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TEMPERATURE DISTRIBUTION IN A METAL CYLINDER CONTAINING A HEAT SOURCE

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SYMBOLS

a radius of cylinder

b half-length of cylinder

B heat production, Btu/hr-ft-$^{\circ}$F

C constant

C' constant

d derivative

∂ partial derivative

J_o Bessel's function, first kind and zero order

k coefficient of thermal conductivity

k_x, k_y, k_z Thermal conductivity in the x, y, and z direction, respectively

L/D length-diameter ratio of cylinder

n coefficient

r radial coordinate

t, T temperature

q, Q heat flow, Btu/hr.

Z axial coordinate

∞ infinity

α constant

β constant

Δ increment

θ temperature difference, $t_{actual} - t_{surface}$

π pi

ρ radial coordinate

τ time

TEMPERATURE DISTRIBUTION IN A METAL CYLINDER CONTAINING A HEAT SOURCE[1]

Leon Pletke and Glenn Murphy

I. ABSTRACT

The object of this report is to describe a method for finding the temperature distribution in a metal cylinder containing a heat source distributed in any manner throughout the cylinder. Specific solutions are given for cylinders with L/D ratios of 1, 2, 3, and ∞.

The solution was developed by writing the general partial differential equation in the form of a difference equation containing a term proportional to the heat production. Solutions were obtained for the difference equation by modifying the known solution for a similar equation containing one less term. This approach was convenient as it eliminated the necessity of a solution by successive approximations.

The results are not given directly in terms of temperature, but are expressed in terms of a coefficient, n. The temperature is equal to $t_{surface} + C'(n)$. The term C' is equal to $\frac{16a^2}{\pi^3} \frac{B}{k}$, where a is the radius of the cylinder, B the heat production term per unit volume, and k the thermal conductivity. The expression of the results in terms of n gives a general solution to the four cases in this paper. To obtain a specific numerical solution where the values of a, B, and k are known, it is necessary to substitute them into C' and then multiply by n. The actual temperature at a point is then equal to the surface temperature $+ C'(n)$.

The curves presented are general for the L/D ratios indicated and may be used with any variation of heat production and thermal conductivity throughout the cylinder.

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[1]This report is based on a Master of Science thesis by Leon Ray Pletke submitted August, 1953 to Iowa State College, Ames, Iowa. This work was done under contract with the Atomic Energy Commission.

II. INTRODUCTION

The prupose of this thesis is to describe a method for determining the temperature distribution in a metal cylinder when heat is liberated within the material of the cylinder. Only the steady state condition will be considered although the general method could be extended to the transient condition. The problem is one of heat conduction in a solid, and the solution is based on the conventional equation of heat flow applied to a metal cylinder. The solution described consists in re-writing the partial differential equation in the finite difference form and obtaining a numerical solution.

There are many applications in which it is desirable to know the temperature distribution within a body. The limiting operating conditions may be determined by the maximum temperature allowable within the body. Some of the important effects of temperature are phase changes of metals resulting in dimensional changes, loss of strength at elevated temperatures, and changes in magnetic and electrical properties. The ability of a metal body to conduct away the heat generated within it establishes the limiting design condition in some cases. Since heat transfer by conduction is dependent on the temperature gradient, it is necessary to know the temperature at various points within the body.

The heat source may be of an electrical, electromagnetic, or nuclear nature. Some of the common examples are resistance heating where the heat production is a constant, and induction heating where the heat production varies as the radial distance. These and other applications make it desirable for the engineer to be able to obtain a numerical solution of the temperature distribution within a body.

III. REVIEW OF LITERATURE

The literature of heat transfer contains many solutions of heat conduction problems in the steady state. The general equation of heat conduction is[4]

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial t}{\partial z}\right) + q''' = 0$$

where q''' is the heat energy developed in a unit volume and a unit time. Some modifications may be made in certain cases to simplify the equation. For example, when the thermal conductivity k is the same in all three directions, and may be assumed constant with temperature, the equation reduces to:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q''''}{k} = 0 .$$

The solutions of partial differential equations of heat conduction must satisfy the boundary conditions of the problem. Many special solutions are found in the literature of one dimensional heat flow. Some of the common problems are the infinitely long hollow cylinder, thin rods, hollow spheres, etc. The problems of two dimensional heat flow are slightly more complicated. In the unsteady state case of a solid cylinder symmetrical about the Z axis, the equation is

$$\frac{\partial t}{\partial t} = k\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2}\right]$$

and the solution given by Boelter is

$$t = Ce^{-a(\alpha^2+\beta^2)t} \begin{matrix}\sin\\ \cos\end{matrix} (\beta Z)\, J_0(\alpha r)$$

where α and β are constants determined by the conditions of the problem.

An alternate to Boelter's solution is to use the relaxation method. The idea of relaxation was attributed to R. V. Southwell. However, the relaxation method only gives a numerical solution to a specific problem involving specific ratios of dimensions and properties. It cannot be made to yield a solution in general terms, so that all problems, even those of a similar type, must be worked out completely.[1] In the article by Billington and Becher the temperature distribution was worked out for various problems including the case of a corner in a wall, an external wall with an internal partition, and several others.

A paper by H. W. Emmons[3] also describes the numerical solution of steady state heat conduction problems by the method of relaxation. Problems of one, two, or three-dimensional heat flow may be solved. The method for a two-dimensional case is as follows.

To solve the problem the continuous body is replaced by a net of conducting rods. The heat Q_0, conducted from any one intersection point at which the temperature is T_0 along any rod is expressed in terms of the thermal conductivity k, the length in the third dimension of the body b, and the temperature T_1 at the other end of the rod.

$$_0Q_1 = kb(T_0 - T_1)$$

The equations for the heat conducted along rods 0-2, 0-3, 0-4 are similar expressions, and the total is

$$\frac{Q}{kb} = T_1 + T_2 + T_3 + T_4 - 4T_0.$$

For steady state and interior points $\frac{Q}{kb} = 0$, and for the surface or boundary $\frac{Q}{kb}$ must be consistent with the boundary conditions. To solve the problem, the two-dimensional region is laid out to scale and a grid of squares established. Values of the temperature at each point are assumed and the magnitude of Q is calculated from the grid equation. Successive calculations are then made to establish a final temperature distribution that will satisfy each grid equation. A finer mesh requires more labor, but gives greater accuracy.

IV. OBJECT OF INVESTIGATION

The object of this investigation is to develop the finite difference method for determining the temperature distribution on a metal cylinder containing a heat source for steady state conditions. To illustrate the method the temperature distribution is found for solid cylinders with L/D ratios of 1, 2, 3, and ∞. The results are left in terms of a general case; i.e., no values are given to length, diameter, thermal conductivity, or heat production term, so that they may be used in a large variety of problems.

A discussion of application is included for the case of constant heat production, and for the case in which heat production varies linearly with the radius.

V. ANALYSIS

A. Derivation

Fourier's law for the conduction of heat states that the instantaneous rate of heat flow is equal to the product of three factors: the area A of the section, taken at right angles to direction of heat flow; the temperature gradient $-\frac{dt}{dx}$; and a proportionality factor k, known as the thermal conductivity of the material conducting the heat.[5] Fourier's law may be expressed as:

$$\frac{dQ}{d\tau} = - kA \frac{dt}{dx} .$$

In the steady state condition the temperature at any point does not vary with time, so the temperature gradient $-\frac{dt}{dx}$ and the rate of heat flow $\frac{dQ}{d\tau}$ are independent of time. The rate of heat flow may then be expressed as $\frac{Q}{\tau}$, and designated by q. The expression for the steady state condition then is

$$q = -kA\frac{dt}{dx} \quad . \tag{1}$$

For the particular case of heat flow in both the axial and radial direction with heat production within the cylinder the various q terms may be equated as follows:

$$q_{\text{net radial}} + q_{\text{net axial}} = q_{\text{production}}. \tag{2}$$

Figure 1 shows an incremental ring of radial thickness $\Delta\rho$ and axial length ΔZ that may be used to develop the equation used as the basis of the method to be described. The heat flow in the radial direction, q_{radial} is equal to the q at $\rho + \Delta\rho$ minus the q at ρ

$$q_{rad} = \left[-k\Delta z\left(2\pi\rho\frac{\partial t}{\partial\rho}\right)_{\rho+\Delta\rho}\right] - \left[-k\Delta z\left(2\pi\rho\frac{\partial t}{\partial\rho}\right)_{\rho}\right] \tag{3}$$

or

$$q_{rad} = -k\,2\pi\Delta z\left[\left(\rho\frac{\partial t}{\partial\rho}\right)_{\rho+\Delta\rho} - \left(\rho\frac{\partial t}{\partial\rho}\right)_{\rho}\right]$$

$$= -k\,2\pi\Delta z\,\frac{\left[\left(\rho\frac{\partial t}{\partial\rho}\right)_{\rho+\Delta\rho} - \left(\rho\frac{\partial t}{\partial\rho}\right)\right]\Delta\rho}{\Delta\rho}$$

$$\doteq -k\,2\pi\,\Delta\rho\,\Delta z\left[\frac{\partial\left(\rho\frac{\partial t}{\partial\rho}\right)}{\partial\rho}\right] \tag{4}$$

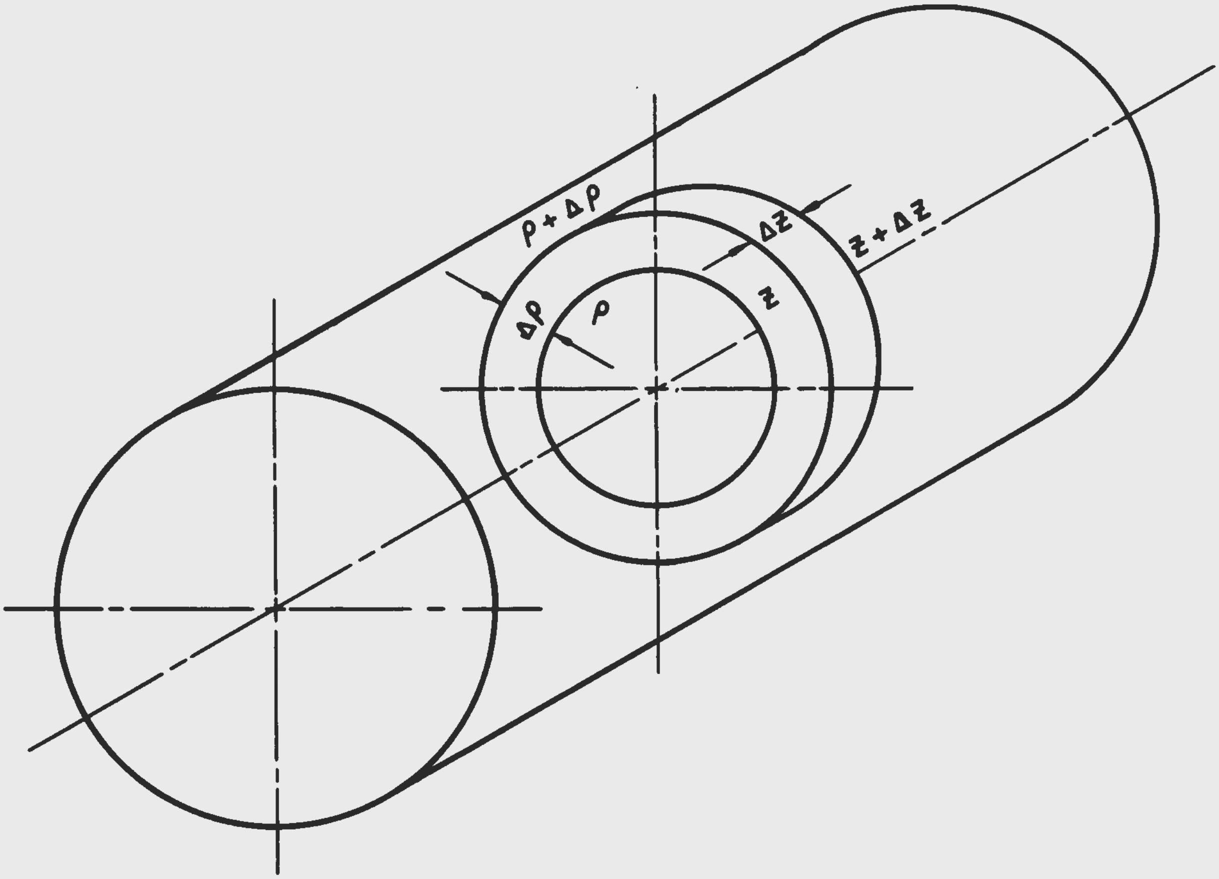

Fig. 1--Incremental ring.

In the same way the axial heat flow term is

$$q_{axial} = \left[-k\,2\pi\rho\Delta\rho\left(\frac{\partial t}{\partial z}\right)_{z+\Delta z}\right] - \left[-k\,2\pi\rho\Delta\rho\left(\frac{\partial t}{\partial z}\right)_{z}\right] \tag{5}$$

$$= -k\,2\pi\rho\Delta\rho\,\frac{\left[\left(\frac{\partial t}{\partial z}\right)_{z+\Delta z} - \left(\frac{\partial t}{\partial z}\right)_{z}\right]}{\Delta z}\Delta z$$

$$= -k\,2\pi\rho\Delta\rho\,\Delta z\left[\frac{\partial\left(\frac{\partial t}{\partial z}\right)}{\partial z}\right] \tag{6}$$

The heat production term will be expressed as B Btu/sec.-vol. times the volume of the unit ring, and is written as

$$q_{produced} = B(2\pi\rho\Delta\rho\Delta z)$$

$$q_{produced} = B(2\pi\rho\Delta\rho\Delta z)\ . \tag{7}$$

The values from equations 4, 6, and 7 are substituted into equation 2 to give

$$-k\,2\pi\Delta\rho\Delta z\left[\frac{\partial\left(\rho\frac{\partial t}{\partial\rho}\right)}{\partial\rho}\right] - k\,2\pi\rho\Delta\rho\Delta z\left[\frac{\partial\left(\frac{\partial t}{\partial z}\right)}{\partial z}\right] = B2\pi\rho\Delta\rho\Delta z. \tag{8}$$

Upon simplification the equation becomes:

$$\frac{1}{\rho}\left[\frac{\partial\left(\rho\frac{\partial t}{\partial\rho}\right)}{\partial\rho}\right] + \left[\frac{\partial\left(\frac{\partial t}{\partial z}\right)}{\partial z}\right] = -\frac{B}{k}$$

or

$$\frac{1}{\rho}\left[\rho\frac{\partial^2 t}{\partial\rho^2} + \frac{\partial t}{\partial\rho}\right] + \frac{\partial^2 t}{\partial z^2} = -\frac{B}{k}. \tag{9}$$

The temperature t expressed in equation 9 is the temperature at a point. Since all temperatures are relative to some base temperature, t is used in this thesis to be the actual temperature at a point, and θ is used to denote the difference in actual and surface temperature. The term θ then is equal to $t_{actual} - t_{surface}$, and the equation written in terms of θ is

$$\frac{\partial^2 \theta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \theta}{\partial \rho} + \frac{\partial^2 \theta}{\partial z^2} = -\frac{B}{k}. \qquad (10)$$

Equation 10 is the final form of the partial differential equation for heat conduction in a metal cylinder of finite length containing a heat source.

B. Procedure

Equation 10 may be converted from a differential equation into a difference equation by standard procedures. With the grid points selected as shown in Figure 2 and grid spacings of $\Delta \rho$ and ΔZ , equation 10 becomes

$$\frac{\theta_2 + \theta_4 - 2\theta_0}{(\Delta\rho)^2} + \frac{1}{\rho}\frac{\theta_0 - \theta_4}{\Delta\rho} + \frac{\theta_3 + \theta_1 - 2\theta_0}{(\Delta z)^2} = -\frac{B}{k}. \qquad (11)$$

If $\Delta \rho$ is chosen equal to ΔZ further simplification gives

$$\frac{\theta_2 + \theta_4 + \theta_1 + \theta_3 - 4\theta_0}{(\Delta\rho)^2} + \frac{1}{\rho}\frac{\theta_0 - \theta_4}{\Delta\rho} = -\frac{B}{k} \qquad (12)$$

The ρ term is the distance from the center line to the θ_0 point. The elevation may also be expressed in terms of the coefficient n, since $\theta = C'(n)$.

$$C' = \frac{16a^2}{\pi^3}\frac{B}{k} \qquad (13)$$

So substituting for C' gives:

$$\frac{n_2 + n_4 + n_1 + n_3 - 4n_0}{\Delta\rho^2} + \frac{1}{\rho}\frac{n_0 - n_4}{\Delta\rho} = -\frac{\pi^3}{16a^2} \qquad (14)$$

The right hand side is now a constant. The final values of n may be found for each point of the grid by application of equation 20. The corresponding θ at each point may be computed by multiplying n by $\frac{16a^2}{\pi^3}\frac{B}{k}$. The temperature at the point is then $\theta + t_{surface}$.

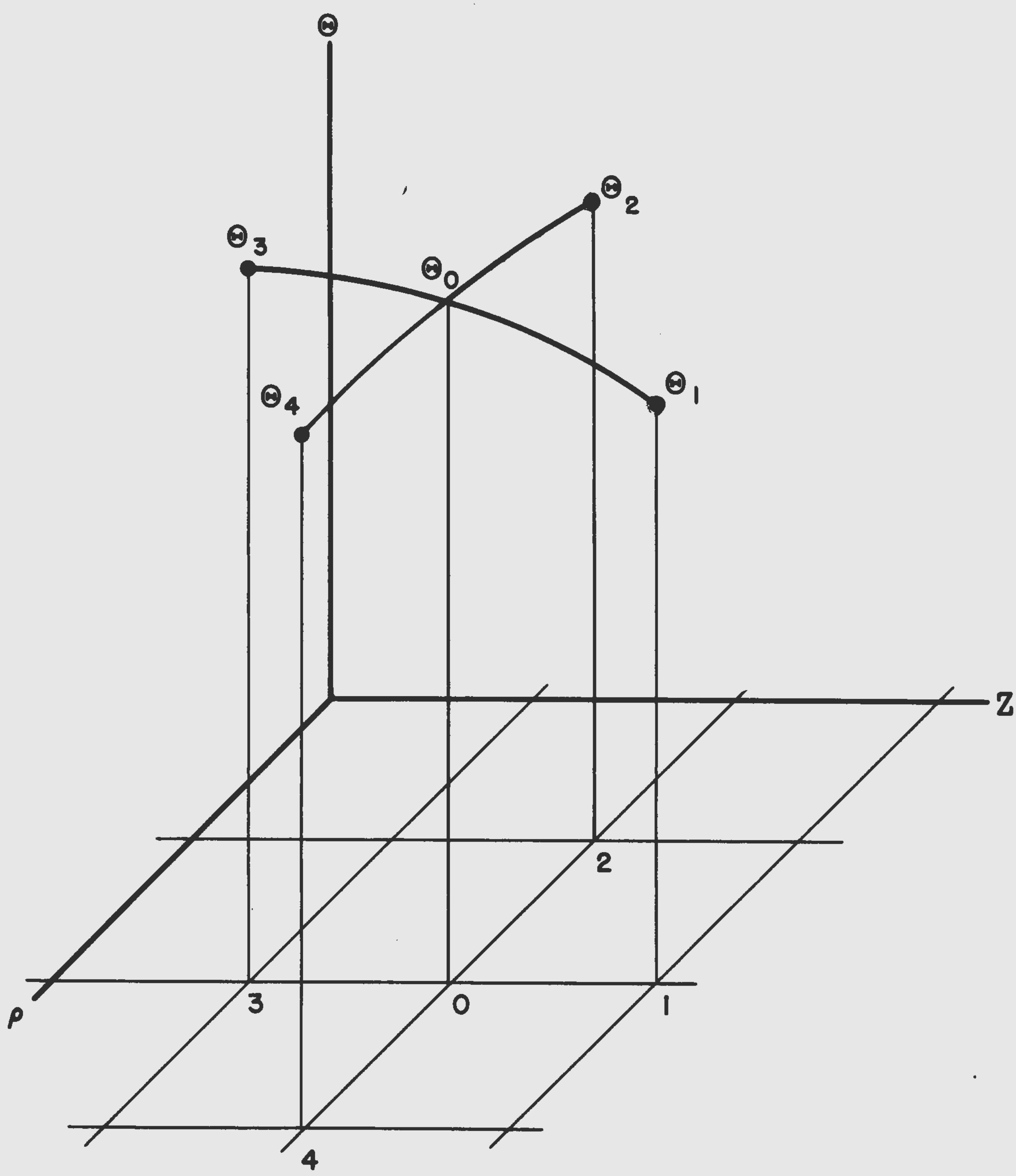

Fig. 2--Rectangular grid for finite difference equation.

In the general case a solution is obtained by assuming values of n and adjusting them until equation 14 is satisfied at every point in the grid. For the problem considered in this report it was possible to start with a known solution[6] for an equation similar to equation 10 except that it does not contain the first-order term. Although these values of n are not correct in general for the problem at hand, the values along the longitudinal axis are correct because of symmetry and thus constitute convenient starting values for a direct point-by-point solution without successive approximation.

The procedure used is to begin at the center of the cylinder with the point $\rho = 0$, $Z = 0$ as n_o. The point to the left and right of n_o at $\rho = 0$ $Z = \pm Z$ are n_1 and n_3. The values of n at the two remaining points n_2 and n_4 are equal, from the symmetry of the grid and are found directly by applying equation 14. The point n_o is then moved one grid square to the right and the process is repeated. After all of the values of one row are found the point n_o is moved one grid square in the radial direction onto the row just solved for. The values of n_o, n_1, n_3, and n_4 are known and so n_2 may be evaluated. The process is repeated until the values of n at all points of the grid have been obtained.

If a graph of n versus ρ indicates that the curve does not pass through 0 at $\rho = a$, the centerline values of n are lowered to make the curve pass through 0. Then with these lowered values of n at centerline points the finite difference method is again applied to all of the points. With the final value of n at each point of the grid it is possible to compute the value of θ, since $\theta = C'(n)$. The C' term is equal to $\frac{16a^2}{\pi^3} \frac{B}{k}$, so

$$t = t_{surface} + \frac{16a^2}{\pi^3} \frac{B}{K} (n) \tag{15}$$

For the case of B/k equal to a constant, the coefficient n at each point of the grid is multiplied by the same value. For the case of B/k varying as a linear function of ρ it is necessary to compute the value of B/k at each different value of ρ used in the grid system.

An illustration of this is the case of linear distribution where B/k equals α_1, at the centerline and α_2 at the outside edge, or $\rho = a$. The α or B/k at any point may be expressed as:

$$\alpha = \left[\alpha_1 + \frac{(\alpha_2 - \alpha_1)\rho}{a}\right. \tag{16}$$

at points $\frac{\rho}{a} = \frac{1}{4}$:

$$\alpha = \frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2$$

at points $\frac{\rho}{a} = \frac{1}{2}$:

$$\alpha = \frac{\alpha_1}{2} + \frac{\alpha_2}{2}$$

at points $\frac{\rho}{a} = \frac{3}{4}$:

$$\alpha = \frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2$$

These values of α or B/k are then substituted into C' and the θ term computed by multiplying C' and the coefficient n.

The method may also be applied to cases where B/k varies as a function of Z, or as a function of both ρ and Z. The function may be in any form; i.e., linear, sinusoidal, or exponential and the method would still apply. It is only necessary to be able to calculate B/k at each point for the particular distribution.

VI. RESULTS OF ANALYSIS

The results are given in general terms for the four cases of cylinders with L/D ratio of 3, 2, 1 and ∞. Due to the large number of values obtained it is desirable to present them in graphical form. The ordinate of the graph in each case is n, and abscissa the position coordinate, i.e., ρ/a or Z/b. The temperature θ is obtained by multiplying n by C'.

$$\theta = n(C') = n\left(\frac{16a^2}{\pi^3}\ \frac{B}{k}\right)$$

The actual temperature inside the cylinder at any point is then: $t = t_s + n(C')$.

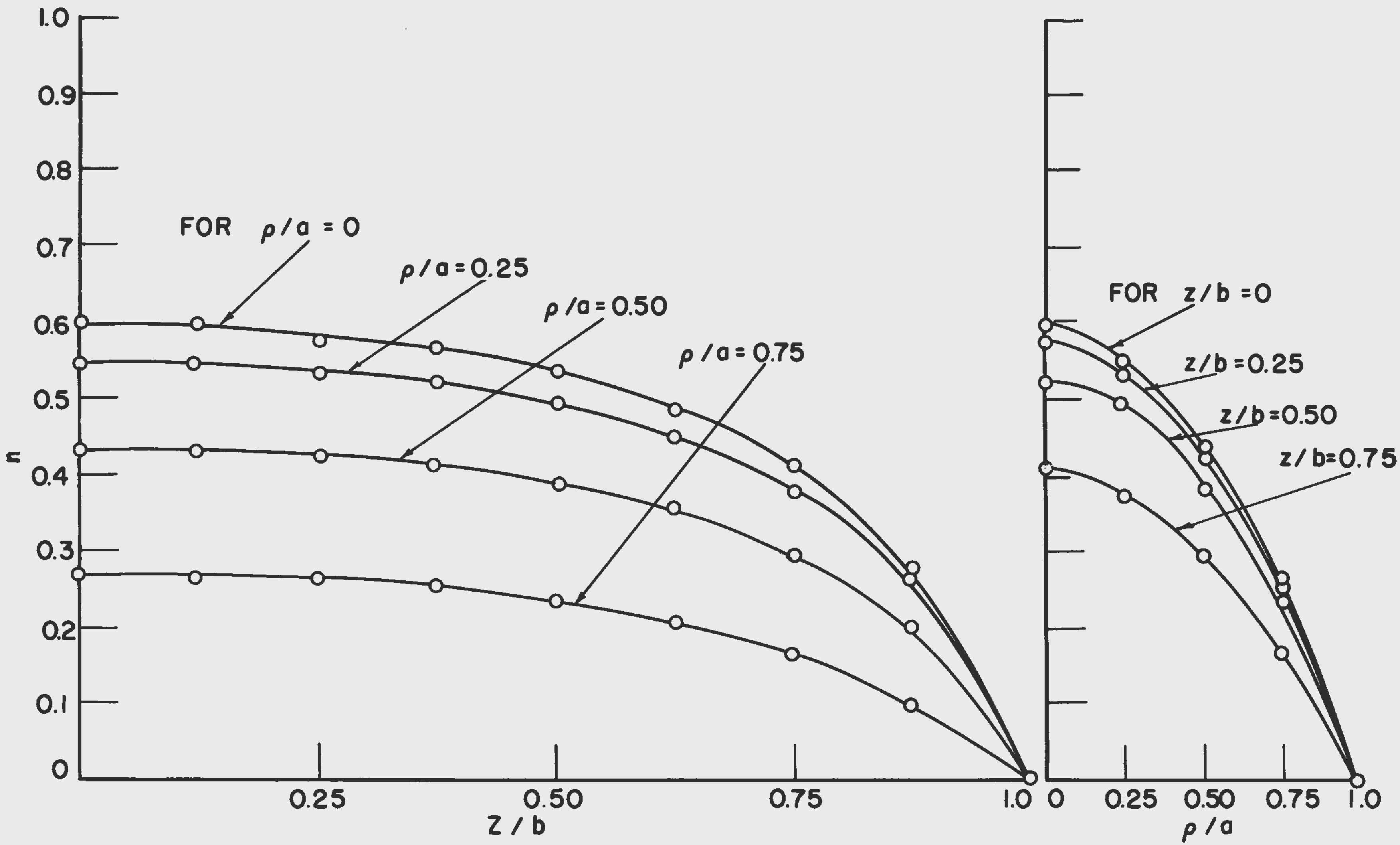

Fig. 3--Plot of n versus position coordinates for a cylinder with L/D ratio of 3.

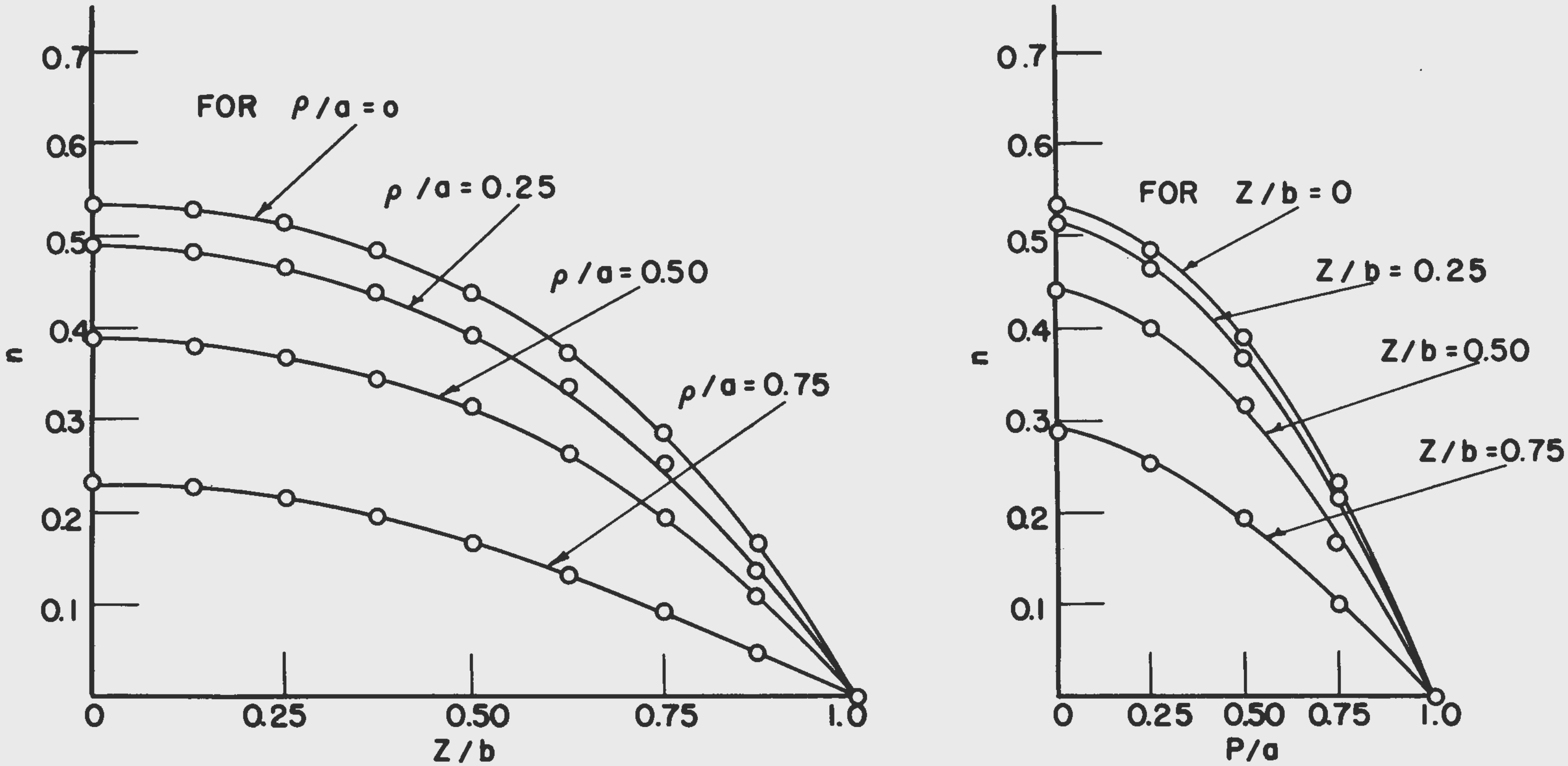

Fig. 4--Plot of n versus position coordinates for a cylinder with L/D ratio of 2.

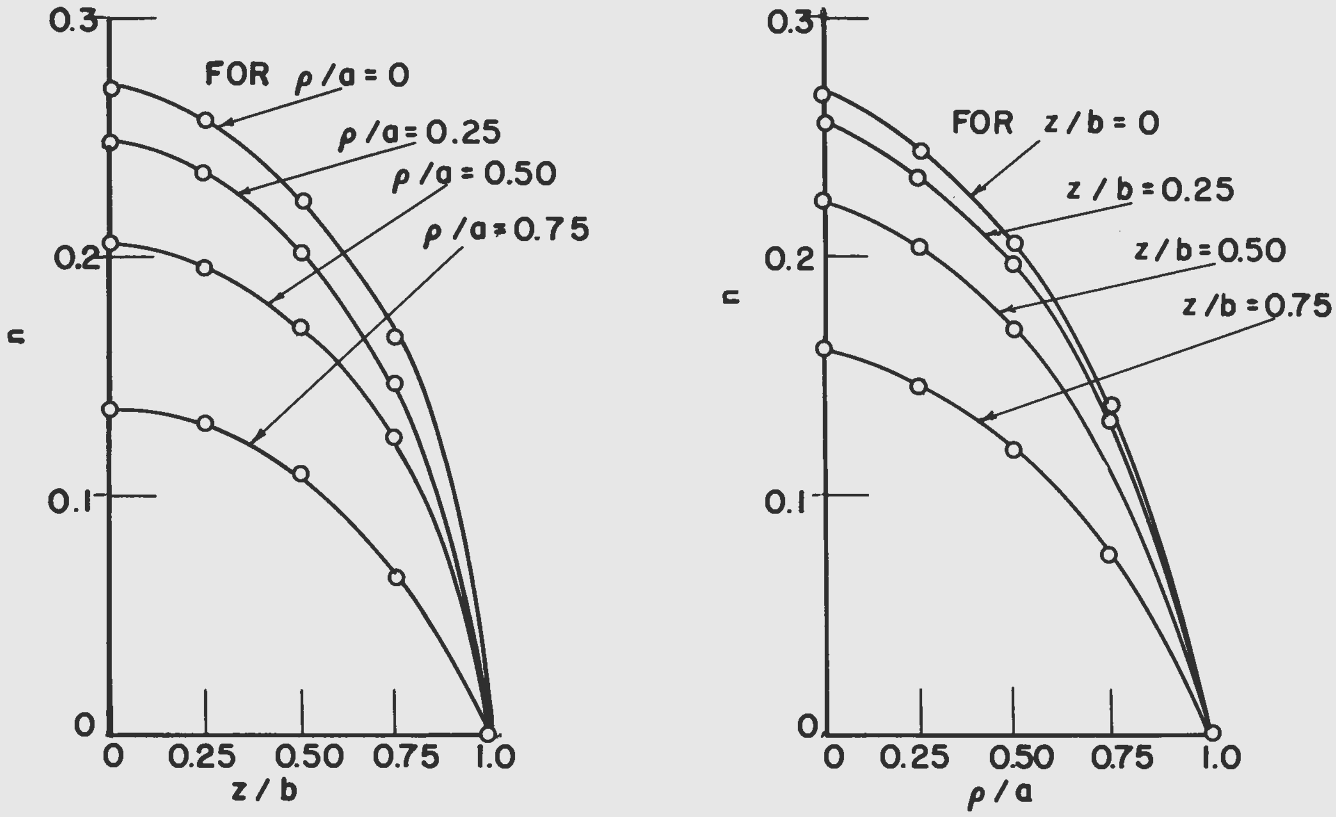

Fig. 5--Plot of n versus position coordinates for a cylinder with L/D ratio of 1.

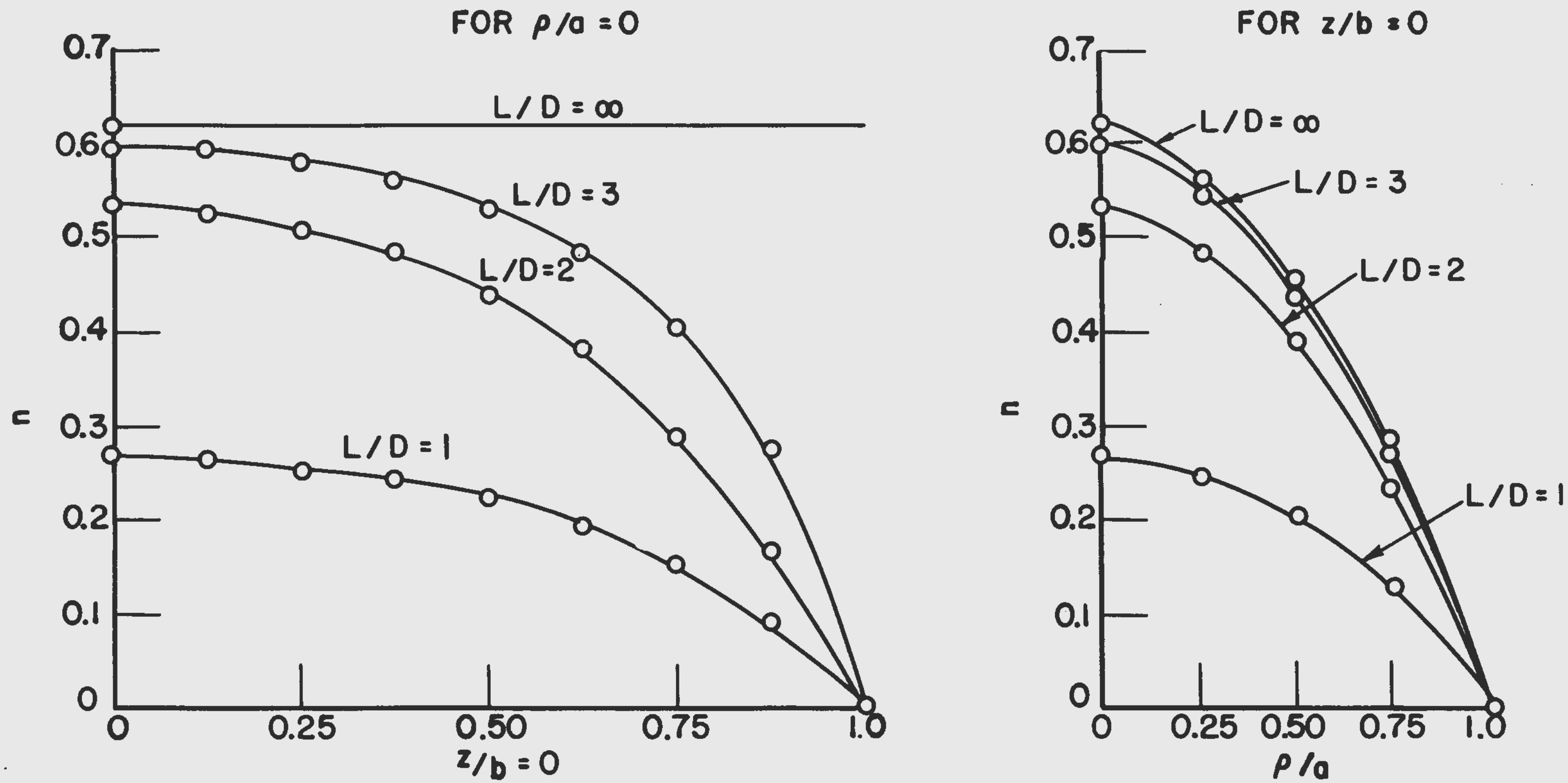

Fig. 6--Comparison of n versus position coordinates for cylinders of L/D equal to 3, 2, 1, and ∞.

VII. DISCUSSION OF RESULTS

The results of the determination of temperature distribution in a cylinder are shown in the graphs of Figures 3, 4, 5, and 6. These results are not expressed as temperatures but are in terms of n. The temperature, $t_{actual} = t_{surface} + C'(n)$. The C' term contains as variables the diameter of the cylinder, a; the thermal conductivity of the metal cylinder, k; and the heat production term, B. To find the actual temperature at any point for a specific case where a, k, and B are known it is only necessary to read off the value of n at that point and multiply it by C' which is $\frac{16a^2}{\pi 3} \frac{B}{k}$ and then add this product to the surface temperature. If the heat production term is some function of radius ρ or longitudinal position Z, the value of B for the point in question may be calculated and used in C' at that point. If neither B nor k vary for a given application, C' will be constant for all points, and all values of n may be multiplied by the same C'. The results were left in general terms to permit the usage of any size cylinder, metal material, and heat production value that might be applicable in general practice.

To check the results obtained by this method used was made of a solution given by Boelter[2] of the equation:

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2} = 0 \qquad (17)$$

The solution is

$$t = C \begin{matrix}\sin\\ \cos\end{matrix} (BZ) J_0 (\alpha r) . \qquad (18)$$

To include the heat production term it is necessary to find a solution that satisfies the following equation:

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2} = -\frac{B}{k} \qquad (19)$$

The complete solution used was

$$t = C \begin{matrix}\sin\\ \cos\end{matrix} (BZ) J_0 (\alpha r) - \frac{B}{k}\frac{r^2}{4} . \qquad (20)$$

To find α, β, and C which are constants in equation 20 the temperature t was found at three points $\rho = 0$, $Z = 0$; $\rho = a$; $Z = 0$, and $\rho = 0$, $Z = b$ from the previous solution $\theta = C'n$.

To obtain a numerical value for t it was necessary to arbitrarily select an L/D ratio, cylinder size, and B/k value. For this check purpose a cylinder with $L/D = 3$, length of four inches, and $B/k = 100{,}000$ Btu/hr-ft^2 was used.

The constants α, β, and C were determined and substituted into equation 20 giving:

$$t = 88.2 \cos\left(\frac{\pi}{4} Z\right) J_0(1.36\, r) - \frac{1 \times 10^5 r^2}{4} \tag{21}$$

Equation 21 was solved for at $\rho = 0.16$, $Z = 0$; $\rho = 0.32$, $Z = 0$; and $\rho = 0.48$ and $Z = 0$. Values of t were obtained also for the three points by using $t = C'(n)$. The comparison is shown in Table I.

Table I

Comparison of Results with Boelter Equation

ρ/a	Temperature, deg. F		% Diff.
	Eqn. 11	Boelter's Eqn.	
0.25	81.0	82.6	1.94
0.50	63.9	64.2	0.47
0.75	39.4	38.2	3.15

This check indicates that the shape of the curves t versus ρ for the two solutions are within a few per cent of each other.

VIII. CONCLUSIONS

On the basis of the investigation conducted the following conclusions appear reasonable:

1. The finite difference method may be used to determine the temperature distribution in a metal cylinder containing a heat source.

2. The method applies to any distribution of heat source, giving it a flexibility not found in conventional solutions such as Boelter's solution.

3. The results as checked against Boelter's solution are within 3 per cent accuracy.

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