MAPPINGS WITH SMALL POINT-INVERSES

by

Thomas John Robinson

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

In Charge of Major Work

Head of Major Department

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1963

TABLE OF CONTENTS

I. INTRODUCTION

In their book, Dimension Theory, Hurewicz and Wallman (7) define an ϵ-mapping of a compact metric space X into a metric space Y as follows.

<u>Definition 1.1</u>: A mapping f, i.e. a continuous function, of a compact metric space X into a metric space Y is an ϵ-mapping if and only if the inverse image of every point of f (X) has diameter less than ϵ.

This concept of ϵ-mapping is then used to prove that a compact separable metric space having dimension $\leq$ n is homeomorphic to a subset of I_{2n+1}. However, to prove that an arbitrary separable metric space of dimension $\leq$ n is homeomorphic to a subset of I_{2n+1}, ϵ-mappings are inadequate. Thus the following generalization arises.

<u>Definition 1.2</u>: Let α be an open covering of a space X and f a mapping of X into a space Y. Then f is an α-mapping if and only if there is an open cover μ of Y such that $f^{-1}(\mu)$ refines α. Equivalently, f is an α-mapping if and only if every point of Y has a neighborhood in Y whose inverse image is entirely contained in some member of α.

This definition is applicable not only to metric spaces, but to more general topological spaces as well, and it is this definition that forms the basis for this paper.

Maxwell (10) defines a partial ordering on topological spaces in terms of α-mappings as given in the following definition.

<u>Definition 1.3</u>: Let X and Y be topological spaces. We say $X \leq Y$ if and only if for every open cover α of X there is an α-mapping f of X onto Y.

It is easy to see that this relation is transitive.

Maxwell investigates certain properties of spaces which are

"inherited" under this relation. Most of them are listed in the theorem below.

<u>Theorem 1.4</u>: If $X \leq Y$, then:

(1) If X or Y is compact, both are.

(2) If X or Y is connected, both are.

(3) If Y is paracompact, then X is paracompact.

(4) If Y is Lindelöf, then X is Lindelöf.

(5) dim $X \leq$ dim Y.

(6) If X is T_1 and Y is T_2, then X is T_2.

(7) If X is T_1 and Y is regular, then X is regular.

(8) If X is T_1 and Y is completely regular, then X is completely regular.

(9) If Y is normal, then X is normal.

A question asked by Ulam in his problem book (14) concerns the invariance of the fixed point property under this order relation. Maxwell proves the following theorem.

<u>Theorem 1.5</u>: If X and Y are metric absolute neighborhood retracts with $X \leq Y$, and Y has the fixed point property, then X has the fixed point property.

To conclude his results, Maxwell gives examples to show that $X \leq Y$ does not imply that X have the same homotopy type nor the same dimension as Y.

Another of Ulam's questions is the following: Does there exist for every $\epsilon > 0$ an ϵ-map of the disk onto the torus? M. K. Fort, Jr. (3) and T. Ganea (4) have both given negative answers to this question using

different methods. Ganea used Cech cohomology and also proves that if Y

is a compact n-dimensional manifold and X is an absolute neighborhood

retract with $X \leq Y$, then X has the same homotopy type as an n-manifold.

Fort uses arcs to obtain his results, and it seems worthy of mention that

he shows for $\epsilon < 1/6$ there is no ϵ-mapping of a unit disk onto a torus.

A third question asked by Ulam with regard to α-maps and the order

relation $\leq$ involves $X \leq Y$ and $Y \leq X$. He asks whether X and Y are then

homeomorp : if they are n-manifolds. In addition to his result noted

above, Fort shows that if $X \leq Y$ and $Y \leq X$ where X and Y are closed, orient-

able, 2-manifolds, then X and Y are homeomorphic. Somewhat along this same

line, Borsuk (2) has given an example of compact spaces X and Y such that

for every $\epsilon > 0$ there is an ϵ-mapping of X into Y and vice versa. X and

Y are both subsets of E_3 with X being homeomorphic to a subspace of the

plane while Y is not.

In this paper we will consider the more general concept of α-mappings

and the relation $\leq$ as given in Definition 1.3. In Chapter II the signifi-

cance of f being an α-mapping for every open cover α of X is investigated.

In the process it is found that if X is a T_1 space and f an α-mapping of

X onto Y for all α, then f is a homeomorphism, while if X is not T_1, then

the topology of Y is not determined by the topology of X.

Chapter III is a study of covering properties of spaces and their

inheritance under the relation $\leq$. The results of Theorem 1.4 are extended

to include metacompactness, countable paracompactness, compactness of de-

gree γ, and γ-reducibility.

Results similar to those of Chapter III and some of those of Theorem

1.4 can be obtained. Thus if $\mathcal{P}$ represents one of the properties of being

compact, Lindelöf, etc., and if for every open cover α of X there is an

α-mapping f_α of X onto a space Y_α having property $\mathcal{P}$, then X has property

$\mathcal{P}$.

The chief results of Chapter IV are $\overline{\prod}_{a \in A} X_a \leq \overline{\prod}_{a \in A} Y_a$ if for each $a \in A$ X_a

is compact and $X_a \leq Y_a$, and a partial converse of this.

Chapter V is divided into two parts, the first dealing with metriza-

tion, generalizations, and the countability axioms, and the second with an

embedding theorem. In the first part it is shown that if $X \leq Y$ the

property of being a uniform space is inherited when X is T_1, but metri-

zability, developability, and the countability axioms are not. The chief

result of the second part is that the α-mappings of a T_1 space induce an

embedding of the domain space into the product of the range spaces. When

applied to theorems of Ponomarev (12), this embedding gives rise to neces-

sary and sufficient conditions for a T_1 space to be paracompact or

Lindelof.

Chapter VI has an application of the embedding of Chapter V which

gives an isomorphism of a compact topological group onto the inverse

limit of an inverse system of factor groups. This is done after proving

that if G is a compact group with arbitrarily small invariant subgroups

$\{G_a\}_{a \in A}$, then for each open cover α of G there is an $a \in A$ such that the

natural mapping of G onto G/G_a is an α-mapping. The first part of Chapter

VI deals with quotient spaces, and a method is given for constructing

spaces X and Y such that $X \leq Y$ and $Y \leq X$ but X and Y are not homeomorphic.

Most of the definitions will be stated as they are needed. The

terminology and notation are similar to what is used in most topology

books. For example, see Hocking and Young (6) or Kelley (8). An exception

to the terminology of Kelley is that the term "neighborhood" as used here

is an "open neighborhood" in his book.

II. α-MAPPINGS FOR ALL α

Let X be a topological space and let $\Sigma(X)$ be the family of open covers of X. For $\alpha, \mu \in \Sigma(X)$ we will write $\mu R \alpha$ if and only if μ refines α. Then R is a partial ordering in the terminology of Kelley (8, p. 13), for he only requires R to be a transitive relation. Other authors have a slightly more restrictive definition of partial ordering (5, p. 275). They require that $\mu R \alpha$ and $\alpha R \mu$ imply $\mu = \alpha$, but of course this does not follow for refinements.

Let $(X, \mathcal{T})$ and (Y, σ) be topological spaces and f a mapping of $(X, \mathcal{T})$ onto (Y, σ). Then since $f^{-1}(\sigma) \subset \mathcal{T}$, it follows that f^{-1} induces an order preserving function, also denoted by f^{-1}, of $\Sigma(Y, \sigma)$ into $\Sigma(X, \mathcal{T})$. Thus if $\alpha, \mu \in \Sigma(Y, \sigma)$ such that $\alpha R \mu$, then $f^{-1}(\alpha) R f^{-1}(\mu)$. f^{-1} also preserves such properties as local finiteness, a concept that will be encountered in Chapter III.

We also note that if f maps a space X onto a space Y and $\mu \in \Sigma(Y)$, then $f^{-1}(\mu) \in \Sigma(X)$. If α is a subcover of $f^{-1}(\mu)$, then we construct a subcover μ' of μ by letting $U \in \mu'$ if and only if $f^{-1}(U) \in \alpha$. Since f is onto, $f(f^{-1}(U)) = U$ for each $U \in \mu$ and hence cardinal $\mu' \leq$ cardinal α. This is used in the proofs of some of the theorems of Chapter III.

Let $X \leq Y$ and $\alpha \in \Sigma(X)$. Then if $\mu \in \Sigma(X)$ such that $\mu R \alpha$, a μ-mapping f of X onto Y is also an α-mapping. Under this particular relation R, if $\alpha, \mu \in \Sigma(X)$, then there is a $\lambda \in \Sigma(X)$ such that $\lambda R \alpha$ and $\lambda R \mu$; i.e. $(\Sigma(X), R)$ is a directed set. Hence it follows that for each $\alpha, \mu \in \Sigma(X)$ there is a mapping f of X onto Y which is both an α-map and a μ-map. It is easily seen that this can be extended to any finite collection of elements of

$\Sigma(X)$. Hence if $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset \Sigma(X)$, there is a mapping f of X onto Y such that f is an α_i-mapping for all $i \leq n$.

We actually get some slightly stronger statements than that mentioned in the last sentence of the preceding paragraph. These are stated as a theorem, but the easy proof will be omitted.

Theorem 2.1: Let $X \leq Y$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any finite subcollection of $\Sigma(X)$ with $f_1, f_2, \ldots, f_n$ corresponding α_i-mappings of X onto Y, $i = 1, 2, \ldots, n$. If $\mu_1, \ldots, \mu_n$ are open covers of Y such that $f_i^{-1}(\mu_i) R \alpha_i$, then there exists a $\mu \, \varepsilon \, \Sigma(Y)$ and a mapping f of X onto Y such that $\mu R \mu_i$ and $f^{-1}(\mu) R \alpha_i$ for each $i \leq n$.

In this hierarchy of refinements and α-mappings there is one important question that will be examined closely. Thus we ask, "What is the significance of the existence of a λ in $\Sigma(X)$ such that $\lambda R \alpha$ for every α in $\Sigma(X)$, or a mapping f which is an α-map for all α in $\Sigma(X)$?" Of course if the former condition is satisfied, then the latter condition will also be satisfied. The converse is not true, however. We will consider a theorem and some examples involving specific questions applied to the domain and range spaces, and functions which are α-mappings for all α in $\Sigma(X)$.

Theorem 2.2: Let $X \leq Y$ where X is T_1, and let f be an α-mapping for all $\alpha \, \varepsilon \, \Sigma(X)$. Then f is a homeomorphism.

Proof: To show that f is one to one, suppose there are distinct points $x_1, x_2 \, \varepsilon \, X$ such that $f(x_1) = f(x_2)$. X is T_1 and hence there is a neighborhood U of x_1 such that $x_2 \notin U$. Let $\alpha = \{U, X - \{x_1\}\}$. Then $\alpha \, \varepsilon \, \Sigma(X)$ and hence there is a $\mu \, \varepsilon \, \Sigma(Y)$ such that $f^{-1}(\mu) R \alpha$. If V is any member of μ that contains $f(x_1)$, then $f^{-1}(V) \subset U$, for $x_1 \notin X - \{x_1\}$. But

since $f(x_1) = f(x_2), x_2 \in f^{-1}(V)$, hence $x_2 \in U$. This is a contradiction and it follows that f is one to one.

Let U be an open set in X. We will show that $f(U)$ is open. Let x be any point of U. Then $\alpha = \{U, X - \{x\}\}$ is in $\Sigma(X)$ and hence there is a $\mu \in \Sigma(Y)$ such that $f^{-1}(\mu) R \alpha$. If V is any member of μ that contains $f(x)$, then $f^{-1}(V) \subset U$, and since $f(f^{-1}(V)) = V, V$ is a subset of $f(U)$. Hence there is a neighborhood V of $f(x)$ that is contained in $f(U)$. Hence $f(U)$ is open and since U is arbitrary, f is an open mapping. Therefore, since f is one to one, onto, and open, it is a homeomorphism.

<u>Corollary 2.3</u>: If $X \leq Y$ and X is discrete, then Y is discrete.

Proof: If α is the member of $\Sigma(X)$ consisting of singleton sets, then $\alpha R \mu$ for all $\mu \in \Sigma(X)$. The proof then follows from Theorem 2.2.

If $X \leq Y$ and Y is discrete, then it doesn't follow that X need be discrete, however. For let X be a two point indiscrete space, Y a singleton set, and f the mapping of X onto Y.

In view of Theorem 2.2, we are led to the following: Let $X \leq Y$ where X is not T_1, and let f be an α-mapping for every $\alpha \in \Sigma(X)$. Is f necessarily open, or closed, or one to one?

A simple example shows that f need not have any of these properties. Let $X = \{a, b, c\}$ with open sets $\emptyset, \{a\}, \{a, b\}$, and X. Let $Y = \{d, e\}$ with open sets $\emptyset$ and Y. Then since $X \in \alpha$ for every $\alpha \in \Sigma(X)$, every mapping f of X onto Y is an α-mapping for all $\alpha \in \Sigma(X)$. It is easily seen that each such mapping f satisfies none of the properties of being open, or closed, or one to one.

Along this same line we consider another question: Let $X \leq Y$ where

X is not T_1, and let f be an α-mapping for every $\alpha \in \Sigma(X)$. Suppose that in addition there is a $\mu \in \Sigma(Y)$ such that $f^{-1}(\mu) R \alpha$ for every $\alpha \in \Sigma(X)$. What is the significance of this latter property?

From the above example we see that this additional condition does not force the mapping f to be open, closed, or one to one. However, we can say that $\Sigma(X)$ has an element which is "minimal" in a certain way, namely $f^{-1}(\mu) R \alpha$ for every $\alpha \in \Sigma(X)$. This minimal element may not be the open cover which contains the most elements, as is also seen in the above example, nor does it have to be the open cover with the fewest elements.

If f is a function on a space X onto a set Y, there are topologies which can be put on Y to make the function f continuous. The largest such topology is called the <u>quotient</u> <u>topology</u>. Thus U is open in the quotient topology if and only if $f^{-1}(U)$ is open in X. If f is a one to one function then the set Y with the quotient topology relative to X and f is just a homeomorphic copy of X.

We have shown in Theorem 2.2 that if $X \leq Y$, X is T_1, and f an α-mapping for all $\alpha \in \Sigma(X)$, then the topology on Y is directly determined by f and X. If $X \leq Y$, X is not T_1, and f is an α-mapping for all $\alpha \in \Sigma(X)$, then by the above example the topology on Y is in general not determined directly by f and X since f need not be a homeomorphism. The question remains, however, whether the topology on Y is indirectly determined via the quotient topology of f and X.

Since the quotient topology is the largest topology for which f is continuous, it is true, of course, that every member of the given topology is also a member of the quotient topology. The following example shows

that this is nearly all that can be said about them in general.

Example 2.4: Let X be the reals with a basis for X the family of all closed right rays, i.e. $U = \{x : x \geq b$ for some $b\}$. Let Y be the reals with a basis the family of all open right rays.

The identity mapping i of X onto Y is an α-mapping for all α in $\Sigma(X)$. Furthermore, the family μ of all open right rays is a refinement of each α in $\Sigma(X)$. As we observed above, the set Y with the quotient topology of i is just X since i is one to one. However, since closed right rays are not open in the given topology on Y, the given topology and the quotient topology are not equivalent.

III. COVERING PROPERTIES

As we have seen in Theorem 1.4, when $X \leq Y$ and Y is compact, paracompact or Lindelöf, then X has the same property. These three concepts are defined in terms of subcovers or refinements of open covers. There are other properties which are somewhat similar to the three listed above, and these are quite easily shown to be inherited under the relation $\leq$.

While countable compactness is not defined explicitly in terms of open covers, we will begin with a theorem on countable compactness. By definition a subset A of a space X is countably compact if and only if every infinite subset of A has at least one limit point in A (5, p. 66).

Theorem 3.1: Let Y be T_0 and countably compact. Suppose Y has the additional property that the intersection of each local base contains at most two points. Then if $X \leq Y$, X is also countably compact.

Remark: An example of a space Y which is not T_1 but has this local base restriction is obtained by considering a three point space $Y = \{a,b,c\}$ with open sets $\emptyset$, $\{a,b\}$, $\{b,c\}$, $\{b\}$, and Y.

Proof of Theorem: Let A be an infinite subset of X and suppose that A has no limit point in X. Then A is closed and X - A is an open set which contains no points of A. Since A has no limit point, then in particular no point of A is a limit point of A. Thus if $x \in A$, there is a neighborhood U(x) of x such that $U(x) \cap A = \{x\}$. Let $\alpha = \{X-A, \{U(x)\}_{x \in A}\}$. Then $\alpha \in \Sigma(X)$ and there is a mapping f of X onto Y and a $\mu \in \Sigma(Y)$ such that $f^{-1}(\mu)R\alpha$.

We now show that $f|A$ is one to one. From the above observations A as a subspace of X is discrete and $\alpha' = \alpha \cap A = \{U : U = V \cap A \text{ for some } V \in \alpha\}$

is the discrete cover of A. Hence f is an α'-map of A onto f(A) for all $\alpha' \in \Sigma(A)$ and by Theorem 2.2, A and f(A) are homeomorphic. Therefore, $B_1 = f(A)$ is an infinite subset of Y and from the above remarks it follows that no point of B_1 is a limit point of B_1, and that no member of μ contains more than one point of B_1.

To complete the proof we will show that there is a sequence of distinct limit points of B_1 which does not itself have a limit point in Y, and hence have a contradiction of the assumption that Y is countably compact.

Since B_1 is infinite and Y is countably compact, there is a point b_{01} in Y which is a limit point of B_1. Then every neighborhood of b_{01} contains at least one point $b_1 \in B_1$ such that $b_{01} \neq b_1$. From our observations above we have that $b_{01} \notin B_1$, and it also follows that b_{01} is a limit point of a singleton set $\{b_1\}$. Since Y is T_0 there is a neighborhood $V(b_1)$ of b_1 which does not contain b_{01}, and hence b_1 is not a limit point of the singleton set $\{b_{01}\}$. This will be used in showing that no point of B_1 is a limit point of the sequence to be constructed.

Now consider the set $B_2 = B_1 - \{b_1\}$. B_2 is infinite and hence has at least one limit point, call it b_{02}. $b_{02} \notin B_1$, and $b_{02} \neq b_{01}$ because there are no points of B_1 other than b_1 in any element of μ that contains b_{01}. It also follows that b_{02} is a limit point of a singleton set $\{b_2\}$ in B_2, while b_2 is not a limit point of the singleton set $\{b_{02}\}$.

Having obtained the distinct limit points b_{01} and b_{02} of $\{b_1\}$ and $\{b_2\}$, respectively, we proceed inductively to obtain $B_{n+1} = B_n - \{b_n\}$, and the point b_{0n+1} which is a limit point of B_{n+1}. By the induction

process we obtain a sequence of distinct points $\{b_{0n}\}_{n=1}^{\infty}$ each of which is a limit point of a singleton set $\{b_n\}$ which is a point of B_n (but not of B_k for $k > n$). The infinite set $\{b_{0n}\}_{n=1}^{\infty}$ must have a limit point also, but we will show that this is impossible and hence have our contradiction.

Let b_0 be a limit point of $\{b_{0n}\}_{n=1}^{\infty}$. Either b_0 is in B_1, or it is not in B_1 but is a limit point of B_1. Suppose first that $b_0 \in B_1$. Every neighborhood of b_0 contains a point b_{0k} in $\{b_{0n}\}_{n=1}^{\infty}$, and because of the form of μ, b_0 must be a limit point of a singleton set $\{b_{0k}\}$. From our previous observations $b_0 \neq b_k$, and hence there is an element of μ that contains the points b_0 and b_k of B_1 which is a contradiction.

Next we assume that b_0 is a limit point of B_1 in addition to being a limit point of $\{b_{0n}\}_{n=1}^{\infty}$. Since b_0 is a limit point of a singleton set $\{b_{0m}\}$ for some m, then given any neighborhood $V(b_0)$ of b_0 the point b_{0m} in $\{b_{0n}\}_{n=1}^{\infty}$ and the corresponding point in B_1, b_m, are in $V(b_0)$. However, by hypothesis there is a neighborhood of each of these three points which contains at most one of the other two. Hence we have a contradiction.

Thus we have shown that the infinite set of distinct points $\{b_{0n}\}_{n=1}^{\infty}$ does not have a limit point in Y which contradicts the assumption that Y is countably compact. Therefore A has a limit point in X, and since A is arbitrary, X is countably compact.

Two results arise due to Theorem 3.1, one from the separation hypothesis and the other from the proof of the theorem. We state them both in the following corollary.

<u>Corollary 3.2</u>: If $X \leq Y$, Y is countably compact and either T_1 or

satisfies the condition that each limit point of a subset is an accumulation point of that subset, then X is countably compact.

The restrictions placed on Y in Theorem 3.1 are sufficient to prove the theorem. It appears that if Y is T_0 and there is an integer n such that the intersection of each local base in Y contains at most n points, then the theorem is still true. We have been unable to construct either a proof or a counterexample for Y only T_0. The following example shows, however, that if Y satisfies the local base condition but is not T_0, then the theorem is false.

Example 3.3: Let Y be the set of positive integers with a basis for Y the sets of the form $\{2n-1,2n\}$ for n any positive integer. Let X be the positive integers with a basis for X the sets of the form $\{2n-1,2n\}$ or $\{2n\}$ for n any positive integer. Define f on X onto Y by letting $f(x) = x$ for all $x \in X$. Then f is an α-mapping for all $\alpha \in \Sigma(X)$, for each set $\{2n-1,2n\}$ is contained in some member of α. Hence $X \leq Y$. Clearly Y is countably compact, for the point 2n is a limit point of the singleton set $\{2n-1\}$ and conversely $2n - 1$ is a limit point of the singleton set $\{2n\}$. However, the infinite set of odd integers, $\{2n - 1\}_{n=1}^{\infty}$, has no limit point in X and thus X is not countably compact.

Definition 3.4: A family of subsets of a space X is point finite if and only if no point of X belongs to more than a finite number of members of the family. A space is metacompact if and only if each open cover of X has a point finite refinement (8, p. 171).

Theorem 3.5: Let $X \leq Y$ and let Y be metacompact. Then X is metacompact.

Proof: Let $\alpha \in \Sigma(X)$. Then there is a mapping f of X onto Y and a

$u \in \Sigma(Y)$ such that $f^{-1}(\mu)R\alpha$. Since Y is metacompact μ has a point finite refinement μ'. But then $f^{-1}(\mu')$ is a point finite refinement of α. α is arbitrary and hence X is metacompact.

Definition 3.6: A family of subsets of a space X is called locally finite if and only if for each $x \in X$ there is a neighborhood $U(x)$ of x such that $U(x)$ intersects only a finite number of members of the family. A space X is (countably) paracompact if and only if each (countable) open cover of X has a locally finite refinement.

Before stating a theorem on countable compactness, it appears convenient to introduce some further concepts involving open covers.

Definition 3.7: Let $\alpha \in \Sigma(X)$ and let $x \in X$ be any point. Then $\alpha^*(x)$, the star of α with respect to x, is defined by $\alpha^*(x) = \bigcup\limits_{x \in U \in \alpha} U$. Then α^*, the α star cover of X, is defined by $\alpha^* = \{U: \text{ for some } x \in U, U = \alpha^*(x)\}$. If f is an α-mapping of X onto Y for some $\alpha \in \Sigma(X)$, and if $\mu \in \Sigma(Y)$ is such that $f^{-1}(\mu)R\alpha$, we define the α-restricted μ star cover, μ_α^*, by $\mu_\alpha^* = \{U: \text{ for some } A \in \alpha, U = \bigcup\limits_{f(A) \supset V \in \mu} V\}$.

Theorem 3.8: Let $X \leq Y$ and Y be countably paracompact. Then X is countably paracompact.

Proof: Let α be any countable open cover of X. There is a mapping f of X onto Y and a $\mu \in \Sigma(Y)$ such that $f^{-1}(\mu)R\alpha$. α is countable and it follows from its construction that μ_α^* is also countable and $f^{-1}(\mu_\alpha^*)R\alpha$.

Since μ is an open cover of Y and $f^{-1}(\mu)R\alpha$, it follows that μ_α^* is an open cover of Y. Y is countably paracompact and so the countable open cover μ_α^* of Y has a locally finite refinement, call it μ'. f^{-1} preserves refinement, openness, and local finiteness, and since α is arbitrary X is

countably paracompact.

A slight variation of the above proof is used to obtain the corresponding theorem for paracompactness.

Definition 3.9: A subset A of a space X is compact of degree γ if and only if every open cover of A contains a subcovering of cardinal $\leq \gamma$ (9, p. 24).

Definition 3.10: A space X is γ-reducible if and only if every $\alpha \in \Sigma(X)$ of cardinal γ has a subcover of cardinal $< \gamma$.

We observe that Lindelöf spaces coincide with spaces which are compact of degree $\aleph_0$, and which are γ-reducible for every uncountable cardinal γ. It is also true that a T_1 space is countably compact if and only if it is γ-reducible for every countably infinite cardinal γ. Furthermore, a space is compact if and only if it is γ-reducible for every infinite cardinal γ, or compact of degree γ for γ finite.

Theorem 3.11: Let $X \leq Y$. Then if either X or Y is compact of degree γ, both of them are.

Proof: Suppose first that Y is compact of degree γ and let $\alpha \in \Sigma(X)$. Then there is a mapping f of X onto Y and a $\mu \in \Sigma(Y)$ such that $f^{-1}(\mu)R\alpha$. Since Y is compact of degree γ, μ has a subcover μ' of cardinal $\leq \gamma$. Since $\mu'R\mu$, $f^{-1}(\mu')R\alpha$.

We construct a subcover α' of α by extracting elements from α in the following manner. For each element V of μ', select exactly one element U from α such that $f^{-1}(V) \subset U$. The subcover α' thus obtained has cardinal $\leq$ cardinal $\mu' \leq \gamma$. α is arbitrary and hence X is compact of degree γ.

Suppose now that X is compact of degree γ and let $\mu \in \Sigma(Y)$ be

arbitrary. Then if f is any mapping of X onto Y, $f^{-1}(\mu) \in \Sigma(X)$. Since X is compact of degree γ, there is a subcovering α of $f^{-1}(\mu)$ of cardinal $\leq \gamma$. But then, as was observed in Chapter II, the subcover μ' of μ consisting of elements U such that $f^{-1}(U) \in \alpha$ has cardinal $\leq \gamma$. Therefore, since μ is arbitrary, Y is compact of degree γ.

Theorem 3.12: Let $X \leq Y$. Then if either X or Y is γ-reducible, they both are.

Proof: Suppose first that Y is γ-reducible, and let α be any member of $\Sigma(X)$ of cardinal γ. Then there is a mapping f of X onto Y and a $\mu \in \Sigma(Y)$ such that $f^{-1}(\mu) R \alpha$. Then μ_α^* has cardinal $\leq \gamma$ and $f^{-1}(\mu_\alpha^*) R \alpha$.

Suppose first that the cardinal of μ_α^* is γ. Y is γ-reducible and hence the open cover μ_α^* has a subcover μ' of cardinal $< \gamma$. Then also $f^{-1}(\mu') R \alpha$. We construct a subcover α' of α by selecting for each member of $f^{-1}(\mu')$ exactly one element from α which contains it. Since the cardinality of $f^{-1}(\mu') < \gamma$, the cardinality of α' is also $< \gamma$. If the cardinality of $\mu_\alpha^* < \gamma$, then for μ' as above we will write $\mu' = \mu_\alpha^*$. In either case a subcover of α of cardinal $< \gamma$ is obtained and, since α is arbitrary, X is γ-reducible.

Suppose next that X is γ-reducible, and let μ be a member of $\Sigma(Y)$ of cardinal γ. Then if f is any mapping of X onto Y, $f^{-1}(\mu) \in \Sigma(X)$ has cardinal γ. X is γ-reducible and hence there is a subcover α of $f^{-1}(\mu)$ of cardinal $< \gamma$. Then let μ' be the subcover of μ such that $U \in \mu'$ if and only if $f^{-1}(U) \in \alpha$. Thus the cardinal of $\mu' < \gamma$ and, since μ is arbitrary, Y is γ-reducible.

IV. PRODUCT SPACES

Definition 4.1: Let f be a function on a set X onto a topological space

Y. Then the product topology of f is the smallest topology on X such that

f is continuous.

Definition 4.2: Let $\prod_{a \varepsilon A} X_a$ be a product of spaces and let $\Delta \subset A$. Define

the projection P_Δ of $\prod_{a \varepsilon A} X_a$ onto $\prod_{a \varepsilon \Delta} X_a$ by $P_\Delta(\{x_a\}_{a \varepsilon A}) = \{x_a\}_{a \varepsilon \Delta}$.

If $\Delta = \{a\}$, then $P_a(\{x_a\}_{a \varepsilon A}) = x_a$. The product $\prod_{a \varepsilon A} X_a$ is given the

smallest topology which includes the product topology of all the P_a's, mak-

ing each P_a open and continuous. Then if α is an open cover of $\prod_{a \varepsilon A} X_a$,

the projection of α onto $\prod_{a \varepsilon \Delta} X_a$ is an open cover α_Δ of $\prod_{a \varepsilon \Delta} X_a$ defined by

$\alpha_\Delta = P_\Delta(\alpha) = \{U: \text{ for some } V \varepsilon \alpha, U = P_\Delta(V)\}$. If $\Delta = \{a\}$, then

$\alpha_a = P_a(\alpha)$ is an open cover of X_a.

Definition 4.3: Let A be an index set and suppose that for each $a \varepsilon A$, f_a

is a mapping of a space X_a onto a space Y_a. The product mapping F of

$\prod_{a \varepsilon A} X_a$ onto $\prod_{a \varepsilon A} Y_a$ is defined by $F(\{x_a\}_{a \varepsilon A}) = \{f_a(x_a)\}_{a \varepsilon A}$ and we

write $F = \prod_{a \varepsilon A} f_a$.

The following lemma gives a property of the inverse of a product

mapping that is used in the proofs of some theorems on product spaces.

Lemma 4.4: Let A be an index set, f_a a mapping of X_a onto Y_a for each

a A, and let $F = \prod_{a \varepsilon A} f_a$. Then if $U_a \subset Y_a$ for each $a \varepsilon A$,

$F^{-1}(\prod_{a \varepsilon A} U_a) = \prod_{a \varepsilon A} f_a^{-1}(U_a)$.

Proof: We will prove the lemma by showing that

$$F^{-1}(\prod_{a \varepsilon A} U_a) \subset \prod_{a \varepsilon A} f_a^{-1}(U_a) \text{ and } \prod_{a \varepsilon A} f_a^{-1}(U_a) \subset F^{-1}(\prod_{a \varepsilon A} U_a).$$

These two inclusions will be obtained by writing a series of equivalent

statements.

$$\{x_a\}_{a \in A} \in F^{-1}(\underset{a \in A}{\textstyle\prod} U_a)$$

if and only if $F(\{x_a\}_{a \in A}) = \{f_a(x_a)\}_{a \in A} \in \underset{a \in A}{\textstyle\prod} U_a$

if and only if $f_a(x_a) \in U_a$ for each $a \in A$

if and only if $x_a \in f_a^{-1}(U_a)$ for each $a \in A$

if and only if $\{x_a\}_{a \in A} \in \underset{a \in A}{\textstyle\prod} f_a^{-1}(U_a)$.

Definition 4.5: Let α_i be an open cover of X_i for $i = 1,2$. Then $\alpha_1 \times \alpha_2$ is an open cover of $X_1 \times X_2$ defined by $\alpha_1 \times \alpha_2 = \{ U \times V : U \in \alpha_1, V \in \alpha_2 \}$.

We observe that if $\alpha_1' R \alpha_1$ and $\alpha_2' R \alpha_2$, then $(\alpha_1' \times \alpha_2') R (\alpha_1 \times \alpha_2)$.

Theorem 4.6: Let $X_1 \leq Y_1$ and $X_2 \leq Y_2$, where X_1 and X_2 are compact. Then $X_1 \times X_2 \leq Y_1 \times Y_2$, and each α-map may be chosen to be a product map with the corresponding μ a product cover.

Proof: Let α be an open cover of $X_1 \times X_2$. Without loss of generality we make the following assumptions about α:(i). No member of α is a subset of any other member of α, since the removal of such a set leaves a refinement of α. (ii)$\alpha = \{ U_a \times V_a \}_{a \in A}$ where A is some index set, for each open cover can be refined by a cover consisting entirely of basis elements. (iii) α may be chosen to be finite since $X_1 \times X_2$ is compact. Thus $\alpha = \{ U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n \}$ for some integer n with each U_i open in X_1 and V_i open in X_2.

We wish to find a refinement of α of the form $\alpha_1 \times \alpha_2$ where $\alpha_1 \in \Sigma(X_1)$ and $\alpha_2 \in \Sigma(X_2)$. We do this starting with the families $\alpha_1' = \{ U_i \}_{i \leq n}$ and $\alpha_2' = \{ V_i \}_{i \leq n}$ of sets which appear as first or second elements of members of α. α_1' and α_2' are, of course, open covers of X_1 and X_2,

respectively.

For each $x_1 \in X_1$ take the intersection of all members of α_1' which contain x_1. α_1' is finite and hence this intersection is an open set $U(x_1)$ containing x_1. The family of open sets formed in this way is a finite collection $\alpha_1 = \{U(x_{1i}) : i \leq n' \text{ for some } n' \text{ and } x_{1i} \in X_1\}$ which is an open cover of X_1. In a similar manner we obtain a finite open cover of X_2, $\alpha_2 = \{V(x_{2i}) : i \leq m \text{ for some } m \text{ and } x_{2i} \in X_2\}$. For $i = 1,2$, we have $\alpha_i R \alpha_i'$.

Clearly $\alpha_1 \times \alpha_2$ is an open cover of $X_1 \times X_2$. Also, if $U(x_1) \in \alpha_1$ and $V(x_2) \in \alpha_2$, then $U(x_1) \times V(x_2)$ is contained in some member of α. For $(x_1, x_2) \in U(x_1) \times V(x_2)$, and $U(x_1) \subset U_i$ for every $U_i \in \alpha_1'$ which contains x_1; and $V(x_2) \subset V_i$ for every $V_i \in \alpha_2'$ which contains x_2. Now $(x_1, x_2) \in U_i \times V_i$ for some i, and since $U(x_1) \subset U_i$, $V(x_2) \subset V_i$, it follows that $U(x_1) \times V(x_2) \subset U_i \times V_i$.

Since $X_1 \leq Y_1$ and $X_2 \leq Y_2$, corresponding to α_1 and α_2 are mappings f_1 and f_2 and open covers $\mu_1 \in \Sigma(Y_1)$, $\mu_2 \in \Sigma(Y_2)$ such that $f_i(X_i) = Y_i$ and $f_i^{-1}(\mu_i) R \alpha_i$ for $i = 1,2$. Then form an open cover μ of $Y_1 \times Y_2$ by letting $\mu = \mu_1 \times \mu_2$, a "product cover", and the product map $F = f_1 \times f_2$ of $X_1 \times X_2$ onto $Y_1 \times Y_2$. It follows from Lemma 4.4 that $f_1^{-1}(\mu_1) \times f_2^{-1}(\mu_2) = F^{-1}(\mu)$, and since $f_i^{-1}(\mu_i) R \alpha_i$ for $i = 1,2$, $F^{-1}(\mu) R(\alpha_1 \times \alpha_2)$ as we observed prior to the theorem. Therefore, since α is arbitrary $X_1 \times X_2 \leq Y_1 \times Y_2$.

Corollary 4.6: Let $X_i \leq Y_i$ for $i = 1, \ldots, n$, where n is any positive integer and each X_i is compact. Then $\prod_{i=1}^{n} X_i \leq \prod_{i=1}^{n} Y_i$, and each α-map may be chosen to be a product map with the corresponding μ a product cover.

Proof: Each open cover α of $\prod_{i=1}^{n} X_i$ may be chosen to be a finite collection of basis elements and has a refinement of the form $\prod_{i=1}^{n} \alpha_i$, where $\alpha_i \in \Sigma(X_i)$ is also finite. The proof then follows by induction.

Theorem 4.7: Let A be an index set with $X_a \leq Y_a$ for each $a \in A$ and each X_a is compact. Then $\prod_{a \in A} X_a \leq \prod_{a \in A} Y_a$ and each α-map may be chosen to be a product map with the corresponding μ a product cover.

Proof: If α is an open cover of $\prod_{a \in A} X_a$, then α can be taken to be a finite collection of basis elements since $\prod_{a \in A} X_a$ is compact. If U is any member of α, then $U = \prod_{a \in A} U_a$, where $U_a = X_a$ for all $a \in A - \Delta'$ with Δ' a finite subset of A and for all $a \in \Delta'$, U_a is an open subset of X_a. Hence there is a finite subset Δ of A such that $P_a(\alpha) = \alpha_a = \{X_a\}$, the trivial open cover of X_a, for all $a \in A - \Delta$; and for $a \in \Delta$, $P_a(\alpha) = \alpha_a$ is a non-trivial open cover of X_a and the proof is thus reduced to working with this finite subset of A. Corollary 4.6 will be used to obtain the result.

We first consider the set Δ. Using Corollary 4.6 we have $\prod_{a \in \Delta} X_a \leq \prod_{a \in \Delta} Y_a$. Thus for the open cover $P_\Delta(\alpha) = \alpha_\Delta$ of $\prod_{a \in \Delta} X_a$ there is an open cover $\mu_\Delta = \prod_{a \in \Delta} \mu_a$ of $\prod_{a \in \Delta} Y_a$ and a product mapping F_Δ of $\prod_{a \in \Delta} X_a$ onto $\prod_{a \in \Delta} Y_a$ such that $F^{-1}(\mu_\Delta) R \alpha_\Delta$. That is, $F_\Delta = \prod_{a \in \Delta} f_a$ for α_a-mappings f_a of X_a onto Y_a. For $a \in A - \Delta$ we have $\alpha_a = \{X_a\}$, and any map f_a of X_a onto Y_a is an α_a-map for $a \in A - \Delta$. Then let F be defined by $F = \prod_{a \in A} f_a$, where f_a is as described above. Form an open cover μ of $\prod_{a \in A} Y_a$ by letting $V \in \mu$ if and only if $P_\Delta(V) \in \mu_\Delta$, and for $a \in A - \Delta$ we need not be particular about $P_a(V)$ as we have observed. It follows from this construction that $F^{-1}(\mu) R \alpha$. α is arbitrary and hence $\prod_{a \in A} X_a \leq \prod_{a \in A} Y_a$.

The last theorem on product spaces is a partial converse of Theorem 4.7. Compactness of the coordinate spaces is not necessary for this theorem, but some other restrictions are necessary.

Theorem 4.8: Let A be an index set and $\prod_{a \in A} X_a \leq \prod_{a \in A} Y_a$. Suppose that for

each open cover α of $\prod_{a \in A} X_a$ there is an α-map which is a product map $F = \prod_{a \in A} f_a$. Then $X_a \leq Y_a$ for all $a \in A$.

Proof: Let $a' \in A$ be arbitrary and α' be an open cover of $X_{a'}$. The collection of open sets $\prod_{a \in A} U_a$, where $U_{a'} \in \alpha'$ and $U_a = X_a$ for $a \neq a'$, is an open cover α of $\prod_{a \in A} X_a$. By hypothesis there is a product map F on $\prod_{a \in A} X_a$ onto $\prod_{a \in A} Y_a$ and an open cover μ of $\prod_{a \in A} Y_a$ such that $F^{-1}(\mu)R\alpha$. Furthermore, the component $f_{a'}$ of F which maps $X_{a'}$ onto $Y_{a'}$ is continuous. If $P_{a'}$ is the projection mapping of $\prod_{a \in A} Y_a$ onto $Y_{a'}$, then $P_{a'}(\mu)$ is an open cover μ' of $Y_{a'}$ since $P_{a'}$ is an open mapping. We may assume that μ consists of basis elements for there is always a refinement of μ of this form. Then since $V \in \mu$ implies $F^{-1}(V)$ is contained in some member of α, $f_{a'}^{-1}(P_{a'}(V))$ is contained in some element of α'. This follows from Lemma 4.4. Thus $f_{a'}^{-1}(\mu')R\alpha$, and since α' and a' are arbitrary, $X_{a'} \leq Y_{a'}$ for all $a' \in A$.

V. METRIZATION AND EMBEDDING

A. Metrization and Generalizations and the Countability Axioms

If X is T_1 and $X \leq Y$ where Y is regular, then, as we have seen, X is regular. Since a regular second countable T_1 space is metrizable, we might ask whether metrizability itself is inherited. Perhaps just as interesting is the same question with regard to the countability axioms and some of the generalizations of metric spaces.

We will examine two generalizations of metric spaces, uniform spaces and developable spaces. We first look at the relation between uniform spaces and metrizable spaces, and obtain our only positive result with regard to the questions in the preceding paragraph. The discussion below on uniform spaces is taken from Kelley(8), Chapter 6.

Definition 5.1: A _uniformity_ for a set X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ such that

(1) each member of $\mathcal{U}$ contains the diagonal $\Delta = \{(x,x) : x \in X\}$;

(2) if $U \in \mathcal{U}$, then $U^{-1} = \{(x,y) : (y,x) \in U\} \in \mathcal{U}$;

(3) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$; ($V \circ V = \{(x, z):$ for some $y, (x,y) \in V$ and $(y, z) \in V\}$)

(4) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$; and

(5) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is a _uniform space_; and the topology $\mathcal{T}$ of the uniformity $\mathcal{U}$, or the _uniform topology_, is the family of all subsets T of X such that for each $x \in T$ there is a $U \in \mathcal{U}$ such that $U[x] = \{y : (x,y) \in U\} \subset T$.

Uniform spaces can be obtained in the following manner. A uniformity

is assigned to each family of pseudo-metrics for a set X, and every

uniformity is derived in this fashion from the family of uniformly con-

tinuous pseudo-metrics. A uniformity can be derived from a single pseudo-

metric if and only if the uniformity has a countable base. Since a T_1

pseudo-metrizable space is metrizable, it follows that a T_1 uniform space

is metrizable if and only if the uniformity has a countable base.

Each uniform space is homeomorphic to a subspace of a product of

pseudo-metric spaces. Then since a space is completely regular if and

only if it is homeomorphic to a subspace of a product of pseudo-metric

spaces, it follows that uniform spaces coincide with completely regular

spaces. Thus we have the following theorem.

__Theorem 5.2__: If X is T_1, Y is a uniform space, and $X \leq Y$, then X is a

uniform space.

Next we look briefly at developable spaces.

__Definition 5.3__: A topological space X is __developable__ if and only if there

exists a sequence $\{G_n\}_{n=1}^{\infty}$ of open coverings of X such that the following

conditions are satisfied: (i) for each $n \in I^+, G_{n+1} R G_n$; (ii) for each $x \in X$

and each open set U containing x, there exists an integer $N = N(x,U) \in I^+$

such that $G_N^*(x) \subset U$. The sequence $\{G_n\}_{n=1}^{\infty}$ is called a __development__ for

X. A space X will be called __uniformly developable__ if and only if X has

a development $\{G_n\}_{n=1}^{\infty}$ such that for each $\alpha \in \Sigma(X)$ there exists an integer

$n(\alpha) \in I^+$ such that $G_{n(\alpha)}^*(x) \subset \alpha^*(x)$ for all $x \in X$.

Bing (1) proves that if a regular developable T_1 space is collection-

wise normal then it is metrizable. He also proves that paracompactness

implies collectionwise normality, and hence a regular developable T_1 space

which is paracompact is metrizable. An unanswered question is whether or not each normal developable T_1 space is metrizable.

The following theorem exhibits an important property of the space that will be used in getting a negative answer to the question of the inheritance of developability.

Theorem 5.4: Let X be the set of real numbers, and let a basis for X be the collection of all half-open intervals, closed on the left. If $\alpha \in \Sigma(X)$, then there is a countable refinement of α consisting of disjoint basis members.

Proof: If $\alpha \in \Sigma(X)$, then there is a refinement α' of α consisting entirely of basis elements. X is Lindelöf and hence there is a countable subcollection of α', $\{I_n\}_{n=1}^{\infty}$, which covers X and each I_n is of the form $[a_n, b_n)$. From $\{I_n\}_{n=1}^{\infty}$ we will construct the desired open cover.

We begin by taking I_1 and noting that then $I_2 - I_1$ is empty or contains at most two disjoint basis elements of X. We can also observe that for each m, $I_m - \bigcup_{k=1}^{m} I_k$ is a finite collection of disjoint basis elements, at most m of them, which we indicate by $I_{m1}, I_{m2}, \ldots, I_{mk(m)}$. Then the collection $\{I_{mn} : m \in I^+, n = 1, \ldots, k(m)\}$ is a countable collection of disjoint basis elements which covers X and is a refinement of α.

Example 5.5: Let X be the space described in Theorem 5.4. We observe that X is first countable, since for each $x \in X$ a countable neighborhood system can be defined by $N_n(x) = [x, x + 1/n)$ for $n \in I^+$. We also note that X is T_2, separable, normal, and paracompact as well as being Lindelöf.

Let Y be the positive integers with the discrete topology and let $\alpha \in \Sigma(X)$. By Theorem 5.4, there is a countable refinement α' of α

consisting of disjoint basic members of X. (If α is finite, then we can

always get an infinite refinement α' from some member of α.) Let f be a

one to one function between members of α' and points of Y. That is, the

members of α' can be ordered by I_1', I_2', ..., and let $f(I_n') = n$ for each

$n \varepsilon I^+$. Clearly f is continuous and if μ is the discrete cover of Y,

$f^{-1}(\mu) R \alpha'$ and hence $f^{-1}(\mu) R \alpha$. α is arbitrary and so $X \leq Y$.

The space Y is metrizable. It also satisfied all the conditions of

Definition 5.3 and is hence uniformly developable. For the collection of

singleton sets of Y can be taken as a development. That is, $G_1 = G_2 = \cdots$.

However, X is not developable. For Sims (13) has shown that X is not

semi-metrizable and a T_1 developable space is semi-metrizable.

Here, of course, the space X is not second countable, for every basis

contains sets of the form [a,b) for each real number a.

We will conclude this section with an example in which the first

axiom of countability is also not inherited under the relation $\leq$.

Example 5.6: Let $X = [0,1]$ and let a basis for X consist of singleton

sets, $\{ x \}$, $x \neq 0$, and all sets containing zero, each of whose complement

is countable. Let Y be the positive integers with the discrete topology.

We note that both X and Y are Lindelöf, T_2, and regular. Y is of course

both first and second countable.

Let $\alpha \varepsilon \Sigma(X)$ and let $0 \varepsilon U \varepsilon \alpha$. Then X - U is countable. If X - U

is infinite, define a function f on X by letting $f(U) = 1$, and let f be

any one to one function on X - U onto Y - $\{ 1 \}$. Clearly f is continuous,

for each singleton set in X - U is open, and likewise the singleton sets

in Y. If X - U is finite, we extract a countably infinite set of points

from U, different from zero, so that we have a new open set $U' \subset U$ which

contains zero and such that X - U' is countably infinite. Thus an

$\alpha' \in \Sigma(X)$ is obtained such that $\alpha' R \alpha$ and also a mapping f as described

above.

It is clear that the mapping f thus constructed is an α-mapping, and

since such a mapping can be constructed for each $\alpha \in \Sigma(X)$, $X \leq Y$. However,

X is not first countable, for there is no countable local base at zero.

B. An Embedding Theorem

Let F be a family of functions such that each $f \in F$ is on a space X

onto a space Y_f. There is a natural mapping of X into the product $\prod_{f \in F} Y_f$

which is defined by mapping a point x of X into the member of the product

whose f-th coordinate is f(x). Thus we have the following definition.

Definition 5.7: The __evaluation map__ e is defined by $e(x) = \{f(x)\}_{f \in F}$;

i.e. $P_f \circ e(x) = f(x)$, where P_f is the projection of $\prod_{f \in F} Y_f$ onto Y_f.

Definition 5.8: A family F of functions on X distinguishes points if and

only if for each pair of distinct points x_1 and x_2 of X there is an $f \in F$

such that $f(x_1) \neq f(x_2)$. The family distinguishes points and closed sets

if and only if for each closed subset A of X and each point x of X - A

there is an $f \in F$ such that $f(x) \notin \overline{f(A)}$.

The above brief discussion and definitions, and the following lemma

are found in Kelley (8, pp. 115-116).

Lemma 5.9: Let F be a family of continuous functions, each member being

on a topological space X to a space Y_f. Then:

(1) The evaluation map e is a continuous function on X to the

product space $\prod_{f \in F} Y_f$.

(2) The function e is an open map of X onto e(X) if F distinguishes points and closed sets.

(3) The function e is one to one if and only if F distinguishes points.

We now return to α-mappings and their relation to the above discussion.

Lemma 5.10: Let X be a T_1 space and suppose that for each $\alpha \in \Sigma(X)$ there is an α-map f_α of X onto a space Y_α. Then the family F of α-mappings of X distinguishes points, and points and closed sets.

Proof: Since X is T_1 we need only show that F distinguishes points and closed sets. Thus let A be a closed subset of X and let $x \in X - A$ be arbitrary. Since X is T_1, $U_1 = X - A$ and $U_2 = X - \{x\}$ are open sets containing x and A, respectively, but $A \cap U_1 = \emptyset$ and $x \notin U_2$. Let $\alpha = \{U_1, U_2, X - (A \cup \{x\})\}$. Then $\alpha \in \Sigma(X)$ and by hypothesis there exists an α-map f_α of X onto a space Y_α, hence an open cover μ of Y_α such that $f_\alpha^{-1}(\mu)R\alpha$.

Let $y \in \overline{f_\alpha(A)}$. Then every neighborhood of y contains a point of $f_\alpha(A)$. Hence if $y \in V \in \mu$, then there is an $a \in A$ such that $f_\alpha(a) \in V$. But $f_\alpha^{-1}(V) \subset U_2$, and since this is true of every $y \in \overline{f_\alpha(A)}$, it follows that $f_\alpha(x) \notin \overline{f_\alpha(A)}$.

Thus from Lemma 5.10 we have that any family F of mappings of a T_1 space X which contains an α-mapping for each $\alpha \in \Sigma(X)$ distinguishes points and also points and closed sets.

Theorem 5.11: Let X be a T_1 space and suppose that for each $\alpha \in \Sigma(X)$ there is an α-map f_α of X onto a space Y_α. Then X is homeomorphic to a subspace

of the product space $\prod_{\alpha \in \Sigma(X)} Y_\alpha$.

Proof: Define the evaluation map e by letting $e(x) = \{f_\alpha(x)\}_{\alpha \in \Sigma(X)}$

for each $x \in X$. By Lemma 5.10 the family F of α-maps distinguishes points,

and points and closed sets. Then by Lemma 5.9, e is one to one, open,

and continuous, hence a homeomorphism.

We note that this embedding is such that the projection to the

coordinates of $e(X)$ is actually onto the spaces Y_α. That is, $P_\alpha(e(X)) = Y_\alpha$.

It is also worthy of note that Theorem 2.2 is actually a corollary of

Theorem 5.11. For in the earlier theorem f was an α-map for all $\alpha \in \Sigma(X)$.

Ponomarev (12) proved the following two theorems.

Theorem 5.12: A necessary and sufficient condition that a T_2 space X be

paracompact is that for each $\alpha \in \Sigma(X)$ there is an α-map f_α of X onto a

metric space Y_α.

Theorem 5.13: A necessary and sufficient condition that a regular space

X be Lindelöf is that for every $\alpha \in \Sigma(X)$ there is an α-map f_α of X onto a

separable metric space Y_α.

In view of our observations in Chapter I with regard to Maxwell's

results we may state the following variants of Ponomarev's theorems.

Theorem 5.12': A necessary and sufficient condition that a space X be

paracompact is that for each $\alpha \in \Sigma(X)$ there is an α-map f_α of X onto a

paracompact space Y_α.

Theorem 5.13': A necessary and sufficient condition that a space X be

Lindelöf is that for each $\alpha \in \Sigma(X)$ there is an α-map f_α of X onto a

Lindelöf space Y_α.

Applying Theorem 5.11 to Theorems 5.12 and 5.13 yields the following

results.

Theorem 5.14: A necessary and sufficient condition that a T_2 space X be

paracompact is that there is a homeomorphism e of X into a product of

metric spaces $\prod_{\alpha \in \Sigma(X)} Y_\alpha$ such that $P_\alpha \circ e$ is an α-mapping of X onto Y_α.

Proof: The existence of the homeomorphism of X if X is paracompact

follows from Theorems 5.11 and 5.12. If the homeomorphism e exists, then

X is paracompact by Theorem 5.12.

Theorem 5.15: A necessary and sufficient condition that a regular T_1 space

X be Lindelöf is that there is a homeomorphism e of X into a product of

separable metric spaces $\prod_{\alpha \in \Sigma(X)} Y_\alpha$ such that $P_\alpha \circ e$ is an α-mapping of X onto

Y_α.

Proof: The existence of the homeomorphism when X is Lindelöf follows

from Theorems 5.11 and 5.13. If the homeomorphism exists, then X is

Lindelöf by Theorem 5.13.

VI. QUOTIENT SPACES AND TOPOLOGICAL GROUPS

A. Quotient Spaces

Definition 6.1: If Q is an equivalence relation on a set X, then X/Q is defined to be the family of equivalence classes. If $A \subset X$, then $Q[A] = \{ y:(x,y) \in Q \text{ for some } x \in A \}$. The quotient map of X onto X/Q is the function P whose value at x is the equivalence class $[x]$ to which x belongs. Thus if $x \in X$, then $p(x) = [x]$ (8, pp. 96-97).

Definition 6.2: If X is a space with topology $\mathcal{T}$, then the set X/Q is given the quotient topology of P, which will be indicated by $\mathcal{T}/Q$. (See the discussion preceding Example 2.4.) Thus if $\mathcal{Q} \subset X/Q$, then $P^{-1}(\mathcal{Q}) = \underset{A \in \mathcal{Q}}{\cup} A$, and $\mathcal{Q}$ is open (closed) relative to $\mathcal{T}/Q$ if and only if $\underset{A \in \mathcal{Q}}{\cup} A$ is open (closed) in $(X, \mathcal{T})$. The pair $(X/Q, \mathcal{T}/Q)$ will be called a quotient space.

If $(X/Q, \mathcal{T}/Q)$ is a quotient space, then there is another topology on X which is obtained from the quotient space in the following manner. If P is the quotient map of the set X onto X/Q, then the product topology of P on X will be indicated by $\mathcal{T}_Q$. Thus $U \in \mathcal{T}_Q$ if and only if $U = P^{-1}(V)$ for some $V \in \mathcal{T}/Q$.

There are instances in which there are equivalence relations Q such that $(X, \mathcal{T}) \leq (X, \mathcal{T}_Q)$ and/or $(X, \mathcal{T}_Q) \leq (X, \mathcal{T})$, and we wish to investigate this occurrence. Before proceeding with this, however, we should note that each $\mathcal{T}_Q$ as defined above is such that $\mathcal{T}_Q \subset \mathcal{T}$. It also should be pointed out that if there is any nondegenerate element $[x]$ in X/Q, then $(X, \mathcal{T}_Q)$ cannot be even a T_0 space. Hence the topologies $\mathcal{T}_Q$ have very

little in the way of separation properties.

The spaces $(X, \mathcal{T}_Q)$ and $(X/Q, \mathcal{T}/Q)$ are related by the following lemma.

Lemma 6.3: If $(X/Q, \mathcal{T}/Q)$ is a quotient space, then $(X, \mathcal{T}_Q) \leq (X/Q, \mathcal{T}/Q)$.

Proof: The quotient map P is an α-map of $(X, \mathcal{T}_Q)$ onto $(X/Q, \mathcal{T}/Q)$ for all α when $\mathcal{T}_Q$ is the product topology of P. For $U \varepsilon \mathcal{T}_Q$ if and only if there is a $V \varepsilon \mathcal{T}/Q$ such that $P^{-1}(V) = U$.

Thus for each equivalence relation Q the pair of spaces $((X, \mathcal{T}_Q),$ $(X/Q, \mathcal{T}/Q))$ belongs to a class in which there is an α-mapping for all α, of the first space onto the second. (Compare this with the discussion in Chapter II.)

The following theorems give some conditions under which $(X, \mathcal{T}_Q) \leq (X, \mathcal{T})$ and/or $(X, \mathcal{T}) \leq (X, \mathcal{T}_Q)$.

Theorem 6.4: Let $(X, \mathcal{T})$ be a topological space and Q an equivalence relation on X. If $(X/Q, \mathcal{T}/Q) \leq (X, \mathcal{T})$, then $(X, \mathcal{T}_Q) \leq (X, \mathcal{T})$.

Proof: By Lemma 6.3, $(X, \mathcal{T}_Q) \leq (X/Q, \mathcal{T}/Q)$. Then by the transitivity of the relation $\leq$ the theorem follows.

Remark: The hypothesis $(X/Q, \mathcal{T}/Q) \leq (X, \mathcal{T})$ is not as restrictive as it might seem in view of a theorem of R. L. Moore. Moore's theorem states that if X/Q is any nontrivial upper semicontinuous decomposition of a plane $(X, \mathcal{T})$ into continua not separating X, then $(X, \mathcal{T})$ and $(X/Q, \mathcal{T}/Q)$ are in fact homeomorphic (15). Much interest is centered on the extension of this theorem to higher dimensional spaces.

Theorem 6.5: Let $(X, \mathcal{T})$ be a topological space, Q an equivalence relation on X, and $\alpha \varepsilon \Sigma(X, \mathcal{T})$. Then $\alpha \subset \mathcal{T}_Q$ if and only if $Q[U] = U$ for all $U \varepsilon \alpha$, and both of these imply that the identity map $i:(X, \mathcal{T}) \longrightarrow (X, \mathcal{T}_Q)$ is an

α-map.

Proof: We will first show that $\alpha \subset \mathcal{T}_Q$ if and only if $Q[U] = U$ for all $U \varepsilon \alpha$. Observe that $\alpha \subset \mathcal{T}_Q$ if and only if $U \varepsilon \alpha$ implies $U \varepsilon \mathcal{T}_Q$; and $U \varepsilon \mathcal{T}_Q$ if and only if $U = P^{-1}(V')$ for some $V' \varepsilon \mathcal{T}/Q$. Then U is a union of equivalence classes and it follows that $U = Q[U]$.

To prove the other half let $Q[U] = U$ for all $U \varepsilon \alpha$. Then U is a union of equivalence classes, and hence there is a $U' \subset X/Q$ such that $P^{-1}(U') = U$. Then $U' \varepsilon \mathcal{T}/Q$. Since $\mathcal{T}_Q$ is the product topology of P it follows that $U \varepsilon \mathcal{T}_Q$. Therefore, $\alpha \subset \mathcal{T}_Q$.

To prove the theorem, note that i is continuous because $\mathcal{T}_Q \subset \mathcal{T}$. If $\alpha \subset \mathcal{T}_Q$, then clearly i is an α-mapping, for $i^{-1}(\alpha)R\alpha$.

Definition 6.6: If $\Sigma(X)$ is the family of open coverings of a space X, then a subcollection $\Sigma'(X)$ is called a cofinal family of coverings of X if and only if for every $\alpha \varepsilon \Sigma(X)$ there is an $\alpha' \varepsilon \Sigma'(X)$ such that $\alpha'R\alpha$.

Corollary 6.7: Let $\Sigma'(X, \mathcal{T})$ be a cofinal family of open coverings of $(X, \mathcal{T})$ and suppose that for each $\alpha \varepsilon \Sigma'(X, \mathcal{T})$ there is an equivalence relation $Q = Q(\alpha)$ such that $\alpha \subset \mathcal{T}_Q$. If each $(X, \mathcal{T}_Q) \leq (Y, \sigma)$, a fixed space, then $(X, \mathcal{T}) \leq (Y, \sigma)$.

Proof: Let $\alpha \varepsilon \Sigma'(X, \mathcal{T})$ be given and Q the corresponding equivalence relation such that $\alpha \subset \mathcal{T}_Q$. Then by Theorem 6.5 the identity mapping $i: (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}_Q)$ is an α-mapping. Hence there is a $\mu \varepsilon \Sigma(X, \mathcal{T}_Q)$ such that $i^{-1}(\mu)R\alpha$. By hypothesis there is a μ-mapping f of $(X, \mathcal{T}_Q)$ onto (Y, σ). Then $f \circ i$ is an α-map of $(X, \mathcal{T})$ onto (Y, σ), and since α is arbitrary, $(X, \mathcal{T}) \leq (Y, \sigma)$.

To illustrate the use of Theorems 6.4 and 6.5, and Corollary 6.7,

consider the following example.

Example 6.8: Let X = [0,1] and define a topology $\mathcal{T}$ on X as follows.

Let $\{x\} \in \mathcal{T}$ if $x \neq 0$. If $0 \in U$, then $U \in \mathcal{T}$ if and only if X - U is finite. Define an equivalence relation Q on X by letting $[x] = \{x\}$ if x is irrational. If r_1 and r_2 are any two distinct rationals, then $[r_1] = [r_2]$. It is easily seen that the quotient space $(X/Q, \mathcal{T}/Q)$ and $(X, \mathcal{T})$ are homeomorphic. Hence by Theorem 6.4, $(X, \mathcal{T}_Q) \leq (X, \mathcal{T})$.

To show that $(X, \mathcal{T}) \leq (X, \mathcal{T}_Q)$, let α' be any open cover of $(X, \mathcal{T})$. Then if $0 \in U \in \alpha'$, X - U is finite. Let α be such that $U \in \alpha$ and if $x \in X - U$, $\{x\} \in \alpha$. Then $\alpha R \alpha'$. Define an equivalence relation $Q(\alpha)$ by letting $[0]$ contain the points of any countably infinite subset A of U such that $0 \in A$, and $[x] = \{x\}$ if $x \in X - A$. Obtaining $\mathcal{T}_{Q(\alpha)}$ from $(X/Q(\alpha), \mathcal{T}/Q(\alpha))$, it is clear that $\alpha \subset \mathcal{T}_{Q(\alpha)}$. Hence by Theorem 6.5, the identity $i:(X, \mathcal{T}) \longrightarrow (X, \mathcal{T}_{Q(\alpha)})$ is an α-map.

For each $\alpha' \in \Sigma(X, \mathcal{T})$ there is an α as above such that $\alpha R \alpha'$, and it is clear that the corresponding space $(X, \mathcal{T}_{Q(\alpha)})$ is homeomorphic to $(X, \mathcal{T}_Q)$. Hence $(X, \mathcal{T}_{Q(\alpha)}) \leq (X, \mathcal{T}_Q)$, and it follows from Corollary 6.7 that $(X, \mathcal{T}) \leq (X, \mathcal{T}_Q)$.

B. Topological Groups

Before proving a theorem on compact topological groups, some background on topological groups is necessary. The discussion that follows is taken from Montgomery and Zippin (11).

Definition 6.9: A topological group G is a space in which G is a group, and the functions f and g defined by $f(x) = x^{-1}$ and $g(x,y) = xy$ are con-

tinuous on G and G×G, respectively. Unless otherwise indicated, when we refer to a group it will be a topological group.

Definition 6.10: When a subset H of a group G is an abstract group, then H will be called a (topological) subgroup of G, H being given the relative topology.

Lemma 6.11: Let G be a group, A an open subset of G, and b an element of G. Then Ab and bA are open.

Theorem 6.12: Let G be a group and let A and B be subsets of G. Then if A or B is open, AB is open.

If G is a group satisfying any of the T_i axioms, then G is T_2 and, in fact, completely regular.

Definition 6.13: A homomorphism of a group G is a group homomorphism which is continuous. An isomorphism is a group isomorphism and a space homeomorphism.

Definition 6.14: If G is a group and A⊂G, let $A^{-1} = \{a^{-1}:a \in A\}$. A set A is called symmetric if and only if A^{-1} = A.

If U is a neighborhood of the identity e, then U^{-1} is a neighborhood of e and $U \cap U^{-1}$ is a symmetric neighborhood of e.

Theorem 6.15: Let G be a group and U a neighborhood of e. Then there exists a symmetric neighborhood W of e such that $W^2 \subset U$.

If B = $\{V\}$ is a local base at e, then xV is an open set containing x for each V ∈ B. Hence xB = $\{xV\}_{V \in B}$ is a local base at x for each x ∈ G.

Let G be a group and H a subgroup of G. The sets xH and yH for x,y ∈ G either coincide or are disjoint. Each set xH is called a left

coset of H. The notation G/H is used to represent the set of all left

cosets of H. If H is an invariant subgroup, i.e. xH = Hx for all x ∈ G,

then G/H is an abstract group, the abstract factor group, with topology to

be specified below.

Definition 6.16: The natural map T of a group G onto the abstract factor

group G/H is defined by T(x) = xH for x ∈ G.

Definition 6.17: Let G be a group and H a subgroup of G. By an open set

in G/H is meant a set whose inverse image under the natural map is an

open set in G.

Thus the factor group G/H is given the quotient topology of T. Note

that for any subset U of G, T(U) = UH and hence T is open. It is also

true that $T^{-1}(T(U)) = UH$.

The necessary machinery to prove the following theorem is now avail-

able.

Theorem 6.18: Let G be a group, and let $\{G_a\}_{a \in A}$ be a family of

arbitrarily small invariant subgroups of G for some index set A. Then if

$\alpha \in \Sigma(G)$, there is an $a \in A$ such that the natural mapping $T_a: G \longrightarrow G/G_a$ is

an α-mapping.

Proof: Let $\alpha \in \Sigma(G)$. Then if $V \in \alpha$ and $x \in V$, $xU \subset V$ for some U in a

local base at e. Furthermore, since U is a neighborhood of e, there is a

symmetric neighborhood of e, call it W, such that $W^2 \subset U$. Then clearly

$W \subset U$ and $xW \subset xU$. Let μ be a refinement of α consisting of open sets of

the form xW, where each W is as described above. G is compact, and hence

there is a finite subfamily of μ, call it μ', which covers G. The family

μ' can be indicated by $\{x_i W_i\}_{i \leq n}$, where n is a positive integer.

Since G has arbitrarily small invariant subgroups, there is an a $\hat{A}$

such that $G_a \subset \bigcap_{i=1}^{n} W_i$. Let $T_a : G \longrightarrow G/G_a$ be the natural map of G onto the

factor group G/G_a. Observe that $T_a(x_i W_i) = x_i W_i G_a$. Since $\{x_i W_i\}_{i \leq n}$

is an open cover of G, it follows that the family $\{x_i W_i G_a\}_{i \leq n} = T_a(\mu')$

is an open cover of G/G_a. Furthermore, $T_a^{-1}(x_i W_i G_a) = x_i W_i G_a \subset x_i W_i^2 \subset x_i U$,

so it follows that $T_a^{-1}(T_a(\mu'))R^\alpha$. Thus T_a is an α-map, and since α is

arbitrary, the theorem is proved.

This theorem and Lemmas 5.9 and 5.10 are used in obtaining a group

G as the inverse limit of an inverse system of factor groups of G. A brief

discussion of inverse systems and inverse limits is given below (6, pp.

91-94).

Definition 6.19: An inverse system of sets $\{X, P\}$ over a directed set A

is a function which attaches to each a $\in$ A a set X_a, and to each pair a,b

such that a < b, a map

$$P_{ab} : X_b \longrightarrow X_a$$

such that

$$P_{aa} = \text{identity}, \ a \in A,$$

$$P_{ab} P_{bc} = P_{ac}, \ a < b < c \text{ in } A.$$

Definition 6.20: Let $\{X, P\}$ be an inverse system of sets over the direct-

ed set A. The inverse limit X_∞ of $\{X, P\}$ is the subset of the product

$\prod_{a \in A} X_a$ consisting of those points $x = \{x_a\}_{a \in A}$ such that for each rela-

tion a < b in A, $P_{ab}(x_b) = x_a$.

If the sets X_a are groups, then the functions P_{ab} are assumed to be

homomorphisms as defined in Definition 6.5. X_∞ is given the relative

topology as a subset of the product space $\prod_A X_a$, and when each X_a is a group then X_∞ is a subgroup of $\prod_{a \in A} X_a$.

Let G be a compact group which is T_0, hence is a T_2 space. Let $\{G_a\}_{a \in A}$ be a family of arbitrarily small closed invariant subgroups of G. Then A can be made into a directed set by writing $a < b$ in A if and only if $G_b \subset G_a$. Then the family $\{H,P\}$, where $H_a = G/G_a$ and for $a < b$, $P_{ab}: G/G_b \longrightarrow G/G_a$ is the natural mapping, is an inverse system of groups. Thus we have the following theorem.

Theorem 6.21: Let G be a compact group which is T_2 and $\{H,P\}$ an inverse system of factor groups of arbitrarily small closed invariant subgroups G_a of G. Then G is isomorphic to G_∞, the inverse limit of $\{H,P\}$.

Proof: Consider the family $\{T_a\}_{a \in A}$ of natural mappings of G onto $H_a = G/G_a$. By Theorem 6.18, for each α $\Sigma(G)$ there is a T_a which is an α-map, and by Lemma 5.10 the family of α-maps distinguishes points, and points and closed sets. Then by Lemma 5.9 the mapping on G defined by $f(g) = \{T_a(g)\}_{a \in A}$ is an isomorphism of G into $\prod_{a \in A} G_a$. Using the facts that all the subgroups G_a are compact, and A is a directed set, it can be shown that each $\{x_a\}_{a \in A}$ in G_∞ is of the form $\{gG_a\}_{a \in A}$ for some $g \in G$. Since these are precisely the elements of $f(G)$, the theorem is proved.

VII. LITERATURE CITED

1. Bing, R. H. Metrization of topological spaces. Canadian Journal of Mathematics 3: 175-186. 1951.

2. Borsuk, K. Remarques sur las quasi-homeomorphie. Colloquium Mathematicum 6: 1-4. 1958.

3. Fort, M. K., Jr. Epsilon maps of a disk onto a torus. Bulletin de L'Academie Polonaise des Sciences Serie des Sciences Mathématique, Astronomiques et Physiques 7: 51-54. 1959.

4. Ganea, T. Epsilon mappings onto manifolds. Fundamenta Mathematicae 47: 35-44. 1959.

5. Hall, D. W. and Spencer, G. L., II. Elementary topology. New York, N. Y., John Wiley and Sons, Inc. 1955.

6. Hocking, J. G. and Young, G. S. Topology. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc. 1961.

7. Hurewicz, W. and Wallman, H. Dimension theory. Princeton, New Jersey, Princeton University Press. 1941.

8. Kelley, J. L. General topology. New York, N. Y., D. Van Nostrand, Inc. 1955.

9. McGrath, F. J. Set properties induced by subsets of lesser cardinality. Unpublished M.S. thesis. Ames, Iowa, Library, Iowa State University of Science and Technology. 1962.

10. Maxwell, C. N. An order relation among topological spaces. Transactions of the American Mathematical Society 99: 201-204. 1961.

11. Montgomery, D. and Zippin, L. Topological transformation groups. New York, N. Y., Interscience Publishers, Inc. 1955.

12. Ponomarev, V. On paracompact and finally-compact spaces. Soviet Mathematics 2: 1510-1512. 1961.

13. Sims, B. T. Some properties and generalizations of semi-metric spaces. Unpublished Ph.D. thesis. Ames, Iowa, Library, Iowa State University of Science and Technology. 1962.

14. Ulam, S. A collection of mathematical problems. New York, N. Y., Interscience Publishers, Inc. 1960.

15. Whyburn, G. T. Decomposition spaces. In Fort, M. K., Jr., ed. Topology of 3-manifolds and related topics. pp. 2-4. Englewood Cliffs, New Jersey, Prentice-Hall, Inc. 1962.

VIII. ACKNOWLEDGMENT

The author would like to take this opportunity to express his appreciation to Dr. D. E. Sanderson for suggesting the topic, and for his invaluable criticisms and suggestions in the preparation of this thesis.