frac{1}{2}\int_{\partial S_1}(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi)^2 K da_X - \\ &\int_S \hat{\lambda}\mathrm{div}_1(J\mathbf{J}^{-1}\mathbf{J}^{-T}\nabla_1\hat{\mathsf{T}})dv_X \end{aligned} \tag{6}$$

In the above $\hat{\lambda}$ plays the role of the Lagrange multiplier. To derive the sensitivities, we solve the variation of the $\delta\hat{G}^+$ which equal $\hat{G}$ since the variation of the augmented term is zero. Taking the design variation of $\hat{G}^+$ yields

$$\begin{aligned} \delta\hat{G}^+ &= \int_{\partial S_1}[(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi))(\nabla_1\delta_2\hat{\mathsf{T}}\cdot\mathbf{J}^*\mathrm{N} + \nabla_1\hat{\mathsf{T}}\cdot\delta_2\mathbf{J}^*\mathrm{N} - \\ &\nabla_1\hat{\mathsf{T}}\cdot\mathbf{J}^*\delta_2 K/K) + \frac{1}{2}(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi))^2\delta_2 K]da_X - \\ &\int_S \hat{\lambda}\mathrm{div}(\delta_2\mathbf{J}^*\nabla_1\hat{\mathsf{T}} + \mathbf{J}^*\nabla_1\delta_2\hat{\mathsf{T}})dv_X \end{aligned} \tag{7}$$

where $\mathbf{J}^* = J\mathbf{J}^{-1}\mathbf{J}^{-T}$ and $\delta_2\mathbf{J}^* = \delta_2 J\mathbf{J}^{-1}\mathbf{J}^{-T} + J\delta_2\mathbf{J}^{-1}\mathbf{J}^{-T} + J\mathbf{J}^{-1}\delta_2\mathbf{J}^{-T}$.

In order to use the boundary element method, the volume integral terms of above equation must be transformed to the boundary. Several relations from continuum mechanics are used to transform the above equation to (see [9])

$$\begin{aligned}
\delta\hat{G}^+ \;=\; & \int_{\partial S_1} [(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi))(-q(f(\mathsf{X},\Phi),\Phi)\delta_2 K) + \\
& \frac{1}{2}(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi))^2 \delta_2 K] da_X + \\
& \int_{\partial S} [\nabla_1 \lambda(f(\mathsf{X},\Phi),\Phi)\cdot\nabla_1 \mathsf{T}(f(\mathsf{X},\Phi),\Phi)\mathbf{v} - \\
& \nabla_1\lambda(f(\mathsf{X},\Phi),\Phi)\cdot\mathbf{v}\nabla_1\mathsf{T}(f(\mathsf{X},\Phi),\Phi) - \nabla_1\mathsf{T}(f(\mathsf{X},\Phi),\Phi)\cdot\mathbf{v}\nabla_1\lambda(f(\mathsf{X},\Phi),\Phi)]\cdot \\
& J\mathbf{J}^{-T}\mathbf{N}da_X - \int_{\partial S_2+\partial S_4+\partial S_5} \lambda(f(\mathsf{X},\Phi),\Phi)q^p\delta_2 K da_X + \\
& \int_{\partial S_1} [(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi)) - \lambda(f(\mathsf{X},\Phi),\Phi)]\nabla_1\delta_2\hat{\mathsf{T}}\cdot\mathbf{J}^*\mathbf{N}da_X + \\
& \int_{\partial S_2+\partial S_4+\partial S_5} (0 + \mathbf{J}^*\nabla_1\hat{\lambda}\cdot\mathbf{N})\delta_2\hat{\mathsf{T}}da_X - \int_S \delta_2\hat{\mathsf{T}}\mathrm{div}_1(\mathbf{J}^*\nabla_1\hat{\lambda})dv_X \qquad (8)
\end{aligned}$$

where $\partial S = \partial S_1 + \partial S_2 + \partial S_3 + \partial S_4 + \partial S_5$ and $\hat{\lambda} = \lambda(f(\mathsf{X},\Phi),\Phi)$.

The appropriate choice of $\hat{\lambda}$ may be used to eliminate the implicit response variations $\delta_2\hat{\mathsf{T}}$ and $\nabla_1\delta_2\hat{\mathsf{T}}$ from $\delta\hat{G}^+$. Upon examination of the above equation, to annihilate the implicit variations, $\hat{\lambda}$ must satisfy the following conditions

$$\begin{aligned}
\mathrm{div}_1(J\mathbf{J}^{-1}\mathbf{J}^{-T}\nabla_1\hat{\lambda}) = 0 \qquad & \mathrm{for}(\mathsf{X},\Phi)\in S \\
\hat{\lambda} = (q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi)) \qquad & \mathrm{for}(\mathsf{X},\Phi)\in \partial S_1 \\
\hat{\lambda} = 0 \qquad & \mathrm{for}(\mathsf{X},\Phi)\in \partial S_3 \qquad (9) \\
q_\lambda(f(\mathsf{X},\Phi),\Phi) = \mathbf{J}^*\nabla_1\hat{\lambda}\cdot\mathbf{N} = 0 \qquad & \mathrm{for}(\mathsf{X},\Phi)\in \partial S_2 + \partial S_4 + \partial S_5
\end{aligned}$$

Note that this adjoint problem is solved with the same stiffness matrix as the original system. Only a different load vector needs to be formed. Thus, only a back substitution is required to evaluate $\hat{\lambda}$.

Substituting the above value for $\hat{\lambda}$ into equation 8 yields desired sensitivity expression

$$\begin{aligned}
\delta\hat{G}^+ \;=\; & \int_{\partial S_1} [(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi))(-q(f(\mathsf{X},\Phi),\Phi)\delta_2 K) + \\
& \frac{1}{2}(q(f(\mathsf{X},\Phi),\Phi) - q_{exp}(f(\mathsf{X},\Phi),\Phi))^2 \delta_2 K] da_X + \\
& \int_{\partial S} [\nabla_1 \lambda(f(\mathsf{X},\Phi),\Phi)\cdot\nabla_1 \mathsf{T}(f(\mathsf{X},\Phi),\Phi)\mathbf{v} - \\
& \nabla_1\lambda(f(\mathsf{X},\Phi),\Phi)\cdot\mathbf{v}\nabla_1\mathsf{T}(f(\mathsf{X},\Phi),\Phi) - \nabla_1\mathsf{T}(f(\mathsf{X},\Phi),\Phi)\cdot\mathbf{v}\nabla_1\lambda(f(\mathsf{X},\Phi),\Phi)]\cdot \\
& J\mathbf{J}^{-T}\mathbf{N}da_X - \int_{\partial S_2+\partial S_4+\partial S_5} \lambda(f(\mathsf{X},\Phi),\Phi)q^p\delta_2 K da_X \qquad (10)
\end{aligned}$$

This sensitivity equation is evaluated using the BEM techniques.

NUMERICAL PROCEDURE

Now we describe the solution procedure for the inverse problem. The objective is to minimize the response function $\hat{G}$ with respect to the ellipse variables X_c, Y_c, a, b, and α. This ensures that the experimental and computed response is the same. Thus, we presume that the assumed void shape in the BEM analysis is identical to that in the physical domain from which the experiments were performed.

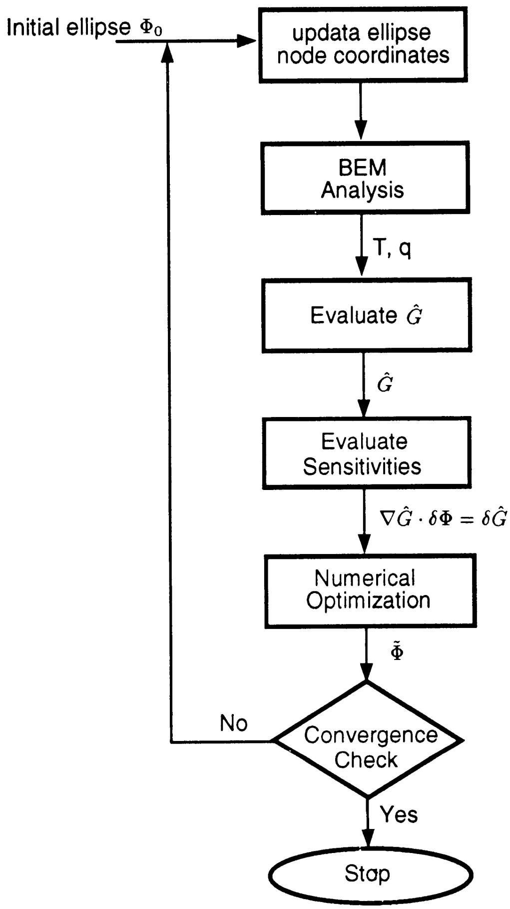

Figure 2. Inverse alogorithm

In general this problem is not unique. Different initial guesses of the assumed void may lead to different results. Therefore we can not guarentee that the assumed void and actual void are identical. Indeed, we surely can not assume that the actual void is the ellipse, and we may actually have many voids in the test specimens.

We use the numerical optimization to obtain the minimum value of $\hat{G}$. In this method, we first supply an assumed ellipse void by selecting a starting value Φ for the void. This input information is used to create the node coordinates. Next a BEM analysis is performed to evaluate T, q, and after which the error function $\hat{G}$ is computed. If the error function value is small, we assume that the assumed and actual voids coalesce. If not, the shape sensitivities $\nabla\hat{G}$ are computed as described in the previous section.

CONCLUSION

A scheme based on BEM, the shape sensitivity analysis, and numerical optimization is proposed for the solution of the inverse problem of the heat conducting solids. The numerical implementation of this procedure is currently in progress. The scheme may be used to determine void size and location in test specimens.

REFERENCES

1. Rizzo, F. J., Shippy, D. J. and Rezayat, M.: A Boundary Integral Equation Method for Radiation and Scattering of Elastic Waves in Three Dimensions. Int. J. for Num. Meth. in Engg., 21 (1985), 115-129

2. Nishimura, N., Kobayashi, S., IABEM-90 (Symposium of the International Association for Boundary Element Methods). October 15-18,1990

3. Zinn, A., On an Optimization Method for the Full- and Limited-Aperture Problem in Inverse Acoustic Scattering for a Sound-Soft obstacle. Inverse Problem 5 (1989), 239 -253

4. Kristensson, G., Vogel, C.R., Inverse Problem for Acoustic Waves Using the Penalised Likelihood Method. Inverse Problems 2 (1986), 461-479

5. Roger.,A., Newton Kantorovitch Algorithm Applied to An Electromagnetic Inverse Problem. IEEE Trans. Ant. Prip. AP-29 (1981), 232-238

6. Colton, D., Monk, P., The Numerical Solution of the Three Dimensional Inverse Scattering Problem for Time-Hamonic Acoustic Waves. SIAM J. Sci. Stat. Comp. 8 (1987),278-291

7. Kirsch, A., Kress, R., Zinn, A., Two Methods for Solving the Inverse Acoustic Scattering Problem. Inverse Problems 4 (1988), 749-770

8. Krishnasamy, G., Schmerr, L. W., Rudolphi, T. J. and Rizzo, F. J.: Hypersingular Boundary Integral Equations: Some Applications in Acoustic and Elastic Wave Scattering. J. Appl. Mech., 57 (1990), 404-414

9. D. A. Tortorelli, Z. X. Wang, A Systematic Approach for Shape Sensitivity Analysis. To Appear