MINIMUM NORM SOLUTION OF A LINEAR EQUATION IN HILBERT SPACE IN TERMS OF SOLUTIONS OF RELATED PROJECTED EQUATIONS

by

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I. INTRODUCTION AND PRELIMINARY DEFINITIONS

A system of m linear equations in n unknowns can be expressed concisely by means of the matrix equation

$$AX = C$$

where A is the m x n matrix of coefficients and X and C are column matrices. The matrix X consists of the n unknowns while C consists of n constants.

Such a system may, or may not, have a solution. However, in any case there exists a set of n unknowns

$$\bar{X} = (\bar{x}_1, \bar{x}_2, - - -, \bar{x}_n)^T \quad ,$$

where T means transpose, such that the norm

$$\| AX - C \|$$

is a minimum. In fact, $\bar{X}$ is the solution of

$$A^T AX = A^T C \quad . \tag{1.1}$$

In case $m > n$, $\bar{X}$ can be expressed in terms of the solutions of certain subsystems. These subsystems each consist of n equations obtained from the given system of m equations. The number of subsystems thus formed is $\binom{m}{n}$, this being the usual binomial coefficient. Let us identify these subsystems with the positive integers $J = 1, 2, - -, \binom{m}{n}$. Let A^J denote the matrix of coefficients and C^J denote the column matrix of constants for the Jth subsystem. Then, for $J = 1, 2, --, \binom{m}{n}$, consider the equation

$$A^J X = C^J \quad . \tag{1.2}$$

If the subsystem J is consistent, then (1.2) will have a solution X^J. Whenever $|A^J|$, the determinant of the matrix A^J, is different from zero then the subsystem has a unique solution X^J. If the subsystem J is inconsistent, then $|A^J| = 0$.

If at least one of the $|A^J|$, $J = 1, 2, --, \binom{m}{n}$, is different from zero, then $|A^T A| \neq 0$. Then, the solution of (1.1) can be written as a weighted sum of the solutions of the $\binom{m}{n}$ subsystems in the preceding paragraph, namely

$$\bar{X} = \sum_{J=1}^{\binom{m}{n}} \frac{|A^J|^2}{|A^T A|} X^J \quad , \tag{1.3}$$

provided any term in the sum for which $|A^J| = 0$ is taken to be zero whether an X^J exists or not.

As an example consider the system of 3 linear equations in 2 unknowns

$$x - 2y = 1$$

$$-x + 4y = 1$$

$$+x - 2y = -1$$

Here, $A = \begin{pmatrix} 1 & -2 \\ -1 & 4 \\ 1 & -2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

There will be 3 subsystems, each consisting of two equations, namely

(1)
$$x - 2y = 1$$
$$-x + 4y = 1$$

with

$$A^1 = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \quad C^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

(2)
$$x - 2y = 1$$
$$x - 2y = -1$$

with

$$A^2 = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}, \quad C^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

(3)
$$-x + 4y = 1$$
$$x - 2y = -1$$

with

$$A^3 = \begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}, \quad C^3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The subsystem 2 is inconsistent. However, subsystems 1 and 3 are consistent and have solutions

$$X^1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad X^3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

respectively. From (1.3)

$$\bar{X} = \sum_{J=1}^{3} \frac{|A^J|^2}{|A^T A|} X^J .$$

Since $|A^T A| = 8$, $|A^1| = 2$, $|A^2| = 0$, and $|A^3| = -2$, one obtains

$$\overline{X} = \frac{1}{8}\left\{4\begin{pmatrix}3\\1\end{pmatrix} + 0 + 4\begin{pmatrix}-1\\0\end{pmatrix}\right\} = \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix} .$$

One may readily verify that this satisfies the equation $A^T AX = A^T C$. In the above sum the term corresponding to $J = 2$ is zero since $|A^2| = 0$, for any X^2.

The purpose of this paper is to obtain an expression similar to (1.3) for an element in a Hilbert space H which is the best solution in the norm sense of an equation in H.

Let f and g be elements of a Hilbert space H. Consider the linear operator T defined for all elements f of H by

$$Tf = \sum_{i=1}^{n} p_i(q_i, f)$$

where the p_i and q_i are elements of H. It will be shown that for an arbitrary fixed point g of H there does exist an element f of H for which the norm expression $\| Tf - g \|$ is a minimum.

We shall introduce a denumerable set of projection operators $\left\{P^J\right\}$ which maps elements of H into subspaces of H determined by n elements of a complete orthonormal set. One gets then for the operator T previously mentioned, the equation

(1.4) $$P^J Tf = P^J g$$

for each $J = 1, 2, - - -$. For certain ones of these subsystems, (1.4) will have a solution, namely when the system of equations arising from (1.4) is consistent.

One can then show that the element in H for which (1.3) is a minimum can be expressed as an average of the solutions of (1.4) for those J's where the system is consistent.

The defining postulates for a Hilbert space will follow after the introduction of some preliminary definitions.

Definition 1.1. The complex-valued function (f, g) defined for every pair f, g of elements of a linear space R is called an inner product if it satisfies the following properties for all elements f, g, h of R and all numbers α, β:

(1) $(g, f) = \overline{(f, g)}$

(2) $(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$

(3) $(f, f) \geqq 0$; $(f, f) = 0$ if and only if $f = 0$.

Definition 1.2. The non-negative real number $(f, f)^{1/2}$, denoted by $\|f\|$, is called the norm of f.

The norm of f, $\|f\|$, satisfies the following properties for every number α and all elements f, g of R:

(1) $\|f\| \geqq 0$; $\|f\| = 0$ if and only if $f = 0$

(2) $\|\alpha f\| = |\alpha| \, \|f\|$

(3) $\|f + g\| \leqq \|f\| + \|g\|$.

The last inequality is Minkowski's inequality, known also as the triangle inequality. Another important inequality is the Schwarz inequality which states, that for all elements f, g of R,

$$|(f, g)| \leqq \|f\| \, \|g\| .$$

The number $\|f - g\|$ is called the distance between the elements f, g. This number satisfies the postulates for a metric since from the properties above for the norm, for all elements f, g, h of R

(1) $\|f - g\| \geqq 0$; $\|f - g\| = 0$ if and only if $f = g$

(2) $\|g - f\| = \|f - g\|$

(3) $\|f - h\| \leqq \|f - g\| + \|g - h\|$.

Definition 1.3. The elements $f_1, f_2, --, f_r$ of a linear space R are called linearly independent if for the numbers $\alpha_1, \alpha_2, ---, \alpha_r$, the relation

$$\sum_{i=1}^{r} \alpha_i f_i = 0$$

implies that

$$\alpha_i = 0 \qquad i = 1, 2, --, r.$$

If there exist numbers $\alpha_1, \alpha_2, --, \alpha_r$, not all zero, for which

$$\sum_{i=1}^{r} \alpha_i f_i = 0$$

then the elements $f_1, f_2, --, f_r$ are called linearly dependent.

Definition 1.4. A linear space R is said to have dimension n if it contains n linearly independent elements but any n + 1 or more elements are linearly dependent. If a linear space R has arbitrarily many linearly independent elements it is called an infinite dimensional space.

Definition 1.5. A sequence $\{f_n\}$, $n = 1, 2, - -$, of elements of R has an element f of R as a limit, or converges to f, when

$$\lim_{n\to\infty} \| f - f_n \| = 0.$$

This will be indicated also by the notation $f_n \to f$.

Definition 1.6. If the sequence $\{f_n\}$, $n = 1, 2, - -$, of elements of R is such that

$$\| f_n - f_m \| \to 0 \qquad \text{as } n, m \to \infty ,$$

the sequence is called a Cauchy sequence, or fundamental sequence.

Definition 1.7. A metric space S is called a complete space when every Cauchy sequence in it converges to an element of S.

Definition 1.8. A linear space H of elements f, g, h, - - is called a Hilbert space if

(1) there is an inner product (f, g) defined for every pair of elements f, g of H.

(2) the space H is infinite dimensional.

(3) there exists a denumerable set of elements of H that is everywhere dense in H, that is, H is separable.

(4) the space H is complete, where the metric is obtained from the norm associated with the inner product.

There are several realizations of the above defined abstract Hilbert space. One of these is the class of real-valued measurable L_2- integrable functions defined on the interval $[0, 1]$. If f and g are two elements of this Hilbert space, called the class L_2, then f = g if and only if they differ on at most a set of measure zero. (See Stone [3] for details.)

We wish next to introduce the definition of an operator on the Hilbert space H.

Definition 1.9. A set of elements $M \subseteq H$ is called a linear manifold if for every pair of elements f, g in M and every pair of numbers α, β, then $\alpha f + \beta g$ is also in M.

Definition 1.10. A function, or mapping, T defined on the elements of a subset M of a Hilbert space H with the range of values Tf in a subset of H is called an operator.

Definition 1.11. Let the operator T be defined on a linear manifold M. Then, T is said to be a linear operator if for every pair of elements f, g of M, and for all numbers α, β, then

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg .$$

The operator T is said to be bounded on M if there exists a constant B such that

$$\|Tf\| \leq B \|f\|$$

for all f in M.

For the space L_2 consider the integral operator

$$Kf = \int_0^1 K(x, t)f(t)dt$$

where $K(x, t)$ is measurable and

$$\int_0^1 \int_0^1 K^2(x, t)\, dx\, dt < +\infty .$$

Now, by the Schwarz inequality,

$$\int_0^1 (Kf)^2\, dx = \int_0^1 \left[\int_0^1 K(x, t)f(t)dt\right]\left[\int_0^1 K(x,s)f(s)ds\right] dx$$

$$\leq \int_0^1 \left[\int_0^1 K^2 dt\right]^{\frac{1}{2}} \left[\int_0^1 f^2 dt\right]^{\frac{1}{2}} \left[\int_0^1 K^2 ds\right]^{\frac{1}{2}} \left[\int_0^1 f^2 ds\right]^{\frac{1}{2}} dx$$

$$= \int_0^1 \int_0^1 K^2\, dx\, dt \quad \|f\|^2 .$$

Now Kf is measurable since both K and f are measurable functions and thus Kf is in L_2. It is clear that the operator K defined above is a linear operator. Also, K is a bounded operator, where a suitable bound B is

$$B = \left[\int_0^1 \int_0^1 K^2(x, t)\, dx\, dt\right]^{\frac{1}{2}} .$$

<u>Definition 1.12</u>. Two elements f, g of H are said to be orthogonal if $(f, g) = 0$.

<u>Definition 1.13</u>. The element g is said to be orthogonal to a subset M of H, written $g \perp M$, if $(f, g) = 0$ for every element f of M.

<u>Definition 1.14</u>. If M is a subset of H, then $M^{\perp}$ will denote the set of all elements f of H such that $f \perp M$.

From now on we shall consider a real abstract Hilbert space with the scalar multipliers being real numbers.

II. MINIMUM NORM SOLUTION OF $Tf = g$

Let T be a linear operator defined over the whole space H. Suppose now that g is an arbitrary element of H. There may not be an element f in H for which $Tf = g$. However, we shall show that for a class of linear operators T there does exist an element f in H which minimizes the quantity

(2.1) $$\|Tf - g\|$$

Theorem 2.1. Let g be an arbitrary fixed element of H. If $\delta = \inf\left\{\|Tf - g\| : f \in H\right\}$ then there exists an element h in H which is either a limit point of elements Tf or which equals Tf for some f, such that

$$\|h - g\| = \delta .$$

Proof: From the definition of the norm it follows that for elements f_1, f_2 of H

(2.2) $$\|f_1 + f_2\|^2 + \|f_1 - f_2\|^2 = 2\|f_1\|^2 + 2\|f_2\|^2 .$$

Either there is an f in H such that $\|Tf - g\| = \delta$ or there is a sequence of elements in H such that $\|Tf_n - g\| \to \delta$. In the first case $h = Tf$ while in the latter, for every m and n, it follows from (2.2) that

$$\|Tf_n - Tf_m\|^2 = \|(Tf_n - g) - (Tf_m - g)\|^2$$

$$= 2\|Tf_n - g\|^2 + 2\|Tf_m - g\|^2 - 4\left\|\frac{T(f_n + f_m)}{2} - g\right\|^2$$

(2.3) $$\leq 2\|Tf_n - g\|^2 + 2\|Tf_m - g\|^2 - 4\delta^2$$

where since $f_n + f_m$ is in H,

$$\| 1/2\ T(f_n + f_m) - g \|^2 \geqq \delta^2 .$$

Then, as m, $n \to \infty$, the right side of (2.3) tends to zero. Thus, $\{Tf_n\}$ is a Cauchy sequence. The space H is complete by Definition 1.8 so there exists an element h in H such that

(2.4) $$\lim_{n \to \infty} \| Tf_n - h \| = 0 .$$

We want to show that $\| h - g \| = \delta$.

According to the definition of δ , for every $\epsilon > 0$ there exist positive integers N_1 and N_2 such that

$$\| Tf_n - g \| < \epsilon/2 + \delta \qquad n > N_1$$

and

$$\| Tf_n - h \| < \epsilon/2 \qquad n > N_2$$

from (2.4). Thus, for n max (N_1, N_2)

$$\| h - g \| \leqq \| Tf_n - h \| + \| Tf_n - g \|$$

$$< \epsilon/2 + \delta + \epsilon/2 = \epsilon + \delta .$$

Since ϵ is an arbitrary positive number, then

$$\| h - g \| \leqq \delta .$$

On the other hand, $\| Tf_n - g \| \geqq \delta$, so

$$\| h - g \| = \lim_{n \to \infty} \| Tf_n - g \| \geqq \delta .$$

Thus,

$$\| h - g \| = \delta .$$

Let us now consider the operator T defined on the space H which maps every element f of H into H according to

$$(2.5) \qquad Tf = \sum_{i=1}^{n} p_i(q_i, f)$$

where the p_i and q_i are elements of H.

Lemma 2.1. If $Tf = \sum_{i=1}^{n} p_i(q_i, f)$ is not identically zero for f in H, this sum can be replaced by one over r terms similar to the above sum, $1 \leq r \leq n$, where the p_i's and q_i's are linearly independent and the p_i's are orthogonal.

Proof: Either the set $\{p_i\}$, or $\{q_i\}$, $i = 1, --, n$, can be reduced to a linearly independent set. Suppose this has been done for the $\{q_i\}$, $i = 1, --, n$. Since $Tf \neq 0$, for some f in H at least one of the p_i is different from zero. Then there exists a subset of the $\{p_i\}$, $i = 1, --, n$, consisting of r linearly independent elements, $1 \leq r \leq n$, such that each of the p_i, $i = 1, --, n$, can be written as a linear combination of elements of this subset. Then using the Schmidt orthogonolization process (see Stone, [3]) if necessary, one can obtain a set of r linearly independent elements that are also orthogonal. Denote this orthogonal set by $\{p_j^*\}$, $j = 1, --, r$. Then there exist real numbers

$$\alpha_{ij} \quad , \quad i = 1, --, n; \; j = 1, --, r$$

such that

$$(2.6) \qquad p_i = \sum_{j=1}^{r} \alpha_{ij} \, p_j^* \qquad i = 1, --, n .$$

One may then write

$$Tf = \sum_{i=1}^{n} p_i(q_i, f) = \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_{ij} p_j^*(q_i, f)$$

$$= \sum_{j=1}^{r} p_j^* \sum_{i=1}^{n} \alpha_{ij} (q_i, f)$$

$$= \sum_{j=1}^{r} p_j^* (\sum_{i=1}^{n} \alpha_{ij} q_i, f) .$$

Denote the quantity

$$\sum_{i=1}^{n} \alpha_{ij} q_i \qquad j = 1, --, r$$

by q_j^* .

Then,

$$Tf = \sum_{j=1}^{r} p_j^* (q_j^*, f) .$$

From (2.6) and the definition of the p_j^* it follows that the matrix (α_{ij}) is of rank r.

It will now be shown that the $\left\{q_j^*\right\}$,

$$(2.7) \qquad q_j^* = \sum_{i=1}^{n} \alpha_{ij} q_i \qquad j = 1, --, r$$

form a linearly independent set. Consider

$$\sum_{j=1}^{r} \beta_j q_j^* = 0$$

which becomes from (2.7)

$$\sum_{j=1}^{r} \beta_j \sum_{i=1}^{n} \alpha_{ij} q_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{r} \alpha_{ij} \beta_j \right) q_i = 0.$$

The elements $\left\{ q_i \right\}$ form a linearly independent set, so that

$$\sum_{j=1}^{r} \alpha_{ij} \beta_j = 0 \qquad i = 1, --, n.$$

Since the matrix (α_{ij}) is of rank r it is clear that

$$\beta_j = 0 \qquad j = 1, --, r.$$

Therefore, the $\left\{ q_i^* \right\}$, $\qquad i = 1, --, r$

also form a linearly independent set of elements.

Henceforth, when the operator

$$Tf = \sum_{i=1}^{n} p_i (q_i, f)$$

is considered, we will assume that the $\left\{ p_i \right\}$, and $\left\{ q_i \right\}$ form linearly independent sets and that the $\left\{ p_i \right\}$ is also an orthogonal set of elements.

Definition 2.1. By the notation M_q we shall mean the closed linear manifold spanned by the $\left\{ q_i \right\}$, $i = 1, --, n$. That is, if $f \in M_q$ then f is a linear combination of the $\left\{ q_i \right\}$.

Definition 2.2. For each $i = 1, 2, --, n$, let

$$M_{q_i}^{\perp} = \left\{ f : f \in H \text{ and } (q_i, f) = 0 \right\}.$$

Lemma 2.2. $M_{q_i}^{\perp}$ is closed, $i = 1, --, n$.

Proof: Suppose that $\left\{f_k{}^i\right\}$ is a sequence of elements of H such that $(f_k{}^i, q_i) = 0$ for all k. We must show that if

$$\lim_{k\to\infty} \| f_k{}^i - f^i \| = 0 \quad ,$$

then also

$$(f^i, q_i) = 0 \; .$$

Now, if $\lim_{k\to\infty} \| f_k{}^i - f^i \| = 0$, then for every $\epsilon > 0$ there exists an N^i_ϵ such that if $k > N^i_\epsilon$, then

$$\| f_k{}^i - f^i \| < \frac{\epsilon}{\| q_i \|}$$

where $\| q_i \| \neq 0$ since q_i is an element of a linearly independent set. By application of the Schwarz inequality

$$\begin{aligned} \left| (q_i, f_k{}^i) - (q_i, f^i) \right| &= \left| (q_i, f_k{}^i - f^i) \right| \\ &\leqq \| q_i \| \; \| f_k{}^i - f^i \| \\ &< \| q_i \| \frac{\epsilon}{\| q_i \|} = \epsilon \end{aligned}$$

for $k > N^i_\epsilon$. Since $(q_i, f_k{}^i) = 0$ for all k, then

$$\left| (q_i, f^i) \right| < \epsilon$$

and since ϵ is arbitrary,

$$\left| (q_i, f^i) \right| = 0.$$

Lemma 2.3. If $Tf = 0$, then $f \epsilon M_q^{\perp}$. (Here clearly $M_q^{\perp} = \bigcap_{i=1}^{n} M_{q_i}^{\perp}$ by virtue of Definitions 1.14 and 2.2)

Proof: If $Tf = 0$, this means

$$\sum_{i=1}^{n} p_i(q_i, f) = 0 .$$

Since the $\left\{ p_i \right\}$ are a linearly independent set of elements of H, then

$$(q_i, f) = 0 \qquad\qquad i = 1, --, n.$$

Therefore, $f \in M_{q_i}^{\perp}$ for each $i = 1, --, n$ and so

$$f \in M_q^{\perp} .$$

Lemma 2.4. If $f \in M_q$ and $Tf = 0$, then $f = 0$.

Proof: From Lemma 2.3

$$Tf = \sum_{i=1}^{n} p_i(q_i, f) = 0$$

implies that $f \in M_q^{\perp}$. But by hypothesis

$$f = \sum_{j=1}^{n} \alpha_j q_j$$

so that

$$(q_i, f) = (q_i, \sum_{j=1}^{n} \alpha_j q_j)$$

$$= \sum_{j=1}^{n} \alpha_j (q_i, q_j) = 0 \qquad i = 1, --, n.$$

Since the $\left\{ q_i \right\}$ are a linearly independent set of elements, the determinant $|(q_i, q_j)|$ will be non-zero. Therefore,

$$\alpha_j = 0 \qquad\qquad j = 1, --, n.$$

and thus $f = 0$.

Lemma 2.5. If $f_1, f_2 \in M_q$, and $Tf_1 = Tf_2$, then $f_1 = f_2$.

Proof: Since $Tf_1 = Tf_2$, then

$$Tf_1 - Tf_2 = T(f_1 - f_2) = 0 .$$

The element $f_1 - f_2 \in M_q$. By Lemma 2.4

$$f_1 - f_2 = 0 , \qquad \text{or} \quad f_1 = f_2 .$$

Lemma 2.6. For every element $f \in H$ one can write

$$f = f' + f^*$$

where $f' \in M_q$ and $f^* \in M_q^{\perp}$.

Proof: For every $f \in H$ one sees that Tf can be written as

$$Tf = \sum_{i=1}^{n} \beta_i p_i \quad , \qquad \beta_i = (q_i, f).$$

Now, let

$$f' = \sum_{i=1}^{n} \alpha_i q_i$$

so that

$$Tf' = \sum_{i,j=1}^{n} \alpha_i (q_j, q_i)\, p_j \quad .$$

Since the Gramian, $G(q) = |(q_i, q_j)| \neq 0$, there exists a unique solution for the α_i, $i = 1, --, n$ such that

$$\sum_{i=1}^{n} (q_j, q_i)\, \alpha_i = \beta_j \qquad j = 1, --, n.$$

For this choice of the α_i's ,

$$Tf - Tf' = T(f - f') = 0 .$$

Therefore, according to Lemma 2.4,

$$f - f' = f^*$$

where $f^* \in M_q^{\perp}$, which proves the lemma.

Theorem 2.2. There exist elements

$$f_i^o \in M_q \qquad i = 1, --, n$$

such that

$$T f_i^o = p_i \qquad i = 1, --, n.$$

Proof: For the n elements of H defined by

$$\sum_{i=1}^{n} \alpha_i^k q_i \qquad k = 1, --, n$$

it follows that

$$T\left(\sum_{i=1}^{n} \alpha_i^k q_i\right) = \sum_{i,j=1}^{n} \alpha_i^k (q_j, q_i) p_j \qquad k = 1, --, n.$$

Consider the system of equations

$$\sum_{i,j=1}^{n} \alpha_i^k (q_j, q_i) p_j = p_k \qquad k = 1, --, n.$$

Since the $\left\{p_i\right\}$ are a linearly independent set, this can be written as

$$\sum_{i=1}^{n} (q_j, q_i) \alpha_i^k = \delta_{jk} \qquad j = 1, --, n$$

for each k, where δ_{jk} is the Kronecker delta. Then, for each k, since $G(q) \neq 0$, one can solve uniquely for

α_i^k $\qquad i = 1, --, n.$

Thus, it is clear that for this choice of the α_i^k, then

$$f_k^o = \sum_{i=1}^{n} \alpha_i^k q_i \qquad k = 1, --, n$$

satisfy the conclusion of the theorem.

Theorem 2.3. The set $\left\{Tf: f \in H\right\}$ forms a complete space.

Proof: If $f \in H$, then

$$Tf = \sum_{i=1}^{n} \beta_i p_i \ , \qquad \beta_i = (q_i, f) .$$

Suppose that $\left\{Tf_k\right\}$ is a Cauchy sequence. Then, for every $\epsilon > 0$ there exists an N_ϵ such that whenever

$n_1 > n_2 > N_\epsilon$, then

$$\| Tf_{n_1} - Tf_{n_2} \| < \epsilon .$$

Now, for the element f_k let

$$Tf_k = \sum_{i=1}^{n} \beta_i^k p_i \qquad k = 1, --, n .$$

Then, since $\left\{p_i\right\}$ are an orthogonal set

$$\| Tf_{n_1} - Tf_{n_2} \|^2 = \| \sum_{i=1}^{n} \beta_i^{n_1} p_i - \sum_{i=1}^{n} \beta_i^{n_2} p_i \|^2$$

$$= \| \sum_{i=1}^{n} (\beta_i^{n_1} - \beta_i^{n_2}) p_i \|^2$$

$$= \sum_{i=1}^{n} |\beta_i^{n_1} - \beta_i^{n_2}|^2 \quad \|p_i\|^2 .$$

From this for each i = 1, - -, n

$$|\beta_i^{n_1} - \beta_i^{n_2}| \, \|p_i\| \leqq \|Tf_{n_1} - Tf_{n_2}\| < \epsilon .$$

Since p_i is an element of a linearly independent set, for each i, $\|p_i\| \neq 0$ so

$$|\beta_i^{n_1} - \beta_i^{n_2}| < \frac{\epsilon}{\|p_i\|} .$$

Thus, for each i = 1, - -, n, $\left\{\beta_i^{\,n}\right\}$ is a Cauchy sequence of real numbers. Denote the limits by $\beta_i^{\,o}$, i = 1, - -, n. From

$$\|Tf_k - \sum_{i=1}^{n} \beta_i^{\,o} p_i\| = \|\sum_{i=1}^{n} \beta_i^{\,k} p_i - \sum_{i=1}^{n} \beta_i^{\,o} p_i\|$$

$$\leqq \sum_{i=1}^{n} |\beta_i^{\,k} - \beta_i^{\,o}| \, \|p_i\|$$

one sees that

$$Tf_k \rightarrow \sum_{i=1}^{n} \beta_i^{\,o} p_i \quad \text{as } k \rightarrow \infty .$$

Now, let

$$f_1^{\,o}, f_2^{\,o}, - -, f_n^{\,o}$$

be elements of H such that

$$Tf_i^{\,o} = p_i \qquad\qquad i = 1, - -, n,$$

the existence of which is assured by Theorem 2.2. Then, if

$$f^o = \sum_{i=1}^{n} \beta_i^o f_i^o$$

one gets for Tf^o

$$Tf^o = T\left(\sum_{i=1}^{n} \beta_i^o f_i^o\right) = \sum_{i=1}^{n} \beta_i^o Tf_i^o$$

$$= \sum_{i=1}^{n} \beta_i^o p_i \quad .$$

Thus, the Cauchy sequence $\{Tf_k\}$ converges to Tf^o and the theorem is proved.

Theorem 2.4. Let g be an arbitrary fixed element of H. If

$$\delta = \inf\left\{\|Tf - g\| : f \in H\right\}$$

then there exists an element $f_o \in H$ such that $\|Tf_o - g\| = \delta$.

Proof: The proof follows from Theorems 2.1 and 2.3. By the meaning of δ there is an element f_o such that $\|Tf_o - g\| = \delta$, or there is a sequence of elements in H such that

$$\|Tf_k - g\| \longrightarrow \delta .$$

The sequence $\{Tf_k\}$ is a Cauchy sequence as was shown in the proof of Theorem 2.1. Since the space $\{Tf : f \in H\}$ is a complete space by Theorem 2.3, there exists an element f_o of H such that

$$\|Tf_k - Tf_o\| \longrightarrow 0 \quad .$$

Then,

$$\|Tf_o - g\| \leq \|Tf_k - Tf_o\| + \|Tf_k - g\| \quad . \tag{2.8}$$

Letting $k \rightarrow \infty$ in (2.8) one obtains

$$\|Tf_o - g\| \leq \delta .$$

By the definition of δ ,

$$\| Tf_o - g \| \geqq \delta .$$

Consequently,

$$\| Tf_o - g \| = \delta .$$

<u>Theorem 2.5</u>. There exists an element f_o' in M_q for which

$$\| Tf_o' - g \| = \delta .$$

Proof: By Theorem 2.4 there exists an element $f_o \in H$ which minimizes $\| Tf - g \|$. By Lemma 2.6 the element f_o can be expressed as $f_o = f_o' + f_o^*$, where $f_o' \in M_q$ and $f_o^* \in M_q^\perp$. Hence, since $Tf_o^* = 0$,

$$Tf_o = Tf_o'$$

and so

$$\| Tf_o' - g \| = \| Tf_o - g \| = \delta .$$

We have shown that there does exist an element f_o in H which minimizes the quantity $\| Tf_o - g \|$ and a corresponding element f_o' in M_q which achieves the same minimum where g is an arbitrary fixed element of H. We shall now find a necessary and sufficient condition that this element f_o' must satisfy.

Let f and η be elements of H where f is an element of H which minimizes $\| Tf - g \|$, g being an arbitrary but fixed element of H. Now, $f + \alpha \eta$ is an element of H for any real number α . Consider the quantity

$$(2.9) \quad I(\alpha) = \| T(f + \alpha \eta) - g \|^2$$

$$= \| Tf - g \|^2 + 2\alpha(Tf - g, T\eta) + \alpha^2 \| T\eta \|^2$$

(H is a real Hilbert space.)

Now, a necessary condition that the quantity $I(\alpha)$ have a minimum at $\alpha = 0$ is that $I'(0) = 0$. From (2.9) one gets

$$I'(\alpha) = 2(Tf - g, T\eta) + 2\alpha \| T\eta \|^2$$

and so

$$I'(0) = 2(Tf - g, T\eta) .$$

Thus, for a minimum

$$(Tf - g, T\eta) = 0 . \tag{2.10}$$

<u>Definition 2.3</u>. The operator T^* is defined for the operator T we are considering by

$$T^*f = \sum_{i=1}^{n} (p_i, f)q_i .$$

One sees that the operator T^* is the adjoint operator of T, for

$$(T^*f, g) = \left(\sum_{i=1}^{n} (p_i, f)q_i, g \right) = \sum_{i=1}^{n} (p_i, f)(q_i, g)$$

$$= (f, \sum_{i=1}^{n} p_i(q_i, g)) = (f, Tg).$$

<u>Lemma 2.7</u>. A necessary and sufficient condition that $(f, \eta) = 0$ for all $\eta \in H$ is that $f = 0$.

Proof: If $(f, \eta) = 0$ for all η, then $(f, f) = 0$ which implies that $f = 0$. On the other hand, if $f = 0$, then $(f, \eta) = 0$ for all $\eta \in H$.

For arbitrary η in H one obtains from (2.10) the condition

$$(T^*(Tf - g), \eta) = 0$$

that the element f minimize $\| Tf - g \|$, g being an arbitrary fixed element. Hence, by Lemma 2.7, from $(T^*(Tf - g), \eta) = 0$

(2.11) $$T^*Tf = T^*g \ .$$

That is, an element f that minimizes $\| Tf - g \|$ is a solution of (2.11). Suppose f_1 is a solution of (2.11) and that ζ is an element of H such that $T\zeta = 0$. Then, $f_2 = f_1 + \zeta$ is also a solution of (2.11).

From Lemma 2.6 recall that the minimizing element f of the expression $\| Tf - g \|$ may be written as $f = f' + f^*$, where $f' \in M_q$ and $f^* \in M_q^\perp$. Since $Tf^* = 0$, then $Tf = Tf'$ so from Theorem 5 it follows that f' minimizes $\| Tf - g \|$, also. From the preceding paragraph, one sees that f' also satisfies (2.11).

<u>Theorem 2.6</u>. The function $f' \in M_q$ which minimizes $\| Tf - g \|$, or which satisfies $T^*Tf = T^*g$, is unique.

Proof: Suppose that f' and f" are elements of M_q such that f' and f" both minimize $\| Tf - g \|$. Any element which minimizes $\| Tf - g \|$ must satisfy the condition

$$T^*Tf = T^*g \ .$$

Thus,

$$T^*Tf' = T^*Tf'' \ .$$

Then, for arbitrary η in H,

(2.12) $$(T^*Tf' - T^*Tf'', \eta) = 0 \ .$$

From (2.12) one gets

(2.13) $$(T(f' - f''), T\eta) = 0 .$$

Since η is an arbitrary element of H, let $\eta = f' - f''$, and we get immediately from (2.13)

$$T(f' - f'') = 0 .$$

Then, from Lemma 2.5 it follows that

$$f' - f'' = 0 , \quad \text{or } f' = f''$$

which proves the theorem.

By this theorem and Theorem 2.5 we see that $T^*Tf = T^*g$ is a necessary and sufficient condition on $f \in M_q$ that $\|Tf - g\|$ be a minimum.

We shall now express this condition in an equivalent form.

Let the unique minimizing element of M_q be represented by

(2.14) $$f' = \sum_{i=1}^{n} c_i q_i \quad .$$

We want to determine the c_i, $i = 1, - -, n$. These will be obtained by considering

$$T^*Tf' = T^*g.$$

From (2.14), one obtains

$$Tf' = \sum_{i=1}^{n} p_i(q_i, f') = \sum_{i=1}^{n} p_i(q_i, \sum_{j=1}^{n} c_j q_j)$$

$$= \sum_{i=1}^{n} p_i \sum_{j=1}^{n} c_j(q_i, q_j) .$$

Then,

$$T^*Tf' = \sum_{\ell=1}^{n} \left[\sum_{i,j=1}^{n} (p_\ell , p_i)(q_i, q_j) c_j \right] q_\ell$$

$$= \sum_{\ell=1}^{n} \left[\sum_{j=1}^{n} \| p_\ell \|^2 (q_\ell , q_j) c_j \right] q_\ell$$

since the $\{p_i\}$ are an orthogonal set of elements of H. Moreover,

$$T^*g = \sum_{\ell=1}^{n} q_\ell \ (p_\ell , g) ,$$

and since the $\{q_i\}$ are a linearly independent set of elements of H, one must have

$$(2.15) \qquad \| p_\ell \|^2 \sum_{j=1}^{n} (q_\ell , q_j) c_j = (p_\ell , g) \qquad \ell = 1, --, n.$$

<u>Definition 2.4</u>. The symbol

$$(C \,\vdots\, \alpha ; m) = ((c_{ij}) \,\vdots\, (\alpha_i) ; m)$$

will mean the matrix obtained by replacing the mth column of the matrix $C = (c_{ij})$ by the column matrix $\alpha = (\alpha_i)$. Also, $|C|$ will mean the determinant of C.

Applying Cramer's rule to (2.15) one gets, for m = 1, 2, --, n ,

$$(2.16) \qquad G(p)\, G(q)\, c_m = |((p_\ell , p_i)(q_i, q_j) \,\vdots\, (p_\ell , g); m)|$$

where G(p) and G(q) are the Gramians $|(p_i, p_j)|$ and $|(q_i, q_j)|$, respectively.

III. EQUATIONS OBTAINED FROM PROJECTIONS OF Tf = g

A linear manifold has been defined (Definition 1.9). If m is a closed linear manifold, then $M + M^{\perp} = H$ [2, p. 25]. Since the only common element of M and $M^{\perp}$ is the zero element, O, every element h of H can be expressed uniquely in the form

$$h = f + g$$

where $f \in M \subseteq H$ and $g \in M^{\perp}$. In the preceding chapter we gave a proof of these facts in the special case $M = M_q$.

Definition 3.1. If $M \subseteq H$ is a closed linear manifold and h of H is written as h = f + g, where $f \in M$, then f is called the projection of h on M along $M^{\perp}$. An operator P such that Ph = f is called a projection operator.

Let $\{\phi_k\}$, k = 1, 2, - -, be a complete orthonormal set in H.

Definition 3.2. Let an element of H be expressed as a linear combination of the complete orthonormal set $\{\phi_k\}$, namely

$$f = \sum_{k=1}^{\infty} \gamma_k \phi_k \quad .$$

Then the projection operator P^J is defined to be an operator such that

$$P^J f = \sum_{i=1}^{n} \gamma_{\ell i} \phi_{\ell i} \quad .$$

The operator P^J thus maps elements of H into the closed linear manifold $[\phi_{\ell 1}, \phi_{\ell 2}, - -, \phi_{\ell n}]$ where $\ell_1, \ell_2, - -,$

ℓ_n are n positive integers. The symbol J corresponds to the set of integers $\left\{\ell_1, \ell_2, --, \ell_n\right\}$. The precise correspondence will be discussed below.

Consider the arrays of n positive integers such that the positive integers are ordered in the usual manner. The first array will be (1, 2, - -, n-1, n) which is made to correspond to J = 1. There will be new arrays having n+1 as the nth element. In fact there are $\binom{n}{n-1}$ of these, where $\binom{r}{s}$ is the usual binomial coefficient. The integers J = 2, 3, - -, n+1 are made to correspond to the arrays having n+1 as the nth element according to the lexicographic ordering. The general ordering of arrays will be taken such that for the arrays

I $(a_1, a_2, ---, a_n)$

II $(b_1, b_2, ---, b_n)$

if $a_n = b_n$, then I precedes II when $a_1 < b_1$, or for $1 < j \leqq n-1$, when

$$a_1 = b_1 ; \quad a_2 = b_2 ; \; --; \; a_{j-1} = b_{j-1} ; \quad a_j < b_j ,$$

while for any two arrays I, II such that $a_n < b_n$, array I is ordered so as to precede array II.

Thus, after the arrays corresponding to J = 1, 2, - -, n+1 one considers array where the nth element is n+2. The remaining n-1 elements can be chosen in $\binom{n+1}{n-1}$ ways. Then, by lexicographic ordering these arrays are made to correspond to

$$J = n+2, \text{ - - -}, \binom{n+2}{n} .$$

This method is continued for n+3, n+4, - -, as the nth element of the array of n integers.

Suppose that the arrays of n positive integers having n, n+1, - -, N as the nth element have already been ordered. There will be $\binom{N}{n-1}$ new arrays formed having N+1 as the nth element. The positive integers J which correspond to these $\binom{N}{n-1}$ arrays are then

$$J = \binom{N}{n} + 1, \text{ - - -}, \binom{N+1}{n} .$$

As an example, with n = 3, then

$$(1, 2, 3) \sim 1$$
$$(1, 2, 4) \sim 2$$
$$(1, 3, 4) \sim 3$$
$$(2, 3, 4) \sim 4$$
$$(1, 2, 5) \sim 5$$
$$(1, 3, 5) \sim 6$$
$$(1, 4, 5) \sim 7$$
$$(2, 3, 5) \sim 8$$
$$(2, 4, 5) \sim 9$$
$$(3, 4, 5) \sim 10$$

and so forth.

<u>Lemma 3.1</u>. The totality of the number of arrays of n positive integers as described above forms a denumerable set.

Proof: Consider the increasing sequence of non-negative integers $\{n+k\}$, $k = 0, 1, ---$. Recall the method for associating the positive integers J with arrays of n positive integers. Let N, $N \geq n$, denote a positive integer. When N is increased from n+k to n+k+1 the number of new arrays added will be

$$\binom{n+k}{n-1} ,$$

all having the property that n+k+1 occurs as the nth element. With each integer $k = 0, 1, 2, ---$, there will be associated the finite number $\binom{n+k}{n-1}$ of arrays of n integers. For every pair of distinct values of k it is clear that the associated sets of arrays are disjoint sets. Therefore, since the non-negative integers form a denumerable set, the totality of arrays of n positive integers forms a denumerable set.

In the Hilbert space H consider an arbitrary fixed element g and the elements $\{p_i\}$, $i = 1, --, n$. As linear combinations of elements ϕ_k of a complete orthonormal set $\{\phi_k\}$, these can be written

$$p_i = \sum_{k=1}^{\infty} p_i^{\,k} \phi_k \qquad i = 1, --, n$$

$$g = \sum_{k=1}^{\infty} g^k \phi_k .$$

In this case the inner product of two elements f, g of H becomes

$$(f, g) = \sum_{i=1}^{\infty} f^i g^i .$$

This, of course, is finite as are also the norms

$$\| p_i \| = \left[\sum_{k=1}^{\infty} (p_i^k)^2 \right]^{1/2} \qquad i=1, - -, n$$

$$\| g \| = \left[\sum_{k=1}^{\infty} (g^k)^2 \right]^{1/2} .$$

We denote by P the infinite matrix which has for its ith column the Fourier coefficients of the element p_i with respect to the complete orthonormal system $\left\{ \phi_k \right\}$, $k = 1, 2, - -$, namely,

$$p_i^{\ 1}, \ p_i^{\ 2}, \ - - - \quad .$$

The matrix P is then

(3.1) $$P = \begin{pmatrix} p_1^1 & p_2^1 & - & - & p^1{}_n \\ p_1^2 & p_2^2 & - & - & p^2{}_n \\ \vdots & \vdots & & & \vdots \end{pmatrix} .$$

Similarly, denote by $\bar{g}$ the column matrix having the Fourier coefficients of g as its elements, namely

(3.2) $$\bar{g} = (g^1, g^2, - - -)^T$$

where here again T means the transpose.

<u>Definition 3.3</u>. The norm $\| A \|$ of a matrix $A = (a_{ij})$ is defined as

$$\|A\| = \left[\sum_i \sum_j (a_{ij})^2\right]^{1/2}$$

if this quantity is finite.

Clearly for a finite matrix the norm is finite.

Lemma 3.2. For an infinite matrix of the form P, or of the form P^T, the transpose of P,

$$\|P\|^2 = \sum_{j=1}^{n} \|p_j\|^2 . \tag{3.3}$$

Proof:

$$\|P\|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{n} (p^i{}_j)^2$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{\infty} (p^i{}_j)^2 = \sum_{j=1}^{n} \|p_j\|^2$$

where the order of summation can be changed since the elements are all non-negative and the limits

$$\sum_{i=1}^{\infty} (p^i{}_j)^2$$

exist for $j = 1, 2, - -, n$.

For any infinite matrix L, (such as P) we shall mean by L_N the matrix obtained from L by replacing all elements in the rows following the Nth row by zeros. Thus

$$(3.4) \qquad P_N = \begin{pmatrix} p_1^1 & p_2^1 & - - & p^1{}_n \\ p_1^2 & p_2^2 & - - & p^2{}_n \\ \vdots & \vdots & & \vdots \\ p_1^N & p_2^N & - - & p^N{}_n \\ 0 & 0 & - - & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix} ;$$

obviously $\| P_N \|$ is finite, and moreover $\| P_N \| \leqq \| P \|$.

Definition 3.4. If A and B are two infinite matrices then we say that the product matrix exists if all of the series

$$\sum_{k=1}^{\infty} a_{ik} b_{kj}$$

converge. Then the element in the ith row and jth column of AB is given by

$$(AB)_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{kj} \quad .$$

It will now be shown that the products $P^T P$ and $P_N{}^T P_N$ exist. First,

$$(P^T P)_{ij} = \sum_{k=1}^{\infty} (P^T)_{ik} P_{kj} = \sum_{k=1}^{\infty} p_i^k p_j^k$$

$$= (p_i, p_j)$$

which is finite. Also,

$$(P_N{}^T P_N)_{ij} = \sum_{k=1}^{\infty} (P_N{}^T)_{ik}(P_N)_{kj}$$

$$= \sum_{k=1}^{N} p_i^k \, p_j^k$$

which is obviously finite.

The infinite matrix Q is defined similarly to P in (3.1).

Lemma 3.3. $P^T Q = (\,(p_i, q_j)\,)$.

Proof:

$$(P^T Q)_{ij} = \sum_{k=1}^{\infty} (P^T)_{ik} Q_{kj}$$

$$= \sum_{k=1}^{\infty} p_i^k \, q_j^k = (p_i, q_j).$$

Thus, each element of $P^T Q$ is finite, and

$$P^T Q = (\,(p_i, q_j)\,).$$

Lemma 3.4. For finite matrices A and B, or if B is finite and A is infinite with $\|A\| < \infty$

$$\|AB\| \leqq \|A\| \quad \|B\| \quad .$$

Proof:

$$\|AB\|^2 = \sum_{i,j} (AB)_{ij}^2$$

$$= \sum_{i,j} \left[\sum_k A_{ik} B_{kj} \right]^2$$

$$\leqq \sum_{i,j} \left(\sum_k A_{ik}^2 \right) \left(\sum_k B_{kj}^2 \right)$$

$$= \left(\sum_{i,k} A_{ik}^2 \right) \left(\sum_{j,k} B_{kj}^2 \right) = \|A\|^2 \|B\|^2$$

using Cauchy's inequality. Since the norm is non-negative, then

$$\|AB\| \leq \|A\| \|B\| .$$

In Lemmas 3.5 and 3.6 the matrices A and B will be infinite matrices with the columns being the Fourier coefficients of a_j, b_j, j = 1, - -, n, which are elements of H, with respect to the complete orthonormal set $\{\phi_k\}$.

Lemma 3.5. $\|A^T B\| \leq \|A\| \|B\|$.

Proof: The element $(A^T B)_{ij}$ is finite for

$$(A^T B)_{ij} = \sum_{k=1}^{\infty} A_{ik}^T B_{kj} = \sum_{k=1}^{\infty} a_i^k b_j^k$$

$$= (a_i, b_j) .$$

Then, by the Schwarz inequality

$$\|A^T B\|^2 = \sum_{ij=1}^{n} (a_i, b_j)^2$$

$$\leq \sum_{i,j=1}^{n} \|a_i\|^2 \|b_j\|^2$$

$$= \|A\|^2 \|B\|^2 .$$

Thus,

$$\|A^T B\| \leq \|A\| \|B\| .$$

<u>Lemma 3.6</u>. $\|A + B\| \leq \|A\| + \|B\|$.

Proof: From Lemma 3.2, applied to A and B, and using the Minkowski inequality and Cauchy's inequality, one obtains

$$\|A + B\|^2 = \sum_{j=1}^{n} \|a_j + b_j\|^2$$

$$\leq \sum_{j=1}^{n} (\|a_j\| + \|b_j\|)^2$$

$$= \sum_{j=1}^{n} \|a_j\|^2 + 2 \sum_{j=1}^{n} \|a_j\| \|b_j\| + \sum_{j=1}^{n} \|b_j\|^2$$

$$\leq \|A\|^2 + 2 \left[\sum_{j=1}^{n} \|a_j\|^2\right]^{\frac{1}{2}} \left[\sum_{j=1}^{n} \|b_j\|^2\right]^{\frac{1}{2}} + \|B\|^2$$

$$= \|A\|^2 + 2 \|A\| \|B\| + \|B\|^2$$

$$= (\|A\| + \|B\|)^2 .$$

Therefore,

$$\|A + B\| \leq \|A\| + \|B\| .$$

<u>Lemma 3.7</u>. For the matrices P and P_N,

$$\lim_{N \to \infty} \|P - P_N\| = 0 .$$

Proof: Since the order of sums can be reversed

$$\|P - P_N\|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{n} (P - P_N)_{ij}^2$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{\infty} (P - P_N)_{ij}^2$$

$$= \sum_{j=1}^{n} \sum_{i=N+1}^{\infty} (p^i{}_j)^2 .$$

Since $\sum_{i=1}^{\infty} (p^i{}_j)^2$ is finite for $j = 1, 2, --, n$, then for every $\epsilon > 0$, there exists an N_j, $j = 1, --, n$, such that whenever $N > N_j$,

$$\sum_{i=N+1}^{\infty} (p^i{}_j)^2 < \frac{\epsilon^2}{n}, \qquad j = 1, --, n.$$

Now, let $N^* = \max_j \{N_j\}$. Then, for $N > N^*$,

$$\| P - P_N \|^2 < \sum_{j=1}^{n} \frac{\epsilon^2}{n} = \epsilon^2 .$$

Therefore, $\| P - P_N \| < \epsilon$ for $N > N^*$, and

$$\lim_{N \to \infty} \| P - P_N \| = 0 .$$

Theorem 3.1. If P and P_N are matrices as previously defined and if C is an nxn matrix, then

$$\lim_{N \to \infty} |(P_N C)^T (P_N C)| = |(PC)^T (PC)| .$$

Proof: One sees that by applying Cauchy's inequality

$$\sum_{i=1}^{\infty} (PC)^2_{ij} = \sum_{i=1}^{\infty} \left[\sum_{k=1}^{n} p_{ik} c_{kj} \right]^2$$

$$\leq \sum_{i=1}^{\infty} \left[\sum_{k=1}^{n} (p^i{}_k)^2 \right] \left[\sum_{k=1}^{n} c^2_{kj} \right]$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{\infty} (p^{i}{}_{k})^2 \sum_{k=1}^{n} (C_{kj})^2$$

$$\leqq \|P\|^2 \ \|C\|^2 .$$

Therefore, the jth column of PC consists of the Fourier coefficients of an element of H in terms of a complete orthonormal set $\left\{ \phi_k \right\}$.

Each element of the matrix $(PC)^T(PC)$ is then finite since it will be the inner product of two elements of H. Thus, by Lemma 3.2 $\|PC\|$ is finite. Clearly the product matrix $(P_NC)^T(P_NC)$ is defined since all rows after the Nth in P_NC are zero. Thus one sees that the jth column of

$$PC - P_NC$$

consists of just the Fourier coefficients of an element of H. Therefore, the element

$$(\ (PC - P_NC)^T(PC)\ )_{ij}$$

is finite, since the inner product of two elements of Hilbert space is finite. Also, the element

$$(\ (P_NC)^T(PC - P_NC)\ )_{ij}$$

is finite since all columns of $(P_NC)^T$ after the Nth are zero.

Then, by Lemmas 3.4, 3.5, and 3.6 as well as

$$\|P_NC\| \ \leqq \ \|PC\|$$

one has

$$\|(PC)^T(PC) - (P_NC)P_NC)\|$$

$$\leq \|(PC)^T(PC) - (P_NC)^T(PC)\| + \|(P_NC)^T(PC) - (P_NC)^T(P_NC)\|$$

$$\leq \|((PC)^T - (P_NC)^T)(PC)\| + \|(P_NC)^T(PC - P_NC)\|$$

$$= \|(PC - P_NC)^T PC\| + \|(P_NC)^T(PC - P_NC)\|$$

$$\leq \|PC - P_NC\| \|PC\| + \|P_NC\| \|PC - P_NC\|$$

$$\leq \|P - P_N\| \|C\| \|PC\| + \|PC\| \|P - P_N\| \|C\|$$

$$= 2 \|C\| \|PC\| \|P - P_N\| .$$

From Lemma 3.7, $\lim_{N \to \infty} \|P - P_N\| = 0$, so clearly

(3.5) $$\lim_{N \to \infty} \|(PC)^T(PC) - (P_NC)^T(P_NC)\| = 0 .$$

Therefore,

$$((PC)^T(PC) - (P_NC)^T(P_NC))_{ij} \to 0 \quad \text{as } N \to \infty .$$

Thus,

(3.6) $$\lim_{N \to \infty} |(P_NC)^T(P_NC)| = |(PC)^T PC|$$

since a determinant is a continuous function of its elements.

<u>Corollary</u>. If P and P_N are matrices as previously defined, then

(3.7) $$\lim_{N \to \infty} |P_N{}^T P_N| = |P^T P| .$$

Proof: The proof is immediate from Theorem 3.1 if one takes C to be the nxn identity matrix in (3.6).

Consider again the projection operator P^J, recalling that the positive integers J correspond to arrays of n positive integers. If J corresponds to the n positive integers $\{\ell_1, \ell_2, --, \ell_n\}$, then the projection operator P^J maps every element h of H into the subspace

$$\left[\phi_{\ell_1}, \phi_{\ell_2}, --, \phi_{\ell_n}\right].$$

In particular for the elements Tf and g, g an arbitrary fixed element of H, and T the operator on H defined in (2.5), one gets

$$(3.7) \qquad P^J Tf = \sum_{i=1}^{n} \sum_{j=1}^{n} p_j^{\ell_i} \phi_{\ell_i} (q_j, f)$$

$$(3.8) \qquad P^J g = \sum_{i=1}^{n} g^{\ell_i} \phi_{\ell_i} .$$

Now, we consider the question of the existence of an element $f \in H$ for which

$$(3.9) \qquad P^J Tf = P^J g.$$

Since the $\{\phi_k\}$ are linearly independent, it follows from considering (3.7) and (3.8) in the expression (3.9) that

$$(3.10) \qquad \sum_{j=1}^{n} p_j^{\ell_i} (q_j, f) = g^{\ell_i} \qquad i = 1, --, n.$$

This may be written as

$$(3.11) \qquad P^J X = \bar{g}^J$$

if we introduce the matrices

(3.12)

$$P^J = \begin{pmatrix} p_1^{\ 1} & - & - & - & p_n^{\ 1} \\ \vdots & & & & \vdots \\ p_1^{\ n} & & & & p_n^{\ n} \end{pmatrix}$$

(3.13)

$$\bar{g}^{\ J} = \begin{pmatrix} g^{\ 1} \\ \vdots \\ g^{\ n} \end{pmatrix} \ ; \quad X = \begin{pmatrix} (q_1, f) \\ \vdots \\ (q_n, f) \end{pmatrix} .$$

<u>Definition 3.5</u>. If A is an nxn matrix, and X and C are nx1 matrices, then the system

$$AX = C$$

is said to be a consistent system if the rank of the augmented matrix formed from A and C is the same as the rank of A.

Since the rank of A $\leq$ rank of (A, C) $\leq$ n, then if the rank of A is n the system will be a consistent one and will have a unique solution, since then $|A| \neq 0$.

A question which arises now is whether there exists an element $f \in H$ that is a solution of (3.10), or (3.11), which result from the equation

$$\rho^{\ J}Tf = \rho^{\ J}g \ .$$

Suppose that there exists an element in M_q which is a solution of (3.10). Denote this solution by f^J and let

(3.14)
$$f^J = \sum_{k=1}^{n} \alpha_k^J q_k \ .$$

Then, for f^J in (3.10), one obtains

$$(3.15) \qquad \sum_{j,k=1}^{n} p_j^{\ell_i}(q_j, q_k) \alpha_k^J = g^{\ell_i} \quad i = 1, --, n \ .$$

Then from (3.15) one obtains

$$(3.16) \qquad P^J Q^T Q \alpha^J = \bar{g}^J$$

where

$$\alpha^J = \begin{pmatrix} \alpha_1^J \\ \vdots \\ \alpha_n^J \end{pmatrix} .$$

By Cramer's rule and since $|Q^T Q| = G(q)$,

$$(3.17) \quad |P^J| \; G(q) \; \alpha_m^J = |(P^J Q^T Q \vdots \bar{g}^J; m)| \quad m = 1, --, n.$$

If the system (3.15) is a consistent one, then there do exist numbers satisfying (3.17)

$$\alpha_m^J \qquad\qquad m = 1, --, n.$$

In this case then there exists an element $f^J \in M_q$ given by (3.14) which is a solution of (3.10). If the system is inconsistent then the matrix P^J in (3.16) is singular and $|P^J| = 0$. In fact $|P^J| = 0$ if the rank of P^J is less than n even though the system might be a consistent one.

IV. MINIMUM NORM SOLUTION OF TF = g AS AN AVERAGE OF THE SOLUTIONS OF PROJECTED EQUATIONS

In a similar manner to the way in which P^J was defined, we define $P_N{}^J$ as the nxn matrix with rows $\ell_1, --, \ell_n$ from P_N, where the positive integer J corresponds to the array of n positive integers in their usual order, $\{\ell_1, \ell_2, --, \ell_n\}$.

If A is of order nxm and B is of order mxn, where $m \geq n$, then the determinant of the matrix AB, namely $|AB|$, is equal to the sum of $\binom{m}{n}$ products made by pairing each minor of order n from n columns of A with the minor of order n from the corresponding rows of B [1].

From the definition of P_N in (3.4), it is seen that the product $P_N{}^T P_N$ may be considered as the product of an nxN matrix and an Nxn matrix. Then, for $N \geq n$ the determinant $|P_N{}^T P_N|$ can be written as

$$(4.1) \qquad |P_N{}^T P_N| = \sum_{J=1}^{\binom{N}{n}} |(P_N{}^J)^T| \, |P_N{}^J| \quad .$$

<u>Theorem 4.1</u> $\quad |P^T P| = \sum_{J=1}^{\infty} |(P^J)^T| \, |P^J| = \sum_{J=1}^{\infty} |P^J|^2 .$

Proof: For the case of P_N, $N \geq n$

$$|P_N{}^T P_N| = \sum_{J=1}^{\binom{N}{n}} |(P_N{}^J)^T| \, |P_N{}^J| = \sum_{J=1}^{\binom{N}{n}} |P^J|^2 .$$

Now, by the corollary to Theorem 3.1,

$$\lim_{N\to\infty} |P_N{}^T P_N| = |P^T P| \quad .$$

Thus, the subsequence of partial sums

$$\sum_{J=1}^{\binom{N}{n}} |P^J|^2$$

which equal $|P_N{}^T P_N|$ converges to $|P^T P|$.

Since the series

$$\sum_{J=1}^{\infty} |P^J|^2$$

has all non-negative terms, the series itself converges, and must then converge to $|P^T P|$. Thus

$$|P^T P| = \sum_{J=1}^{\infty} |P^J|^2 \quad .$$

<u>Theorem 4.2</u>.

$$(4.2) \quad |(PQ^T Q \vdots \bar{g};m)^T (PQ^T Q \vdots \bar{g};m)| = \sum_{J=1}^{\infty} |(P^J Q^T Q \vdots \bar{g}^J; m)|^2 .$$

Proof: The matrix $(PQ^T Q \vdots \bar{g}; m)$ has the same properties as P in that they were used in the proof of Theorem 4.1. The proof is similar to that of Theorem 4.1.

<u>Lemma 4.1</u>. Consider the bracketed series

$$S = \sum_{i=1}^{\infty} c_i \qquad \text{where } c_i = \sum_{j=L_{i-1}+1}^{L_i} a_j b_j$$

where the a_j and b_j are real,

$$\sum_{j=1}^{\infty} a_j^{\,2} \quad \text{and} \quad \sum_{j=1}^{\infty} b_j^{\,2} \quad \text{are finite,}$$

and $1 = L_0 < L_1 < L_2 < - - -$, $L_i \to \infty$ as $i \to \infty$. If $S_N = \sum_{i=1}^{N} c_i$,

and $\lim_{N \to \infty} S_N = S$, then $\lim_{M \to \infty} T_M = S$, where $T_M = \sum_{j=1}^{M} a_j b_j$.

Proof: Since $\sum_{j=1}^{\infty} a_j^{\,2}$ and $\sum_{j=1}^{\infty} b_j^{\,2}$ are convergent series, there exist positive integers h_1 and h_2 such that for every $\epsilon > 0$,

$$\sum_{j=h+1}^{\infty} a_j^{\,2} < \epsilon \qquad \text{whenever } h \geqq h_1$$

$$\sum_{j=h+1}^{\infty} b_j^{\,2} < \epsilon \qquad \text{whenever } h \geqq h_2 \; .$$

Let $M' = \max (h_1, h_2)$.

Consider

$$|T_{M_1} - T_{M_2}| = \left| \sum_{j=M_2+1}^{M_1} a_j b_j \right| \leqq \sum_{j=M_2+1}^{M_1} |a_j b_j|$$

$$\leqq \left[\sum_{j=M_2+1}^{M_1} a_j^2 \right]^{\frac{1}{2}} \left[\sum_{j=M_2+1}^{M_1} b_j^2 \right]^{\frac{1}{2}}$$

by the Cauchy inequality. Then, for $M_1 > M_2 \geqq M'$, M' defined as above,

$$|T_{M_1} - T_{M_2}| < \epsilon \; .$$

Therefore, $\left\{T_M\right\}$ is a Cauchy sequence. Thus, $\lim_{M \to \infty} T_M$ exists, and must equal S, since the subsequence, $T_{L_N} = S_N$, converges to S.

Since the matrix $P^T PQ^T Q$ appears in a later theorem it will be necessary to show that the associative law applies to this matrix. That is,

$$(4.3)\quad (P^T P)(Q^T Q)=(P^T(PQ^T))Q=P^T((PQ^T)Q)=((P^T P)Q^T)Q=P^T(Q^T Q)) \; .$$

For all positive integers M and N, such an equality holds for the partial sums, namely

$$(4.4)\quad \sum_{\ell=1}^{n} \sum_{k=1}^{N} \sum_{s=1}^{M} p_i^k p_\ell^k q_\ell^s q_j^s = \sum_{s=1}^{M} \sum_{k=1}^{N} \sum_{\ell=1}^{n} p_i^k p_\ell^k q_\ell^s q_j^s$$

$$(4.5)\quad = \sum_{k=1}^{N} \sum_{s=1}^{M} \sum_{\ell=1}^{n} p_i^k p_\ell^k q_\ell^s q_j^s$$

$$(4.6)\quad = \sum_{s=1}^{M} \sum_{\ell=1}^{n} \sum_{k=1}^{N} p_i^k p_\ell^k q_\ell^s q_j^s$$

$$(4.7)\quad = \sum_{k=1}^{N} \sum_{\ell=1}^{n} \sum_{s=1}^{M} p_i^k p_\ell^k q_\ell^s q_j^s \; .$$

Now consider

$$\sum_{\ell=1}^{n} \sum_{k=1}^{N} \sum_{s=1}^{M} |p_i^k p_\ell^k q_\ell^s q_j^s| = \sum_{\ell=1}^{n} \sum_{k=1}^{N} \sum_{s=1}^{M} |p_i^k p_\ell^k| |q_\ell^s q_j^s|$$

$$= \sum_{\ell=1}^{n} \sum_{k=1}^{N} |p_i^k p_\ell^k| \sum_{s=1}^{M} |q_\ell^s q_j^s|$$

$$\leq \sum_{\ell=1}^{n} \left[\sum_{k=1}^{N} (p_i^k)^2\right]^{\frac{1}{2}} \left[\sum_{k=1}^{N} (p_\ell^k)^2\right]^{\frac{1}{2}} \left[\sum_{s=1}^{M} (q_\ell^s)^2\right]^{\frac{1}{2}} \left[\sum_{s=1}^{M} (q_j^s)^2\right]^{\frac{1}{2}}$$

$$\leq \|p_i\| \; \|q_j\| \sum_{\ell=1}^{n} \|p_\ell\| \; \|q_\ell\| \quad .$$

Therefore the left side of (4.4) is convergent since it is absolutely convergent. Because of the absolute convergence any rearrangement is permissible so as $M, N \to \infty$, each of the right hand sides of (4.4), (4.5), (4.6) and (4.7) converge to a common limit, namely

$$\sum_{\ell=1}^{n} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} p_i^k p_\ell^k q_\ell^s q_j^s = (P^T P)(Q^T Q) \ .$$

The expressions on the right in (4.4), (4.5), (4.6) and (4.7) converge respectively to $(P^T(PQ^T))Q$, $P^T((PQ^T)Q)$, $((P^TP)Q^T)Q$ and $P^T(P(Q^TQ))$. This establishes the equality (4.6).

It is clear from Definition 2.4 that

(4.8) $(P^TPQ^TQ \vdots P^T\bar{g};\ m) = P^T(PQ^TQ \vdots \bar{g};\ m)$

where m ranges from 1 to n.

<u>Theorem 4.3</u>. For $\bar{g}$, P, Q as defined previously

(1) $|P^T(PQ^TQ \ \vdots\ \bar{g};\ m)| = \lim_{N \to \infty} |P_N{}^T(P_NQ^TQ \vdots\ \bar{g};\ m)|$

(2) $|(P^T(PQ^TQ \ \vdots\ \bar{g};\ m)| = \sum_{J=1}^{\infty} |P^J| \ |(P^JQ^TQ \ \vdots\ \bar{g}^J;\ m)|$

for m = 1, - -, n.

Proof of (1): It has been shown that the products P^TPQ^TQ and $P^T\bar{g}$ are defined. Also, the elements of the matrices $P_N{}^TP_NQ^TQ$ and $P_N\bar{g}$ are clearly finite.

It follows from addition of matrices that

$$(4.9)\quad (P^TPQ^TQ \,\vdots\, P^T\bar{g};\, m) - (P_N{}^TP_NQ^TQ \,\vdots\, P_N{}^T\bar{g};\, m)$$

$$= (P^TPQ^TQ - P_N{}^TP_NQ^TQ \,\vdots\, (P^T - P_N{}^T)\,\bar{g};\, m)$$

and, for each m,

$$\|(P^TPQ^TQ - P_N{}^TP_NQ^TQ \,\vdots\, (P - P_N)^T\,\bar{g};\, m)\|$$

$$\leqq \|(P^TP - P_N{}^TP_N)\,Q^TQ\| + \|(P - P_N)^T\,\bar{g}\|$$

$$\leqq \|P^TP - P_N{}^TP_N\|\ \|Q^TQ\| + \|P - P_N\|\ \ \|\bar{g}\|$$

by Lemma 3.5. According to Lemma 3.7

$$\lim_{N\to\infty}\|P - P_N\| = 0$$

and from (3.7)

$$\lim_{N\to\infty}\|P^TP - P_N{}^TP_N\| = 0\ .$$

Therefore, for each m, from (4.12) it follows that

$$(P^TPQ^TQ \,\vdots\, P^T\bar{g};\, m)_{ij} - (P_N{}^TP_NQ^TQ \,\vdots\, P_N{}^T\,\bar{g};\, m)_{ij} \to 0$$

as $N\to\infty$.

Since a determinant is a continuous function of its elements, then, for m = 1, - -, n

$$|P^T(PQ^TQ \,\vdots\, \bar{g};\, m)| = \lim_{N\to\infty}|P_N{}^T(P_NQ^TQ \,\vdots\, \bar{g};\, m)|\ .$$

Proof of (2): In order to prove part (2), Lemma 4.1 will be used. From Theorems 4.1 and 4.2,

$$\sum_{J=1}^{\infty} |P^J|^2 \quad ; \quad \sum_{J=1}^{\infty} |(P^J Q^T Q \vdots \bar{g}^J;\ m)|^2 \quad m = 1, --, n$$

are both finite. The quantities

$$|P^J| \qquad \text{and} \qquad |(P^J Q^T Q \vdots \bar{g}^J;\ m)|$$

correspond to the a_j and b_j in Lemma 4.1. The binomial coefficient $\binom{N}{n}$, $N \geq n$, corresponds to the L_i.

Again according to the expansion of the determinant of the product of two matrices as described in the second paragraph of this chapter, the determinant $|P_N^T(P_N Q^T Q \vdots \bar{g};\ m)|$ can be written for m = 1, - -, n, as

$$|P_N^T(P_N Q^T Q \vdots \bar{g};\ m)| = \sum_{J=1}^{\binom{N}{n}} |(P^J)^T|\ |(PQ^T Q \vdots \bar{g};\ m)^J|$$

$$(4.10) \qquad = \sum_{J=1}^{\binom{N}{n}} |P^J|\ |(P^J Q^T Q \vdots \bar{g}^J;\ m)| \quad .$$

From part (1) of this theorem the limit of partial sums for the subsequence in (4.10) exists, namely

$$|P^T(PQ^T Q \vdots \bar{g};\ m)| = \lim_{N \to \infty} \sum_{J=1}^{\binom{N}{n}} |P^J|\ |(P^J Q^T Q \vdots \bar{g}^J;\ m)| \ .$$

Since the hypotheses of Lemma 4.1 are satisfied, then the limit as $M \to \infty$ of

$$\sum_{J=1}^{M} |P^J| \, |(P^J Q^T Q \vdots \bar{g}^J ;\, m)| \qquad m = 1, \text{--}, n$$

exists. In fact, then

$$\lim_{M \to \infty} \sum_{J=1}^{M} |P^J| \, |(P^J Q^T Q \vdots \bar{g}^J ;\, m)|$$

$$= |P^T (PQ^T Q \vdots \bar{g};\, m)| \qquad m = 1, \text{--}, n$$

which establishes (2).

Let us consider now (2.16), which can now be written for $m = 1, \text{--}, n$, as

$$(4.11) \qquad G(p)G(q)c_m = |(P^T PQ^T Q \vdots P^T \bar{g};\, m)| \,.$$

Then, applying Theorem 4.3, from (4.11) one obtains for $m = 1, \text{--}, n$

$$(4.12) \qquad G(p)G(q)c_m = \sum_{J=1}^{\infty} |P^J| \, |(P^J Q^T Q \vdots \bar{g}^J ;\, m)| \,.$$

In those systems (3.16) for which $|P^J| \neq 0$ there will be a unique solution α^J. In this case, from (3.17) one gets by substitution in (4.12)

$$(4.13) \qquad G(p)G(q)c_m = \sum_{J=1}^{\infty} |P^J|^2 G(q) \, \alpha_m^J \qquad m = 1, \text{--}, n$$

where for those J such that $|P^J| = 0$, one may take α_m^J to be anything. Now since $G(q) \neq 0$, it follows from (4.13) that

$$(4.14) \qquad c_m = \sum_{J=1}^{\infty} \frac{|P^J|^2}{G(p)} \, \alpha_m^J \qquad m = 1, 2, \text{--}, n \,.$$

From (2.14) and (3.14), f' and f^J are elements of M_q, namely

$$f' = \sum_{i=1}^{n} c_i q_i$$

$$f^J = \sum_{i=1}^{n} \alpha_i^J q_i \qquad J = 1, 2, - - .$$

Theorem 4.4. For f' and f^J as defined above

$$(4.15) \qquad f' = \sum_{J=1}^{\infty} \frac{|P^J|^2}{G(p)} f^J .$$

The element f' may be considered as a weighted average of the f^J, $J = 1, 2, - -$ since

$$(4.16) \qquad \sum_{J=1}^{\infty} \frac{|P^J|^2}{G(p)} = 1 .$$

Proof: The proof of (4.19) follows from Theorem 4.1. That is,

$$\sum_{J=1}^{\infty} \frac{|P^J|^2}{G(p)} = \frac{1}{G(p)} \sum_{J=1}^{\infty} |P^J|^2$$

$$= \frac{1}{G(p)} |P^T P| = 1 .$$

Now, from (4.17), it follows that

$$\sum_{m=1}^{n} c_m q_m = \sum_{m=1}^{n} \sum_{J=1}^{\infty} \frac{|P^J|^2}{G(p)} \alpha_m^J q_m .$$

We can interchange the order of summation here, for

$$\| \sum_{m=1}^{n} c_m q_m - \sum_{J=1}^{M} \sum_{m=1}^{n} \frac{|P^J|^2}{G(p)} \alpha_m^J q_m \|$$

$$= \left\| \sum_{m=1}^{n} \left[c_m - \sum_{J=1}^{M} \frac{|p^J|^2}{G(p)} \alpha_m^J \right] q_m \right\|$$

$$\leqq \sum_{m=1}^{n} \left| c_m - \sum_{J=1}^{M} \frac{|p^J|^2}{G(p)} \alpha_m^J \right| \| q_m \|$$

using properties of the norm. Thus,

$$\lim_{M \to \infty} \left\| \sum_{m=1}^{n} c_m q_m - \sum_{J=1}^{M} \sum_{m=1}^{n} \frac{|p^J|^2}{G(p)} \alpha_m^J q_m \right\| = 0$$

which can be written as

$$\sum_{m=1}^{n} c_m q_m = \sum_{J=1}^{\infty} \sum_{m=1}^{n} \frac{|p^J|^2}{G(p)} \alpha_m^J q_m ,$$

or, in fact

$$f' = \sum_{J=1}^{\infty} \frac{|p^J|^2}{G(p)} f^J .$$

This completes the proof of the theorem.

V. SUMMARY

Let T be the operator defined for every element f of a Hilbert space H by

$$Tf = \sum_{i=1}^{n} p_i(q_i, f)$$

where the p_i and q_i are elements of H. For any fixed g in H it was shown that even though there is not in general a solution to the equation Tf = g there does exist an element f of H which minimizes the norm expression

$$\| Tf - g \| .$$

Projection operators P^J, J = 1, 2, - - were introduced which map elements of H into n-dimensional subspaces spanned by n elements of a complete orthonormal set $\{\phi_k\}$ in H. A one-to-one correspondence between the positive integers J and the subsets $(\phi_{k_1}, \phi_{k_2}, --, \phi_{k_n})$ was set up and the projected equations

$$P^J Tf = P^J g \qquad J = 1, 2, - -$$

were examined.

The principal result of this thesis is that an element of H that minimizes $\| Tf - g \|$ can be written as a weighted average of solutions of those projected equations which are consistent.

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