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ON THE EFFECTS OF SPIN, SYMMETRY, AND POLARIZATION WHEN AN ISOSINGLET RESONANCE DECAYS TO FOUR PIONS

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Terence John Nelson

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I. INTRODUCTION

The purpose of this paper is to provide a phenomenological treatment of the decay of an isosinglet resonance into four pions. This problem is also under active investigation by Dotson, Nyborg, and Good (1) and by Weller and Dotson[a]. Pais (2) has provided a general treatment of states of many pions. A special result is that states of four pions that observe Bose statistics and have isospin zero divide into two symmetry types.

There are, in general, eight phase-space variables which characterize the four-particle decay problem. Three of the variables may be interpreted as Euler angles which orient the decay products, the other five can be taken as relativistically invariant masses of combinations of the particles. An eight-dimensional distribution would require a large number of events for meaningful comparison with experiment. Nor is it desirable to calculate merely isotropic distributions, which don't depend on the Euler angles, because in certain situations anisotropy may be the most sensitive test of the quantum numbers of the resonance. Even isotropic distributions in five variables are not likely to be practical. In this work distributions will be calculated for both choices of parity and symmetry type for spins zero and one. The scalar distributions are necessarily isotropic and will be three-dimensional after integrations are carried out over two of the invariant masses. The vector distributions are calculated for the unpolarized decaying resonance and for the case when the resonance is in a pure state of spin-projection zero along an external direction. The reason for calculating the vector distributions in this way is developed below. The

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[a] Weller, G. and Dotson, A. Western Michigan University, Kalamazoo, Michigan. Private communication. 1967.

zero spin-projection vector distributions retain two of the Euler angles and integrations are performed over the same two invariant masses as for the scalar distributions. The vector distributions will be exhibited in five-dimensional form. All the distributions to be calculated possess a certain flexibility in that only two of the remaining invariant mass variables require numerical techniques if integration is performed on them. In some cases even these integrations do not need to be actually performed because of symmetry in the distributions. The other variables are all angles, one of the remaining three invariant masses being replaced by an angle between two planes defined by pairs of the momenta.

Multipion resonances have been observed in proton-antiproton annihilation. Kallen (3) has reviewed the theoretical and experimental situation of the ω-particle, which is a three-pion effect. Since the effect is observed in a five-pion final state, it is reasonable to assume that the ω decays from an unpolarized state. There are only two independent invariant mass variables in the three particle decay. The two-dimensional distributions can be plotted in a triangle such that the distances to the three sides are linearly related to the energies of the three pions. There is a kinematically determined boundary inscribed in the triangle and the triangle plus inscribed boundary are known as a Dalitz diagram. A discussion of Dalitz diagrams is also given by Hagedorn (4).

The effect is only observed in the neutral combination $\pi^+\pi^-\pi^o$. Its isotopic spin is therefore taken to be zero. Since the isotopic spin of a pion is one, there is only one way to couple the three pions to isotopic spin zero, and that is completely antisymmetric in the three sets of isotopic spin labels. Since pions are spinless Bosons, the resonance decays into a state that is completely antisymmetric in the three momenta. A T-matrix is then conjectured, for each spin and parity, to be the amplitude for the state of three pions simulating the quantum numbers of the decaying resonance to be observed with a prescribed labelling of charge and momentum of the pions. Kallen apologizes for taking the simplest possible T-matrix in each case but in such a phenomenological calculation one should not do otherwise except for a good reason. These simple phenomenological T-matrices resulted in two-dimensional distributions that identified the ω-particle as a vector meson. The success of this approach encourages a similar attempt for resonances that decay to four pions.

Indeed four-pion effects are not lacking. Kernan, Lyon, and Crawley (5) have observed an enhancement at 1610 MeV in the neutral combination $2\pi^+2\pi^-$ in the five-pion final state of proton-antiproton annihilation. The enhancement is not seen in the charged combinations, indicating that its isotopic spin is zero. A curious and potentially profitable feature of this experiment is that the resonance plus pion system is preferentially aligned with the beam direction. At forward or backward production of the resonance, conservation of angular momentum requires the spin-projection of the resonance along the beam direction to be plus one, zero, or minus one. Of course if the resonance is spinless, the distribution of pions will be isotropic in the rest frame of the resonance. But if the resonance has spin greater than zero, and if production takes place equally for each of the four spin states in the initial state, then the relative weights for spin-projections along the beam direction (plus one/zero/minus one) would be (one/two/one). Thus calculating the unpolarized vector distribution and the distribution for zero spin-projection along an external direction permits the sum of the two distributions to be compared with this simplest form of anisotropy in the decay. If the effect is observed in final states of six or more pions, the unpolarized distributions should be compared with experiment.

The phenomenological treatment of the decay of an isotopic singlet resonance to four pions is complicated in two ways. There are five invariant masses in the four pion system compared to only two in the three-pion system. Moreover there are three different ways to couple four isotopic spin one systems to isotopic spin zero, corresponding to intermediate couplings pairwise to isotopic spin zero, one, or two. The next two sections clarify these problems.

II. FOUR-PARTICLE PHASE SPACE

A. The Goldhaber Triangle

In the decay to four particles, there are only five independent invariants of the form:

$$q_i \circ q_j = \omega_i \omega_j - \vec{k}_i \circ \vec{k}_j \tag{2a.1}$$

because of conservation of energy and momentum. Equation (2a.1) introduces the present notation which uses q_i, $\vec{k}_i$, and ω_i for the four-momentum, three-momentum, and energy, respectively, of particle i. Most of the calculations to follow are carried out in the rest frame of the decaying resonance, which is the zero-momentum frame of the four particles. In that frame, squaring the equation of conservation of four-momentum results in:

$$\omega^2 = \sum_{i=1}^{4} m_i^2 + 2 \sum_{i<j}^{4} q_i \circ q_j \quad . \tag{2a.2}$$

This limits the independent invariants to five, where ω is the total energy, or rest energy of the resonance. The invariant masses are combinations of the form:

$$m_{ij}^2 = (q_i + q_j)^2$$

$$m_{ijk}^2 = (q_i + q_j + q_k)^2 \quad . \tag{2a.3}$$

Later on, the distributions will be calculated using the four invariant masses m_{12}^2, m_{34}^2, m_{124}^2, and m_{134}^2. All of these are independent of the angle between the plane defined by $\vec{k}_1$ and $\vec{k}_2$ and the plane defined by $\vec{k}_3$ and $\vec{k}_4$. That angle, denoted by $\phi_{(12)3}$, will be used instead of a fifth independent mass variable. This type of description has been discussed with reference to phase space by Nyborg, Song, Kernan, and Good (6) and with reference to $\phi_{(12)3}$ dependence for several spin and parity assignments by Nyborg and Skjeggestad (7). Anticipating that integrations will be performed over the variables m^2_{124} and m^2_{134}, it is of interest to see what is the kinematically allowed region in the m_{12} m_{34}-plane. Using $\vec{k}_{12}$ for the sum of the momenta of particles 1 and 2, it is easy to see that the following conditions must be satisfied:

$$m_{12} \geq m_1 + m_2$$

$$m_{34} \geq m_3 + m_4$$

$$[k^2_{12} + m^2_{12}]^{1/2} + [k^2_{12} + m^2_{34}]^{1/2} = \omega \quad . \tag{2a.4}$$

These three conditions limit the allowed region to a right triangle known as the Goldhaber triangle (8, 9). The allowed region is the shaded area of Figure 1. It will be seen that on the diagonal boundary $\vec{k}_{12}$ vanishes while particles 1 and 2 are at rest with respect to one another along the horizontal boundary and similarly particles 3 and 4 have the same rest frame along the vertical boundary.

B. Distribution in Particle Momenta

The object of dynamical theories is to calculate an S-matrix which is the amplitude for finding the system in the final state f at the end of the reaction when it was known to be in the initial state i at the beginning. A T-matrix is defined from the S-matrix by the relation:

$$\langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4\, \delta^4(q_f - q_i)\, N_f N_i\, \langle f|T|i\rangle \quad . \tag{2b.1}$$

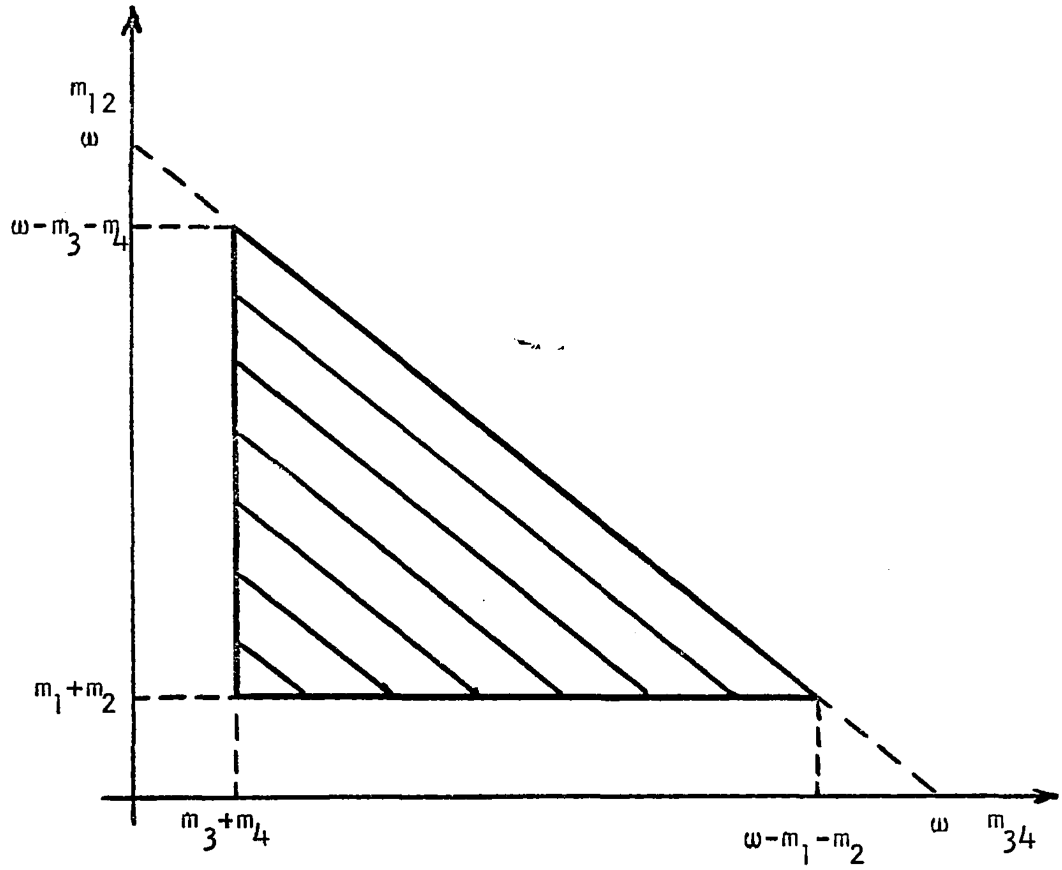

Figure 1. The Goldhaber Triangle

The object in defining the T-matrix is to pull out the four-dimensional delta function expressing conservation of energy and momentum. If the state f contains n particles, the normalization factor N_f is defined by:

$$N_f = \prod_{\alpha=1}^{n} (2V\omega_\alpha)^{-1/2} \tag{2b.2}$$

where V is a fictitious volume used in normalizing single-particle plane waves. The product ranges over all the particles in the state f, and the normalization factor N_i is a similar product ranging over the particles in the initial state.

The probability of observing the final state f is the absolute magnitude squared of the S-matrix. The square of the delta function is handled by remembering that $(2\pi)^4\delta^4(q_f-q_i)$ is the integral over the volume and duration of the reaction of $e^{i(q_f-q_i)\cdot x}$. Therefore the probability per unit volume and unit time of observing the final state $f \neq i$ is:

$$\begin{aligned}&(\text{prob } i \rightarrow f)/\text{unit vol.-unit time}\\ &\qquad = (2\pi)^4\, \delta^4(q_f-q_i)\; N_f^2 N_i^2\; |\langle f|T|i\rangle|^2 \quad .\end{aligned} \tag{2b.3}$$

In the present problem a resonance r decays to four pions. The probability of one resonance decaying, per unit time, is obtained by dividing by the number of resonances per unit volume 1/V and summing over all final states. There are two parts to this sum over final states. The first part is performed as an integration over d^3k_i weighted by the density of states $V/(2\pi)^3$ for each pion. The second part is performed as a sum over different final states leading to the charge configurations $2\pi^+2\pi^-$, $\pi^+\pi^-2\pi^0$, and $4\pi^0$. Thus the probability of decay per unit time is:

$$W = \frac{1}{(2\pi)^8 2\omega} \iiiint \frac{d^3k_1}{2\omega_1} \frac{d^3k_2}{2\omega_2} \frac{d^3k_3}{2\omega_3} \frac{d^3k_4}{2\omega_4} \cdot$$

$$\cdot\ \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)\ \delta(\omega - \omega_1 - \omega_2 - \omega_3 - \omega_4) \cdot$$

$$\cdot \left[\begin{array}{l} \sum_f |\langle f;\pi^+(\vec{k}_1),\pi^+(\vec{k}_2),\pi^-(\vec{k}_3),\pi^-(\vec{k}_4)|T|r\rangle|^2 \\ + \sum_f |\langle f;\pi^+(\vec{k}_1),\pi^-(\vec{k}_2),\pi^0(\vec{k}_3),\pi^0(\vec{k}_4)|T|r\rangle|^2 \\ + \sum_f |\langle f;\pi^0(\vec{k}_1),\pi^0(\vec{k}_2),\pi^0(\vec{k}_3),\pi^0(\vec{k}_4)|T|r\rangle|^2 \end{array} \right] . \qquad (2b.4)$$

This method of writing the decay probability avoids an arbitrary labelling of the pions with subscripts by associating the four vectors with prescribed charge states. The branching ratios $2\pi^+2\pi^-/\pi^+\pi^-2\pi^0/4\pi^0$ will be calculated when the isotopic spin coupling problem is investigated. The twelve-fold distribution is written neglecting the factor $(2\pi)^8(2\omega)$ since the distribution is unnormalized anyway. Thus the twelve-fold distribution in particle momenta, for $2\pi^+2\pi^-$ for example, is:

$$d^{12}R(4) = \frac{d^3k_1}{2\omega_1} \frac{d^3k_2}{2\omega_2} \frac{d^3k_3}{2\omega_3} \frac{d^3k_4}{2\omega_4} \cdot$$

$$\cdot\ \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\ \delta(\omega-\omega_1-\omega_2-\omega_3-\omega_4) \cdot$$

$$\cdot \sum_f |\langle f;\pi^+(\vec{k}_1),\pi^+(\vec{k}_2),\pi^-(\vec{k}_3),\pi^-(\vec{k}_4)|T|r\rangle|^2 . \qquad (2b.5)$$

C. *Distribution in Angles and Invariant Mass Variables*

The twelve-fold distribution in the particle momenta contains delta functions for conservation of energy and momentum. The first integration to be performed is over d^3k_4 which uses up the delta function $\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$

replacing $\vec{k}_4$ by $-\vec{k}_1-\vec{k}_2-\vec{k}_3$:

$$d^9\, R(4) = \frac{d^3k_1}{2\omega_1}\ \frac{d^3k_2}{2\omega_2}\ \frac{d^3k_3}{2\omega_3}\ \frac{1}{2\omega_4}\ \delta(\omega-\omega_1-\omega_2-\omega_3-\omega_4)\cdot$$

$$\cdot\sum_f |\langle T\rangle|^2 \,/\, \vec{k}_4 = -\vec{k}_1-\vec{k}_2-\vec{k}_3 \quad . \qquad (2c.1)$$

The energy delta function can be used by expressing the differential volume d^3k_3 in a particular coordinate frame defined by the vectors $\vec{k}_1$ and $\vec{k}_2$. Figure 2 shows this coordinate frame, which is called the body frame because the Euler angles which orient it with respect to a spatial coordinate frame will be discussed. In this frame d^3k_3 is $k_3^2dk_3\ d(\cos\theta_{(12)3})\ d\phi_{(12)3}$. At constant $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ one has:

$$\omega_4^{\ 2} = m_4^{\ 2} + k_{12}^{\ 2} + k_3^{\ 2} + 2k_{12}k_3\ \cos\theta_{(12)3}$$

$$\omega_4 d\omega_4 = k_{12}k_3 d(\cos\theta_{(12)3}) \quad . \qquad (2c.2)$$

Performing the integration over ω_4 leads to:

$$d^8R(4) = \frac{d^3k_1}{2\omega_1}\ \frac{d^3k_2}{2\omega_2}\ \frac{k_3dk_3}{2\omega_3}\ \frac{1}{2k_{12}}\ d\phi_{(12)3}\ \cdot$$

$$\cdot\theta([2k_{12}k_3]^2 - [(\omega-\omega_1-\omega_2-\omega_3)^2 - m_4^2-k_{12}^2-k_3^2]^2)\cdot$$

$$\cdot\ \sum_f |\langle T\rangle|^2 / \vec{k}_4 = -\vec{k}_1-\vec{k}_2-\vec{k}_3$$

$$\omega_4 = \omega-\omega_1-\omega_2-\omega_3 \qquad (2c.3)$$

The θ function in equation (2c.3) is unity if its argument is positive and zero otherwise. It occurs in equation (2c.3) because ω_4^2 must exceed $m_4^{\ 2} + (k_{12}-k_3)^2$ and must be less than $m_4^{\ 2} + (k_{12}+k_3)^2$ as can be seen from

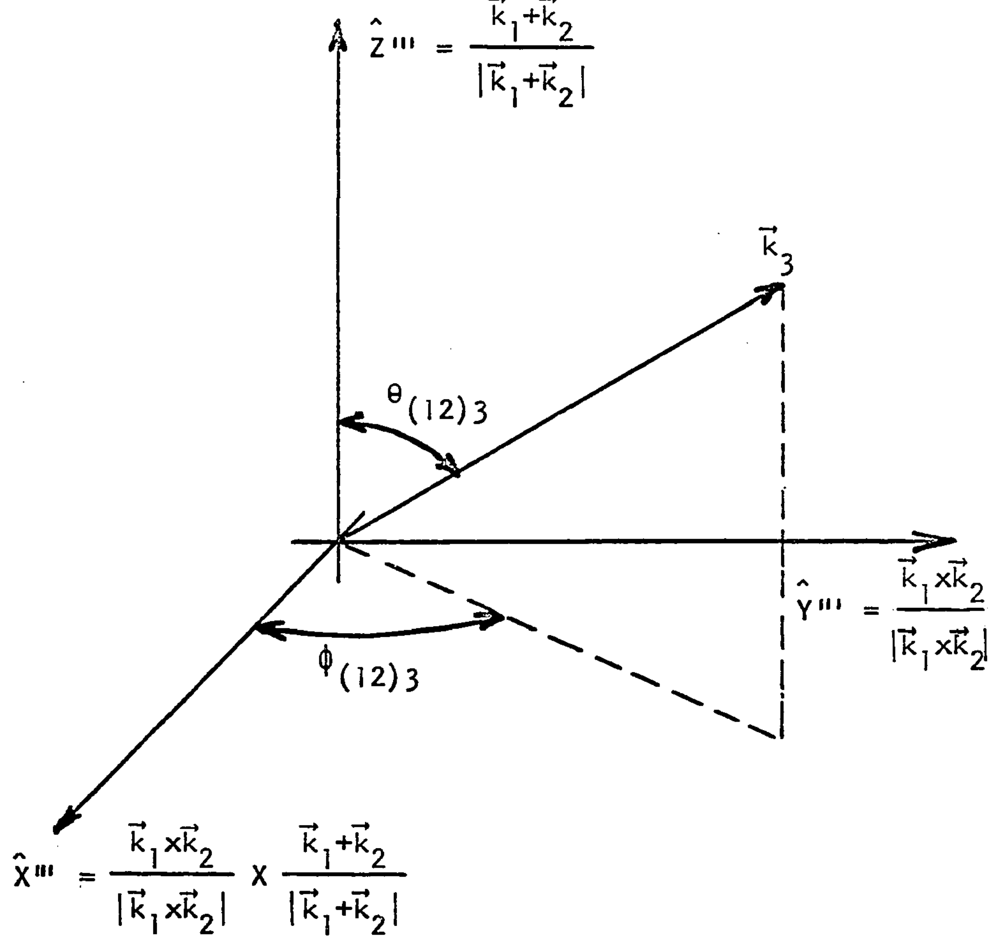

Figure 2. The body coordinate frame

equation (2c.2). Because a constant positive factor of 4 is of no consequence in the argument of the step-function, equation (2c.3) may be re-written as:

$$d^8R(4) = \frac{d^3k_1}{2\omega_1}\ \frac{d^3k_2}{2\omega_2}\ \frac{k_3dk_3}{2\omega_3}\ (\frac{1}{2k_{12}})\ d\phi_{(12)3}\cdot$$
$$\cdot\theta(k_{12}^2k_3^2\sin^2\theta_{(12)3})\cdot$$
$$\cdot\sum_f|\langle T\rangle|^2\ /\vec{k}_4 = -\vec{k}_1-\vec{k}_2-\vec{k}_3$$
$$\omega_4 = \omega-\omega_1-\omega_2-\omega_3 \quad . \tag{2c.4}$$

The boundary condition expressed by the step-function is important and will be investigated in detail below.

The next revision of equation (2c.4) is to re-express the differential volume $d^3k_1d^3k_2$. Introducing the linear combinations:

$$\vec{k}_{12} = \vec{k}_1 + \vec{k}_2$$
$$\vec{K}_{12} = -\vec{k}_1 + \vec{k}_2 \tag{2c.5}$$

the Jacobian of the transformation defined by equation (2c.5) is easily computed and shows that:

$$d^3k_1d^3k_2 = \frac{1}{8}\ d^3k_{12}d^3K_{12} \quad . \tag{2c.6}$$

The differential volume d^3K_{12} will be expressed in the doubly primed coordinate frame shown in Figure 3. The Z''-axis is taken in the direction of $\vec{k}_{12}$. The doubly primed coordinate frame is achieved by rotating the spatial frame through α about the Z-axis and then through β about the Y'-axis. The angles α and β are obviously polar angles for $\vec{k}_{12}$. Now $\vec{K}_{12}$ lies

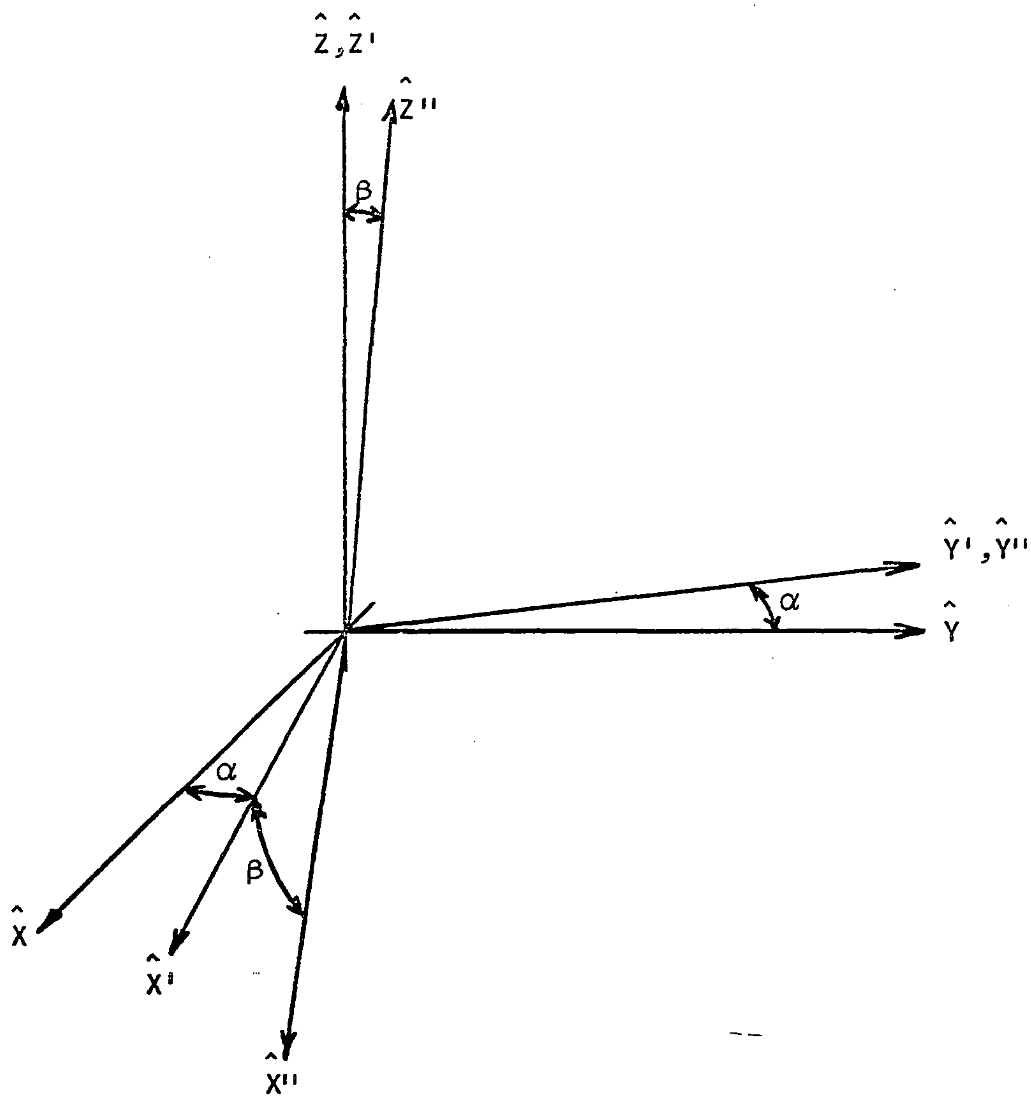

Figure 3. The coordinate frame chosen for d^3K_{12}

in the $X'''Z'''$-plane in the first quadrant. Thus the azimuthal angle γ of $\vec{K}_{12}$ in the doubly primed coordinate frame is just the angle which that frame must be rotated, about the Z''-axis, to achieve coincidence with the body coordinate frame of Figure 2. Thus α, β, and γ are Euler angles that orient the body frame with respect to the spatial frame and:

$$d^3k_1 d^3k_2 = \frac{1}{8}\, d\alpha\, d(\cos\beta)\, d\gamma\, k_{12}^2 dk_{12} \cdot$$
$$\cdot d(\cos\theta'')\, K_{12}^2 dK_{12} \quad . \qquad (2c.7)$$

In equation (2c.7), θ'' is the angle between $\vec{k}_{12}$ and $\vec{K}_{12}$. It is easy to see that K_{12} and $\cos\theta''$ are related to k_1 and k_2 by:

$$K_{12} = [2k_1^2 + 2k_2^2 - k_{12}^2]^{1/2}$$
$$\cos\theta'' = (-k_1^2 + k_2^2)/k_{12}K_{12} \quad . \qquad (2c.8)$$

So holding k_{12} constant, a change of variables can be made to k_1 and k_2, by computing the Jacobian of the transformation of equation (2c.8):

$$K_{12}^2 dK_{12} d(\cos\theta'') = 8(k_1k_2/k_{12})\, dk_1 dk_2 \quad . \qquad (2c.9)$$

There is another boundary condition to be imposed in the distribution in terms of k_1, k_2, and k_{12} because:

$$(k_1-k_2)^2 \leq k_{12}^2 \leq (k_1+k_2)^2 \quad . \qquad (2c.10)$$

This boundary condition is expressable by the step-function $\theta((2k_1k_2)^2 - (k_{12}^2-k_1^2-k_2^2)^2)$ which is equivalent to $\theta(k_1^2k_2^2 \sin^2\theta_{12})$ where θ_{12} is the angle between $\vec{k}_1$ and $\vec{k}_2$. The distribution can thus be written:

$$d^8R(4) = \frac{1}{8}\, d\alpha\ d(\cos\beta)\ d\gamma\ \frac{k_{12}^2 dk_{12}}{2\omega_1} \cdot 8\left(\frac{k_1k_2}{k_{12}}\right) dk_1 dk_2 \cdot$$

$$\cdot \frac{1}{2\omega_2}\ \frac{k_3 dk_3}{2\omega_3}\ \left(\frac{1}{2k_{12}}\right)\ d\phi_{(12)3} \cdot$$

$$\cdot\ \theta(k_1^2k_2^2 \sin^2\theta_{12})\ \theta(k_{12}^2k_3^2 \sin^2\theta_{(12)3}) \cdot$$

$$\cdot \sum_f |\langle T\rangle|^2 \Big/ \begin{array}{l}\vec{k}_4 = -\vec{k}_{12}-\vec{k}_3 \\ \omega_4 = \omega-\omega_1-\omega_2-\omega_3\end{array}$$

$$= \frac{1}{2^4}\ d\alpha d(\cos\beta) d\gamma\ dk_{12} d\omega_1 d\omega_2 d\omega_3 d\phi_{(12)3} \cdot$$

$$\cdot\theta(k_1{}^2k_2{}^2 \sin^2\theta_{12})\ \theta(k_{12}^2k_3{}^2 \sin^2\theta_{(12)3}) \cdot$$

$$\cdot \sum_f |\langle T\rangle|^2 \Big/ \begin{array}{l}\vec{k}_4 = -\vec{k}_{12}-\vec{k}_3 \\ \omega_4 = \omega-\omega_1-\omega_2-\omega_3\end{array} \quad . \qquad (2c.11)$$

The next important change of variables replaces k_{12}, ω_1, ω_2, and ω_3 by the invariant masses squared m_{12}^2, m_{34}^3, m_{134}^2 and m_{124}^2. These variables will be seen to be given by:

$$\begin{aligned}
m_{12}^2 &= (\omega_1+\omega_2)^2 - k_{12}^2 \\
m_{34}^2 &= (\omega-\omega_1-\omega_2)^2 - k_{12}^2 \\
m_{134}^2 &= \omega^2 - 2\omega\omega_2 + m_2^2 \\
m_{124}^2 &= \omega^2 - 2\omega\omega_3 + m_3^2 \quad .
\end{aligned} \qquad (2c.12)$$

It should be observed again that these variables are independent of the angle $\phi_{(12)3}$. This isolation of the angle between the plane of $\vec{k}_1$ and $\vec{k}_2$

and the plane of $\vec{k}_3$ and $\vec{k}_4$ simplifies the whole problem considerably. The system of equations given by equation (2c.12) can be inverted easily and the result is:

$$\omega_1 = \frac{1}{2\omega}(\omega^2+m_1{}^2 - [\omega^2+m_1{}^2+m_2{}^2-m_{12}^2+m_{34}^2-m_{134}^2])$$

$$\omega_2 = \frac{1}{2\omega}(\omega^2+m_2{}^2-m_{134}^2)$$

$$\omega_3 = \frac{1}{2\omega}(\omega^2+m_3^2-m_{124}^2)$$

$$\omega_4 = \omega-\omega_1-\omega_2-\omega_3$$

$$= \frac{1}{2\omega}(\omega^2+m_4^2-[\omega^2+m_3{}^2+m_4{}^2+m_{12}^2-m_{34}^2-m_{124}^2])$$

$$k_{12} = \frac{1}{2\omega}\left[\begin{array}{l} m_{12}^4 + m_{34}^4 + \omega^4 \\ -2m_{12}^2m_{34}^2 - 2m_{12}^2\omega^2 - 2m_{34}^2\omega^2 \end{array}\right]^{1/2} \quad . \qquad (2c.13)$$

These results have been written in a suggestive manner for future reference where it will be helpful to introduce the functions of the variables m_{12}^2, m_{34}^2 and the particle masses:

$$C_{134} = \omega^2 + m_1{}^2 + m_2{}^2 - m_{12}^2 + m_{34}^2$$

$$C_{124} = \omega^2 + m_3{}^2 + m_4{}^2 + m_{12}^2 - m_{34}^2 \quad . \qquad (2c.14)$$

These expressions are interchanged under the interchange of particle labels $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. The Jacobian of the transformation defined by equation (2c.13) is easily computed to be $(2(2\omega)^3 k_{12}(m_{12}^2, m_{34}^2))^{-1}$. Thus the new form of the eight-fold distribution is given by:

$$d^8R(4) = \left[\frac{1}{2^5(2\omega)^3}\right] d\alpha d(\cos\beta) d\gamma \frac{1}{k_{12}} dm_{12}^2 dm_{34}^2 d\phi_{(12)3} \cdot$$

$$\cdot dm_{134}^2 \, dm_{124}^2 \, \theta(k_1^2 k_2^2 \sin^2\theta_{12}) \, \theta(k_{12}^2 k_3^2 \sin^2 \theta_{(12)3}) \cdot$$

$$\cdot \sum_f |\langle T\rangle|^2 / \vec{k}_4 = -\vec{k}_{12} - \vec{k}_3$$

$$\omega_4 = \omega - \omega_1 - \omega_2 - \omega_3 \quad . \qquad (2c.15)$$

Before investigating the boundary conditions it is worth mentioning that coordinates of a vector in the body frames are determined by those in the spatial frame through the transformation $R_{z''}(\gamma)\ R_{y'}(\beta)\ R_z(\alpha)$:

$$R_{z''}(\gamma)\ R_{y'}(\beta)\ R_z(\alpha) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix} . \qquad (2c.16)$$

In particular, the body frame components of a unit vector pointing in the spatial $\hat{z}$-direction are given by:

$$((\hat{z})_{x'''},\ (\hat{z})_{y'''},\ (\hat{z})_{z'''}) = (-\cos\gamma\sin\beta,\ \sin\gamma\sin\beta,\ \cos\beta) \ . (2c.17)$$

D. The Boundary Conditions

The study of the boundary conditions is important for determining the ranges of the integrations that are to be performed over m^2_{134} and m^2_{124}. Aside from this, the study of the boundary conditions suggests convenient changes of variables that facilitate the integrations and the arguments of the boundary condition step functions are themselves important functions that will have to be integrated when definite forms of the T-matrix are conjectured. The particle energies, with the definitions of equation (2c.14), are given by:

$$\omega_1 = \frac{1}{2\omega}(\omega^2+m_1^2-[C_{134}-m^2_{134}])$$

$$\omega_2 = \frac{1}{2\omega}(\omega^2+m_2^2-m^2_{134})$$

$$\omega_3 = \frac{1}{2\omega}(\omega^2+m_3^2-m^2_{124})$$

$$\omega_4 = \frac{1}{2\omega}(\omega^2+m_4^2-[C_{124}-m^2_{124}]) \quad . \qquad (2d.1)$$

Using the function $\lambda(x,y,z)$ of Kallen:

$$\lambda(x,y,z) = x^2+y^2+z^2 - 2xy-2xz-2yz \qquad (2d.2)$$

it is easy to see that the momenta are given by:

$$k_1 = \frac{1}{2\omega}\lambda^{1/2}(C_{134}-m^2_{134},\ m_1^2,\ \omega^2)$$

$$k_2 = \frac{1}{2\omega}\lambda^{1/2}(m^2_{134},\ m_2^2,\ \omega^2)$$

$$k_3 = \frac{1}{2\omega}\lambda^{1/2}(m^2_{124},\ m_3^2,\ \omega^2)$$

$$k_4 = \frac{1}{2\omega}\lambda^{1/2}(C_{124}-m^2_{124},\ m_4^2,\ \omega^2)$$

$$k_{12} = \frac{1}{2\omega}\lambda^{1/2}(m^2_{12},\ m^2_{34},\ \omega^2) \quad . \qquad (2d.3)$$

The components of the three vectors $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ in the body frame are given by:

$$((\vec{k}_1)_{x'''},(\vec{k}_1)_{y'''},(\vec{k}_1)_{z'''}) = (-\frac{1}{k_{12}}k_1k_2 \sin\theta_{12},\ 0,\ \frac{1}{2}k_{12}+\frac{1}{2k_{12}}(k_1^2-k_2^2))$$

$$((\vec{k}_2)_{x'''},(\vec{k}_2)_{y'''},(\vec{k}_2)_{z'''}) = (+\frac{1}{k_{12}}k_1k_2 \sin\theta_{12},\ 0,\ \frac{1}{2}k_{12}-\frac{1}{2k_{12}}(k_1^2-k_2^2))$$

$$((\vec{k}_3)_{x'''},(\vec{k}_3)_{y'''},(\vec{k}_3)_{z'''}) = (k_3 \sin\theta_{(12)3} \cos\phi_{(12)3}, k_3 \sin\theta_{(12)3} \cdot \sin\phi_{(12)3}, k_3 \cos\theta_{(12)3})) \quad . \tag{2d.4}$$

Also, $\cos\theta_{12}$ and $\cos\theta_{(12)3}$ may be expressed by:

$$\cos\theta_{12} = \frac{1}{2k_1k_2}(k_{12}^2 - k_1^2 - k_2^2)$$

$$\cos\theta_{(12)3} = \frac{1}{2k_{12}k_3}(k_4^2 - k_{12}^2 - k_3^2) \quad . \tag{2d.5}$$

From equation (2d.5) it will be seen that the arguments of the boundary-condition step-functions are:

$$k_1^2k_2^2 \sin^2\theta_{12} = -\frac{1}{4}\lambda(k_{12}^2, k_1^2, k_2^2)$$

$$k_{12}^2k_3^2 \sin^2\theta_{(12)3} = -\frac{1}{4}\lambda(k_{12}^2, k_4^2, k_3^2) \quad . \tag{2d.6}$$

By referring to equations (2d.3) and (2c.14) it will be seen that these two functions of the invariant masses transform into one another under the simultaneous interchange of labels $1\leftrightarrow4$ and $2\leftrightarrow3$. For this reason only one of the boundary conditions needs to be investigated in detail.

The function $k_1^2k_2^2 \sin^2\theta_{12}$ will be investigated in detail. The object is to factor it so that the roots of m_{134}^2 are apparent. This will be done by using the simple identity:

$$\lambda(z+x+y,x,y) = z^2-4xy \tag{2d.7}$$

twice and identifying the expression:

$$P_{12} = \left(\frac{1}{2m_{12}}\right) \lambda^{1/2}(m_{12}^2, m_1^2, m_2^2) \quad . \tag{2d.8}$$

Of course P_{12} is the momentum of either particle 1 or 2 in their combined zero-momentum frame, using equations (2d.6), (2d.3), and (2c.14), $k_1^2 k_2^2 \sin^2\theta_{12}$ may be written:

$$k_1^2 k_2^2 \sin^2\theta_{12} = -\frac{1}{4}\left(\frac{1}{2\omega}\right)^4 \lambda\Big((2\omega k_{12})^2,\ \lambda\big((\omega^2+m_2^2-m_{134}^2) - (\omega^2+m_{12}^2-m_{34}^2) + m_1^2+\omega^2, m_1^2, \omega^2\big),\ \lambda(m_{134}^2, m_1^2, \omega^2)\Big)$$

$$= -\frac{1}{4}\left(\frac{1}{2\omega}\right)^4 \lambda\Big((2\omega k_{12})^2,\ (2\omega k_{12})^2 + \lambda(m_{134}^2, m_2^2, \omega^2) - 2(\omega^2+m_2^2-m_{134}^2)(\omega^2+m_{12}^2-m_{34}^2) - 4\omega^2(m_1^2-m_2^2-m_{12}^2),\ \lambda(m_{134}^2, m_2^2, \omega^2)\Big)$$

$$= -\left(\frac{1}{2\omega}\right)^2 \Big[m_{12}^2(\omega^2+m_2^2-m_{134}^2)^2 + [(2\omega k_{12})^2 + 4\omega^2 m_{12}^2]^{1/2} [(2m_{12}P_{12})^2 + 4m_{12}^2 m_2^2]^{1/2} \cdot (\omega^2+m_2^2-m_{134}^2) + (2\omega k_{12})^2 m_2^2 + \omega^2[(2m_{12}P_{12})^2 + 4m_{12}^2 m_2^2] \Big]$$

$$= \left(\frac{m_{12}}{2\omega}\right)^2 \left[m_{134}^2 - \frac{(m_1^2-m_2^2)}{2m_{12}^2}(\omega^2-m_{34}^2) - \frac{1}{2}C_{134} + 2\omega k_{12}\frac{P_{12}}{m_{12}}\right] \cdot \left[\frac{m_1^2-m_2^2}{2m_{12}^2}(\omega^2-m_{34}^2) + \frac{1}{2}C_{134} + 2\omega k_{12}\frac{P_{12}}{m_{12}} - m_{134}^2\right] \quad . \tag{2d.9}$$

It would be natural, but quite unecessary, to make a great mess of algebra out of this derivation. The last step in equation (2d.9) is merely a factorization of the quadratic expression in $\omega^2+m_2^2-m_{134}^2$ followed by some simple shuffling of the terms. The argument of the other boundary-condition step-function is given by:

$$k_{12}^2 k_3^2 \sin^2\theta_{(12)3} = \left(\frac{m_{34}}{2\omega}\right)^2 \left[m_{124}^2 + \frac{(m_3^2-m_4^2)}{2m_{34}^2}(\omega^2-m_{12}^2) - \frac{1}{2}C_{124} + 2\omega k_{12}\frac{P_{34}}{m_{34}}\right] \cdot$$

$$\cdot\left[-\left(\frac{m_3^2-m_4^2}{2m_{34}^2}(\omega^2-m_{12}^2) + \frac{1}{2}C_{124} + 2\omega k_{12}\frac{P_{34}}{m_{34}} - m_{124}^2\right]. \quad (2d.10)$$

The expression P_{34} is obtained from P_{12} by the interchange on particle labels $1\leftrightarrow 4$ and $2\leftrightarrow 3$ on the right hand side of equation (2d.8). It is the momentum of the particles in their combined zero-momentum frame.

E. The Equal Mass Case

When the four particle masses are equal there are some further simplifications that merit observation. When $m_1=m_2=m_3=m_4=m$, it is efficient to define:

$$\xi = m_{134}^2 - \frac{1}{2}C_{134}$$

$$\eta = m_{124}^2 - \frac{1}{2}C_{124}$$

$$\alpha_{12} = \omega^2 + m^2 - \frac{1}{2}C_{134}$$

$$= \frac{1}{2}(\omega^2 + m_{12}^2 - m_{34}^2)$$

$$\alpha_{34} = \omega^2 + m^2 - \frac{1}{2} c_{124}$$

$$= \frac{1}{2} (\omega^2 - m_{12}^2 + m_{34}^2) \quad . \qquad (2e.1)$$

With these changes the individual particle energies may be expressed in the important combinations:

$$\omega_1 + \omega_2 = 2\alpha_{12}/2\omega$$

$$\omega_1 - \omega_2 = 2\xi/2\omega$$

$$\omega_3 + \omega_4 = 2\alpha_{34}/2\omega$$

$$\omega_3 - \omega_4 = -2\eta/2\omega \quad . \qquad (2e.2)$$

The arguments of the boundary-condition step-functions expressed in terms of ξ and η are:

$$k_1^2 k_2^2 \sin^2 \theta_{12} = \left(\frac{m_{12}}{2\omega}\right)^2 \left[\left(2\omega k_{12} \frac{P_{12}}{m_{12}}\right)^2 - \xi^2\right]$$

$$k_{12}^2 k_3^2 \sin^2 \theta_{(12)3} = \left(\frac{m_{34}}{2\omega}\right)^2 \left[\left(2\omega k_{12} \frac{P_{34}}{m_{34}}\right)^2 - \eta^2\right] \quad . \qquad (2e.3)$$

The problem of a resonance decaying to four charged pions meets the condition of equal masses exactly, so far as anyone knows. Even with neutral pions in the final state, it is legitimate to ignore their slightly lower mass as long as Bose statistics are imposed on both the momentum and isotopic spin dependencies of the final state wave function anyway. The theory is then approximate as well as phenomenological. The distributions will be calculated with the aid of the notation of equations (2e.1) and (2e.3). This is useful because odd functions in ξ or η vanish when integrated over the symmetric limits. In the next section, the problem of isotopic spin coupling and Bose statistics will be investigated.

III. ISOTOPIC SPIN AND BOSE STATISTICS

A. Coupling and Recoupling of Isotopic Spins

The four isotopic spin one systems may be coupled together with Clebsch-Gordon coefficients to isotopic spin zero in three different ways:

$$|\pi_1\pi_2(t),\pi_3\pi_4(t),0> = \sum_{\substack{\mu_1\mu_2\\ \mu_3\mu_4\\ \mu\,\mu'}} C(tt0;\mu\mu'0)\; C(11t;\mu_1\mu_2\mu)\cdot C(11t;\mu_3\mu_4\mu')\,|\mu_1\mu_2\mu_3\mu_4> \quad . \tag{3a.1}$$

The three different states are those with t equal to 0,1, and 2. The symbol $C(j_1j_2j;m_1m_2m)$ is used by Rose (10) to denote the Clebsch-Gordan coefficient. The Clebsch-Gordan coefficient is related to the Wigner 3j-symbol by:

$$C(j_1j_2j;m_1m_2m) = (-)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \quad . \tag{3a.2}$$

Brink and Satchler (11) give tables of 3j-symbols while Rose tabulates the Clebsch-Gordan coefficients directly. A useful feature of the 3j-symbol is that two of its columns may be interchanged or the signs of the entries in the bottom row may be changed, if the symbol is multiplied by $(-)^{j_1+j_2+j_3}$. Thus the state of equation (3a.1) becomes multipled by $(-)^t$ under either interchange: $1\leftrightarrow 2$ or $3\leftrightarrow 4$.

To impose Bose statistics, it will be necessary to know how the state of equation (3a.1) is affected by all interchanges among the isotopic spin labels. One way to discuss this problem is in terms of vector analysis since the individual pions have isotopic spin 1. The problem has been treated in this way by Dotson, Nyborg and Good. Another way to treat the problem is with the use of Racah coefficients. The Racah coefficient W(abcd,ef) is designed to facilitate recouplings among three angular momenta and is defined by:

$$[(2j'+1)(2j''+1)]^{1/2}\, W(j_1 j_2 j j_3; j' j'')$$
$$= \langle j_1, j_2 j_3(j''), jm | j_1 j_2(j'), j_3 jm \rangle \quad . \qquad (3a.3)$$

Of course the Racah coefficient sometimes enters into other problems, and is useful here because of its property:

$$\sum_{f\phi} [(2e+1)(2f+1)]^{1/2}\, W(abcd;ef)\; C(bdf;\beta\delta\phi)\; C(afc;\alpha\phi\gamma)$$
$$= \sum_{\epsilon} C(abe,\alpha\beta\epsilon)\; C(edc;\epsilon\delta\gamma) \quad . \qquad (3a.4)$$

Using equations (3a.1) and (3a.4) and the fact that C(tt0,mm'0) is equal to $(-)^{t-m}\delta_{m,-m'}(2t+1)^{-1/2}$ it is not difficult to show that:

$$|\pi_1\pi_2(t),\pi_3\pi_4(t),0\rangle = \sum_f (-)^{t+f}[(2f+1)(2t+1)]^{1/2}\; W(11\;11;tf)\cdot$$
$$\cdot|\pi_1\pi_3(f),\pi_2\pi_4(f),0\rangle$$
$$= \sum_g (-)^g[(2g+1)(2t+1)]^{1/2}\; W(11\;11;tg)\cdot$$
$$\cdot|\pi_1\pi_4(g),\pi_2\pi_3(g),0\rangle \quad . \qquad (3a.5)$$

The Racah coefficients required for the subsequent discussion are:

$$W(1111;0f) = (-)^{2-f}/3$$

$$W(1111;1f) = (-)^{1-f}[4-f(f+1)]/12$$

$$W(1111;2f) = 4/(2-f)!\;(3+f)! \qquad (3a.6)$$

The first two of these results are tabulated by Rose and by Brink and Satchler. The third result has been specialized from a more general algebraic formula given by Brink and Satchler.

B. Type 1 - The Completely Symmetric Isotopic Spin State

There is one linear combination of states in the different intermediate couplings that is completely symmetric in the isotopic spin labels. Let A_o, A_1, A_2 be the coefficients for the three values of t and consider the expansion:

$$|\alpha> = \sum_t A_t \; |\pi_1\pi_2(t), \pi_3\pi_4(t), 0>$$

$$= \sum_f \{\sum_t (-)^{t+f} [(2f+1)(2t+1)]^{1/2} \, W(1111;tf) A_t\} \cdot$$

$$\cdot \; |\pi_1\pi_3(f), \pi_2\pi_4(f), 0> \quad . \qquad (3b.1)$$

A completely symmetric state $|\alpha>$ can be acheived by setting A_1 equal to zero and choosing A_o and A_2 so that the coefficient for f = 1 in the second form of equation (3b.1) also vanishes. The isotopic spin states of equation (3a.1) are orthogonal for different values of t so the completely symmetric normalized isotopic spin state $|\alpha>$ is determined by:

$$|\alpha> = A_o|\pi_1\pi_2(0), \pi_3\pi_4(0), 0> + A_2|\pi_1\pi_2(2), \pi_3\pi_4(2), 0>$$

$$|A_o|^2 + |A_2|^2 = 1$$

$$A_o/\sqrt{3} - \sqrt{5}\, A_2/2\sqrt{3} = 0 \quad . \qquad (3b.2)$$

Solving the equations for A_o and A_2 results in $\sqrt{5}/3$ and 2/3 respectively. The state $|\alpha>$ may be expanded in terms of $|\mu_1\mu_2\mu_3\mu_4>$ by explicit evaluation of the Clebsch-Gordan coefficients. The result of this is:

$$|\alpha\rangle = \frac{1}{3\sqrt{5}}\begin{bmatrix} 2\begin{bmatrix} |+-+-\rangle + |+--+\rangle + |-++-\rangle \\ +|-+-+\rangle + |++--\rangle + |--++\rangle \end{bmatrix} \\ -\begin{bmatrix} |+-00\rangle + |-+00\rangle + |00+-\rangle + |00-+\rangle \\ +|+0-0\rangle + |-0+0\rangle + |0+0-\rangle + |0-0+\rangle \\ +|0+-0\rangle + |0-+0\rangle + |+00-\rangle + |-00+\rangle \end{bmatrix} \\ +3\,|0000\rangle \end{bmatrix} . \quad (3b.3)$$

Since the isotopic spin state $|\alpha\rangle$ is already completely symmetric, the state of four pions that obeys Bose statistics of type I is:

$$|\Psi_I\rangle = A(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)|\alpha\rangle \quad (3b.4)$$

where $A(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ must be invariant to interchanges among the momenta and exhibits the spin and parity of the resonance. The probability that four pions in the state $|\Psi_I\rangle$ will be observed with two positive charges with momenta $\vec{k}_1$ and $\vec{k}_2$ and two negative charges with momenta $\vec{k}_3$ and $\vec{k}_4$ is:

$$\sum_f |\langle f;\pi^+(\vec{k}_1),\pi^+(\vec{k}_2),\pi^-(\vec{k}_3),\pi^-(\vec{k}_4)|T|r\rangle|^2$$

$$= \sum_f |\langle f;\pi^+(\vec{k}_1),\pi^+(\vec{k}_2),\pi^-(\vec{k}_3),\pi^-(\vec{k}_4)|\Psi_I\rangle|^2$$

$$= \frac{4}{45}\begin{bmatrix} |A(\vec{k}_1,\vec{k}_3,\vec{k}_2,\vec{k}_4)|^2 + |A(\vec{k}_1,\vec{k}_3,\vec{k}_4,\vec{k}_2)|^2 + |A(\vec{k}_3,\vec{k}_1,\vec{k}_2,\vec{k}_4)|^2 \\ +|A(\vec{k}_3,\vec{k}_1,\vec{k}_4,\vec{k}_2)|^2 + |A(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)|^2 + |A(\vec{k}_3,\vec{k}_4,\vec{k}_1,\vec{k}_2)|^2 \end{bmatrix}$$

$$= 8|A(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)|^2/15 \quad . \quad (3b.5)$$

The sum over f denotes a sum over final states, not to be confused with intermediate couplings, and is accomplished with the aid of equation (3b.3). The calculation is just as trivial for final states with two charged pions and no charged pions. It will be seen that the branching ratio for type 1 is:

$$\text{b.r.}(4\text{ charged}/2\text{ charged}/0\text{ charged}) = 8/4/3 \quad . \tag{3b.6}$$

C. Type 2 - The Mixed Symmetry Isotopic Spin States

There are two more independent isotopic spin states to be considered besides $|\alpha>$. These states are necessarily antisymmetric to some interchanges and it will be shown that the two states themselves are interchanged under some permutations of the particle labels. Since some antisymmetry is inevitable, the two states $|\beta>$ and $|\gamma>$ will be taken to be pure t = 1 and pure f = 1 states, respectively. The parameters A_o,A_1, and A_2 for the state $|\gamma>$are determined from the set of equations:

$$(1/3)A_o \ +(1/\sqrt{3})A_1 \ +(\sqrt{5}/3)A_2 = 0$$

$$(1/\sqrt{3})A_o +(1/2)A_1 \ -(\sqrt{5}/2\sqrt{3})A_2 = 1$$

$$(\sqrt{5}/3)A_o -(\sqrt{5}/2\sqrt{3})A_1 + 1/6\ A_2 = 0 \quad . \tag{3c.1}$$

Solving this set of equations results in values of $1/\sqrt{3}$, 1/2, and $-\sqrt{5}/2\sqrt{3}$ for A_o,A_1, and A_2, respectively. Thus the two states under consideration are:

$$|\beta> = |\pi_1\pi_2(1),\pi_3\pi_4(1),0>$$

$$|\gamma> = |\pi_1\pi_3(1),\pi_2\pi_4(1),0>$$

$$= 1/\sqrt{3}\ |\pi_1\pi_2(0),\pi_3\pi_4(0),0>$$

$$+ 1/2|\pi_1\pi_2(1),\pi_3\pi_4(1),0>$$

$$-\sqrt{5}/2\sqrt{3} \quad |\pi_1\pi_2(2),\pi_3\pi_4(2),0> \quad . \tag{3c.2}$$

The states $|\alpha>$, $|\beta>$, and $|\gamma>$ are linearly independent of one another, and $|\alpha>$ is orthogonal to $|\beta>$ and $|\gamma>$. It is worth noting that:

$$\begin{aligned}|\beta> - |\gamma> &= -1/\sqrt{3} \quad |\pi_1\pi_2(0),\pi_3\pi_4(0),0> \\ &\quad + 1/2 \quad |\pi_1\pi_2(1),\pi_3\pi_4(1),0> \\ &\quad + \sqrt{5}/2\sqrt{3} \quad |\pi_1\pi_2(2),\pi_3\pi_4(2),0> \\ &= -|\pi_1\pi_4(1),\pi_2\pi_3(1),0> \quad .\end{aligned} \tag{3c.3}$$

The last step in equation (3c.3) is made by noticing that the coefficients in $|\beta> - |\gamma>$ differ from those in $|\gamma>$ itself by a factor of $-(-)^t$, and comparing the result with equation (3a.5), the point of these observations is that the effect of interchanging particle labels on the states $|\beta>$ and $|\gamma>$ may now be specified. Let τ_{ij} be an operator acting on the isotopic labels that interchanges particles i and j. Then the complete set of such transpositions is:

$$\begin{aligned}
&\tau_{12}|\beta> = -|\beta> && \tau_{12}|\gamma> = -|\beta> + |\gamma> \\
&\tau_{13}|\beta> = |\beta> - |\gamma> && \tau_{13}|\gamma> = -|\gamma> \\
&\tau_{14}|\beta> = |\gamma> && \tau_{14}|\gamma> = |\beta> \\
&\tau_{23}|\beta> = |\gamma> && \tau_{23}|\gamma> = |\beta> \\
&\tau_{24}|\beta> = |\beta> - |\gamma> && \tau_{24}|\gamma> = -|\gamma> \\
&\tau_{34}|\beta> = -|\beta> && \tau_{34}|\gamma> = -|\beta> + |\gamma> \quad .
\end{aligned} \tag{3c.4}$$

For future reference it may be noted that the transpositions are determined by the following facts: first that $|\beta>$ is antisymmetric to the interchange $1 \leftrightarrow 2$ and to the interchange $3 \leftrightarrow 4$; second that $|\beta>$ is symmetric to the simultaneous interchange of $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$; third that $\tau_{23}\ |\beta>$ is equal to $|\gamma>$; and finally that $\tau_{13}\ |\beta>$ is equal to $|\beta> - |\gamma>$.

It is obvious that Bose statistics cannot be satisfied with the states $|\beta>$ and $|\gamma>$ by themselves. Let $B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ and $C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ be two functions of the particle momenta with the same properties under the interchange of the particle labels as $|\beta>$ and $|\gamma>$, respectively. Then a general linear combination of the states $|\beta>$ and $|\gamma>$ using the functions B and C as coefficients, apart from an over-all normalization factor, is:

$$|\Psi_2> = B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)(|\beta> + x\,|\gamma>) + C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)(y|\beta> + z\,|\gamma>) \quad . \tag{3c.5}$$

Requiring $|\Psi_2>$ to be symmetric under the interchange $2 \leftrightarrow 3$ leads to $z = 1$ and $x = y$, and then requiring $|\Psi_2>$ to be symmetric under $1 \leftrightarrow 2$ determines $x = -1/2$. Thus the state of type 2 is given by:

$$\begin{aligned} |\Psi_2> &= B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)(|\beta> - 1/2\ |\gamma>) \\ &\quad + C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)(-\ 1/2\ |\beta> + |\gamma>) \\ &= 1/4(B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) + C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4))(|\beta> + |\gamma>) \\ &\quad + 3/4(B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4))(|\beta> - |\gamma>) \ . \end{aligned} \tag{3c.6}$$

By referring to equations (3c.2) and (3a.1), and tables of the Clebsch-Gordan coefficients, the state $|\Psi_2>$ can be expanded in terms of $|\mu_1\mu_2\mu_3\mu_4>$. It is some small saving in this to write B_{1234} and C_{1234} as a shorthand for $B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ and $C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$. Collecting coefficients with the same set of charges, $|\Psi_2>$ may be written:

$$|\Psi_2> = \frac{1}{4\sqrt{3}} \begin{bmatrix} (B_{1234} - 2C_{1234})(|+ + - 0> + |- - + +>) \\ +(B_{1234} + C_{1234})(|+ - - +> + |- + + ->) \\ +(-2B_{1234} + C_{1234})(|+ - + -> + |- + - +>) \end{bmatrix}$$

$$+ \frac{1}{4\sqrt{3}} \begin{bmatrix} (B_{1234} - 2C_{1234})(|+-00> + |-+00> + |00+-> + |00-+>) \\ +(B_{1234} + C_{1234})(|+00-> + |0-+0> + |0+-0> + |-00+>) \\ +(-2B_{1234} + C_{1234})(|+0-0> + |-0-0> + |0+0-> + |0-0+>) \end{bmatrix}. \quad (3c.7)$$

The probability that four pions in the state $|\Psi_2>$ will be observed with two positively charged particles with momenta $\vec{k}_1$ and $\vec{k}_2$ and two negatively charged particles with momenta $\vec{k}_3$ and $\vec{k}_4$ is:

$$\sum_f |<f; \pi^+(\vec{k}_1), \pi^+(\vec{k}_2), \pi^-(\vec{k}_3), \pi^-(\vec{k}_4)|T|r>|^2$$

$$= \sum_f |<f; \pi^+(\vec{k}_1), \pi^+(\vec{k}_2), \pi^-(\vec{k}_3), \pi^-(\vec{k}_4)|\Psi_2>|^2$$

$$= \frac{1}{48} \begin{bmatrix} |(B_{1234} - 2C_{1234}|^2 & +|B_{3412} - 2C_{3412}|^2 \\ +|B_{1342} + C_{1342}|^2 & +|B_{3124} + C_{3124}|^2 \\ +|-2B_{1324} + C_{1324}|^2 & +|-2B_{3142} + C_{3142}|^2 \end{bmatrix}$$

$$= \frac{1}{8} |B_{1234} - 2C_{1234}|^2 \quad . \quad (3c.8)$$

The last step in equation (3c.8) is made by using equation (3c.4) since B_{1234} and C_{1234} have the same properties under interchange of particle labels as $|\beta>$ and $|\gamma>$. The probability that the four pions in the state $|\Psi_2>$ will be observed with a positive and negative pion with momenta $\vec{k}_1$ and $\vec{k}_2$

and two neutral pions with momenta $\vec{k}_3$ and $\vec{k}_4$ is:

$$\sum_f |\langle f;\ \pi^+(\vec{k}_1),\pi^-(\vec{k}_2),\ \pi^0(\vec{k}_3),\pi^0(\vec{k}_4)|T|r\rangle|^2$$

$$= \sum_f |\langle f;\ \pi^+(\vec{k}_1),\pi^-(\vec{k}_2),\pi^0(\vec{k}_3),\pi^0(\vec{k}_4)|\Psi_2\rangle|^2$$

$$= \frac{1}{48}\left[\begin{array}{l} |B_{1234}-2C_{1234}|^2+|B_{2134}-2C_{2134}|^2+|B_{3412}-2C_{3412}|^2+|B_{3421}-2C_{3421}|^2 \\ +|B_{1342}+C_{1342}|^2+|B_{3214}+C_{3214}|^2+|B_{3124}+C_{3124}|^2+|B_{2341}+C_{2341}|^2 \\ +|-2B_{1324}+C_{1324}|^2+|-2B_{2314}+C_{2314}|^2+|2B_{3142}+C_{3142}|^2+|-2B_{3241}+C_{3241}|^2 \end{array}\right]$$

$$= \quad |B_{1234} - 2C_{1234}|^2/4 \quad . \qquad (3c.9)$$

Again, the last step is made by extensive use of the symmetry properties of B_{1234} and C_{1234}. The state of type 2 does not admit four neutral pions at all, so the branching ratio is:

$$\text{b.r. (4 charged/2 charged/o charged)} = 1/2/0 \quad . \qquad (3c.10)$$

The functions $B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ and $C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ must have the conjectured spin and parity of the resonance as well as the symmetry properties discussed. It has been shown that the distribution for type 2 is determined by the function of momenta $|B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)-2C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)|^2$ whereas for type 1 the distribution is determined by $|A(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)|^2$. In the next two sections the simplest suitable functions will be integrated over the two invariant masses m^2_{134} and m^2_{124} for the four scalars and the four vectors.

IV. SCALAR DISTRIBUTIONS

A. Preliminary Remarks

Since the scalar distributions are necessarily isotropic, the integration over Euler angles can be performed without loss of information. Since the equal mass case applies, the integrations will be performed with the change of variables from m_{134}^2 and m_{124}^2 to ξ and η. Thus the distributions are given by:

$$d^3R(4) = \frac{\pi^2}{(2\omega)^3} dm_{12} dm_{34} d\phi_{(12)3} \frac{m_{12}m_{34}}{k_{12}} \cdot$$

$$\cdot \iint d\xi d\eta \sum_f |\langle T\rangle|^2 / \vec{k}_4 = -\vec{k}_{12} - \vec{k}_3 \quad \omega_4 = \omega - \omega_1 - \omega_2 - \omega_3 \tag{4a.1}$$

where the limits of the integrations are from $-2\omega k_{12} P_{12}/m_{12}$ to $+ 2\omega k_{12} P_{12}/m_{12}$ for ξ and from $-2\omega k_{12} P_{34}/m_{34}$ to $+2\omega k_{12} P_{34}/m_{34}$ for η. The distributions will be expressed in terms of m_{12} and m_{34} instead of m_{12}^2 and m_{34}^2 as in equation (4a.1) because of the simple geometry of the Goldhaber triangle. The sum over final states has been performed in the previous section, for type 1:

$$\sum_f |\langle T\rangle|^2 = |A(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)|^2 \tag{4a.2}$$

and for type 2:

$$\sum_f |\langle T\rangle|^2 = |B(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) - 2C(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)|^2 \quad . \tag{4a.3}$$

Constant factors will be ignored in $\sum_f |\langle T\rangle|^2$, essentially because this involves the coupling constant for the decay. If coupling constants were known, then presumably something of the form of the T-matrix would also be known, and this approach would not be required. The procedure followed here will be to drop constant factors from the functions of momenta $A(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ and $B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)-2C(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ after conservation of momentum has been imposed. The constant factors already included in equation (4a.1) will be retained. The simplest functions for a given spin and parity will be taken to be those with the simplest combination of the momenta with that spin and parity combined with the simplest functions of the energies required by the symmetry.

B. Distributions for $J^P = 0^+$

For type 1 the simplest function $A(0^+)$ which is a true scalar and which is completely symmetric in the four momenta is just a constant. The constant is taken to be unity here so:

$$d^3 R(4,,0^+) = \frac{4\pi^2}{2\omega} dm_{12} dm_{34} d\phi_{(12)3}\ k_{12}\ P_{12}\ P_{34} \quad . \tag{4b.1}$$

It will be noticed that this distribution vanishes on all three boundaries of the Goldhaber triangle and is uniform in the angle $\phi_{(12)3}$.

For type 2 the dependence on the momenta is no longer trivial. With the criterion of simplicity stated above, the functions involved are:

$$\begin{aligned} B(0^+) &= (\omega_1-\omega_2)(\omega_3-\omega_4) \\ C(0^+) &= (\omega_1-\omega_3)(\omega_2-\omega_4) \quad . \end{aligned} \tag{4b.2}$$

It may be verified easily that these functions have the correct symmetry properties and:

$$B(0^+) - 2C(0^+) = (\omega_1-\omega_2)(\omega_3-\omega_4) - 2(\omega_1-\omega_3)(\omega_2-\omega_4)$$

$$= \frac{1}{2}(\omega_1-\omega_2)^2 + \frac{1}{2}(\omega_3-\omega_4)^2 - \frac{1}{2}(\omega_1+\omega_2-\omega_3-\omega_4)^2$$

$$= 2(\frac{1}{2\omega})^2 [\xi^2 + \eta^2 - (\alpha_{12}-\alpha_{34})^2] \quad . \qquad (4b.3)$$

The integrations over ξ and η are not difficult and the distribution is given by:

$$d^3R(4,{}_2 0^+) = \frac{2^4\pi^2}{(2\omega)^5} dm_{12}dm_{34}d\phi_{(12)3}\; k_{12}\, P_{12}\, P_{34} \cdot$$

$$\cdot \begin{bmatrix} (\alpha_{12}-\alpha_{34})^4 \\ -2/3(\alpha_{12}-\alpha_{34})^2\;(2\omega k_{12})^2\;[(\frac{P_{12}}{m_{12}})^2 + (\frac{P_{34}}{m_{34}})^2] \\ +(2\omega k_{12})^4\;[\frac{1}{5}\;(\frac{P_{12}}{m_{12}})^4 + \frac{2}{9}\;(\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^2 + \frac{1}{5}\;(\frac{P_{34}}{m_{34}})^4] \end{bmatrix} \quad . \qquad (4b.4)$$

Since $\alpha_{12}-\alpha_{34}$ is equal to $m_{12}^2-m_{34}^2$, the first two terms in equation (4b.4) vanish along the line bisecting the right angle of the Goldhaber triangle. Near the diagonal boundary, where k_{12} is small, the type 2 distribution will show a trough along the line $m_{12} = m_{34}$. Like the type 1 distribution, this case is uniform in the angle $\phi_{(12)3}$.

C. Distributions for $J^P = 0^-$

The pseudoscalar functions must be a triple product in the momenta multipled by appropriate functions of the energies. For type 1, the function $A(0^-)$ is rather complicated because the triple product is antisymmetric. Consider the vectors:

$$\vec{V}_a = \vec{k}_1\omega_1 + \vec{k}_2\omega_2 + \vec{k}_3\omega_3 + \vec{k}_4\omega_4$$

$$\vec{V}_b = \vec{k}_1\omega_1^2 + \vec{k}_2\omega_2^2 + \vec{k}_3\omega_3^2 + \vec{k}_4\omega_4^2$$

$$\vec{V}_c = \vec{k}_1\omega_2\omega_3\omega_4 + \vec{k}_2\omega_3\omega_4\omega_1 + \vec{k}_3\omega_4\omega_1\omega_2 + \vec{k}_4\omega_1\omega_2\omega_3 \quad . \qquad (4c.1)$$

These vectors are the simplest three which are nonvanishing and completely symmetric in the momenta. The third vector $\vec{V}_c$ could also be constructed with the cubes of the energies, but it doesn't matter which one is used in the present development. A pseudovector can be constructed by forming the cross-product $\vec{V}_a \times \vec{V}_b$. After imposing conservation of momentum this is found to be:

$$\vec{V}_a \times \vec{V}_b = -\vec{k}_1 \times \vec{k}_2(\omega_1-\omega_2)(\omega_1-\omega_4)(\omega_2-\omega_4)$$

$$-\vec{k}_1 \times \vec{k}_3(\omega_1-\omega_3)(\omega_1-\omega_4)(\omega_3-\omega_4)$$

$$-\vec{k}_2 \times \vec{k}_3(\omega_2-\omega_3)(\omega_2-\omega_4)(\omega_3-\omega_4) \quad . \qquad (4c.2)$$

A pseudoscalar may then be formed by taking the dot product with $\vec{V}_c$. This results in:

$$A(0^-) = \vec{V}_a \times \vec{V}_b \cdot \vec{V}_c$$

$$= \vec{k}_1 \times \vec{k}_2 \cdot \vec{k}_3 \; (\omega_1-\omega_2)(\omega_3-\omega_4)(\omega_1-\omega_3) \cdot$$

$$\cdot \; (\omega_1-\omega_4)(\omega_2-\omega_3)(\omega_2-\omega_4) \quad . \qquad (4c.3)$$

This function for the pseudoscalar of type 1 was given by Gell-Mann (12).

The pseudoscalar function of type 1 is one of the more complicated ones dealt with in this treatment. The triple product has a simple form. It will be seen from equations (2d.4) and (2e.3) that:

$$\vec{k}_1 \times \vec{k}_2 \cdot \vec{k}_3 = \frac{1}{k_{12}} (k_1 k_2 \sin\theta_{12})(k_{12} k_3 \sin\theta_{(12)3}) \sin\phi_{(12)3}$$

$$= \frac{1}{k_{12}} \frac{m_{12} m_{34}}{(2\omega)^2} [(2\omega k_{12} \frac{P_{12}}{m_{12}})^2 - \xi^2]^{1/2} .$$

$$\cdot [(2\omega k_{12} \frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2} \sin\phi_{(12)3} \quad . \qquad (4c.4)$$

After changing variables with the aid of equation (2e.2), the energy factor of equation (4c.3) becomes:

$$(\omega_1-\omega_2)(\omega_3-\omega_4)(\omega_1-\omega_3)(\omega_1-\omega_4)(\omega_2-\omega_3)(\omega_2-\omega_4)$$

$$= 2^2 (\frac{1}{2\omega})^6 \xi \eta [(\alpha_{12}-\alpha_{34})^4 - 2(\alpha_{12}-\alpha_{34})^2(\xi^2+\eta^2) + (\xi^2-\eta^2)^2] . \quad (4c.5)$$

Thus the integrand for the integrations over ξ and η is:

$$|A(0^-)|^2 = \frac{2^4}{(2\omega)^{16}} (\frac{m_{12} m_{34}}{k_{12}})^2 [(2\omega k_{12} \frac{P_{12}}{m_{12}})^2 - \xi^2][(2\omega k_{12} \frac{P_{34}}{m_{34}})^2 - \eta^2] \cdot$$

$$\cdot \xi^2 \eta^2 \begin{bmatrix} (\alpha_{12}-\alpha_{34})^8 \\ -4 \ (\alpha_{12}-\alpha_{34})^6 (\xi^2+\eta^2) \\ +2 \ (\alpha_{12}-\alpha_{34})^4 (3\xi^4+2\xi^2\eta^2+3\eta^4) \\ -4 \ (\alpha_{12}-\alpha_{34})^2 (\xi^6-\xi^4\eta^2-\xi^2\eta^4+\eta^6) \\ + \ (\xi^8-4\xi^6\eta^2+6\xi^4\eta^4-4\xi^2\eta^6+\eta^8) \end{bmatrix} \sin^2 \phi_{(12)3} \quad . \qquad (4c.6)$$

The integrations are performed with the aid of the simple formula:

$$\int_{-a}^{+a} dx\,[a^2 - x^2]\, x^{2n} = \frac{4a^{2n+3}}{(2n+1)(2n+3)} \quad . \tag{4c.7}$$

Therefore the distribution for the pseudoscalar of type 1 is given by:

$$d^3R(4,{}_1 0^-) = \frac{2^8\pi^2}{(2\omega)^9}\; dm_{12}dm_{34}d\phi_{(12)3}\; \frac{k_{12}^7 P_{12}^5 P_{34}^5}{m_{12}^2 m_{34}^2} \sin^2\phi_{(12)3} \cdot$$

$$\cdot \Bigg[(1/3^2\cdot 5^2)(\alpha_{12}-\alpha_{34})^8$$
$$- (4/5^2\cdot 7^2)(\alpha_{12}-\alpha_{34})^6(2\omega k_{12})^2\Big[\left(\frac{P_{12}}{m_{12}}\right)^2 + \left(\frac{P_{34}}{m_{34}}\right)^2\Big]$$
$$+ (2/5\cdot 7)(\alpha_{12}-\alpha_{34})^4(2\omega k_{12})^4 \cdot$$
$$\cdot\Big[\frac{1}{9}\left(\frac{P_{12}}{m_{12}}\right)^4 + \frac{2}{5\cdot 7}\left(\frac{P_{12}}{m_{12}}\right)^2\left(\frac{P_{34}}{m_{34}}\right)^2 + \frac{1}{9}\left(\frac{P_{34}}{m_{34}}\right)^4\Big]$$
$$- (4/5\cdot 9)(\alpha_{12}-\alpha_{34})^2(2\omega k_{12})^6 \cdot$$
$$\cdot\Big[\frac{1}{3\cdot 7}\left(\frac{P_{12}}{m_{12}}\right)^6 - \frac{1}{7^2}\left(\frac{P_{12}}{m_{12}}\right)^4\left(\frac{P_{34}}{m_{34}}\right)^2 - \frac{1}{7^2}\left(\frac{P_{12}}{m_{12}}\right)^2\left(\frac{P_{34}}{m_{34}}\right)^4 + \frac{1}{3\cdot 7}\left(\frac{P_{34}}{m_{34}}\right)^6\Big]$$
$$+ (2\omega k_{12})^8\Big[\frac{1}{3\cdot 5\cdot 11\cdot 13}\left(\frac{P_{12}}{m_{12}}\right)^8 + \frac{1}{5\cdot 7\cdot 9\cdot 11}\left(\frac{P_{12}}{m_{12}}\right)^6\left(\frac{P_{34}}{m_{34}}\right)^2$$
$$+ \frac{1}{7^2\cdot 9^2}\left(\frac{P_{12}}{m_{12}}\right)^4\left(\frac{P_{34}}{m_{34}}\right)^4 + \frac{1}{5\cdot 7\cdot 9\cdot 11}\left(\frac{P_{12}}{m_{12}}\right)^2\left(\frac{P_{34}}{m_{34}}\right)^6$$
$$+ \frac{1}{3\cdot 5\cdot 11\cdot 13}\left(\frac{P_{34}}{m_{34}}\right)^8\Big]\Bigg] . \tag{4c.8}$$

This distribution is noticeable by the strong tendency for $\vec{k}_1$ and $\vec{k}_2$ to lie in a plane perpendicular to the plane of $\vec{k}_3$ and $\vec{k}_4$. A depression, at small k_{12}, along the line $m_{12}=m_{34}$ on the Goldhaber triangle is caused by the factors of $(\alpha_{12}-\alpha_{34})$ and the distribution vanishes much faster at the boundaries of the triangle than do the scalar distributions.

For type 2, it is again convenient to construct a pseudovector first. It is easy to see that the pseudovectors:

$$\vec{B}(1^+) = \vec{k}_1 \times \vec{k}_3(\omega_1-\omega_3) - \vec{k}_1 \times \vec{k}_4(\omega_1-\omega_4)$$
$$- \vec{k}_2 \times \vec{k}_3(\omega_2-\omega_3) + \vec{k}_2 \times \vec{k}_4(\omega_2-\omega_4)$$

$$\vec{C}(1^+) = \vec{k}_1 \times \vec{k}_2(\omega_1-\omega_2) - \vec{k}_1 \times \vec{k}_4(\omega_1-\omega_4)$$
$$- \vec{k}_2 \times \vec{k}_3(\omega_2-\omega_3) + \vec{k}_3 \times \vec{k}_4(\omega_3-\omega_4) \qquad (4c.9)$$

possess the correct properties under interchange of the momenta. Imposing conservation of momentum and dropping a constant factor of -3, the pseudovector distribution is governed by:

$$B(1^+) - 2C(1^+) = \vec{k}_1 \times \vec{k}_2(\omega_1-\omega_2)$$
$$+\vec{k}_1 \times \vec{k}_3(\omega_3-\omega_4)$$
$$+\vec{k}_2 \times \vec{k}_3(\omega_3-\omega_4) \quad . \qquad (4c.10)$$

The pseudoscalar functions may be formed by contracting the vectors of equation (4c.9) with a completely symmetric vector, namely, $\vec{V}_a$ of equation (4c.1) as this does not affect their symmetry properties. Imposing conservation of momentum, it will be seen that the pseudoscalar distribution of type 2 is governed by:

$$B(0^-) - 2C(0^-) = \vec{k}_1\cdot\vec{k}_2 \times \vec{k}_3(\omega_1-\omega_2)(\omega_3-\omega_4) \quad . \qquad (4c.11)$$

A constant factor of 2 has been dropped in arriving at equation (4c.11). The function that must be integrated is given by:

$$|B(0^-) - 2C(0^-)|^2$$

$$= \frac{2^4}{(2\omega)^8} \left(\frac{m_{12}m_{34}}{k_{12}}\right)^2 \left[\left(2\omega k_{12} \frac{P_{12}}{m_{12}}\right)^2 - \zeta^2\right]\left[\left(2\omega k_{12} \frac{P_{34}}{m_{34}}\right)^2 - \eta^2\right] \cdot$$

$$\cdot \zeta^2\eta^2 \sin^2 \phi_{(12)3} \quad . \tag{4c.12}$$

Thus the pseudoscalar distribution is given by:

$$d^3R(4,{}_20^-) = \frac{2^8\pi^2}{3^2\cdot 5^2(2\omega)} dm_{12}dm_{34}d\phi_{(12)3} \sin^2 \phi_{(12)3} \cdot$$

$$\cdot \frac{k_{12}^7 P_{12}^5 P_{34}^5}{m_{12}^2 m_{34}^2} \quad . \tag{4c.13}$$

Like the type 1 distribution, there is a strong tendancy for $\phi_{(12)3}$ to be near $\pi/2$. The depression at 45^o on the Goldhaber plot is not present in the type 2 distribution.

V. VECTOR DISTRIBUTIONS

A. Preliminary Remarks

The vector distributions will be exhibited in five-dimensional form. The unpolarized distribution and the distribution for zero spin-projection along the spatial Z-axis will be calculated. Since the body coordinates of the spatial Z-axis do not depend on α, integrations are to be performed over m_{134}^2, m_{124}^2, and α. Thus both distributions for each parity and symmetry type will be exhibited in five-dimensional form. The sum of the two distributions for a given parity and symmetry type applies to production of the resonance and a single pion in the forward or backward direction when production takes place equally from the four initial spin states of protons bombarded by antiprotons. The integrations are most efficiently performed with the aid of the change of variables from m_{134}^2 and m_{124}^2 to ξ and η. Thus, referring to equation (2c.15), it will be seen that the distributions to be calculated have the form:

$$d^5R(4) = \frac{2\pi}{2^3(2\omega)^3}\, d\,(\cos\beta)\, d\gamma\, \frac{m_{12}m_{34}}{k_{12}}\, dm_{12}dm_{34}d\phi_{(12)3}\;\cdot$$

$$\cdot\iint d\xi d\eta \sum_f |\langle T\rangle|^2 \,/\, \vec{k}_4 = -\vec{k}_{12}-\vec{k}_3$$

$$\omega_4 = \omega-\omega_1-\omega_2-\omega_3 \quad . \tag{5a.1}$$

For type 1 the two cases are:

$$\left(\sum_f |\langle T\rangle|^2\right)_{unpol} = \vec{A}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\cdot\vec{A}^*(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) \tag{5a.2}$$

and

$$\left(\sum_f |\langle T\rangle|^2\right)_o = |\vec{A}(k_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\cdot\hat{z}|^2 \quad . \tag{5a.3}$$

For type 2 the integrands in equation (5a.1) will be:

$$(\sum_f |\langle T\rangle|^2)_{unpol} = (\vec{B}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - 2\vec{C}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4))\cdot$$

$$\cdot(\vec{B}^*(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - 2\vec{C}^*(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)) \qquad (5a.4)$$

and

$$(\sum_f |\langle T\rangle|^2)_o = |[\vec{B}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - 2\vec{C}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)]\cdot\hat{z}|^2 \quad . \qquad (5a.5)$$

The practice of dropping constant multipliers from these functions after $\vec{k}_4$ has been set equal to $-\vec{k}_{12}-\vec{k}_3$ will be followed again.

B. Distributions for $J^P = 1^-$

The vector distribution for type 1 is derived from the simplest completely symmetric vector which is:

$$\vec{A}(1^-) = \vec{k}_1\omega_1 + \vec{k}_2\omega_2 + \vec{k}_3\omega_3 + \vec{k}_4\omega_4 \quad . \qquad (5b.1)$$

The sum of the momenta without the energy factors is symmetric, but zero, so equation (5a.6) is the simplest suitable non-trivial vector. It is convenient to express $\vec{A}(1^-)$ in terms of its components in the body frame. After setting $\vec{k}_4$ equal to $-\vec{k}_{12}-\vec{k}_3$ and using equations (2d.4) and (2d.5) and the change of variables of equation (2e.2), the result is:

$$(\vec{A}(1^-))_{x'''} = -\frac{2}{k_{12}(2\omega)^2}\left[\begin{array}{l} m_{12}[(2\omega k_{12}\frac{P_{12}}{m_{12}})^2 - \xi^2]^{1/2}\xi \\ + m_{34}[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2}\eta \cos\phi_{(12)3} \end{array}\right]$$

$$(\vec{A}(1^-))_{y'''} = -\frac{2}{k_{12}(2\omega)^2}\, m_{34}[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2}\eta \sin \phi_{(12)3}$$

$$(\vec{A}(1^-))_{z'''} = \frac{1}{k_{12}(2\omega)^3}\begin{bmatrix} (2\omega k_{12})^2(\alpha_{12}-\alpha_{34}) \\ +4(\alpha_{12}\xi^2 - \alpha_{34}\eta^2) \end{bmatrix} . \qquad (5b.2)$$

The unpolarized distribution is given by the sum of the squares of these terms. Integration over ξ and η causes the cross term from the X''' component to vanish. The coefficients of $\cos^2 \phi_{(12)3}$ and $\sin^2 \phi_{(12)3}$ are the same so these terms add together and the dependence of the distribution on $\phi_{(12)3}$ drops out. Thus the unpolarized distribution is given by:

$$d^5R(4,{}_1 1^-)_{unpol} = \frac{\pi}{(2\omega)^3}\, d(\cos\beta)\; d\gamma\; dm_{12} dm_{34} d\phi_{(12)3} k_{12}^3 P_{12} P_{34} \cdot$$

$$\cdot[(2^3/3\cdot 5)(2\omega)^2 [m_{12}^2(\frac{P_{12}}{m_{12}})^4 \quad m_{34}^2(\frac{P_{34}}{m_{34}})^4]$$

$$+\begin{bmatrix} (\alpha_{12} - \alpha_{34})^2 \\ +(2^3/3)(\alpha_{12}-\alpha_{34})[\alpha_{12}(\frac{P_{12}}{m_{12}})^2 - \alpha_{34}(\frac{P_{34}}{m_{34}})^2] \\ +2^4/5\; \alpha_{12}^2(\frac{P_{12}}{m_{12}})^4 - 2^5/9\; \alpha_{12}\alpha_{34}(\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^2 \\ +\; 2^4/5(\frac{P_{34}}{m_{34}})^4 \end{bmatrix}] . \qquad (5b.3)$$

Now the vector $\vec{A}$ may be converted to a spherical basis and the part with zero angular momentum along the Z-axis is just $\vec{A}(1^-)\cdot\hat{Z}$. The components of the unit vector in the spatial Z-direction were given by equation (2c.17) so:

$$\vec{A}(1^-)\cdot\hat{z} = +\frac{2}{k_{12}(2\omega)^2}\, m_{12}[(2\omega k_{12}\frac{P_{12}}{m_{12}})^2-\xi^2]^{1/2}\,\xi \cos\gamma \sin\beta$$

$$+\frac{2}{k_{12}(2\omega)^2}\, m_{34}[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2-\eta^2]^{1/2}\,\eta \cos(\phi_{(12)3}+\gamma)\sin\beta$$

$$+\frac{1}{k_{12}(2\omega)^3}\begin{bmatrix}(2\omega k_{12})^2(\alpha_{12}-\alpha_{34})\\ +4(\alpha_{12}\xi^2-\alpha_{34}\eta^2)\end{bmatrix}\cos\beta \quad . \qquad (5b.4)$$

The cross terms in the square of this quantity all contain odd powers of ξ or η or both and so they do not contribute to the distribution after the integrations are performed. Thus, keeping track of the origin of the terms in equation (5b.3) it will be seen that the distribution when the resonance decays from an eigenstate of spin projection zero along the z-axis is:

$$d^5R(4,{}_1 1^-)_o = \frac{\pi}{(2\omega)^3}\, d(\cos\beta)\, d\gamma\, dm_{12}dm_{34}d\phi_{(12)3}k_{12}^3P_{12}P_{34}\,\cdot$$

$$\cdot[(2^3/3\cdot5)(2\omega)^2\begin{bmatrix}m_{12}^2(\frac{P_{12}}{m_{12}})^4\cos^2\gamma\sin^2\beta\\ +m_{34}^2(\frac{P_{34}}{m_{34}})^4\cos^2(\phi_{(12)3}+\gamma)\sin^2\beta\end{bmatrix}$$

$$+\begin{bmatrix}(\alpha_{12}-\alpha_{34})^2\\ +2^3/3\,(\alpha_{12}-\alpha_{34})[\alpha_{12}(\frac{P_{12}}{m_{12}})^2-\alpha_{34}(\frac{P_{34}}{m_{34}})^2]\\ +\frac{2^4}{5}\alpha_{12}^2(\frac{P_{12}}{m_{12}})^4-\frac{2^5}{9}\alpha_{12}\alpha_{34}(\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^2+\frac{2^4}{5}(\frac{P_{34}}{m_{34}})^4\end{bmatrix}\cos^2\beta]. \qquad (5b.5)$$

If an integration is performed over m_{12} and m_{34} that is, over the Goldhaber triangle, the coefficients of $\cos^2 \gamma \sin^2 \beta$ and $\cos^2(\phi_{(12)3} + \gamma) \sin^2 \beta$ will be equal.

For the vector distribution of type 2, the simplest vectors with the required symmetry are:

$$\vec{B}(1^-) = (\vec{k}_1-\vec{k}_2)(\omega_3-\omega_4) + (\vec{k}_3-\vec{k}_4)(\omega_1-\omega_2)$$

$$\vec{C}(1^-) = (\vec{k}_1-\vec{k}_3)(\omega_2-\omega_4) + (\vec{k}_2-\vec{k}_4)(\omega_1-\omega_3) \quad . \qquad (5b.6)$$

Imposing conservation of momentum, dropping a factor of 2, and shuffling some terms, it is seen that

$$\vec{B}(1^-) - 2\vec{C}(1^-) = \vec{A}(1^-) - \frac{3}{2}\vec{k}_{12}(\omega_1+\omega_2-\omega_3-\omega_4) \quad . \qquad (5b.7)$$

Thus, in the body frame, $\vec{B}(1^-) - 2\vec{C}(1^-)$ differs from $\vec{A}(1^-)$ only in the third term:

$$(\vec{B}(1^-) - 2\vec{C}(1^-))_{z'''} = \frac{1}{k_{12}(2\omega)^3}\begin{bmatrix}(2\omega k_{12})^2(\alpha_{12}-\alpha_{34}) \\ +4(\alpha_{12}\xi^2-\alpha_{34}\eta^2)\end{bmatrix}$$

$$= \frac{1}{k_{12}(2\omega)^3}\left[3(2\omega k_{12})^2(\alpha_{12}-\alpha_{34})\right]$$

$$= \frac{1}{k_{12}(2\omega)^3}\begin{bmatrix}(2\omega k_{12})^2(-2(\alpha_{12}-\alpha_{34})) \\ +4(\alpha_{12}\xi^2-\alpha_{34}\eta^2)\end{bmatrix} \quad . \qquad (5b.8)$$

Thus the desired distributions for type 2 are rather easily obtained from those of type 1:

$$d^5R(4,{}_21^-)_{unpol} = \frac{\pi}{(2\omega)^3} d(\cos\beta)\, d\gamma\, dm_{12} dm_{34} d\phi_{(12)3} k_{12}^3 P_{12} P_{34} \cdot$$

$$\cdot [(2^3/3\cdot 5)(2\omega)^2 [m_{12}^2 (\frac{P_{12}}{m_{12}})^4 + m_{34}^2 (\frac{P_{34}}{m_{34}})^4]$$

$$+ \begin{bmatrix} 4(\alpha_{12}-\alpha_{34})^2 \\ -(2^4/3)(\alpha_{12}-\alpha_{34})[\alpha_{12}(\frac{P_{12}}{m_{12}})^2 - \alpha_{34}(\frac{P_{34}}{m_{34}})^2] \\ +\frac{2^4}{5}\alpha_{12}^2(\frac{P_{12}}{m_{12}})^4 - \frac{2^5}{9}\alpha_{12}\alpha_{34}(\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^2 + \frac{2^4}{5}\alpha_{34}^2(\frac{P_{34}}{m_{34}})^4 \end{bmatrix}]$$

(5b.9)

and:

$$d^5R(4,{}_21^-)_0 = \frac{\pi}{(2\omega)^3} d(\cos\beta)\, d\gamma\, dm_{12} dm_{34} d\phi_{(12)3} k_{12}^3 P_{12} P_{34} \cdot$$

$$\cdot [(2^3/3\cdot 5)(2\omega)^2 \begin{bmatrix} m_{12}^2 (\frac{P_{12}}{m_{12}})^4 \cos^2\gamma \sin^2\beta \\ + m_{34}^2 (\frac{P_{34}}{m_{34}})^4 \cos^2(\phi_{(12)3}+\gamma) \sin^2\beta \end{bmatrix}$$

$$+ \begin{bmatrix} 4(\alpha_{12}-\alpha_{34})^4 \\ -2^4/3(\alpha_{12}-\alpha_{34})[\alpha_{12}(\frac{P_{12}}{m_{12}})^2 - \alpha_{34}(\frac{P_{34}}{m_{34}})^2] \\ +\frac{2^4}{5}\alpha_{12}^2(\frac{P_{12}}{m_{12}})^4 - \frac{2^5}{9}\alpha_{12}\alpha_{34}(\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^2 + \frac{2^4}{5}\alpha_{34}^2(\frac{P_{34}}{m_{34}})^4 \end{bmatrix} \cos^2\beta].$$

(5b.10)

The difference between the type 1 and type 2 vector distributions is not dramatic. It may be difficult to distinguish between them unless the data fits the criterion for adding the unpolarized and Z-component of angular momentum zero distributions in which case the coefficient of $\cos^2\beta$ compared to the coefficients of the other terms would make the distinction.

C. Distributions for $J^P = 1^+$

For type 1, the pseudovector $\vec{A}(1^+)$ has been determined in connection with the discussion of the type 1 pseudoscalar distributions. It is given by:

$$\begin{aligned}\vec{A}(1^+) = \; & \vec{k}_1 \times \vec{k}_2 \; (\omega_1-\omega_2)(\omega_1-\omega_4)(\omega_2-\omega_4) \\ & + \vec{k}_1 \times \vec{k}_3 \; (\omega_1-\omega_3)(\omega_1-\omega_4)(\omega_3-\omega_4) \\ & + \vec{k}_2 \times \vec{k}_3 \; (\omega_2-\omega_3)(\omega_2-\omega_4)(\omega_3-\omega_4) \quad . \end{aligned} \tag{5c.1}$$

The vector $\vec{k}_1 \times \vec{k}_2$ points in the body Y''' direction. The body components of $\vec{k}_1 \times \vec{k}_3$ are:

$$(\vec{k}_1 \times \vec{k}_3)_{x'''} = -\frac{1}{2k_{12}^2}(k_{12}k_3\sin\theta_{(12)3})(k_{12}^2+\omega_1^2-\omega_2^2)\sin\phi_{(12)3}$$

$$\begin{aligned}(\vec{k}_1 \times \vec{k}_3)_{y'''} = & -\frac{1}{2k_{12}^2}(k_1k_2\sin\theta_{12})(k_{12}^2+(\omega_3^2-\omega_4^2)) \\ & + \frac{1}{2k_{12}^2}(k_{12}k_3\sin\theta_{(12)3})(k_{12}^2+\omega_1^2-\omega_2^2)\cos\phi_{(12)3}\end{aligned}$$

$$(\vec{k}_1 \times \vec{k}_3)_{z'''} = -\frac{1}{k_{12}^2}(k_1k_2\sin\theta_{12})(k_{12}k_3\sin\theta_{(12)3})\sin\phi_{(12)3} \tag{5c.2}$$

The body components of $\vec{k}_2 \times \vec{k}_3$ are given by:

$$(\vec{k}_2 \times \vec{k}_3)_{x'''} = -\frac{1}{2k_{12}^2} (k_{12}k_3 \sin\theta_{(12)3})(k_{12}^2 - (\omega_1^2 - \omega_2^2)\sin\phi_{(12)3}$$

$$(\vec{k}_2 \times \vec{k}_3)_{y'''} = \frac{1}{2k_{12}^2} (k_1k_2 \sin\theta_{12}) \; (k_{12}^2 + (\omega_3^2 - \omega_4^2))$$

$$+ \frac{1}{2k_{12}^2} (k_{12}k_3 \sin\theta_{(12)3})(k_{12}^2 - (\omega_1^2 - \omega_2^2)) \cos\phi_{(12)3}$$

$$(\vec{k}_2 \times \vec{k}_3)_{z'''} = \frac{1}{k_{12}^2} (k_1k_2 \sin\theta_{12})(k_{12}k_3 \sin\theta_{(12)3})\sin\phi_{(12)3} \quad . \qquad (5c.3)$$

The components of $\vec{A}(1^+)$ in the body frame, with the substitutions of equation (2e.2) are found to be:

$$(\vec{A}(1^+))_{x'''} = +\frac{2m_{34}}{k_{12}^2(2\omega)^6} [(2\omega k_{12} \frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2} \cdot \begin{bmatrix} (2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2 \\ + [(2\omega k_{12})^2 + 2^3\alpha_{12}(\alpha_{12}-\alpha_{34})]\xi^2 \\ - (2\omega k_{12})^2 \eta^2 \end{bmatrix} \eta \sin\phi_{(12)3}$$

$$(\vec{A}(1^+))_{y'''} = \frac{2m_{12}}{k_{12}^2(2\omega)^6} [(2\omega k_{12} \frac{P_{12}}{m_{12}})^2 - \xi^2]^{1/2} \cdot \begin{bmatrix} (2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2 \\ - (2\omega k_{12})^2 \xi^2 \\ + [(2\omega k_{12})^2 - 2^3\alpha_{34}(\alpha_{12}-\alpha_{34})] \eta^2 \end{bmatrix} \xi$$

$$- \frac{2m_{34}}{k_{12}^2(2\omega)^6} [(2\omega k_{12} \frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2} \cdot$$

$$\cdot \begin{bmatrix} (2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2 \\ + [(2\omega k_{12})^2 - 2^3\alpha_{12}(\alpha_{12}-\alpha_{34})] \xi^2 \\ - (2\omega k_{12})^2 \eta^2 \end{bmatrix} \eta \cos\phi_{(12)3}$$

$$(\vec{A}(1^+))_{z'''} = + \frac{2^3 m_{12} m_{34}}{k_{12}^2(2\omega)^5} [(2\omega k_{12} \frac{P_{12}}{m_{12}})^2 - \xi^2]^{1/2} [(2\omega k_{12} \frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2} \cdot$$

$$\cdot (\alpha_{12}-\alpha_{34}) \xi\eta \sin\phi_{(12)3} \quad . \qquad (5c.4)$$

The unpolarized distribution is derived by integrating the sum of the squares of the components over ξ and η. The cross term in the square of the second component integrates out to zero. Thus the unpolarized distribution is given by:

$$d^5R(4,{}_1 1^+)_{unpol} = \frac{2^3\pi}{(2\omega)^5} d(\cos\beta)\, d\gamma\, dm_{12} dm_{34} d\phi_{(12)3}\, k_{12}^5\, P_{12}\, P_{34} \cdot$$

$$\cdot [m_{34}^2 \begin{bmatrix} (1/3\cdot5)(\alpha_{12}-\alpha_{34})^4 (\frac{P_{34}}{m_{34}})^4 \\ +(2/3^2\cdot5)(\alpha_{12}-\alpha_{34})^2[(2\omega k_{12})^2+2^3\alpha_{12}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^4 \\ -(2/5\cdot7)(2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2(\frac{P_{34}}{m_{34}})^6 \\ +(1/3\cdot5^2)[(2\omega k_{12})^2+2^3\alpha_{12}(\alpha_{12}-\alpha_{34})]^2(\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^4 \\ -(2/3\cdot5\cdot7)(2\omega k_{12})^2[(2\omega k_{12})^2+2^3\alpha_{12}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^6 \\ +(1/7\cdot9)(2\omega k_{12})^4(\frac{P_{34}}{m_{34}})^8 \end{bmatrix}$$

$$+ m_{12}^2 \begin{bmatrix} (1/3\cdot5)(\alpha_{12}-\alpha_{34})^4(\frac{P_{12}}{m_{12}})^4 \\ +(2/3^2\cdot5)(\alpha_{12}-\alpha_{34})^2[(2\omega k_{12})^2-2^3\alpha_{34}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^2 \\ -(2/5\cdot7)(2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2(\frac{P_{12}}{m_{12}})^6 \\ +(1/3\cdot5^2)[(2\omega k_{12})^2-2^3\alpha_{34}(\alpha_{12}-\alpha_{34})]^2(\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^4 \\ -(2/3\cdot5\cdot7)(2\omega k_{12})^2[(2\omega k_{12})^2-2^3\alpha_{34}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^6(\frac{P_{34}}{m_{34}})^2 \\ +(1/7\cdot9)(2\omega k_{12})^4(\frac{P_{12}}{m_{12}})^8 \end{bmatrix}$$

$$+ (2^5/3^2\cdot5^2)m_{12}^2\, m_{34}^2\, (2\omega)^2(\alpha_{12}-\alpha_{34})^2(\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^4 \sin\phi_{(12)3}] \cdot \qquad (5c.5)$$

When the resonance decays from the eigenstate with angular momentum zero along the Z-axis, the function which must be squared and integrated is:

$$\vec{A}(1^+)\cdot\hat{z} = -\frac{2m_{34}}{k_{12}^2(2\omega)^6}[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2} \cdot$$

$$\cdot \begin{bmatrix} (2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2 \\ +[(2\omega k_{12})^2+2^3\alpha_{12}(\alpha_{12}-\alpha_{34})]\xi^2 \\ -(2\omega k_{12})^2\eta^2 \end{bmatrix} \eta \sin(\gamma+\phi_{(12)3})\sin\beta$$

$$+ \frac{2m_{12}}{k_{12}^2(2\omega)^6}[(2\omega k_{12}\frac{P_{12}}{m_{12}})^2 - \xi^2]^{1/2} \cdot$$

$$\cdot\begin{bmatrix}(2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2\\ -(2\omega k_{12})^2\,\xi^2\\ +[(2\omega k_{12})^2-2^3\alpha_{34}(\alpha_{12}-\alpha_{34})]\,\eta^2\end{bmatrix}\xi\,\sin\gamma\,\sin\beta$$

$$+\frac{2^3 m_{12}m_{34}}{k_{12}^2(2\omega)^5}\left[(2\omega k_{12}\frac{P_{12}}{m_{12}})^2-\xi^2\right]^{1/2}\left[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2-\eta^2\right]^{1/2}\cdot$$

$$\cdot(\alpha_{12}-\alpha_{34})\,\xi\eta\,\sin\phi_{(12)3}\,\cos\beta \quad . \qquad (5c.6)$$

In the square of this expression, all of the cross terms are odd functions of ξ or η or both. The cross terms therefore drop out of the integration and the process of integrating is the same as followed in deriving equation (5c.5). Therefore the distribution for decay from the state with the Z-component of angular momentum equal to zero is:

$$d^5R(4,{}_1 1^+)_o=\frac{2^3\pi}{(2\omega)^5}\,d(\cos\beta)\,d\gamma\;dm_{12}dm_{34}d\phi_{(12)3}\;k_{12}^5\,P_{12}\,P_{34}\;\cdot$$

$$\cdot\,[m_{34}^2\begin{bmatrix}(1/3\cdot5)(\alpha_{12}-\alpha_{34})^4(\frac{P_{34}}{m_{34}})^4\\ +(2/3^2\cdot5)(\alpha_{12}-\alpha_{34})^2[(2\omega k_{12})^2+2^3\alpha_{12}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^4\\ -(2/5\cdot7)(2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2(\frac{P_{34}}{m_{34}})^6\\ +(1/3\cdot5^2)[(2\omega k_{12})^2+2^3\alpha_{12}(\alpha_{12}-\alpha_{34})]^2(\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^4\\ -(2/3\cdot5\cdot7)(2\omega k_{12})^2[(2\omega k_{12})^2+2^3\alpha_{12}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^2(\frac{P_{34}}{m_{34}})^6\\ +(1/7\cdot9)(2\omega k_{12})^4(\frac{P_{34}}{m_{34}})^8\end{bmatrix}\sin^2\beta\cdot\sin^2(\gamma+\phi_{(12)3}) \quad .$$

$$+ m_{12}^2 \begin{bmatrix} (1/3\cdot5)(\alpha_{12}-\alpha_{34})^4(\frac{P_{12}}{m_{12}})^4 \\ +(2/3^2\cdot5)(\alpha_{12}-\alpha_{34})^2[(2\omega k_{12})^2-2^3\alpha_{34}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^2 \\ -(2/5\cdot7)(2\omega k_{12})^2(\alpha_{12}-\alpha_{34})^2(\frac{P_{12}}{m_{12}})^6 \\ +(1/3\cdot5^2)[(2\omega k_{12})^2-2^3\alpha_{34}(\alpha_{12}-\alpha_{34})]^2(\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^4 \\ -(2/3\cdot5\cdot7)(2\omega k_{12})^2[(2\omega k_{12})^2-2^3\alpha_{34}(\alpha_{12}-\alpha_{34})](\frac{P_{12}}{m_{12}})^6(\frac{P_{34}}{m_{34}})^2 \\ +(1/7\cdot9)(2\omega k_{12})^4(\frac{P_{12}}{m_{12}})^8 \end{bmatrix} \sin^2\beta\cdot \\ \cdot\sin^2\gamma$$

$$+ (2^5/3^2\cdot5^2)m_{12}^2\, m_{34}^2\, (2\omega)^2(\alpha_{12}-\alpha_{34})^2(\frac{P_{12}}{m_{12}})^4(\frac{P_{34}}{m_{34}})^4\sin^2\phi_{(12)3}\cos^2\beta] \quad . \tag{5c.7}$$

If an integration is performed over the Goldhaber triangle, the coefficients of $\sin^2(\gamma+\phi_{(12)3})\sin^2\beta$ and $\sin^2\gamma\sin^2\beta$ will be equal.

For the pseudovector of type 2, the simplest functions were discussed in connection with the pseudoscalar of type 2. Equation (4c.10) shows that the pseudovector distribution is obtained by squaring and integrating:

$$\vec{B}(1^+) - 2\vec{C}(1^+) = \vec{k}_1 \times \vec{k}_2\ (\omega_1-\omega_2) + \vec{k}_{12} \times \vec{k}_3\ (\omega_3-\omega_4) \quad . \tag{5c.8}$$

The components of this pseudovector in the body frame are:

$$(\vec{B}(1^+)-2\vec{C}(1^+))_{x''} = + \frac{2m_{34}}{(2\omega)^2}[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2 - \eta^2]^{1/2}\,\eta\,\sin\phi_{(12)3}$$

$$(\vec{B}(1^+)-2\vec{C}(1^+))_{y'''} = \frac{2m_{12}}{(2\omega)^2}\left[(2\omega k_{12}\frac{P_{12}}{m_{12}})^2 - \xi^2\right]^{1/2}\xi$$

$$-\frac{2m_{34}}{(2\omega)^2}\left[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2 - \eta^2\right]^{1/2}\eta\cos\phi_{(12)3}$$

$$(\vec{B}(1^+)-2\vec{C}(1^+))_{z'''} = 0 \quad . \tag{5c.9}$$

Equation (5c.9) shows that the pseudovector distribution of type 2 is of a rather simple sort. The cross term in the square of the second component integrates out, as usual. The unpolarized distribution is given by:

$$d^5R(4,{}_21^+)_{unpol} = \frac{2^3\pi}{(2\omega)}\, d(\cos\beta)\, d\gamma\, dm_{12}dm_{34}d\phi_{(12)3}k_{12}^5\, P_{12}\, P_{34}\cdot$$

$$\cdot\begin{bmatrix}(1/3\cdot5)m_{34}^2\,(\frac{P_{34}}{m_{34}})^4\\ +(1/3\cdot5)m_{12}^2\,(\frac{P_{12}}{m_{12}})^4\end{bmatrix} \tag{5c.10}$$

The component of $\vec{B}(1^+)-2\vec{C}(1^+)$ along the spatial Z-axis is given by:

$$(\vec{B}(1^+)-2\vec{C}(1^+))\cdot\hat{z} = -\frac{2m_{34}}{(2\omega)^2}\left[(2\omega k_{12}\frac{P_{34}}{m_{34}})^2 - \eta^2\right]^{1/2}\eta\,\sin(\gamma+\phi_{(12)3})\cdot$$
$$\cdot\sin\beta$$

$$+\frac{2m_{12}}{(2\omega)^2}\left[(2\omega k_{12}\frac{P_{12}}{m_{12}})^2 - \xi^2\right]^{1/2}\xi\,\sin\gamma\,\sin\beta \quad . \tag{5c.11}$$

The distribution for the resonance decaying from its eigenstate of zero angular momentum along the Z-axis is:

$$d^5R(4,{}_2 1^+)_o = \frac{2^3\pi}{(2\omega)} d(\cos\beta)\, d\gamma\, dm_{12}\, dm_{34}\, d\phi_{(12)3}\, k_{12}^5\, P_{12}\, P_{34} \cdot$$

$$\cdot \left[\begin{array}{l} (1/3\cdot 5) m_{34}^2 \left(\frac{P_{34}}{m_{34}}\right)^4 \sin^2(\gamma+\phi_{(12)3}) \sin^2\beta \\ +(1/3\cdot 5) m_{12}^2 \left(\frac{P_{12}}{m_{12}}\right)^4 \sin^2\gamma \sin^2\beta \end{array} \right] . \qquad (5c.12)$$

It will be noticed that the coefficients of $\sin^2(\gamma+\phi_{(12)3})\sin^2\beta$ and $\sin^2\gamma$ $\sin^2\beta$ are the same in this distribution, after integration over m_{12} and m_{34}. The type 2 distributions are much simpler than the type 1 distributions when the resonance is a pseudovector. The type 1 distributions show the trough along $m_{12} = m_{34}$ near the diagonal boundary of the Goldhaber triangle.

VI. CONCLUSIONS

The simplest distributions for the decay of an isotopic singlet resonance into four pions have been calculated. The calculations have been performed for each parity and symmetry type for spin zero and spin one. The vector and pseudovector distributions have been calculated assuming first that the resonance is produced unpolarized and second that it is in the eigenstate with the Z-component of angular momentum equal to zero.

If the resonance is produced with a single extra pion in proton-antiproton annihilation, and if the resonance is produced with good statistics in the backward or forward direction, then the data should be tested for anisotropy in the zero-momentum frame of the resonance. The sum of $d^5R(4)_{unpol}$ and $d^5R(4)_o$ for the four types of vector distributions apply when the resonance is produced in the forward or backward direction if production takes place equally from the four initial spin states. If there is such anisotropy, the spin of the resonance is not zero. To test for spin one, the data can be compared with the distributions:

$$d^3R(4,{}_1 1^-)_{unpol+o} = d(\cos\beta)\,d\gamma\,d\phi_{(12)3}\begin{bmatrix} 1 + a\cos^2\beta \\ + b[\cos^2\gamma+\cos^2(\gamma+\phi_{(12)3})]\sin^2\beta \end{bmatrix}$$

$$d^3R(4,{}_2 1^-)_{unpol+o} = d(\cos\beta)\,d\gamma\phi_{(12)3}\begin{bmatrix} 1 + a'\cos^2\beta \\ + b'[\cos^2\gamma+\cos^2(\gamma+\phi_{(12)3})]\sin^2\beta \end{bmatrix}$$

$$d^3R(4,{}_1 1^+)_{unpol+o} = d(\cos\beta)\,d\gamma\,d\phi_{(12)3}\begin{bmatrix} 1 + a''\sin^2\phi_{(12)3}(1+\cos^2\beta) \\ +1/2[\sin^2\gamma+\sin^2(\gamma+\phi_{(12)3})]\sin^2\beta \end{bmatrix}$$

$$d^3R(4,{}_21^{\pm})_{unpol+o} = d(\cos\beta)\,d\gamma\, d\phi_{(12)3} \quad [1+1/2(\sin^2\gamma+\sin^2(\gamma+\phi_{(12)3}))\sin^2\beta] \ . \qquad (6.1)$$

The most practical final state is $2\pi^+2\pi^-$ because each decay product leaves a track. Figure 4 shows how the angles $\phi_{(12)3}$, β, and γ are defined for this case. It will be easy to differentiate between positive and negative parity for spin one. The distinction will also be clear between the two symmetry types for $J^P=1^+$. If the distribution fits that for 1^-, the constants a, a', b, and b' can be calculated and differentiation between the two symmetry types may be possible.

If the resonance is not produced significantly in the forward or backward directions, or is produced in a final state with six or more pions, then recourse will be made to the isotropic distributions. The eight isotropic distributions for each parity and symmetry type for spins zero and one can be examined for dependence on the angle $\phi_{(12)3}$ between the plane of the $2\pi^+$ and the plane of the $2\pi^-$. The distributions for both symmetry types for $J^P=0^+$ and 1^- are independent of the angle $\phi_{(12)3}$. The remaining distributions are:

$$dR(4,{}_10^-) = \sin^2\phi_{(12)3}\ d\phi_{(12)3}$$

$$dR(4,{}_20^-) = \sin^2\phi_{(12)3}\ d\phi_{(12)3}$$

$$dR(4,{}_11^+)_{unpol} = [1+a\sin^2\phi_{(12)3}]\ d\phi_{(12)3}$$

$$dR(4,{}_21^+)_{unpol} = d\phi_{(12)3} \quad . \qquad (6.2)$$

This analysis will distinguish $J^P=0^-$ from the rest but won't distinguish between the symmetry types. Type 1 symmetry with $J^P=1^+$ will be uniquely

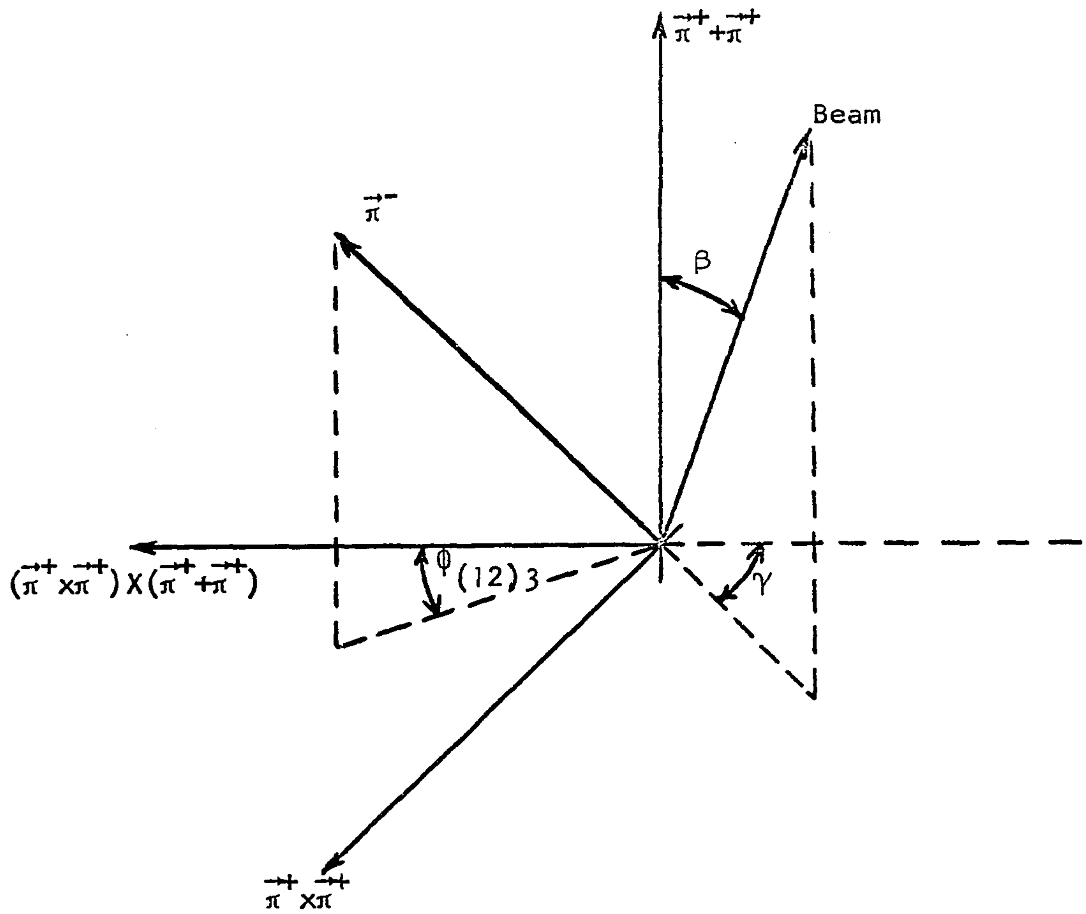

Figure 4. Definition of the angles $\phi_{(12)3}$, β and γ

identified. Those cases left uncertain must be investigated in their dependence on the variables m_{12} and m_{34}. In particular the formation of a trough along part of the line $m_{12}=m_{34}$ on the Goldhaber triangle has been mentioned. Finally there is the possibility that none of the distributions are applicable because nature chooses to obey something other than the simplest possibilities.

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