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Analysis and control of nonlinear flexible manipulator

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1 INTRODUCTION

A robot is often considered as the assembly of many rigid links. The control of a robot generally involves the response of the end-effector of the manipulator to the joint input commands. Once the control at the joints are determined, the dynamic response of the robot to control may be obtained by solving a set of ordinary differential equations of motion numerically. In general, the robot control system is designed according to the rigid robot model, i.e., the individual links are assumed rigid. However, the performance of robot control system designed based on simple rigid-body assumption may not be satisfactory if the members of the robot undergo elastic deformation. If the heavy, high-stiffness robot links are used to preserve the rigid-body assumption, the resulting control system may spend as high as ninety percent of the actuator output to lift the weight of the arm itself, leaving very little for the payload. Alternatively, when the deflections of the links are not negligible, instead of using the heavy members to suppress the elastic deformation, we may include the link flexibility in the system model and controller design. However, modeling and analysis of flexible systems is not trivial. The introduction of flexibility into a kinetic model generally results in a very complicated dynamical system. Analysis and control of such systems are still under extensive study [1][2][3][4][5].
1.1 Literature Review

A flexible manipulator is a deforming continuum: its equations of motion are mixed partial and ordinary differential equations. These differential equations usually contain terms in the integral form \([6][7]\). With few exceptions, the closed-form solution of PDEs is not practical. Hence, motion prediction usually relies on approximation. In many cases, a set of admissible space functions is chosen to eliminate the spatial dependence of the PDEs, which leaves the time functions corresponding to these functions as generalized coordinates. The distributed response of a structure to a control or any other force or torque is approximated by finding the coordinates' time response and multiplying these with their individual shapes. These shape functions are obtained analytically or numerically. In the analytically approach, for instance, mode shapes of a fixed-free cantilever beam are used frequently [8]. In the numerical approach, the shape functions are found by the finite element techniques [9] or assumed modes methods [10].

In general, two distinct approaches exist in formulating the equations of motion for the approximate system. The first method uses the relative joint coordinates and relative joint velocities in equation derivation [11]. For both tree structures and systems with closed kinematic loops, a set of ordinary differential equations (ODEs) are found as the equations of motion. The second approach uses the generalized Cartesian coordinates. Motions of the mass center of the bodies are always referred to as an inertial reference frame. The variables that describe the deformation of bodies are next added to the mass center variables. The dynamic constraints between the system components are shown in terms of the mass center variables and the generalized coordinates associated with flexibility [12]. The resulting equations are
mixed differential and algebraic equations (DAEs).

The flexible manipulator is first investigated by Book et al. [10]. A flexible arm is assembled by connecting two flexible links together. The flexible links are modeled as uniform Euler-Bernoulli beam and the undamped free vibrational modes of fixed-free end conditions are chosen as the basis functions for discretization. The design of closed-loop control systems are investigated extensively. A method which utilizes the eigenvectors obtained from the finite element analysis as approximation functions is developed by Sunada and Dubowsky [9]. NASTRAN is used to produce the eigenvectors of an industrial manipulator. Their study shows that even for a fairly rigid industrial robot arm, the effects of flexibility are still significant. In the above-mentioned studies, the responses of flexible manipulators to the control are obtained by forward-integrating a set of ordinary differential equations. However, including the flexibility in designing a control system for a flexible robot is a much more demanding task.

Two distinct control methodologies are commonly employed in controlling a robot: model-based control and non-model-based control. The model-based control requires a detailed, carefully predetermined kinematic and dynamic model of the actual system [13]. The feedforward control technique is a very popular scheme of this category. The feedforward controller computes the torque required to track the desired joint acceleration, velocity and position using the constructed robot dynamics. In theory, when the model is an exact representation of the real system, the robot is controllable and an end-effector can be positioned precisely as commanded. Practically, there is a position and derivative (PD) control parallel to the forward computation to prevent the arm from deviating away the preplanned trajectory due
to the unmodeled dynamics and perturbation. There are a number of disadvantages in controlling a flexible arm using the model-based control:

1. The effects of the disturbances and joint friction on a mechanical system are never perfectly modeled. Therefore, the end-effector of a robot driven by a pre-determined torque is usually misplaced at a location away from the desired position.

2. The method itself has been shown to be very sensitive to the variation of system parameters [1], such as variation of payload. Therefore, an on-line parameter identification method is necessary to insure satisfactory performance of a model-based closed-loop control system.

3. Flexible arm dynamics is so complicated that the demand of the computer CPU is high. Consequently, the control scheme might be infeasible.

4. For the feedback control system, the on-line estimation of the time function corresponding to the generalized coordinates for the elastic deformation is essential and extremely difficult.

5. The instability for systems of this type is severe and difficult to predict.

A common non-model-based control is the independent joint control (IJC). In the IJC method, the individual link is driven by a PD, or by a position, derivative and integral (PID) control at the base with only its own joint coordinate and the rate of the joint coordinate being fed back to the controller. The advantages of using the non-model-based control are:
1. It does not have the difficult state estimation problem from which the model-based control suffers.

2. The instability of the control system is less likely.

However, for a multi-link arm, IJC may perform poorly because the control at one joint might become the disturbance for the other links. Due to this undesirable interaction, the robot arm controlled by the IJC tends to vibrate more, and it usually takes longer to position the robot at the desired location.

The control problem for a flexible manipulator is also studied by Book et al. [10]. The IJC and two other schemes are tested on a two-link flexible arm while the joint torques are produced by PD controllers. Cannon and Schmitz [8] show that a noncolocated sensor and controller system is controllable and the end-tip positioning of a one-link flexible arm can be achieved using linear quadratic Gaussian approach. Later, Schmitz develops the equations of motion for a two-link arm using Kane's method and shows the variation of system frequencies due to the change of equilibrium position of the two-link arm [14]. A computed torque technique is presented by Bayo [2], in which the end-tip positioning of a flexible arm is addressed. Other than [10] and [9], the effect of gravity is not considered, and, except for [10] and [14], only one-link arm model is studied. In these research presentations, however, a number of mistakes are found: unproper shape functions were used in [10] to approximate the deformation of the elastic links; and gravity is not treated properly in the stability analysis [9].
1.2 Statement of the Problem

The objective of this study is to investigate the stability characteristics and performance of the closed-loop one-link and two-link flexible manipulators. A flexible robot is considered as the assembly of several flexible link elements. A flexible link element is composed of a flexible member, a rigid hub at one end, and a mass attached at the opposite end of the member. The flexible member is modeled as an Euler-Bernoulli beam. No stretching or shortening of the flexible link is considered. An energy approach is used to derive the governing differential equations. Two types of differential equations are derived. We first may express the Lagrangian of the flexible systems in terms of a set of hybrid coordinates, i.e., the angular rotation of the rigid bases and the spatially varying deflection of the links, and then invoke the extended Hamilton’s Principle to obtain the partial differential equations (PDEs) and boundary conditions (BCs). These PDEs and BCs are nonlinear and usually very complicated. The closed-form solutions of these PDEs are usually not feasible. No attempt will be made to solve for the responses of the arms subjected to control or disturbance by integrating these PDEs. When the stability of the arm is of interest, these PDEs and BCs are used to obtain the linear PDEs and BCs which are valid only for the selected equilibrium configurations about which the original nonlinear equations are linearized. The resulting linearized locally-valid equations and boundary conditions are used to form a number of transfer functions relating the control to the response of the arm. These functions, in turn, are utilized to solve for the poles of the open- or closed-loop systems. We notice that the PDEs cannot be used easily for the purpose of nonlinear transient-response simulation. Therefore, it is clear that a discretization scheme is necessary to produce a set of ordinary differential equations which can be
integrated for the simulation. One of the methods to obtain the ODEs for this purpose is to invoke the Ritz method and express the elastic deformation of the flexible members using a finite number of basis functions and their corresponding generalized coordinates. The reduced-order deflection distribution is substituted into the Lagrangian and Lagrange's equations are applied to give a set of ODEs. Since these ODEs are strongly nonlinear, the analytical solutions are still not feasible. However, the nonlinear ODEs can be integrated forward numerically for the transient responses very conveniently [15]. Similar to that for the distributed-coordinates approach, the stability characteristics of the flexible system at the chosen equilibrium positions are analyzed by studying the linearized system. The resulted local system frequencies are compared with those acquired by using the PDEs to determine the accuracy of the approximate system. As will be shown later, while it is still possible to derive the ODEs of the two-link flexible arms manually, the PDEs and BCs of the same model become unmanageable very quickly. A symbolic mathematical manipulation program, Macsyma [16], is utilized to manipulate the linear PDEs and BCs to obtain the local system transfer functions which, in turn, are used to find the characteristics of the linearized system in the Laplace transform domain.

The use of PDEs is valuable when the viscoelastic damping of the flexible member is considered [17]. By invoking the Correspondence Principle, the identical solution procedure used for the undamped flexible arm is equally applicable to the arm with viscoelastic material damping. For the ODE approach, except for systems with linear visco-elastic damping, the stability analysis of visco-damped structure is relatively complicated [18]. In particular, when the high-order time derivatives or fractional derivatives are used to model the material damping, the stability characteristics of
the visco-damped arm are very difficult to predict. For this investigation, a linear viscoelastic damping model, Kelvin material, is used. The time-domain behavior of the visco-damped nonlinear flexible systems are shown to be significantly different from those obtained by neglecting material damping. Finally, the stability problem is examined using the Liapunov direct method. The results from Liapunov's method are shown to be consistent with the linear analysis of the local system for the PD control system.
2 EQUATIONS OF MOTION

In this chapter, we will discuss the formulation of the governing differential equations of flexible robots. The equations of motion of a multiple-axis rigid robot arm generally are extremely complicated and strongly nonlinear [13]. When the flexibility of the links are considered, a larger number of new terms, in additional to those of the rigid arm, are introduced in the equations of motion. These terms account for the coupling of the elastic links, the elastic deformation due to gravitational force, etc. Depending upon which of the methods we choose to describe the motion of the flexible links, differential equations of different types are obtained. Two distinct methods of describing the deflection of flexible arm are discussed in this study: infinite-dimensional, spatially varying distributed coordinates, and finite-dimensional coordinates. In the distributed-coordinates approach, the deflection of flexible links is described using a distributed variable along the length of the arm without any discretization. For the discrete-coordinates method, the elastic deflections are modeled using a set of admissible functions and their generalized coordinates. After the method for describing the elastic deformation is chosen, the corresponding position and velocity expressions can be used to derive the system potential and kinetic energy. The Lagrangian of the flexible system, which is defined as the difference of the system kinetic and potential energy, is subsequently employed to construct the differential
equations. Partial and ordinary differential equations result for the two modeling methods. Later in this study, these differential equations are shown to inherit different advantages and disadvantages when used in conjunction with other techniques to analyze the flexible manipulators.

2.1 Flexible Arm Model

A robot can be considered as the assembly of many flexible or rigid links. A model of a basic flexible link is shown in Figure 2.1. Generally, it is composed of a flexible member, a rigid hub at one end, and a mass attached at the opposite end of the member. The adjacent links, if any, are connected to the flexible link at their ends. The flexible member is assumed to be slender such that the Euler-Bernoulli beam assumption is valid. We further assume that the link is maneuvered in the vertical plane and that the out-of-plane deflection is negligible. The constants $m_b$ and $I_b$ are the mass and the mass moment of inertia of the rigid hub respectively. The constant $I$ is the area moment inertia of the cross section about the neutral axis. The constant $\rho$ is the mass per unit length of the flexible member. The end-tip mass is denoted as $m_e$ and the mass moment of inertia of this portion of the arm is assumed to be negligible. The vectors $\bar{r}_b$ and $\bar{v}_b$ are the position and velocity of the hub referred to the ground reference frame. If we assume that the deformation in the axial direction of the link is negligible, and neglect the thickness of the member itself, then the position of any point along the member can be written as

$$\bar{r}_i = \bar{r}_b + \bar{z} + \bar{\gamma}$$  \hspace{1cm} (2.1)
Figure 2.1: Flexible Arm Element
where

\[ z = z \dd{z} \]  \hspace{1cm} (2.2)
\[ y = y(z, t) \dd{y} \]  \hspace{1cm} (2.3)

and where \( y(z, t) \) is the deflection of the elastic member measured from its undeformed configuration. The \( \dd{y} \) and \( \dd{z} \) are unit vectors in the \( y \) and \( z \) directions respectively as shown in Figure 2.1. Differentiating the expression for the position vector with respect to time, we obtain the velocity of any point along the member.

\[ \dd{v}_i = \dd{v}_b + \dd{\omega} \times \dd{z} + \dd{y} \]  \hspace{1cm} (2.4)

where \( \dd{\omega} \) is the angular velocity of the rigid hub measured with respect to the inertial frame. The symbol \( \times \) denotes the vector cross product and \( (') \) denotes a derivative with respect to time. Using the velocity expression, we write the kinetic energy of a flexible link as

\[ T_e = \frac{1}{2} I \dd{\omega}^2 + \frac{1}{2} \int_{0}^{L} \rho^* \dd{v}_i \cdot \dd{v}_i dz \]  \hspace{1cm} (2.5)

where the symbol, \( \cdot \), is the dot product or inner product and \( \rho^* = \rho(z) + m_b \delta(z) + m_e \delta(z - L) \), and \( \delta \) is the dirac delta function. Using the position vector, we write the potential energy of a cross-sectional element as

\[ V_e = V_s + V_g = \frac{1}{2} \int_{0}^{L} EI (y''')^2 dz + \dd{g} \cdot \int_{0}^{L} \rho \dd{^2}_i dz \]  \hspace{1cm} (2.6)

where \( V_s \) is the elastic strain energy and \( V_g \) is the gravitational potential energy. The \( (\cdot')' \) denotes the second derivative with respect to the spatial variable \( z \). Using the kinetic and potential energy given above, we form the Lagrangian of a flexible link
element according to

$$L_e = T_e - V_e$$  \hspace{1cm} (2.7)$$

The Lagrangian of the entire flexible system is obtained by summing the Lagrangian of all the links of the flexible robot together.

$$L = \sum_{i=1}^{m} L_e$$  \hspace{1cm} (2.8)$$

where $m$ is the number of the links. Once the Lagrangian is derived, we can invoke the extended Hamilton's Principle \cite{19} to derive the governing differential equations. Depending on the selected modeling method for the flexible components, we may obtain either PDEs or ODEs as the equations of motion for the flexible system.

### 2.2 Partial Differential Equations

When the distributed coordinate $y(z, t)$ is used to describe the elastic deflection of the flexible link, the extended Hamilton's Principle is applicable for deriving the system governing partial differential equations.

$$\delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0$$  \hspace{1cm} (2.9)$$

Since the torque at the base of an individual link is supplied by the member or ground that the rigid hub is fixed upon, the work done by the nonconservative control torque is found by multiplying the applied torque at the rigid hub with the variation of the angular rotation of the hub relative to the previous link or ground.

$$\delta W_{nc} = \sum_{i=1}^{m} T_{qi} \delta \theta_i$$  \hspace{1cm} (2.10)$$
Generally, for each link, an equation in the integral form and a partial differential equation with four boundary conditions would be obtained. These differential equations are nonlinear. For the one-link flexible arm, the nonlinearity arises from the presence of gravitation. For the multi-link flexible arm, the nonlinearity is contributed by gravity and the coupling of the links. These nonlinear differential equations are extremely complicated and no apparent closed-form solution is available. However, if the only concern is the system stability characteristics, the problem can be studied by linearizing the nonlinear PDEs about chosen equilibrium positions, and solving for the poles of the transfer functions which relate the angular rotations of the rigid base and torques acting at the joints. The solutions from this method are exact. However, there are several difficulties in using the the distributed-coordinate approach. First, the complexity of the nonlinear PDEs, local linear PDEs and BCs increase drastically as the number of flexible links increases. The equations of motion for the flexible arms composed of two links are very difficult to derive manually. Secondly, the numerical methods for solving the PDEs are not as readily available as are those for ODEs. Hence, the time-domain behavior of the flexible arm described by the PDEs may not be determined easily. Finally, the distributed-coordinate approach is valid only for a flexible arm with uniform mass and cross section distribution. For robots with complicated shapes, the PDEs may not be practical to determine.

2.3 Ordinary Differential Equations

Alternatively, we may approximate the continuous deflection of a flexible link by a set of assumed shape functions and their time-dependent generalized coordinates.
The reduced-order model of a flexible member is commonly expressed as

\[ y(z,t) = \sum_{i=1}^{n} \phi_i(z)q_i(t) = \{\phi\}^T\{q\} \quad (2.11) \]

where \(\{\phi(z)\}\) are a set of basis functions, \(\{q(t)\}\) are the time functions corresponding to the basis functions, and \(n\) is the number of functions. Equation (2.11) is substituted into Equations (2.5) and (2.6) to form the approximation to the kinetic and potential energy of the system. We define the augmented variables \(\{\psi\}\) as the collection of the \(\theta\)'s and \(\{q\}\)'s of all the links. By using the extended Hamilton's Principle in the form of Lagrange's equations, a set of nonlinear ODEs result.

\[ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}_i}\right) - \frac{\partial L}{\partial \psi_i} = T\psi_i \quad (2.12) \]

The closed-form solution of the nonlinear system ODEs is still not possible, and the motion of the flexible system is solved numerically. If the stability characteristics of a flexible robot is of interest, the nonlinear equations are linearized and the linearized equations are used to form an eigenvalue problem. The resulting eigenvalues may be utilized to obtain the frequencies of the linearized system.

It is noted that the costs for numerical integration and eigenvalue solution are directly related to the dimension of solution matrix. Increase in the dimension of the solution matrix will generally lead to higher CPU cost. From the preceding brief description, it is apparent that the number of equations used for the forward integration and the size of the eigenvalue problem both increase when the order of the approximation functions increases. When an analyst increases the order of approximation functions at the expense of CPU time, a better solution to the problem is anticipated, but, as will be shown later in this study, will not always occur. Hence, a proper selection method for the comparison functions is needed that will improve the
accuracy the solution asymptotically by increasing the number of the approximation
functions. To achieve the above objective, the criteria of the Ritz method for choosing
the basis functions are used.

2.4 The Ritz Method

When the Ritz method is invoked, the basis functions must be chosen to satisfy
the following minimal requirements:

1. The basis functions must be linearly independent.

2. The basis functions must be chosen from a complete set.

3. The basis functions must satisfy the geometric boundary conditions of the sys­
tem they approximate.

If the basis functions are selected accordingly, the approximate solution theoretically
would approach the exact solution as the order of the approximation is increased.
However, in practice, the series of basis functions is truncated when the error of the
approximation becomes insignificant.

Four sets of functions of two types are employed in this study, eigenvectors of
cantilever beams and polynomials. Under special circumstances, such as a flexible
element with a uniform cross section, eigenvectors can be derived analytically. For
flexible arms consisted of links with complex shapes, the eigenvectors for the individ­
ual links are usually obtained by finite element analysis. If the individual flexible links
are approximated using the chosen functions, the global nonlinear system differential
equations remain coupled due to the coupling of the inertias. This applies to the
case when the individual links are discretized using their own eigenvectors as well. In the available literature, due to the highly coupled differential equations they produce, polynomials are not commonly utilized for approximating the deformable link. Instead, the eigenvectors of the individual elastic components are frequently used as the approximation functions. For models with special configurations, as for instance in the case of the spinning satellite and one-link flexible arm, the use of eigenvectors of the elastic component as approximation functions simplifies the system equations substantially. However, for the multi-link robots undergoing large-angle maneuvers, the use of eigenfunctions as basis functions will not simplify the system of equations to a great extend. Furthermore, in the examples shown later it is demonstrated that less accurate results are obtained when eigenvectors are used for approximation.
3 FLEXIBLE ROBOT ARMS

The procedures of deriving the system PDEs and ODEs discussed in the last chapter are applied to the one- and two-link flexible arms. For the distributed-coordinates approach, the PDEs and BCs are employed to derive the transfer functions which relate the joint responses with the inputs to the flexible arm. The poles of these transfer functions are found for the stability analysis. For the discrete coordinates approach, the formulation of ODEs allows for the use of various approximation functions, i.e., when different functions are used for the approximation, the required changes to form the equations of motion are limited to a small number of invariant coefficient matrices. Where system stability is concerned, the nonlinear ODEs are linearized and the linear ODEs are utilized to form an eigenvalue problem. The resulted eigenvalues may be used to obtain the frequencies of the open- or closed-loop systems.

3.1 One-Link Flexible Arms

The one-link flexible arm is constructed by directly connecting the base of the flexible element in Figure 2.1 to the ground. Hence the position and velocity vectors, \( \bar{r}_b \) and \( \bar{v}_b \), both are equal to zero for this model. This simplifies the position and
velocity vectors of the point along the axial direction of the flexible arm

\[ \bar{r}_i = z\bar{e}_z + y\bar{e}_y \]
\[ \bar{v}_i = (z\dot{\theta} + \dot{y})\bar{e}_y \]  

(3.1)

where the \( z \) axis is along the length of the arm and the in-plane \( y \) axis is perpendicular to the \( z \) axis and measured from the undeformed configuration of the arm. As one notices, a term, \( y\dot{\theta}\bar{e}_z \), in the velocity expression has been dropped. This term, which is small for low angular velocities, is eliminated to avoid a so called 'softening effect' [20]. The behavior of a flexible system may become unrealistic when this term is considered, as for instance in the case of a spinning satellite. Unless a dynamic stiffening effect is introduced [21], the solutions resulting from Equation (3.1) are believed to correspond more closely with physical reality than those obtained by retaining the \( y\dot{\theta} \) term. Using the position and velocity vectors, we can write the system potential and kinetic energy respectively as

\[ V = \frac{1}{2} \int_0^L EIy^2 \, dz + \int_0^L \rho^* g[z\sin \theta + y\cos \theta] \, dz \]  

(3.2)

\[ T = \frac{1}{2} I_b \dot{\theta}^2 + \frac{1}{2} \int_0^L \rho^*[z\dot{\theta} + \dot{y}]^2 \, dz \]  

(3.3)

By substituting the above two energy expressions into Equations (2.8) and (2.9) and performing integration by parts, we obtain the system governing partial differential equations (PDEs) by setting the collected terms associated with the variation of \( \theta \), and \( y \) to zero. The geometric boundary conditions are obtained based on the physical constraints of the flexible link, i.e., \( \delta y(0,t) = 0 \) and \( \delta y'(0,t) = 0 \). The natural boundary conditions are derived by setting the coefficients of \( \delta y(L,t) \) and
\( \delta y'(L, t) \) equal to zero.

\[
I_t \ddot{\theta} + g \cos \theta \int_0^L \rho^* dz + \int_0^L \rho^* [z \ddot{y} - g \sin \theta y] dz = T_q(t)
\]  

(3.4)

\[
\frac{\partial^2}{\partial z^2} \left( E I \frac{\partial y^2}{\partial z^2} \right) + \rho (z \ddot{\theta} + \ddot{y}) + g \cos \theta = 0
\]  

(3.5)

\[
y(0, t) = 0
\]  

(3.6)

\[
y'(0, t) = 0
\]  

(3.7)

\[
y''(L, t) = 0
\]  

(3.8)

\[
y'''(L, t) = \frac{m e}{E I} [\ddot{y}(L) + L \ddot{\theta} + g \cos \theta]
\]  

(3.9)

where

\[
I_t = I_b + \int_0^L \rho^* z^2 dz
\]  

(3.10)

The analytical solution of Equations (3.4) and (3.5) is available only when the gravitational force is omitted and when the flexible arm has uniform stiffness and mass distribution. Even though the presence of gravity in the model would make the analytical solution for the global motion of a one-link flexible arm unfeasible, it must be included in our analysis. Without the modeling gravity, the results obtained would have limited applicability. In the examples given later, the flexible links are assumed to be uniform in cross section and mass distribution. These assumptions are imposed in order to compare results from the infinite- and finite-dimensional coordinates.

The nonlinear PDEs and BCs are used to obtain the linearized, locally valid PDEs and BCs by perturbing the original nonlinear equations of motion about various static equilibrium configurations. Let \( \theta_0 \) denote the inclined angle corresponding to
the equilibrium configuration about which dynamic, closed-loop stability is of interest. And let $y_0(z)$ and $u_0$ denote the static deflection and static holding torque at the angle $\theta_0$. The perturbed equations are derived by substituting the following definitions into Equations (3.4), (3.5) and (3.6) - (3.9).

\begin{align}
\theta &= \theta_0 + \varepsilon \\
y &= y_0 + w \\
T_q &= u_0 + u
\end{align}

(3.11)  
(3.12)  
(3.13)

After some modest manipulation, the linearized equations can finally be written as

\begin{align}
I_\varepsilon \varepsilon - \left(\frac{g\sin \theta_0}{2} \int_0^L \rho^* dz\right) \varepsilon + \int_0^L \rho^* [z\ddot{w} - g y_0 \cos \theta_0 \varepsilon - g \sin \theta_0 w] dz = u \\
EI w^{IV} + \rho (\ddot{w} + z\dddot{\varepsilon} - g \sin \theta_0 \varepsilon) &= 0
\end{align}

(3.14)  
(3.15)

\begin{align}
w(0,t) &= 0 \\
w'(0,t) &= 0 \\
w''(L,t) &= 0 \\
w'''(L,t) &= -(\frac{m_\varepsilon g \sin \theta_0}{EI}) \varepsilon
\end{align}

(3.16)  
(3.17)  
(3.18)  
(3.19)

For this case, the exact closed-form solution to the static equilibrium control torque $u_0$ and member deflection $y_0(z)$ is found by first letting the acceleration terms in Equations (3.5) and (3.6) to (3.9) be zero to obtain the static governing differential equations.

\begin{align}
g \cos \theta \int_0^L \rho^* dz - g \sin \theta \int_0^L \rho^* y dz = T_q(t) \\
\frac{\partial^2}{\partial z^2} (EI \frac{\partial y^2}{\partial z^2}) + g \cos \theta &= 0
\end{align}

(3.20)  
(3.21)
Observing the Equation (3.21), we assume that the static deflection has the form of

\[ y_o(z) = -(\frac{g m_e \cos \theta_o}{24 E I}) z^4 + A z^3 + B z^2 + C z + D \]  

(3.26)

Substituting Equation (3.26) into the BCs, we obtain the unknowns A, B, C and D.

\[ A = \frac{g (m_e + L \rho)}{6 E I} \cos \theta_o \]  

(3.27)

\[ B = \frac{-g L (2 m_e + L \rho)}{4 E I} \cos \theta_o \]  

(3.28)

\[ C = 0 \]  

(3.29)

\[ D = 0 \]  

(3.30)

The static equilibrium holding torque is obtained by embedding the \( y_o \) expression into Equation (3.20).

\[ u_o = m_e g L \cos \theta_o (1 + \sin \theta_o \frac{g L^3 \rho}{4 E I} + \sin \theta_o \frac{g m_e L^2 \rho}{3 E I}) \]  

(3.31)

\[ + \rho g L^2 \cos \theta_o (\frac{1}{2} + \sin \theta_o \frac{g L^3 \rho}{20 E I}) \]

In order to analyze the stability of the linearized system, the PDEs and BCs are transformed into the Laplace domain to eliminate their time dependence. We define the transformed domain variables \( \bar{w} \), \( \bar{\varepsilon} \) and \( \bar{u} \) as follows:

\[ \bar{w} = L\{w\} \]  

(3.32)

\[ \bar{\varepsilon} = L\{\varepsilon\} \]  

(3.33)

\[ \bar{u} = L\{u\} \]  

(3.34)
where \( L \) is the Laplace transform operator [22]. The transformed domain equations and boundary conditions are obtained by simply taking the Laplace transform of both sides of Equations (3.14), (3.15) and (3.16) - (3.19). An open-loop transfer function \( G(s) \) is defined according to

\[
G(s) = \frac{\bar{v}}{u} \quad \text{(3.35)}
\]

This transfer function is obtained using a procedure which is similar to, but, due to the modeled gravitation, more complicated than, the one presented in [7]. The expression for \( G(s) \) is usually very long and tedious. To arrive at the expression for the \( G(s) \), we first assume the solution form of the \( \bar{w}(z) \) as

\[
\bar{w}(z) = e^{z\beta} [A\cos(z\beta) + B\sin(z\beta)] + e^{-z\beta} [C\cos(z\beta) + D\sin(z\beta)] + \left[ \frac{g\sin(z\beta)}{s^2} - z\bar{\varepsilon} \right]
\]

where \( \beta^2 = s^2 \frac{\rho}{4EI} \). The transformed domain deflection \( \bar{w}(z) \) shown above is used in the transformed boundary conditions to solve for the unknown coefficients \( A, B, C \) and \( D \) in terms of \( \bar{\varepsilon} \). The transfer function \( G(s) \) is finally derived by substituting the deflection \( \bar{w}(z) \) to Equation (3.14). For the special case when the base mass and end-tip mass both equal zero, \( G(s) \) has a relatively compact form as shown in Equation (3.37).

\[
G(s) = \frac{1}{(a) - (b) + (c) + (d) - (e)}
\]

Where

\[
A = \frac{1 + \cos(2\beta L)}{2\beta(2 + \cosh(2\beta L) + \cos(2\beta L))} \quad \text{(3.37)}
\]
\[
2 + \cos(2\beta L) - \sin(2\beta L) + e^{-\beta L} \frac{g\sin(\beta L)}{s^2} \frac{2 + \cosh(2\beta L) + \cos(2\beta L)}{2(2 + \cosh(2\beta L) + \cos(2\beta L))}
\]

\[
B = \frac{1 + \sin(2\beta L) + e^{-\beta L}}{2\beta(2 + \cosh(2\beta L) + \cos(2\beta L))}
\]

\[
+ \frac{2 - \cos(2\beta L) - \sin(2\beta L) + e^{-\beta L}}{2(2 + \cosh(2\beta L) + \cos(2\beta L))} \frac{g\sin(\beta L)}{s^2}
\]

\[
C = \frac{-1 - \cos(2\beta L)}{(2 + \cosh(2\beta L) + \cos(2\beta L))}
\]

\[
+ \frac{2 - \cos(2\beta L) - \sin(2\beta L) - e^{-\beta L}}{2(2 + \cosh(2\beta L) + \cos(2\beta L))} \frac{g\sin(\beta L)}{s^2}
\]

\[
D = \frac{1 + \sin(2\beta L) + e^{2\beta L}}{(2 + \cosh(2\beta L) + \cos(2\beta L))}
\]

\[
+ \frac{\cos(2\beta L) - \sin(2\beta L) - e^{-\beta L}}{(2 + \cosh(2\beta L) + \cos(2\beta L))} \frac{g\sin(\beta L)}{s^2}
\]

\[
(a) = I_t s^2
\]

\[
(b) = \frac{\rho g L \sin \beta}{2}
\]

\[
(c) = \frac{(\rho g^2)}{2\beta} \left[ e^{\beta L} L \left( (A + B) \sin(\beta L) + (A - B) \cos(\beta L) \right) + e^{-\beta L} L \left( (C - D) \sin(\beta L) - (C + D) \cos(\beta L) \right) + \frac{e^{\beta L}}{\beta} (B - A) + \frac{e^{-\beta L}}{\beta} (C - D) - \frac{1}{\beta} (B - D) \right]
\]

\[
+ \left( \frac{g\sin(\beta L)L^2}{2s^2} \right) - \left( \frac{L^3}{3} \right)
\]

\[
(d) = \frac{(\rho g \cos \theta_0)^2 L^5}{20EI}
\]

\[
(e) = \frac{(\rho g \sin \theta_0)^2}{2\beta} \left[ e^{\beta L} \left( (A + B) \sin(\beta L) + (A - B) \cos(\beta L) \right) + e^{-\beta L} \left( (C - D) \sin(\beta L) - (C + D) \cos(\beta L) \right) - (A - B) + (C + D) \right]
\]

\[
+ \left( \frac{g\sin(\beta L)L}{s^2} \right) - \left( \frac{L^2}{2} \right)
\]
In the absence of gravity, all the poles are imaginary. If gravitation is considered, then depending on the value of the angle $\theta_0$, two poles otherwise would be located at the origin of the s-plane may relocate. For the lower values of $\theta_0$, the two poles would depart from each other, both move along the imaginary axis. At higher values of $\theta_0$, the two poles would depart from each other along the real axis; one would move to the right of the imaginary axis while the other would move to the left. When the real part of any of the poles vanishes, the local system is marginally stable, i.e., the perturbed motion will oscillate about the equilibrium position with a constant amplitude in the steady state. When the real part of any of the poles is positive, the flexible arm moves away from the equilibrium angle for any disturbance. Depending on $\theta_0$, then, the linearized open-loop system may be unstable or marginally stable.

We may achieve an asymptotically stable system by introducing feedback control at the joint. For PID control, the form of the control is

\[ T_q = K_p(\theta_d - \theta) + K_i(\int_0^t (\theta_d(\eta) - \theta(\eta))d\eta + \psi_0) + K_v(\dot{\theta}_d - \dot{\theta}) \]  

(3.46)

where the $\theta_d(t)$ is a prescribed reference input. The stability of the closed-loop system is examined by locating the poles of the closed-loop transfer function. In order to express this transfer function, we define the perturbed reference input as

\[ \theta_d = \theta_{do} + \varepsilon_d \]  

(3.47)

Once the open-loop transfer function $G(s)$ is available, the closed-loop transfer function $C(s)$ as defined in Equation (3.48) is determined. A block diagram for the closed-loop input/output relationship is shown in Figure 3.1.

\[ C(s) = \frac{\bar{\varepsilon}}{\varepsilon_d} \]  

(3.48)
Figure 3.1: Block Diagram of Controlled One-Link Flexible Arm

\[
\begin{align*}
\ddot{\xi}_d & \rightarrow - \frac{G(K_v s^2 + K_p s + K_i)}{G(K_i + s K_p) + s(1 + G K_v s)}
\end{align*}
\]

The same nonlinear equation solver used for the open-loop poles is applicable here to solve for the closed-loop poles. For the transfer function approach, the material damping is introduced into the analysis through the open-loop transfer function. When the frequencies of the damped system are of interest, the open-loop system transfer function is obtained according to the Correspondence Principle [23], and the poles of the open- or/and closed-loop transfer function are solved using the same equation solver just mentioned. But, when the time domain response is required, the transfer function approach becomes ineffective because the inverse Laplace transformation of the transfer function is very difficult and the resulting impulse response is valid only in a limited region.

The exact transfer function approach gives the exact value of the local system frequencies with the restriction that the flexible arm must be uniform in mass and stiffness distributions. In reality, very few arms possess such properties. Hence the solution for an actual arms would usually be obtained using approximation.
3.2 One-Link Flexible Arms by Ritz Approximation

The Ritz method is employed to approximate the deflection of the flexible link according to

\[ y(z, t) = \sum_{i=1}^{n} \phi_i(z)q_i(t) = \{q\}^T \{\phi\} \]  \hspace{1cm} (3.49)

The spatial and time derivatives of the reduced-order approximation are substituted into the kinetic and potential energy in Equations (3.2) and (3.3) to obtain

\[ V = \frac{1}{2} \{q\}^T \{\bar{K}\} \{q\} + m_ay_0 \sin \theta + \cos \theta \{q\}^T \{B\} \]  \hspace{1cm} (3.50)

\[ T = \frac{1}{2} (I_b + I_{bb}) \dot{\theta}^2 + \frac{1}{2} \{\dot{q}\}^T \{\bar{M}\} \{\dot{q}\} + \dot{\theta} \{\dot{q}\}^T \{A\} \]  \hspace{1cm} (3.51)

where \( m_a, \) \( y_a, \) \( I_{bb} \) and the elements of \( \{A\}, \{B\}, \{\bar{M}\} \) and \( \{\bar{K}\} \) are given by

\[ m_a = \int_0^L \rho^* dz \]  \hspace{1cm} (3.52)

\[ y_a = \int_0^L \rho^* z dz / m_a \]  \hspace{1cm} (3.53)

\[ I_{bb} = \int_0^L \rho^* z^2 dz \]  \hspace{1cm} (3.54)

\[ A_i = \int_0^L \rho^* z \phi_i dz \]  \hspace{1cm} (3.55)

\[ B_i = \int_0^L \rho^* g \phi_i dz \]  \hspace{1cm} (3.56)

\[ \bar{M}_{ij} = \int_0^L \rho^* \phi_i \phi_j dz \]  \hspace{1cm} (3.57)

\[ \bar{K}_{ij} = \int_0^L EI \phi_i'' \phi_j'' dz \]  \hspace{1cm} (3.58)

The resulting governing nonlinear system of ODEs are obtained using Lagrange's equations.

\[ I_t \ddot{\theta} + \{A\}^T \{\dot{q}\} + m_ay_0 \cos \theta - \{B\}^T \{q\} \sin \theta = T_q \]  \hspace{1cm} (3.59)
The above dynamic equations are combined with the kinematic equations to form a set of first-order state equations. Once the initial conditions and joint torques are given, an ODE integrator, such as DVERK of IMSL [15], can be used to integrate the first order differential equations for the response of the flexible arm.

Similar to the exact transfer function approach, we may study the linearized system characteristics by perturbing the nonlinear ODEs about the equilibrium configuration corresponding to a given angle $\theta_o$. Since the perturbation is taken about the static equilibrium position, the static deflection is necessary. The holding torque and elastic deflection corresponding to $\theta_o$ are found by solving the static equilibrium equations

$$\{q_o\} = -[\tilde{K}]^{-1} \{B\} \cos \theta_o$$  \hspace{1cm} (3.61)

$$u_o = g \cos \theta \int_0^L \rho^* z dz - \{B\}^T \{q_o\} \sin \theta_o$$  \hspace{1cm} (3.62)

We define the perturbed generalized coordinates as

$$q_i = q_{oi} + \xi_i$$  \hspace{1cm} (3.63)

Using Equations (3.11), (3.13) and (3.63), we have the linearized ODEs as

$$I_t \ddot{\xi} + \{A\}^T \{\xi\} - (m agy \sin \theta_o + \{B\}^T \{q_o\} \cos \theta_o) \varepsilon$$  \hspace{1cm} (3.64)

$$-\sin \theta_o \{B\}^T \{\xi\} = u$$

$$[\tilde{M}] \{\ddot{\xi}\} + \{A\} \ddot{\xi} + [\tilde{K}] \{\xi\} - \sin \theta_o \{B\} \varepsilon = 0$$  \hspace{1cm} (3.65)

This set of equations may be transformed into the Laplace domain to solve for the poles of the approximation to the transfer function $G(s)$ as previously defined. Alternatively, the local system frequencies can be obtained by solving a standard eigenvalue
problem. Let's define the augmented variable \( \{ \psi \} \) for the nonlinear ODEs as

\[
\{ \psi \} = \begin{bmatrix}
\theta \\
\{ q \}
\end{bmatrix}
\]  

(3.66)

The ODEs may be rewritten in terms of \( \{ \psi \} \) as follows

\[
[M]\{ \ddot{\psi} \} + [K]\{ \psi \} + \frac{\partial V_g}{\partial \{ \psi \}} = \{ T_q \}
\]

(3.67)

where

\[
[M] = \begin{bmatrix}
I_t & \{ A \}^T \\
\{ A \} & [\ddot{M}]
\end{bmatrix}
\]

(3.68)

\[
[K] = \begin{bmatrix}
0 & \{ 0 \}^T \\
\{ 0 \} & [\ddot{K}]
\end{bmatrix}
\]

(3.69)

The perturbed variables are related to \( \{ \psi \} \) according to

\[
\{ \psi \} = \{ \psi_0 \} + \{ \delta \psi \}
\]

(3.70)

where

\[
\{ \delta \psi \} = \begin{bmatrix}
\varepsilon \\
\{ \xi \}
\end{bmatrix}
\]

(3.71)

and where \( V_g \) is the gravitational potential energy. The eigenvalue problem is formulated by assuming that the solution form of the perturbed angle and elastic deflection are

\[
\{ \delta \psi \} = \{ Z \} e^{st}
\]

(3.72)

Equation (3.72) is substituted into the linearized ODEs to give

\[
(s^2[M^*] + [K^*])\{ Z \} e^{st} = \{ 0 \}
\]

(3.73)
where

\[
[M^*] = [M] \quad (3.74)
\]

\[
[K^*] = \begin{bmatrix}
ma g y a s i n \theta_o + \cos \theta_o (B)^{T} q_o & \sin \theta_o (B)^{T} \\
\sin \theta_o (B) & [K]
\end{bmatrix} \quad (3.75)
\]

If the linear viscoelastic damping is considered, a damping matrix \([C]\) is introduced into the equations of motion:

\[
[M]\ddot{\psi} + [C]\dot{\psi} + [K]\psi + \frac{\partial V_q}{\partial \psi} = \{T_q\} \quad (3.76)
\]

where

\[
[C] = \begin{bmatrix}
0 & \{0\}^T \\
\{0\} & [\tilde{C}]
\end{bmatrix} \quad (3.77)
\]

and where \([\tilde{C}]\) is the damping matrix associated with material energy dissipation.

The eigenvalue problem is expanded to accommodate this change [24]. When PID control and material damping are both considered, the stiffness and damping matrices for the expanded eigenvalue problem are

\[
[K^*] = \begin{bmatrix}
K_p + ma g y a s i n \theta_o + \cos \theta_o (B)^{T} q_o & \sin \theta_o (B)^{T} \\
\sin \theta_o (B) & [K]
\end{bmatrix} \quad (3.78)
\]

\[
[C^*] = \begin{bmatrix}
K_v & \{0\}^T \\
\{0\} & [\tilde{C}]
\end{bmatrix} \quad (3.79)
\]

When integral control is employed, an additional state variable is defined as the integral of the difference of the reference angular position and the actual joint angular position.

\[
\psi_v = \int^t (\theta_d(\tau) - \theta(\tau))d\tau + \psi_{eo} \quad (3.80)
\]
In addition to the changes in \([K^*]\) and \([C^*]\), the linearized equation of Equation (3.80) is added to the expanded eigenvalue problem.

3.3 Two-Link Flexible Arm

The flexible arm model consists of two links. The link connected to the ground is referred to as "link one" and the link attached to the tip of the link one is "link two". The notations employed to describe the two-link arm are similar to those of the one-link arm. In addition, a subscript is added to all the variables and link properties to denote where these quantities reside. For example, the distributed deflections, \(y_1\) and \(y_2\), are the deflections of the first link and second link as shown in Figure 3.2. These deflections are measured from their undeformed configurations. The angle \(\theta_1\) is measured from the \(x\) axis of the ground reference frame. The angle \(\theta_2\) is the joint rotation of the rigid base on the second link measured from the tangent line at the end-tip of the first link. Later, we assume the control applied at the intermediate joint of the links is generated by an actuator located at the end of the first link which acts upon the rigid base of the second link. Let \(\vec{r}_1\) and \(\vec{r}_2\) denote the position vectors of a point on link one and link two respectively.

\[
\vec{r}_1 = z_1 \vec{e}_z + y_1 \vec{e}_y = (z_1 \cos \theta_1 - y_1 \sin \theta_1) \vec{e}_x + (z_1 \sin \theta_1 + y_1 \cos \theta_1) \vec{e}_y
\]

\[
\vec{r}_2 = L_1 \vec{e}_z + y_1(L_1) \vec{e}_y + x_2 \vec{e}_x + y_2 \vec{e}_y = (L_1 \cos \theta_1 - y_1(L_1) \sin \theta_1 + x_2 \cos(\theta_1 + \theta_2 + y_1(L_1))) - y_2 \sin(\theta_1 + \theta_2 + y_1(L_1)) \vec{e}_x + (L_1 \sin \theta_1 + y_1(L_1) \cos \theta_1 + x_2 \sin(\theta_1 + \theta_2 + y_1(L_1))) \vec{e}_y
\]
Figure 3.2: Two-Link Flexible Arm Model
These position and velocity vectors are employed to express the system kinetic and potential energy as follows. In order to express the kinetic energy, we need the square of the magnitude of the velocities, i.e. \( \vec{v}_1 \cdot \vec{v}_1 \) and \( \vec{v}_2 \cdot \vec{v}_2 \). These quantities are found by taking the dot product of the velocity vectors. For link one, the velocity vector is relatively compact. The resulted kinetic energy expression is identical to that of the one-link arm. For link two, the dot product of the velocity vectors is available once all the unit vectors in the expressions for the velocity are referred to the same coordinate system. Typically, we may either relate the unit vectors of the links to the ground reference frame or express the unit vectors of one link in terms of unit vectors of the other. A transformation matrix is used for this purpose, as, for example, where the unit vectors on link one are related to those of the link two through the transform matrix \([A^{12}]\).

\[
[A^{12}] = \begin{bmatrix}
\cos(\theta_2 + y_1'(L_1)) & \sin(\theta_2 + y_1'(L_1)) \\
\sin(\theta_2 + y_1'(L_1)) & \cos(\theta_2 + y_1'(L_1))
\end{bmatrix}
\]

After all the unit vectors are referred to the same reference frame, we find the square of the velocity magnitude of link two as

\[
\vec{v}_2 \cdot \vec{v}_2 = (L_1 \dot{\theta}_1 + \dot{y}_1(L_1))^2 + [z_2(\dot{\theta}_1 + y'_1(L_1) + \dot{\theta}_2) + \dot{y}_2]^2
\]

(3.87)
For this planar problem, we may also obtain this expression directly using the law of cosines. With the position and velocity vectors, we write the potential and kinetic energy of link one and link two as

\[ V_1 = \frac{1}{2} \int_{0}^{L_1} EI_1 (y_1')^2 dx + \int_{0}^{L_1} \rho_1 \dot{\theta}_1 \sin \theta_1 + y_1 \cos \theta_1 dx \] (3.88)

\[ V_2 = \frac{1}{2} \int_{0}^{L_2} EI_2 (y_2')^2 dx + \int_{0}^{L_2} \rho_2 \dot{\theta}_2 \sin \theta_2 + y_2 \cos \theta_1 + y_1 (L_1) \cos \theta_1 \]

\[ + z \sin (\theta_1 + y_1 (L_1) + \theta_2) + y_2 \sin (\theta_1 + y_1 (L_1) + \theta_2)] dx \] (3.89)

\[ T_1 = \frac{1}{2} I_{b1} \dot{\theta}_1^2 + \frac{1}{2} \int_{0}^{L_1} \rho_1 \dot{y}_1 \dot{\theta}_1 + \dot{y}_1 \dot{\theta}_1 \] (3.90)

\[ T_2 = \frac{1}{2} I_{b2} (\dot{\theta}_2 + \dot{y}_1(L_1) + \dot{\theta}_2)^2 + \frac{1}{2} \int_{0}^{L_2} \rho_2 \dot{\theta}_2 (L_1 \dot{\theta}_1 + \dot{y}_1(L_1))^2 dx \] (3.91)

\[ + \frac{1}{2} \int_{0}^{L_2} \rho_2 \dot{y}_2 [z(\dot{\theta}_1 + \dot{y}_1(L_1) + \dot{\theta}_2) + \dot{y}_2] dx \]

\[ + \int_{0}^{L_2} \rho_2 \dot{y}_2 (L_1 \dot{\theta}_1 + \dot{y}_1(L_1)) [z(\dot{\theta}_1 + \dot{y}_1(L_1) + \dot{\theta}_2) + \dot{y}_2] dx \]

The Lagrangian of the complete system is formed using Equations (2.8) and (3.88) to (3.91). The work done by the nonconservative torque acting at the joints is

\[ \delta W_{nc} = T_{q1} \delta \theta_1 + T_{q2} \delta \theta_2 \] (3.92)

The variational principle is utilized to produce four nonlinear governing differential equations. Two of the four differential equations are in integral form and involve the control torques, while the remaining two equations have both partial derivatives with respect to time and partial derivatives with respect to \( z \). There are four boundary
conditions associated with each of the mixed ordinary and partial differential equations. Among the BCs, the two natural boundary conditions at the tip of link one are extremely complicated. The full expression of the differential equations will be omitted, while the equations used for the static analysis and stability analysis of the linearized system will be given when appropriate. No attempt is made to numerically solve for the system response using the nonlinear PDEs.

For the two-link arm, the equilibrium configuration is found by solving the following ODEs.

\[ u_2 + \int_0^{L_1} \rho_1^* g [z \cos \theta_1 - y_1 \sin \theta_1] dz = u_1 \quad (3.93) \]

\[ \int_0^{L_2} \rho_2^* g [L_1 \cos \theta_1 - y_1(L_1) \sin \theta_1 + z \cos (\theta_1 + y_1'(L_1) + \theta_2)] - y_2 \sin (\theta_1 + y_1'(L_1) + \theta_2)] dz - u_2 = 0 \quad (3.94) \]

\[ EI_1 y_1'''' + \rho_1 g \cos \theta_1 = 0 \quad (3.95) \]

\[ EI_2 y_2'''' + \rho_2 g \cos (\theta_1 + \theta_2 + y_1'(L_1)) = 0 \quad (3.96) \]

\[ y_1(0) = 0 \quad (3.97) \]

\[ y'_1(0) = 0 \quad (3.98) \]

\[ y''_1(L_1) = -\frac{g}{EI_1} \int_0^{L_2} \rho_2^* g [z \cos (\theta_1 + y_1'(L_1) + \theta_2)] - y_2 \sin (\theta_1 + y_1'(L_1) + \theta_2)] dz \quad (3.99) \]

\[ y''''_1(L_1) = \frac{g \cos \theta_1}{EI_1} (m e + \int_0^{L_2} \rho_2^* dz) \quad (3.100) \]

\[ y_2(0) = 0 \quad (3.101) \]

\[ y'_2(0) = 0 \quad (3.102) \]

\[ y''_2(L_2) = 0 \quad (3.103) \]
Observing Equations (3.95) and (3.96), we assume the static deflection of the flexible links as

\[
y_{1o}(z) = -\frac{g\rho_1 \cos \theta_{1o}}{24EI_1} z^4 + A_1 z^3 + B_1 z^2 + C_1 z + D_1 \tag{3.105}
\]
\[
y_{2o}(z) = -\frac{g\rho_2 \cos(\theta_{1o} + \theta_{2o} + y'_{1o}(L_1))}{24EI_2} z^4 + A_2 z^3 + B_2 z^2 + C_2 z + D_2 \tag{3.106}
\]

where the subscript \( o \) denotes the static solution. The deflection expressions above are substituted into the boundary conditions to solve for the eight unknown coefficients. Four of the eight undetermined coefficients vanish due to the zero geometric boundary conditions at the base of the links. The remaining four coefficients \( A_1, B_1, A_2 \) and \( B_2 \) can be solved iteratively using Newton's method. Alternatively, we can linearize the nonlinear natural boundary conditions by assuming that the deflection and slope of the elastic deflection are both small. Hence we may expand the boundary conditions according to

\[
\cos(\theta_{1o} + \theta_{2o} + y'_{1o}(L_1)) = \cos(\theta_{1o} + \theta_{2o}) - \sin(\theta_{1o} + \theta_{2o}) y'_{1o}(L_1) \tag{3.107}
\]
\[
\sin(\theta_{1o} + \theta_{2o} + y'_{1o}(L_1)) = \sin(\theta_{1o} + \theta_{2o}) + \cos(\theta_{1o} + \theta_{2o}) y'_{1o}(L_1) \tag{3.108}
\]

After substituting the equations above into the BCs, we now eliminate the high order terms such as the product of the deflection and slope of the links. The undetermined coefficients can be obtained by solving four linear equations. For the test cases shown later, the linear solution is proved to be very accurate for the flexible links with approximately 10% deflection. For the stability analysis of the linearized system, we
perturb the PDEs according to the following definitions.

\[ \theta_1 = \theta_{10} + \varepsilon_1 \]  
\[ y_1 = y_{10} + w_1 \]  
\[ T_{q1} = u_{10} + u_1 \]  
\[ \theta_2 = \theta_{20} + \varepsilon_2 \]  
\[ y_2 = y_{20} + w_2 \]  
\[ T_{q2} = u_{20} + u_2 \]  

The resulting PDEs are as follows.

\[ EI_1 w_1 IV + \rho_1 \ddot{w}_1 + \rho_1 \dot{z} \ddot{\varepsilon}_1 - \rho_1 gs_\alpha \varepsilon_1 = 0 \]  
\[ EI_2 w_2 IV + \rho_2 \ddot{w}_2 + \rho_2 [z + L_1 c_\beta] \ddot{\varepsilon}_1 + \rho_2 \dot{z} \ddot{\varepsilon}_2 + \rho_2 z \ddot{\varepsilon}_1(L_1) \]  
\[ + \rho_2 c_\beta \ddot{w}_1(L_1) = \rho_2 gs_\gamma (\varepsilon_1 + \varepsilon_2 + w_1(L_1)) = 0 \]  
\[ [I_{1b} + I_{2b} + I_{1x} + I_{2x} + L_1^2 m_2a + 2L_1 c_\beta m_2a Y_2a] \theta_1 \]  
\[ - g[s_\alpha (m_1a Y_{1a} + L_1 m_2a)] + c_\alpha(\int_0^{L_1} \rho_1^* y_{10} dz + y_{10}(L_1) m_2a) \]  
\[ s_\gamma m_2a Y_2a + c_\gamma(\int_0^{L_2} \rho_2^* y_{20} dz) \theta_1 + [I_{2b} + I_{2x} + L_1 c_\beta m_2a Y_2a] \theta_2 \]  
\[ - g[s_\gamma m_2a Y_2a + c_\gamma(\int_0^{L_2} \rho_2^* y_{20} dz) \theta_2] + \int_0^{L_1} \rho_1^* z \ddot{w}_1 dz - g s_\alpha \int_0^{L_1} \rho_1^* w_{1} dz + m_2a w_{1}(L_1) \]  
\[ + [L_1 m_2a Y_2a + c_\beta I_{2x} \ddot{w}_1(L_1)] + [I_{2b} + I_{2x} + L_1 c_\beta m_2a Y_2a] \ddot{w}_1(L_1) \]  
\[ - g[s_\gamma m_2a Y_2a + c_\gamma(\int_0^{L_2} \rho_2^* y_{20} dz) \ddot{w}_1(L_1)] - g s_\gamma \int_0^{L_2} \rho_2^* w_{2} dz = u_1 \]
\[ I_2b + I_1x + I_2x + L_1c\beta m_2aY_2a \varepsilon_1 \] (3.118)
\[-g[s\gamma I_2x + c\gamma \int_0^{L_2} \rho_2^* y_{20} dz] \varepsilon_1 + [I_2b + I_2x] \varepsilon_2 \]
\[-g[s\gamma m_2a Y_2a + c\gamma \int_0^{L_2} \rho_2^* y_{20} dz] \varepsilon_2 + c\beta m_2a Y_2a \ddot{w}_1(L_1) \]
\+[I_2b + I_2x] \ddot{w}_1(L_1) - g[s\gamma m_2a Y_2a + c\gamma \int_0^{L_2} \rho_2^* y_{20} dz] w_1'(L_1) \]
\[ + \int_0^{L_2} \rho_2^* \ddot{w}_2 dz = u_2 \]

\[ EI_1 w_1''(t, L_1) = [I_2b + I_2x + L_1c\beta m_2a Y_2a] \varepsilon_1 \] (3.119)
\[-g[s\gamma m_2a Y_2a + c\gamma \int_0^{L_2} \rho_2^* y_{20} dz] \varepsilon_1 \]
\+[I_2b + I_2x] \varepsilon_2 - g[s\gamma m_2a Y_2a + c\gamma \int_0^{L_2} \rho_2^* y_{20} dz] \varepsilon_2 \]
\+[c\beta m_2a Y_2a] \ddot{w}_1(L_1) + [I_2b + I_2x] \ddot{w}_1'(L_1) \]
\[-g[s\gamma m_2a Y_2a + c\gamma \int_0^{L_2} \rho_2^* y_{20} dz] w_1'(L_1) \]
\[ + \int_0^{L_2} \rho_2^* \ddot{w}_2 dz - g\gamma \int_0^{L_2} \rho_2^* w_2 dz \]

\[ EI_1 w_1'''(t, L_1) = [m_2a L_1 + c\beta m_2a Y_2a - m_1 c L_1] \varepsilon_1 \] (3.120)
\[-g\gamma [m_2a + m_1 c] \varepsilon_1 \]
\+[c\beta m_2a Y_2a] \ddot{w}_1 + [m_2a + m_1 c] \ddot{w}_1(L_1) \]
\[ + c\beta \int_0^{L_2} \rho_2^* z dz \ddot{w}_1(L_1) - c\beta \int_0^{L_2} \rho_2^* \ddot{w}_2 dz \]

\[ EI_2 w_2''(t, L_2) = m_2b[L_2 + L_1 c \beta + y_1'(L_1)] \varepsilon_1 \] (3.121)
\[ + m_2b \gamma c \varepsilon_1 + m_2b L_2 \ddot{c}_2 + m_2b \gamma c \ddot{\varepsilon}_2 \]
\[ - m_2b L_2 \ddot{w}_1'(L_1) + m_2b \gamma c \ddot{w}_1'(L_1) \]
\[ - m_2b c \beta \ddot{w}_1(L_1) - m_2b \ddot{w}_2(L_2) \]

\[ w_1(t, 0) = w_1'(t, 0) = w_2(t, 0) = w_2'(t, 0) = w_2''(t, L_2) = 0 \] (3.122)
where

\[ c_\alpha = \cos \theta_{1o} \quad (3.123) \]
\[ s_\alpha = \sin \theta_{1o} \quad (3.124) \]
\[ c_\beta = \cos(\theta_{2o} + \gamma_{1o}(L_1)) \quad (3.125) \]
\[ s_\beta = \sin(\theta_{2o} + \gamma_{1o}(L_1)) \quad (3.126) \]
\[ s_\gamma = \cos(\theta_{1o} + \theta_{2o} + \gamma_{1o}(L_1)) \quad (3.127) \]
\[ s_\gamma = \sin(\theta_{1o} + \theta_{2o} + \gamma_{1o}(L_1)) \quad (3.128) \]
\[ I_{1x} = \int_0^{L_1} \rho_{1z}^2 dz \quad (3.129) \]
\[ I_{2x} = \int_0^{L_2} \rho_{2z}^2 dz \quad (3.130) \]
\[ m_{1a} = \int_0^{L_1} \rho_{1z}^* dz \quad (3.131) \]
\[ m_{2a} = \int_0^{L_2} \rho_{2z}^* dz \quad (3.132) \]
\[ Y_{1a} = \int_0^{L_1} \rho_{1z}^* dz/m_{1a} \quad (3.133) \]
\[ Y_{2a} = \int_0^{L_2} \rho_{2z}^* dz/m_{2a} \quad (3.134) \]

Observing the PDEs, one can assume that the solution form of \( w_1 \) and \( w_2 \) in the Laplace transformed domain is

\[ \bar{w}_1(z) = e^{\beta_1z}(A_1 \cos \beta_1 z + B_1 \sin \beta_1 z) \quad (3.135) \]
\[ + e^{-\beta_1z}(C_1 \cos \beta_1 z + D_1 \sin \beta_1 z) \]
\[ + \left( \frac{g \sin \theta_1 z}{s^2 - z} \right) \bar{v}_1 \]
\[ \bar{w}_2(z) = e^{\beta_2z}(A_2 \cos \beta_2 z + B_2 \sin \beta_2 z) \quad (3.136) \]
\[ + e^{-\beta_2z}(C_2 \cos \beta_2 z + D_2 \sin \beta_2 z) \]
By substituting $\bar{w}_1$ and $\bar{w}_2$ into the boundary conditions, the eight coefficients in the expressions of $\bar{w}_1$ and $\bar{w}_2$ are solved in terms of $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$.

\[
\begin{align*}
\bar{w}_1 &= D_{11}\bar{\varepsilon}_1 + D_{12}\bar{\varepsilon}_2 \\
\bar{w}_2 &= D_{21}\bar{\varepsilon}_1 + D_{22}\bar{\varepsilon}_2
\end{align*}
\] (3.139) (3.140)

The open-loop transfer functions shown in the next two equations are found by substituting the relationship in between $\bar{w}_1$, $\bar{w}_2$, $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ into the Laplace transformed Equations (3.118) and (3.119).

\[
\begin{align*}
\bar{\varepsilon}_1 &= G_{11}\bar{u}_1 + G_{12}\bar{u}_2 \\
\bar{\varepsilon}_2 &= G_{21}\bar{u}_1 + G_{22}\bar{u}_2
\end{align*}
\] (3.141) (3.142)

The closed-loop PID control system is formed by placing a controller at the base of each link. The PID controllers have the form of

\[
T_{q1} = K_{1p}(\theta_{1d} - \theta_1) + K_{1u}(\dot{\theta}_{1d} - \dot{\theta}_1)
\] (3.143)
Using the open-loop transfer functions and the control gains, one can derive the PID control closed-loop transfer functions. To illustrate the relationship between the control and joint angular response, a block diagram relating the system variables is given. From the the block diagram shown in Figure 3.3, we write the following two equations.

\[ T_{q2} = K_{2p}(\theta_{2d} - \theta_2) + K_{2v}(\dot{\theta}_{2d} - \dot{\theta}_2) \]
\[ + K_{2i}(\int^t_0 (\theta_{2d} - \theta_2)dr + \psi_1e) \]  

\[ (3.144) \]

\[ \ddot{\theta}_1 = \ddot{\psi}_1 + \frac{K_{1u}}{s} + sK_{1i}G_{11}(\ddot{\theta}_{1d} - \ddot{\theta}_1) \]
\[ + \frac{K_{2u}}{s} + sK_{2i}G_{12}(\ddot{\theta}_{2d} - \ddot{\theta}_2) \]  

\[ (3.145) \]

\[ \ddot{\theta}_2 = \ddot{\psi}_2 + \frac{K_{1u}}{s} + sK_{1i}G_{21}(\ddot{\theta}_{1d} - \ddot{\theta}_1) \]
\[ + \frac{K_{2u}}{s} + sK_{2i}G_{22}(\ddot{\theta}_{2d} - \ddot{\theta}_2) \]  

\[ (3.146) \]

The closed-loop transfer functions are obtained by rearranging the two equations above to the final form.

\[ \ddot{\theta}_1 = C_{11}\ddot{\theta}_{1d} + C_{12}\ddot{\theta}_{2d} \]  

\[ (3.147) \]

\[ \ddot{\theta}_2 = C_{21}\ddot{\theta}_{1d} + C_{22}\ddot{\theta}_{2d} \]  

\[ (3.148) \]

Both the expressions for the \( G_s \) and \( C_s \) are very lengthy, and they are derived partially using Macsyma [16]. Due to the complexity of these expressions, we will not write them explicitly here.
Figure 3.3: Block Diagram of Two-Link Flexible Arm
3.4 The Ritz Approximation of Two-Link Flexible Arm

Similar procedures as those we use for the one-link arm are employed to approximate the elastic deflection of the two-link flexible arm. The distributed deflections of the two-link flexible arms are approximated by

\[ y_1(z,t) = \sum_{i=1}^{n_1} \phi_{1i}(z)q_{1i}(t) = \{\phi_1\}^T\{q_1\} \quad (3.149) \]

\[ y_2(z,t) = \sum_{i=1}^{n_2} \phi_{2i}(z)q_{2i}(t) = \{\phi_2\}^T\{q_2\} \quad (3.150) \]

where \{\phi_1\} and \{\phi_2\} are two sets of admissible functions. These functions need not be chosen from the same class of functions. In other words, one can use polynomials for link one while using another type of function for link two. However, in order for the approximated system to reproduce behaviors of the original system, the comparison functions must satisfy the three criteria previously stated. Based on the approximated deflections in Equation (3.149), we write

\[ y'_1(z,t) = \sum_{i=1}^{n_1} \phi'_{1i}(z)q_{1i}(t) = \{\phi'_1\}^T\{q_1\} \quad (3.151) \]

\[ y'_2(z,t) = \sum_{i=1}^{n_2} \phi'_{2i}(z)q_{2i}(t) = \{\phi'_2\}^T\{q_2\} \quad (3.152) \]

\[ \dot{y}_1(z,t) = \sum_{i=1}^{n_1} \phi_{1i}(z)\dot{q}_{1i}(t) = \{\phi_1\}^T\{\dot{q}_1\} \quad (3.153) \]

\[ \dot{y}_2(z,t) = \sum_{i=1}^{n_2} \phi_{2i}(z)\dot{q}_{2i}(t) = \{\phi_2\}^T\{\dot{q}_2\} \quad (3.154) \]

These reduced-order approximations are substituted for the distributed deflections in the kinetic and potential energy shown previously to give

\[ V_1 = \frac{1}{2}\{q_1\}^T[K^{-1}]\{q_1\} + m_1aY_{1a}g\sin \theta_1 + \cos \theta_1 \{q_1\}\{B^1\}^T \quad (3.155) \]
\[ V_2 = \frac{1}{2} \{ q_2 \}^T \{ K^2 \} \{ q_2 \} + m_{2a} Y_{2a} g \sin \theta_1 \tag{3.156} \]
\[ + m_{2a} g \cos \theta_1 \{ q_1 \}^T \{ \phi_1 (L_1) \} \]
\[ + m_{2a} Y_{2a} g \sin (\theta_1 + \theta_2 + \{ \phi'_1 (L_1) \})^T \{ q_1 \} \]
\[ + \cos (\theta_1 + \theta_2 + \{ \phi'_1 (L_1) \})^T \{ q_2 \}^T \{ B^2 \} \]
\[ T_1 = \frac{1}{2} (I_{1b} + I_{1x}) \dot{\theta}_1^2 + \frac{1}{2} \{ \dot{q}_1 \}^T \{ M^1 \} \{ \dot{q}_1 \} + \dot{\theta}_1 \{ \dot{q}_1 \}^T \{ A^1 \} \tag{3.157} \]
\[ T_2 = \frac{1}{2} (I_{2b} + I_{2x} + m_{2b} L_1^2 \]
\[ + 2 m_{2a} \cos (\theta_2 + \{ \phi'_1 (L_1) \})^T \{ q_1 \} \} \dot{\theta}_1^2 \]
\[ + \frac{1}{2} \{ \dot{q}_2 \}^T \{ M^2 \} \{ \dot{q}_1 \} + \dot{\theta}_2 \{ \dot{q}_2 \}^T \{ A^2 \} + \dot{\theta}_1 \{ \dot{q}_2 \}^T \{ A^2 + A^3 \cos (\theta_2 + \{ \phi'_1 (L_1) \})^T \{ q_1 \} \} \]
\[ + \dot{\theta}_2 \{ \dot{q}_1 \}^T \{ \{ \phi'_1 (L_1) \} \{ I_{2b} + I_{2x} \}
\[ + m_{2a} Y_{2a} \{ \phi_1 (L_1) \} \cos (\theta_2 + \{ \phi'_1 (L_1) \})^T \{ q_1 \} \} \]
\[ \} \{ A^2 \}^T \]
\[ + \frac{1}{L_1} \{ \phi_1 (L_1) \}^T \{ A^3 \} \cos (\theta_2 + \{ \phi'_1 (L_1) \})^T \{ q_1 \} \} \{ \dot{q}_2 \} \]

where the elements of the \{ A^1 \}, \{ A^2 \}, \{ A^3 \}, \{ A^4 \}, \{ B^1 \}, \{ B^2 \}, \{ M^1 \}, \{ M^2 \}, \{ M^3 \} are

\[ A_i^1 = \int_0^{L_1} \rho_1 \hat{z} \phi_1 i dz \tag{3.159} \]
\[ A_i^2 = \int_0^{L_2} \rho_2 \hat{z} \phi_2 i dz \tag{3.160} \]
\[ B_i^1 = \int_0^{L_1} \rho_1 \hat{z} \phi_1 i dz \tag{3.161} \]
\[ B_i^2 = \int_0^L \rho_{2g} \phi_2 i dz \]  \hspace{1cm} (3.162)

\[ A_i^3 = L_1 \int_0^L \rho_{2p} \phi_2 i dz \]  \hspace{1cm} (3.163)

\[ A_i^4 = [I_{2b} + \int_0^L \rho_{2p} z^2 dz] \]  \hspace{1cm} (3.164)

\[ + L_1 \cos(\theta_2 + \{\phi_1'(L_1)\}^T \{q_1\}) \int_0^L \rho_{2p}^2 z dz \phi_1 i(L_1) \]

\[ + [m_{2b} L_1 + \cos(\theta_2 + \{\phi_1'(L_1)\}^T \{q_1\}) \int_0^L \rho_{2p}^2 z dz] \phi_1 i(L_1) \]

\[ M_{i,j}^1 = \int_0^L \rho_1^* \phi_{i1} \phi_{1j} dz \]  \hspace{1cm} (3.165)

\[ M_{i,j}^2 = \int_0^L \rho_{2p}^* \phi_{2i} \phi_{2j} dz \]  \hspace{1cm} (3.166)

\[ M_{i,j}^3 = m_{2b} [\phi_{1i}(L_1) \phi_{1j}(L_1) + \phi_{1i}(L_1) \phi_{1j}(L_1)] \]  \hspace{1cm} (3.167)

\[ + [I_{2b} + \int_0^L \rho_{2p}^* z^2 dz + \cos(\theta_2 + \{\phi_1'(L_1)\}^T \{q_1\}) \int_0^L \rho_{2p}^* z dz] \]

\[ \phi_{1i}(L_1) \phi_{1j}(L_1) + \phi_{1i}(L_1) \phi_{1j}(L_1) \]

\[ K_{i,j}^1 = \int_0^L E I_1 \phi_{1i}'' \phi_{1j}'' dz \]  \hspace{1cm} (3.168)

\[ K_{i,j}^2 = \int_0^L E I_2 \phi_{2i}'' \phi_{2j}'' dz \]  \hspace{1cm} (3.169)

For convenience, we define the collection the all the variables of the system as

\[ \{\psi\} = \left\{ \begin{array}{c}
\theta_1 \\
\theta_2 \\
\{q_1\} \\
\{q_2\}
\end{array} \right\} \]  \hspace{1cm} (3.170)

and by using the Lagrange's equations, the system ODEs can be written as

\[ [M]{\ddot{\psi}} + [M]{\dot{\psi}} - \frac{1}{2} \frac{\partial \{\dot{\psi}\}^T [M]\{\dot{\psi}\}}{\partial \{\psi\}} + [K]{\psi} + \frac{\partial V_{eq}}{\partial \{\psi\}} = \{T_q\} \]  \hspace{1cm} (3.171)
Noting that
\[ \frac{\partial((\dot{\psi})^T[M][\dot{\psi}])}{\partial{\psi}} = [\dot{M}][\dot{\psi}] \]  

(3.172)

Hence the equations of motion can finally be written as
\[ [M][\ddot{\psi}] + \frac{1}{2}[\dot{M}][\dot{\psi}] + [K][\psi] + \frac{\partial V_g}{\partial{\psi}} = \{T_q\} \]  

(3.173)

Both mass and stiffness matrices are symmetric. And, the order of both matrices are \((1 + n_1 + n_2)\). The elements of mass matrix, \(M_{ij}\), are formed by superposing the contributions from link one and link two. For instance, the \(M_{11}\) is the sum of \(M_{11}^1\) from link one and \(M_{11}^2\) from link two,

\[ M_{11} = M_{11}^1 + M_{11}^2 \]  

(3.174)

where

\[ M_{11}^1 = I_1b + I_1x \]  

(3.175)

\[ M_{11}^2 = I_2b + I_2x \]  

(3.176)

\[ + L_1(m_2bL_2 + m_2aY_2a\cos(\theta_2 + \{\phi'(L_1)\}^T\{q_1\})) \]

The mass matrix is usually dense regardless of what type of basis functions are chosen. The global stiffness matrix, \([K]\) is composed of the stiffness matrices of the flexible members. For the two-link arm, it has the form as shown below

\[ [K] = \begin{bmatrix}
[0]_{2 \times 2} & [0]_{2 \times n_1} & [0]_{2 \times n_2} \\
[0]_{n_1 \times 2} & [K^1]_{n_1 \times n_1} & [0]_{n_1 \times n_2} \\
[0]_{n_2 \times 2} & [0]_{n_2 \times n_1} & [K^2]_{n_2 \times n_2}
\end{bmatrix} \]  

(3.177)

Since the stiffness matrix \([K]\) composed of only the stiffness matrix of the individual links, the characteristics of \([K]\) are determined by its submatrices. If the submatrices...
are diagonal, \([K]\) is diagonal. The joint torques and generalized forces due to the 
gravity are found by taking the partial derivative of the gravitational potential en-
dergy with respect to the augmented system variables, \(\{\psi\}\). The resulting torques or 
generalized forces are

\[
\frac{\partial V_g}{\partial \theta_1} = \frac{\partial V_g}{\partial \theta_2} + (m_1aY_1a + m_2aL_1)g\cos \theta_1 \\
- \sin \theta_1 \{q_1\}^T \{B_1\} - m_2ag\sin \theta_1 \{q_1\}^T \{\phi_1(L_1)\}
\]

\[
\frac{\partial V_g}{\partial \theta_2} = m_2aY_2ag\cos(\theta_1 + \theta_2 + \{\phi_1(L_1)\})^T \{q_1\}
- \sin(\theta_1 + \theta_2 + \{\phi_1(L_1)\})^T \{\phi_1(L_1)\}\{q_2\}^T \{B_2\}
\]

\[
\frac{\partial V_g}{\partial \{q_1\}} = \cos \theta_1 \{B_1\}^T + \cos \theta_1 m_2a \{\phi_1(L_1)\}
+ m_2aY_2ag\cos(\theta_1 + \theta_2 + \{\phi_1(L_1)\})^T \{q_1\}\{\phi_1(L_1)\}
- \sin(\theta_1 + \theta_2 + \{\phi_1(L_1)\})^T \{\phi_1(L_1)\}\{q_2\}^T \{B_2\}
\]

\[
\frac{\partial V_g}{\partial \{q_2\}} = \cos(\theta_1 + \theta_2 + \{\phi_1(L_1)\})^T \{q_1\}\{B_2\}
\]

Once all the matrices are available, the simulation of the two-link flexible arm can 
be performed by forward integrating the nonlinear ODEs. Control torques or/and 
friction can be introduced into the analysis through the loading array, \(\{T_q\}\). Since 
the mass matrix is not a diagonal matrix regardless of the type of comparison function 
chosen, the system differential equations are always coupled. Again, when the linear 
system analysis is desired for the design purpose, the linearized ODEs can be shown 
to be

\[
[M]\{\delta \psi\} + [K]\{\delta \psi\} + \frac{\partial^2 V_g}{\partial \{\psi\}^2} \{\delta \psi\} = \{u\}
\]
where

\[
\{\delta \psi\} = \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\{\xi_1\} \\
\{\xi_2\}
\end{bmatrix}
\] (3.183)

If we let

\[
\{\delta \psi\} = \{Z\} e^{st}
\] (3.184)

and substitute this expression into the linear differential equations, an eigenvalue problem is formed for evaluating the open-loop system characteristics.

\[
([K^*] + s^2[M^*])\{Z\} e^{st} = \{0\}
\] (3.185)

where the elements of \([K^*]\) and \([M^*]\) matrices are in the form of

\[
M_{ij}^* = M_{ij}|_{\theta_o,\psi_o}
\] (3.186)

\[
K_{ij}^* = (K_{ij} + \frac{\partial^2 V_g}{\partial \psi_i \partial \psi_j})|_{\theta_o,\psi_o}
\] (3.187)

If closed-loop PID control and damping are considered, the stiffness and damping matrices shown below are used in the expanded eigenvalue problem.

\[
K_{ij}^{**} = (K_{ij} + \frac{\partial^2 V_g}{\partial \psi_i \partial \psi_j} - \frac{\partial u_i}{\partial \psi_j})|_{\theta_o,\psi_o}
\] (3.188)

\[
C_{ij}^{**} = (C_{ij} - \frac{\partial u_i}{\partial \psi_j})|_{\theta_o,\psi_o}
\] (3.189)
where the damping matrix, $[C]$, represents material damping of the flexible links. Similar to the case of the one-link arm, an additional state variable in Equation (3.80) is defined for the integral control for each link and their linearized equations are included in the eigenvalue analysis.
4 CASE STUDIES

In this chapter, the influence of link flexibility and gravitation on the stability characteristics and behavior of the flexible systems are investigated. Specifically, we will examine the following:

1. Effects of the choice of comparison functions on the accuracy of the static deflection and dynamic response of the flexible arms.

2. Effect of link flexibility on the stability characteristics of the flexible arms.

3. Effect of gravity on the stability characteristics of the flexible arms.

4. Importance of the stability analysis in designing the PID controller at the joints.

For both the static and dynamic analysis, the position of the end-tip mass is used as a measurement of accuracy. Among many methods in determining the quality of the results we obtain, one way to access the degree of accuracy of the approximation is to examine the frequency content of the flexible systems. Two approaches are used here to determine the frequency content. In general, the system frequencies are found using exact transfer function first. The results from the approximation schemes are next evaluated and compared with those derived from the exact transfer function approach. This comparison shows how well the system is approximated in the context of frequency content. As we progress, it is observed that the availability of
exact solutions is somewhat limited but nevertheless extremely useful. The stability characteristics of the traditional rigid model and the flexible model subject to an identical control are shown to be substantially different. It is noticed that even for the same flexible arm model, differences in the static and dynamic solutions are also observed when the basis functions are different.

4.1 Comparison Functions

Four sets of comparison functions of two different types are used in approximating the deformation of the flexible links including polynomials and free vibrational mode shapes of cantilever beams. Since the Ritz method requires that all the basis functions satisfy the essential boundary conditions of the flexible member, we choose the polynomial comparison functions

$$\phi_i(z) = \left(\frac{z}{L}\right)^{i+1}, \quad i = 1, n \quad (4.1)$$

Clearly, the geometric boundary conditions of the flexible links are satisfied since the polynomials in Equation (4.1) always lead to $\phi_i(0) = \phi_i'(0) = 0$. Additionally, three different sets of shape functions are generated according to eigenfunctions of the cantilever beams shown in Figures 4.1 to 4.3. These shape functions are determined by using the roots in $k$ of the following characteristic equation along with Equation (4.6).

$$1 + \cos(kL)\cosh(kL) + (kL)^4 \alpha \beta [\cos(kL)\cosh(kL) - 1] \quad (4.2)$$

$$+ (kL)^3 \beta [\sin(kL)\cosh(kL) + \cos(kL)\sinh(kL)]$$

$$+ kL\alpha [\sin(kL)\cosh(kL) - \cos(kL)\sinh(kL)] = 0$$
Figure 4.1: Free Vibrational Modes with One End Fixed and the Other End Free

\[ \rho = 0.15 \text{Kg/m}, \quad EI = 45 \text{N/m}^2, \quad L = 1 \text{m} \]

\( M_0 = 0.0 \text{Kg} \]

\( I_0 = 0.0 \text{Kg} \cdot \text{m}^2 \)

---

Figure 4.2: Free Vibrational Cantilever Beam with One End Fixed and A Mass, 0.15 Kg, Attached to the Other End

\[ \rho = 0.15 \text{Kg/m}, \quad EI = 45 \text{N/m}^2, \quad L = 1 \text{m} \]

\( M_0 = 0.15 \text{Kg} \]

\( I_0 = 0.0 \text{Kg} \cdot \text{m}^2 \)

---

Figure 4.3: Free Vibrational Cantilever Beam with One End Fixed and A Mass, 0.25 Kg, Attached to the Other End

\[ \rho = 0.15 \text{Kg/m}, \quad EI = 45 \text{N/m}^2, \quad L = 1 \text{m} \]

\( M_0 = 0.25 \text{Kg} \]

\( I_0 = 0.0 \text{Kg} \cdot \text{m}^2 \)
where

\[ \alpha = \frac{m_e}{\rho L} \]  
\[ \beta = \frac{I_e}{\rho L^3} \]  
\[ \omega = k^2 \left( \frac{EI}{\rho} \right)^{\frac{1}{2}} \]  

\[ \phi_i = \cosh(k_i x) + \cos(k_i x) - \frac{\cosh(k_i L) - \cos(k_i L)}{\sinh(k_i L) - \sin(k_i L)} [\sinh(k_i x) + \sin(k_i x)] \]  

In the order that they appear, the three sets of eigenfunctions corresponding to the cantilever beams in Figures 4.1, 4.2 and 4.3 are identified, respectively, as Set A, Set B and Set C. Once these eigenfunctions are obtained, they are substituted into the ODEs for the static, dynamic and frequency domain analyses. Since the end-tip position of a robot arm is a common maneuvering objective, it will be used as the measurement of the accuracy of the static and dynamic solutions that individual sets of functions can achieve. For the static analysis, the exact solution of the end-tip displacement can be obtained by solving, for instance, the ODEs in Equations (3.95)-(3.104). Hence, it is used as the basis for monitoring the error due to the approximation. We define the percent error of the end-tip displacement as

\[ \text{\% Error} = 100 \left| \frac{\Delta \text{Cantilever mode}}{\Delta \text{exact}} \right| \]  

For the dynamic analysis, no exact solution is available. However, the solution obtained using the polynomials, in theory, will converge to the exact solution. Therefore, it is used as the basis for monitoring the error due to the use of eigenfunctions as the shape functions. For this case, the percent error is defined as

\[ \text{\% Error} = 100 \left| \frac{\Delta \text{Cantilever mode}}{\Delta \text{polynomial}} \right| \]
Table 4.1: Two-Link Arm Properties

<table>
<thead>
<tr>
<th>No of Link</th>
<th>$L$</th>
<th>$\rho$</th>
<th>$EI$</th>
<th>$m_b$</th>
<th>$m_e$</th>
<th>$I_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link One</td>
<td>1 m</td>
<td>0.3 $Kg/m$</td>
<td>45 $N/m^2$</td>
<td>0.00 $Kg$</td>
<td>0.15 $Kg$</td>
<td>0.2 $Kgm^2$</td>
</tr>
<tr>
<td>Link Two</td>
<td>1 m</td>
<td>0.1 $Kg/m$</td>
<td>5 $N/m^2$</td>
<td>0.00 $Kg$</td>
<td>0.10 $Kg$</td>
<td>0.067 $Kgm^2$</td>
</tr>
</tbody>
</table>

Table 4.2: Frequencies of the Flexible Links with Cantilever End Condition

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link One</td>
<td>24.69</td>
<td>207.00</td>
<td>663.20</td>
<td>1298.94</td>
<td>2206.05</td>
<td>3354.72</td>
</tr>
<tr>
<td>Link Two</td>
<td>11.01</td>
<td>114.91</td>
<td>359.89</td>
<td>743.86</td>
<td>1267.36</td>
<td>2930.40</td>
</tr>
</tbody>
</table>

The two-link flexible arm with the properties listed in Table 4.1 is utilized as the physical model. The frequencies of the flexible links, in radians per second, with cantilever end condition are listed in Table 4.2. For convenience, the flexible arm is maneuvered in the $xy$ plane of the ground frame. Hence the deflection at the end tip of link two, $\Delta$, is calculated according to

$$\Delta = [(X^R - X^D)^2 + (Y^R - Y^D)^2]^{1/2}$$

(4.9)

where $X^R$ and $Y^R$ are the coordinates, in the ground frame, of the end-tip mass of the rigid arm and $X^D$ and $Y^D$ are the coordinates, in the ground frame, of the end-tip mass of the flexible arm.
4.2 Static Equilibrium Solution

The static configuration of the two-link flexible arm is examined because the static equilibrium position generally is the starting and terminal configuration of a maneuver. Due to the assumptions of uniform cross-section and density, the two-link arm possesses an exact static solution. All four sets of comparison functions are tested and used to obtain the end-tip position of the two-link arm. We notice that, during the development of the eigenfunctions, a zero moment condition at one end of the flexible link is imposed. However, when this zero-moment end is connected to the base of another link to construct the two-link robot arm, a static torque is required to hold both links together. It is obvious that none of the eigenfunctions is capable of developing such a static torque. Mathematically, this difficulty arises because the eigenfunctions can never satisfy the natural boundary conditions of the flexible arm. For instance, if we substitute any one or any combination of the three sets of eigenfunctions into the moment boundary condition of the two-link arm, Equation (3.120), the equality is never satisfied. Since the eigenfunctions always violate the moment natural boundary condition, they do not form a complete set. Theoretically, when the functions chosen from a incomplete set are used for approximation, convergence of the approximation solution to the exact solution is not assured [19]. However, the existence and error of the solution due to this modeling deficiency have to be determined case by case [25]. Let's examine the end-tip position of link two for the flexible arm at $\theta_1 = 0$ and $\theta_2 = 0$. The exact value of the magnitude of the deflection is equal to 0.2207 meters. For the approximation methods, various magnitudes are obtained and shown in Table 4.3. In Table 4.3, we observe that the end-tip displacement of link two converges to the exact solution only for the polynomial functions. For the
Table 4.3: Static End Tip Displacement of Two-Link Flexible Arm

<table>
<thead>
<tr>
<th>No of functions</th>
<th>Type of functions</th>
<th>$\Delta_{Poly}$ (Error%)</th>
<th>$\Delta_{SetA}$ (Error%)</th>
<th>$\Delta_{SetB}$ (Error%)</th>
<th>$\Delta_{SetC}$ (Error%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0.1873 (15.13)</td>
<td>0.2008 (9.030)</td>
<td>0.2111 (4.332)</td>
<td>0.2091 (5.273)</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.2207 (0.003)</td>
<td>0.2126 (3.664)</td>
<td>0.2164 (1.927)</td>
<td>0.2159 (2.175)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.2207 (0.000)</td>
<td>0.2157 (2.253)</td>
<td>0.2182 (1.160)</td>
<td>0.2179 (1.267)</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.2207 (0.000)</td>
<td>0.2171 (1.637)</td>
<td>0.2189 (0.840)</td>
<td>0.2187 (0.900)</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.2207 (0)</td>
<td>0.2179 (1.277)</td>
<td>0.2193 (0.651)</td>
<td>0.2192 (0.690)</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0.2207 (0)</td>
<td>0.2184 (1.049)</td>
<td>0.2195 (0.534)</td>
<td>0.2195 (0.561)</td>
</tr>
</tbody>
</table>

rest of the comparison functions, the end displacement does not converge to the exact solution as the number of basis functions increases. If we convert the displacement magnitude into the percent error as defined in Equation (4.8) and shown in the Table 4.3, we find the error ranges from 0.5 % to 1.0 %. The violation of both shear and moment boundary conditions may account for the higher percent error of the results obtained using Set A.

The identical static analysis can be performed for the one-link flexible arm. We may obtain a one-link flexible arm by removing link two of the two-link arm. The
same types of comparison functions used for the two-link arm are utilized. For an incline of zero, the end tip deflections measured from the undeformed configuration are shown in Table 4.4. The closed-form solution for this configuration is $1.9075 \times 10^{-2}$ meter. Only the eigenfunctions from Set A do not satisfy the shear natural boundary condition of this one-link arm model, hence, all the comparison functions, except for the Set A, would converge to the exact solution. Practically, when the number of comparison functions is six, the percent error of the end-tip displacement for all the approximate systems are in the order of $10^{-4}$ or less. If we further increase the number of comparison functions, the predicted end-tip position for all the eigenfunctions converge to the exact solution or a value very closed to the exact solution. At this point, all the solutions are satisfactory.

4.3 Frequency Domain Analysis

When motion of a deformable body is approximated with discretizing functions, the frequency content of the approximation can be used as a criterion to test the accuracy of the approximation. One of the reasons for choosing the frequency content as the measure of a successful approximation is that the mode shapes of an approximate system are less accurate than the approximated frequency values. In other words, if a frequency of the approximated system has significant error compared with the exact value, the mode shape corresponding to that frequency is also inaccurate. This implies that when a set of comparison functions fails to estimate the frequency components of a flexible system accurately above a particular frequency value, for instance, $\omega_n$, then the response of the flexible system is likely incorrect if any eigenvector corresponding to the frequencies which are higher than $\omega_n$ participates significantly in the motion.
Table 4.4: Static End Tip Displacement of One-Link Flexible Arm

<table>
<thead>
<tr>
<th>No of functions</th>
<th></th>
<th>Type of functions</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \Delta_{Poly} ) (Error%)</td>
<td>( \Delta_{SetA} ) (Error%)</td>
<td>( \Delta_{SetB} ) (Error%)</td>
<td>( \Delta_{SetC} ) (Error%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>1.8865 ( 10^{-2} ) (1.101 ( 10^{-2} ))</td>
<td>1.9131 ( 10^{-2} ) (2.926 ( 10^{-3} ))</td>
<td>1.9114 ( 10^{-2} ) (2.037 ( 10^{-3} ))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.9075 ( 10^{-2} ) (0.000)</td>
<td>1.9021 ( 10^{-2} ) (2.798 ( 10^{-3} ))</td>
<td>1.9073 ( 10^{-2} ) (1.084 ( 10^{-4} ))</td>
<td>1.9069 ( 10^{-2} ) (3.003 ( 10^{-4} ))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.9075 ( 10^{-2} ) (0.000)</td>
<td>1.9061 ( 10^{-2} ) (7.027 ( 10^{-4} ))</td>
<td>1.9075 ( 10^{-2} ) (1.224 ( 10^{-5} ))</td>
<td>1.9075 ( 10^{-2} ) (1.128 ( 10^{-6} ))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.9075 ( 10^{-2} ) (0)</td>
<td>1.9068 ( 10^{-2} ) (3.483 ( 10^{-4} ))</td>
<td>1.9075 ( 10^{-2} ) (2.407 ( 10^{-6} ))</td>
<td>1.9075 ( 10^{-2} ) (9.444 ( 10^{-6} ))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.9075 ( 10^{-2} ) (0)</td>
<td>1.9071 ( 10^{-2} ) (1.654 ( 10^{-4} ))</td>
<td>1.9075 ( 10^{-2} ) (6.599 ( 10^{-7} ))</td>
<td>1.9075 ( 10^{-2} ) (5.246 ( 10^{-7} ))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.9075 ( 10^{-2} ) (0)</td>
<td>1.9073 ( 10^{-2} ) (1.022 ( 10^{-4} ))</td>
<td>1.9075 ( 10^{-2} ) (2.255 ( 10^{-7} ))</td>
<td>1.9075 ( 10^{-2} ) (1.134 ( 10^{-6} ))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.3.1 Convergence in Frequency Domain

The two-link flexible arm for the static analysis is employed here. The arm is at rest with joint angles $\theta_1 = 0$ and $\theta_2 = 0$. Three comparison functions are used for each link and gravity is omitted. Both exact frequencies which are obtained from the exact transfer function, and approximate solutions which are obtained from eigenvalue analysis, are shown in Table 4.5. The selected cantilever modes are Sets A and B. In addition to the consistent mass approximation, we also generate a lumped-mass model for the two-link arm. This lumped-mass approach is employed in a general-purpose multi-body dynamic simulation package, ADAMS/MODAL. Refer to [26], [27], [28], and [29] for the detailed information regarding the ADAMS/MODAL formulation. The lumped-mass approach can be shown to be very effective in creating the flexible arm model. The previously mentioned 'softening effect' also is eliminated since the lumped-mass model is capable to capture the nonlinearity due to the shortening in the z axis of the beam. In the ADAMS model, each link is modeled by four beam elements. It is obvious that, at the high frequencies, the cantilever modes give the best results despite the fact that they are taken from an incomplete set of functions. The polynomial solutions are improved by raising the order of the approximation functions. If we increase the number of polynomials to six, the original ill-predicted higher frequencies approach the exact values very closely. The improved results are shown in Table 4.6. Improvement on the lumped-mass model may also be achieved by raising the number of rigid bodies used to discretize the continuous links. The improved results shown in Table 4.7 are obtained if the number of rigid bodies for each link is raised to five. Similar results are observed for both one- and two-link, open- and closed-loop systems, regardless of the set points for $\theta_1$ and $\theta_2$. 
Figure 4.4: Two-link Flexible Arm ADAMS Model
Table 4.5: Poles of Open-Loop Two-Link Flexible Arm

<table>
<thead>
<tr>
<th>Mode</th>
<th>Exact</th>
<th>Set B</th>
<th>Polynomial</th>
<th>Set A</th>
<th>Lumped Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2</td>
<td>(0, 0.00)</td>
<td>(0, 0.00)</td>
<td>(0, 0.00)</td>
<td>(0, 0.00)</td>
<td>(0, 0.00)</td>
</tr>
<tr>
<td>3</td>
<td>(0, 19.15)</td>
<td>(0, 19.15)</td>
<td>(0, 19.15)</td>
<td>(0, 19.15)</td>
<td>(0, 19.05)</td>
</tr>
<tr>
<td>4</td>
<td>(0, 33.47)</td>
<td>(0, 33.47)</td>
<td>(0, 33.47)</td>
<td>(0, 33.45)</td>
<td>(0, 32.48)</td>
</tr>
<tr>
<td>5</td>
<td>(0, 121.79)</td>
<td>(0, 121.79)</td>
<td>(0, 121.79)</td>
<td>(0, 122.08)</td>
<td>(0, 115.78)</td>
</tr>
<tr>
<td>6</td>
<td>(0, 207.36)</td>
<td>(0, 207.39)</td>
<td>(0, 208.08)</td>
<td>(0, 208.31)</td>
<td>(0, 160.49)</td>
</tr>
<tr>
<td>7</td>
<td>(0, 367.62)</td>
<td>(0, 367.67)</td>
<td>(0, 556.56)</td>
<td>(0, 376.76)</td>
<td>(0, 318.65)</td>
</tr>
<tr>
<td>8</td>
<td>(0, 631.57)</td>
<td>(0, 632.62)</td>
<td>(0, 973.63)</td>
<td>(0, 645.64)</td>
<td>(0, 327.09)</td>
</tr>
</tbody>
</table>

Table 4.6: Improved Approximation Open-loop Frequencies of Two-Link Flexible Arm with Polynomial Functions as Basis Functions

<table>
<thead>
<tr>
<th>Mode</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>(0, 207.36)</td>
<td>(0, 367.62)</td>
<td>(0, 631.57)</td>
</tr>
<tr>
<td>Polynomials</td>
<td>(0, 207.36)</td>
<td>(0, 367.70)</td>
<td>(0, 631.81)</td>
</tr>
</tbody>
</table>
Table 4.7: Improved Approximation Open-Loop Frequencies of Two-Link Flexible Arm using Lumped-Mass Method

<table>
<thead>
<tr>
<th>Mode</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>(0, 207.36)</td>
<td>(0, 367.62)</td>
<td>(0, 631.57)</td>
</tr>
<tr>
<td>Lumped Mass</td>
<td>(0, 203.69)</td>
<td>(0, 345.20)</td>
<td>(0, 595.20)</td>
</tr>
</tbody>
</table>

4.3.2 Linear Stability of Two-Link Flexible Arm

Two root locus plots are given to demonstrate the effects of flexibility and control on the two-link flexible-arm PID control systems. The root locus plot in Figure 4.5 is generated using the following control gains: $K_{1v} = 3$, $K_{1i} = 1$, $K_{2p} = 5$, $K_{2u} = 1$ and $K_{2i} = 1$; and varying $K_{1p}$, the proportional gain for link one, from 0 to infinity at three angular positions of $(\theta_1, \theta_2)$: $(−90^\circ, 30^\circ)$, $(0^\circ, 30^\circ)$ and $(90^\circ, 30^\circ)$. Parallel to the analysis of the flexible arm, the same procedures are performed on the rigid arm using the same inertial properties. Examining Figure 4.6, the influences of angular position about which linearization occurs on the system characteristics are clearly important for the rigid case as well. A stable control system at $(−90^\circ, 30^\circ)$ may become unstable at $(90^\circ, 30^\circ)$. Comparing the results from the flexible and rigid models, the importance of flexibility is evident. The marginal stability, for example, which occurs in a certain region corresponding to a high position gain is missing for the rigid model. For the models we tested, all the higher poles remain to the left of the imaginary axis once the lowest poles associated with the rigid-body motion are stabilized.
Figure 4.5: Two-Link Flexible Arm PID Control Root Locus

Figure 4.6: Two-Link Rigid Arm PID Control Root Locus
4.3.3 Linear Stability of One-Link Flexible Arm

Similar to the stability analysis for the two-link flexible arm, the one-link arm with the physical properties shown in Table 4.8 is tested. With the properties as listed, the transverse deflection of end-tip mass of the arm is about 10% of its length. For the open-loop analysis, the arm is held at the equilibrium position for the joint angle of 45 degrees from the horizontal, and no feedback torque. The eigenfunctions for the cantilever beam as shown in Figure 4.7 and polynomials are chosen as the comparison functions. For the case that six comparison functions are used for both polynomials and eigenfunctions, the resulting poles of the open-loop system are shown in Table 4.9. The predicted frequencies obtained using the eigenfunctions are clearly closer to the exact solutions than are those from polynomials. Similar to the analysis we perform previously, improved results for the polynomials can be obtained by increasing the order of approximation. For instance, if we increase the order of approximation by two, the significant differences of the fifth to seventh poles in Table 4.9 reduce substantially. By comparing the original and improved results in Tables 4.9 and 4.10, we find that the error of the seventh pole drops from the original 230 percent to only 2 percent. It is also observed that the approximated frequencies approach the exact values from the upper bound, which is the consequence of using the Ritz method.

We next demonstrate the influence of structural flexibility as well as the influence of control gains on the local stability characteristics of the one-link arm. Two types of controllers are employed. First, the one-link flexible arm is subjected to position and derivative control at the rigid base. The angle of inclination is again chosen as 45 degrees. The following root locus plots are generated by holding the derivative control gain fixed and letting the position gain vary from zero to infinity. We repeat
Figure 4.7: Free Vibrational Cantilever Beam with One End Fixed and the Other End Free

the same procedure for both of the rigid and flexible arms. By comparing the root locus plots, we observe many obvious differences between the flexible and rigid models. The rigid model is unable to predict any higher poles due to its model limitation and also predicts quite different poles corresponding to the rigid body motion for the high position gains.

In the root locus plots of the flexible arm model, we also find that the higher frequencies approach the imaginary axis when the value of the position gain is very large. Physically, this is consistent with the expectation that a high-gain system usually tends to be more oscillatory. Except for the lowest pole associated with the rigid body motion, the rest of the poles remain on the left half plane of the root locus plot and never cross the imaginary axis. Once the rigid body motion is stabilized, the flexible arm is stable at the equilibrium position. For all the test cases, we always
can achieve a stable system by choosing a sufficiently large position gain to force the rigid body pole to stay on or to the left side of the imaginary axis. For the flexible arm with other physical properties, results similar to those described above can be obtained.

If only a PD controller is used at the rigid base, problems occur during the course of the simulation. In order for the base of the arm to track a reference input and remain at the desired angular position, the position gain of the PD controller is so high that the control system is physically unrealizable. Hence, integral control is also necessary for the one-link arm. Two root locus plots as shown in Figures 4.10 and 4.11 are generated by positioning the arm at angles $\theta_o$ of $-90^\circ$, $0^\circ$ and $90^\circ$ in sequence. The position and integral control gains are both are 1335 while the velocity control gain is allowed to vary. Similar to the observation in the two-link arm, a stable system at an inclined angle of a lower value may become unstable and no higher pole crosses the imaginary axis when the pole for the rigid-body motion is stabilized.

Since the presence of gravity gives rise to nonlinearity for the one-link flexible arm, we will show a root locus plot with the one-link arm under different magnitude of gravitational constant: $0 \text{ g}$, $\frac{1}{2} \text{ g}$ and $\text{ g}$, where $\text{ g}$ denotes the earth's gravitational constant. The arm is held statically at the 15 degrees angle of inclination. PID control is employed with the position gain and integral control gains respectively fixed at 1355 and 407 and the velocity control gain varying from 0 to infinity. It is clear from Figure 4.12, that under the higher magnitude of gravity, the control system requires a higher velocity gain to maintain stable, i.e., a stable flexible arm under a lower gravitational constant might turn unstable as the gravitational constant increases.
Figure 4.8: One-Link Flexible Arm PD Control Root Locus

<table>
<thead>
<tr>
<th>Curve</th>
<th>$K_{pd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>814</td>
</tr>
<tr>
<td>B</td>
<td>847</td>
</tr>
<tr>
<td>C</td>
<td>915</td>
</tr>
<tr>
<td>D</td>
<td>1017</td>
</tr>
</tbody>
</table>

Figure 4.9: One-Link Rigid Arm PD Control Root Locus
Figure 4.10: One-Link Flexible Arm PID Control Root Locus

Figure 4.11: One-Link Rigid Arm PID Control Root Locus
Figure 4.12: The Influence of Gravitation on the Stability of One-Link Flexible Arm
4.4 Nonlinear Dynamic Simulation

The flexible arm is driven by a PID controller at the base of each link. The respective reference inputs to the joints are

\[ \theta_{1d} = \begin{cases} (\theta_{1f} - \theta_{1i})[10\left(\frac{t}{t_{1f}}\right)^3 - 15\left(\frac{t}{t_{1f}}\right)^4 + 6\left(\frac{t}{t_{1f}}\right)^5] + \theta_{1i} & t \leq t_{1f} \\ \theta_{1f} & t > t_{1f} \end{cases} \] (4.10)

\[ \theta_{2d} = \begin{cases} (\theta_{2f} - \theta_{2i})[10\left(\frac{t}{t_{2f}}\right)^3 - 15\left(\frac{t}{t_{2f}}\right)^4 + 6\left(\frac{t}{t_{2f}}\right)^5] + \theta_{2i} & t \leq t_{2f} \\ \theta_{2f} & t > t_{2f} \end{cases} \] (4.11)

where \( \theta_{1i} \) and \( \theta_{1f} \) are the initial and final angular positions of link one respectively. The time \( t_{1f} \) represents the instant at which the reference input of link one reaches \( \theta_{1f} \). The same convention applies to link two. The \( \theta_{2i} \) and \( \theta_{2f} \) are the initial and final angular position of link two, and \( t_{2f} \) is the time at which the reference input of the link two reaches \( \theta_{2f} \). Two sets of control gains in Table 4.11 are tested for the closed-loop system. The set with higher control gains is named Group I, and the one with lower values is Group II. The angular positions about which the differential equations are linearized are \( \theta_1 = 30^\circ \) and \( \theta_2 = 30^\circ \). The poles corresponding to these gains are listed in Table 4.12. If the linearized solution is valid, the flexible arm driven by the control specified by the second set of gains would suffer from the relatively long response time due to the large time constant associated with the lowest frequency of the closed-loop system. In other words, if the control excites the motion which contains the lowest frequencies, the joint angular responses is likely oscillatory and the elapsed time for the flexible arm to approach the desired position will be fairly large. The system controlled by the first controller, however, does not possess the difficulty just described because the largest time constant is about one second.
Two dynamic simulations are performed to validate this. In these analyses, both the $t_1f$ and $t_2f$ are set to one second. The arm is initially at rest with the angular positions $\theta_1 = 0$ and $\theta_2 = 0$. The desired final angular positions are identical with the values used for the linear analysis, i.e. $\theta_1f = 30^\circ$ and $\theta_2f = 30^\circ$. The plots of angular responses are shown in Figures 4.13 and 4.14. It is clear that the second control system approaches the final reference position much slower compared with the first system. This observation is very well predicted by the Laplace domain analysis of the linearized systems. If, in lieu of the polynomials, the cantilever modes are used for approximation, the five-term approximation results have relatively more error, compared to those of the polynomials-in the range of 2 to 10%. Depending on the type of cantilever mode shapes used, the magnitude of the error differs. In general, Sets B and C result in solutions closer to those of the polynomials than does Set A. The rate of convergence is also different depending on the type of comparison function chosen. The end-tip displacement obtained using the four-term and five-term polynomial functions have relative error in the order of $10^{-3}\%$, while the results obtained using four-term and five-term cantilever beam shape functions still differ by 10% or higher.
Figure 4.13: Joint Angular Position of Two-Link Flexible Arm Subjected to High Gain PID Control

Figure 4.14: Joint Angular Position of Two-Link Flexible Arm Subjected to Low Gain PID Control
Figure 4.15: The Percent Difference of the End-Tip Displacement of Two-link Flexible Arm Using Five Polynomial and Set B Functions
Figure 4.16: The Percent Difference of the End-Tip Displacement of Two-link Flexible Arm Using Four and Five Polynomial Functions

Figure 4.17: The Percent Difference of the End-Tip Displacement of Two-link Flexible Arm Using Four and Five Set A Functions
Table 4.8: One-Link Arm Properties for Stability Analysis

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>$\rho_1$</th>
<th>$EI_1$</th>
<th>$m_{1b}$</th>
<th>$m_{1c}$</th>
<th>$I_{1b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9144 m</td>
<td>0.2617 Kg/m</td>
<td>0.8354 N/m²</td>
<td>0 Kg</td>
<td>0 Kg</td>
<td>3.863 Kg - m²</td>
</tr>
</tbody>
</table>

Table 4.9: One-link Flexible Arm Open-Loop Frequencies

<table>
<thead>
<tr>
<th>Mode/Pole</th>
<th>Exact (Rad/Sec)</th>
<th>Set A (Rad/Sec)</th>
<th>Polynomial (Rad/Sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2.08, 0.00)</td>
<td>(2.08, 0.00)</td>
<td>(2.08, 0.00)</td>
</tr>
<tr>
<td>2</td>
<td>(0. , 17.08)</td>
<td>(0. , 17.08)</td>
<td>(0. , 17.08)</td>
</tr>
<tr>
<td>3</td>
<td>(0. , 63.60)</td>
<td>(0. , 63.60)</td>
<td>(0. , 63.60)</td>
</tr>
<tr>
<td>4</td>
<td>(0. , 174.16)</td>
<td>(0. , 174.16)</td>
<td>(0. , 174.21)</td>
</tr>
<tr>
<td>5</td>
<td>(0. , 340.48)</td>
<td>(0. , 340.48)</td>
<td>(0. , 361.60)</td>
</tr>
<tr>
<td>6</td>
<td>(0. , 562.54)</td>
<td>(0. , 562.54)</td>
<td>(0. , 836.77)</td>
</tr>
<tr>
<td>7</td>
<td>(0. , 840.19)</td>
<td>(0. , 840.20)</td>
<td>(0. , 2830.78)</td>
</tr>
</tbody>
</table>

Table 4.10: Improved Approximation of the One-Link Flexible Arm using Polynomial Functions

<table>
<thead>
<tr>
<th>Mode</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>(0, 340.48)</td>
<td>(0, 562.54)</td>
<td>(0, 840.19)</td>
</tr>
<tr>
<td>Polynomials</td>
<td>(0, 339.95)</td>
<td>(0, 562.37)</td>
<td>(0, 853.28)</td>
</tr>
</tbody>
</table>
Table 4.11: Two-Link Flexible Arm PID Control Gains

<table>
<thead>
<tr>
<th>Group</th>
<th>$K_{1p}$</th>
<th>$K_{1v}$</th>
<th>$K_{1i}$</th>
<th>$K_{2p}$</th>
<th>$K_{2v}$</th>
<th>$K_{2i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>50</td>
<td>30</td>
<td>10</td>
<td>50</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>II</td>
<td>10</td>
<td>15</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 4.12: Two-Link Flexible Arm Closed-Loop Poles

<table>
<thead>
<tr>
<th>Mode</th>
<th>Group I</th>
<th>Group II</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2</td>
<td>Real axis</td>
<td>Real axis</td>
</tr>
<tr>
<td>3,4</td>
<td>(-0.968, ± 0.572)</td>
<td>Real axis</td>
</tr>
<tr>
<td>5,6</td>
<td>(-0.224, ± 0.539)</td>
<td>Real axis</td>
</tr>
<tr>
<td>7,8</td>
<td>(-2.244, ± 7.493)</td>
<td>(-1.115, ± 7.238)</td>
</tr>
<tr>
<td>9,10</td>
<td>(-2.119, ± 24.217)</td>
<td>(-1.083, ± 23.767)</td>
</tr>
<tr>
<td>11,12</td>
<td>(-6.193, ± 62.427)</td>
<td>(-4.684, ± 61.256)</td>
</tr>
<tr>
<td>13,14</td>
<td>(-0.021, ± 120.088)</td>
<td>(-0.019, ± 120.084)</td>
</tr>
<tr>
<td>15,16</td>
<td>(-5.836, ± 304.079)</td>
<td>(-4.604, ± 304.025)</td>
</tr>
<tr>
<td>17,18</td>
<td>(-0.058, ± 369.265)</td>
<td>(-0.048, ± 369.265)</td>
</tr>
</tbody>
</table>
In the previous chapter, we have studied the stability characteristics and performance of one- and two-link flexible arms subject to two types of controllers. In particular, we find that the higher poles for the closed-loop systems are critical in most cases. These poles are situated close to the imaginary axis, which implies the high frequency motion of the flexible arm is oscillatory with a constant amplitude. Once the rapid vibrational motion of the arm is activated, it will take the arm a very long time to reach the desired joint coordinates. However, physically, the motion corresponding to the higher modes always tends to damp out very quickly. Hence, a realistic flexible arm model should include a damping mechanism to account for this observation. In this chapter, we will study the effects of a linear viscoelastic damping model, Kelvin material, on the behavior of the flexible arms. It is revealed later that Kelvin damping is a proportional damping and the magnitude of the resulting generalized forces are directly relative to the stiffness of the flexible link. Even though Kelvin damping might not be as important for the time domain behavior of a arm with little vibrational motion, it always contributes to stability. For the flexible arm constructed with Kelvin material, the poles with higher values are located away from the imaginary axis and in the left half plane of the root locus plot. Finally, the stability characteristics of PD controlled flexible arms are re-examined using Liapunov's
direct method. The conclusion from Liapunov's method is compared with those obtained by solving the eigenvalue problem. The results from both methods are shown to be consistent.

5.1 Visco-Elastic Damping: Kelvin Material

In this section, we derive the differential equations for the flexible arm with Kelvin material using a variational principle. Since the damping force is nonconservative, its effect is introduced into the equations of motion through the work done by the nonconservative force. We denote the moment introduced by the viscoelastic damping as $M^*$. The work done by $M^*$ is found by using the elementary beam theory. The $M^*$ and $(M^* + \partial M^*/\partial z \, dz)$ are the moments due to Kelvin damping. The $\partial y/\partial z$ is the rotational displacement at the end that $M^*$ is acting upon and $(\partial y/\partial z + \partial^2 y/\partial z^2 \, dz)$ is the angular displacement at the other end. From the equilibrium condition, we have

$$ v^* = \frac{\partial M^*}{\partial z} \quad (5.1) $$

The energy stored within the element is

$$ dW = -M^* \frac{\partial y}{\partial z} + (M^* + \frac{\partial M^*}{\partial z} \, dz)(\frac{\partial y}{\partial z} + \frac{\partial^2 y}{\partial z^2} \, dz) $$

$$ + v^* y - v^* (y + \frac{\partial y}{\partial z} \, dz) \quad (5.2) $$

Expanding this expression, we have

$$ dW = M^* \frac{\partial^2 y}{\partial z^2} \, dz + \frac{\partial M^*}{\partial z} \frac{\partial y}{\partial z} \, dz + \frac{\partial M^*}{\partial z} \frac{\partial^2 y}{\partial z^2} \, dz^2 $$

$$ - v^* \frac{\partial y}{\partial z} \, dz \quad (5.3) $$

By using Equation (5.1) and neglecting the high order terms in $dz$, we have

$$ dW = M^* \frac{\partial^2 y}{\partial z^2} \, dz \quad (5.4) $$
Hence, the virtual work done by the $M^*$ can be written as

$$\delta W = \int_0^L M^* \frac{\partial^2 \delta y}{\partial z^2} \, dz$$

$$= \int_0^L M^* \delta y'' \, dz$$

(5.5)

For the Kelvin material, the non-conservative moment acting in the flexible link has a special form and is derived using the following stress-strain relationship

$$\sigma_z = E(\epsilon_z + a \frac{d\epsilon_z}{dt}) = E(\epsilon_z + a\epsilon_z)$$

(5.6)

This expression is used in conjunction with the element beam theory to give the total moment acting on the beam as

$$M = EIy'' + EIay''$$

(5.7)

The effect of damping is included by adding terms to the work done by the non-conservative force in Equation (2.9).

$$\delta W_{nc} = \int_0^L EIa'y'' \frac{\partial^2 \delta y}{\partial z^2} \, dz$$

$$= \int_0^L EIa'y'' \delta y'' \, dz$$

(5.8)

For the purpose of this study, the regeneration of system PDEs is not necessary due to the following:

1. Damping does not change the static configuration.

2. The exact transfer functions can be re-evaluated using the Correspondence Principle [23].
The use of the Correspondence Principle is relatively straightforward and requires only the following substitutions into the transfer functions

\begin{align}
EI_1 &= EI_1(1 + a_1 s) \\
EI_2 &= EI_2(1 + a_2 s)
\end{align}

where \( a_1 \) and \( a_2 \) are the coefficients as defined in Equation (5.6). The procedure for obtaining the poles of the open- or closed-loop transfer functions are identical with that we use for the undamped systems. For the ODEs, the change occurs in the damping matrix \([C]\), for instance, the equation of motion for a two-link arm becomes

\[
[M]\{\ddot{\psi}\} + \frac{1}{2}[M]\{\dot{\psi}\} + [K]\{\psi\} + [C]\{\dot{\psi}\} + \frac{\partial V_q}{\partial \{\psi\}} = \{T_q\}
\]

where

\[
[C] = \begin{bmatrix}
[0]_{2 \times 2} & [0]_{2 \times n_1} & [0]_{2 \times n_2} \\
[0]_{n_1 \times 2} & [C^1]_{n_1 \times n_1} & [0]_{n_1 \times n_2} \\
[0]_{n_2 \times 2} & [0]_{n_2 \times n_1} & [C^2]_{n_2 \times n_2}
\end{bmatrix}
\]

where the components of the \([C^1]\) and \([C^2]\) are

\begin{align}
C^1_{ij} &= a_1 K^1_{ij} \\
C^2_{ij} &= a_2 K^2_{ij}
\end{align}

A root locus plot which includes the effect of Kelvin damping is shown in Figure 5.1. This root locus plot is generated for a two-link arm at \( \theta_1 = \theta_2 = 0 \). Gravity is omitted and control at the joint is deactivated. By increasing the damping coefficients \( a_1 \) and \( a_2 \), the open-loop poles are driven away from the imaginary axis. As the coefficients become large, the poles move onto the negative real axis. In order to demonstrate
Figure 5.1: Open-Loop Frequencies of Two-Link Flexible Arm with Kelvin Materials
the effects of the Kelvin damping in the time domain, two simulations are given. In the first simulation, no material damping is considered. The motion is generated by releasing the two-link arm from the static equilibrium position at \((\theta_1, \theta_2) = (30^\circ, 30^\circ)\). Since there is no nonconservative force/torque applied to the system, the total system energy is conserved, i.e., the sum of the kinetic and potential energy remains constant.

The simulation results at selected time steps, with five polynomial basis functions, are shown in Figure 5.2. The trace of the end-tip position of link two is also plotted for the first five seconds. The deformation of the arm is noticeable particularly after 0.6 sec. The total energy, potential energy, elastic energy, and kinetic energy are plotted in Figure 5.3. We observe in this figure that the total energy is conserved and the elastic energy is not negligible. The noticeable elastic vibration of the flexible links may be reduced if material damping is introduced to the arm. We choose the damping coefficients according to

\[
\begin{align*}
a_1 & = 0.005 \\
a_2 & = 0.005
\end{align*}
\] (5.14) (5.15)

The simulation results and energy plot are given in Figures 5.4 and 5.5. Comparing the results here with those in Figures 5.2 and 5.3, we find little difference in the first few seconds of the solution. However, fairly large amount of energy loss is observed near 5.5 seconds for the arm with Kelvin damping. The strain energy stored in the flexible links of an undamped system, as shown in Figure 5.3, dissipates very quickly in Figure 5.5. We can explain this observation by plotting the arm configuration of the undamped two-link flexible arm for the time from 5.5 to 6.0 seconds. In Figure 5.6, the quick change of the deformed shapes imply the large time rate of change of
the generalized coordinates of the basis functions. Due to the viscoelastic damping modeling, the large magnitude of velocities eventually leads to substantial energy dissipation. For this example, the high rate of the generalized coordinates for the lower order comparison functions contribute to the energy loss mostly.

5.2 Liapunov's Stability

In the previous chapters, we investigated the stability characteristics of the one-link and two-link flexible arm by solving for the poles of the transfer functions in the Laplace transform domain or by solving an eigenvalue problem. However, in order to form a transfer function or an eigenvalue problem, linearization of the nonlinear differential equations is necessary. In this section, we would like to relax the assumption used for linearizing the differential equations by retaining the nonlinear terms, such as the products of the velocities, in the stability analysis. The method we employ is Liapunov’s direct method [30]. The use of Liapunov’s direct method requires the selection of a trial function which equals zero and has a local minimum value at the equilibrium position under investigation. Once the trial function is selected, if the total time rate change of the trial function is negative definite for all the possible values of the system variables, the system is stable. For our flexible arms, the two requirements on the trial function, \( H(\{\psi\}, \{\dot{\psi}\}) \), may be written as

\[
H(\{\psi_0\}, \{0\}) = 0
\]  

(5.16)

\[
H(\{\psi_0\} + \{\delta \psi\}, \{\delta \dot{\psi}\}) \geq 0
\]  

(5.17)
Figure 5.2: Free-Falling Undamped Two-Link Flexible Arm
Figure 5.3: Energy Plot of Free-Falling Undamped Two-Link Flexible Arm
Figure 5.4: Free-Falling Damped Two-Link Flexible Arm
Figure 5.5: Energy Plot of Free-Falling Damped Two-Link Flexible Arm
Figure 5.6: Undamped Two-Link Flexible Arm: 5.5 to 6.0 Second
where \( \{ \psi \} \) is the vector contains the variables of the system as defined in Equations (3.66) or (3.170). The \( \{ \delta \dot{\psi} \} \) of a two-link arm, for instance, is defined as

\[
\{ \delta \dot{\psi} \} = \begin{bmatrix}
\dot{\epsilon}_1 \\
\dot{\epsilon}_2 \\
\{ \xi_1 \} \\
\{ \xi_2 \}
\end{bmatrix}
\]  

(5.18)

We will introduce one matrix definition and two frequently used matrix properties for later use. The definition we need is associated with the positive definite matrix. The definition of a positive definite matrix is as follows: An \( n \) by \( n \) real symmetric matrix \([A]\) is positive definite if \( x^T[A]x > 0 \) unless each element of \( \{ x \} \) is zero. The two frequently used matrix properties are:

1. If all the principal minors of a matrix \([A]\) are positive, \([A]\) is positive definite.

2. For a matrix which consists of only submatrices on the diagonal, the determinant of the matrix is equal to the product of the determinants of its submatrices.

For this study, the trial energy function for the PD control system is selected as

\[
H = \frac{1}{2} \{ \dot{\psi} \}^T [M] \{ \dot{\psi} \} + \frac{1}{2} \{ \psi \}^T [K] \{ \psi \} - \frac{1}{2} \{ \psi_o \}^T [K] \{ \psi_o \} + \frac{1}{2} (\{ \psi \} - \{ \psi_o \})^T [K_p] (\{ \psi \} - \{ \psi_o \}) + V_g(\{ \psi \}) - V_g(\{ \psi_o \})
\]

(5.19)

It is apparent that the energy function shown above satisfies the condition \( H(\{ \psi_o \}, \{ 0 \}) = 0 \). We will next find the sufficient condition such that the energy function has a relative minimum value at the equilibrium position \( \{ \psi_o \} \). We define the augmented variables \( \{ \eta \} \) as follows

\[
\{ \eta \} = \begin{bmatrix}
\{ \psi \} \\
\{ \dot{\psi} \}
\end{bmatrix}
\]

(5.20)
The Taylor series expansion of $H$ about the equilibrium position $\{\psi_0\}$ can be shown as

$$H(\{\eta_0 + \delta \eta\}) = H(\{\eta_0\}) + \nabla H^T|_{\{\eta_0\}} \{\eta - \eta_0\}$$

$$+ \frac{1}{2} \{\eta - \eta_0\}^T \nabla^2 H|_{\{\eta_0\}} \{\eta - \eta_0\} + \text{high order terms}$$

where $\{\psi_0\} = \{0\}$

$$\{\delta \eta\} = \{\delta \psi\}$$

and where the components of $\nabla H$ and $\nabla^2 H$ are in the form of

$$\nabla H_i = \frac{\partial H}{\partial \eta_i}$$

$$\nabla^2 H_{ij} = \frac{\partial^2 H}{\partial \eta_i \partial \eta_j}$$

Using the conditions that $\{\psi_0\} = \{0\}$ and $\frac{\partial H}{\partial \{\psi\}}|_{\{\eta_0\}} = \{0\}$, the second term of the Taylor series expansion of $H$, $\nabla H^T|_{\{\eta_0\}} \{\delta \eta\}$, vanishes. As a result, the $\nabla^2 H$ shown below must be positive definite for the energy function to satisfy the inequality in Equation (5.17).

$$\nabla^2 H|_{\eta=\eta_0} = \begin{bmatrix} [M](\{\eta_0\}) & 0 \\ 0 & [K' + K_p + \nabla^2 V_g|_{\{\eta_0\}}] \end{bmatrix}$$
where the elements of $\nabla^2 V_g$ are of the form
\[ \nabla^2 V_{gij} = \frac{\partial^2 V_g}{\partial \psi_i \partial \psi_j} \quad (5.28) \]
and where $[K_p]$ is the matrix for the position feedback. Observing the form of $\nabla^2 H$ and using the two matrix properties we described previously, one can conclude that, if $[K^*]$ as shown following is positive definite, then the $\nabla^2 H$ would also be positive definite.

\[ [K^*] = [K + K_p + \nabla^2 V_g|\{\eta\} = \{\eta_0\}] \quad (5.29) \]

The $[K^*]$ matrix is exactly the same as the stiffness matrix we use for the eigenvalue analysis. The rank of $[K]$ is equal to the order of the matrix less two. The characteristics of $\nabla^2 V_g$ depend on the static equilibrium configuration under study. For instance, for the one-link arm, it is positive-definite for lower values of $\theta$ and negative-definite for higher values of $\theta$. Only by choosing the proportion control gains properly, we may have a positive definite $[K^*]$ for all the possible static equilibrium configurations.

Once the energy function satisfies the positive definite requirement, we continue the stability analysis by taking the total time derivative of the function $H$,

\[ \dot{H} = \{\dot{\psi}\}^T \{[M]\{\ddot{\psi}\} + \frac{1}{2}[\dot{M}]{\dot{\psi}} + [K]{\psi} + [K_p]{\psi - \psi_0} + \frac{\partial V_g}{\partial \{\psi\}}\} \quad (5.30) \]

For the system with only PD controllers, we have
\[ [M]{\ddot{\psi}} + \frac{1}{2}[\dot{M}]{\dot{\psi}} + [K]{\psi} + [K_p]{\psi - \psi_0} + \frac{\partial V}{\partial \{\psi\}} \]
\[ = -[K_v]{\dot{\psi}} - [C]{\dot{\psi}} \quad (5.31) \]

This expression is substituted into $\dot{H}$ to give
\[ \dot{H} = -\{\dot{\psi}\}^T \{[K_v]{\dot{\psi}} + [C]{\dot{\psi}}\} \]
\[ = -\{\dot{\psi}\}^T[K^*]{\dot{\psi}} \quad (5.32) \]
If the damping matrix, \([C^*]\) is positive definite, the following inequality holds:

\[
\dot{H} = -\{\dot{\psi}\}^T[C^*]\{\dot{\psi}\} < 0
\]  

(5.33)

Hence, the linearized PD control system is Liapunov stable. Physically, Liapunov stability ensures that the system will return to the equilibrium position following a disturbance.

The same stability conclusion may also be obtained when we solve for the stability characteristics via the eigenvalue solution. A theorem in [31] is used for this purpose. It is proved in [31] that, for two positive definite real symmetrical matrices, \([M^*]\) and \([K^*]\), the eigenvalues \(\lambda\) are real and positive.

\[
\lambda[M^*]\{Z\} = [K^*]\{Z\}
\]  

(5.34)

Comparing the equation above with Equation (3.185), we conclude that the \(s^2\) in Equation (3.185) must be real and negative if both \([M^*]\) and \([K^*]\) are positive definite. This implies that all the poles in \(s\) are pure imaginary. Therefore the system is marginally stable. Since the kinetic energy \(\frac{1}{2}\{\dot{\psi}\}^T[M^*]\{\dot{\psi}\}\) is always greater than zero for nonzero \(\{\dot{\psi}\}\), \([M^*]\) is, by definition, positive definite. However, due to the degrees of freedom associated with the rigid body motion and the presence of gravitation, \([K^*]\) may be positive-definite, semi-definite or negative-definite. Only by choosing proper position feedback gains \([K_p]\), may \([K^*]\) be positive-definite. If \([K^*]\) is positive-definite, the linearized system is marginally stable. Hence, we have the stability conclusion identical with the one that we obtain using Liapunov's direct method.
For the PID control system, no stability conclusion similar to that of a PD system is obtained. Likewise, no suitable energy function $H$ is found in order to analyze a PID system using Liapunov’s method.
6 CONCLUSIONS

In this dissertation, both partial and ordinary differential equations of one- and two-link flexible arms are derived and used to obtain static solutions and Laplace transform domain characteristics. Additionally, the time-domain behavior of the one- and two-link arms are obtained by integrating a set of nonlinear ordinary differential equations. The conclusions from the static, dynamic and frequency domain analyses of the one-link and two-link flexible arm are summarized as following:

1. The choice of comparison functions for approximating the elastic deformation of the flexible links is very important. It has been shown that less accurate solutions may result when the basis functions are chosen improperly. If functions selected from an incomplete set are used for approximating the deflection of the flexible arm, the Laplace domain information may be fairly accurate. However, the time domain results could be less accurate. This observation implies that order truncation based on the frequency content may be misleading if improper basis functions are chosen.

2. The link flexibility is shown to affect the motion and frequency domain characteristics of the flexible arms. The importance of considering the link flexibility is clear when the frequency content of the closed-loop system for the model with rigid links is compared with that for the model with flexible links. The
rigid model is shown to predict different stability characteristics not only for the frequencies associated with the flexible structure but also for those of rigid body motion. The effect of material damping is significant when flexibility is considered. The Kelvin damping model is shown to dissipate the high strain energy in the flexible link quickly when the time rate of the deflection is large. This is significant especially if the control at the joints cannot suppress the rapid, oscillatory motion of the flexible link.

3. The stability characteristics of the flexible arm systems are studied using the transfer function approach, eigenvalue formulation, and Liapunov's direct method. The solutions from the eigenvalue formulation are shown to converge to those obtained using the transfer function approach for both open-loop and closed-loop systems. The stability prediction from the eigenvalue solution and Liapunov's direct method are also shown to be consistent for the independent PD control.

4. A PID controller generally results in oscillatory motion and the amplitude of the oscillation depends on the selection of the feedback control gains. Two sets of PID control gains which are selected based on the linear analysis are tested by large-angle, nonlinear dynamic simulations. It is shown that the time-domain response of the flexible arm at the terminal angular position of the large-angle maneuver is very well predicted by the linearized frequency-domain analysis when the reference inputs to the control are chosen properly. This observation suggests the possibility of improving the control system by fine tuning the control gains using the pole placement technique.

Future research areas may include
1. Extending the current two-dimensional flexible arm models to three dimensions.

2. Extending the current flexible arm model to include the deformation due to other effects, such as shear strain effect and rotary inertia.

3. Developing easy-to-use approximation functions for elastic systems with complex shapes.

4. Developing a suitable method to allow the direct use of the finite element method for flexible-arm models.

5. Designing a controller to reduce the oscillatory motion of the flexible arms near equilibrium.
7 BIBLIOGRAPHY


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