Greco-Latin squares as bijections

by

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CHAPTER 1. INTRODUCTION

In [Jam64], I.M. James defines a quasigroup in a category as an object A with a morphism f such that $(\pi_1, f) : A \times A \to A \times A$ and $(f, \pi_2) : A \times A \to A \times A$ are isomorphisms for projections π_1 and π_2. Two quasigroups (A, f), (A, g) are orthogonal if (f, g) is an isomorphism. This raises the question: What sort of isomorphism is (f, g)? Given the equivalence of a finite quasigroup and a Latin square, and the equivalence of a certain number of orthogonal Latin squares and a projective plane (demonstrated in Chapter 2), in particular we might ask what sort of isomorphism (f, g) is in this context, what it can tell us in relation to projective planes. These are the main questions that drove the research presented here.

In Chapter 2 we present all but a very little of the background information needed for the rest of this thesis. The main exception is in Appendices A and B, where the preliminary material needed only in the appendices is presented there.

In Chapter 3 the isomorphism (f, g) is treated as a transformation of Latin squares. This transformation is a generalization of several other transformations given in Chapter 2. In this role the isomorphism is used in a theorem giving a necessary and sufficient condition such that two ternary rings (defined in Chapter 2) are associated with the same projective plane. This theorem is motivated by a theorem due to Grari in [Gra04] which gives another such condition. Our theorem is an improvement in that it has fewer cases and requires fewer transformations in the most complicated case. We also include in Chapter 3 a discussion of the equivalence classes of ternary rings for Hall planes of order at least 16, and show how to obtain a member of one equivalence class from a member of another.

In Chapter 4 the isomorphism (f, g) is shown to be an orthomorphism, a lesser-known concept that, despite this fact, has proven to be useful is several contexts. Specifically, the

isomorphism is an orthomorphism with respect to the quasigroup (A, g) and a left isomorphism with respect to (A, f). We also show that (f, g) is an isomorphism with respect to another quasigroup when (A, g) is linear in that quasigroup, and that for a linear ternary ring all (f, g) are orthomorphisms. We end the chapter with a proof that for a certain non-linear ternary ring associated with the Hughes planes, all (f, g) are orthomorphisms.

One other proposition that does not fit in these chapters is provided in Appendix A.

CHAPTER 2. BACKGROUND

2.1 Latin Squares

Definition 2.1.1. A *Latin square of order* n is an $n \times n$ array of n symbols, which we will take to be $0, \ldots, n-1$, such that every symbol appears exactly once in each row and in each column.

We will use $0, 1, \ldots, n-1$ as row and column labels as well. The uppermost row will be the zeroth row, the next the first row, and so on, and likewise for the columns.

Example 2.1.2. The square

0	1	2
1	2	0
2	0	1

is a Latin square of order 3, and has as the $(2, 0)$ entry the symbol 2.

We will always use a capital ell, L, to denote a Latin square.

2.1.1 Orthogonality and MAXMOLS

Let L_1 and L_2 be two Latin squares of the same order. Form an $n \times n$ square $L_{1,2}$ by placing $(a_{i,j}, b_{i,j})$ in the (i, j) position, where $a_{i,j}$ is the (i, j) entry of L_1 and $b_{i,j}$ is the (i, j) entry of L_2, as in Figure 2.1.

Definition 2.1.3. Two Latin squares L_1, L_2 are *orthogonal* if no ordered pair is repeated in $L_{1,2}$, i.e., $|\{(a_{i,j}, b_{i,j}) : 0 \leq i, j \leq n-1\}| = n^2$. In this case $L_{1,2}$ is called a *Greco-Latin square.*

$$L_1 = \begin{array}{|c|c|c|c|} \hline a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ \hline a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \\ \hline \end{array} \quad L_2 = \begin{array}{|c|c|c|c|} \hline b_{0,0} & b_{0,1} & \cdots & b_{0,n-1} \\ \hline b_{1,0} & b_{1,1} & \cdots & b_{1,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline b_{n-1,0} & b_{n-1,1} & \cdots & b_{n-1,n-1} \\ \hline \end{array}.$$

$$L_{1,2} = \begin{array}{|c|c|c|c|} \hline (a_{0,0}, b_{0,0}) & (a_{0,1}, b_{0,1}) & \cdots & (a_{0,n-1}, b_{0,n-1}) \\ \hline (a_{1,0}, b_{1,0}) & (a_{1,1}, b_{1,1}) & \cdots & (a_{1,n-1}, b_{1,n-1}) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline (a_{n-1,0}, b_{n-1,0}) & (a_{n-1,1}, b_{n-1,1}) & \cdots & (a_{n-1,n-1}, b_{n-1,n-1}) \\ \hline \end{array}$$

Figure 2.1 Constructing $L_{1,2}$ from L_1, L_2.

If two Latin squares are orthogonal, then we can think of the square $L_{1,2}$ as a bijection on the set of n^2 ordered pairs $\{(i,j) : 0 \leq i,j \leq n-1\}$ with the action $L_{1,2} : (i,j) \mapsto (a_{i,j}, b_{i,j})$. We will invariably use the symbol φ for this permutation.

Example 2.1.4. Below are two orthogonal Latin squares of order three, their composition square, and the permutation in cycle notation.

$$L_1 = \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \quad L_2 = \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array}$$

$$L_{1,2} = \begin{array}{|c|c|c|} \hline (0,0) & (1,1) & (2,2) \\ \hline (1,2) & (2,0) & (0,1) \\ \hline (2,1) & (0,2) & (1,0) \\ \hline \end{array}$$

$$\varphi_{1,2} = \Big((0,1)\ (1,1)\ (2,0)\ (2,1)\ (0,2)\ (2,2)\ (1,0)\ (1,2)\Big)$$

Unless stated otherwise, we will assume that every Latin square's zeroth row will be in increasing order, that the $(0,i)$ entry is i for $i = 0, \ldots, n-1$. Such a square is in *natural order*. If two squares are orthogonal and not in natural order we can permute the symbols in each square separately to put them in natural order. The resulting squares are still orthogonal, for

if θ is a bijection of $\{0, \ldots, n-1\}$, and

$$L_{1,2} = \begin{array}{|c|c|c|c|} \hline (a_{0,0}, b_{0,0}) & (a_{0,1}, b_{0,1}) & \cdots & (a_{0,n-1}, b_{0,n-1}) \\ \hline (a_{1,0}, b_{1,0}) & (a_{1,1}, b_{1,1}) & \cdots & (a_{1,n-1}, b_{1,n-1}) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline (a_{n-1,0}, b_{n-1,0}) & (a_{n-1,1}, b_{n-1,1}) & \cdots & (a_{n-1,n-1}, b_{n-1,n-1}) \\ \hline \end{array}$$

a Greco-Latin square, then

$$L'_{1,2} = \begin{array}{|c|c|c|c|} \hline (a_{0,0}\theta, b_{0,0}) & (a_{0,1}\theta, b_{0,1}) & \cdots & (a_{0,n-1}\theta, b_{0,n-1}) \\ \hline (a_{1,0}\theta, b_{1,0}) & (a_{1,1}\theta, b_{1,1}) & \cdots & (a_{1,n-1}\theta, b_{1,n-1}) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline (a_{n-1,0}\theta, b_{n-1,0}) & (a_{n-1,1}\theta, b_{n-1,1}) & \cdots & (a_{n-1,n-1}\theta, b_{n-1,n-1}) \\ \hline \end{array}$$

also contains each ordered pair exactly once as well, and likewise so would

$$L''_{1,2} = \begin{array}{|c|c|c|c|} \hline (a_{0,0}\theta, b_{0,0}\psi) & (a_{0,1}\theta, b_{0,1}\psi) & \cdots & (a_{0,n-1}\theta, b_{0,n-1}\psi) \\ \hline (a_{1,0}\theta, b_{1,0}\psi) & (a_{1,1}\theta, b_{1,1}\psi) & \cdots & (a_{1,n-1}\theta, b_{1,n-1}\psi) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline (a_{n-1,0}\theta, b_{n-1,0}\psi) & (a_{n-1,1}\theta, b_{n-1,1}\psi) & \cdots & (a_{n-1,n-1}\theta, b_{n-1,n-1}\psi) \\ \hline \end{array}$$

for any bijection ψ. (Here we are using suffix notation for functions, placing the operator symbol to the right of the operand symbol, as in $a_{0,0}\theta$. In prefix notation this would be $\theta(a_{0,0})$.)

It is not hard to see that the maximum number of mutually orthogonal Latin squares of order n is $n-1$. Notice that for any two orthogonal Latin squares L_1, L_2, the first row of $L_{1,2}$ is the ordered pairs (i, i). Hence if the $(1, 0)$ position of L_1 is z then the $(1, 0)$ position of any square orthogonal to L_1 cannot be z or 0 (if it were 0, then the $(0, 0)$ and $(1, 0)$ entries would both be 0 and the square would not be Latin), leaving $n-2$ choices, and thus at most $n-2$ squares orthogonal to L_1.

Definition 2.1.5. A maximal set $\mathcal{M}$ of $n-1$ mutually orthogonal Latin squares of order n is

called a MAXMOLS *of order* n.

For our purposes, the order of the squares in the MAXMOLS is of no consequence; we assume the MAXMOLS is an unordered set.

Example 2.1.6. A MAXMOLS of order 4:

L_1

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

L_2

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

L_3

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

The Greco-Latin squares are

$L_{1,2}$

$(0,0)$	$(1,1)$	$(2,2)$	$(3,3)$
$(1,2)$	$(0,3)$	$(3,0)$	$(2,1)$
$(2,3)$	$(3,2)$	$(0,1)$	$(1,0)$
$(3,1)$	$(2,0)$	$(1,3)$	$(0,2)$

$L_{1,3}$

$(0,0)$	$(1,1)$	$(2,2)$	$(3,3)$
$(1,3)$	$(0,2)$	$(3,1)$	$(2,0)$
$(2,1)$	$(3,0)$	$(0,3)$	$(1,2)$
$(3,2)$	$(2,3)$	$(1,0)$	$(0,1)$

$L_{2,1}$

$(0,0)$	$(1,1)$	$(2,2)$	$(3,3)$
$(2,1)$	$(3,0)$	$(0,3)$	$(1,2)$
$(3,2)$	$(2,3)$	$(1,0)$	$(0,1)$
$(1,3)$	$(0,2)$	$(3,1)$	$(2,0)$

$L_{2,3}$

$(0,0)$	$(1,1)$	$(2,2)$	$(3,3)$
$(2,3)$	$(3,2)$	$(0,1)$	$(1,0)$
$(3,1)$	$(2,0)$	$(1,3)$	$(0,2)$
$(1,2)$	$(0,3)$	$(3,0)$	$(2,1)$

$L_{3,1}$

$(0,0)$	$(1,1)$	$(2,2)$	$(3,3)$
$(3,1)$	$(2,0)$	$(1,3)$	$(0,2)$
$(1,2)$	$(0,3)$	$(3,0)$	$(2,1)$
$(2,3)$	$(3,2)$	$(0,1)$	$(1,0)$

$L_{3,2}$

$(0,0)$	$(1,1)$	$(2,2)$	$(3,3)$
$(3,2)$	$(2,3)$	$(1,0)$	$(0,1)$
$(1,3)$	$(0,2)$	$(3,1)$	$(2,0)$
$(2,1)$	$(3,0)$	$(0,3)$	$(1,2)$

,

and the first few equivalent bijections are

$$\varphi_{1,2} = \Big((0,1)\ (1,1)\ (0,3)\ (3,3)\ (0,2)\ (2,2)\Big)\Big((1,0)\ (1,2)\ (3,0)\ (3,1)\ (2,0)\ (2,3)\Big)\Big((1,3)\ (2,1)\ (3,2)\Big)$$

$$\varphi_{1,3} = \Big((0,1)\ (1,1)\ (0,2)\ (2,2)\ (0,3)\ (3,3)\Big)\Big((1,0)\ (1,3)\ (2,0)\ (2,1)\ (3,0)\ (3,2)\Big)\Big((1,2)\ (3,1)\ (2,3)\Big)$$

$$\varphi_{2,1} = \Big((0,1)\ (1,1)\ (3,0)\ (1,3)\ (1,2)\ (0,3)\ (3,3)\ (2,0)\ (3,2)\ (3,1)\ (0,2)\ (2,2)\Big)\Big((1,0)\ (2,1)\ (2,3)\Big)$$

2.1.2 Quasigroups and Loops

Definition 2.1.7. A *quasigroup* $(Q, \cdot)$ is a set Q with a binary operation $\cdot$ defined on Q such that any two of x, y, z uniquely determines the third in $x \cdot y = z$. The *order* of a quasigroup $(Q, \cdot)$ is the order of the set Q.

Example 2.1.8. Any group is a quasigroup. The quasigroup operation is the group operation.

Let $(Q, \cdot)$ be a quasigroup of order n. Let L be the multiplication table of Q, the square with z in the (x, y) position if and only if $xy = z$ (we will often omit the operation symbol $\cdot$ and write xy for $x \cdot y$). If $xy = z$ then, by the definition of a quasigroup, x and z uniquely determine y and y and z uniquely determine x. Thus every symbol z appears exactly once in each row x and once in each column y, so L is a Latin square. Conversely, each Latin square determines a quasigroup by setting $xy = z$ if and only if the (x, y) position is z. We will henceforth associate quasigroups and Latin squares in this way.

If L_1 and L_2 are two Latin squares of the same order, then they are associated with two quasigroups of the same order. In order to differentiate the quasigroup operations, we affix a subscript: $(Q, \cdot_1)$, $(Q, \cdot_2)$. For example, for the Latin squares

$$L_1 = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array}, \quad \text{and} \quad L_2 = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array},$$

$1 \cdot_1 1 = 2$ and $1 \cdot_2 1 = 0$. We will later make use of the quasigroup operations $\backslash$ and $/$, which satisfy $x\backslash z = y$ if and only if $xy = z$ if and only if $z/y = x$, and sometimes affix subscripts to differentiate these as well: $/_1$, $/_2$.

We will later also make use of two sets of functions associated with the quasigroup, from the next definition.

Definition 2.1.9. For Latin square L_1 or quasigroup $(Q, \cdot_1)$, for any element x, *left multiplication by* x is the function $\mathbb{L}_1(x) : y \mapsto x \cdot_1 y$ and *right multiplication by* y is the function $\mathbb{R}_1(x) : y \mapsto x \cdot_1 y$. (Notice that the subscript identifies the Latin square.)

These functions are bijections by the rule that any two of x, y, z from $x \cdot y = z$ uniquely determines the third.

Definition 2.1.10. A *loop* is a quasigroup $(Q, \cdot)$ with an element id such that $id \cdot x = x = x \cdot id$ for all $x \in Q$. The element id is the *identity* of the loop.

Example 2.1.11. A group is again a loop whose identity element is the identity element of the group.

Example 2.1.12. The quasigroup associated with square (a) below is a loop whose identity element is 0 (the zeroth row and column are in natural order, i.e., $0 \cdot x = x = x \cdot 0$).

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array}$$

(a) (b)

The quasigroup associated with square (b) is not a loop (no i^{th} row and column are both in natural order or even the same order).

2.2 Ternary Rings

Definition 2.2.1. [HP73] [Ste72] A *ternary ring* is a set R with a ternary operation $(\cdot,\cdot,\cdot)$ on R such that

i. if $a,b,c,d \in R$, $a \neq c$, then there is a unique $x \in R$ such that $(x,a,b) = (x,c,d)$,

ii. if $a,b,c \in R$, then there is a unique $z \in R$ such that $(a,b,z) = c$,

iii. if $a,b,c,d \in R$, $a \neq c$, then there is a unique ordered pair $y,z \in R$ such that $(a,y,z) = b$ and $(c,y,z) = d$.

The ternary ring *has a zero* if it satisfies

iv. there is an element $0 \in R$ such that $(0,b,c) = (b,0,c) = c$ for all $b,c \in R$.

The ternary ring *has a one* if it satisfies iv. and

v. there is an element $1 \in R$ such that $(1,b,0) = (b,1,0) = b$ for all $b \in R$.

All of the ternary rings we consider will have set $R = \{0,1,\ldots,n-1\}$ but the use of the symbols $0,1$ will not necessarily imply that these elements satisfy properties iv. and v. above. Almost all ternary rings we consider here will have a zero (i.e., satisfy property iv.), and many will have a one (satisfy property v.). The main exception will be in subsection 2.5.2 and discussion relating to the contents of it.

Example 2.2.2. Let $\mathbb{F}$ be any finite field. Then $\mathbb{F}$ is a ternary ring with ternary operation $(x,y,z) = x \cdot y + z$. For instance, the unique x satisfying property i., $xa + b = xc + d$, $a \neq c$, is $x = (d-b)(a-c)^{-1}$. $\mathbb{F}$ has a zero and one, the elements 0, 1.

We will often neglect to mention the set R when discussing the ternary ring and simply refer to the ternary operation $(\cdot,\cdot,\cdot)$ as the ternary ring, and in fact often shorten $(\cdot,\cdot,\cdot)$ to $(\ )$. A variety of types of brackets will be used to refer to a ternary operation: $(\ ), [\ ], \{\ \}, \langle\ \rangle$.

Let $(R, (\ ))$ be a ternary ring with zero and consider the binary operation $+ : (y, z) \mapsto (e, y, z)$ for a chosen element $e \in R$. By property ii. of Definition 2.2.1, $(e, y, z) = (e, y, z')$ if and only if $z = z'$. Suppose $(e, y, z) = (e, y', z)$ and $y \neq y'$. By property i., e is the unique element for which equality holds, but by iv., equality holds if $e = 0$. Hence $(e, y, z) = (e, y', z)$ with $y \neq y'$ if and only if $e = 0$.

Definition 2.2.3. For a ternary ring with zero, the operation $+$ is defined as $y + z := (1, y, z)$, without assuming the element 1 satisfies property v. of Definition 2.2.1.

By the previous paragraph, $y + z = k = y + z'$ if and only if $z = z'$, and $y + z = k = y' + z$ if and only if $y = y'$, since $1 \neq 0$. Thus any two of y, z, k determine the third, so $(R, +)$ is a quasigroup by Definition 2.1.1. Now, by property iv. $0 + x := (e, 0, x) = x$, and if $(\ )$ has a one, then property v. implies $x + 0 := (1, x, 0) = x$. Thus, if $(\ )$ is a ternary ring with both zero, 0, and one, 1, then 0 is an identity element of $(R, +)$, so $(R, +)$ is a loop.

Let $(R, (\ ))$ have a zero, and define a binary operation $\cdot$ as $\cdot : (x, y) \mapsto (x, y, 0)$. We will show that $(R - \{0\}, \cdot)$ is a quasigroup, and if the element 1 satisfies property v. above, 1 is an identity element, so $(R - \{0\}, \cdot)$ is a loop. Suppose $(x, y, 0) = (x, y', 0)$, and $x, y \neq 0$. If $y \neq y'$, then by property i., x is the unique element for which $(x, y, 0) = (x, y', 0)$, and thus $x = 0$ by property iv. Since $x \neq 0$ by choice, we must have $y = y'$. Suppose $(x, y, 0) = (x', y, 0) = k$ with $y \neq 0$. Then $(x, y, 0) = (x', y, 0) = (x, 0, k) = (x', 0, k')$, and by property iii., $x = x'$. Hence $(R - \{0\}, \cdot)$ is a quasigroup. Property v. clearly implies that 1 is an identity element.

Definition 2.2.4. For a ternary ring with zero, the operation $\cdot$ is defined as $x \cdot y := (x, y, 0)$ for all $x, y \in R$.

We have proven the following.

Theorem 2.2.5. *If $(\ )$ is a ternary ring with zero, then $(R, +)$ and $(R - \{0\}, \cdot)$ are quasigroups. If the element 1 satisfies property v. of Definition 2.2.1, then $(R, +)$ and $(R - \{0\}, \cdot)$ are loops with identities 0 and 1 respectively.*

Definition 2.2.6. If $(\ )$ is a ternary ring with zero, then $(\ )$ is *linear* if $(x, y, z) = xy + z$. In other words, $(\ )$ is linear if $(x, y, z) = (1, (x, y, 0), z)$.

Example 2.2.7. The finite field $\mathbb{F}$ with the ternary operation $(x, y, z) = xy + z$ is obviously linear, where the loops are the usual multiplication and addition groups.

Example 2.2.8. We construct here the right Hall systems. Let $\mathbb{F}$ be a finite field of order q greater than 2, and let $xf = x^2 - rx - s$ be an irreducible polynomial over $\mathbb{F}$. Let R be the set of ordered pairs of elements from $\mathbb{F}$. Define addition on R by $(x, y) + (x', y') = (x + x', y + y')$, where addition on the left is in R and addition on the right is in $\mathbb{F}$. Define multiplication by the following:

$$(x, y) \cdot (x', y') = \begin{cases} (xx', yx') & \text{if } y' = 0 \\ (xx' - yy'^{-1}(yf), xy' - x'y + xr) & \text{if } y' \neq 0 \end{cases},$$

where again multiplication on the left is in R and multiplication on the right is in $\mathbb{F}$. (See [Ste72].)

Now define a ternary operation $(\ )$ on R as $(x, y, z) = xy + z$, hence making $(R, (\ ))$ a ternary ring of order q^2. Obviously, as defined this way $(R, (\ ))$ is linear. One can check that $(\ )$ has both zero $(= (0, 0))$ and one $(= (1, 0))$, and that $(R, +)$ is a group and $(R - \{0\}, \cdot)$ a loop.

Definition 2.2.9. If $(\ )$ is a linear ternary ring with zero such that $(R, +)$ is a group, then $(R, (\ ))$ is called a *cartesian group.*

Definition 2.2.10. If $(\ )$ is a cartesian group that satisfies the left distributive law: $a(b+c) = ab+ac$ for all $a, b, c \in R$, then $(R, (\ ))$ is a *left quasifield* or simply, *quasifield.* $(R, (\ ))$ is a *right quasifield* if it is a cartesian group that satisfies the right distributive law: $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Example 2.2.11. The Hall system from the previous example is a right quasifield. If we replace the multiplication there by

$$(x, y) \cdot (x', y') = \begin{cases} (xx', xy') & \text{if } y = 0 \\ (xx' - y^{-1}y'(xf), yx' - xy' + y'r) & \text{if } y \neq 0 \end{cases},$$

then the resulting ternary ring is a left quasifield. (See [HP73].)

Definition 2.2.12. A quasifield with associative multiplication is called a *nearfield*.

We will need the following definition and theorem to construct a certain class of projective planes in the next section.

Definition 2.2.13. The *kernel* of a (left) quasifield is the set of all elements k such that

i. $(x+y)k = xk + yk$ and

ii. $(xy)k = x(yk)$.

The *kernel* of a right quasifield is the set of all elements k such that

i. $k(x+y) = kx + ky$ and

ii. $k(xy) = (kx)y$.

Theorem 2.2.14. *For any odd prime power q there exists a nearfield of order q^2 such that the kernel is the finite field of order q, and every element of the kernel commutes with all elements of the nearfield.*

2.3 Projective Planes

Definition 2.3.1. A *projective plane* is a set of lines and a set of points satisfying three incidence rules:

i. Every two points lie on a unique line.

ii. Every two lines intersect at a unique point.

iii. There exist four distinct points, no three of which lie on the same line.

Example 2.3.2. The Fano plane has seven points and lines, three points per line and three lines through every point. (Figure 2.2. The circle counts as a line.)

Theorem 2.3.3. *Let π be a projective plane with point set $P = \{P_i : i \in I\}$ and line set $\ell = \{\ell_j : j \in J\}$. Then let π_d have point set $\ell = \{\ell_j : j \in J\}$, line set $P = \{P_i : i \in I\}$, and incidence defined as ℓ_j is on P_i if and only if P_i is on ℓ_j in π. Then π_d is a projective plane.*

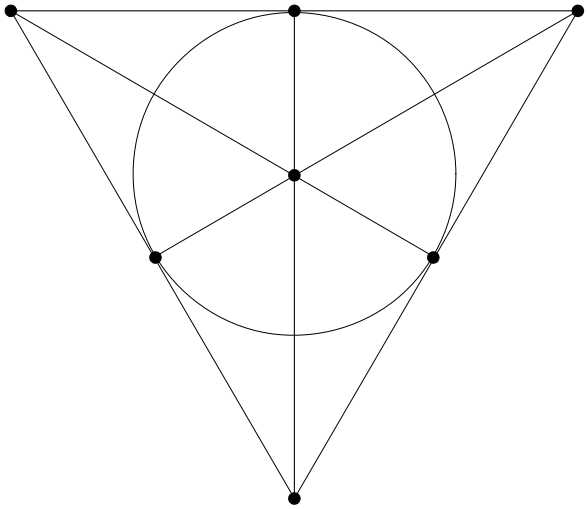


Figure 2.2 The Fano plane.

Definition 2.3.4. The plane π_d from the previous theorem is called the *dual* of π.

Definition 2.3.5. Let π and π' be two projective planes with point set $\mathbf{P}$ and $\mathbf{P'}$, and line sets $\mathbf{L}$ and $\mathbf{L'}$, respectively. If $\gamma : \pi \to \pi'$ is a bijection $\gamma : \mathbf{P} \to \mathbf{P'}$ and $\gamma : \mathbf{L} \to \mathbf{L'}$ such that $P_0 \in \mathbf{P}$ and $L_0 \in \mathbf{L}$ are incident if and only if $P_0\gamma$ and $L_0\gamma$ are incident, then γ is an *isomorphism* of the planes. If an isomorphism exists between two planes then they are *isomorphic*.

Definition 2.3.6. An isomorphism γ is a *collineation* if $\gamma : \pi \to \pi$.

We will be concerned exclusively with projective planes whose sets of points and lines are finite.

Theorem 2.3.7. *Let π be a projective plane whose point set and line set are finite. Then there is an n such that π contains $n^2 + n + 1$ points, $n^2 + n + 1$ lines, the number of points on each line is $n + 1$, and the number of lines through each point is $n + 1$.*

Definition 2.3.8. The number n from the previous theorem is the *order* of the plane.

Example 2.3.9. The Fano plane from Example 2.3.2 has order 2: there are $7 = 2^2 + 2 + 1$ points and lines each, and $3 = 2 + 1$ points on each line and lines through each point.

Example 2.3.10. Let $\mathbb{F}$ be a finite field of order n. We construct a projective plane as follows. Consider the set $\{(x, y, z) : x, y, z \text{ not all } 0\}$. Define an equivalence relation on this set by $(x, y, z) \sim (x', y', z')$ if and only if there is a $k \neq 0$ in $\mathbb{F}$ such that $(x, y, z) = (kx', ky', kz')$. We will use (x, y, z) to refer to the equivalence class containing the element (x, y, z). Let the set of lines be the set of all such equivalence classes. Let the set of points be the same set of equivalence classes considered as a separate set. Define incidence by (x, y, z) is on (r, s, t) if and only if $xr + ys + zt = 0$. We can check that with this incidence, these sets of points and lines form a projective plane of order n. We will call this the *field plane* or *Desarguesian plane*.

Example 2.3.11. We construct the Hughes planes from [Hug57].

In order to construct these planes we need to know the following.

1. For every odd prime p and positive integer n there is a nearfield $(N, (\ ))$ of order p^{2n} such that the kernel of N is the field $\mathbb{F}$ of order p^n, and every member of the kernel commutes with every member of N (as in Theorem 2.2.14).

2. There is a 3-by-3 matrix A over $\mathbb{F}$ such that A^z is a multiple of the identity matrix if and only if $z = p^{2n} + p^n + 1$.

The Hughes planes are constructed as follows. The points are constructed as for a field plane. Let V be the set of ordered triples of elements of N other than the triple $(0, 0, 0)$. Let two elements $(x, y, z),\ (x', y', z')$ of V be equivalent if there is a $k \in N$ such that $k(x, y, z) := (kx, ky, kz) = (x', y', z')$. We will simply refer to the equivalence class containing (x, y, z) as (x, y, z). The set of points of the Hughes plane will be the equivalence classes of V.

Let t be an element of N, and $t = 1$ or $t \notin \mathbb{F}$. Let $\ell_t = \{(x, y, z) \mid x + yt + z = 0\}$. (If $(x, y, z) = (x', y', z')$ then $x + yt + z = 0$ if and only if $x' + y't + z' = 0$.) The lines of the Hughes plane will be all members of the set $\{\ell_t A^m \ :\ t = 1 \text{ or } t \notin \mathbb{F},\ 1 \leq m \leq p^{2n} + p^n + 1\}$. Any such line can be identified as $x_0\alpha + y_0\beta + z_0\gamma + (x_0\alpha' + y_0\beta' + z_0\gamma')t = 0$ with $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{F}$, so that, for instance, $(1, 0, -1)$ lies on line ℓ_s if and only if $\alpha - \gamma + (\alpha' - \gamma')s = 0$.

2.4 Equivalence of Projective Planes, MAXMOLS, and Ternary Rings

2.4.1 Projective Planes and MAXMOLS

The correspondence between MAXMOLS and projective planes will be through the following construction.

We first label all points and lines of the plane. Choose a line to label $[\infty]$, called the line at infinity, and label the points on this line as $(\infty), (0), (1), \ldots, (n-1)$ where n is the order of the plane. Label the lines through (∞) other than $[\infty]$ as $[0], \ldots, [n-1]$ arbitrarily and the lines through (0) other than $[\infty]$ as $[0,1], [0,2], \ldots, [0,n-1]$ arbitrarily. In Figure 2.3, we illustrate the labeling up to this point for the plane of Example 2.3.2, and in Figure 2.4 for a generic plane, where for convenience the top line has been chosen as $[\infty]$.

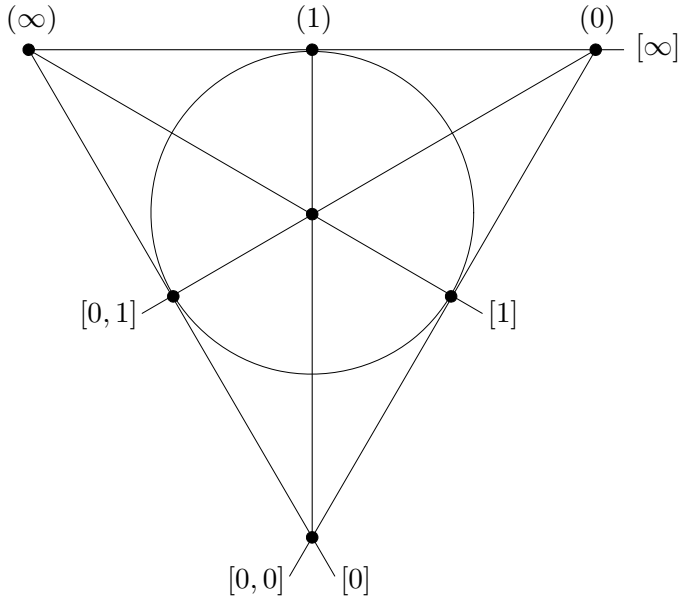


Figure 2.3 Labeling the Fano plane.

Recall there are n^2+n+1 points, with $n+1$ on $[\infty]$ or any other line. Hence there are n^2 points not on $[\infty]$. Since any two points in a projective plane determine a unique line, any of the n^2 points not on $[\infty]$ are on a unique line $[i]$ through (∞) and a unique line $[0,j]$ through (0), and this point will then be labeled (i,j). The lines other than $[\infty]$ through any (i) we label

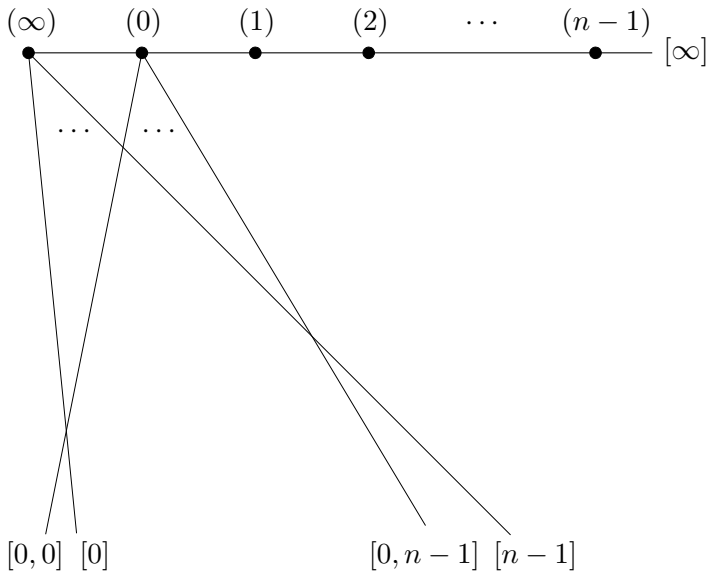


Figure 2.4 Labeling a generic plane.

as $[i,0],\ldots,[i,n-1]$, with $i=0,1,\ldots,n-1$. These latter labels, rather than being chosen arbitrarily, are fixed in the following way. The line through (i) which gets the label $[i,k]$ will be that line which also passes through the point $(0,k)$.

The Latin squares L_i of the MAXMOLS will correspond to the points (i), $i=1,\ldots,n-1$. To construct the Latin squares, the point (i,j) lies on a unique line through (k) and if this line is $[k,m]$, then the (i,j) entry of L_k is m. In Figure 2.5, position (i,j) of Latin square L_2 is filled in this way.

In this way, all of the n^2 points not on $[\infty]$ correspond to the n^2 coordinates of each Latin square. Notice that 1) the lines through (∞) correspond to the rows of each square, and those through (0) to the columns, and 2) all of the coordinates in a single square L_k which have the same entry m are on the same line $[k,m]$.

The MAXMOLS we produce with this construction is necessarily in natural order, i.e., the zeroth row of each is $0,1,\ldots,n-1$. This is due to the rule that the line $[2,k]$, for instance, had to pass through $(0,k)$. In other words, the entry in position $(0,k)$ was chosen to be k.

The three axioms of projective planes from Definition 2.3.1 insure that the resulting set of squares are in fact Latin and are all mutually orthogonal. We omit the details, which can be

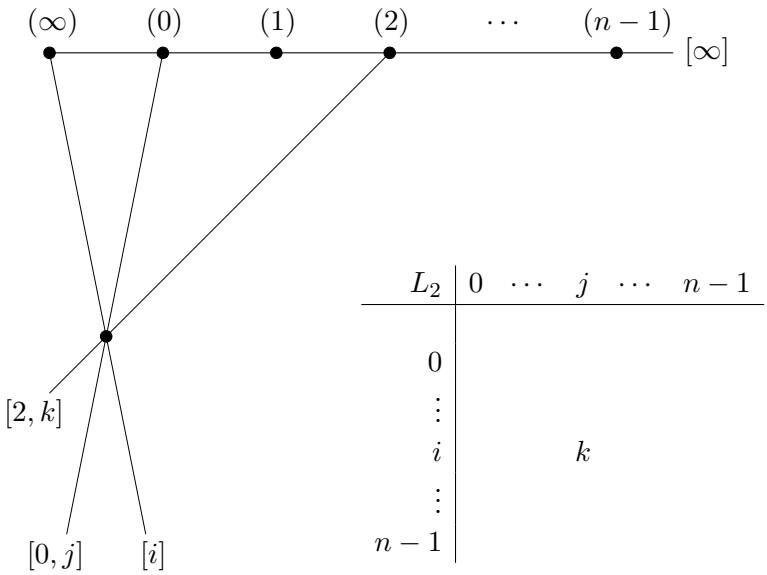


Figure 2.5 Constructing Latin squares from a plane.

L_0

0	1	⋯	$n-1$
0	1	⋯	$n-1$
⋮	⋮		⋮
0	1	⋯	$n-1$

L_∞

0	0	⋯	0
1	1	⋯	1
⋮	⋮		⋮
$n-1$	$n-1$	⋯	$n-1$

Figure 2.6 L_0 and L_∞ squares.

found, for example, in [HP73] or [DK74]. It is easy to see, though, that the squares we have constructed have the right dimension and that there are in fact $n-1$ of them.

If we were to construct Latin squares in the same way for the point (∞), the square would be the column-Latin square L_∞ whose (i,j) entry is i. The square for (0) would be the row-Latin square L_0 whose (i,j) entry is j. We will call these squares the adjunct squares.

Given a MAXMOLS, we can, of course, assume these adjuncts and reverse this process to produce the projective plane. We sketch this construction more briefly.

Construct a line $[\infty]$ with $n+1$ points (∞), (0), ... $(n-1)$. Through (∞) and (0) construct a grid of lines, with n new lines through each of the two points, such that each new line $[i]$ through (∞) and each new line $[0,j]$ through (0) intersect. Call (i,j) the intersection of $[i]$

and $[0, j]$. Now construct new lines $[\ell, k]$ through (ℓ) consisting of all the points (m, b) where the (m, b) entry of L_ℓ is k.

2.4.2 MAXMOLS and Ternary Rings

The equivalence between a MAXMOLS and ternary ring is easier to construct and verify. Specifically, we will show the equivalence of a MAXMOLS in natural order to a ternary ring with zero.

To a MAXMOLS $L_1, \ldots, L_{n-1}$, which we assume to be in natural order, we add the adjunct square L_0 given above. We construct a ternary operation $(\ )$ where $(x, y, z) = k$ if and only if the (y, z) entry of square L_x is k. Hence, for instance, $(0, y, z) = z$ for any y, z by construction of the square L_0, and $(x, 0, z) = z$ for any x, z since the zeroth row of each square is assumed to be in natural order, with z as the $(0, z)$ entry. Thus property iv. of Definition 2.2.1 is established, the ternary operation has a zero.

$$\underline{\text{MAXMOLS} \cup L_0 \leftrightarrow \text{Ternary Ring}}$$

$$\begin{array}{ccccccccc} & \text{square} & & \text{row} & & \text{column} & & & \text{entry} \\ & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ \langle & x & , & y & , & z & \rangle & = & k \end{array}$$

Figure 2.7 Correspondence between MAXMOLS and ternary ring.

We recall properties i.-iii. of Definition 2.2.1 and verify that the construction produces a ternary ring:

i. if $a, b, c, d \in R$, $a \neq c$, then there is a unique $x \in R$ such that $(x, a, b) = (x, c, d)$,

ii. if $a, b, c \in R$, then there is a unique $z \in R$ such that $(a, b, z) = c$,

iii. if $a, b, c, d \in R$, $a \neq c$, then there is a unique ordered pair $y, z \in R$ such that $(a, y, z) = b$, $(c, y, z) = d$.

Since all squares are orthogonal, their composite square must have (b, d) in only one entry. This is equivalent to property iii., which is consequently satisfied. Property ii. is satisfied since all of the squares $L_0, \ldots, L_{n-1}$ are row-Latin (contain each of the symbols $0, \ldots, n-1$ exactly once in each row). The easiest way to show property i. is to appeal to the projective plane associated with the MAXMOLS, and use the fact that every two points, in this case the points (a, b) and (c, d), determine a unique line ℓ. The line ℓ must cross $[\infty]$, and if $a \neq c$, then the intersection must be some (x) other than (∞) (the points on line $[m]$, say, through (∞) are all the points (m, k), $k = 0, \ldots, n-1$). Thus there must be some square L_x with the same line $[x, k]$ through (a, b) and (c, d), which is property i.

Example 2.4.1. The MAXMOLS given in Example 2.1.6 are determined by the ternary ring for fields of Example 2.2.2. For instance, square L_1 has (i, j) entry $1 \cdot i + j$.

2.4.3 Projective Planes and Ternary Rings

The equivalence of projective planes and ternary rings follows from the equivalences of the last two sections, and the method of constructing one from the other follows as well. However, it might be worthwhile to cover the construction briefly, as all of the correspondences will be used frequently later.

Let the projective plane be labeled as in section 2.4.1. Then the ternary ring is $(R, (\ ))$ where $(x, y, z) = k$ if and only if the point (y, z) is on line $[x, k]$ through point (x).

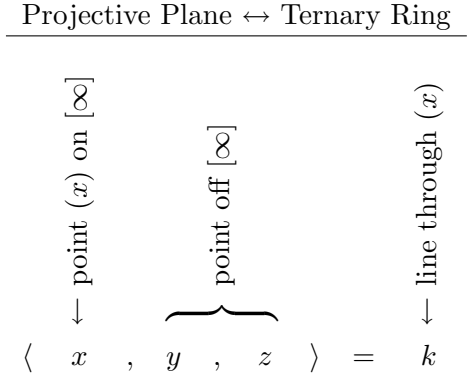


Figure 2.8 Correspondence between proj. plane and ternary ring.

Recall the point (y, z) is the point where the line $[y]$ through (∞) and the line $[0, z]$ through (0) intersect.

The ternary ring given in Example 2.2.2 for a finite field determines a plane by this construction. This plane is isomorphic to the plane given in Example 2.3.10.

Definition 2.4.2. A plane determined by a ternary ring from Example 2.2.11 is called a *Hall plane*.

2.5 Transformations of MAXMOLS and Ternary Rings

2.5.1 Transformations for MAXMOLS

Given the equivalence of MAXMOLS and ternary rings, we may sometimes refer to a MAXMOLS by the ternary operation associated with it: $(w, x, y) = z$ if and only if position (x, y) of square L_w is z for $w \neq 0$, and $(0, x, y) = y$. We will use $\mathcal{M}$ to denote a MAXMOLS and $\mathcal{M}^+$ a MAXMOLS with the addition of the adjunct squares L_0 and L_∞ (Figure 2.6). Abusing the definition, we will refer to $\mathcal{M}, \mathcal{M} \cup \{L_0\}$, and $\mathcal{M}^+$ all as MAXMOLS.

The mapping $[x, y, \cdot] : z \mapsto [x, y, z]$ is a bijection for any x, y, the action being left multiplication by y in Latin square L_x. We set $\rho_{xy} := [x, y, \cdot]$. Let R be the $n \times n$ array whose (i, j) entry is ρ_{ij}, $0 \leq i, j \leq n - 1$. R is called the *representation array* for the MAXMOLS.

[Owe92] defines five transformations on a MAXMOLS which convert it into another MAXMOLS for the same plane or its dual. We want to stress that it is the labels which are moved by the transformations, while the underlying geometry is fixed. The transformations are

($T1.\theta$) Apply the permutation θ to the row labels in all of the Latin squares, then permute the symbols in each square separately to return each to natural order.

Geometrical effect: The lines through (∞) are renumbered according to θ, i.e., the line which was formerly labeled as $[k]$ is labeled as $[k\theta]$. Returning the squares to natural order relabels the lines through each (i), $i = 1, \ldots, n - 1$.

($T2.\phi$) Apply the permutation ϕ to the column symbols in all of the Latin squares, then permute the symbols in each square to return each to natural order.

Geometrical effect: The lines through (0) are renumbered according to ϕ. The line formerly labeled as $[0, k]$ is labeled as $[0, k\phi]$. Again, returning the squares to natural order relabels the lines through each (i), $i = 1, \ldots, n-1$.

($T3$) Replace each Latin square by its transpose, then permute the symbols in each square separately to return each to natural order.

Geometrical effect: The point which was formerly labeled as (∞) is labeled as (0), and vice versa. The lines are relabeled as well, so that the line $[k]$ is relabeled as $[0, k]$, and vice versa.

($T4.r$) For a chosen r, $1 \leq r \leq n-1$, replace $R = (\rho_{ik})$ with $R' = (\rho_{rk}^{-1}\rho_{ik})$ for all (i, k).

Geometrical effect: The point formerly labeled (r) and the lines formerly labeled as $[r, j]$ are labeled as (0) and $[0, j]$ and vice versa. No other labels of points on $[\infty]$ (points labeled (∞) or (i)) need change. Since lines $[r, j]$ and $[0, j]$ always intersect at point $(0, j)$ (as this is how the label $[r, j]$ was chosen for this line), the labels for points $(0, 0), \ldots, (0, n-1)$ do not change. Every other label $(\cdot, \cdot)$ changes due to moving the labels $[0, j]$.

($T5$) Replace $R = (\rho_{ik})$ with R^*, where $R^*_{ik} = \rho_{ki}^{-1}$.

Geometrical effect: The MAXMOLS obtained is a MAXMOLS for the dual plane obtained by switching points and lines $(\cdot) \leftrightarrow [\cdot]$, $(\cdot, \cdot) \leftrightarrow [\cdot, \cdot]$. For instance, the point (i, j), which is the intersection of lines $[i]$ and $[0, j]$, becomes the line $[i, j]$, which passes through the points (i) and $(0, j)$.

We may combine the first two transformations into one.

($T\theta\phi$) Apply the permutation θ to the row labels and ϕ to the column labels in all of the Latin squares, then permute the symbols in each square separately to return each to natural order.

Definition 2.5.1. Two MAXMOLS are *equivalent*, denoted $\mathcal{M}_1 \sim \mathcal{M}_2$, if they belong to the same $T\theta\phi$ equivalence class, that is, if one MAXMOLS is obtained from the other by a $T\theta\phi$ transformation. We will use the notation $\mathbb{M}$ for an equivalence class of MAXMOLS.

The above transformations were used in [OP95] to study the equivalence classes of MAXMOLS for the four projective planes of order 9, to find the numbers of equivalence classes, maps to follow to transform one MAXMOLS to another, and other details. The numbers of equivalence classes and maps are given in Table 2.1 and Figure 2.9.

Plane	# of classes	Classes
Desaurgesian	1	$\mathbb{M}_1$
Hall	5	$\mathbb{M}_2 - \mathbb{M}_6$
Dual Hall	5	$\mathbb{M}_7 - \mathbb{M}_{11}$
Hughes	8	$\mathbb{M}_{13} - \mathbb{M}_{19}$

Table 2.1 Equivalence classes for planes of order 9.

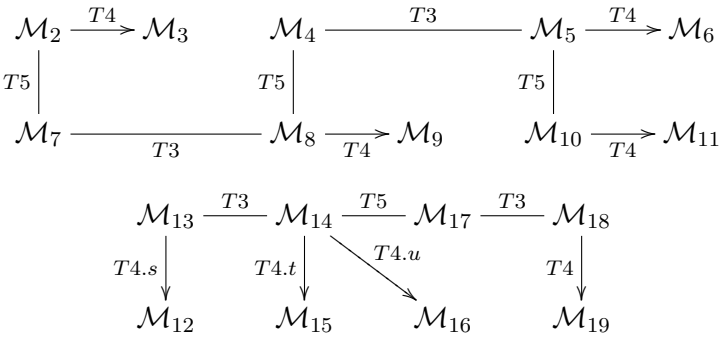


Figure 2.9 Maps for the non-Desarguesian planes of order 9.

Notice that the first map is for both the Hall plane (bottom line) and Dual Hall plane (top line), $\mathcal{M}_2 - \mathcal{M}_{11}$. The $T4.r$ transformation is labeled simply as $T4$ when the value of r is immaterial – any choice $r = 1, \ldots, n-1$ will have the desired effect. For the second map, the choices for s, t, u need to be specific values, but these depend on the ordering of squares after the $T4.r$ transformation (this is discussed in Section 3.2). Of course, the maps are not helpful without a starting point. Representative MAXMOLS $\mathcal{M}_8$ and $\mathcal{M}_{14}$ for classes $\mathbb{M}_8$ and $\mathbb{M}_{14}$ are given in [OP95] and in Appendix C.

2.5.2 Transformations for Ternary Rings

Theorem 2.5.6 below, from [Gra04], gives a necessary and sufficient condition such that two ternary rings correspond to the same plane. To our knowledge it is the only statement of its kind concerning all ternary rings of a projective plane. This condition uses several transformations of ternary rings, defined below.

Of course, given the equivalence of ternary rings and MAXMOLS, the transformations in the previous subsection imply transformations of ternary rings. Later we will compare the transformations from the previous subsection to those given below.

[Gra04] gives a more general labeling of the projective plane than that given in Section 2.3.1. This labeling can be obtained from the labeling above by allowing the lines through any point labeled (i) (including $i = 0$) to be labeled arbitrarily. In other words, the lines passing through (i) formerly labeled $[i, k]$ in the more restricted labeling can be labeled $[i, k\theta]$ for an arbitrary permutation θ of $\{0, \ldots, n-1\}$, and the points labeled (m, b) can be labeled $(m, b\psi)$ for an arbitrary permutation ψ. In particular, the point labeled (m, b) still lies on line $[m]$ but need not lie on $[0, b]$ as above.

The difference in labeling implies that the ternary ring need not have a 0. If a ternary ring does not have a zero, and if squares are constructed from the ternary ring as in the previous section, then these squares need not be in natural order or even be Latin squares. In fact, the most that we can say is that the squares $L_0, \ldots, L_{n-1}$ are orthogonal, row-Latin squares, and L_0 need not be the square whose (i, j) entry is j. By row-Latin we mean that each symbol appears exactly once in each row. We will assume this looser construction when discussing the transformations from [Gra04].

The construction of the ternary ring from the projective plane also differs in [Gra04] from the construction given in Section 2.4. In fact, the ternary ring by the construction given in that paper is the ternary ring corresponding to the *dual* plane by our construction. It will not be necessary to use this construction explicitly, however, so we will not go into detail.

Definition 2.5.2. Let $\langle\ \rangle$ and $[\ ]$ be two ternary rings on the same set S. $\langle\ \rangle$ is called a *comparative* of $[\ ]$ if there exist two permutations σ, δ on S and two families $(\sigma_x)_{x\in S}, (\delta_x)_{x\in S}$

of permutations on S such that $\langle x, m, b\rangle\sigma_x = [x\sigma, m\delta, b\delta_m]$ for all $x, m, b \in S$.

The geometric effect of tranforming to a comparative is to permute the labels of the points of $[\infty]$ by δ, labels on lines through $(m\delta)$ by δ_m, the labels on lines through (∞) by σ, and the points on $[x\sigma]$ by σ_x.

Example 2.5.3. Let u be an arbitrary element of S. We may choose δ_m and σ_x such that

$$\langle x, m, b\delta_m\rangle = \langle u, u, b\rangle \quad \text{and} \quad y\sigma_x = \langle x, u, y\rangle \quad \text{for all } x, m, b, y \in S.$$

The resulting comparative, denoted $\langle\ \rangle_u$, for which $\langle x, m, b\rangle_u\sigma_x = \langle x, m, b\delta_m\rangle$ is a ternary ring with zero u.

Definition 2.5.4. Let $\langle\ \rangle$ be a ternary ring with 0 on S. The ternary ring with zero $\langle\ \rangle(i)$ is the *inverse* of $\langle\ \rangle$ if for all $x, m, \in S$ and $x, m \neq 0$, $\langle x, 0, b\rangle(i) = b$ and $\langle\langle x, m, b\rangle(i), m, d\rangle = x$, where d is the element such that $\langle b, m, d\rangle = 0$.

Definition 2.5.5. [Mar67] The ternary ring $[\ ] = \langle\ \rangle(d)$ is the *dual* of $\langle\ \rangle$ if $y = \langle x, m, [m, x, y]\rangle$, or equivalently, $y = [x, m, \langle m, x, y\rangle]$ for all $x, m, b, y \in S$.

Notation:

1. $\langle\ \rangle_{(u)}$ will denote $\langle\ \rangle_u(i)$.

2. $\langle\ \rangle_{[u]}$ will denote $(\langle\ \rangle(d))_u(i)(d)$ (the ternary operation obtained from $\langle\ \rangle$ by performing the following: dualizing, introducing a zero u, inverting, dualizing).

After the work of defining these transformations, we obtain the following, which to our knowledge is the state of the art.

Theorem 2.5.6. *Let $\langle\ \rangle$ and $[\ ]$ be two ternary rings on the same set S. Then they induce isomorphic projective planes if and only if*

1. *$[\ ]$ is a comparative of $\langle\ \rangle$, or*

2. *$[\ ]$ is a comparative of one of the following ternary rings with zero:*

(a) $\langle\ \rangle_{(u)}$

(b) $\langle\ \rangle_{[u]}$

(c) $\langle\ \rangle_{(u)[v]}$

(d) $\langle\ \rangle_{[u](v)}$

(e) $\langle\ \rangle_{(u)[v](w)}$

for some $u, v, w \in S$.

Theorem 2.5.7. *(Lemma 4.2 of [Gra04]) The point and line labeled* (u) *and* $[\infty]$ *in* $\langle\ \rangle$ *are labeled* $(\infty), [\infty]$ *in* $\langle\ \rangle_{(u)}$. *The point and line labeled* (∞) *and* $[u]$ *in* $\langle\ \rangle$ *are labeled* $(\infty), [\infty]$ *in* $\langle\ \rangle_{(u)}$.

Hence the effect of $\langle\ \rangle_{(u)}$ is to move the label (∞) to the point previously labeled (u) on the line $[\infty]$, much like $T4.r$ which moves the label (0). The effect of $\langle\ \rangle_{[u]}$ is to move the label $[\infty]$ to the line through (∞) previously labeled $[u]$. Compare this to $T5T4.rT5$, which moves the label $[0]$.

2.6 Orthomorphisms

Complete mappings of groups were first defined in [Man42] and used there to find orthogonal Latin squares *based on a group*, i.e., squares L_x where the maps $\mathbb{L}_x(y)$ for all rows y form a group (see also [DK74]). We use an expanded definition.

Definition 2.6.1. A bijection ϕ of a set Q closed under a binary operation $\cdot$ is a *(left) complete mapping* if $x \mapsto x(x\phi)$ is also a bijection of Q.

It will often be useful to have a right-hand version of this definition. When we wish to make a distinction, we will call the previous a left complete mapping, while the following we will usually just call a complete mapping.

Definition 2.6.2. A bijection ϕ of a set Q closed under a binary operation $\cdot$ is a *(right) complete mapping* if $x \mapsto x\phi \cdot x$ is also a bijection of Q.

A related concept was introduced in [JDM61].

Definition 2.6.3. A bijection ϕ of a set Q closed under a binary operation $+$ (not necessarily abelian) is a *(right) orthomorphism* if $x \mapsto x\phi - x$ is also a bijection of Q.

We can of course make a similar, left-handed definition of a left orthomorphism, and we will assume that the term orthomorphism, without further qualification, will refer to right orthomorphism.

It is easy to see that if ϕ is a bijection of $(Q, +)$ and $x\theta = x\phi - x$, then ϕ is an orthomorphism if and only if θ is a complete mapping. Likewise, if θ is a bijection and $x\theta + x = x\phi$, then θ is a complete mapping if and only if ϕ is an orthomorphism.

An often useful notion when working with orthomorphisms is that of orthogonality.

Definition 2.6.4. Two orthomorphisms ϕ, θ on the same group, quasigroup, etc., $(Q, +)$, are *orthogonal* if $\phi - \theta$ is a bijection.

CHAPTER 3. TRANSFORMATIONS

3.1 Some Remarks on Equivalence Classes

Recall the definition of equivalence of MAXMOLS: $\mathcal{M}_1 \sim \mathcal{M}_2$ if related by a $T\theta\phi$ transformation. A more common definition of equivalence of MAXMOLS is by isotopism.

Definition 3.1.1. Two MAXMOLS $[\cdot,\cdot,\cdot]$ and $(\cdot,\cdot,\cdot)$ are *isotopic* if there exist bijections f, g, h, i such that $[x, y, z] = (xf, yg, zh)^i$. The ordered quadruple of bijections f, g, h, i is called an *isotopism.*

Of course, it is natural to ask what the relationship is between the present notion of equivalence and the more common notion of equivalence by isotopism.

Proposition 3.1.2. *If $\mathcal{M}_1 \cup \{L_0\}$ and $\mathcal{M}_2 \cup \{L_0\}$ are isotopic* MAXMOLS *in natural order, then they are equivalent.*

Proof. Let the ternary operation of $\mathcal{M}_1$ be $[\cdot,\cdot,\cdot]$, that of $\mathcal{M}_2$ be $(\cdot,\cdot,\cdot)$, and let $[x, y, z] = (xf, yg, zh)^i$.

Notice that, in terms of the Latin squares, the effect of f is to rearrange the squares (including the zero square L_0). But the order of squares in the MAXMOLS is immaterial, so we may, without loss of generality, assume $f = 1$.

Similarly, g rearranges the rows of the squares, and h the columns, so we may take $\theta = g$, $\phi = h$. Thus $\mathcal{M}_1 \sim \mathcal{M}_2$ by a $T\theta\phi$ transformation if the action of i is to return the squares to natural order. But this must be the action of i (once we associate θ with g and ϕ with h) since we assume the squares are in natural order. (By the way, the fact that i does not depend on x forces $0g = 0$.) Hence, $\mathcal{M}_1 \sim \mathcal{M}_2$.

Alternately, the proof follows from Proposition 3.3.1 below. □

Though not ever explicitly stated, the guiding principle behind the search for equivalence classes in [OP95] is the following. In this proposition, we take labels such as P_1, P_2 to be fixed identifiers of the points and lines of a projective plane, in contradistinction to labels such as $(0), [\infty], [m, k]$, which are assumed to be temporary and transferable. These labels will be used this way whenever they arise.

Proposition 3.1.3. *Let P_1, P_2, P_3, P_4 be four points of a projective plane π. The* MAXMOLS *with P_1, P_2 labeled $(\infty), (0)$ and the* MAXMOLS *with P_3, P_4 labeled $(\infty), (0)$ are equivalent if and only if there is a collineation γ such that $P_1\gamma = P_3$ and $P_2\gamma = P_4$.*

Indeed, if there is such a collineation then the projective plane must have the same structural relationship with respect to the points P_1, P_2 and P_3, P_4, and conversely, with the only freedom being in the numbering of the lines through (∞) and (0) (which are rearranged by a $T\theta\phi$ transformation), and the labeling of the other points on $[\infty]$ (which is immaterial in the MAXMOLS).

Definition 3.1.4. Let Γ be a collineation group of a projective plane π and let (P_1, P_2) be an ordered pair of points of π. The *2-orbit* of (P_1, P_2) is the set of all images of (P_1, P_2) under Γ. In other words, (P_1', P_2') is in the 2-orbit of (P_1, P_2) if and only if there is a collineation $\gamma \in \Gamma$ such that $P_1\gamma = P_1'$ and $P_2\gamma = P_2'$. Γ is *doubly transitive* if there is only one 2-orbit, that is, if for any two ordered pairs of points there is a collineation that sends the first ordered pair to the second.

Corollary 3.1.5. *Let π be a projective plane with collineation group Γ. Then the number of equivalence classes for π is equal to the number of 2-orbits of Γ.*

In [OP95] it is shown that the Desaurgesian plane (field plane) of order 9 has only one equivalence class. In fact, this characteristic uniquely distinguishes Desaurgesian planes.

Theorem 3.1.6. *[OW59] If Γ is the collineation group of a projective plane π, then Γ is doubly transitive on the points of π if and only if π is Desarguesian.*

Corollary 3.1.7. *π is Desarguesian if and only if π has exactly one equivalence class of* MAXMOLS.

3.2 Equivalence Classes of Hall Planes

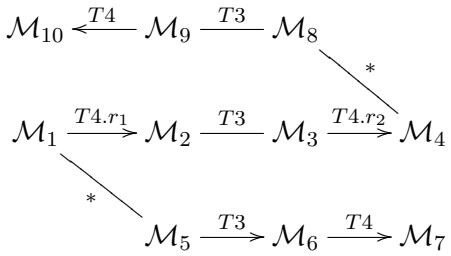


Figure 3.1 Map of MAXMOLS for Hall planes of order at least 16.

[OP95] gave a map for finding all equivalence classes for the Hall, dual Hall, and Hughes planes of order 9, Figure 2.9. Here we compose the map in Figure 3.1 for the equivalence classes of Hall planes of order at least 16. We use the following facts, which can be found in Chapter X of [HP73] or in [Hug59]. These facts hold for all Hall planes of order at least 16.

1. The Hall plane contains a unique line t called the translation line, which is fixed by every collineation.

2. There is a partition of the points on the translation line into two sets, S_1, S_2, such that $|S_1| = q + 1$, and S_1,S_2 are fixed by every collineation.

From facts 1 and 2 we infer the existence of (at least) ten distinct equivalence classes based on the choice of points labeled (∞), (0).

1. (∞), (0) in S_1

2. (∞), (0) in S_2

3. (∞) in S_1, (0) in S_2

4. (0) in S_1, (∞) in S_2

5. (∞) in S_1, (0) off t (the translation line)

6. (0) in S_1, (∞) off t

7. (∞) in S_2, (0) off t

8. (0) in S_2, (∞) off t

9. (∞), (0) off t, and $[\infty] \cap t$ in S_1 (where, as usual, $[\infty]$ is the line passing through both (∞) and (0)).

10. (∞), (0) off t, and $[\infty] \cap t$ in S_2.

In fact, these are exactly the equivalence classes, but because the proof of this fact is not very illuminating and probably overly complicated, we relegate it to Appendix B. For now, we will assume that these are the equivalence classes and show how the transformations $T3 - T5$ are used to compose the map in Figure 3.1. First, we rearrange the classes in a more convenient order.

1. (∞), (0) in S_1
2. (∞) in S_1, (0) in S_2
3. (0) in S_1, (∞) in S_2
4. (∞), (0) in S_2
5. (∞) in S_1, (0) off t
6. (0) in S_1, (∞) off t
7. (∞), (0) off t, $[\infty] \cap t$ in S_1
8. (∞) in S_2, (0) off t
9. (0) in S_2, (∞) off t
10. (∞), (0) off t, $[\infty] \cap t$ in S_2

We suppose that we have a MAXMOLS $\mathcal{M}_1$ from equivalence class 1 above, that S_1 is the set of points labeled $(\infty), (0), (1), \ldots, (q-1)$, and that S_2 the points $(q), \ldots, (q^2-1)$.

A transformation $T4.r$ moves (0) to S_2 if $q \leq r \leq q^2 - 1$ and fixes (∞), giving $\mathcal{M}_2$ for equivalence class 2. Now, at the completion of this and every $T4.r$ transformation we reconstruct the Latin squares from the representational array in whichever order we choose. If P_2 corresponds to row i of the representational array before the transformation, then P_2 corresponds to row i of the transformed array, but we may assign whatever label we wish to square made from this row (except the r^{th} row of the array must correspond to (0)). Thus the order of the non-adjunct squares is arbitrary. Whatever the choice, we assume that it is recorded which squares correspond to points in S_1 and which to S_2.

Using $T3$ on $\mathcal{M}_2$ switches (∞) and (0), giving a MAXMOLS $\mathcal{M}_3$ in class 3. Another $T4.r$ transformation moves (0) to S_2, giving $\mathcal{M}_4$. The specific r here depends upon the choice of order of squares after the previous $T4.r$ transformation.

To move from $\mathcal{M}_1$ to $\mathcal{M}_5$ we need to move (0) off t. The effect of $T5$ is to interchange $[\infty] \leftrightarrow (\infty)$ and $[0] \leftrightarrow (0)$, and the effect of $T3$ is to switch (0) and (∞). Hence if we transform with $T5T3T5$ (marked as ' * ' in the figure), then the net effect is to switch the labels $[0]$ and $[\infty]$, as illustrated in Figure 3.2. The label (∞) is fixed by the $T5T3T5$ transformation since it is the intersection of $[\infty]$ and $[0]$, but (0) moves since this label must stay on $[\infty]$, which is no longer t, and so (0) is off t.

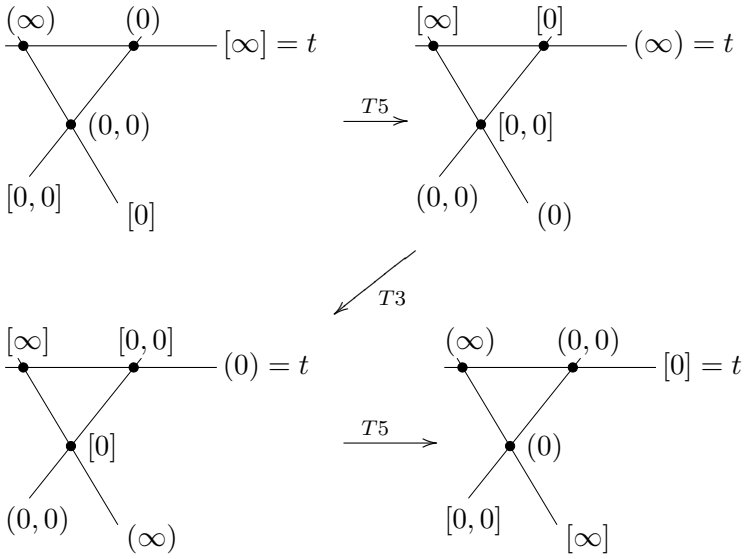


Figure 3.2 The transformation $T5T3T5$.

The MAXMOLS $\mathcal{M}_6$ has (0) on t and (∞) off, so $T3$ transforms $\mathcal{M}_5$ to $\mathcal{M}_6$. For any r, $T4.r$ will move (0) off t, giving $\mathcal{M}_7$.

To get $\mathcal{M}_8$ from $\mathcal{M}_4$ we use the $T5T3T5$ again. Likewise, the steps from $\mathcal{M}_8$ to $\mathcal{M}_9$ and $\mathcal{M}_{10}$ mirror those from $\mathcal{M}_5$ to $\mathcal{M}_6$ and $\mathcal{M}_7$.

In order to use the map we need to be able to construct a suitable $\mathcal{M}_1$. The MAXMOLS corresponding to the ternary ring in Example 2.2.11 is one such MAXMOLS ([Hug59]).

3.3 Comparison of Transformations

Since the transformations $T1$-$T5$ always produce a MAXMOLS in natural order, the corresponding ternary ring always has a zero if we choose that $(0, y, z) = z$ for all y, z: being in natural order means $(x, 0, z) = 0$ for all x, z, and a ternary ring has a zero if $(0, y, z) = (x, 0, z) = 0$. Thus the transformations $T1$-$T5$ give a method for finding all ternary rings with zero corresponding to a given plane and its dual.

[Gra04] gives a set of algebraic transformations through which any ternary ring for a plane may be found from any other. These ternary rings will sometimes have zero and sometimes not. Since it may conceivably be advantageous to find all ternary rings with zero and only ternary rings with zero we compare the transformations due to Owens from [Owe92] to those of Grari from [Gra04].

We remind the reader of some notation: $(Q, \cdot_x)$ is the quasigroup associated with Latin square L_x. $\mathbb{L}_x(y)$ denotes left multiplication by y in square L_x, i.e., the permutation $z \to y \cdot_x z$. $\mathbb{R}_x(y)$ denotes right multiplication, $z \to z \cdot y$. Notice $(x, y, z) = z\mathbb{L}_x(y) = y\mathbb{R}_x(z)$, and the function ρ_{ik} from the representation array (see subsection 2.5.1) is exactly the function $\mathbb{L}_i(k)$.

Proposition 3.3.1. *Let $[\cdot, \cdot, \cdot]$ be a ternary ring before a transformation $T\theta\phi, T3, T4.r$, or $T5$ is applied and $(\cdot, \cdot, \cdot)$ the ternary ting after. Let $\mathbb{L}$ and $\mathbb{R}$ denote left and right multiplication in the $[\cdot, \cdot, \cdot]$ ternary ring. Then*

($T\theta\phi$) $(x, y, z) = [x, y\theta^{-1}, z\phi^{-1}]^{\delta_{x,\theta\phi}}$ *where* $\delta_{x,\theta} = \mathbb{L}_x^{-1}(0\theta^{-1})$.
Hence $(x, y, z) = z\phi^{-1}\mathbb{L}_x(y\theta^{-1})\mathbb{L}_x^{-1}(0\theta^{-1})\phi$.

($T3$) $(x, y, z) = [x, z, y]^{\sigma_{x,z}}$, *where* $\sigma_{x,z}^{-1} = \mathbb{R}_x(z)$.
Hence $(x, y, z) = y\mathbb{L}_x(z)\mathbb{R}_x^{-1}(z)$.

($T4.r$) $(x, y, z) = [x, y, z\mathbb{L}_r^{-1}(y)] = z\mathbb{L}_r^{-1}(y)\mathbb{L}_x(y)$.

($T5$) $(x, y, z) = z\mathbb{L}_y^{-1}(x)$.

Proof.

($T\theta\phi$) Let L_x be a square before and L'_x the corresponding square after applying the transformation. The row labeled y in L_x is labeled $y\theta$ in L'_x, and the column labeled z in L_x is labeled $z\phi$ in L'_x. Thus, if we want the row y and column z in L'_x, we want row $y\theta^{-1}$ and column $z\phi^{-1}$ in L_x, and before rearranging the symbols we have $(x, y, z) = [x, y\theta^{-1}, z\phi^{-1}]$.

Now, to rearrange symbols, notice that the $(0, z)$ entry of L'_x is the $(0\theta^{-1}, z\phi^{-1})$ entry of L_x, or $w := 0\theta^{-1} \cdot_x z\phi^{-1}$. Thus $w\mathbb{L}_x^{-1}(0\theta^{-1})\phi = z$, and $(x, y, z) = [x, y\theta^{-1}, z\phi^{-1}]^{\delta_{x,\theta}\phi}$ where $\delta_{x,\theta} = \mathbb{L}_x^{-1}(0\theta^{-1})$.

($T3$) Obviously, transposing the square changes $[x, y, z]$ to $[x, z, y]$. Before reordering, the zeroth row of L_x^T (the transpose of L_x) is the zeroth column of L_x. The $(y, 0)$ entry of L_x is $y \cdot_x 0$. The $(y, 0)$ entry of $L_x\mathbb{R}_x^{-1}(0)$, then, is y. Hence $L_x^T\mathbb{R}_x^{-1}(0)$ is in natural order. (By $L_x\mathbb{R}_x^{-1}(0)$ we mean the square whose (m, b) entry is $w\mathbb{R}_x^{-1}(0)$, or $w/_x0$, if the (m, b) entry of L_x is w.)

($T4.r$) The transformation is $z\sigma_{xy} := (x, y, z) = z\rho_{ry}^{-1}\rho_{xy}$, where σ has the same meaning in relation to $(\cdot, \cdot, \cdot)$ that ρ does in relation to $[\cdot, \cdot, \cdot]$. Since, as pointed out above, $\rho_{ik} = \mathbb{L}_i(k)$, we have $(x, y, z) = z\mathbb{L}_r^{-1}(y)\mathbb{L}_x(y)$.

($T5$) $z\sigma_{xy} = (x, y, z) = z\rho_{yx}^{-1} = z\mathbb{L}_y^{-1}(x)$.

□

If we wish, we could guarantee that the MAXMOLS we generate has a 1 as well as zero. If a MAXMOLS (ternary ring) has a zero, the condition for the MAXMOLS to have a 1 is $(1, x, 0) = x = (x, 1, 0)$ for all x (Definition 2.2.1 v.).

Proposition 3.3.2. *If $[\cdot, \cdot, \cdot]$ is a ternary ring with 0, then there are θ, ψ such that $(x, y, z) = [x\psi^{-1}, y\theta^{-1}, z]$ is a ternary ring with 0 and 1.*

Proof. Choose θ such that the first *column* of the square labeled as L_1 is in natural order when the rows are relabeled according to θ (i.e., the $(y, 0)$ entry will be y). (The chosen θ is $\mathbb{R}_1^{-1}(0)$.) Applying this θ to all of the squares, no reordering is necessary to put the first row in natural

order since the $(0,0)$ entry was 0 before (and after) applying θ. We have $(1,x,0) = x$ for the ternary operation resulting from this step.

We now relabel the squares according to which symbol appears in the $(1,0)$ spot, so we choose the label L_x for the square that has x as the $(1,0)$ entry. (Notice that the square previously chosen as L_1 keeps the label L_1.) We have $(x,1,0) = x$ after this step. If this relabeling of squares is called ψ, then $(x,y,z) = [x\psi^{-1}, y\theta^{-1}, 0]$ is the resulting ternary ring with 0 and 1. □

We now put the transformations from [Gra04] into a form similar to Proposition 3.3.1. Though the construction of a ternary ring from a projective plane is different in [Gra04], we assume the same construction of squares from the ternary ring. The squares constructed need not be Latin since the ternary ring need not have a zero. The squares are row-Latin, however, with no symbol appearing more than once in any row. This means that left multiplication $\mathbb{L}_x(y)$ is still a permutation.

Proposition 3.3.3. *Let $\mathbb{L}_x$ denote left multiplication in square L_x of ternary ring $\langle\ \rangle$, and let $\chi_{y,z} = \langle \cdot, y, z\rangle$. Then*

1. *[ ] is a comparative of $\langle\ \rangle$ if $[x,y,z] = z\delta_y \mathbb{L}_{x\sigma}(y\delta)\sigma_x^{-1}$.*
2. *[ ] is the comparative $\langle\ \rangle_u$ of $\langle\ \rangle$ if $[x,y,z] = z\mathbb{L}_u(u)\mathbb{L}_x^{-1}(u)$.*
3. *[ ] is the inverse of $\langle\ \rangle$ if, for $x, y \neq 0$, $[x,y,z] = x\chi_{y,d}^{-1}$ where $d = 0\mathbb{L}_z^{-1}(y)$, and $[x,0,z] = z = [0,y,z]$.*
4. *[ ] is the dual of $\langle\ \rangle$ if $[x,y,z] = z\mathbb{L}_y^{-1}(x)$.*

Proof.

1. Comparative: The result is immediate from the fact that $\mathbb{L}_x(y) = \langle x, y, \cdot\rangle$ and the definition of comparative: [ ] is a comparative of $\langle\ \rangle$ if

$$[x,y,z] = \langle x\sigma, y\delta, z\delta_y\rangle\sigma_x^{-1} = z\delta_y\mathbb{L}_{x\sigma}(y\delta)\sigma_x^{-1}.$$

2. $\langle\ \rangle_u$ comparative: If $[\ ]$ is the comparative $\langle\ \rangle_u$ from Example 2.5.3, then

$$\langle x, m, b\delta_m\rangle = \langle u, u, b\rangle \quad \text{and} \quad y\sigma_x = \langle x, u, y\rangle \quad \text{for all } x, m, b, y \in S,$$

or, with $\rho_{ij} = \langle i, j, \cdot\rangle$,

$$b\delta_m\rho_{xm} = b\rho_{uu} \quad \text{and} \quad y\sigma_x = y\rho_{xu}.$$

Thus $\delta_m = \rho_{uu}\rho_{xm}^{-1}$, $\sigma_x = \rho_{xu}$ and we have

$$\begin{aligned}\langle x, y, z\rangle_u &= \langle x, y, z\delta_y\rangle\sigma_x^{-1} \\ &= z\rho_{uu}\rho_{xy}^{-1}\mathbb{L}_x(y)\rho_{xu}^{-1} \\ &= z\mathbb{L}_u(u)\mathbb{L}_x^{-1}(y)\mathbb{L}_x(y)\mathbb{L}_x^{-1}(u), \\ &= z\mathbb{L}_u(u)\mathbb{L}_x^{-1}(u),\end{aligned}$$

where we have again made use of the fact that $\rho_{ij} = \mathbb{L}_i(j)$.

3. Inverse: That $[x, 0, z] = z = [0, y, z]$ is part of the definition of inverse. The definition for $x, y \neq 0$ is $\langle\langle x, y, z\rangle(i), y, d\rangle = x$, where d is the element such that $\langle z, y, d\rangle = 0$. Thus $\langle x, y, z\rangle(i)\chi_{y,d} = x$ where $d\mathbb{L}_z(y) = 0$, or $\langle x, y, z\rangle(i) = x\chi_{y,d}^{-1}$ where $d = 0\mathbb{L}_z^{-1}(y)$.

4. Dual: Let $\gamma_{ij} = [i, j, \cdot]$. Then $[\ ]$ is the dual of $\langle\ \rangle$ if $z = z\gamma_{yx}\rho_{xy}$ and $z = z\rho_{yx}\gamma_{xy}$. Thus $\gamma_{xy} = \rho_{yx}^{-1}$, and this transformation is the same as the $T5$ transformation above: $[x, y, z] = z\mathbb{L}_y^{-1}(x)$.

□

The two advantages of Grari's transformations are in giving every ternary ring of the plane, and in giving a necessary and sufficient condition such that two ternary rings correspond to the same plane. Said another way, the two possible drawbacks of Owens' transformations are the facts that they produce only ternary rings with zero, and that the derivable necessary and sufficient condition such that two ternary rings correspond to the same plane is not very strong: one can be derived from the other by consecutive application of transformations $T1$-$T5$. In the next section this latter point is remedied. Here we discuss how the former point can be.

Proposition 3.3.2 could clearly be used to modify the transformations $T1$-$T5$ to always produce ternary rings with both 0 and 1. Notice that this transformation is a comparative, by Proposition 3.3.1.

Alternately, since the $T\theta\phi$ transformation is a comparative as well (comparing Proposition 3.3.1 to Proposition 3.3.3), if we wish to alter transformations $T1$-$T5$ to include *all* ternary rings, not just ternary rings with zero, we could loosen the definition of equivalence, Definition 2.5.1, to say that two MAXMOLS are equivalent if one MAXMOLS is a comparative of the other. The transformations $T3$-$T5$ were written with the expectation that the MAXMOLS be in natural order. However, the transformations have the same effect for any set of squares corresponding to a ternary ring without zero, and so could be used on any comparative of a MAXMOLS. Despite these facts, we will continue to use the original definition of equivalence, restricting ourselves to ternary rings with zero.

3.4 A New Transformation

As always, let φ_{xy} be the permutation equivalent to the Greco-Latin square $L_{x,y}$ obtained from Latin squares L_x and L_y.

Proposition 3.4.1. *As a map of points in π, φ_{xy} maps the pair of points $((x),(y))$ to the pair $((\infty),(0))$, the lines $[x,k]$ to $[k]$, and the lines $[y,k]$ to $[0,k]$. (Here x and y may be any of the symbols $\infty, 0, 1, \ldots, n-1$.)*

Proof. Let $1 \leq k \leq n$ and let $(m_1, b_1), \ldots, (m_n, b_n)$ be the n coordinates for which the entry in square L_x is k, that is, all of the points on line $[x,k]$ (if x is the symbol ∞, then take $[\infty,k]$ to mean $[k]$). Then φ_{xy} maps these n coordinates to $(k,b'_1), \ldots, (k,b'_n)$ for some $b'_1, \ldots, b'_n$. The points with labels $(k,b'_1), \ldots, (k,b'_n)$ are exactly the points other than (∞) on line $[k]$. Thus φ_{xy} maps the lines $[x,k]$ to $[k]$ and, by a similar argument, the lines $[y,k]$ to $[0,k]$.

Since (x) is the one point on all of the lines $[x,k]$, it must be sent to the one point on all of the lines $[k]$, which is (∞). Likewise (y) is sent to (0). □

We can define a transformation $T6$ based on φ_{xy}. Though the result of this transformation

can be obtained by combinations of transformations $T1$-$T5$, $T6$ is useful as a shortcut.

($T6.xy$) Create a new MAXMOLS $\mathcal{M}^+ = \{L'_\infty, L'_0, \ldots, L'_{n-1}\}$ as follows: Make the (m, b) entry of square L'_i the $(m, b)\varphi_{xy}^{-1}$ entry of L_i. Then permute the symbols of each square to put in natural order.

Geometrical effect: The labels $((\infty), (0))$ are moved to the points previously labeled $((x), (y))$ and labels $[k], [0, k]$ are moved to the lines previously labeled $[x, k], [y, k]$, respectively.

Notice that $T6$ includes the possibility of not moving one or both of the labels (∞),(0), or even interchanging them. Hence, in particular, $T4.r$ and $T3$ are both examples of $T6$ transformations. ($T4.r$ is $T6.\infty r$, and $T3$ is $T6.0\infty$. The identity is $T6.\infty 0$.)

If the squares $L_0, \ldots, L_{n-1}$ are those of a ternary ring without zero, then the squares $L_1, \ldots, L_{n-1}$ are not necessarily Latin. They are row-Latin and still orthogonal, though, and the $T6$ transformation can still be used on them. (In this case $T6.\infty 0$ is not necessarily the identity transformation. The squares resulting from $T6$ are always Latin by the proof of Proposition 3.4.3, below.) Were the propositions in this section to include the possibility that the ternary ring is without zero, the proofs would be the same or change only very slightly.

Lemma 3.4.2. *If L_a, L_b are two orthogonal squares and ϕ any permutation of the n^2 coordinates, then if L_a and L_b are both rearranged according to ϕ, then the resulting squares are again orthogonal.*

Proposition 3.4.3. *The result of performing $T6$ on a* MAXMOLS *$\mathcal{M}_1^+$ is another* MAXMOLS*, $\mathcal{M}_2^+$, of the same plane, with the given geometrical effect.*

Proof. As before, relabeling the symbols at the end of the transformation does not influence orthogonality nor have any geometric effect beyond relabeling the lines through each point on $[\infty]$. Hence we will ignore this relabeling.

If $(i, j)\varphi_{xy} = (i', j')$ then the (i', j') entry of L'_x and L'_y is the (i, j) entry of L_x and L_y, respectively. But, by the definition of φ_{xy}, the (i, j) entry of L_x and L_y is i' and j'. Thus the

(i', j') entry of L'_x is i' and that of L'_y is j'. In other words, L'_x is the row-Latin adjunct and L'_y is the column-Latin adjunct.

By the lemma, since $L_\infty, L_0, \ldots, L_{n-1}$ are mutually orthogonal, the squares $L'_\infty, L'_0, \ldots, L'_{n-1}$ are mutually orthogonal. The fact that the non-adjunct squares are each orthogonal to both L'_x and L'_y means that each is also Latin. (Being orthogonal to a row-Latin square implies no entry is repeated in any row, and being orthogonal to a column-Latin square implies no entry is repeated in any column.) Hence the set of squares $L'_\infty, L'_0, \ldots, L'_{n-1}$ is a MAXMOLS with adjuncts.

That the geometrical effect is as stated follows from Proposition 3.4.1 once we establish that the MAXMOLS obtained does, in fact, correspond to the same plane. What we need to establish is that the mapping preserves incidence.

Let $P_1 = (a_1, b_1), P_2 = (a_2, b_2)$ be two points on any line $[i, j]$ before the transformation. In other words the (a_1, b_1) and (a_2, b_2) entries of square L_i are both j. After the transformation the $(a_1, b_1)\varphi_{x,y}$ and $(a_2, b_2)\varphi_{x,y}$ entries of L'_i are j. Thus the line $[i, j]$ is preserved but with points on it relabeled as $(a_1, b_1)\varphi_{x,y}$ and $(a_2, b_2)\varphi_{x,y}$. Likewise the lines through any point on $[\infty]$ are preserved. It is also clear that if the point labeled (a, b) is on several lines $[i], [j, k], [\ell, m]$ then the same point, now labeled $(a, b)\varphi_{x,y}$ is on the same lines, though they may be labeled differently. □

It is easy to see that $T6.\infty 0$ is the identity mapping and that $T6.0\infty$ is $T3$ by inspection of the square $L_{\infty,0}$, which has (i, j) in position (i, j), and $L_{0,\infty}$, which has (j, i) in position (i, j). It is also easy to see that $T6.\infty r$ has the same effect on the labels (0) and (r) as $T4.r$, but perhaps not so obvious is the fact that the two are algebraically equivalent. This is what the next proposition shows.

Proposition 3.4.4. *$T6.\infty r$ is exactly the transformation $T4.r$.*

Proof. Let $(\ )$ be the ternary ring resulting from $[\ ]$ by applying $T6.\infty r$. Then $(x, y, z) = k$ if and only if $[x, y', z'] = k$ where $(y', z')\varphi_{\infty,r} = (y, z)$, because the transformation moves the entry (y', z') of square L_x to the $(y', z')\varphi_{\infty,r}$ entry. Now, the first component of $(y', z')\varphi_{\infty,r}$ is

the (y', z') entry of square L_∞, which is y', hence $y' = y$. The second component is $[r, y', z'] = [r, y, z'] = z$. If $\mathbb{L}_i(j) = [i, j, \cdot]$, then this last equality is $z'\mathbb{L}_r(y) = z$, or $z' = z\mathbb{L}_r^{-1}(y)$. Then we have $(x, y, z) = [x, y, z\mathbb{L}_r^{-1}(y)] = z\mathbb{L}_r^{-1}(y)\mathbb{L}_x(y)$. This is the transformation $T4.r$ by Proposition 3.3.1. □

In Figure 3.3 we remake Figure 3.1, the map for Hall planes of order at least 16, by adding $T6$ transformations. The double lines are $T5T3T5$ (with $T3 = T6.0\infty$), and all of the others are $T6$. The values of x, y chosen in the $T6$ transformations are important in every case, but since the labeling is arbitrary we leave these off. The correct values to use can be inferred from the discussion given in Section 3.2, and the labeling chosen after each transformation.

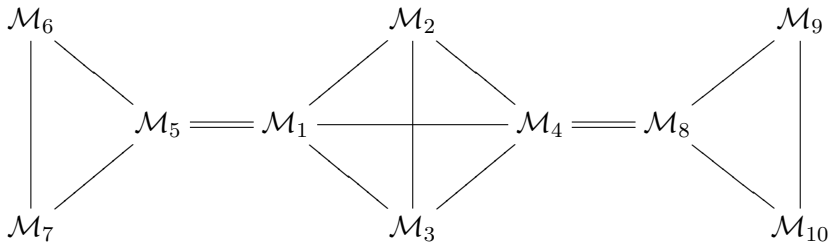


Figure 3.3 Map for Hall planes, order ≥ 16, using $T6$.

3.5 A New Necessary and Sufficient Condition Such That Two Ternary Rings Correspond to the Same Plane

Lemma 3.5.1. *The transformations $T5, T6, T5, T6$ can be used to move the labels $(\infty), (0)$ to any two points on any line through (∞) (see Figure 3.4).*

Proof. Let the points of the plane π to which we wish to send the labels $(\infty), (0)$ be P_1, P_2 and the line through these points be ℓ. Use transformation $T5$ to transform to the dual plane π^*, and let us use labels $(\infty)^*, (0)^*$, etc. to denote labels in the dual plane and labels without asterisks to denote the labels in the original plane. After $T5$, $(\infty)^*$ is (the dual of) $[\infty]$ and $(0)^*$ is the dual of $[0]$. If $(x)^*$ is the label on the dual of ℓ, use transformation $T6.x0$ to put label $(\infty)^*$ on the dual of ℓ. Use $T5$ to transform back to π, where now line ℓ has label $[\infty]$. Thus

P_1, P_2 are on the line labeled $[\infty]$ and a $T6$ transformation can be used put labels $(\infty), (0)$ on these points. □

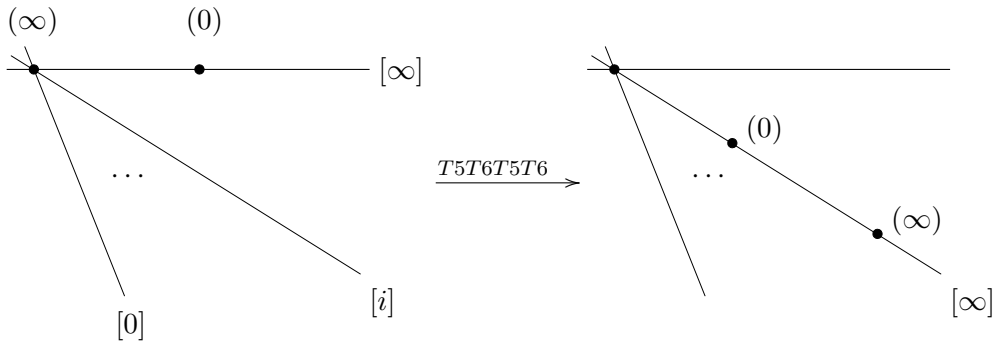


Figure 3.4 The transformation $T5T6T5T6$ in Lemma 3.5.1.

Suppose [ ] and ⟨ ⟩ are two ternary rings with zero, corresponding to the same plane. If points P_1, P_2 and P_3, P_4 are labeled as $(\infty), (0)$ in [ ] and ⟨ ⟩, respectively, then there are three possible relationships between the points:

1. $P_1 = P_3$ and $P_2 = P_4$.

2. All of the points are on the same line, but $P_1 \neq P_3$ or $P_2 \neq P_4$.

3. P_1, P_2 and P_3, P_4 determine two different lines.

Proposition 3.5.2. *Let* [ ] *and* ⟨ ⟩ *be two ternary rings with zero corresponding to the same plane. Then* [ ] *can be obtained from* ⟨ ⟩ *by successively applying* $T6$, $T5$, $T6$, $T5$, $T6$, $T\theta\phi$.

Proof. Let the points labeled $(\infty), (0)$ in [ ] and ⟨ ⟩ be P_1, P_2 and P_3, P_4, respectively.

1. If $P_1 = P_3$ and $P_2 = P_4$, then [ ] and ⟨ ⟩ are equivalent (or more exactly, their MAXMOLS are), so they differ by a $T\theta\phi$ transformation, and

$$T\theta\phi = (T6.\infty 0)(T5)(T6.\infty 0)(T5)(T6.\infty 0)(T\theta\phi)$$

since $T6.\infty 0$ and $T5T5$ are both the identity map.

2. If all of P_1, P_2, P_3, P_4 are on the same line, a $T6$ transformation on $\langle\ \rangle$ will move the labels $(\infty), (0)$ from P_3, P_4 to P_1, P_2. The resulting ternary ring is equivalent to $[\ ]$, so differs by a $T\theta\phi$ transformation. Hence $[\ ]$ is obtained from $\langle\ \rangle$ by using $T6$ then $T\theta\phi$, and

$$T6.xyT\theta\phi = (T6.\infty 0)(T5)(T6.\infty 0)(T5)(T6.xy)(T\theta\phi).$$

3. If P_1, P_2, and P_3, P_4 determine two different lines, let P_5 be the intersection of the two lines. P_5 is on the line labeled $[\infty]$ for the ternary ring $\langle\ \rangle$ (P_5 is on the line $\overline{P_3P_4}$, which is $[\infty]$ for $\langle\ \rangle$), so let the label on P_5 be (x_1) for this ternary ring. Then a $T6.x_1y_1$ transformation for any y_1 moves the label (∞) to P_5. By Lemma 3.5.1 we can use $T5T6.xyT5T6$ for some x, y to move the labels (∞) and (0) to P_1, P_2. The resulting ternary ring will be equivalent to $[\ ]$. Hence $[\ ]$ is obtained by successively using $T6$, $T5$, $T6$, $T5$, $T6$, $T\theta\phi$.

□

While the third possibility in the proof of Proposition 3.5.2 is fairly complicated, needing as many as 6 transformations, notice that it less complicated than 2(e) of Theorem 2.5.6, which requires as many as 9. Hence, Proposition 3.5.2 is decidedly simpler overall. This proposition, though, seems to be less general than Theorem 2.5.6 since it applies only to ternary rings with zero and not all ternary rings. This is true for the proposition as stated, but if the $T\theta\phi$ transformations are replaced by comparatives, then the result holds for any ternary rings $[\ ]$ and $\langle\ \rangle$ with or without zero.

CHAPTER 4. ORTHOMORPHISMS

Recall Definition 2.6.3 for a quasigroup $(G, \cdot)$: a permutation f of G is an *orthomorphism* if $(xf)/x$ is a permutation.

Let $(R, (\ ))$ be a ternary ring and let $\varphi_{x,y}$ be the permutation on the set $R \times R$ given by the conjunction of mutually orthogonal Latin squares L_x, L_y. Recall the (m, b) coordinate of the Latin square for x has symbol $xm + b$ if the ternary ring is linear, or symbol (x, m, b) generally. Thus the action of $\varphi_{x,y}$ is $(m, b)\varphi_{x,y} = (xm + b, ym + b)$ if the ternary ring is linear, $((x, m, b), (y, m, b))$ generally.

4.1 Greco-Latin Squares as Orthomorphisms

Proposition 4.1.1. *Let $(R, (\ ))$ be a linear ternary ring. Then $\varphi_{x,y}$ is an orthomorphism of $(R^2, +)$.*

Proof. Let $(m, b)h = (m, b)\varphi_{x,y} - (m, b)$ (here '$-$' is used for '$/$'). Then

$$(m, b)h = (xm + b - m, ym + b - b) = (xm + b - m, ym).$$

If $(m, b)h = (m', b')h$, then we must have $ym = ym'$, so, since $(R^*, \cdot)$ is a quasigroup, $(m, b)h$ uniquely determines m. Now suppose $(m, b)h = (m, b')h$. Then

$$(xm + b' - m, ym) = (xm + b - m, ym),$$

so $xm + b' = xm + b$, which forces $b = b'$. Thus $(m, b)h$ uniquely determines (m, b), implying h is a permutation and φ_{xy} is an orthomorphism. □

In searching for further properties of these orthomorphisms, the first to consider is orthogonality, Definition 2.6.4: two permutations f, g of a quasigroup are *orthogonal* if f/g is also a permutation. Unfortunately, no two of the $\varphi_{x,y}$ are orthogonal in a ternary ring with 0.

If the ternary ring has a 0, then the squares are in natural order, the $(0, a)$ position in each has symbol $a = (x, 0, a)$, so that the first rows of all Latin squares are the same. Hence $(0, a)\varphi_{w,x} = (a, a)$ for each and

$$(0, a)\varphi_{w,x}/(0, a)\varphi_{y,z} = (0, a)\varphi_{w,x} - (0, a)\varphi_{y,z} = (a, a) - (a, a) = (0, 0)$$

for each a. Thus, no $\varphi_{w,x}/\varphi_{y,z}$ is a permutation.

If a ternary ring is linear, then the rows of all of the Latin squares for the ternary ring are the same up to rearrangement. This is easy to see, for row m of square L_y is $ym + b$ for all b, which is row ym of square L_1. Thus the rows of L_y are permuted from those of L_1 by $\tau_y : m \mapsto ym$. We use this fact to define linearity in Latin squares generally, i.e., not necessarily squares from a MAXMOLS.

Definition 4.1.2. A Latin square L_x is *linear in* a second Latin square L_y if the rows of L_x are those of L_y but rearranged.

Proposition 4.1.3. *Let L_x and L_y be orthogonal Latin squares. If L_y is linear in a Latin square L_z, then $\varphi_{x,y}$ is an orthomorphism of the quasigroup $(Q^2, +_z)$ (where $(Q, +_z)$ is the quasigroup associated with L_z).*

Proof. Let the quasigroups associated with L_x and L_y be $(Q, \cdot_x)$ and $(Q, \cdot_y)$. Let $(a, b) -_z (c, d) = (e, f)$ if and only if $e +_z c = a,\ f +_z d = b$ (the usual interpretation of '$-$' on Q^2), and take '$-$' to mean '$-_z$'. The claim is that θ, defined by

$$(m, b)\theta = (m \cdot_x b, m \cdot_y b) - (m, b) = (m \cdot_x b - m, m \cdot_y b - b),$$

is a permutation on $Q \times Q$. Now, L_y has the same rows as L_z but rearranged. In terms of the quasigroup operations, $m \cdot_y b$ is the same as $m + b$ up to a permutation on m. In other words $m \cdot_y b = m\tau + b$ for some permutation τ of Q. Thus,

$$(m, b)\theta = (m \cdot_x b, m \cdot_y b) - (m, b) = (m \cdot_x b - m, (m\tau + b) - b) = (m \cdot_x b - m, m\tau).$$

If $(m,b)\theta = (m'b')\theta$ then $(m \cdot_x b - m, m\tau) = (m' \cdot_x b' - m', m'\tau)$ and $m = m'$. But then, if $(m,b)\theta = (m',b')\theta$, from the first component we have

$$\begin{aligned} m \cdot_x b - m &= m' \cdot_x b' - m' \\ &= m \cdot_x b' - m, \qquad (\text{since } m = m') \\ m \cdot_x b &= m \cdot_x b' \qquad (\text{since } (Q,+) \text{ is a quasigroup}) \\ b &= b' \qquad (\text{since } (Q,\cdot_x) \text{ is a quasigroup}). \end{aligned}$$

Hence $(m,b) = (m',b')$, and θ is a permutation. □

Of course, it isn't necessary to have three distinct Latin squares.

Corollary 4.1.4. *Let L_x and L_y be any orthogonal Latin squares. Then $\varphi_{x,y}$ is an orthomorphism of $(Q^2, \cdot_y)$.*

The proof below is just the proof of the last proposition, simplified slightly for the present case.

Proof. Let the quasigroups associated with L_x and L_y be $(Q,\cdot_x)$ and $(Q,\cdot_y)$. The claim is that θ, defined by $(m,b)\theta = ((m,b)\varphi_{x,y})\,/_y(m,b)$, is a permutation on $Q \times Q$.

Now,

$$\begin{aligned} (m,b)\theta &= ((m,b)\varphi_{x,y})\,/_y(m,b) \\ &= (m \cdot_x b, m \cdot_y b)\,/_y(m,b) \\ &= ((m \cdot_x b)/_y m, (m \cdot_y b)/_y b) \\ &= ((m \cdot_x b)/_y m, m)\,. \end{aligned}$$

Hence if $(m,b)\theta = (m',b')\theta$, then $((m \cdot_x b)/_y m, m) = ((m' \cdot_x b')/_y m', m')$, and $m = m'$. Thus we have

$$\begin{aligned} (m \cdot_x b)/_y m &= (m \cdot_x b')/_y m \\ m \cdot_x b &= m \cdot_x b' \\ b &= b', \end{aligned}$$

the first simplification being due to the fact that $(Q, /_y)$ is a quasigroup, the second that $(Q, /_x)$ is.

Hence θ is a permutation. □

Likewise, if we define a left orthomorphism as a permutation φ such that $m\backslash(m\varphi)$ is a permutation, then $\varphi_{x,y}$ is a left orthomorphism with respect to $(Q^2, \cdot_x)$. Restating this:

Proposition 4.1.5. *Let L_x and L_y be orthogonal Latin squares. Then $\varphi_{x,y}$ is a left orthomorphism of $(Q^2, \cdot_x)$.*

Proof. The proof follows the same steps as the previous proof. □

Recall the squares L_0 and L_∞, from Figure 2.6:

L_0

0	1	$\cdots$	$n-1$
0	1	$\cdots$	$n-1$
$\vdots$	$\vdots$		$\vdots$
0	1	$\cdots$	$n-1$

L_∞

0	0	$\cdots$	0
1	1	$\cdots$	1
$\vdots$	$\vdots$		$\vdots$
$n-1$	$n-1$	$\cdots$	$n-1$

Proposition 4.1.6.

1. *In a linear ternary ring $\varphi_{0,x}$ is an orthomorphism with respect to $(R^2, +)$ for any $x \neq 0, \infty$.*

2. *In any ternary ring with 0, $\varphi_{0,1}$ is an orthomorphism and $\varphi_{1,\infty}$ is a left orthomorphism with respect to $(R^2, +)$.*

3. *If L_x is any Latin square, then $\varphi_{0,x}$ is an orthomorphism and $\varphi_{x,\infty}$ is a left orthomorphism with respect to $(Q^2, \cdot_x)$.*

Proof.

1. Linear:

   $((m,b)\varphi_{0,x})/(m,b) = (b, xm+b)/(m,b) = (b/m, (xm+b)/b) = (b/m, xm)$.

   If $(b/m, xm) = (b'/m', xm')$, then $m = m'$ and $b = b'$.

2. Ternary ring with 0,1:

   (a) $((m,b)\varphi_{0,1})/(m,b) = (b, m+b)/(m,b) = (b/m, (m+b)/b) = (b/m, m)$.
   If $(b/m, m) = (b'/m', m')$, then $m = m'$, forcing $b = b'$.

   (b) $(m,b)\backslash((m,b)\varphi_{1,\infty}) = (m,b)\backslash(m+b, m) = (m\backslash(m+b), b\backslash m) = (b, b\backslash m)$.
   If $(b, b\backslash m) = (b', b'\backslash m')$, then $b = b'$ and $m = m'$.

3. Any L_x (with quasigroup operations those of $(Q, \cdot_x)$):

   (a) $((m,b)\varphi_{0,x})/(m,b) = (b, mb)/(m,b) = (b/m, mb/b) = (b/m, m)$.
   If $(b/m, m) = (b'/m', m')$, then $m = m'$ and $b = b'$.

   (b) $(m,b)\backslash((m,b)\varphi_{x,\infty}) = (m,b)\backslash(mb, m) = (m\backslash(mb), b\backslash m) = (b, b\backslash m)$.
   If $(b, b\backslash m) = (b', b'\backslash m')$, then $b = b'$ and $m = m'$.

□

4.2 Characterizations of $\varphi_{x,y}$ as an Orthomorphism

Since the Latin square for each nonzero element x is determined by setting the (m,b) element to be (x,m,b), the Latin square L_1 is the addition table of $(R,+)$ (when R has a 0). If we consider the orthomorphisms $\varphi_{x,y}$ in terms of these Latin squares, then the mapping $(m,b) \mapsto (m,b)\varphi_{x,y} - (m,b)$ is determined by three squares: L_x, L_y, obviously, and L_1 since the operation '$-$' is determined by L_1.

The failure of a particular $\varphi_{x,y}$ to be an othormorphism in a ternary ring with 0 translates to the existence of the following in the Latin squares.

Proposition 4.2.1. *$\varphi_{x,y}$ is not an othormorphism if and only if there exist coordinates $(m,b), (m'b')$ of L_x and L_y, $m \neq m'$ and $b \neq b'$, and rows c, d of L_1 such that the (m,b) and $(m'b')$ entries of L_x are the (c,m) and (c,m') entries of L_1, respectively, and the (m,b) and $(m'b')$ entries of L_y are the (c,b) and (c,b') entries of L_1, respectively. This scenario is illustrated below.*

L_x	b	b'
m	z	
m'		z'

L_y	b	b'
m	w	
m'		w'

L_1	m	m'	b	b'
c	z	z'		
d			w	w'

$m \neq m',\ b \neq b'$

Proof. We prove one direction; the converse is proved similarly. Suppose $\varphi_{x,y}$ is not an orthomorphism, that is, that $\varphi_{x,y} - 1$ fails to be a permutation. Then $\varphi_{x,y} - 1$ fails to be one-to-one and for some $(m, b) \neq (m', b')$ we have

$$(m, b)\varphi_{x,y} - (m, b) = (m', b')\varphi_{x,y} - (m', b') = (c, d),$$

say. Then

$$((x.m, b) - m, (y, m, b) - b) = (c, d) = ((x, m', b') - m', (y, m', b') - b'),$$

so

$$(x, m, b) - m = c = (x, m', b') - m',$$
$$(y, m, b) - b = d = (y, m', b') - b',$$

which implies

$$\begin{aligned}(x,m,b) &= c+m\\ (x,m',b') &= c+m'\\ (y,m,b) &= d+b\\ (y,m',b') &= d+b' \ .\end{aligned}$$

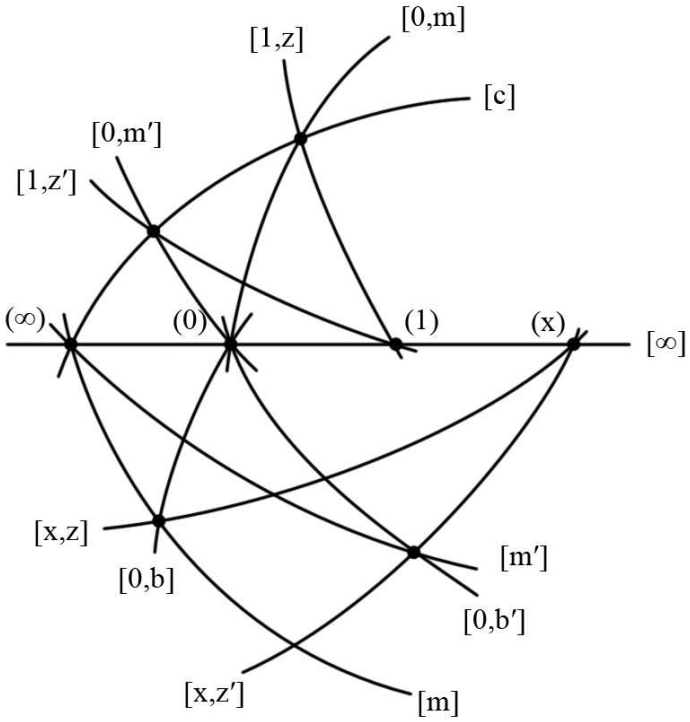
Since, for example, (x,m,b) is the (m,b) entry of L_x, and $c+m$ is the (c,m) entry of L_1, the result follows, once we show we must have $m \neq m'$ and $b \neq b'$.

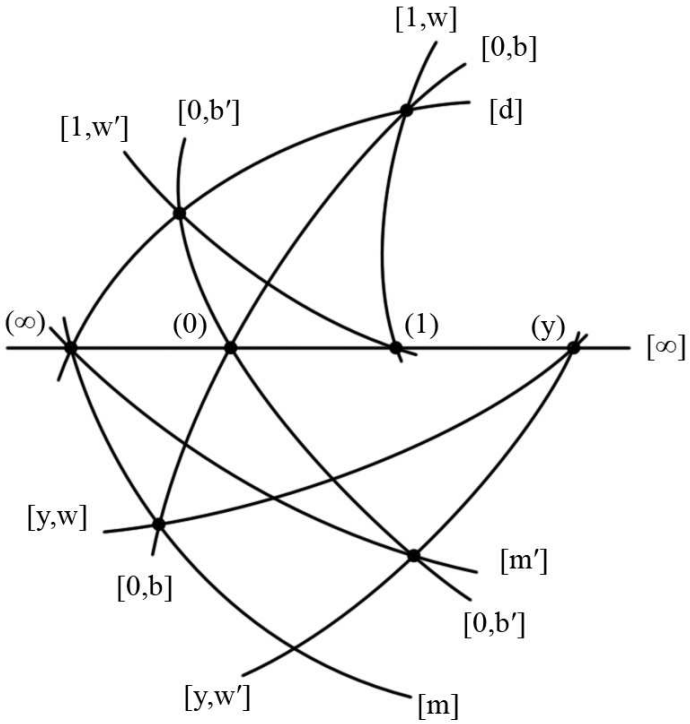
Suppose $m = m'$. Then $(x,m,b) - m = c = (x,m',b') - m'$ becomes $(x,m,b) - m = c = (x,m,b') - m$, so $(x,m,b) = (x,m,b')$. Thus row m will contain the same symbol in column b and b'. If L_x is Latin, we must have $b = b'$, and hence $(m,b) = (m',b')$. This contradicts our choice of $(m,b) \neq (m',b')$. The proof that $b \neq b'$ is similar, using $(y,m,b) - b = d = (y,m',b') - b'$. □

This proposition could be used to give another proof that $\varphi_{x,y}$ is an orthomorphisms for a linear ternary ring. If the proposition holds, then row d of L_1 cannot be any row of L_y. Suppose it is and that it is row d' of L_y. If $d' \neq m$, then the entries of both (m,b) and (d',b) are w, contradicting the fact that L_y is Latin. Hence, $d' = m$. Likewise, looking at column b' we must have $d' = m'$, so $d' = m = m'$, which is a contradiction, since $m \neq m'$. Thus the ternary ring is not linear. Actually, since this only uses the squares L_1 and L_y (and not L_x), we have proven the stronger result of Proposition 4.1.3.

If we translate the conditions of Proposition 4.2.1 into geometric terms we get the following.

Proposition 4.2.2. *φ_{xy} is not an orthomorphism if and only if the following configuration exists, with $m \neq m'$, $b \neq b'$.*





(The two pictures are parts of the same configuration, and are assumed to hold simultaneously.)

Notice the following subtle difference in the two pictures. The lines $[0, b], [0, b']$ are employed both in top and bottom halves of the second picture. The first picture uses $[0, b], [0, b']$ only on the bottom half, and uses two other lines, $[0, m]$ and $[0, m']$, on the top half.

Proof. We reference the Latin squares in Proposition 4.2.1.

Recall that the (m, b) position in Latin square L_x holds symbol z if and only if line $[x, z]$ passes through point (m, b) of the projective plane. In other words, if and only if the three lines $[m], [0, b], [x, z]$ meet at a single point. Likewise, the (m', b') position in Latin square L_x holds symbol z' if and only if $[m'], [0, b'], [x, z']$ meet at single point. This explains the bottom half of the first picture and the corresponding statements for the L_y square explain the bottom half of the second picture.

For the top half of the first picture, we have z in position (c, m) and z' in position (c, m') of square L_1. Thus the lines $[c], [0, m], [1, z]$ must meet in a single point, and $[c], [0, m'], [1, z']$ must meet in a single point.

For the top half of the second picture we must have $[d], [0, b], [1, w]$ meet in a single point and $[d], [0, b'], [1, w']$ in a single point. □

4.3 Hughes Plane Orthomorphisms

In the first section we showed that $\varphi_{x,y}$ is an orthomorphim for every linear ternary ring. Linearity is not a necessary condition, however. The Hughes planes admit no linear ternary ring, but we show that for at least one ternary ring, all $\varphi_{x,y}$ are orthomorphisms.

In this section, as in Example 2.3.11, $(\cdot, \cdot, \cdot)$ denotes an equivalence class of ordered triples, and not a ternary ring. It may be helpful at this point for the reader to review the construction of Hughes planes, Example 2.3.11.

The ternary ring we examine is the ternary ring first constructed from the plane in [Hug57]. For this ternary ring we have the following: the point (∞) is $(0, 0, 1)$ and each (i) is $(1, 0, -i)$ for $i = 0, \ldots, p^{2n} - 1$. The point (u, v) (that is, the intersection of lines $[u]$ and $[0, v]$) is the point $(u, 1, v)$. The ternary ring of the proposition is the one inferred from these labels. We will also need the fact that if $(R, (\ ))$ is a nearfield, then $(R, +)$ is abelian.

Proposition 4.3.1. *For the given ternary ring for the Hughes plane, all φ_{xy} are orthomorphisms ($x \neq y$ and $1 \leq x, y \leq p^{2n-1}$).*

Proof. Suppose some φ_{xy} is not an orthomorphism. Then we have the situation in Propositions 4.2.1 and 4.2.2. Then the (c, b) entry of L_1 is z, and $c+b = z$. Also, that the (m, b) entry of L_x is z means that line $[x, z]$ passes through point (m, b). But the line $[x, z]$ is exactly that line which passes through (x) and $(0, z)$ so we have that $(x), (m, b)$, and $(0, z)$ are collinear, or in other words $(1, 0, -x), (m, 1, b), (0, 1, z)$ all satisfy $x_0\alpha + y_0\beta + z_0\gamma + (x_0\alpha' + y_0\beta' + z_0\gamma')t = 0$.

Thus we have

(1) $(1, 0, -x)$: $\alpha - x\gamma + (\alpha' - x\gamma')t = 0$

(2) $(0, 1, z)$: $\beta + z\gamma + (\beta' + z\gamma')t = 0 = \beta + (c + b)\gamma + (\beta' + (c + b)\gamma')t$ (since $z = c + b$)

(3) $(m, 1, b)$: $m\alpha + \beta + b\gamma + (m\alpha' + \beta' + b\gamma')t = 0$.

We will show that $mx = c$.

Case 1: $t = 1$. Using the facts that the nearfield satisfies left distributivity and the nearfield addition is abelian, we get

$$\begin{aligned} \alpha + \alpha' &= x(\gamma + \gamma') \\ \beta + \beta' &= -(c + b)(\gamma + \gamma') \\ m(\alpha + \alpha') + (\beta + \beta') + b(\gamma + \gamma') &= 0, \end{aligned}$$

Now, if $\gamma = -\gamma'$, then we have $\alpha = -\alpha'$ and $\beta = -\beta'$. The equation for the line would then be $x_0 0 + y_0 0 + z_0 0 = 0$, which every point of the plane would satisfy, meaning all points would be on this line. Clearly this cannot happen, so $\gamma \neq -\gamma'$.

Now, making substitutions,

$$\begin{aligned} m(\alpha + \alpha') + (\beta + \beta') + b(\gamma + \gamma') &= 0, \\ &= mx(\gamma + \gamma') - (c + b)(\gamma + \gamma') + b(\gamma + \gamma') \\ &= mx(\gamma + \gamma') - (\gamma + \gamma')(c + b) + b(\gamma + \gamma') \\ &= mx(\gamma + \gamma') - (\gamma + \gamma')c - (\gamma + \gamma')b + b(\gamma + \gamma') \\ &= (mx - c)(\gamma + \gamma'), \end{aligned}$$

where in several planes we have used the fact that the members of the kernel commute with all members of the nearfield. Since $\gamma \neq -\gamma'$, we have $mx = c$ when $t = 1$.

Case 2: $t \neq 1$. We have

$$\begin{aligned} \alpha + \alpha' t &= x(\gamma + \gamma' t) \\ \beta + \beta' t &= -(c+b)(\gamma + \gamma' t) \\ m(\alpha + \alpha' t) + (\beta + \beta' t) + b(\gamma + \gamma' t) &= 0. \end{aligned}$$

(Here we had to use the associativity of nearfield multiplication in addition to left distributivity and commutativity of nearfield addition.) Substituting and rearranging again,

$$\begin{aligned} 0 &= mx(\gamma + \gamma' t) - (c+b)(\gamma + \gamma' t) + b(\gamma + \gamma' t) \\ &= mx\gamma + mx\gamma' t + (-c-b)\gamma + (-c-b)\gamma' t + b\gamma + b\gamma' t \\ &= \gamma mx + \gamma' mxt + \gamma(-c-b) + \gamma'(-c-b)t + \gamma b + \gamma' bt \\ &= \gamma(mx - c - b + b) + \gamma'(mxt - ct - bt + bt) \\ &= \gamma(mx - c) + \gamma'(mxt - ct), \end{aligned}$$

where, for instance, in the third equality $mx\gamma = \gamma mx$ because $\gamma \in \mathbb{F}$, and all elements of $\mathbb{F}$ commute with all elements of the nearfield (see Example 2.3.11). Now, if both γ and γ' are 0, then the line is again $x_0 0 + y_0 0 + z_0 0 = 0$, so at least one of γ, γ' is not 0.

If $\gamma \neq 0$ then $0 = \gamma(mx - c) + \gamma'(mxt - ct)$ becomes

$$mx - c + \gamma^{-1}\gamma'(mxt - ct) = 0$$

or

$$mx(1 + \gamma^{-1}\gamma' t) = c(1 + \gamma^{-1}\gamma' t).$$

Since $t \neq 1$, i.e., $t \notin F$, and $\gamma^{-1}\gamma' \in F$, we know $\gamma^{-1}\gamma' t \neq -1$. Thus we again have $mx = c$.

If $\gamma' \neq 0$ then

$$(\gamma')^{-1}\gamma(mx - c) + mxt - ct = 0$$

or

$$mx((\gamma')^{-1}\gamma + t) = c((\gamma')^{-1}\gamma + t).$$

Now $t \notin F$, so $(\gamma')^{-1}\gamma \neq -t$. Again, $mx = c$.

Thus in every case we have $mx = c$. Repeating all of these steps for the condition that the (m', b') position of L_x is z' and the (c, b') position of L_1 is z' (i.e., the points $(1, 0, -x), (0, 1, z'), (m', 1, b')$ are collinear and $c + b' = z'$) will show that $m'x = c$. But multiplication in the set of nonzero elements of the nearfield is a loop. Hence either $m = m'$ or one of m, m' is zero. If one of m, m' is zero, then $mx = 0$ or $m'x = 0$, either of which forces the other, so $m = m'$. In either case, then, $m = m'$ which contradicts the condition that $m \neq m'$ in Proposition 4.2.1. □

APPENDIX A. MISCELLANEOUS RESULT

We record here a proposition that does not fit in the main of the thesis, along with the requisite background material, which comes from [BJJ01].

Definition A.0.2. A linear ternary ring $Q = (V, +, \cdot)$, with finite set V is a *prequasifield* if

(1) $(V, +)$ is an Abelian group with identity 0,

(2) $(V - \{0\}, \cdot)$ is a quasigroup,

(3) for all $x \in V,\ x \cdot 0 = 0$,

(4) for all $x, y, z \in V,\ x \cdot (y + z) = x \cdot y + x \cdot z$.

(These conditions imply $0 \cdot x = 0$.)

Definition A.0.3. The *slopeset* of a prequasifield is

$$\tau := \{\tau_a \in \mathrm{Hom}(V, +) | \tau_a : x \mapsto ax,\ a \in V\}.$$

The *external kernel* K of a prequasifield is the centralizer of τ in the ring $\mathrm{Hom}(V, +)$ (the ring of homomorphisms of $(V, +)$).

Theorem A.0.4. *K is a field (in the finite case) and $(V, +)$ is a (right) vector space over K.*

Prequasifield and external kernel of a prequasifield are relatively new generalizations of quasifield and kernel of a quasifield. A quasifield is a prequasifield that satisfies

$(2')$ $(V - \{0\}, \cdot)$ is a loop.

Proposition A.0.5. *If $k \in K$ is identified with the map $x \mapsto xk$, then for a quasifield the external kernel and the kernel coincide.*

We prove the following new proposition. In this proposition x and y represent any two different squares L_x and L_y of the MAXMOLS (without adjuncts), or in other words, any two of the points $(1), (2), \ldots, (n-1)$ on the line $[\infty]$. Let $\varphi_{x,y}$ be the permutation on the set $V \times V$ of a prequasifield given by the Greco-Latin square $L_{x,y}$. The (m, b) coordinate of the Latin square for x has symbol $xm + b$, thus the action of $\varphi_{x,y}$ is $(m, b)\varphi_{x,y} = (xm + b, ym + b)$.

Proposition A.0.6. *Let $(V, +, \cdot)$ be a prequasifield. $\varphi_{x,y}$ is an automorphism of the vector space $(V \times V, +)$ over the external kernel K.*

Proof.

$$\begin{aligned}(m_1 + m_2, b_1 + b_2)\varphi_{x,y} &= (x(m_1 + m_2) + (b_1 + b_2), y(m_1 + m_2) + (b_1 + b_2)) \\ &= (xm_1 + xm_2 + b_1 + b_2, ym_1 + ym_2 + b_1 + b_2) \\ &= (m_1, b_1)\varphi_{x,y} + (m_2, b_2)\varphi_{x,y}.\end{aligned}$$

The second equality follows from identity (4) of Definition A.0.2. Now let $\psi \in K$.

$$\begin{aligned}(m, b)\psi\varphi_{x,y} &= (m\psi, b\psi)\varphi_{x,y} \\ &= (x(m\psi) + b\psi, y(m\psi) + b\psi) \\ &= (m\psi\tau_x + b\psi, m\psi\tau_y + b\psi) \\ &= (m\tau_x\psi + b\psi, m\tau_y\psi + b\psi) \qquad (\text{since } \psi \in K) \\ &= ((xm)\psi + b\psi, (ym)\psi + b\psi) \\ &= ((xm + b)\psi, (ym + b)\psi) \qquad (\text{since } \psi \in \mathrm{Hom}(V, +)) \\ &= (m, b)\varphi_{x,y}\psi.\end{aligned}$$

Thus $\varphi_{x,y}$ is an endomorphism. The Latin squares x, y are orthogonal, so $\varphi_{x,y}$ is a permutation, hence an automorphism. □

APPENDIX B. PROOF FOR SECTION 3.2

B.1 Background

For all of the following, the projective plane under consideration will be the Hall plane of order $q^2 \geq 16$ with kernel $K = \mathbb{F}_q$, the finite field of order q. The points of $[\infty]$ are the elements of the Hall ternary ring $(R, (\ ))$ (constructed using an irreducible polynomial $xf = x^2 - px - q$) plus the point (∞). Points off $[\infty]$ will be called finite points. S_1 is the set of points $(\infty), (0), \ldots, (q-1)$ and S_2 the points $(q), \ldots, (q^2-1)$.

All of the following background material is from [Hug59].

Let $S = S(a, b, c, d)$ be the following mapping of elements $(x, y) \in R$ (elements of R are ordered pairs of elements of $\mathbb{F}_q$):

$$(x, y)S = (ax + by, cx + dy) = (x, y) \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

and let s be the unique element of R such that $sS = 1$. If $x, y, u, v \in R$, $y \neq 0$, $v \neq 0$, $x \notin Ky$, $u \notin Kv$, then there is a unique mapping S such that $xS = u$ and $yS = v$. In other words, if x and y are not K-multiples of each other, then for any u, v which are not K-multiples of each other, there is a unique S such that $x \mapsto u,\ y \mapsto v$. Henceforth, ordered pairs will be ordered pairs of elements of R as labels on finite points.

Let $\sigma = \sigma(S)$ be the mapping

$(x, y) \mapsto (xS, yS)$ $\qquad$ $[m, k] \mapsto [(ms)S, kS]$

$(x) \mapsto ((xs)S)$ $\qquad$ $[k] \mapsto [kS]$

$(\infty) \mapsto (\infty)$ $\qquad$ $[\infty] \mapsto [\infty]$,

and let Σ be the set of all such maps. Then Σ is a collineation group which fixes S_1 pointwise and is transitive on S_2.

Let δ be the mapping

$(x, y) \mapsto (-x, px + y)$ $\quad$ $[m, k] \mapsto [-m + p, k]$

$(x) \mapsto (-x + p)$ $\quad$ $[k] \mapsto [-k]$

$(\infty) \mapsto (\infty)$ $\quad$ $[\infty] \mapsto [\infty]$.

Then δ is a collineation of order 2.

Let $\theta = \theta(a, b)$ for any $a, b \in R$ be the mapping

$(x, y) \mapsto (x + a, y + b)$ $\quad$ $[m, k] \mapsto [m, k + ma + b]$

$(x) \mapsto (x)$ $\quad$ $[k] \mapsto [k + a]$

$(\infty) \mapsto (\infty)$ $\quad$ $[\infty] \mapsto [\infty]$,

and let Θ be the set of all such mappings. Then Θ is a collineation group which is transitive on lines through any point on t, transitive on points not on t and fixes each point of t.

Let a be a member of K^*. Also denote by a the mapping

$(x, y) \mapsto (xa, ya)$ $\quad$ $[m, k] \mapsto [m, ka]$

$(x) \mapsto (x)$ $\quad$ $[k] \mapsto [ka]$

$(\infty) \mapsto (\infty)$ $\quad$ $[\infty] \mapsto [\infty]$,

and let A be the set of all such mappings. Then A is a collineation group.

In addition to these groups, there is a group Λ of collineations that fixes S_2 and $(0, 0)$ pointwise and is transitive on the points of S_1. (This is the group $\mathfrak{L}$ in [Hug59].)

Theorem B.1.1. *For any order finite field, there exists an irreducible polynomial $xf = x^2 - px - q$ with $p \neq 0$.*

Theorem B.1.2. *Any choice of points labeled $(\infty), (0), (1)$ in S_1 lead to a Hall ternary ring.*

Theorem B.1.3. *Any Hall plane of the same order is isomorphic.*

We will use Theorems B.1.1 and B.1.3 together to assume below that $p \neq 0$.

B.2 Proof

Lemma B.2.1. *For any two finite points $(0, 0), (a, b)$, the line $\overline{(0,0)(a,b)}$ intersects $[\infty]$ in S_1 if and only if a and b are K-multiples. Furthermore, $\overline{(0,0)(a,b)} = [0]$ if $a = 0$, and $\overline{(0,0)(a,b)} = [m, 0]$ if $-ma = b$, $a \neq 0$.*

Proof. We have $a = 0$ (so $a = 0b$) if and only if $\overline{(0,0)(0,b)}$ is the line $[0]$, which intersects $[\infty]$ at $(\infty) \in S_1$.

If $a \neq 0$ then $\overline{(0,0)(a,b)}$ is some line $[m,k]$. Then $m0 + 0 = k$ and $ma + b = k$. The first equation implies $k = 0$, so that $ma = -b$ or $-ma = b$. Now, the intersection of $[m,k]$ with $[\infty]$ is (m), which is in S_1 if and only if $m \in K$. □

Lemma B.2.2. *The collineation group is doubly transitive on finite points $(a,b),(c,d)$ such that the line $\overline{(a,b)(c,d)}$ intersects $[\infty]$ in S_1.*

Proof. Let $(a,b),(c,d)$ and $(a',b'),(c',d')$ be two pairs of finite points such that $\overline{(a,b)(c,d)}$ and $\overline{(a',b')(c',d')}$ intersect $[\infty]$ in S_1. We need to show that there is some collineation such that $(a,b) \mapsto (a',b')$ and $(c,d) \mapsto (c',d')$.

We may use $\theta(-c,-d)$ to transform the first pair of points to $(a-c,b-d),(0,0)$ and $\theta(-c',-d')$ to transform the second pair to $(a'-c',b'-d'),(0,0)$. Then if we find a collineation γ which maps $(a-c,b-d),(0,0)$ to $(a'-c',b'-d'),(0,0)$, the collineation which maps $(a,b),(c,d)$ to $(a',b'),(c',d')$ is $\theta(-c,-d)\gamma\theta^{-1}(-c',-d')$. Hence, without loss of generality we assume $(c,d) = (c',d') = (0,0)$.

Case 1: $a = 0,\ a' = 0$ (hence $b \neq 0$ and $b' \neq 0$)

If $b' = kb$ for some $k \in K^*$, then use collineation $k \in A$ to map $(0,b) \mapsto (0,kb)$ and $(0,0) \mapsto (0,0)$.

If $b' \neq kb$ for any $k \in K^*$, thus use a collineation $\sigma(S) \in \Sigma$ to map $(0,b) \mapsto (0,b')$ and $(0,0) \mapsto (0,0)$.

Case 2: $a = 0,\ a' \neq 0$

The line $\overline{(0,0),(a',b')}$ intersects $[\infty]$ in some point (m). Since Λ is transitive on points of S_1 while fixing $(0,0)$, there is a collineation $\lambda \in \Lambda$ which maps (m) to (∞), and hence maps $(0,0) \mapsto (0,0)$ and $(a',b') \mapsto (a'',b'') = (0,b'')$ (if $\overline{(0,0)(a'',b'')}$ is a line through (∞), then it must be $[0]$ and hence $a'' = 0$). Now apply case 1, and the desired collineation is the composition of λ with the collineation in case 1.

Case 3: $a \neq 0, a' = 0$

Use $\lambda \in \Lambda$ to map $(a, b) \mapsto (0, b'')$ and $(0, 0) \mapsto (0, 0)$ as in the last case. Again, this puts us in case 1.

Case 4: $a \neq 0, a' \neq 0$

Use $\lambda_1 \in \Lambda$ to map $(a, b) \mapsto (0, b'')$ and $(0, 0) \mapsto (0, 0)$ and $\lambda_2 \in \Lambda$ to map $(a', b') \mapsto (0, b''')$ and $(0, 0) \mapsto (0, 0)$. Again, from case 1 there is a γ which maps $(0, b'') \mapsto (0, b''')$ and $(0, 0) \mapsto (0, 0)$, and the desired collineation is $\lambda_1 \gamma \lambda_2^{-1}$.

□

Lemma B.2.3. *The collineation group is doubly transitive on the points of S_2.*

Proof. Let (x), (y) be two distinct points of S_2. We wish to show that if $(x'), (y')$ are any two distinct points of S_2, then there is a collineation mapping $(x) \mapsto (x')$, $(y) \mapsto (y')$.

Claim: If $x', y' \in R$ are K-multiples, $x' \neq y'$, then $-x' + p, -y' + p$ are not K-multiples for $p \in K^*$.

Let $kx' = y'$ ($k \neq 1$ since $x' \neq y'$) and let x', y' be written as ordered pairs of elements of K as $x' = (x'_1, x'_2), y' = (y'_1, y'_2)$. Suppose $-x' + p, -y' + p$ are K-multiples, say $-k'x' + k'p = -y' + p$ ($k' \neq 1$ since $x' \neq y'$). Then $k'x' = y' + (k' - 1)p$ and $k'x'$ and y' differ by an element of $K = \mathbb{F}_q$. Hence $(k'x'_1, k'x_2) = (y'_1 + (k' - 1)p, y'_2)$, so $k'x'_2 = y'_2$. This implies $k' = k$, since $kx'_2 = y'_2$, which implies $p = 0$ or $k' = 1$, which is a contradiction.

Case 1: x, y are not K-multiples.

If x', y' are not K-multiples, then there is an S such that $x \mapsto x',\ y \mapsto y'$, hence $\sigma(S)$ is the needed collineation. If x', y' are K-multiples, then $-x' + p, -y' + p$ are not. With $S : x \mapsto -x' + p,\ y \mapsto -y' + p$, the needed collineation is $\sigma(S)\delta$.

Case 2: x, y are K-multiples. Then $-x + p, -y + p$ are not, so we apply the above collineations in Case 1 after applying δ:

If x', y' are not K-multiples, then there is an S such that $-x+p \mapsto x',\ -y+p \mapsto y'$, hence $\delta\sigma(S)$ is the needed collineation. If x', y' are K-multiples, then $-x'+p, -y'+p$ are not. With $S : -x+p \mapsto -x'+p,\ -y+p \mapsto -y'+p$, the needed collineation is $\delta\sigma(S)\delta$.

□

Proposition B.2.4. *There are exactly 10 equivalence classes for each Hall plane of order at least 16:*

1. (∞), (0) *in* S_1	*6.* (0) *in* S_1, (∞) *off* t
2. (∞), (0) *in* S_2	*7.* (∞) *in* S_2, (0) *off* t
3. (∞) *in* S_1, (0) *in* S_2	*8.* (0) *in* S_2, (∞) *off* t
4. (0) *in* S_1, (∞) *in* S_2	*9.* (∞), (0) *off* t, $[\infty] \cap t$ *in* S_1
5. (∞) *in* S_1, (0) *off* t	*10.* (∞), (0) *off* t, $[\infty] \cap t$ *in* S_2

Proof. The discussion in Section 3.2 shows that there are at least 10 equivalence classes. We will show that there are no more than 10 by showing that if P_1, P_2 and P_3, P_4 are two pairs of points that satisfy the descripition of a class above, then there is a collineation γ such that $P_1\gamma = P_3$ and $P_2\gamma = P_4$.

1. This follows from Theorems B.1.2 and B.1.3.

2. This follows from Lemma B.2.3.

3 & 4. Let P_1, P_3 be any points in S_1. Λ (the collineation group from Section B.1) is transitive on S_1 (and fixes S_2 pointwise), so let $\lambda \in \Lambda$ be such that $P_1\lambda = P_3$. Let P_2, P_4 be any points in S_2. Let $\sigma \in \Sigma$ be such that $P_2\sigma = P_4$. Then for $\gamma := \lambda\sigma$, $\gamma : P_1 \mapsto P_3; P_2 \mapsto P_4$.

5 & 6. Let P_1, P_3 be any points in S_1. Let $\lambda \in \Lambda$ be such that $P_1\lambda = P_3$. Let P_2, P_4 be any points not on t. Let $\theta \in \Theta$ be such that $P_2\theta = P_4$ (recall Θ is transitive on finite points while fixing every point of t). Then for $\gamma := \lambda\theta$, $\gamma : P_1 \mapsto P_3; P_2 \mapsto P_4$.

7 & 8. Similar to 5 & 6, using Σ in place of Λ.

9. This follows from Lemma B.2.2.

10. Let P_1, P_2 be labeled $(a, b), (c, d)$. Map $(a, b), (c, d)$ to $(0, 0), (c-a, d-b)$ by $\theta(-a, -b)$. By Lemma B.2.1, $c-a$ and $d-b$ are not K-multiples, so may be mapped to any other points $c' - a', d' - b'$ which are not K-multiples. Now use $\theta(a', b')$ to map $(0, 0), (c' - a', d' - b')$ to $(a', b'), (c', d')$, the labels of P_3, P_4.

□

APPENDIX C. MAXMOLS OF ORDER NINE USED IN [OP95]

C.1 The MAXMOLS $\mathcal{M}_8$

L_1

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	4	5	6	7	8	0	1	2
4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	2	0	1
6	7	8	0	1	2	3	4	5
7	8	6	1	2	0	4	5	3
8	6	7	2	0	1	5	3	4

L_2

0	1	2	3	4	5	6	7	8
2	0	1	5	3	4	8	6	7
1	2	0	4	5	3	7	8	6
6	7	8	0	1	2	3	4	5
8	6	7	2	0	1	5	3	4
7	8	6	1	2	0	4	5	3
3	4	5	6	7	8	0	1	2
5	3	4	8	6	7	2	0	1
4	5	3	7	8	6	1	2	0

L_3

0	1	2	3	4	5	6	7	8
6	5	4	0	7	8	3	2	1
3	8	7	6	2	1	0	4	5
1	2	0	8	6	4	5	3	7
5	4	6	1	3	7	8	0	2
8	7	3	5	0	2	1	6	4
2	0	1	7	5	6	4	8	3
4	6	5	2	8	3	7	1	0
7	3	8	4	1	0	2	5	6

L_4

0	1	2	3	4	5	6	7	8
8	3	5	7	0	6	2	1	4
4	7	6	1	8	2	5	3	0
7	6	4	2	3	0	8	5	1
1	2	0	5	7	8	4	6	3
3	5	8	6	1	4	0	2	7
5	8	3	4	2	7	1	0	6
6	4	7	0	5	1	3	8	2
2	0	1	8	6	3	7	4	5

L_5

0	1	2	3	4	5	6	7	8
7	4	3	8	6	0	1	5	2
5	6	8	2	1	7	4	0	3
4	3	7	5	8	1	2	6	0
6	8	5	0	2	4	3	1	7
1	2	0	7	3	6	8	4	5
8	5	6	1	0	3	7	2	4
2	0	1	4	7	8	5	3	6
3	7	4	6	5	2	0	8	1

L_6

0	1	2	3	4	5	6	7	8
3	8	7	6	2	1	0	4	5
6	5	4	0	7	8	3	2	1
2	0	1	7	5	6	4	8	3
7	3	8	4	1	0	2	5	6
4	6	5	2	8	3	7	1	0
1	2	0	8	6	4	5	3	7
8	7	3	5	0	2	1	6	4
5	4	6	1	3	7	8	0	2

L_7

0	1	2	3	4	5	6	7	8
5	6	8	2	1	7	4	0	3
7	4	3	8	6	0	1	5	2
8	5	6	1	0	3	7	2	4
3	7	4	6	5	2	0	8	1
2	0	1	4	7	8	5	3	6
4	3	7	5	8	1	2	6	0
1	2	0	7	3	6	8	4	5
6	8	5	0	2	4	3	1	7

L_8

0	1	2	3	4	5	6	7	8
4	7	6	1	8	2	5	3	0
8	3	5	7	0	6	2	1	4
5	8	3	4	2	7	1	0	6
2	0	1	8	6	3	7	4	5
6	4	7	0	5	1	3	8	2
7	6	4	2	3	0	8	5	1
3	5	8	6	1	4	0	2	7
1	2	0	5	7	8	4	6	3

C.2 The MAXMOLS $\mathcal{M}_{14}$

L_1

0	1	2	3	4	5	6	7	8
2	0	1	4	5	3	7	8	6
1	2	0	5	3	4	8	6	7
4	3	5	8	6	7	1	0	2
5	4	3	6	7	8	0	2	1
3	5	4	7	8	6	2	1	0
8	7	6	2	1	0	4	5	3
6	8	7	1	0	2	5	3	4
7	6	8	0	2	1	3	4	5

L_2

0	1	2	3	4	5	6	7	8
1	2	0	5	3	4	8	6	7
2	0	1	4	5	3	7	8	6
7	6	8	1	0	2	5	3	4
6	8	7	2	1	0	4	5	3
8	7	6	0	2	1	3	4	5
5	4	3	7	8	6	2	1	0
4	3	5	6	7	8	0	2	1
3	5	4	8	6	7	1	0	2

L_3

0	1	2	3	4	5	6	7	8
6	4	5	0	7	8	3	1	2
3	7	8	6	1	2	0	4	5
1	2	0	4	8	6	7	5	3
4	5	6	7	2	3	1	8	0
7	8	3	1	5	0	4	2	6
2	0	1	5	6	7	8	3	4
5	6	4	8	3	1	2	0	7
8	3	7	2	0	4	5	6	1

L_4

0	1	2	3	4	5	6	7	8
8	5	3	7	0	6	1	2	4
4	6	7	2	8	1	5	3	0
6	7	4	0	1	3	2	8	5
1	2	0	8	5	7	3	4	6
5	3	8	4	6	2	7	0	1
3	8	5	1	7	4	0	6	2
7	4	6	5	2	0	8	1	3
2	0	1	6	3	8	4	5	7

L_5

0	1	2	3	4	5	6	7	8
7	3	4	8	6	0	2	5	1
5	8	6	1	2	7	4	0	3
3	4	7	2	5	8	0	1	6
8	6	5	4	0	1	7	3	2
1	2	0	6	7	3	5	8	4
6	5	8	0	3	2	1	4	7
2	0	1	7	8	4	3	6	5
4	7	3	5	1	6	8	2	0

L_6

0	1	2	3	4	5	6	7	8
3	8	7	6	2	1	0	4	5
6	5	4	0	7	8	3	2	1
2	0	1	5	3	4	8	6	7
7	3	8	1	6	2	5	0	4
4	6	5	8	0	7	1	3	2
1	2	0	4	5	3	7	8	6
8	7	3	2	1	6	4	5	0
5	4	6	7	8	0	2	1	3

L_7

0	1	2	3	4	5	6	7	8
5	6	8	2	1	7	4	0	3
7	4	3	8	6	0	1	5	2
8	5	6	7	2	1	3	4	0
3	7	4	0	8	6	2	1	5
2	0	1	5	3	4	8	6	7
4	3	7	6	0	8	5	2	1
1	2	0	4	5	3	7	8	6
6	8	5	1	7	2	0	3	4

L_8

0	1	2	3	4	5	6	7	8
4	7	6	1	8	2	5	3	0
8	3	5	7	0	6	2	1	4
5	8	3	6	7	0	4	2	1
2	0	1	5	3	4	8	6	7
6	4	7	2	1	8	0	5	3
7	6	4	8	2	1	3	0	5
3	5	8	0	6	7	1	4	2
1	2	0	4	5	3	7	8	6

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