# ON THE HAUSDORFF HYPERSPACE OF A COMPACT METRIC SPACE

by

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TABLE OF CONTENTS

# INTRODUCTION

The collection of non-empty closed subsets of a compact metric space X when metricized by the Hausdorff metric yields an interesting topological space $2^X$. Application of the structure of this space has been found most useful in the study of such topics as Knaster continua, local separating points, and linear ordering of topological spaces.

In this paper, a survey is made of the results obtained by various authors investigating the properties of $2^X$. In addition to these known results, the following theorem has been established: If X and Y are compact metric spaces, f and g are functions of X to Y, and H is a homotopy of f to g, then $H^*$ defined by $H^*(E,t) = H(E,t)$ is a homotopy of $f^*$ to $g^*$. Also a few examples of $2^X$ for familiar X are given.

## PRELIMINARIES

The following notation and definitions will be used throughout this discussion. $(X,d)$ will denote a compact, metric space. For a point $x$ in $X$ and a positive real number $r$, $B(x,r)$ will denote $\{y \text{ in } X: d(x,y) < r\}$. $D$ will denote $\{B(x,r): x \text{ is in } X \text{ and } r > 0\}$, i.e., the standard d-ball base. $T$ will denote the topology for $X$ that is generated by base $D$.

The Hausdorff hyperspace of $X$ can be generated in the following manner.

<u>Definition</u> Let $2^X$ denote the class of closed, non-empty subsets of $X$.

<u>Definition</u> For $x$ in $X$ and $B$ a subset of $X$, let $d(x,B)$ denote $\inf\{d(x,b): b \text{ in } B\}$.

<u>Definition</u> Let $s(A,B)$ denote the real-valued function defined on $2^X \times 2^X$ by $s(A,B) = \sup\{d(x,B): x \text{ is in } A\}$.

<u>Definition</u> Let $h(A,B)$ denote the real-valued function defined on $2^X \times 2^X$ by $h(A,B) = \sup\{s(A,B), s(B,A)\}$.

Note that for $A$ and $B$ in $2^X$ and any point $a$ of $A$, there exists a point $b$ in $B$ such that $d(a,b) \leq h(A,B)$. This is true because either (1) point $a$ is in $B$ whence $a = b$ and $d(a,b) = 0 \leq h(A,B)$ or (2) $\{a\}$ and $B$ are disjoint compact sets and there exists a point $b$ in $B$ such that $d(a,B) = d(a,b)$. Then $h(A,B) \geq s(A,B) \geq d(a,B) = d(a,b)$.

An equivalent definition for the function $h$ is given by

Kelley (4). The equivalence is shown in Theorem 1.

<u>Definition</u> For E a member of $2^X$ and r a positive real number, let $V_r(E)$ denote $\bigcup_{x \text{ in } E} B(x,r)$.

<u>Theorem 1</u> $h(A,B) = \inf\{e: A \subset V_e(B) \text{ and } B \subset V_e(A)\}$.

<u>Proof</u> $h(A,B) = \sup\{\sup\{d(a,B): a \text{ in } A\}, \sup\{d(b,A): b \text{ in } B\}\} = \inf\{e: d(a,B) \leq e \text{ for each } a \text{ in } A \text{ and } d(b,A) \leq e \text{ for each } b \text{ in } B\} = \inf\{e: d(a,B) < e \text{ for each } a \text{ in } A \text{ and } d(b,A) < e \text{ for each } b \text{ in } B\} = \inf\{e: A \subset V_e(B) \text{ and } B \subset V_e(A)\}$.

The following lemma is an immediate consequence of Theorem 1 and will be used subsequently.

<u>Lemma 1</u> For any pair A,B in $2^X$, $h(A,B) < r$ implies that (1) for each a in A, there exists a point $b_a$ in B such that $d(a,b_a) < r$ and (2) for each b in B, there exists a point $a_b$ in A such that $d(b,a_b) < r$.

<u>Proof</u> Using Theorem 1, $h(A,B) < r$ implies that $A \subset V_r(B)$ and $B \subset V_r(A)$. Thus (1) and (2) hold.

For A and B in $2^X$, note that there exists a point a in A and a point b in B such that $d(a,b) = h(A,B)$. This is true since $h(A,B)$ equals $s(A,B)$ or $s(B,A)$. If $s(A,B) \geq s(B,A)$, then suppose for each point a in A that $d(a,B) < h(A,B)$. Let $q_a = (d(a,B) + h(A,B))/2$. Then $\{V_{q_a}(B): a \text{ in } A\}$ is an open cover for compact A. Thus, there exists a finite subcover for A. Let $r = \max\{q_a: V_{q_a}(B) \text{ is a member of the finite subcover}\}$. Then $A \subset V_r(B)$ and $r = (d(a_i,B) + h(A,B))/2 < h(A,B)$. This is a contradiction; hence there exists a point a in A such that $d(a,B) \geq h(A,B)$. As noted previously, there exists

a point b in B such that $d(a,b) \leq h(A,B)$. Thus $d(a,b) = h(A,B)$.

The function h is a metric for $2^X$ as is seen in Theorem 2 below.

<u>Theorem 2</u> h is a metric for $2^X$.

<u>Proof</u> (i) For each pair A,B in $2^X$, $d(x,y) \geq 0$ for all pairs x,y where point x is in A and point y is in B. Hence for each x in A, $d(x,B) = \inf\{d(x,y)\colon y \text{ is in } B\} \geq 0$. This implies that $s(A,B) \geq 0$ which in turn implies that $h(A,B) \geq s(A,B) \geq 0$.

(ii) If $A = B$, then $s(A,B) = 0 = s(B,A)$. Thus $h(A,B) = 0$. If $h(A,B) = 0$, then $s(A,B) = 0 = s(B,A)$ so that for each x in A, $d(x,B) = 0$. Hence $A \subset \overline{B} = B$. Likewise, for all y in B, $d(y,A) = 0$ and $B \subset \overline{A} = A$. Thus $A = B$.

(iii) For all A,B in $2^X \times 2^X$, $h(A,B) = \max\{s(A,B), s(B,A)\} = \max\{s(B,A), s(A,B)\} = h(B,A)$.

(iv) Let A,B,C be any triple in $2^X$. Let $x_i$ be any point in A. Since C and B are closed subsets of compact space X, both are compact so there is a point $y_i$ in B such that $d(x_i,B) = d(x_i,y_i)$ and there is a point $c_i$ in C such that $d(y_i,C) = d(y_i,c_i)$. Hence $d(x_i,C) \leq d(x_i,c_i) \leq d(x_i,y_i) + d(y_i,c_i) = d(x_i,B) + d(y_i,C) \leq \sup\{d(x,B)\colon x \text{ is in } A\} + \sup\{d(y,C)\colon y \text{ is in } B\} = s(A,B) + s(B,C)$. Since $x_i$ was an arbitrary point of A, $s(A,C) = \sup\{d(x,C)\colon x \text{ is in } A\} \leq s(A,B) + s(B,C)$. Likewise $s(C,A) \leq s(C,B) + s(B,A)$. Thus $h(A,C) = \max\{s(A,C), s(C,A)\} \leq \max\{s(A,B), s(B,A)\} + \max\{s(B, C), s(C,B)\} =$

$h(A,B) + h(B, C)$.

<u>Definition</u> For $E$ in $2^X$ and $r$ a positive real number, let $B(E,r)$ denote $\{F \text{ in } 2^X: h(E,F) < r\}$.

<u>Definition</u> Let $H$ denote $\{B(E,r): E \text{ is in } 2^X \text{ and } r > 0\}$ i.e., the standard h-ball base.

<u>Definition</u> Let $2^h$ denote the topology for $2^X$ that is generated by base $H$.

The function $h$ is called the Hausdorff metric for $X$. $(2^X, 2^h)$ is called the Hausdorff hyperspace of $X$. $X$ is shown to be isometric to a subset of its Hausdorff hyperspace in the following theorem.

<u>Theorem 3</u> The function $i$ from $(X,d)$ into $(2^X,h)$ defined by $i(x) = \{x\}$ is an isometry.

<u>Proof</u> $h(i(x),i(y)) = h(\{x\},\{y\}) = \max\{\sup\{d(x,y): x \text{ is in } \{x\}\}, \sup\{d(y,x): y \text{ is in } \{y\}\}\} = \max\{d(x,y),d(y,x)\} = d(x,y)$.

## COMPACTNESS PROPERTY OF $2^X$

In this section $(2^X,h)$ will be shown to be a compact metric space. This is an important result for numerous applications are made of it in the establishment of inheritance properties of $2^X$.

Whyburn (8), Dugundji (1), and Micheal (5) all outline proofs of the compactness property of $2^X$. In this paper, Micheal's outline will be followed.

To begin, Micheal defines an equivalent topology $2^T$ for $2^X$ called the finite topology. The definition of $2^T$ and its equivalence to $2^h$ are given below.

<u>Definition</u> For $\{O_1,\ldots,O_n\}$, a finite collection of subsets of X, let $/O_1,\ldots,O_n/$ denote $\{E$: E is in $2^X$, $E \subset \bigcup_{i=1}^{n} O_i$, and $E \cap O_i \neq \emptyset$ for all $i = 1,\ldots,n\}$.

<u>Definition</u> Let B denote $\{/U_1,\ldots,U_n/$: for $i = 1,\ldots,n$, $U_i$ is in $T\}$.

<u>Definition</u> For F a subset of $2^X$, let $F^*$ denote $\bigcup_{E \text{ in } F} E$.

<u>Theorem 4</u> B is a basis for a topology for $2^X$.

<u>Proof</u> (i) X is in T so $/X/$ is in B and $/X/ \subset B^*$. But $2^X = /X/$. Thus $B^* = 2^X$.

(ii) Let $W = /W_1,\ldots,W_n/$ and $V = /V_1,\ldots,V_m/$ be in B. Let E be in $W \cap V$. Note that $U = /W_1 \cap (\bigcup_{i=1}^{m} V_i),\ldots,W_n \cap (\bigcup_{i=1}^{m} V_i),$ $V_1 \cap (\bigcup_{i=1}^{n} W_i),\ldots,V_m \cap (\bigcup_{i=1}^{n} W_i)/$ is in B. Now $E \subset \bigcup_{i=1}^{n} W_i$ and

$E \subset \bigcup_{i=1}^{m} V_i$ so $E \subset (\bigcup_{i=1}^{n} W_i) \cap (\bigcup_{i=1}^{m} V_i)$. Also $E \cap W_i \neq \emptyset$ for all $i = 1,\ldots,n$ and $E \cap V_i \neq \emptyset$ for all $i = 1,\ldots,m$. Thus $E \cap (W_i \cap (\bigcup_{i=1}^{m} V_i)) \neq \emptyset$ for all $i = 1,\ldots,n$ and $E \cap (V_i \cap (\bigcup_{i=1}^{n} W_i)) \neq \emptyset$ for all $i = 1,\ldots,m$. Hence E is in U.

Let F be any member of U. Then $F \cap W_i \supset F \cap (W_i \cap (\bigcup_{i=1}^{m} V_i)) \neq \emptyset$ for all $i = 1,\ldots n$, and $F \cap V_i \supset F \cap (V_i \cap (\bigcup_{i=1}^{n} W_i)) \neq \emptyset$ for all $i = 1,\ldots,m$. Also $F \subset (\bigcup_{i=1}^{n} W_i) \cap (\bigcup_{i=1}^{m} V_i)$ so $F \subset \bigcup_{i=1}^{n} W_i$ and $F \subset \bigcup_{i=1}^{m} V_i$. Thus F is in $W \cap V$ which implies $U \subset W \cap V$. Hence for any two sets W and V in B and any set E in $W \cap V$, there exists a U in B such that E is in U and $U \subset W \cap V$.

Thus B is a base for a topology for $2^X$.

<u>Definition</u> Let $2^T$ denote the topology for $2^X$ that is generated by base B.

<u>Theorem 5</u> If $U = /U_1,\ldots,U_n/$ is in B and E is a member of U, then there is a $B(E,r)$ in H such that $B(E,r) \subset U$.

<u>Proof</u> E is in U implies that $E \subset \bigcup_{i=1}^{n} U_i$. Hence $E \cap (\bigcup_{i=1}^{n} U_i)' = \emptyset$ ( ' denotes complement). Since E and $(\bigcup_{i=1}^{n} U_i)'$ are disjoint, closed, compact subsets of metric space X,

$d(E,(\bigcup_{i=1}^{n} U_i)') > 0$. Let $q = (d(E,(\bigcup_{i=1}^{n} U_i)')/3$. Then $E \subset V_q(E)$ and $V_q(E) \cap (\bigcup_{i=1}^{n} U_i)' = \emptyset$ so that $V_q(E) \subset \bigcup_{i=1}^{n} U_i$. E is in U also implies that $E \cap U_i \neq \emptyset$ for all $i = 1,\ldots,n$. For all $i = 1,\ldots,n$, choose an $x_i$ in $E \cap U_i$. Since D is a base for T, for each $i = 1,\ldots,n$ there is a $B(x_i,b_i)$ in D such that $B(x_i,b_i) \subset U_i$. Let $r = \{\min b_1,\ldots,b_n,q\}$. Then for all $i = 1,\ldots,n$, $B(x_i,r) \subset U_i$.

Let A be a member of $B(E,r)$. Then by Lemma 1 above, $h(A,E) < r$ implies (1) and (2) below.

(1) For each $x_i$ above, there exists an $a_i$ in A such that $d(x_i,a_i) < r$. Thus point $a_i$ is in $B(x_i,r) \subset U_i$ so that $A \cap U_i \neq \emptyset$ for all $i = 1,\ldots,n$.

(2) For each point a in A, there exists a point $x_a$ in E such that $d(a,x_a) < r$. Then point a is in $B(x_a,r) \subset B(x_a,q) \subset V_q(E) \subset \bigcup_{i=1}^{n} U_i$. Thus $A \subset \bigcup_{i=1}^{n} U_i$. This implies A is in U. Since A was an arbitrary member of $B(E,r)$, $B(E,r) \subset U$.

<u>Theorem 6</u> For each $B(E,r)$ in H, there exists a $U = /U_1,\ldots, U_n/$ in B such that E is in U and $U \subset B(E,r)$.

<u>Proof</u> Let $B(E,r)$ be a member of H. For each x in E, x is in $B(x,r/3)$. E is a closed subset of compact metric space X so there exists a finite set $\{B(x_i,r/3)\}_{i=1}^{n}$ covering E. Then E is in $U = /B(x_1,r/3),\ldots,B(x_n,r/3)/$.

Let A be any member of U. Then $A \subset \bigcup_{i=1}^{n} B(x_i,r/3)$, so for each point a in A, there exists a point $x_i$ in E such that

$d(a,x_i) < r/3$. Hence $d(a,E) \leq r/3$ for all points a in A so $s(A,E) < r$. Now for each point $x_j$ in E, there exists a $B(x_i,r/3)$ containing point $x_j$. Thus $d(x_j,x_i) < r/3$. Since $A \cap B(x_i,r/3) \neq \emptyset$, there exists a point a in A such that $d(x_i,a) < r/3$. Hence $d(x_j,a) < 2r/3$ which implies that $d(x_j,A) \leq 2r/3$ for all points $x_j$ in E so $s(E,A) < r$. Thus $h(A,E) < r$ and A is in $B(E,r)$. Since A was an arbitrary member of U, $U \subset B(E,r)$.

<u>Theorem 7</u> $2^h$ and $2^T$ are equivalent topologies for $2^X$.

<u>Proof</u> From Theorem 5 above, for each U in B and each E in U, there exists a $B(F,r)$ in H such that E is in $B(F,r)$ and $B(F,r) \subset U$.

If A is in $B(E,r)$ and $A \neq E$, then $h(A,E) > 0$. Let $q = h(A,E)$. Then $B(A,(r - q)/3) \subset B(E,r)$. So for each $B(E,r)$ in H and each A in $B(E,r)$, there exists a $B(A,p)$ such that $B(A,p) \subset B(E,r)$. Then from Theorem 6 above, there exists a U in B such that U contains A and $U \subset B(A,p) \subset B(E,r)$.

Therefore base H and base B are equivalent. Thus $2^h$ and $2^T$ are equivalent topologies for $2^X$.

The following definition and three lemmas are stated by Dugundji (1, p. 253). They are useful in proving Theorem 8 which in turn is useful in proving that $2^X$ is compact.

<u>Definition</u> For O a subset of X, let J(O) denote $\{E \text{ in } 2^X: E \cap O \neq \emptyset\}$ and let I(O) denote $\{E \text{ in } 2^X: E \subset O\}$.

<u>Lemma 2</u> For O a subset of X, $J(O) = (I(O'))'$.

<u>Proof</u> Let E be in J(O). Then $E \cap O \neq \emptyset$ which implies $E \not\subset O'$

and so E is not in $I(O)'$. Thus E is in $(I(O'))'$.

Let E be a member of $2^X$ such that E is not in $J(O)$. Then $E \cap O = \emptyset$, so $E \subset O'$. Thus E is in $I(O')$ and not in $(I(O'))'$.

<u>Lemma 3</u> If O is an open subset of $(X,T)$, $I(O)$ and $J(O)$ are open subsets of $(2^X, 2^T)$.

<u>Proof</u> (i) $I(O) = \{E \text{ in } 2^X : E \subset O\} = /O/$ which is an open basis element in $(2^X, 2^T)$.

(ii) Let A be in $J(O)$. $\{B(a,1): a \text{ in } A\}$ is an open cover for compact set A in X. Thus there exists a finite subcover $\{B(a_i,1)\}_{i=1}^{n}$ for A. Let $V_i = B(a_i,1)$ for $i = 1,\ldots,n$ and $V_{n+1} = O$. Then A is in $/V_1,\ldots,V_{n+1}/$ which is in B. By Theorem 5, there exists a $B(A,r)$ in H such that $B(A,r) \subset /V_1,\ldots,V_{n+1}/$. Suppose there is a member C of $2^X$ in $(J(O))' \cap B(A,r)$. Then C is not in $J(O)$ so $C \cap O = \emptyset$. But C is in $B(A,r) \subset /V_1,\ldots,V_{n+1}/$, so $C \cap V_{n+1} = C \cap O \neq \emptyset$. This is a contradiction; hence $B(A,r) \subset J(O)$. Since A was an arbitrary member of $J(O)$, no member of $J(O)$ is a limit point of $(J(O))'$. Thus $J(O)$ is open.

<u>Lemma 4</u> If O is a closed subset of $(X,T)$, $J(O)$ and $I(O)$ are closed subsets of $(2^X, 2^T)$.

<u>Proof</u> O is closed in $(X,T)$ implies that $O'$ is open in $(X,T)$.

(i) By Lemma 3, $I(O')$ is open in $(2^X, 2^T)$. This implies $(I(O'))'$ is closed in $(2^X, 2^T)$ but by Lemma 2, $J(O) = (I(O'))'$. Thus $J(O)$ is closed in $(2^X, 2^T)$.

(ii) Also by Lemma 3, $J(O')$ is open in $(2^X, 2^T)$ which implies

$(J(O'))'$ is closed in $(2^X, 2^T)$. As a consequence of Lemma 2, $(J(O'))' = I(O)$. Thus $I(O)$ is closed in $(2^X, 2^T)$.

<u>Definition</u> For A and F in $2^X$ where $A \subset F$, let $K(A,F)$ denote $\{E \text{ in } 2^X: E \cap A \neq \emptyset \text{ and } E \subset F\}$.

Note that $K(A,F) = J(A) \cap I(F)$. Thus $K(A,F)$ is closed.

<u>Definition</u> Let S denote $\{K(A,F)$: A and F are in $2^X$ and $A \subset F\}$.

If A is any base for a topology for $2^X$ and c is any closed set in $2^X$, then for some $A_1 \subset A$, $c = (c')' = (\bigcup_{a \text{ in } A_1} a)' = \bigcap_{a \text{ in } A_1} a'$. Therefore, $C = \{a'$: a is in $A\}$ is a base for the closed sets in $2^X$. Noting this fact, the next theorem follows easily.

<u>Theorem 8</u> S is a sub-basis for the closed sets in $(2^X, h)$.

<u>Proof</u> B is a basis for the open sets in $(2^X, h)$ implies that $W = \{/U_1, \ldots, U_n/'$: $/U_1, \ldots, U_n/$ is in $B\}$ is a basis for the closed sets in $(2^X, h)$. Let $/U_1, \ldots, U_n/'$ be a member of W. Then $/U_1, \ldots, U_n/' = (J(U_1) \cap \ldots \cap J(U_n) \cap I(\bigcup_{i=1}^{n} U_i))' = I(U_1') \cup \ldots \cup I(U_n') \cup J((\bigcup_{i=1}^{n} U_i)') = K(U_1', U_1') \cup \ldots \cup K(U_n', U_n') \cup K((\bigcup_{i=1}^{n} U_i)', X)$. Hence each member of W is a finite union of members of S. Thus S is a sub-basis for the closed subsets of $(2^X, h)$.

Both Theorem 9 and Theorem 10 are due to Frink (2).

<u>Theorem 9</u> Every subset of S having the f.i.p. (finite intersection property) has a non-empty intersection.

Proof Let L be a subset of S having the f.i.p. Let $C = \{F$: $K(A,F)$ is in $L\}$. Then if $\{F_i\}_{i=1}^{n}$ is a finite subset of C,

there is a finite subset $\{K(A_i,F_i)\}_{i=1}^{n}$ of L corresponding to the $F_i$'s. Now there exists an E in $2^X$ such that E is a member of each $K(A_i,F_i)$, $i = 1,\ldots,n$. Thus $E \subset F_i$ for each $i = 1,\ldots,n$, $\bigcap_{i=1}^{n} F_i \neq \emptyset$, and C has the f.i.p. Since C is a collection of closed subsets of compact space X, $\bigcap C \neq \emptyset$. Now $\bigcap C$ is closed so $\bigcap C$ is in $2^X$. Also $\bigcap C \subset F$ for all F such that K(A,F) is in L.

Let K(A,F) be any member of L. Let $\{F_i\}_{i=1}^{n}$ be any finite subset of C. Let $\{K(A_i,F_i)\}_{i=1}^{n}$ be a corresponding subset of L. Then $M = \{K(A,F), K(A_1,F_1),\ldots,K(A_n,F_n)\}$ is a finite subset of L, so there exists an E in $2^X$ such that E is in $\bigcap M$. Then $E \bigcap A \neq \emptyset$. Also E is a subset of each $F_i$ so E is a subset of $\bigcap_{i=1}^{n} F_i$. Thus $A \bigcap (\bigcap_{i=1}^{n} F_i) \neq \emptyset$. Since $\{F_i\}_{i=1}^{n}$ was an arbitrary finite subset of C, $\{A\} \bigcup C$ is a class of closed subsets of compact X possessing the f.i.p. Hence $\bigcap (C \bigcup \{A\}) = (\bigcap C) \bigcap A \neq \emptyset$. Since K(A,F) was an arbitrary member of L, $(\bigcap C)$ intersects each A. Therefore $\bigcap C$ is in each member of L, so $\bigcap L \neq \emptyset$.

<u>Theorem 10</u> $(2^X,h)$ is a compact space.

<u>Proof</u> Let C be any collection of basic closed subsets of $(2^X,h)$ having the f.i.p. Consider the partially ordered set O of all collections of basic closed subsets of $2^X$ having the f.i.p. and containing C ordered by set inclusion. C is in O so $O \neq \emptyset$. Each chain M in O has $\bigcup_{Y \text{ in } M} Y$ as an upper bound because (1) each $Y \subset \bigcup_{Y \text{ in } M} Y$, (2) for any Y in M, $C \subset Y \subset$

$\bigcup_{Y \text{ in } M} Y$, and (3) for any finite subset $\{b_1,\ldots,b_n\}$ of $\bigcup_{Y \text{ in } M} Y$, each $b_i$ is a member of a $Y_i$. Since M is a chain, one of the $Y_i$'s contains $\bigcup_{i=1}^{n} Y_i$ and $\{b_1,\ldots,b_n\} \subset Y_i$. $Y_i$ has the f.i.p. so $\bigcap_{i=1}^{n} b_i \neq \emptyset$. Thus $\bigcup_{Y \text{ in } M} Y$ has the f.i.p. and is an upper bound for M. By Zorn's Lemma, there exists a maximal collection A in O. Let Z be any member of A; then $Z = \bigcup_{i=1}^{n} S_i$ where $S_i$ belongs to S. For each Z in A, one of the $S_i$'s is also in A. If this were not the case, then for each $S_i$, there exists a finite subset $D_i = \{E_{i_1},\ldots,E_{i_{m_i}}\}$ of A such that $\bigcap_{n=1}^{m_i} (E_{i_n} \cap S_i) = \emptyset$. But this yields the contradiction that $\{Z, E_{1_1},\ldots,E_{1_{m_1}}, E_{2_1},\ldots,E_{2_{m_2}},\ldots,E_{n_1},\ldots,E_{n_{m_n}}\}$ is a finite subset of A whose intersection is empty.

Let $W = \{S_i \text{ in } S\colon\ S_i \text{ is in } A\}$. $W \subset A$ so W is a subset of S having the f.i.p. and $\bigcap W \neq \emptyset$. Therefore there exists a P in $\bigcap W$ which implies P is in each Z in A. Hence P is in each Z in C so $\bigcap C \neq \emptyset$. Thus $(2^X, h)$ is compact.

## CONNECTIVITY PROPERTIES OF $2^X$

In this section $2^X$ is shown to inherit the following connectivity properties of X: connectedness, local connectedness, total disconnectedness, discreteness, and possession of isolated points. If X does not have one of the above mentioned properties, then $2^X$ is shown to likewise lack the property. Another result established in this section is that the Hausdorff hyperspaces of two homeomorphic compact metric spaces are homeomorphic. Finally, the set K of connected closed subsets of X is shown to be closed and for connected X, K and $2^X$ are shown to be arcwise connected.

The theorems concerning connectivity are due, for the most part, to Micheal (5). Connectedness will be considered first.

<u>Definition</u> For any compact metric space Y and any positive integer n, let F(Y) denote $\{E$ in $2^Y$: E is a finite subset of $Y\}$ and let $F_n(Y)$ denote $\{E$ in $2^Y$: E has at most n elements$\}$.

<u>Theorem 11</u> If (X,d) is connected, then $(2^X,h)$ is connected.

<u>Proof</u> Let n be any positive integer. Then (X,d) is connected implies the Cartesian product $X^n$ with the product topology is connected. Let f be the function from $X^n$ onto $F_n(X)$ defined by $f(x_1,\ldots,x_n) = \{x_1,\ldots,x_n\}$. Let $(x_1,\ldots,x_n)$ be any element in $X^n$ and $/U_1,\ldots,Um/\cap F_n(X)$ be any open basis element in $(F_n(X),h)$ which contains $f(x_1,\ldots,x_n)$. Then $\{x_1,\ldots,x_n\}$ intersects each $U_i$ and is a subset of $\bigcup_{i=1}^{m} U_i$.

For each $j = 1,\ldots,n$ let $O_j = \{i: x_j \cap U_i \neq \emptyset\}$ and $R_j = \bigcap_{i \text{ in } O_j} U_i$. Then $O = \prod_{j=1}^{n} R_j$ is open in $X^n$, $(x_1,\ldots,x_n)$ is in $O$, and $f(O) \subset /U_1,\ldots,U_n/ \cap F_n(X)$. Thus f is a continuous function and $F_n(X)$ is connected. Since n was an arbitrary positive integer, $F_n(X)$ is connected for each n.

Let A and C be members of F(X). Then there exist positive integers a and c such that A is in $F_a(X)$ and C is in $F_c(X)$. Let $d = \max\{a,c\}$; then both A and C belong to $F_d(X)$, a connected subset of F(X). Since A and C were arbitrary members of F(X), F(X) is connected.

Let $U = /U_1,\ldots,U_n/$ be a member of base B. Select one $x_i$ from each $U_i$ and denote the set of these $x_i$'s by V. Since V is a finite subset of X, V is closed and belongs to F(X). V also is a member of U since $V \cap U_i = \{x_i\} \neq \emptyset$ and $V \subset \bigcup_{i=1}^{n} U_i$. Thus each open set in $(2^X,h)$ contains a member of F(X). Hence $\overline{F(X)} = 2^X$. Since F(X) is connected, $\overline{F(X)} = 2^X$ is connected.

<u>Theorem 12</u> If (X,d) is not connected, then $(2^X,h)$ is not connected.

<u>Proof</u> X is not connected implies there exists a proper subset A of X which is both open and closed in (X,d). Then I(A) is both open and closed in $(2^X,h)$. Since $\{X\}$ is in $2^X$ and $X \not\subset A$, I(A) is a proper subset of $2^X$. Thus $(2^X,h)$ is not connected.

Proceeding next to local connectedness, the following

lemma is instrumental in the proof of Theorem 13.

<u>Lemma 5</u> If B is a connected subset of $2^X$ and there exists a C in B such that C is a connected subset of X, then B* is a connected subset of X.

<u>Proof</u> Suppose B* is not connected in X. Then $B^* = H \cup F$ where H and F are mutually separated subsets of X. Let $A = I(H) \cap B$ and $G = B - A$. Then $A \cap G = \emptyset$. Let E be any member of A. Then $E \subset H$ and it follows that each point x of E is not a limit point of F. Hence there exists an open set $O_x$ containing x such that $O_x \subset F'$. Let $O_E = \bigcup_{x \text{ in } E} O_x$. Then $E \subset O_E$ so E is in $/O_E/$. Let C be any member of $/O_E/ \cap B$. Then $C \subset O_E \subset F'$; therefore $C \subset F' \cap B^* = H$ and C is in A and not in G. Thus $/O_E/ \cap B \cap G = /O_E/ \cap G = \emptyset$ and each E in A is contained in an open set $/O_E/$ whose intersection with G is empty. Hence no point of A is a limit point of G.

Let L be any member of G. Then $L \not\subset H$ so there exists an x in L such that x is in F. Since H and F are mutually separated, x is not a limit point of H; so there exists an open set $O_x$ containing x such that $O_x \subset H'$. Then $J(O_x)$ is an open set in $2^X$ such that L is in $J(O_x)$. Let C be any member of $J(O_x) \cap B$. Now $C \cap O_x \neq \emptyset$ and $C \cap H' \neq \emptyset$. Hence $C \not\subset H$ and C is not in A. Thus each L in G is contained in an open set $J(O_x)$ whose intersection with A is empty. Therefore, no point of G is a limit point of A.

It follows then that G and A are mutually separated subsets of $2^X$ whose union is B. But this implies B is not

connected which is a contradiction. Hence $B^*$ must be connected.

<u>Theorem 13</u> If $(X,d)$ is not locally connected, then $(2^X,h)$ is not locally connected.

<u>Proof</u> Let $(2^X,h)$ be locally connected. Let $x$ be in $X$ and let $U$ be any open set in $(X,d)$ containing $x$. Then $\{x\}$ is in $2^X$, $/U/$ is in $2^T$, and $\{x\}$ is in $/U/$. Since $2^X$ is locally connected, there exists a connected open set $N$ such that $\{x\}$ is in $N$ and $N \subset /U/$. $\{x\}$ is connected in $X$, so by Lemma 5, $N^*$ is connected in $X$. Now there exists an open basis element $/V_1,\ldots,V_n/$ such that $\{x\}$ is in $/V_1,\ldots,V_n/ \subset N$. Then $\{x\} \subset \bigcup_{i=1}^{n} V_i \subset N^* \subset U$. Hence $N^*$ is a connected neighborhood of $x$ which is a subset of $U$. It follows that $(X,d)$ is locally connected.

The converse of Theorem 13 will require three preliminary lemmas.

<u>Lemma 6</u> For $i = 1,\ldots,n$, let $U_i$ be a subset of $X$. Then $\overline{/U_1,\ldots,U_n/} = /\overline{U_1},\ldots,\overline{U_n}/$.

<u>Proof</u> Let $E$ be any member of $\overline{/U_1,\ldots,U_n/}$. Suppose $E$ is not in $/\overline{U_1},\ldots,\overline{U_n}/$. Then either (1) $E \not\subset \bigcup_{i=1}^{n} \overline{U}_i$ or (2) for some $i$, $1 \leq i \leq n$, $E \cap \overline{U}_i = \emptyset$. If (1) holds, then $E \not\subset \overline{(\bigcup_{i=1}^{n} U_i)} = \bigcup_{i=1}^{n} \overline{U}_i$. This is a contradiction. If (2) holds, then for some $i$ such that $1 \leq i \leq n$, $E \subset (\overline{U_i})'$ so $E$ is in $/(\overline{U_i})'/$. Since $E$ is in $\overline{/U_1,\ldots,U_n/}$, $/(\overline{U_i})'/ \cap /U_1,\ldots,U_n/ \neq \emptyset$. This is contradictory for no $J$ in $2^X$ is such that $J \subset (\overline{U_i})'$ and $J \cap U_i \neq \emptyset$. Thus $E$ is in $/\overline{U_1},\ldots,\overline{U_n}/$.

Suppose E is in $/\overline{U_1},\ldots,\overline{U_n}/$; then for all $i = 1,\ldots,n$, $E \cap \overline{U}_i \neq \emptyset$ and $E \subset \bigcup_{i=1}^{n} \overline{U}_i$. Let $/V_1,\ldots,V_m/$ be any open basis set containing E. Then for all $j = 1,\ldots,m$, $E \cap V_j \neq \emptyset$ and $E \subset \bigcup_{j=1}^{m} V_j$. For each $V_j$, there exists an $x_j$ in $E \cap V_j$. Now $x_j$ is in $E \subset \bigcup_{i=1}^{n} \overline{U}_i$ implies there exists a $U_i$ such that $x_j$ is a limit point or a point of $U_i$. This implies $V_j \cap U_i \neq \emptyset$ so there exists a $v_j$ in $V_j \cap U_i$. For each $\overline{U}_i$, there exists an $x_i$ in $E \cap \overline{U}_i$. Now $x_i$ is in $E \subset \bigcup_{j=1}^{m} V_j$ implies $x_i$ is in some $V_j$. Hence $U_i \cap V_j \neq \emptyset$. Thus there exists a $u_i$ in $U_i \cap V_j$. Let $J = \{x: x = v_j \text{ for } j = 1,\ldots,m \text{ or } x = u_i \text{ for } i = 1,\ldots,n\}$. Then J is in $/V_1,\ldots,V_m/ \cap /U_1,\ldots,U_n/ \neq \emptyset$. Thus E is a limit point of $/U_1,\ldots,U_n/$ and hence E is in $\overline{/U_1,\ldots,U_n/}$. Thus $\overline{/U_1,\ldots,U_n/} = /\overline{U}_1,\ldots,\overline{U}_n/$.

<u>Definition</u> Let $2^{2^X}$ denote the Hausdorff hyperspace of compact metric space $2^X$.

<u>Lemma 7</u> The function $\sigma$ from $2^{2^X}$ to $2^X$ defined by $\sigma(A) = \bigcup_{E \text{ in } A} E$ is continuous.

<u>Proof</u> $\sigma(A)$ is in $2^X$ for $\sigma(A)$ is a compact subset of X and hence is closed. To see that $\sigma(A)$ is a compact subset of X, let U be any open covering of $\sigma(A)$ in X. Then U is also an open covering of each E in A so for each E in A, there exists a finite subcover $\{U_i\}_{i=1}^{n_E}$ of U for E. Then each E is in $/U_1,\ldots,U_{n_E}/$; thus $\{/U_1,\ldots,U_{n_E}/: E \text{ is in } A\}$ is an open cover of A in $2^X$. Since A is in $2^{2^X}$, A is a closed and compact subset of $2^X$ so there exists a finite subcover $\{/U_{1_j},\ldots,$

$U_{n_{E_j}}\}_{j=1}^{m}$ for A. Hence $\{U_{1_1},\ldots,U_{n_{E_1}},\ldots,U_{1_m},\ldots,U_{n_{E_m}}\}$ is a finite subcover of U for $\sigma(A)$. Thus $\sigma(A)$ is a compact subset of X.

To see that $\sigma$ is continuous, let A be any member of $2^{2^X}$ and let $/U_1,\ldots,U_n/$ be any open basis element containing $\sigma(A)$. Now $I(\bigcup_{i=1}^{n} U_i)$ is open in $2^X$ and also for each i, $/U_i, \bigcup_{j=1}^{n} U_j/$ is open in $2^X$ and $U = /I(\bigcup_{j=1}^{n} U_j), /U_1, \bigcup_{j=1}^{n} U_j/,\ldots,$ $/U_n, \bigcup_{j=1}^{n} U_j//$ is open in $2^{2^X}$.

Now for each i, $1 \leq i \leq n$, there exists an $x_i$ in $\sigma(A) \cap U_i$. Therefore there exists an $E_i$ in A such that $x_i$ is in $E_i$ which implies that $E_i \cap U_i \neq \emptyset$. Since $E_i \subset \sigma(A) \subset \bigcup_{j=1}^{n} U_j$, $E_i$ is in $/U_i, \bigcup_{j=1}^{n} U_j/$. Thus $A \cap /U_i, \bigcup_{j=1}^{n} U_j/ \neq \emptyset$ for each i.

Suppose E is any point of A. Then $E \subset \sigma(A) \subset \bigcup_{j=1}^{n} U_j$ so E is in $I(\bigcup_{j=1}^{n} U_j)$. Thus $A \subset I(\bigcup_{j=1}^{n} U_j)$. Therefore A is in U.

Next let W be any member of U and let x be any point of $\sigma(W)$. Then there exists an E in W such that x is in E. Since W is in U, E is in $I(\bigcup_{j=1}^{n} U_j)$ or in one of the sets of the form $/U_i, \bigcup_{j=1}^{n} U_j/$. In any case, $E \subset \bigcup_{j=1}^{n} U_j$. Thus x is in and $\sigma(W)$ is a subset of $\bigcup_{j=1}^{n} U_j$. Since W is in U, there exists an $E_i$ in $W \cap /U_i, \bigcup_{j=1}^{n} U_j/$. This implies that $E_i \cap U_i \neq \emptyset$. Since $E_i$ is in W, $E_i \subset \sigma(W)$. Thus $\sigma(W) \cap U_i \neq \emptyset$. Thus $\sigma(W)$ is in $/U_1,\ldots,U_n/$. Hence $\sigma(U) \subset /U_1,\ldots,U_n/$.

Thus $\sigma$ is continuous.

<u>Lemma 8</u> If $V_1,\ldots,V_n$ are connected subsets of X, then $/V_1,\ldots,V_n/$ is a connected subset of $2^X$.

<u>Proof</u> Let i be any integer between 1 and n. As in Theorem 12, $V_i$ is connected implies that $(V_i)^n$ is connected which implies $F_n(V_i)$ is connected which implies $F(V_i)$ is connected. Thus $\prod_{i=1}^{n} F(V_i)$ is connected. Let g be the function from $\prod_{i=1}^{n} F(V_i)$ into $2^X$ defined by $g(E_1,\ldots,E_n) = \bigcup_{i=1}^{n} E_i$. Then $g(\prod_{i=1}^{n} F(V_i)) = F(X) \cap /V_1,\ldots,V_n/$. Let p be the projection map from $(2^X)^n$ into $F_n(2^X)$ defined by $p(E_1,\ldots,E_n) = \{E_1,\ldots,E_n\}$. Let $\sigma$ be the map defined in Lemma 7 above. Then g is the restriction of $\sigma \cdot p$ to $\prod_{i=1}^{n} F(V_i)$. Therefore $F(X) \cap /V_1,\ldots,V_n/$ is the continuous image of a connected set and hence is connected. Now $\overline{F(X) \cap /V_1,\ldots,V_n/} = \overline{F(X)} \cap \overline{/V_1,\ldots,V_n/} = 2^X \cap \overline{/V_1,\ldots,V_n/} = \overline{/V_1,\ldots,V_n/}$. Thus $F(X) \cap /V_1,\ldots,V_n/ \subset /V_1,\ldots,V_n/ \subset \overline{F(X) \quad /V_1,\ldots,V_n/}$. Hence $/V_1,\ldots,V_n/$ is connected.

<u>Theorem 14</u> If X is locally connected, then $2^X$ is locally connected.

<u>Proof</u> Let A be any member of $2^X$ and $/U_1,\ldots,U_n/$ be any open basis element containing A. For each $i = 1,\ldots,n$, let $Z_i$ = set of components of $U_i$. Let $M = \bigcup_{i=1}^{n} Z_i$. X is locally connected implies that M is a collection of open sets. Since A is in $2^X$, A is compact. Since M covers A, there exists a finite subcover $\{C_i\}_{i=1}^{m}$ of M for A where each $C_i \cap A \neq \emptyset$. For $i = 1,\ldots,m$, let $V_i = C_i$. Since $A \cap U_i \neq \emptyset$, for each

$i = 1,\ldots,n$ there exists a component $P_i$ of $U_i$ such that $A \cap P_i \neq \emptyset$. For $i = 1,\ldots,n$ let $V_{m+1} = P_i$. Then $/V_i,\ldots,V_{m+n}/$ is an open basis element containing A and such that $/V_1,\ldots, V_{m+n}/ \subset /U_1,\ldots,U_n/$. From Lemma 8 above, since each $V_i$ is a component and hence connected, $/V_1,\ldots,V_{m+n}/$ is connected. Thus $2^X$ is locally connected.

<u>Theorem 15</u> $2^X$ is a Peano continuum if and only if X is a Peano continuum.

<u>Proof</u> Follows from Theorems 11,12,13, and 14.

<u>Theorem 16</u> X has an isolated point if and only if $2^X$ has an isolated point.

<u>Proof</u> Let x be an isolated point of X. Then $\{x\}$ is an open subset of X so $I(\{x\})$ is an open subset of $2^X$. But $I(\{x\}) = \{\{x\}\}$. Thus $\{x\}$ is an isolated point of $2^X$.

Let E be an isolated point of $2^X$. Then $\{E\}$ is open in $2^X$. Since B is a basis for the open sets in $2^X$, there exists a $/U_1,\ldots,U_n/$ in B such that $\{E\} \subset /U_1,\ldots,U_n/ \subset \{E\}$, i.e., $/U_1,\ldots,U_n/ = \{E\}$. For $i = 1,\ldots,n$, choose $x_i$ in $E \cap U_i$. Then $\{x_i\}_{i=1}^{n}$ is a finite subset of X and hence is in $2^X$. This implies $\{x_i\}_{i=1}^{n}$ is in $/U_i,\ldots,U_n/ = \{E\}$ which implies $E = \{x_i\}_{i=1}^{n}$. Suppose there exists an i such that $U_i \neq \{x_i\}$. Then there exists a $y_i$ in $U_i$ such that $y_i \neq x_i$. Then $\{x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n\}$ is in $/U_1,\ldots,U_n/$ and therefore equals E. This is a contradiction and hence for each i, $U_i = \{x_i\}$. Thus each $\{x_i\}$ is an open set in X and each $x_i$ is an isolated point of X.

<u>Definition</u> For $x$ any point of an arbitrary topological space $X$, let $C(x)$ denote the component of $X$ containing $x$.

<u>Theorem 17</u> $X$ is totally disconnected if and only if $2^X$ is totally disconnected.

<u>Proof</u> Suppose $X$ is not totally disconnected. Then there exists an $x$ in $X$ such that $C(x) \neq \{x\}$. Therefore there exists a $y$ in $C(x)$ such that $y \neq x$. From Lemma 8, $C(x)$ is connected implies that $/C(x)/ = I(C(x))$ is connected. Now $\{y\}$ and $\{x\}$ are both members of $I(C(x))$ and $I(C(x)) \subset C(\{x\})$. Thus $C(\{x\}) \neq \{\{x\}\}$ and $2^X$ is not totally disconnected.

Let $X$ be totally disconnected. Let $E$ and $F$ be in $2^X$ and such that there exists an $x$ in $F$ which is not in $E$. Then for each $y$ in $E$, there exists an open-closed set $O_y$ containing $y$ but not $x$. $\{O_y : y \text{ in } E\}$ is therefore an open cover for closed and hence compact set $E$. Hence there exists a finite subcover $\{O_{y_i}\}_{i=1}^{n}$ for $E$. Then $\bigcup_{i=1}^{n} O_{y_i}$ is an open-closed subset of $X$ so $I(\bigcup_{i=1}^{n} O_{y_i})$ is open-closed in $2^X$ containing $E$ but not $F$. Thus $2^X$ is totally disconnected.

<u>Theorem 18</u> $X$ is discrete if and only if $2^X$ is discrete.

<u>Proof</u> Let $X$ be discrete. Then for each $x$ in $X$, $\{x\}$ is open-closed. It follows that $X$ is a finite space for $\{\{x\} : x \text{ in } X\}$ is an open cover of $X$ which is a compact space. $2^X$ is then a finite space so for each $E$ in $2^X$, since $2^X$ is a metric space, $\{E\}$ is open. Therefore each subset $A$ of $2^X$ is the union of open sets $\{E\}$ and hence is open. Thus $2^X$ is

discrete.

Let $2^X$ be discrete. Since $2^X$ is a compact space, $2^X$ is a finite space. Thus X is a finite space. Since X is a metric space, X is discrete.

Micheal states the following lemma and theorem.

<u>Lemma 9</u> Let f be a continuous function of compact metric space X into compact metric space Y. Then the function f* from $2^X$ to $2^Y$ defined by $f^*(E) = f(E)$ is continuous.

<u>Proof</u> Let E be in $2^X$ and let $/U_1,\ldots,U_n/$ be any basis element containing $f^*(E)$. Then $f(E) \subset \bigcup_{i=1}^{n} U_i$. Since f is continuous, for each x in X there exists an open set $O(x)$ containing x such that $f(O(x)) \subset \bigcup_{i=1}^{n} U_i$. E is compact, thus there exists a finite cover $\{O(x_i): i = n+1,\ldots,n+m\}$ for E. Since $f(E) \cap U_i \neq \emptyset$, there exists an $x_i$ in E such that $f(x_i)$ is in $U_i$. Hence there exists an open set $O(x_i)$ containing $x_i$ such that $f(O(x_i)) \subset U_i$. Now $O = /O(x_i,\ldots,O(x_{n+m})/$ is an open set in $2^X$ containing E. Let B be in $f^*(O) = f(O)$. Then there exists an A in O such that $f^*(A) = B$. $A \subset \bigcup_{i=1}^{n+m} O(x_i)$ implies $f(A) = B \subset f(\bigcup_{i=1}^{n+m} O(x_i)) \subset \bigcup_{i=1}^{n} U_i$. For $i = 1,\ldots,n$, $A \cap O(x_i) \neq \emptyset$. Thus $\emptyset \neq f(A \cap O(x_i)) = f(A) \cap f(O(x_i)) \subset B \cap U_i$. Therefore B is in $/U_1,\ldots,U_n/$. Hence $f^*(O) \subset /U_1,\ldots,U_n/$ and f* is continuous.

<u>Theorem 19</u> If compact metric space X is homeomorphic to compact metric space Y, then $2^X$ is homeomorphic to $2^Y$.

<u>Proof</u> If f is a reversibly continuous mapping of X onto Y, then by Lemma 9, f* is a reversibly continuous mapping of $2^X$

onto $2^Y$.

<u>Definition</u> Let K denote $\{E$ in $2^X$: E is a connected subset of $X\}$.

<u>Theorem 20</u> K is a closed subset of $2^X$.

<u>Proof</u> Let F be any member of $2^X$ which is not in K. Then $F = D_1 \cup D_2$ where $D_1$ and $D_2$ are non-empty, mutually separated sets in X. Since F is a closed subset of X, $D_1$ and $D_2$ are closed in X and hence compact. Thus $d(D_1,D_2) > 0$. Hence there exists open sets $O_1$ containing $D_1$ and $O_2$ containing $D_2$ such that $O_1 \cap O_2 = \emptyset$. (For example, let $O_i = \bigcup_{x \text{ in } D_i} B(x, d(D_1,D_2)/3)$.) Then $/O_1,O_2/$ is open in $2^X$, $F \subset O_i \cup O_2$, and $F \cap O_i \neq \emptyset$ for $i = 1,2$; so F is in $/O_1,O_2/$. Suppose there exists an E in K which is in $/O_1,O_2/$. Since E is connected, $E \subset O_1 \cup O_2$ implies that $E \subset O_1$ or $E \subset O_2$ which implies $E \cap O_2 = \emptyset$ or $E \cap O_1 = \emptyset$. This is a contradiction. Thus $K \cap /O_1,O_2/ = \emptyset$ and F is not a limit point of K. Since F was arbitrary, K contains all of its limit points. Thus K is closed.

Throughout the remainder of this section on connectivity and through the next on contraction, let X denote a nondegenerate connected set and $\{a_1,a_2,\ldots\}$ a fixed countable dense subset of X. (A countable dense subset of X exists for X is a compact metric space and hence separable.) The structure of the arguments in these sections is Kelley's (4).

The remaining theorem in this section is that X is connected implies $2^X$ and K are both arcwise connected. In the proof of this theorem, a critical item is the existence of a

real-valued function defined on $2^X$ having the properties stated in Lemma 10 below.

<u>Lemma 10</u> There exists a real-valued function v on $2^X$ having the following properties:

(1) $0 \leq v(A) \leq 1$ for all A in $2^X$.

(2) $v(A) = 0$ if and only if A is a singleton subset of X.

(3) $v(A) = 1$ if and only if $A = X$.

(4) v is continuous.

(5) If A,B are in $2^X$, $A \subset B$, and $A \neq B$, then $v(A) < v(B)$.

Whitney (7) defines a function u which with a slight modification yields a function v satisfying Lemma 10.

<u>Definition</u> For $\{a_1, a_2, a_3, \ldots\}$ the countable dense subset of X, let $f_i$ denote the real-valued function defined on X by $f_i(x) = 1/(1 + d(a_i,x))$. Let $u_i$ be the real-valued function defined on $2^X$ by $u_i(A) = \text{l.u.b.}\{|f_i(p) - f_i(q)| : p \text{ and } q \text{ are in } A\}$. Let u be the real-valued function defined on $2^X$ by $u(A) = \sum_{i=1}^{\infty} u_i(A)/2^i$.

<u>Lemma 11</u> The functions $f_1, f_2, \ldots, u_1, u_2, \ldots,$ and u have the following properties.

(1) $|f_i(p) - f_i(q)| \leq d(p,q)$ since $|f_i(p) - f_i(q)| =$

$$\left|\frac{(1 + d(a_i,q)) - (1 + d(a_i,p))}{(1 + d(a_i,p))(1 + d(a_i,q))}\right| = \left|\frac{d(a_i,q) - d(a_i,p)}{(1 + d(a_i,p))(1 + d(a_i,q))}\right|$$

$\leq |d(a_i,q) - d(a_i,p)| \leq |d(p,q)| = d(p,q)$.

(2) $u_i(A) \leq \delta(A)$ since $u_i(A) = \text{l.u.b.}\{|f_i(p) - f_i(q)| : p \text{ and } q \text{ are in } A\} \leq \text{l.u.b.}\{d(p,q) : p \text{ and } q \text{ are in } A\} = \delta(A)$.

(3) $u(A) \leq \delta(A)$ since $u(A) = \sum_{i=1}^{\infty} u_i(A)/2^i \leq \sum_{i=1}^{\infty} \delta(A)/2^i =$

$\delta(A)\sum_{i=1}^{\infty} 1/2^i = \delta(A)\ 1 = \delta(A)$.

(4) $0 < f_i(p) \leq 1$ since $d(a_i,p) \geq 0$.

(5) $u_i(A) \leq 1$ since for all p,q in A, $|f_i(p) - f_i(q)| \leq 1$.

(6) $u(A) \leq 1$ since $u(A) = \sum_{i=1}^{\infty} u_i(A)/2^i \leq \sum_{i=1}^{\infty} 1/2^i = 1$.

(7) If $A = \{x\}$, then $\delta(A) = 0$ and hence $u(A) = 0$.

(8) If $u(A) = 0$, then $u_i(A) = 0$ for all i, and hence $|f_i(p) - f_i(q)| = 0$ for all p,q in A. Thus $f_i(p) = f_i(q)$. Thus $d(a_i,p) = d(a_i,q)$ for all i and thus $p = q$. Hence each p in A equals q and hence $A = \{q\}$.

(9) u is continuous for let E be in $2^X$ and $B(u(E),2e)$ be any open set containing $u(E)$. Let A be any member of $B(E,e)$. Then $h(A,E) < e$. For each p in A, Lemma 1 implies there exists a $q_p$ in E such that $d(p,q_p) < e$. Thus $|f_i(p) - f_i(q_p)| < e$. Hence each $f_i(p) \leq f_i(q_p) + e \leq$ l.u.b.$\{f_i(q)$: q is in E$\}$ + e. Thus l.u.b. $\{f_i(p)$: p is in A$\} \leq$ l.u.b. $\{f_i(q)$: q is in E$\}$ + e. Also g.l.b.$\{f_i(q)$: q is in E$\}$ - $e \leq f_i(q_p) - e \leq f_i(p)$ for each p in A. Thus g.l.b.$\{f_i(q)$: q is in E$\}$ - $e \leq$ g.l.b.$\{f_i(p)$: p is in A$\}$. Thus $u_i(A)$ = l.u.b.$\{|f_i(p) - f_i(r)|$ : p and r are in A$\}$ = l.u.b.$\{f_i(p)$: p is in A$\}$ - g.l.b.$\{f_i(p)$: p is in A$\} \leq$ l.u.b.$\{f_i(q)$: q is in E$\}$ + e - g.l.b.$\{f_i(q)$: q is in E$\}$ - $(-e) = u_i(E) + 2e$. Then $u(A) = \sum_{i=1}^{\infty} u_i(A)/2^i \leq \sum_{i=1}^{\infty} (u_i(E) + 2e)/2^i = u(E) + 2e$. Thus $u(B(E,e)) \subset B(u(E),2e)$. Hence u is continuous.

(10) If $A \neq E$ and $A \subset E$, then $u(A) < u(E)$ for A and E are closed compact subsets of X and thus there exists a point b in E such that $d(b,A) > 0$. Since $\{a_1,a_2,a_3,\ldots\}$ is dense in

X, there exists an $a_i$ such that $d(a_i,b) \leq d(b,A)/3$. Let $s = d(b,A)$. Then $f_i(b) \geq 1/(1 + s/3) = 3/(3 + s)$. Also $d(a_i,A) > 2/3s$. Thus for all q in A, $f_i(q) < 1/(1 + 2s/3) = 3/(3 + 2s) < f_i(b)$. Hence l.u.b.$\{|f_i(p) - f_i(q)|$ :p and q are in A$\} <$ l.u.b.$\{|f_i(b) - f_i(q)|$ : q is in A$\} <$ l.u.b.$\{|f_i(p) - f_i(q)|$ : p and q are in E$\}$. Thus $u_i(A) < u_i(E)$ and hence $u(A) < u(E)$.

Proof of Lemma 10 follows immediately from the properties of u when the following definition of v is made.

<u>Definition</u> For X not a singleton set, let v denote the real-valued function defined on $2^X$ by $v(A) = u(A)/u(X)$.

The next Lemma follows from the properties of v.

<u>Lemma 12</u> For each positive number e, there exists a positive number r such that if A and B are in $2^X$, $A \subset B$, and $v(B) - v(A) < r$, then $h(A,B) < e$.

<u>Proof</u> Let $e > 0$ be given. If $h(A,B) \geq e$, then there exists a b in B such that $d(b,A) > e/2$. Now there exists an $a_i$ in the countable dense subset of compact X such that $d(b,a_i) < e/8$. Now for each p in A, $d(p,a_i) > e/4$. Thus l.u.b.$\{f_i(q)$: q is in B$\} \geq f_i(b) > 1/(1 + e/8) > (1/(1 + e/4)) > f_i(p)$ for all p in A. Thus l.u.b.$\{f_i(q)$: q is in B$\} > f_i(p) + (1/(1 + e/8)) - f_i(p) > f_i(p) + (1/(1 + e/8)) - (1/(1 + e/4))$ for all p in A. Let $t = (1/(1 + e/8)) - (1/(1 + e/4))$. Then l.u.b.$\{f_i(q)$: q is in B$\} \geq$ l.u.b.$\{f_i(p)$: p is in A$\} + t$. Since $A \subset B$, g.l.b.$\{f_i(q)$: q is in B$\} \leq$ g.l.b.$\{f_i(p)$: p is in A$\}$. Thus $u_i(B) - u_i(A) \geq t$. Hence $u(B) - u(A) \geq t/2^i$ and

$v(B) - v(A) \geq t/2^i(u(X))$. Thus choose $r < t/2^i(u(X))$.

A second critical item in the proof of the arcwise connectedness of $2^X$ and K is the notion of a segment. Its definition and a lemma on its existence follow. Since the proof of the lemma is long, it is divided into parts.

<u>Definition</u> For $A_0, A_1$ in $2^X$, let a segment from $A_0$ to $A_1$ denote a continuous mapping $A_t$ from the interval $[0,1]$ into $2^X$ which satisfies the following two conditions: (1) $v(A_t) = (1-t)v(A_0) + tv(A_1)$ and (2) If $t_1 < t_2$, then $A_{t_1} \subset A_{t_2}$.

<u>Lemma 13</u> For $A_0$, $A_1$ in $2^X$, there exists a segment from $A_0$ to $A_1$ if and only if $A_0 \subset A_1$ and every component of $A_1$ intersects $A_0$.

<u>Part 1</u> Existence of segment from $A_0$ to $A_1$ implies $A_0 \subset A_1$ and every component of $A_1$ intersects $A_0$.

<u>Proof</u> Suppose there exists a segment from $A_0$ to $A_1$. By condition two in the definition of segment, $A_0 \subset A_1$. If there exists a component $B_1$ of $A_1$ such that $A_0 \subset (A_1 - B_1)$, then by Moore (6, p.21) there exists an open set O such that $B_1 \subset O$, $\overline{O} \cap A_0 \neq \emptyset$, and $(\overline{O} - O) \cap A_1 = \emptyset$. Then $\overline{O} \cap A_1 = O \cap A_1$ and so is an open subset of $A_1$. This implies $A_1 - \overline{O}$ is a closed subset of $A_1$ which is a closed subset of X, so $A_1 - \overline{O}$ is closed in X and $I(A_1 - \overline{O})$ is closed in $2^X$. Then $J(\overline{O})$ and $I(A_1 - \overline{O})$ are disjoint closed subsets of $2^X$ covering $A_1$. Since $A_t$ is continuous, $A_t^{-1}(J(\overline{O}))$ and $A_t^{-1}(I(A_1 - \overline{O}))$ are disjoint closed subsets of $[0,1]$ covering $[0,1]$. But $[0,1]$ is connected so this is a contradiction and hence no such component $B_1$ exists.

<u>Part 2</u> Let $A_0 \subset A_1$ and every component of $A_1$ intersect $A_0$. Let Y be the collection of all subsets of E of $2^X$ which have the following two properties:

(1) If W is in E then $A_0 \subset W \subset A_1$ and every component of W intersects $A_0$.

(2) If $W_0, W_1$ are in E then either $W_0 \subset W_1$ or $W_1 \subset W_0$.

Let Y be ordered by set inclusion. Then Y has a maximal element.

<u>Proof</u> $\{A_0, A_1\}$ is in Y so Y is non-empty. Each chain C in Y has $\bigcup_{E \text{ in } C} E$ has an upper bound because of the following. (1) Each W in $\bigcup_{E \text{ in } C} E$ is in an E and thus $A_0 \subset W \subset A_1$ and every component of W intersects $A_0$. (2) If $W_1, W_2 \in \bigcup_{E \text{ in } C} E$, then $W_1$ is in $E_1$, $W_2$ is in $E_2$, and C is a chain so $E_1 \subset E_2$ or $E_2 \subset E_1$ so $W_1, W_2$ both belong to either $E_1$ or $E_2$, thus either $W_1 \subset W_2$ or $W_2 \subset W_1$. (3) Each E in C is a subset of $\bigcup_{E \text{ in } C} E$. Applying Zorn's Lemma, Y has a maximal subset $E_0$.

<u>Part 3</u> Let the conditions of Part 2 apply. Then the maximal element $E_0$ of Y is a closed subset of $2^X$.

<u>Proof</u> Let $W_0$ be any element of $\overline{E}_0$. (1) If there exists an x in $A_0$ such that x is not in $W_0$, then $d(x, W_0) = w > 0$. Since x is in each W in $E_0$, $h(W, W_0) \geq w$ for each W in E and $B(W_0, w/2)$ is then an open set containing $W_0$ which doesn't intersect $E_0$. This is a contradiction so $A_0 \subset W_0$ for all $W_0$ in $\overline{E}_0$. (2) If there exists an x in $W_0$ such that x is not in $A_1$, then $d(x, A_1) = g > 0$. Each W in $E_0$ is a subset of $A_1$, so $d(x, W) \geq g$ and $h(W, W_0) \geq g$. Then $B(W_0, g/2)$ is an open set containing $W_0$ and

not intersecting $E_0$. This is a contradiction so $W_0 \subset A_1$ for all $W_0$ in $\overline{E}_0$. (3) If there exists a component P of $W_0$ which does not intersect $A_0$, then again by Moore (6, p.21) there exists an open set O such that $P \subset O$, $A_0 \cap \overline{O} = \emptyset$ and $(\overline{O} - O) \cap W_0 = \emptyset$. Then $/(\overline{O})', O/$ is an open set containing $W_0$. Suppose there exists a W in $E_0 \cap /(\overline{O})', O/$. Then W contains a component N intersecting both O and $(\overline{O})'$ and $N = (O \cap N) \cup ((\overline{O})' \cap N)$ so N is not connected. This is absurd so $E_0 \cap /(\overline{O})', O/ = \emptyset$. But this is a contradiction for $W_0 \in \overline{E}_0 \cap /(O)', O/$; therefore every component of every set in $\overline{E}_0$ intersects $A_0$. (4) If W and Z are in $\overline{E}_0$ and there exists an x in W - Z and a y in Z - W, let $2K = \min\{d(x,W), d(y,Z)\}$. There exists a finite cover $\{B(w_i,K)\}_{i=1}^{n} \cup \{B(x,K)\}$ for W and a finite cover $\{B(z_i,K)\}_{i=1}^{m} \cup \{B(y,K)\}$ for Z. W is in $/B(x,K), B(w_i,K),\ldots,B(w_n,K)/ = U_1$ and Z is in $/B(y,K), B(z_1,K),\ldots,B(z_m,K)/ = U_2$. There must be an E in $U_1 \cap E_0$ and an F in $U_2 \cap E_0$ and $E \subset F$ or $F \subset E$. If $E \subset F$, $F \cap B(x,K) \neq \emptyset$; but $F \subset V_K(Z)$ so $d(x,F) \geq K$. Similarly, if $F \subset E$, $E \cap B(y,K) \neq \emptyset$; but $E \subset V_K(W)$ so $d(y,E) \geq K$. This is a contradiction so $W \subset Z$ or $Z \subset W$.

Therefore $\overline{E}_0$ is in Y but since $E_0$ is maximal, $E_0 = \overline{E}_0$.

<u>Part 4</u> Let the conditions of Part 3 apply. For each t in $[0,1]$ let $A_t$ be that element of $E_0$ (if it exists) such that $v(A_t) = (1-t)v(A_0) + tv(A_1)$. Then $A_t$ is defined for each t in $[0,1]$.

<u>Proof</u> Note that $A_0$ and $A_1$ are in $E_0$. Suppose there exists

a t such that $A_t$ is not defined. Then let $L = \{A_{t_0} : A_{t_0}$ is defined and $t > t_0\}$ and let $A_{t'} = \sup L$. L is not empty since $A_0$ is in $E_0$. Either (1) $\sup L = \max L$ and $A_{t'}$ is in L so $A_{t'}$ is defined or (2) $\sup L$ = limit of a sequence $\{A_{t_i}\}_{i=1}^{\infty}$ in $E_0$. Since $E_0 = \overline{E}_0$ is maximal, the limit of the sequence $= \sup L = A_{t'}$ is in $E_0$ so $A_{t'}$ is defined. Similarly, there exists an $A_{t''}$ in $E_0$ such that $A_{t''} = \inf\{A_{t_0} : A_{t_0}$ is defined and $t < t_0\}$. Since $A_t$ is not defined, $t' \lneq t \lneq t''$ so $A_{t'} \subsetneq A_{t''}$ and there is no set W in $E_0$ such that $A_{t'} \subset W \subset A_{t''}$.

Now there exists an $e > 0$ such that $\overline{V_e(A_{t'})} \not\supset A_{t''}$ for there exists an x in $A_{t''}$ such that $d(x,A_{t'}) > 0$; let $2e = d(x,A_{t'})$. Let $Q = A_{t''} \cap \overline{V_e(A_{t'})}$. Q is closed and Q is a proper subset of $A_{t''}$ for x is not in Q. Let A consist of the components of Q which intersect $A_{t'}$. $A \subset Q \subsetneq A_{t''}$. Is $A_{t'} \subsetneq A$? If not, then every component of Q intersecting $A_{t'}$ is a subset of $A_{t'}$. But for the above x in $A_{t''} \cap Q'$, the component of $A_{t''}$ containing $x = C(x)$ must be such that $C(x) \cap A_0 \neq \emptyset$. Let $y_0$ be in $C(x) \cap A_0$. Then $y_0$ is in $A_{t'}$; also there exists a component C of Q containing $y_0$ so $C \subset A_{t'}$ and $d(C,\overline{V_e(A_{t'})} - V_e(A_{t'})) \geq e$. But C is also a component of $V_e(A_{t'}) \cap C(x)$ and thus by Moore (6, p.18) since C(x) contains a point of $V_e(A_{t'})$ and a point of $(V_e(A_{t'}))'$, C must have a point of $\overline{V_e(A_{t'})} - V_e(A_{t'})$ as a limit point. This is a contradiction so $A_{t'} \subsetneq A$.

Now $v(A_{t'}) < v(A) < v(A_{t''})$. Is A in $E_0$? Yes, for consider the following: (1) $A_0 \subset A_{t'} \subset A \subset A_{t''} \subset A_1$. (2) Each

component C of A intersects $A_{t'}$ and contains a point y. y is in a component of $A_{t'}$ which intersects $A_0$ and this component is a connected subset of A so is a subset of C. Thus $C \cap A_0 \neq \emptyset$. (3) Let $W = A_s$ be any set in $E_0$. Then either $s \leq t'$ and $A_s \subset A_{t'} \subset A$ or $s \geq t''$ and $A_s \supset A_{t''} \supset A$. A is therefore in Y and if not in $E_0$, then $E_0 \subset E_0 \cup \{A\}$. But $E_0$ is maximal so A is in $E_0$. $E_0$ contains no set W such that $A_{t'} \subsetneq W \subsetneq A_{t''}$. Therefore, the supposition that there exists a t in $[0,1]$ such that $A_t$ is not defined is false. Thus $A_t$ is defined for each t in $[0,1]$.

<u>Part 5</u> Let the conditions of Part 4 apply. Then $A_t$ is a segment from $A_0$ to $A_1$.

<u>Proof</u> Property 2 of the collection Y assures that $A_t$ is 1-1. Is $A_t$ continuous?

Now $\{v|_{E_0}^{-1}(0):\ 0 \text{ is open in } [v(A_0),v(A_1)]\}$ is a base for the open sets in $E_0$ for let U be an open subset of $E_0$. For each $A_x$ in U, there exists an open basis element of the form $B(A_x,e) \cap E_0$. By Lemma 12, there exists an r such that $v|_{E_0}^{-1}(B(v(A_x),r)) \subset B(A_x,e) \subset U$. $v \circ A_t(t) = (1-t)v(A_0) + tv(A_1)$ so $v \circ A_t$ is a continuous function. Thus for each open basis element $U = v|_{E_0}^{-1}(0)$, $A_t^{-1}(U) = A_t^{-1}(v|_{E_0}^{-1}(0)) = (v \circ A_t)^{-1}(0)$ is open in $[0,1]$. Hence $A_t$ is continuous.

This establishes that $A_t$ is a segment from $A_0$ to $A_1$ and the lemma is proved.

$2^X$ could now be established to be arcwise connected; but, for K, one additional lemma is needed.

<u>Lemma 14</u> If $E$ is in $K$ and $A_t$ is any segment with $A_0 = E$, then $A_t$ is in $K$ for each $t$ in $[0,1]$.

<u>Proof</u> $A_1$ is in $K$ since $A_0 \subset A_1$ and every component of $A_1$ intersects $A_0$. Let $t'$ be in $(0,1)$. For each $t$ in $[0,t']$, let $A_t = B_{t/t'}$. Then $B_{t/t'}$ is a segment from $B_0 = A_0 = E$ to $B_1 = B_{t'/t'}$ for (1) $v(B_{t/t'}) = v(A_t) = (1 - t/t' + t/t'(1 - t'))$ $v(A_0) + (t/t')t'v(A_1) = (1 - t/t')v(A_0) + (t/t')v(A_{t'}) =$ $(1 - t/t')v(B_0) + (t/t')v(B_1)$ and (2) if $t_1/t' < t_2/t'$, then $t_1 < t_2$ so $v(A_{t_1}) = v(B_{t_1/t'}) < v(B_{t_2/t'}) = v(A_{t_2})$. Since $A_t$ is continuous, $B_{t/t'}$ is continuous. Since $B_{t/t'}$ is a segment, $B_0 = E \subset B_1 = A_{t'}$. From Lemma 13 every component of $B_1$ intersects $B_0$, thus $B_1 = A_{t'}$ is connected.

<u>Theorem 20</u> If $X$ is connected, then $2^X$ and $K$ are arcwise connected.

<u>Proof</u> Let $E$ and $F$ be any members of $2^X$. Since $X$ is connected, by Lemma 13 there exists a segment $A^1$ from $E$ to $X$ and a segment $A^2$ from $F$ to $X$. Let $f$ be a function from $[0,1]$ to $2^X$ defined by $f(t) = A^1_{2t}$ for $0 \le t \le \frac{1}{2}$ and $f(t) = A^2_{2-2t}$ for $\frac{1}{2} \le t \le 1$. $f$ is a continuous function and $f([0,1])$ contains an arc in $2^X$ from $E$ to $F$. Thus $2^X$ is arcwise connected.

Let $E$ and $F$ be any members of $K$. By Lemma's 13 and 14, there exist segments $A^3$ and $A^4$ from $E$ to $X$ and $F$ to $X$, respectively, such that $A^3_t$, $A^4_t$ are in $K$ for each $t$ in $[0,1]$. Then $f$ will take $[0,1]$ into $K$ continuously if $f(t) = A^3_{2t}$ for $0 \le t \le \frac{1}{2}$ and $f(t) = A^4_{2-2t}$ for $\frac{1}{2} \le t \le 1$. Hence $f([0,1])$ contains an arc from $E$ to $F$ in $K$. Thus $K$ is arcwise connected.

## CONTRACTION PROPERTIES OF $2^X$

Bearing in mind that X is restricted to connected sets in this section, the main results are that $2^X$ is contractible in itself implies K is contractible in itself and that X is a Peano continuum implies $2^X$ is contractible in itself.

<u>Lemma 15</u> X is a Peano continuum implies that for each $e > 0$ there exists a positive number r such that if points a and c are in X, $d(a,c) < r$, and a is in A which is a member of K, then there exists a C in K containing c with $h(A,C) < e$.

<u>Proof</u> Suppose the Lemma is false. Then there exists an $e > 0$ such that for all integers n there exists points $a_n$ and $c_n$ in X where $d(a_n,c_n) < 1/n$ and an $A_n$ in K containing $a_n$ such that there does not exist a $C_n$ in K containing $c_n$ with $h(A_n,C_n) < e$. Since X is compact, $\{a_n\}_{n=1}^{\infty}$ contains a convergent subsequence $\{a_{n_i}\}_{i=1}^{\infty}$ with limit point p. X is locally connected at p implies that $B(p,e/2)$ contains a connected open set O containing p. It follows that $\delta(O) < e$. Since open balls are a base for metric X, there exists an $r(e) > 0$ such that $B(p,r(e)) \subset O$. Now there exists an integer N such that for all $n_i > N$, $a_{n_i}$ and $c_{n_i}$ are in $B(p,r(e)) \subset O$. Let $C_{n_i} = A_{n_i} \cup O$. Since $a_{n_i}$ is in $O \cap A_{n_i}$, $C_{n_i}$ is in K. Now $c_{n_i}$ is in O so $c_{n_i}$ is in $C_{n_i}$. Also $h(A_{n_i},C_{n_i}) < e$ for $A_{n_i} \subset C_{n_i}$ and thus for all a in $A_{n_i}$, $d(a,C_{n_i}) = 0$. Also since c is in $C_{n_i}$, either c is in $A_{n_i}$ and $d(A_{n_i},c) = 0$ or c is in O and $d(A_{n_i},c) \leq d(A_{n_i},a_{n_i}) + d(a_{n_i},c) < 0 + \delta(O) < e$. This is a

contradiction and thus the Lemma is true.

<u>Definition</u> Let $h^2$ denote the Hausdorff metric for $2^{2^X}$.

<u>Lemma 16</u> X is a Peano continuum implies that $i(X) = \{\{x\}: x \text{ in } X\}$ is contractible in K.

<u>Proof</u> Let W be a mapping of $i(X) \times [0,1]$ into $2^{2^X}$ defined by $W(\{x\},t) = \{E: x \text{ is in } E \text{ which is in } K \text{ and } v(E) = t\}$. $W(\{x\},t) = v^{-1}(t) \cap J(\{x\}) \cap K$ so $W(\{x\},t)$ is a closed subset of $2^X$ and W indeed does map into $2^{2^X}$. $W(\{x\},0) = \{\{x\}\}$ and $W(\{x\},1) = \{X\}$.

W is continuous if for a fixed t, $W(\{x\},t)$ is continuous in $\{x\}$ and if given $e > 0$, there is an $r(e) > 0$ such that for $\{x\}$ in $i(X)$ and $t_1,t_2$ in $[0,1]$, where $|t_2 - t_1| < r(e)$, $h^2(W(\{x\},t_1),W(\{x\},t_2)) < e$.

Suppose $e' > 0$ is given. Then $e = e'/2 > 0$ and from Lemma 12, there exists a $r(e)$ such that if A and C are in $2^X$, $A \subset C$, and $|v(A) - v(C)| < r(e)$, then $h(A,C) < e$. Since v is continuous, there exists a $q(r(e)) > 0$ such that if $h(A,C) < q(r(e)$, then $|v(A) - v(C)| < r(e)$. From Lemma 14, since X is a Peano continuum, there exists a $p(q(r(e))) > 0$ such that if x,y are in X with $d(x,y) < p(q(r(e)))$ and x is in A which is in K, then there exists a C in K containing y with $h(A,C) < q(r(e))$. Let t be fixed. For all $e' > 0$, let $p(q(r(e)))$ be as described above. Let x be any point of X and A any member of K containing x. Then for all y in $B(x,p(q(r(e))))$, there exists a C in K such that y is in C and $h(A,C) < q(r(e))$. Therefore $|v(A) - v(C)| < r(e)$. If

$v(A) = v(C) = t$, then $C$ is in $W(\{y\},t)$. If $v(A) < v(C)$, then there exists a $C_s$ on the segment from $\{y\}$ to $C$ such that $v(C_s) = v(A)$. Then $C_s \subset C$ and $v(C) - v(C_s) < r(e)$ so $h(C,C_s) < e$. Hence $h(A,C_s) < e'$ and $C_s$ is a member of $W(\{y\},t)$. If $v(C) < v(A)$, then there exists a $C_s$ on the segment from $C$ to $X$ such that $v(C_s) = v(A)$. Since $C \subset C_s$ and $v(C_s) - v(C) < r(e)$, $h(C,C_s) < e$ so $h(A,C_s) < e'$ and $C_s$ is in $W(\{y\},t)$. It follows that $h^2(W(\{x\},t),W(\{y\},t)) = \max\{\sup\{h(E_x,W(\{y\},t)): E_x \text{ is in } W(\{x\},t)\}, \sup\{h(E_y,W(\{x\},t): E_y \text{ in } W(\{y\},t)\}\} < e'$. Thus $W(\{x\},t)$ is continuous for a fixed $t$.

Again let $e > 0$ be given. For any $\{x\}$ in $i(X)$ and any pair $t',t''$ where $0 \leq t' \leq t'' \leq 1$, let $A_{t'}$ be a member of $W(\{x\},t')$. Then $A_{t'}$ and $X$ are in $K$ and $A_{t'} \subset X$. By Lemma 13, there exists a segment from $A_t$ to $X$, so there exists an $A_{t''}$ such that $A_{t'} \subset A_{t''}$ and $A_{t''}$ is in $W(\{x\},t'')$. Similarly if $A_{t''}$ is in $W(\{x\},t'')$, then $\{x\} \subset A_{t''}$ so there exists a segment from $\{x\}$ to $A_{t''}$. Thus there exists an $A_{t'}$ such that $A_{t'} \subset A_{t''}$ and $A_{t'}$ is in $W(\{x\},t')$. Now $|t' - t''| < r(e)$ implies that for all $A_{t_i}$ in $W(\{x\},t_i)$, there exists an $A_{t_j}$ in $W(\{x\},t_j)$ such that $|v(A_{t_i})-v(A_{t_j})| < r(e)$. Hence $h^2(W(\{x\},t'),W(\{x\},t'')) < e$. Therefore $W(\{x\},t)$ is uniformly continuous in $t$.

Let $Z = \sigma \cdot W$. Since both $\sigma$ and $W$ are continuous, $Z$ is continuous. Also $Z(\{x\},0) = \{x\}$ and $Z(\{x\},1) = X$. $Z(\{x\},t)$ is a union of connected setswhose intersection contains $x$ so $Z(\{x\},t)$ is connected. By Lemma 7, $\sigma(W(\{x\},t)) = Z(\{x\},t)$ is

in $2^X$; therefore $Z(\{x\},t')$ is in K. This establishes that Z is a contraction of i(X) in K.

<u>Lemma 17</u> i(X) is contractible in $2^X$ implies $2^X$ is contractible in itself.

<u>Proof</u> i(X) is contractible in $2^X$ implies there exists a continuous mapping Y of i(X) x $[0,1]$ into $2^X$ where $Y(\{x\},0) = \{x\}$ and $Y(\{x\},1) = A$, a fixed member of $2^X$. Define the mapping W of $2^X$ x $[0,1]$ into $2^{2^X}$ by $W(E,t) = \{Y(\{x\},t)$: x is in E$\}$. W(E,t) is in $2^{2^X}$ for let $U = \{/U_1,\ldots,U_{n_i}/\}$ be any open basis cover in $2^X$ for W(E,t). Then each $Y(\{x\},t)$ in W(E,t) is a member of one of the $/U_1,\ldots,U_{n_i}/$. Since Y is continuous, there exists an open set $/V_1,\ldots,V_m/ = V_{\{x\}}$ containing $\{x\}$ such that $Y(\{x\},t)$ is in $Y(V_{\{x\}},t) \subset /U_1,\ldots,U_{n_i}/$. Now x is in $\bigcap_{i=1}^{m} V_i = V_x$ which is open in X. E is closed so there exists a finite subcover $\{V_{x_i}\}_{i=1}^{n}$ of the cover $\{V_x$: x is in E$\}$ for E. Thus $W(E,t) \subset W(\bigcup_{i=1}^{n} V_{\{x_i\}},t) \subset$ finite union of $/U_1,\ldots,U_{n_i}/$.

$W(E,0) = \{\{x\}$: x in E$\}$ and $W(E,1) = \{A\}$.

W is continuous for let (E,t) be any member of $2^X$ x $[0,1]$ and let B(W(E,t),e) be an open set containing $\{Y(\{x\},t)$: x in E$\}$. For each $\{x\} \subset E$, there exists an $r_x$ such that $Y(B(\{x\},r_x),t) \subset B(Y(\{x\},t),e)$. E is closed implies there exists a finite subcover of $\{B(x,r_x)$: x in E$\}$. Let $r = \min\{r_x$: $B(x,r_x)$ is in the finite subcover$\}$. Then $\bigcup_{x \text{ in } E} B(x,r)$ is an open set containing E so E is in

$\bigcup_{x \text{ in } E} B(\{x\},r)$. Thus $W(E,t) \subset W(\bigcup_{x \text{ in } E} B(\{x\},r),t) \subset \bigcup_{x \text{ in } E} B(Y(\{x\},t),e) \subset B(W(E,t),e)$.

Then let $Z = \sigma \circ W$. $Z$ is continuous since $\sigma$ and $W$ are, $Z$ maps $2^X \times [0,1]$ into $2^X$, and $Z(E,0) = \sigma(W(E,0)) = \sigma(\{\{x\} : x \text{ in } E\}) = E$ and $Z(E,1) = \sigma(W(E,1)) = \sigma(\{A\}) = A$.

Thus $2^X$ is contractible in itself.

<u>Theorem 22</u> $X$ is a Peano continuum implies $2^X$ is contractible in itself.

<u>Proof</u> By Lemma 14, $X$ is a Peano continuum implies that $i(X)$ is contractible in $K \subset 2^X$. This implies by Lemma 16 that $2^X$ is contractible in itself.

<u>Theorem 23</u> $2^X$ is contractible implies $K$ is contractible in itself.

<u>Proof</u> $2^X$ is contractible implies there exists a continuous mapping $Z$ of $2^X \times [0,1]$ into $2^X$ where $Z(E,0) = E$ and $Z(E,1) =$ a constant $A$. Since $X$ is connected, $2^X$ is arcwise connected and there exists an arc from $A$ to $X$ in $2^X$. That is, there is a continuous mapping $g$ of $[0,1]$ into $2^X$ with $g(0) = A$ and $g(1) = X$. Define the mapping $W$ of $2^X \times [0,1]$ into $2^X$ by $W(E,t) = g(t)$. Then $W(E,0) = A$ and $W(E,1) = X$. Define the mapping $Y$ of $2^X \times [0,1]$ into $2^X$ by $Y(E,t) = Z(E,2t)$ for $0 \le t \le \frac{1}{2}$ and $Y(E,t) = W(E,2t - 1)$ for $\frac{1}{2} \le t \le 1$. Since $Z(E,2(\frac{1}{2})) = Z(E,1) = A = W(E,0) = W(E,2(\frac{1}{2}) - 1)$, $Y(E,t)$ is well-defined. Since $Z$ and $W$ are continuous, $Y$ is continuous.

Let $F$ be a mapping defined by $F(E,t) = \{Y(E,t') : 0 \le t' \le t\}$. $F(E,t)$ is the image of the compact set $E \times [0,t]$ under the

continuous map Y and hence is closed. Therefore F maps $2^X \times [0,1]$ into $2^{2^X}$. F is a continuous mapping, for let $B(F(E,t),e)$ be an open set containing $F(E,t)$. For each $t'$ in $[0,t]$, there exists a $B((E,t'),r')$ such that $Y(B((E,t'),r')) \subset B(Y(E,t'),e)$. There exists a finite subcover $\{B((E,t_i'),r_i')\}_{i=1}^{n}$ for $(E,[0,t])$; hence $F(E,t) = Y(E,[0,t]) \subset$

$$Y(\bigcup_{i=1}^{n} B((E,t_i'),r_i') \subset \bigcup_{i=1}^{n} B(Y(E,t_i'),e) \subset B(F(E,t),e).$$

Let $G = \sigma \cdot F$. Then G is continuous, $G(E,0) = Y(E,0) = E$, $G(E,1) = \bigcup_{t \text{ in } [0,1]} Y(E,t) = X$, if $0 \leq t' \leq t'' \leq 1$, then $G(A,t') \subset G(A,t'')$, and G maps $2^X \times [0,1]$ into $2^X$.

For each $(E,t)$ in $2^X \times [0,1]$, $v(E) \leq t + v(E)(1 - t) \leq 1$ and $[v(E),1]$ is the image of $[0,1]$ under the continuous 1-1 mapping $v \cdot G(E,[0,1])$. Therefore for each $(E,t)$ in $2^X \times [0,1]$, there exists an s in $[0,1]$ such that $v(G(E,s)) = t + v(E)(1-t)$. Define H to be the mapping from $2^X \times [0,1]$ to $2^X$ by $H(E,t) = G(E,s)$. Since $t = 0$ implies $s = 0$ and $t = 1$ implies $s = 1$, $H(E,0) = E$ and $H(E,1) = X$. Also $t \leq t'$ implies $s \leq s'$ so $H(E,t) \subset H(E,t')$. This fact together with G being continuous yield H being continuous. Finally $v(H(E,t)) = v(G(E,s)) = t + v(E)(1 - t) = t(v(H(E,1))) + (1 - t)(v(H(E,0)))$ so H is a segment from $(E,0)$ to $(E,1)$.

For E in K, by Lemma 14, $H(E,[0,1]) \subset K$. Thus the restriction of H to $K \times [0,1]$ maps into K so K is contractible in itself.

<u>Theorem 24</u> X is a Peano continuum implies K is contractible in itself.

<u>Proof</u> Immediate from Theorems 22 and 23.

HOMOTOPY THEOREM

<u>Theorem 25</u> If X and Y are compact metric spaces, f and g are functions of X to Y, and H is a homotopy of f to g, then $H^*$ defined by $H^*(E,t) = H(E,t)$, is a homotopy of $f^*$ to $g^*$.

<u>Proof</u> Let $(E,t)$ be any point of $2^X$ and $/U_1,\ldots,U_n/$ be any open basis set in $2^Y$ containing $H^*(E,t) = H(E,t)$. Then $H(E,t) \subset \bigcup_{i=1}^{n} U_i$ so for each x in E, $H(x,t)$ is in some $U_i$. Since H is continuous, there exists an open set of the form $O(x) \times (a_x,b_x)$ in $X \times [0,1]$ containing $(x,t)$ and such that $H(O(x) \times (a_x,b_x)) \subset U_i$. Now $\{O(x): x \text{ in } E\}$ is an open cover for compact E, thus there exists a finite subcover $\{O(x_i): i = 1,\ldots,m\}$ for E. Note since $H(E,t)$ intersects each $U_i$, the finite subcover can be chosen such that for $i = 1,\ldots,n$, $H(O(x_i) \times (a_{x_i},b_{x_i})) \subset U_i$. Let $T = \bigcap_{i=1}^{m} (a_{x_i},b_{x_i})$. Then T is an open subset of $[0,1]$ containing t. Let $O = /O(x_1),\ldots,O(x_m)/$. Then O is an open subset of $2^X$ containing E. Consequently $O \times T$ is an open subset of $2^X \times [0,1]$ containing $(E,t)$. Let A be any member of $H^*(O,T)$. There must exist a $B = (C,t_j)$ in $O \times T$ such that $A = H(B)$. Now $C \subset \bigcup_{i=1}^{m} O(x_i)$ implies that $A = H(C,t_j) \subset H(\bigcup_{i=1}^{m} O(x_i),t_j) \subset \bigcup_{i=1}^{m} H(O(x_i),(a_{x_i},b_{x_i})) \subset \bigcup_{i=1}^{n} U_i$. Also $C \cap O(x_i) \neq \emptyset$, so $H(C \cap O(x_i),t_j) \neq \emptyset$. Hence $A \cap U_i \supset H(C,t_j) \cap H(O(x_i),(a_{x_i},b_{x_i})) \supset H(C \cap O(x_i),t_j) \neq \emptyset$. Thus A is in $/U_1,\ldots,U_n/$ and $H^*(O,T) \subset /U_1,\ldots,U_n/$. $H^*$ therefore is continuous.

Now $H^*(E,0) = H(E,0) = f(E) = f^*(E)$. Also $H^*(E,1) =$

$H(E,1) = g(E) = g^*(E)$. It follows that $H^*$ is a homotopy of $f^*$ to $g^*$.

## EXAMPLES OF $2^X$

<u>Example 1</u> Let X be the Cantor set. Then X is a totally disconnected, perfect, compact metric space. Theorems 2, 10, 16, and 17 show that $2^X$ is a metric space, compact, perfect, and totally disconnected. Thus $2^X$ is a Cantor set.

<u>Example 2</u> Let $X = \{0,1,1/2,1/3,1/4,\ldots\}$. Let S be the set of all subsets of X containing zero. Then $2^X - S$ is the set of all finite subsets of X not containing zero. Theorem 18 implies $2^X - S$ is countably discrete. S is a closed subset of $2^X$ so it is a compact metric space. X is totally disconnected implies $2^X$ is totally disconnected; hence S is totally disconnected.

Suppose S contained an isolated point A. Let $A_n = \{1/n, 1/n+1, 1/n+2,\ldots\} \cup \{0\} \cup A$ for each positive integer n. Then $\{A_n\}_{n=1}^{\infty}$ is a sequence in $2^X$ converging to A. This implies A is not an open subset of $2^X$ which contradicts the supposition. Hence S has no isolated points.

$2^X - S$ is dense in $2^X$ for let $U = /U_1,\ldots,U_n/$ be any member of base B. Since $\{0\}$ is not open, $U_i \neq \{0\}$. Select one $x_i$ from each $U_i$ such that $x_i \neq 0$. Then $\{x_1,\ldots,x_n\} \subset 2^X - S$ and $\{x_1,\ldots,x_n\}$ is in $/U_1,\ldots,U_n/$. Thus each open set in $(2^X,h)$ contains a member of $2^X - S$ so $\overline{2^X - S} = 2^X$.

In summary, $2^X$ = Cantor set plus a countable, dense, discrete set.

<u>Example 3</u> Let $X = [0,1]$. Then each member of K is a closed interval or a point and so can be uniquely named by the ordered pair of its endpoints. Thus as a set, $K = \{(x,y): y \geq x,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$. Furthermore the topology for K as a subspace of $(2^X,h)$ is equivalent to the usual Euclidean metric m topology for the set. This last statement can be seen by considering the following.

Let i be the function from (K,m) to (K,h) defined by $i((a,b)) = (a,b)$. Since (K,m) and (K,h) are both compact spaces and i is a bijection, i is a homeomorphism if i is continuous. Let (a,b) be any point in (K,m) and let $B_h((a,b),e)$ be any open basis element in (K,h) containing $i((a,b))$. Let (c,f) be any point of $B_h((a,b),e)$. Let $j = h((a,b),(c,f))$. Then $B_m((c,f),e - j) \subset B_h((a,b),e)$ for let (p,q) be any point of $B_m((c,f),e - j)$. Then $m((c,f),(p,q)) = ((c - p)^2 + (f - q)^2)^{\frac{1}{2}} < e - j$. Hence $\max\{(c - p)^2, (f - q)^2\} < (e-j)^2$ so $h((c,f),(p,q)) < e - j$. Thus $h((a,b),(p,q)) \leq j + h((c,f),(p,q)) < e$. Therefore, i is continuous and (K,h) is homeomorphic to a disc in the plane.

<u>Example 4</u> Let X be the circle determined in polar coordinates by the equation $r = 1$. Each member of $K - \{X\}$ is a point or a closed simple arc and so can be uniquely named by the ordered pair (a,b) where $a = 2\pi$ - length of the arc and $b = \theta$-coordinate of the midpoint of the arc. Let $X = (0,0)$. Note that $(0,\theta_1) = (0,\theta_2)$ for any $\theta_1$ and $\theta_2$. K is then $\{(r,\theta): 0 \leq r \leq 2\pi,\ 0 \leq \theta < 2\pi\}$; so the mapping i of (K,h) to

(K, normal Euclidean metric m) can be shown to be a bijection.

Let $U = \{(r,\vartheta): \vartheta_1 < 0 < \vartheta_2,\ r_1 < r < r_2\}$ be an open basis element in (K,m) containing $i(r,\vartheta)$. Let $p = \min\{\vartheta_2 - \vartheta, \vartheta - \vartheta_1,\ r - r_1,\ r_2 - r\}$. Let $a_1$ = interior of the arc from $\vartheta + (2\pi - r)/2 - p/4$ to $\vartheta + (2\pi - r)/2 + p/4$. Let $a_2$ = interior of the arc from $\vartheta - (2\pi - r)/2 - p/4$ to $\vartheta - (2\pi - r)/2 + p/4$. Let $a_3$ = interior of the arc from $\vartheta_1 - (2\pi - r_1)/2$ to $\vartheta_2 + (2\pi - r_1)/2$. Then $(r,\vartheta)$ is a member of $/a_1, a_2, a_3/$ and $i(/a_1, a_2, a_3/)$ is a subset of U so i is continuous. Since K with each metric is compact, i is therefore a homeomorphism. Thus K is a disc in the plane.

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