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ERRATUM: SINGULARITY FORMATION IN CHEMOTAXIS—A CONJECTURE OF NAGAI*

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Abstract. In [H. A. Levine and J. Rencławowicz, *SIAM J. Appl. Math.*, 65 (2004), pp. 336–360] we considered the problem $u_t = u_{xx} - (uv_x)_x, v_t = u - av$ on the interval $I = [0, 1]$, where $u_x, v_x = 0$ at the end points, $u(x, 0), v(x, 0)$ are prescribed, and $a > 0$. (It was claimed in that article that there were solutions that blow up in finite time in every neighborhood of the spatially homogeneous steady state $(u, v) = (\mu, \mu/a)$ if $\mu > a$.) Here we correct an estimate and reduce Nagai's conjecture to the following statement. Let $\sigma = a/(\mu - a), \rho_1 = 1$. If $\lim_{n\to+\infty} \rho_n$ exists, where for $n \geq 2$, $\rho_n^n \equiv 1/(n-1)\sum_{j=1}^{n-1}(1 + \sigma/j)\rho_j^j\rho_{n-j}^{n-j}$, then the blow up assertion holds.

Key words. chemotaxis, finite time singularity formation, Keller–Segel model

AMS subject classifications. 35K55, 92C17

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1. Introduction.

In [1] we studied the system $u_t = u_{xx} - (uv_x)_x, v_t = u - av$ on the interval $I = [0, 1]$, where $u_x, v_x = 0$ at the end points, $u(x, 0), v(x, 0)$, are prescribed, and $a > 0$. Nagai and Nakaki [2] showed that there are solutions that are unbounded in finite *or* in infinite time.[1] We claimed that there were initial conditions for which solutions failed to exist for all time. In our proof we used a differential inequality, the derivation of which was unfortunately flawed. We correct this and make more precise the statement proved in [1].

2. Approximate solution.

The notation of [1] is in force here. Because system $u_t = u_{xx} - (uv_x)_x, v_t = u - av$ is autonomous, we can assume the initial values are prescribed at $t = 0$ and that the blow up time, when it exists, is positive. As in [1], define, for any sequence $z(t) = \{z_n(t)\}_{n=1}^{\infty}$, $\mathcal{G}_n(z, z') = (1/2)C^2n\{(\mathcal{M}z * z')_n + n\frac{a}{2}(z*z)_n\}$ and $\mathcal{H}_n(z, z') = (1/2)C^2n\{[(T_n\mathcal{M}z, z') - (\mathcal{M}z, T_nz')] + an(z, T_nz)\}$, where $\mathcal{M}z(t) = \{nz_n(t)\}_{n=1}^{\infty}$ and $T_kz(t) = \{z_{n+k}(t)\}_{n=1}^{\infty}$. Here $|z| = \{|z_n|\}_{n=1}^{\infty}$ and $(z*w)_n = \sum_{k=1}^{n-1} z_kw_{n-k}$. (The sum is zero if $n = 1$.)

The infinite system of ordinary differential equations for the cosine coefficients $h(t) = \{h_n(t)\}_{n=1}^{\infty}$ is[2]

$$\mathfrak{L}_nh_n \equiv h_n'' + (C^2n^2 + a)h_n' - (\mu - a)C^2n^2h_n = \mathcal{G}_n(h, h') + \mathcal{H}_n(h, h').$$

The infinite system of ordinary differential equations satisfied by the cosine coefficients for the approximate problem, $g(t) = \{g_n(t)\}_{n=1}^{\infty}$, satisfies $\mathfrak{L}_ng_n = \mathcal{G}_n(g, g')$. The

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[1]The Nagai conjecture states that if $\mu > a$, there are spatially nonhomogeneous solutions beginning in every small neighborhood of $(\mu, \mu/a)$ which cannot exist for all time.

[2]The spatially homogeneous solution is given by $V(t) = \mu/a + (v_0 - \mu/a)\exp(-at)$, $U(t) = \mu$. One sets $\psi(x, t) = v(x, t) - V(t)$, $u(x, t) = \mu + \psi_t + a\psi$. Then $h(t)$ is the sequence of cosine coefficients for $\psi(x, t)$.

particular sequence $g(t) \equiv \{g_n(t) = a_n e^{n\lambda t}\}_{n=1}^{\infty}$ satisfies this system for $a_1 > 0$, and for $n \geq 2$ and any integer $M > 0$ with $C = 2\pi M$, $\mu > a$ if

$$2\lambda[n - a/(4\pi^2 M^2)]a_n = \frac{1}{n-1}\sum_{k=1}^{n-1}[\lambda(n-k)k + ak]a_k a_{n-k}, \tag{2.1}$$

where λ is the positive root of $\lambda^2 + (4\pi^2M^2 + a)\lambda - (\mu - a)4\pi^2M^2 = 0$. There are positive constants a, b, ϵ, δ with $a\epsilon^n \leq na_n \leq b\delta^n$ for all positive integers [1]. From this, it follows that $\liminf_{n\to+\infty}[(-\ln na_n)/(n\lambda)] \equiv \underline{T_b}$ and $\limsup_{n\to+\infty}[(-\ln na_n)/(n\lambda)] \equiv \overline{T_b}$ are finite. Hence there is a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to+\infty}[(-\ln n_k a_{n_k})/(n_k\lambda)] \equiv \underline{T_b}$. For this sequence, $\lim_{k\to+\infty} n_k a_{n_k} \exp(n_k\lambda\underline{T_b}) = 1$. Set $a_n = (A_n/n)\exp(-n\lambda\underline{T_b})$. On the subsequence, $A_{n_k} \to 1$ and

$$\lim_{t\uparrow\underline{T_b}}\sum_{k=1}^{\infty}A_{n_k}e^{-n_k\lambda(\underline{T_b}-t)} = +\infty \text{ and } \lim_{t\uparrow\underline{T_b}}\sum_{k=1}^{\infty}\frac{A_{n_k}e^{-n_k\lambda(\underline{T_b}-t)}}{{n_k}^{1+\delta}} < +\infty \tag{2.2}$$

(for any $\delta > 0$).

Now $\underline{T_b}$ must be the blow up time for the approximate solution $g(t)$ in the space $\ell_1^1(0,\underline{T_b}) \times \ell^1(0,\underline{T_b})$. (A sequence $\{a_n\}$ is in ℓ_1^1 if $\{na_n\}$ is in ℓ^1.) To see this, note that as long as t is in the existence interval,

$$\begin{aligned}\|\mathcal{M}g(t)\|_{\ell^1} + \|g'(t)\|_{\ell^1} &= \sum_{n=1}^{\infty}na_n(1+\lambda)e^{n\lambda t} \geq (1+\lambda)\sum_{k=1}^{\infty}n_k a_{n_k}e^{n_k\lambda t}\\ &= (1+\lambda)\sum_{k=1}^{\infty}A_k e^{-n_k\lambda(T_b-t)}.\end{aligned} \tag{2.3}$$

Consequently, from the first equation in (2.2), $g(\cdot)$ must blow up at some time, possibly earlier than $\underline{T_b}$. If $t < \underline{T_b}$, then $\liminf_{n\to+\infty}[(-\ln na_n)/(n\lambda)] \equiv \underline{T_b} > \underline{T_b} - \delta > t$ for some positive δ. Therefore, for sufficiently large N, $\sum_{n=N}^{\infty} na_n e^{n\lambda t} \leq \sum_{n=N}^{\infty} ne^{-n\lambda(\underline{T_b}-\delta-t)} < \infty$.

Set $\sigma = a/\lambda$. Let $\{\ln[na_n/(2a_1^n)]/n\}_{n=1}^{\infty} = \{\ln A_n/n\}_{n=1}^{\infty} \equiv \{p_n/n\}_{n=1}^{\infty}$. The p_n satisfy $p_1 = -\ln 2$, and for $n \geq 2$, $[1 - a/(4\pi^2M^2n)]e^{p_n} = \frac{1}{n-1}\sum_{j=1}^{n-1}(1 + \sigma/j)e^{(p_j+p_{n-j})}$. Then we have the following theorem.

THEOREM 1 (Nagai's conjecture). *Let $\lim_{n\to+\infty}\frac{p_n}{n}$ exist. The corresponding solution of the Nagai problem for which $h_n(0) = g_n(0)$ and $h_n'(0) = g_n'(0)$ for all n cannot both exist and be ℓ^1 regular on $[0,\infty)$. (A solution of the Nagai–Nakaki problem is ℓ^1 regular on an interval $I = [0, T_b)$ if it exists there and if $(\|\mathcal{M}h(s)\|_{\ell^1} + \|h'(s)\|_{\ell^1})$ is uniformly bounded on compact subsets I.)*

3. Estimate. Inequality (7.5) of [1] is incorrect. The correct form of the upper bound for the norm of $g - h \equiv w$, $\|\mathcal{M}w(t)\|_{\ell^1} + \|w'(t)\|_{\ell^1}$, is based on the following (infinite) system of ordinary differential equations:

$$\mathfrak{L}_n w_n = \mathcal{G}_n(h-g,h') + \mathcal{G}_n(g,h'-g') + \mathcal{H}_n(h,h') = \mathcal{G}_n(w,h') + \mathcal{G}_n(g,w') + \mathcal{H}_n(h,h') \tag{3.1}$$

and, for some $B > 0$ depending perhaps on τ but not on w, w', h, h', g, g', is given by

(3.2)

$$\|\mathcal{M}w(t)\|_{\ell^1} + \|w'(t)\|_{\ell^1} \le I(t) + J(t) + B\int_0^t \frac{(\|\mathcal{M}h(s)\|_{\ell^1} + \|h'(s)\|_{\ell^1})^2}{\sqrt{t-s}}\,ds$$
$$+ B\int_0^t \frac{(\|\mathcal{M}w(s)\|_{\ell^1} + \|w'(s)\|_{\ell^1})(\|\mathcal{M}h(s)\|_{\ell^1} + \|h'(s)\|_{\ell^1})}{\sqrt{t-s}}\,ds,$$

where

$$I(t)+J(t) \equiv \int_0^t \sum_{n=1}^{\infty} \mathcal{M}(|g'|*\mathcal{M}|w|)_n e^{-dn^2(t-s)}\,ds + \int_0^t \sum_{n=1}^{\infty} \mathcal{M}^2(|g|*|w|)_n e^{-dn^2(t-s)}\,ds,$$

and where $d > 0$ is the positive constant in [1, Lemma 1]. We have

$$\begin{aligned}
I(t) &= \int_0^t \sum_{n=1}^{\infty}\sum_{k=1}^{n-1} n(n-k)|g'_k||w_{n-k}|e^{-dn^2(t-s)}\,ds \\
&= \int_0^t \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} n(n+k)|g'_k||w_n|e^{-d(n+k)^2(t-s)}\,ds \\
&\le \int_0^t \sum_{k=1}^{\infty} |g'_k|e^{-(d/2)k^2(t-s)}\left[\sum_{n=1}^{\infty}(n+k)e^{-(d/2)(n+k)^2(t-s)}n|w_n|\right] ds \\
&\le c\int_0^t \frac{\sum_{k=1}^{\infty}|g'_k|e^{-(d/2)k^2(t-s)}}{\sqrt{t-s}}\|\mathcal{M}w(s)\|_{\ell_1}\,ds \\
&\le c\int_0^t \sum_{k=1}^{\infty} A_k e^{-(d/2)k^2(t-s)-\lambda k(T_b-s)}\frac{\|\mathcal{M}w(s)\|_{\ell_1}}{\sqrt{t-s}}\,ds \\
&\le c\int_0^t \left\{\sum_{k=1}^{\infty} A_k e^{-[(d/2)k^2+k\lambda](t-s)}\right\}\frac{\|\mathcal{M}w(s)\|_{\ell_1}}{\sqrt{t-s}}\,ds \equiv c\int_0^t \mathcal{W}(t-s)\frac{\|\mathcal{M}w(s)\|_{\ell_1}}{\sqrt{t-s}}\,ds.
\end{aligned}$$

In the same manner,

$$\begin{aligned}
J(t) &= \int_0^t \sum_{n=1}^{\infty}\sum_{k=1}^{n-1} n^2|g_k||w_{n-k}|e^{-dn^2(t-s)}\,ds \\
&\le \int_0^t \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} (n+k)^2|g_k||w_n|e^{-d(n+k)^2(t-s)}\,ds \\
&\le \int_0^t \sum_{k=1}^{\infty} |kg_k|e^{-(d/2)k^2(t-s)}\left[\sum_{n=1}^{\infty}\frac{(n+k)^2}{kn}e^{-(d/2)(n+k)^2(t-s)}n|w_n|\right] ds.
\end{aligned}$$

From the inequality $(k+l)/kl \le 2$,

$$J(t) \le c'\int_0^t \left\{\sum_{k=1}^{\infty} A_k e^{-[(d/2)k^2+k\lambda](t-s)}\right\}\frac{\|\mathcal{M}w(s)\|_{\ell_1}}{\sqrt{t-s}}\,ds \equiv c'\int_0^t \mathcal{W}(t-s)\frac{\|\mathcal{M}w(s)\|_{\ell_1}}{\sqrt{t-s}}\,ds.$$

In view of (2.2), $\lim_{t\uparrow T_b}\sum_{k=1}^{\infty} A_k k^{-2}e^{-k\lambda(T_b-t)} < +\infty$. Thus $\mathcal{W}(t)$ is in every $L^p[0,T_b]$ space for $1 \le p < \infty$. With $f(t) = \|\mathcal{M}w(t)\|_{\ell^1} + \|w'(t)\|_{\ell^1}$, we see $f(t) \le \int_0^t \mathcal{W}(t-s)f(s)/\sqrt{t-s}\,ds + \Phi(h(t))$. From Hölder's inequality with $1/p+1/r+1/q=1$

and $1 < r < 2$, there is a constant $K > 0$ such that $f(t) \leq K[\int_0^t f(s)^q \, ds]^{1/q} + \Phi(h(t))$ on $[0, T_b)$. From Gronwall's inequality, if h is global, $f(t)$ is bounded on $[0, T_b)$. From the first sum in (2.2) and the triangle inequality, this is impossible.

Other minor errors in [1]. Page 345, equation (5.1): Replace $k(w_k g_{n+k} + h_k w_{n+k})$ by $n(w_k g_{n+k} + h_k w_{n+k})$. Page 349, equation in line 13: $c\sqrt{t}$ should be replaced by $c \sup_{[0,T]} \sqrt{t}$.

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