TIME DEPENDENT PULSE PROPAGATION AND SCATTERING IN ELASTIC SOLIDS; AN ASYMPTOTIC THEORY

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INTRODUCTION

Predictive modeling of ultrasonic pulse propagation in elastic solids is usually formulated in the frequency domain. Tractable solutions can then be obtained by using, for example, the powerful technique of geometrical elastodynamics and ray theory for wavefront propagation [1]. Recent advances [2,3] allow us to incorporate the finite pulse width by means of Gaussian profiles. However, a more realistic model should also include the fact that the pulse is of limited duration and therefore spatially localized in all directions. This paper outlines a theory for pulses in the form of a localized disturbance with a Gaussian envelope. The theory is valid if the associated carrier wavelength is short in comparison with typical length scales encountered in the solid. The method provides results explicitly in the time domain without the necessity of intermediate FFTs required by frequency domain methods. Applications to pulse propagation in smoothly varying inhomogeneous media, interface scattering and edge diffraction are discussed. The present theory contains an extra degree of freedom not explicitly considered before, i.e., the temporal width or duration of the pulse. An extensive treatment of the related problem for the scalar wave equation can be found in reference 4.

MATHEMATICAL THEORY

For the sake of generality, the theory is presented for a smoothly varying, inhomogeneous, isotropic elastic medium. The equations of motion are

$$(\lambda u_{k,k})_i + [\mu(u_{i,j}+u_{j,i})]_j - \rho u_{i,tt} = 0, \tag{1}$$

where the elastic moduli λ, μ and the density ρ can vary with position. In Eq. (1), u_i is the displacement in the x_i direction, subscript j denotes the derivative in the x_j direction and the summation convention is assumed. Also, the subscript t means the time derivative. The key to the present theory is the assumption that the displacement is of the form

$$u_i(\underset{\sim}{x},t) = V_i(\underset{\sim}{x},t)\, e^{i\omega\phi(\underset{\sim}{x},t)}, \tag{2}$$

where ω is the central or characteristic frequency of the disturbance. In the standard time harmonic theory, it is implicitly assumed that $\phi(\underset{\sim}{x},t)$ =

$\Phi(\underset{\sim}{x})$-t, but this is not the case here. The frequency ω is assumed to be large, so that when (2) is substituted into (1), we can treat (1) as a sequence of distinct asymptotic equations. The first such equation comes from the terms that are of order ω^2:

$$(\lambda+\mu)V_k\phi_k\phi_i + \mu V_i\phi_k\phi_k - \rho\phi_t^2 V_i = 0 \tag{3}$$

This is the eikonal equation of elastodynamics, and can be simplified by writing $\underset{\sim}{p} = \nabla\phi$, $\underset{\sim}{V} = \underset{\sim}{V}^{(1)} + \underset{\sim}{V}^{(2)}$, where $\underset{\sim}{V}^{(1)}$ and $\underset{\sim}{V}^{(2)}$ are the parts of $\underset{\sim}{V}$ parallel and perpendicular to $\underset{\sim}{p}$, respectively. Then (3) becomes

$$(c_L^2p^2-\phi_t^2)\underset{\sim}{V}^{(1)} + (c_T^2p^2-\phi_t^2)\underset{\sim}{V}^{(2)} = 0, \tag{4}$$

where c_L and c_T are the longitudinal and transverse wave speeds, $c_L^2 = (\lambda+2\mu)/\rho$ and $c_T^2 = \mu/\rho$. Thus, either $\phi_t^2 = c_L^2p^2$ and $\underset{\sim}{V}$ is parallel to $\underset{\sim}{p}$, or $\phi_t^2 = c_T^2p^2$ and $\underset{\sim}{V}$ is perpendicular to $\underset{\sim}{p}$. The former corresponds to a longitudinal pulse and the latter a transverse pulse.

The next equation in the asymptotic sequence is the transport equation for the pre-exponential amplitude. Let the pulse center be at $\bar{\underset{\sim}{x}}_0$ at time t = 0. The subsequent path of the center is at $\bar{\underset{\sim}{x}}(t)$ where $\bar{\underset{\sim}{x}}(t)$ is the solution to the ray equation defined by the eiconal equation. The rays are just straight lines in homogeneous media, but in general satisfy

$$\dot{\bar{x}}_i = c(\bar{\underset{\sim}{x}}(t))e_i(t) \tag{5}$$

$$\dot{e}_i = e_ie_j\,\frac{\partial c}{\partial x_j} - \frac{\partial c}{\partial x_i} \tag{6}$$

where $\underset{\sim}{e}(t)$ is the unit direction vector of the ray and the overdot denotes the total time derivative along the ray. The speed c in (5) and (6) can be either c_L or c_T, and the system is completely described by the initial conditions $\bar{\underset{\sim}{x}}(0) = \bar{\underset{\sim}{x}}_0$ and $\underset{\sim}{e}(0) = \underset{\sim}{e}_0$, the initial ray direction. The transport equation can be solved along the ray in the form

$$\underset{\sim}{V}(\bar{\underset{\sim}{x}}(t),t) = \underset{\sim}{V}_0\,\frac{c(\bar{\underset{\sim}{x}}(t))}{c(\bar{\underset{\sim}{x}}_0)}\left[\frac{(\rho\ c^2\ \det \underset{\sim}{A})(0)}{(\rho\ c^2\ \det \underset{\sim}{A})(t)}\right]^{1/2} \tag{7}$$

where $\underset{\sim}{V} = V\,\underset{\sim}{e}(t)$ for longitudinal motion and $\underset{\sim}{V} = V\,\underset{\sim}{n}(t)$ for transverse motion, where $\underset{\sim}{n}$ is a unit vector orthogonal to $\underset{\sim}{e}(t)$. An equation for the rotation of $\underset{\sim}{n}(t)$ about $\underset{\sim}{e}(t)$ can be obtained, but we defer the details until a later publication. The matrix $\underset{\sim}{A}(t)$ is a complex, 3x3 matrix which satisfies

$$\dot{A}_{ij}(t) = e_i\,\frac{\partial c}{\partial x_k}\,A_{kj} + C^2(B_{ij}-e_ie_kB_{kj}) \tag{8}$$

$$\dot{B}_{ij}(t) = \frac{-1}{c}\,\frac{\partial^2 c}{\partial x_i\partial x_k}\,A_{kj} - \frac{\partial c}{\partial x_i}\,e_kB_{kj}\quad . \tag{9}$$

Initial condition on $\underset{\sim}{A}$ and $\underset{\sim}{B}$ must be specified. It turns out that the combination $\underset{\sim}{B}\underset{\sim}{A}^{-1} \equiv \underset{\sim}{M}(t)$ is equal to the matrix of second derivatives of ϕ along the ray, i.e.,

$$M_{ij}(t) = \frac{\partial^2\phi}{\partial x_i\partial x_j}\,(\bar{\underset{\sim}{x}}(t),t)\quad . \tag{10}$$

It follows from the eikonal equation that $\dot{\phi} = 0$ along the ray, or $\phi(\bar{x}(t),t) = \phi_0$, a constant which can be taken as zero without loss of generality. Also, the gradient of ϕ along the ray, $\nabla\phi(\bar{x}(t),t) = p(\bar{x}(t),t)$ is equal to $e(t)/c(\bar{x}(t))$. Therefore, a local Taylor series expansion of $\phi(x,t)$ about the ray position up to second order in distance gives the paraxial approximation, $\phi \equiv \phi_p$,

$$\phi_p(x,t) = \frac{1}{c}\, e(t)\cdot(x-\bar{x}(t)) + \frac{1}{2} M_{ij}(t)(x_i-\bar{x}_i(t))(x_j-\bar{x}_j(t)) \qquad (11)$$

Equations (2), (7) and (11) define what we call a Gaussian wave packet or GWP for brevity. The GWP will be localized in space if and only if the imaginary part of $M(t)$ is positive definite. It can be shown that if $M(0)$ satisfies this requirement, then (8) and (9) will automatically generate an $M(t)$ that has this property. Without loss of generality, we can specify $A(0) = I$, the identity matrix. Then the subsequent evolution of the GWP is completely prescribed by the initial parameters $\bar{x}_0$, e_0 and $M(0)$.

PROPERTIES OF GAUSSIAN WAVE PACKETS

The key quantity in the theory is the 3x3 complex matrix $M(t)$ which describes the Gaussian envelope of the packet about the pulse center $x = \bar{x}(t)$. The matrix $A(t)$ is like the ray tube area of classical geometrical optics. However, since $M(0)$ is assumed to have a positive definite imaginary part, and $A(0) = I$, the subsequent values of $A_{ij}(t)$ as determined from (8) and (9) are in general complex. Thus, det(A) is really a complex ray tube area. In fact, it can be shown that det(A) is always non-zero, even at the geometrical caustics and loci of classical geometrical optics. Thus, the GWP solution has no singularities. This is one of its most advantageous features; it obviates the necessity of patching up the solution near caustics and loci. The price paid is actually very little in comparison with what one has to do in classical geometric optics to describe the propagation of a ray bundle of time harmonic form. The real set of equations for the ray tube cross-section, the curvature matrix, is a simple projection of the 3x3 complex Eqs. (8) and (9) onto a very similar 2x2 system. The extra algebra in the present theory is really quite trivial.

One example of quite general significance is for a material in which the gradient of $c(x)$, which is either c_L or c_T, is constant. A more generally inhomogeneous medium can be considered by splitting it up into regions of constant ∇c. It is well known in classical geometrical optics that the rays become arcs of circles and the real ray tube cross-section curvatures can be obtained explicitly. Similar explicit solutions can be obtained for all of the GWP parameters, including $M(t)$. A trivial special case of interest is the homogeneous medium, c = constant. Then,

$e(t) = e_o$, $\bar{x}(t) = \bar{x}_0 + cte_0$, and

$$M(t) = M(0)\ [I+c^2t\ PM(0)]^{-1} \qquad (12)$$

where P is the projection matrix $(I-ee^T)$ onto the plane perpendicular to the ray direction. The amplitude follows from Eq. (7), where now

$$\left[\frac{\det A(0)}{\det A(t)}\right]^{1/2} = [\det(I+c^2t\ PM(0))]^{-1/2} \quad . \qquad (13)$$

Various limiting cases of pulses are obtained by taking appropriate values for the initial complex matrix $M(0)$. (1) A very short, thin initial pulse results by letting $\mathrm{Im}(m_s(0)) \to \infty$, where $m_s(t) \equiv e_i(t)e_j(t)M_{ij}(t)$ is the element that determines the length of the pulse in its direction of propagation. (2) Similarly, if $Im(m_s(0)) \to 0$, a very long pulse is obtained. This limit is equal to the case of a time harmonic Gaussian beam, (3) The orthogonal width of the GWP depends upon the 2x2 submatrix $\underset{\sim}{M}_n = \underset{\sim}{P}\underset{\sim}{M}$, where $\underset{\sim}{P} = (\underset{\sim}{I}-\underset{\sim}{e}\underset{\sim}{e}^T)$. Thus, an initially plane pulse corresponds to $\underset{\sim}{M}_n \to 0$, (4) Finally, a GWP of zero initial wavefront curvature, or a point source, is obtained by letting $Im(\underset{\sim}{M}_n(0)) \to \infty$.

REFLECTION AND TRANSMISSION OF GWPs

When a GWP strikes a surface of material discontinuity, e.g., a free surface or the surface of an inclusion, it splits up into reflected and transmitted GWPs, in the same way that a classical ray does. The precise formulation of the interface jump conditions is beyond the scope of the present article, but the general procedure is in the same spirit as, for example, time harmonic geometrical elastodynamics [1]. The properties, i.e., amplitude V(t) and envelope matrix $\underset{\sim}{M}(t)$ of the transmitted and reflected GWPs are completely determined by the material properties of the interface in the neighborhood of the point where the central ray of the incident GWP strikes the interface. The subsequent propagation of the split GWPs is described by the same theory as before.

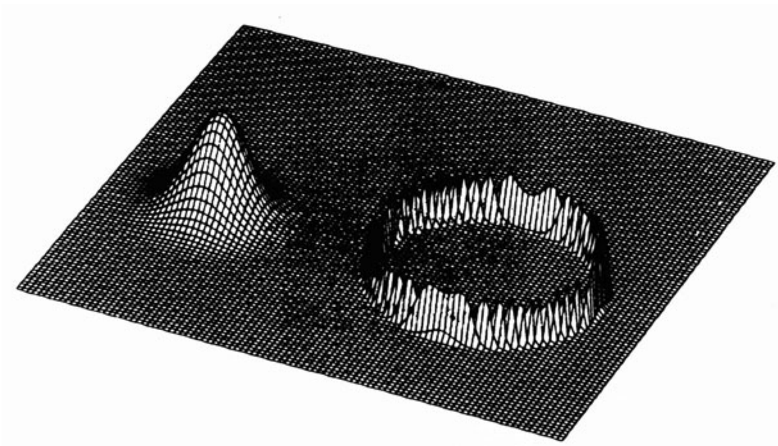

Fig. 1. The incident GWP at t = 0. The circular rim defines the region of slower speed, $c_1/c_0 = 1/4$.

These ideas are made apparent by a specific illustration. The sequence of four figures shows the scattering picture for a pressure wave in an acoustic medium incident upon a region of lower wave speed. The model is two-dimensional for simplicity of presentation. The first figure shows the incident GWP at time t = 0. The GWP is indicated by its envelope function, which we define as $(\mathrm{Re}V)|e^{i\omega\phi}|$. The edge of the "inclusion" is depicted by the circular rim. The initial pulse is circular but broadens in the orthogonal direction as it approaches the interface. The reflection and transmission process occurs between Figs. 2 and 3. Both the reflected and transmitted GWPs are visible in Figs. 3 and 4. Note that the reflected GWP appears as a depression. This is due to the choice of the envelope function, and illustrates that the amplitude V has gone through a 180° phase shift, as we would expect for a soft target, since V defines the displacement in the direction of propagation. This effect would not be apparent for an envelope function like $|Ve^{i\omega\phi}|$.

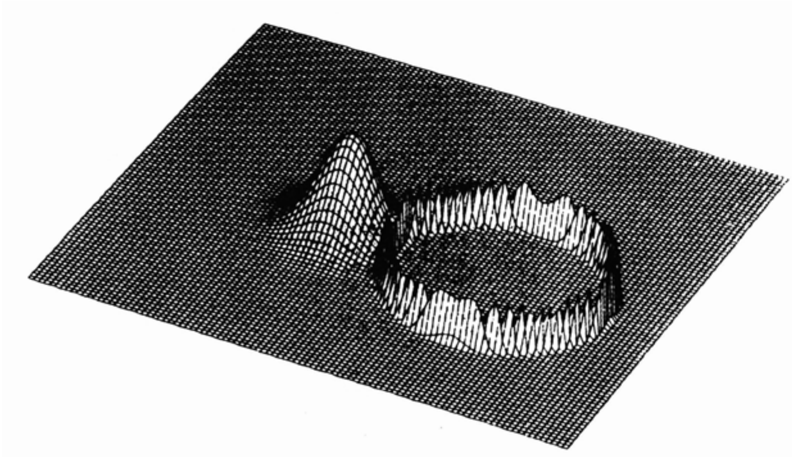

Fig. 2. The incident GWP at t = 1, shortly before it strikes the interface.

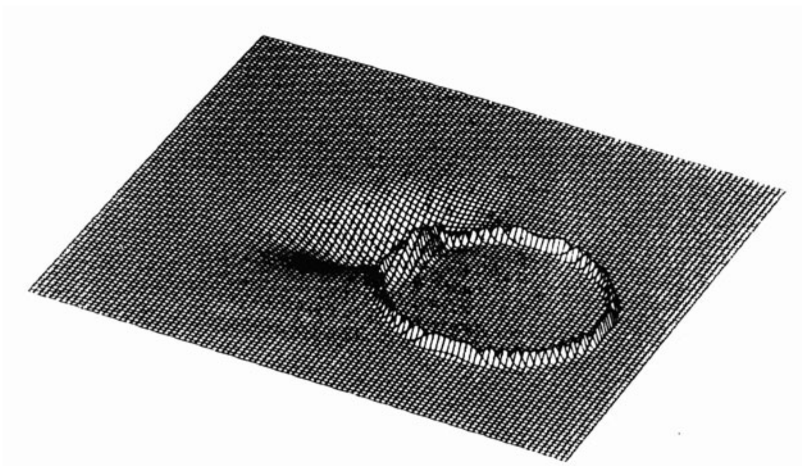

Fig. 3. The reflected and transmitted GWPs at t = 2.

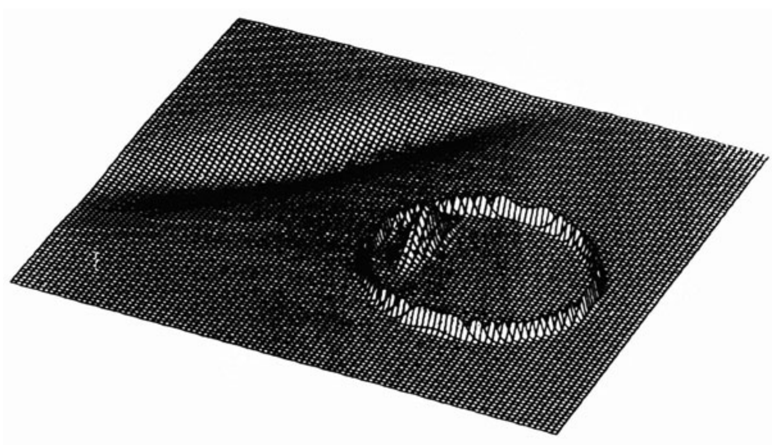

Fig. 4. The transmitted and reflected GWPs at t = 3. Note the significant effects due to the interface curvature, causing focusing and defocusing, respectively.

EDGE DIFFRACTION AND CRITICAL ANGLE REFLECTION

When a GWP strikes the edge of a crack there is a diffraction effect similar to that for a plane wave [1]. Energy is diffracted in all directions away from the edge and mode conversion into both body waves and surface waves on the crack faces occurs. The specific form and amplitude of the diffracted waves depends critically on which part of the GWP strikes the edge. Thus, the diffraction is maximum if the central ray hits the edge. This case is most similar to plane wave diffraction theory. However, in general, the edge is not reached by the central ray, but by a point in the waist of the pulse. The incident field at the crack tip behaves locally like an evanescent wave. This can be treated by an extension of the Geometrical Theory of Diffraction (GTD) [1] that allows for complex angles of incidence. We do not give any details of the procedure here, but note that it does not present any conceptual difficulties.

Critical angle Rayleigh wave phenomena occur when ultrasound is incident at a liquid-solid interface at the angle which couples the acoustic wave to the pseudo-Rayleigh wave in the solid. The most significant effect is the beam displacement phenomenon which has been explained by Bertoni and Tamir [5]. Their model considers a CW beam of Gaussian profile. Using their approach, we have treated the same problem for transient GWP incidence. The solution can be obtained in closed form, and has the advantage of being fully time dependent. It contains both the specularly reflected GWP and a surface leaky Rayleigh wave part.

CONCLUSIONS AND FUTURE APPLICATIONS

The GWP idea offers a simple procedure for modeling pulse propagation and scattering in solids and fluids. It is particularly suitable to piecewise continuous media, where interface reflection/transmission and edge diffraction can be treated by well-known techniques. The main advantage of using GWPs is that they are explicitly time dependent: no transforms are necessary. They also have many physically important features built in. For example, it can be shown that a GWP conserves energy. In the area of ultrasonics in composites, GWPs may be very useful in helping NDE researchers and practitioners understand the ways in which pulses propagate. Future research will focus on the propagation of GWPs in anisotropic materials, with possible application to fiber and ply reinforced composites.

REFERENCES

1. J.D. Achenbach, A.K. Gautesen and H. McMaken, Ray Methods for Waves in Elastic Solids, Pitman, Boston (1982).

2. R.B. Thompson and E.F. Lopes, "A Model for the Effects of Aberrations on Refracted Ultrasonic Fields," Review of Progress in Quantitative NDE5, D.O. Thompson, D.E. Chimenti, Eds., Plenum Press, NY (1986).

3. M.M. Popov, "A New Method of Computation of Wave Fields Using Gaussian Beams," Wave Motion, 4, 85-97 (1982).

4. A.N. Norris, B.S. White and J.R. Schrieffer, "Gaussian Wave Packets in Inhomogeneous Media with Curved Interfaces" (unpublished).

5. H.L. Bertoni and T. Tamir, "Unified Theory of Rayleigh-Angle Phenomena for Acoustic Beams at Liquid-Solid Interfaces," Appl. Phys., 2, 157-172 (1973).