Orthogonal representations, minimum rank, and graph complements

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Abstract

Orthogonal representations are used to show that complements of certain sparse graphs have (positive semidefinite) minimum rank at most 4. This bound applies to the complement of a 2-tree and to the complement of a unicyclic graph. Hence for such graphs, the sum of the minimum rank of the graph and the minimum rank of its complement is at most two more than the order of the graph. The minimum rank of the complement of a 2-tree is determined exactly.

Keywords. minimum rank, orthogonal representation, 2-tree, unicyclic graph, graph complement.
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1 Introduction

A *graph* is a pair $G = (V, E)$, where V is the (finite, nonempty) set of vertices (usually $\{1, \ldots, n\}$ or a subset thereof) and E is the set of edges (an edge is a two-element subset of vertices); what we call a graph is sometimes called a simple undirected graph. The *order* of a graph G, denoted $|G|$, is the number of vertices of G.

The set of $n \times n$ real symmetric matrices will be denoted by S_n. For $A \in S_n$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : \ a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$.

The set of real symmetric matrices of G is

$$\mathcal{S}(G) = \{A \in S_n : \ \mathcal{G}(A) = G\}$$

and the set of real positive semidefinite matrices of G is

$$\mathcal{S}_+(G) = \{A \in S_n : \ A \text{ positive semidefinite and } \mathcal{G}(A) = G\}.$$

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The *minimum rank* of a graph G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}(G)\},$$

and the *positive semidefinite minimum rank* of G is

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}_+(G)\}.$$

Clearly

$$\mathcal{S}_+(G) \subseteq \mathcal{S}(G) \qquad \text{and} \qquad \mathrm{mr}(G) \leq \mathrm{mr}_+(G).$$

The *minimum rank problem* (of a graph, over the real numbers) is to determine $\mathrm{mr}(G)$ for any graph G. See [10] for a survey of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there. A minimum rank graph catalog [2] is available on-line, and will be updated routinely. Positive semidefinite minimum rank, both of real symmetric matrices as just defined, and of possibly complex Hermitian matrices, has been studied in [6], [7], [8], [11], [14].

For vertices $u, v \in V$, if u is adjacent to v (i.e., $\{u, v\} \in E$), then we write $u \sim v$; otherwise $u \not\sim v$. A subset U of vertices is *independent* if no two vertices in U are adjacent. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element sets of V that are not in E. The subgraph $G[R]$ of $G = (V, E)$ *induced* by $R \subseteq V$ is the subgraph with vertex set R and edge set $\{\{i, j\} \in E \mid i, j \in R\}$. The *join* $G_1 \vee G_2$ of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the union of $G_1 \cup G_2$ and the complete bipartite graph with with vertex set $V_1 \cup V_2$ and partition $\{V_1, V_2\}$.

At the AIM workshop [3] the relationship between the minimum rank of a graph and its complement was explored. The following question was asked:

Question 1.1. [9, Question 1.16] *How large can* $\mathrm{mr}(G) + \mathrm{mr}(\overline{G})$ *be?*

It was noted there that for the few graphs for which the minimum rank of both the graph and its complement was known,

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + 2 \tag{1}$$

and equality in this bound is achieved by a path.

In [1] it was shown that the (positive semidefinite) minimum rank of the complement of a tree is at most 3, and thus a tree satisfies the bound (1). We will show that some other families of sparse graphs, including unicyclic graphs (a graph is *unicyclic* if it contains exactly one cycle) and 2-trees (defined below) also satisfy the bound (1).

Unfortunately, there are conflicting uses of the term 2-tree in the literature. Here we follow [16] in defining a *k-tree* to be a graph that can be built up from a k-clique by adding one vertex at a time adjacent to exactly the vertices in an existing k-clique. Thus a tree is a 1-tree and a 2-*tree* can be thought of as a graph built up one triangle at a time by identifying an edge of a new triangle with an existing edge. A 2-tree is *linear* if it has exactly two vertices of degree 2. Thus a linear 2-tree is a “path” of triangles built up one triangle at a time by identifying an edge of a new triangle with an edge that has a vertex of degree 2. In [15] a *linear singly edge-articulated cycle graph* or *LSEAC graph* is (essentially) defined to be a "path" of cycles built up one cycle at a time by identifying an edge of a new cycle with an edge (that has a vertex of degree 2) of the most recently added cycle. Such a graph can be obtained by deleting interior edges from a outerplanar drawing of a linear 2-tree. In [12], the term "linear 2-tree" was used for an LSEAC graph, although an equivalent definition using the dual of an outerplanar drawing was given.

Examples of a linear 2-tree, an LSEAC graph that is not a 2-tree, and a 2-tree that is not linear are shown in Figure 1.

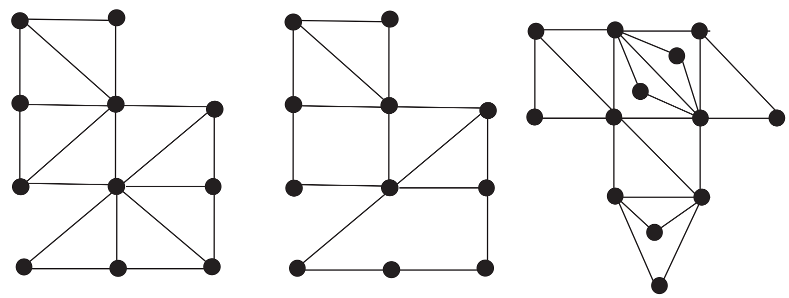

Figure 1: Linear 2-tree, an LSEAC graph that is not a 2-tree, and a nonlinear 2-tree

In [12] (and independently in [15]) it is shown that a 2-connected graph L is an LSEAC graph if and only if $\mathrm{mr}(L) = |L| - 2$. Hence the minimum rank of the left and center graphs shown in Figure 1 is 9.

For a tree, unicyclic graph, or 2-tree G, the number of edges of G is $|G| - 1$, $|G|, 2|G| - 3$, respectively, so trees, unicyclic graphs, and 2-trees are all sparse.

Suppose $G = (V, E)$ is a graph. Then a *d-dimensional orthogonal representation of* G is a function $v \to \vec{v}$ from V to $\mathbb{R}^d$ such that $\vec{u}$ and $\vec{v}$ are orthogonal if and only if u and v are nonadjacent vertices. For a subspace U of $\mathbb{R}^d$, let $U^\perp$ be the subspace of $\mathbb{R}^d$ of vectors orthogonal to U, and $\mathbf{v}^\perp = \mathrm{Span}(\mathbf{v})^\perp$. The following observations will be used repeatedly.

Observation 1.2. *Let $d(G)$ denote the smallest dimension d over all orthogonal representations of G. Then $d(G)$ is equal to* $\mathrm{mr}_+(G)$.

Observation 1.3. *No subspace W of $\mathbb{R}^d$ is a union of a finite number of proper subspaces of W.*

Observation 1.4. *For any three pairwise independent vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^4$, the following are equivalent.*

1. $\dim \mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 3$.
2. $\mathbf{u}^\perp \cap \mathbf{v}^\perp \not\subseteq \mathbf{w}^\perp$.

2 Orthogonal representations of dense graphs

In [1] orthogonal representations were used to prove that the complement of a tree has positive semidefinite minimum rank at most 3. A complement of a tree can be constructed by adding one vertex at a time, with each new vertex adjacent to all but one of the prior vertices. In this section we extend this technique to certain (very) dense graphs constructed by adding vertices adjacent to all but one or two prior vertices. These results will be used in the next section to study complements of certain sparse graphs and the relationship between $\mathrm{mr}(G)$ and $\mathrm{mr}(\overline{G})$.

The following is an easy generalization of the proof of [1, Theorem 3.16].

Theorem 2.1. *Let $Y = (V_Y, E_Y)$ be a graph of order at least two such that there is an orthogonal representation in $\mathbb{R}^d, d \geq 3$ satisfying*

$$\vec{v} \notin \mathrm{Span}(\vec{u}) \qquad \text{for each pair of distinct vertices } u, v \text{ in } V_Y. \tag{2}$$

Let X be a graph that can be constructed by starting with Y and adding one vertex at a time, such that the newly added vertex is adjacent to all prior vertices except at most one vertex. Then there is d-dimensional orthogonal representation of X satisfying (2); in particular, $\mathrm{mr}(X) \leq \mathrm{mr}_+(X) \leq d$.

Proof. Let $V_Y = \{v_1, \ldots, v_k\}$. Let X be constructed from Y by adding vertices $v_{k+1}, \ldots, v_n$ such that for $m > k$, v_m is adjacent to all but at most one of $v_1, \ldots, v_{m-1}$. Assuming that an orthogonal representation of $X[v_1, \ldots, v_{m-1}]$ in $\mathbb{R}^d$ has been constructed satisfying (2), we show there is an orthogonal representation of $X[v_1, \ldots, v_m]$ in $\mathbb{R}^d$ satisfying (2). If v_m is adjacent to $v_1, \ldots, v_{m-1}$ then choose as $\vec{v}_m$ any vector in $\mathbb{R}^d$ that is not in $\left(\bigcup_{i=1}^{m-1} \vec{v}_i^{\perp}\right) \cup \left(\bigcup_{i=1}^{m-1} \mathrm{Span}(\vec{v}_i)\right)$.

Otherwise, let v_s be the only vertex of $X[v_1, \ldots, v_{m-1}]$ not adjacent to v_m in $X[v_1, \ldots, v_m]$. We want to choose a vector $\vec{v}_m$ such that

$$\begin{aligned} \vec{v}_m &\in \vec{v}_s^{\perp} & & \\ \vec{v}_m &\notin \vec{v}_i^{\perp} & & \forall i \neq s, i < m \\ \vec{v}_m &\notin \mathrm{Span}(\vec{v}_i) & & \forall i < m. \end{aligned}$$

By applying Observation 1.3 to $W = \vec{v}_s^{\perp}$ and subspaces

$$\begin{aligned} A_i &= W \cap \vec{v}_i^{\perp} & & i \neq s, i < m \\ B_i &= W \cap \mathrm{Span}(\vec{v}_i) & & i \neq s, i < m \end{aligned}$$

we can conclude the desired vector exists, since clearly none of the subspaces A_i, B_i is equal to W. Thus we have constructed an orthogonal representation of X in $\mathbb{R}^d$ such that $\vec{u}$ and $\vec{v}$ are linearly independent for any distinct vertices u, v of X. $\square$

Theorem 2.2. *Let $Y = (V_Y, E_Y)$ be a graph such that the order of Y is at least two, V_Y does not contain a set of four independent vertices, and there is an orthogonal representation of Y in $\mathbb{R}^4$ satisfying*

$$\vec{v} \notin \operatorname{Span}(\vec{u}) \qquad \textit{for } u \neq v \tag{3}$$

$$\dim \operatorname{Span}(\vec{u}, \vec{v}, \vec{w}) = 3 \qquad \textit{for all distinct } u, v, w \textit{ such that } v \not\sim u \tag{4}$$

for all vertices in V_Y. Let X be a graph that can be constructed by starting with Y and adding one vertex at a time, such that the newly added vertex is adjacent to all prior vertices except at most two nonadjacent vertices. Then there is an orthogonal representation of X satisfying (3) *and* (4)*; in particular,* $\operatorname{mr}(X) \leq \operatorname{mr}_+(X) \leq 4$.

Proof. Let $V_Y = \{v_1, \ldots, v_k\}$. Let X be constructed from Y by adding vertices $v_{k+1}, \ldots, v_n$ such that for $m > k$, v_m is adjacent to all but at most two nonadjacent vertices in $\{v_1, \ldots, v_{m-1}\}$. Assuming that an orthogonal representation of $X[v_1, \ldots, v_{m-1}]$ has been constructed satisfying (3) and (4), we show there is an orthogonal representation of $X[v_1, \ldots, v_m]$ satisfying (3) and (4).

If v_m is adjacent to all vertices except v_s, v_t (with $v_s \not\sim v_t$), it suffices to choose $\vec{v}_m$ such that

$$\vec{v}_m \in \vec{v}_s^{\perp} \cap \vec{v}_t^{\perp} \tag{5}$$

$$\vec{v}_m \notin \vec{v}_i^{\perp} \qquad \forall i \neq s, t, i < m \tag{6}$$

$$\vec{v}_m \notin \operatorname{Span}(\vec{v}_r, \vec{v}_i) \qquad r = s, t, \ \forall i \neq r, i < m \tag{7}$$

$$\vec{v}_m \notin \operatorname{Span}(\vec{v}_i, \vec{v}_j) \qquad \forall i, j \text{ such that } 1 \leq i < j < m \text{ and } v_i \not\sim v_j. \tag{8}$$

We will show that it is always possible to make such a choice.

By applying Observation 1.3 to

$$W = \vec{v}_s^{\perp} \cap \vec{v}_t^{\perp}$$

and subspaces

$$\begin{aligned} A_i &= W \cap \vec{v}_i^{\perp} && i \neq s, t, i < m \\ B_i &= W \cap \operatorname{Span}(\vec{v}_r, \vec{v}_i) && r = s, t, \ i \neq r, i < m \\ C_{i,j} &= W \cap \operatorname{Span}(\vec{v}_i, \vec{v}_j) && i < j < m \text{ such that } v_i \not\sim v_j \end{aligned}$$

we can conclude the desired vector exists, provided none of the subspaces $A_i, B_i, C_{i,j}$ is equal to W.

By condition (4) and Observation 1.4, $\vec{v}_s^{\perp} \cap \vec{v}_t^{\perp} \not\subseteq v_i^{\perp}$ for all $i \neq s, t$, so $A_i \neq W$.

By elementary linear algebra, $\dim W = 2 = \dim \operatorname{Span}(\vec{v}_r, \vec{v}_i)$, so if $W = B_i$, $W = \operatorname{Span}(\vec{v}_r, \vec{v}_i)$. But $\vec{v}_r \in \operatorname{Span}(\vec{v}_r, \vec{v}_i)$ and $\vec{v}_r \notin W$ (since $r = s$ or $r = t$). Thus $B_i \neq W$.

Finally, consider $C_{i,j}$. If $W = C_{i,j}$, then again both W and $\operatorname{Span}(\vec{v}_i, \vec{v}_j)$ are of dimension two, so

$$\operatorname{Span}(\vec{v}_i, \vec{v}_j) = W = \vec{v}_s^{\perp} \cap \vec{v}_t^{\perp} = \operatorname{Span}(\vec{v}_s, \vec{v}_t)^{\perp}.$$

Hence, none of v_i, v_j, v_s, v_t is adjacent to any of the others. Since V_Y does not contain an independent set of four vertices, $\{v_i, v_j, v_s, v_t\} \not\subseteq V_Y$. But then if $p = \max\{i, j, s, t\}$, when v_p was added, it would have been nonadjacent to three prior vertices, which is prohibited.

If v_m is adjacent to all vertices except v_s, choose $\vec{v}_m$ satisfying

$$\begin{aligned} \vec{v}_m &\in \vec{v}_s^{\perp} && \\ \vec{v}_m &\notin \vec{v}_i^{\perp} && \forall i \neq s \\ \vec{v}_m &\notin \mathrm{Span}(\vec{v}_s, \vec{v}_i) && \forall i \neq s, i < m \\ \vec{v}_m &\notin \mathrm{Span}(\vec{v}_i, \vec{v}_j) && \forall i, j \text{ such that } 1 \leq i < j < m \text{ and } v_i \not\sim v_j. \end{aligned}$$

The existence of an acceptable choice is guaranteed by using $W = \vec{v}_s^{\perp}$ in the previous argument and examining dimensions to show none of $A_i, B_i, C_{i,j}$ equals W. If v_m is adjacent to all vertices the choice is even easier. □

The next example shows that the hypothesis, *the two vertices to which the new vertex will not be adjacent must themselves be nonadjacent*, is necessary for Theorem 2.2.

Example 2.3. Let G be the graph shown in Figure 2. It is straightforward to verify that if G is constructed by adding vertices in order, each vertex added is adjacent to all but at most two prior vertices. Since G is a linear 2-tree, $\mathrm{mr}(G) = |G| - 2 = 5$.

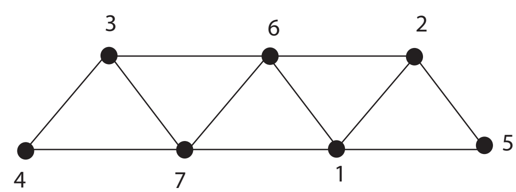

Figure 2: A graph of minimum rank 5 constructed with each added vertex adjacent to all but at most two prior vertices.

The next example shows that the hypothesis, V_Y *does not contain a set of four independent vertices*, is necessary for Theorem 2.2.

Example 2.4. Let X be the graph shown in Figure 3 and let $Y = X[1, 2, 3, 4]$.

The 4×4 identity matrix is a representation of Y that satisfies (3) and (4) of Theorem 2.2. X can be constructed by first adding vertex 5 adjacent to vertices 1 and 2 (and nonadjacent only to nonadjacent vertices 3 and 4), and then adding vertex 6 adjacent to vertices 3, 4, and 5 (and nonadjacent only to nonadjacent vertices 1 and 2). But X cannot have an orthogonal representation in $\mathbb{R}^4$, because X is a tree, so $\mathrm{mr}_+(X) = |X| - 1 = 5$ (cf. [13])

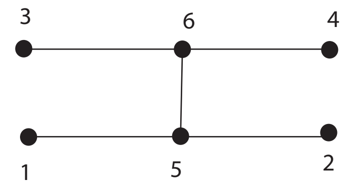

Figure 3: A graph of positive semidefinite minimum rank 5 constructed by adding on to a set of four independent vertices.

3 Minimum rank of graph complements

Theorems 2.1 and 2.2 can be applied to the complements of several families of sparse graphs. Since the complement of a 2-tree satisfies the hypotheses of Theorem 2.2, we have

Corollary 3.1. *If H is a 2-tree, then* $\mathrm{mr}_+(\overline{H}) \leq 4$.

In fact, we can determine exactly what the minimum rank of the complement of a 2-tree is. A *dominating* vertex of a graph H is a vertex adjacent to every other vertex of H, or equivalently, a vertex of degree $|H| - 1$.

Theorem 3.2. *If H is a 2-tree, then*

$$\mathrm{mr}(\overline{H}) = \begin{cases} 0 & \text{if } |H| \leq 3; \\ 1 & \text{if } |H| \geq 4 \text{ and } H \text{ has two dominating vertices}; \\ 3 & \text{if } |H| \geq 5 \text{ and } H \text{ has exactly one dominating vertex}; \\ 4 & \text{if } |H| \geq 6 \text{ and } H \text{ does not have a dominating vertex.} \end{cases}$$

Proof. Let H be a 2-tree. The complement of $\overline{K_r}$ is rK_1, so if $|H|$ is 2 or 3, then $\mathrm{mr}(\overline{H}) = 0$. If $|H| \geq 4$ and H has two dominating vertices, then $H = (|H|-2)K_1 \vee K_2$. and the complement of $(|H|-2)K_1 \vee K_2$ is $K_{|H|-2} \cup 2K_1$, so in this case $\mathrm{mr}(\overline{H}) = 1$.

If H does not have two dominating vertices, then H contains an induced L_3, shown in Figure 4. (The existence of an L_3 is fairly clear, and a statement about the existence of an L_3 with additional properties is established below.) Since $\overline{L_3} = P_4 \cup K_1, \mathrm{mr}(\overline{H}) \geq 3$. Thus it remains to distinguish the cases $\mathrm{mr}(\overline{H}) = 3$ and $\mathrm{mr}(\overline{H}) = 4$.

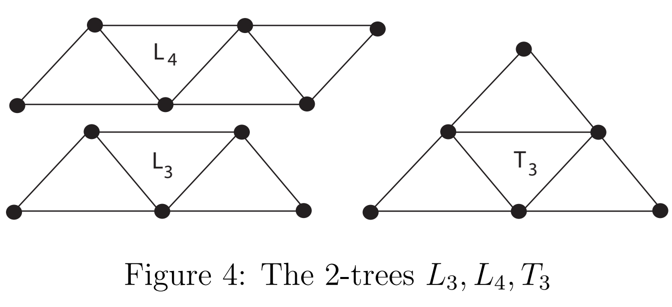

Figure 4: The 2-trees L_3, L_4, T_3

If H has exactly one dominating vertex v, then $H - v$ is a tree, because $H - v$ has $(2|H| - 3) - (|H| - 1) = |H - v| - 1$ edges, and $H - v$ is connected. Since $\overline{H} = \overline{H - v} \cup \{v\}$, $\mathrm{mr}(\overline{H}) = \mathrm{mr}(\overline{H - v}) \leq 3$ by [1, Theorem 3.16].

Let H be a 2-tree that has exactly one dominating vertex v, and let H be constructed from vertices $1, 2, \ldots, n$, in that order. Note that the order of $1, 2, 3$ is irrelevant, so without loss of generality we may assume that 1 is a dominating vertex of H (and all its induced subgraphs) and 2 is a dominating vertex of $H[1, 2, 3, 4]$. We establish the following statement by induction on t.

$$\text{If } s \sim t \text{ and } s < t, \text{ then } \exists R \subseteq V_H \text{ such that } H[R] = L_3 \text{ and } 1, s, t \in R. \qquad (9)$$

(Note that $s = 1$ is permitted.)

Let m be the first index such that $H[v_1, \ldots, v_m]$ does not have two dominating vertices, so $H[v_1, \ldots, v_{m-1}]$ has dominating vertices 1 and 2. Then the neighbors of m are 1 and r, with $2 < r < m$. Initially we show that (9) is true for $t \leq m$. Since $t \sim s$ and $s < t \leq m$, $s \in \{1, 2, r\}$. If we choose R to contain $1, 2, r, t, m$ (and one additional vertex if necessary to obtain a set of five) then $H[R] = L_3$.

Now assume (9) is true for $H[1, \ldots, q - 1], m \leq q - 1 < n$. Let the neighbors of q be $1, r$ (recall 1 is the dominating vertex). By the induction hypothesis, there exists R such that $H[R] = L_3$ and $1, r \in R$. There are two vertices of degree two in $H[R]$. If one of these is r, let x denote the other one. If neither degree two vertex is r, then one, which we denote by x, has r as a neighbor. Let R' be obtained from R by replacing x by q. Then $H[R'] = L_3$. Since (9) is already established when q is omitted, this completes the proof of (9).

Now suppose H does not have a dominating vertex. Let p be the first vertex such that $H[1, \ldots, p]$ does not have a dominating vertex, let 1 be the dominating vertex of $H[1, \ldots, p - 1]$, and let s, t be the two neighbors of p in $H[1, \ldots, p]$, with $1 < s < t < p$; note $s \sim t$. There exists R such that $H[R] = L_3$ and $1, s, t \in R$. When p is added to $H[R]$, it is not adjacent to 1. Thus $H[R \cup \{p\}] = L_4$ or $H[R \cup \{p\}] = T_3$ (see Figure 4). The complements of L_4 and T_3 are the unicyclic graphs shown in Figure 5. In [5] it was shown that $\mathrm{mr}(\overline{L_4}) = 4$ and $\mathrm{mr}(\overline{T_3}) = 4$ ($\overline{L_4}$ is a partial 4-sun and $\overline{T_3}$ is the 3-sun). □

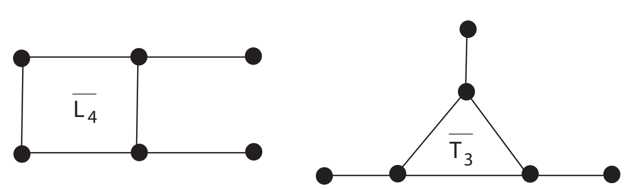

Figure 5: Complements of the linear 2-trees L_4 and T_3

To apply Theorem 2.1 to complements of unicyclic graphs, we need to show that for the complement of any cycle there is an orthogonal representation of dimension at most 4.

Theorem 3.3. *For all $n \geq 3$, there is an orthogonal representation of $\overline{C_n}$ in $\mathbb{R}^4$ satisfying condition* (2).

Proof. By Theorem 2.2, we can find an orthogonal representation of $\overline{P_{n-1}}$ in $\mathbb{R}^4$ satisfying conditions (3) and (4). Then arguing as in the proof of Theorem 2.2 we can add the remaining vertex adjacent to all but two vertices. Note that the hypothesis that these two vertices are not adjacent in $\overline{C_n}$ is not needed to obtain this representation, since this hypotheses is not needed to establish the existence of a vector meeting criteria (5) and (6), which is all that is necessary here. □

Corollary 3.4. *Let H be a unicyclic graph. Then* $\mathrm{mr}_+(\overline{H}) \leq 4$.

Proof. A unicyclic graph can be constructed from a cycle by adding one vertex at a time, with the new vertex adjacent to at most one prior vertex. Thus the complement of a unicyclic graph has an orthogonal representation of dimension at most 4 by Theorems 3.3 and 2.1. □

To ensure condition (2) for $\overline{C_4}$, $\mathbb{R}^4$ is needed, even though $\mathrm{mr}(\overline{C_4}) = 2$, because $\overline{C_4} = 2K_2$, which does not have an orthogonal representation satisfying (2) in $\mathbb{R}^3$. Furthermore, there are examples of unicylic graphs whose complements have minimum rank 4:

Example 3.5. The unicyclic graph $\overline{L_4}$ shown in Figure 5 has the linear 2-tree L_4 (shown in Figure 4) as its complement, and $\mathrm{mr}(L_4) = 6 - 2 = 4$.

We have established the bound (1) for 2-trees and unicyclic graphs.

Corollary 3.6. *Let G be graph such that* $\mathrm{mr}(\overline{G}) \leq 4$. *Then*

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + 2.$$

In particular, trees, unicyclic graphs and 2*-trees satisfy this bound.*

Proof. If $G = P_n$ is a path, it was shown in [1] that $\mathrm{mr}(\overline{P_n}) \leq 3$, so

$$\mathrm{mr}(P_n) + \mathrm{mr}(\overline{P_n}) \leq n - 1 + 3 = n + 2.$$

If G is not a path,

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| - 2 + 4 = |G| + 2. \quad \square$$

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