Computable structure theory of continuous logic

by

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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DEDICATION

To the One who is All.

Soli Deo gloria.

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NOMENCLATURE

a, b, c Constants/Points

d Metric/Pseudometric

e, i, j Indices

f, g, h Functions/Maps

i, j, k, ℓ, m, n Variables for natural numbers

p, q Rational/Dyadic numbers

r, s Variables for real numbers

s, t Computation steps

t Terms

x, y, z Variables

A, B, C, D, E Subsets of natural numbers

L Language/Signature

P, R Predicates/Relations

T Theory

X, Y Oracles/Names

$\mathcal{C}$ Set of constants

$\mathcal{F}$ Set of functions

$\mathcal{P}$ Set of predicates

α, β, γ Ordinals

$\varphi, \psi, \theta, \gamma$ Well-formed formulas

η Arity

σ Variable assignments

Δ Modulus of continuity

Γ Sets of well-formed formulas

Φ, Ψ Turing machines

$\mathbb{C}$ Complex numbers

$\mathbb{N}$ Natural numbers, including 0

$\mathbb{Q}$ Rational numbers

$\mathbb{R}$ Real numbers

$\mathfrak{M}, \mathfrak{N}, \mathfrak{K}, \mathfrak{X}, \mathfrak{A}, \mathfrak{B}$ Models/Structures

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ABSTRACT

This dissertation examines computable structure theory relative to continuous logic. Due to the continuous nature of the relevant structures and space of truth values, special care is required to translate and modify results given in classical computable structure theory to the continuous setting. Three primary results are proven: (1) a generalized effective completeness theorem for continuous logic and computable presentations, (2) the existence of numerals for hyperarithmetical real numbers coded by computable infinitary sentences, and (3) upper and lower bounds on the complexity of the various diagram levels of the finitary and infinitary theories of a computably presented metric structure. Also given are basic model-theoretic results, an explicit formulation of the computable infinitary formulas of continuous logic, propositions concerning infinitary connectives, and a novel combinatorial result which allows for the encoding of a quantifier via a series inequality.

CHAPTER 1. INTRODUCTION

1.1 For the Philosopher

This section is for the layperson and philosopher; the curious mind who seeks to understand *what* we have done but not necessarily *how* we have done it. The remainder of this manuscript is for the mathematician.

Our experience of reality is, plausibly, a relationship between the continuous and the discrete, and the finite and the infinite. The mathematics which best describes the physical world, at least macroscopically, is the differential geometry of general relativity. Spacetime, as constructed, is grounded upon the real numbers. It is, therefore, a continuous structure. The resulting Newtonian mechanics of commonplace events follow suit.

On the other hand, assuming a broadly physicalist ontology, phenomenal experiences are grounded upon electric signals between neurons. Importantly, however, there are only finitely many neurons in a brain. It follows that internal experiences are finite representations of a continuous external reality.

Such representations are precisely what Turing had in mind (pun intended) when he first described what we now call Turing machines in (33). Our descriptions of continuous events are given in a finite language, yet these descriptions function unreasonably well. How is this possible? And what are the limitations on such representations? Understanding this relationship so that we can better use computers to model and reason about continuous reality is the focus of this work.

It is debated among philosophers of mind whether the internal states of the human mind track the states of a Turing machine, but if we relegate ourselves to the "experiences" of a classical computer, the internal states *exactly* track those of Turing machines. Just as information is given to a human mind by its external surroundings, information may also be given to computers from the external world.

Consider a cryptographic computer in Silicon Valley capturing a digital image of a wall of lava lamps to generate (presumably) random digits. This "randomness" needs to be qualified since the complexity of that information is unknown. For reasons such as this, computability theorists not only study computability at its most basic level, but also seek to understand what operations computers can perform when given access to arbitrarily complex information.

From our study of computation relative to continuous structures, we have three main results, summarized for the philosopher here.

First result.

Suppose one writes down a (possibly infinite) list of sentences using the "language of continuity". Suppose, moreover, that none of the sentences disagree with one another. It seems intuitive, then, that those sentences ought to describe a possible continuous "reality". (The basic idea is this: if any two of those sentences *were* contradictory, then surely anything which "realized" all of those sentences would be impossible!) This is called *completeness* in logic. That is, if a set of sentences is internally consistent, then there really is some possible "thing" which they describe. In (5), Ben Yaacov and Pedersen proved this very fact of completeness for continuous structures.

Suppose further, now, that the list of sentences was not arbitrary, but of some specific *complexity*. For example, finite lists of sentences have very low complexity, infinite lists which a computer could print out on its own (given an infinite amount of time) are slightly more complex, and infinite lists that a computer could only print out given some external data from the world around it are of a higher complexity. Maybe if the list is very simple, or finite, the sentences only describe the details of a possible world to a very low degree, while more complex lists describe the details more intricately. It seems intuitive, then, that a possible continuous "reality" *of a similar complexity* ought to exist to witness that none of those sentences are contradictory.

To give another example, "There is a sphere" is a very simple list of sentences (indeed a list containing only *one* sentence). On the other hand, a list of all the sentences describing a globe labeled with oceans and mountain ranges is considerably more complex. And while a complex (spherical) globe also satisfies the sentence that "There is a sphere", a simple, smooth, unlabeled sphere would *not* satisfy the latter list. As the list becomes more complex, so does the "simplest" object which satisfies it.

My first result confirms this intuition. If a computer can write out a list of noncontradictory sentences using the language of continuity, then a computer can also construct a model of an object that those sentences are describing. This is called *effective completeness.*

Second result.

Some sentences are always true. A common example used among laypeople is "Two plus two will always equal four." (Although whether this is really a *necessary* truth is up for debate!) Similarly some sentences are always false, like "There exists a married bachelor." But are there sentences which universally hold *partial* truth values somewhere in between?

Once one has understood the basics of the language of continuity, it is simple to formulate sentences which are half-true, three-quarters true, and so on. But it turns out that, at least in the basic form of the language of continuity, there aren't any more ... unless we allow sentences to be infinitely long, that is. When we do allow infinitely long sentences with no restrictions, then *any* amount of truth or falsity is universally representable!

But what if, as before, we only look at infinitely long sentences of some given *complexity*? The idea is that there may be some special code which allows us to write out one class of infinite sentences, but which doesn't work for other classes. My second result says that the complexity of the *amount of truth* a sentence can universally represent *exactly* corresponds to the complexity of writing out that sentence.

Third result.

What if, lastly, we work in the opposite direction? What if, instead of beginning with the language of continuity and searching for which realities and amounts of truth it can represent, we begin with a continuous reality and try to represent it in the language of continuity? In other words, can the language of continuity really capture *everything* we want it to from continuous realities? The answer is a qualified yes. While the language *can* describe everything, it's a bit harder for it to describe continuous realities than it is for the language of discreteness to describe discrete realities.

The basic idea is this: consider all the sentences of a given complexity which may or may not describe a “simple” continuous reality. How hard is it to tell *which* sentences describe that reality? In the discrete case the complexity scales directly. But in the continuous case, certain sentences look arbitrarily similar to each other. Not actually identical, but *as similar* as one can imagine. Distinguishing exactly *which* of these “supersimilar” sentences describe the structure and which do not is a hard problem. Indeed, it’s just *slightly* harder than what a computer can “easily” do. This is my final main result.

But this final result technically has two parts to it. First, it proves that it’s *no harder* than this level of “difficult” computation which is slightly harder than regular computation. And second, it shows that for *at least one* continuous reality, it’s *exactly* as hard as this low-level of “difficult” computation. At the end of the day, though, this level of difficult computation is still fairly reasonable, just not as simple as the discrete case where regular computation suffices.

1.2 For the Mathematician

Until Church’s work in (11) and Turing’s in (33), it was unclear whether any mathematical problem *couldn’t* be solved by an algorithm, or indeed what an “algorithm” even *was.* Nonetheless, the *Entscheidungsproblem* (decision problem) was given as follows.

Provide an algorithm which, given a sentence in the language of first-order logic, determines if that sentence is valid.

Church and Turing were the first to show, in those seminal papers, that no such algorithm exists. Thus began both the study of the *computable* and the *incomputable.* In other words, thus began the study of determining which mathematical problems *could* be solved by algorithms, and which could *not.*

Just previous to that work by Church and Turing, in (32), Alfred Tarski provided the foundation for what would eventually become the discipline of *model theory.* Model theory is the study of which mathematical structures satisfy which sentences in a given language. In other words, it is the study of truth-makers under a specified correspondence relation.

But a formal language, as was shown by Kurt Gödel in (18), may itself be viewed as a mathematical object. Now that Turing had shown there was a distinction between mathematical objects which were computable and those which were not, three natural questions arose.

(i) Which languages are computable? (Recursive language theory)

(ii) Which mathematical structures are computable? (Computability theory)

(iii) What is the relationship between the two? (Computable structure theory)

Since recursive languages are, themselves, mathematical objects, recursive language theory is often viewed as a subset of computability theory, a good introduction of which is Cooper (12), and further study of which can be found in Soare (30) and Sacks (29). Model theory, however, with computable structure theory following suit, has largely focused on discrete mathematical structures. Summaries of this work and related results can be found in Harizanov (21), Ash and Knight (1), and Montalbán (25). But for mathematicians of the analytic persuasion, a lack of computable model theory has been completed until only recently.

In Ben Yaacov *et al.* (3), the authors provide an extension of the fuzzy Łuckasiewicz logic, called first-order *continuous logic*, which functions extremely well as a logic for continuous structures. Indeed, in Ben Yaacov and Pedersen (5), they actually prove it is a *complete* logic for metric structures. Very briefly, notions of effectivity on such structures were considered in Calvert (8), relating metric structures to probabilistic computation via an effective completeness theorem. However, this project yielded minimal fruit.

Soon after, in Melnikov (23) and his work with Nies (24), techniques from traditional computable model theory were adapted for use in metric structures. The work combines classical model theory with ideas from computable analysis, a primer of which is Weihrauch (34). The *computable presentations* of structures introduced served as continuous correlates of the *computable copies* of classical computable model theory. The success of this method can be found more recently in McNicholl's work with Brown and Melnikov (7) and Franklin (16).

In the 2010s, three projects were undertaken on effectivity with respect to continuous logic and metric structures. Didehvar and Pourmahdian completed joint work with Ghasemloo (13) and Tavana (28), and Moody wrote a Ph.D. thesis (26). The first two of these implicitly used computable presentations, while the last did so somewhat explicitly. In (13), an effective completeness theorem with respect to linearly complete theories was proven, and in (28), an effective omitting types theorem followed. Lastly, the dissertation of (26) is presented in a somewhat scattered state, with arcane notation. This thesis attempts to construct a more firm foundation for the computable structure theory of continuous logic.

Also related to this paper is infinitary continuous logic and its computable fragment. Infinitary continuous logic was first introduced in Ben Yaacov and Iovino (4), with further work completed by Eagle in (14) and (15). Classical infinitary logic was used in relation to classical computable structure theory in Ash *et al.* (2) and Chisholm (10), and summarized in Ash and Knight (1). Though briefly sketched in Moody (26), our manuscript is the first to formally construct the computable fragment of infinitary continuous logic.

The three main results of this work are as follows.

Theorem 8. *There is an effective procedure which, given a name $X \in \mathbb{N}^{\mathbb{N}}$ of an L-theory T, produces a presentation of an L^+-structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$.*

In this case, L is a continuous signature with a metric, theories are consistent sets of sentences in that signature, L^+ is an extension of L which includes Henkin witnesses, and $\mathfrak{M}$ is a metric structure. This result is a very strong version of effective completeness, generalizing and strengthening the result found in (13).

Theorem 9. (C., McNicholl, 2022+) *There are partial computable functions f and g from $(\mathrm{O} \setminus \{1\}) \times \mathbb{N}$ to the set of computable infinitary sentences such that the following hold, for every $1 \leq \alpha < \omega_1^{\mathrm{CK}}$.*

- *If $\langle a, i\rangle$ is a Π^0_α index of a right Dedekind cut of a real number s, then $f(a,i)$ is a code of a Π^c_α sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.*

- *If $\langle a, i\rangle$ is a Σ^0_α index of a right Dedekind cut of a real number s, then $g(a,i)$ is a code of a Σ^c_α sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.*

This theorem shows that computable infinitary sentences in continuous logic are robust enough to track hyperarithmetical real numbers. Moreover, such real numbers are actually representable as numerals in *every* metric structure.

Theorems 11, 12, 13, and 14. (C., Goldbring, McNicholl, 2021) *Let $\mathfrak{M}$ be a computably presentable L-structure, and let N be a positive integer.*

1. *The closed Π_N diagram of $\mathfrak{M}$ is Π^0_N, and the open Π_N diagram of $\mathfrak{M}$ is Σ^0_{N+1}.*

2. *The closed Σ_N diagram of $\mathfrak{M}$ is Π^0_{N+1}, and the open Σ_N diagram of $\mathfrak{M}$ is Σ^0_N.*

These results hold uniformly in the sense that from N and an index for a computable presentation for $\mathfrak{M}$, one can compute an index for any of the above diagrams. Moreover, there is an L-structure in which these bounds are optimal, and similar correlates hold in the infinitary case.

These theorems prove fundamental results concerning the complexity of the theory of a computably presented metric structure and have already seen application by Goldbring and Hart (19).

1.3 Map of the Manuscript

The remainder of this thesis is organized as follows. Chapter 2 introduces the syntax and semantics for finitary and infinitary continuous logic, and discusses important results on completeness which will be used throughout this manuscript. Chapter 3 reviews key ideas from computability theory, including the hyperarithmetical hierarchy, computable analysis, and computable presentations. At the end of Chapter 3, previous results on limited versions of effective completeness for continuous logic are also summarized. In Chapter 4, we introduce our first main result on generalized effective completeness (Theorem 8). We begin by contrasting the result with previous results, and then move to building up the result by proving model-theoretic and effective extension-based lemmas. Chapter 5 introduces the computable fragment of continuous logic and proves Theorem 9 concerning hyperarithmetical numerals. Theorems 11, 12, 13, and 14, related to the complexity of the theory of a computably presented structure, are proven in Chapter 6. Notably, in that chapter, we also introduce diagrams of metric structures and prove an important combinatorial result (Theorem 10) which allows for the encoding of a single quantifier via a series inequality. Lastly, Chapter 7 offers a conclusion, along with notes about applications and further research.

The results presented in Chapters 2 and 3 are those of other mathematicians, while all the work given in Chapters 4, 5, and 6 was completed by myself, with assistance from Drs. Timothy McNicholl and Isaac Goldbring. Each theorem is labeled with all those who contributed to its proof.

This manuscript includes exercises which are folklore results but nonetheless useful for the reader to prove for an adequate understanding of the material. Solutions to selected exercises can be found in the appendix.

At certain points, largely in Chapter 3, *heuristics* are given in the place of formal definitions and notations. These are given for ease of understanding on the reader's part, as formal definitions and notations can often get in the way of a useful and flexible view of the material. For more formal definitions concerning computability-theoretic concepts, the reader is directed toward Cooper (12) and Weihrauch (34).

CHAPTER 2. PRELIMINARIES ON CONTINUOUS LOGIC

2.1 Continuous Logic

Every logic begins with a language. Some philosophers take symbols to be brutely given. On the other hand, symbols may just be considered as distinguished sets. In our case, we begin with the alphabet $\{0,1\} = \{\emptyset, \{\emptyset\}\}$ and freely generate all finite strings. We then take each of the following symbols to formally correspond to some designated such string, though we proceed to give them naïvely.

Definition 1. The *logical symbols* of continuous logic (CL) consist of the following.

- (and) are the *parentheses*.
- $x, y, z, \ldots$ are the *variable symbols* ($\mathcal{V}$).
- $\neg$, $\frac{1}{2}$, and $\dot{-}$ are the *connectives*.
- sup and inf are the *quantifiers*.

Remark 1. In some versions of continuous logic, the set of connectives contains a distinguished symbol $\underline{u}$ for each continuous map $u : [0,1]^{\eta(u)} \to [0,1]$. The resulting set of well-formed formulas of such a logic, however, is uncountable and thus fails to operate effectively. Our choice of $\neg$, $\frac{1}{2}$, and $\dot{-}$ as the connectives was made for four reasons.

1. $\neg$ plays precisely the role of classical negation ($\neg$) and $\dot{-}$ of reverse implication ($\leftarrow$)[1]. The interpretation of the $\frac{1}{2}$ operator is similarly intuitive.
2. Ben Yaacov and Usyatsov (6) showed that, after interpretation, combinations of $\neg$, $\frac{1}{2}$, and $\dot{-}$ are dense in the set of all continuous maps on $[0,1]$. Thus finitary well-formed formulas in

[1]That this corresponds to reverse and not standard implication will become clear after we have introduced the interpretation of the connective.

these connectives can approximate those in the wider set of connectives arbitrarily well. Such an approximation is sufficient even for completeness (as seen in Ben Yaacov and Pedersen (5)).

3. When a signature (Definition 2) is effectively numbered (Definition 48), the sentences and well-formed formulas of that signature may be effectively enumerated.

4. Because of 1, 2, and 3, $\neg$, $\frac{1}{2}$, and $\dot{-}$ have become a somewhat canonical set of connectives for continuous logic.

Definition 2. A *signature* is a quintuple $L = \left(\mathcal{P}, \mathcal{F}, \mathcal{C}, \Delta, \eta\right)$ such that each of the following hold.

- $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$ are mutually disjoint, and contain no logical symbols.
- $\Delta : \mathcal{P} \cup \mathcal{F} \to \mathbb{N}^{\mathbb{N}}$.
- $\eta : \mathcal{P} \cup \mathcal{F} \to \mathbb{N} \setminus \{0\}$.
- There is some $\underline{d} \in \mathcal{P}$ such that $\Delta(\underline{d}) = \mathrm{id}_{\mathbb{N}}$ and $\eta(\underline{d}) = 2$.

$\mathcal{P}$ is the set of *predicate symbols*, $\mathcal{F}$ the set of *function symbols*, and $\mathcal{C}$ the set of *constant symbols*. Δ is the *modulus map* and η the *arity map*. Each predicate (or function) symbol F is an $\eta(F)$*-ary* predicate (or function) symbol.

For the remainder of this chapter, unless stated otherwise, we will assume we have a fixed signature L. We proceed to generate the terms.

Definition 3. The *terms* of L are generated recursively as follows.

- Every constant symbol and variable symbol is a term.
- If f is a function symbol and $t_0, ..., t_{\eta(f)-1}$ are terms, then $f(t_0, ..., t_{\eta(f)-1})$ is a term.

As an example, if one considers the language of arithmetic, terms include phrases like "0+1", "9×8", "$x - 7$", and "$4, 100, 323$". For the remainder of this manuscript, we will leave out the use of quotations as syntactic designators. We now generate the well-formed formulas.

Definition 4. The *well-formed formulas* (wffs) of L are generated recursively as follows.

- If P is a predicate symbol and $t_0, ..., t_{\eta(P)-1}$ are terms, then $P(t_0, ..., t_{\eta(P)-1})$ is a wff.
- If φ is a wff then both $\neg\varphi$ and $\frac{1}{2}\varphi$ are wffs.
- If φ and ψ are wffs then $\varphi \dot{-} \psi$ is a wff.
- If φ is a wff and x is a variable symbol then both $\sup_x \varphi$ and $\inf_x \varphi$ are wffs.

Free and *bound* variables are defined as in classical logic. *Correct* substitution is also given as in the classical case, where $[t/x]$ means t is substituted for x.

Again, if we work in the language of arithmetic but use $\underline{d}$ in the place of $\neq$, we find that each of $\underline{d}(0, x)$, $\underline{d}(1 + 1, 2)$, and $\sup_x \underline{d}(12178, 6 \times x)$ are wffs of this signature. Notably, x is free in $\underline{d}(0, x)$, while it is bound in $\sup_x \underline{d}(12178, 6 \times x)$.

Heuristic 1. The following syntax maps are used as shorthand.

Shorthand	**String**
$\varphi \vee \psi$	$\neg\big((\neg\varphi) \dot{-} \psi\big)$
$\varphi \wedge \psi$	$\varphi \dot{-} (\varphi \dot{-} \psi)$
$\varphi \leftrightarrow \psi$	$(\varphi \dot{-} \psi) \vee (\psi \dot{-} \varphi)$
$\vec{x}$	$(x_0, ..., x_n)$
$\sup_{x_0,...,x_n} \varphi$	$\sup_{x_0} \dots \sup_{x_n} \varphi$
$\inf_{x_0,...,x_n} \varphi$	$\inf_{x_0} \dots \inf_{x_n} \varphi$
$\underline{0}$	$\sup_x \underline{d}(x, x)$
$\underline{1}$	$\neg\, \underline{0}$
$\varphi \dot{+} \psi$	$\neg\big((\underline{1} \dot{-} \varphi) \dot{-} \psi)\big)$
$m\varphi$	$\underbrace{(...(\varphi \dot{+} \varphi) \dot{+} \cdots \dot{+} \varphi)}_{m\text{-many}}$
$\underline{2^{-k}}$	$\underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{k\text{-many}}\ \underline{1}$
$\underline{\frac{\ell}{2^k}}$	$\underbrace{(...(\underline{2^{-k}} \dot{+} \underline{2^{-k}}) \dot{+} \cdots \dot{+} \underline{2^{-k}})}_{\ell\text{-many}}$

Some of the above syntax may seem loaded; this is for good reason. Exercise 1 explains why.

Once the well-formed formulas of a logic are given, one needs to know how theorems are proven. Generally speaking, the provable theorems are given via axioms and rules of inference. Continuous logic follows this general format.

Definition 5. The *axiom schemata* of continuous logic are as follows. In each of the following, φ, ψ, and θ range over arbitrary wffs. The first four schemata correspond to the classical propositional axioms.

I. $(\varphi \dot{-} \psi) \dot{-} \varphi$.

II. $\big((\theta \dot{-} \varphi) \dot{-} (\theta \dot{-} \psi)\big) \dot{-} (\psi \dot{-} \varphi)$.

III. $\big(\varphi \dot{-} (\varphi \dot{-} \psi)\big) \dot{-} \big(\psi \dot{-} (\psi \dot{-} \varphi)\big)$

IV. $(\varphi \dot{-} \psi) \dot{-} (\neg\varphi \dot{-} \neg\varphi)$.

The next four correspond to the classical first-order axiom schemata. For every $x \in \mathcal{V}$ and term t,

V. $(\sup_x \psi \dot{-} \sup_x \varphi) \dot{-} \sup_x(\psi \dot{-} \varphi)$.

VI. $\varphi[t/x] \dot{-} \sup_x \varphi$, when this substitution is correct.

VII. $\sup_x \varphi \dot{-} \varphi$, when x is not free in φ.

VIII. $\inf_x \varphi \leftrightarrow \neg(\sup_x \neg\varphi)$.

These two define the $\frac{1}{2}$ connective.

IX. $\frac{1}{2}\varphi \dot{-} (\varphi \dot{-} \frac{1}{2}\varphi)$.

X. $(\varphi \dot{-} \frac{1}{2}\varphi) \dot{-} \frac{1}{2}\varphi$.

The following define the predicate symbol $\underline{d}$. For every $x, y \in \mathcal{V}$,

XI. $\underline{d}(x, x)$.

XII. $\underline{d}(x,y) \dot{-} \underline{d}(y,x)$.

XIII. $\big(\underline{d}(x,z) \dot{-} \underline{d}(x,y)\big) \dot{-} \underline{d}(y,z)$.

The next schema defines interaction between $\underline{d}$ and function symbols. For every $f \in \mathcal{F}$, $n \in \mathbb{N}$, (possibly empty) tuples of terms $\vec{t_0}, \vec{t_1}$, and $x, y \in \mathcal{V}$,

XIV. $\big(\underline{2^{-\Delta(f;n)}} \dot{-} \underline{d}(x,y)\big) \wedge \big(\underline{d}\big(f(\vec{t_0}, x, \vec{t_1}), f(\vec{t_0}, y, \vec{t_1})\big) \dot{-} \underline{2^{-n}}\big)$.

Lastly, the following defines interaction between $\underline{d}$ and other predicate symbols. For every $P \in \mathcal{P}$, $n \in \mathbb{N}$, (possibly empty) tuples of terms $\vec{t_0}, \vec{t_1}$, and $x, y \in \mathcal{V}$,

XV. $\big(\underline{2^{-\Delta(P;n)}} \dot{-} \underline{d}(x,y)\big) \wedge \big(\big(P(\vec{t_0}, x, \vec{t_1}) \dot{-} P(\vec{t_0}, y, \vec{t_1})\big) \dot{-} \underline{2^{-n}}\big)$.

There is a slight discrepancy between how schemata XIV and XV are given here as opposed to their presentation in (5). The above formulation is not only more parsimonious, but requires no use of access to metric structure semantics (which is a bit circular). Moreover, the above description lends itself nicely to effective constructions.

Definition 6. The *rules of inference* of continuous logic are as follows, where φ and ψ are wffs and x is a variable symbol.

- *Modus ponens*

$$\frac{\varphi,\ \psi \dot{-} \varphi}{\psi}$$

- *Generalization*

$$\frac{\varphi}{\sup_x \varphi}$$

As in classical logic, once the axioms and rules of inference are given, we may define the provable wffs.

Definition 7. The set of *provable* wffs ($\vdash$) is defined as in the classical case; it is the smallest subset of wffs that contains all the axioms and is closed under modus ponens and generalization. As is convention, when φ is a wff, we write $\vdash \varphi$ when $\varphi \in \vdash$ and $\nvdash \varphi$ when $\varphi \notin \vdash$.

Definition 8. Let Γ be a set of wffs. Γ is *inconsistent* if there is some finite set $\{\varphi_0, ..., \varphi_n\} \subseteq \Gamma$ such that for some $k \in \mathbb{N}$,

$$\vdash \underline{2^{-k}} \dot{-} (\varphi_0 \vee ... \vee \varphi_n).$$

Γ is *consistent* if it is not inconsistent. A consistent set of sentences is called a *theory.*

The set of *consequences* of Γ ($\Gamma \vdash$) is the smallest subset of wffs that includes $\Gamma \cup \vdash$ and is closed under modus ponens. Again, as is convention, we write $\Gamma \vdash \varphi$ when $\varphi \in \Gamma \vdash$ and $\Gamma \nvdash \varphi$ when $\varphi \notin \Gamma \vdash$.

We will soon see that the set of consequences of a set of wffs does not act in the same way as it does in classical logic. This has to do with the continuous nature of the space of truth values.

2.2 Metric Structures

There is, arguably, no structure more important to mathematical analysis and spatial reasoning than a (pseudo)metric space.[2] These are the structures continuous logic was developed to represent. Importantly, however, a signature must be able to speak about the continuity of maps on such structures.

Definition 9. Let $(|\mathfrak{M}|, d)$ and $(|\mathfrak{M}'|, d')$ be pseudometric spaces of diameter 1 and let $f : |\mathfrak{M}| \to |\mathfrak{M}'|$. A map $\Delta(f) : \mathbb{N} \to \mathbb{N}$ is called a *modulus of continuity* for f if for every $a, b \in |\mathfrak{M}|$, $d(a, b) < 2^{-\Delta(f;n)}$ implies that $d'\big(f(a), f(b)\big) \leq 2^{-n}$.

We are now able to define our semantics.

Definition 10. An *interpretation* of a signature L is a map $\cdot^{\mathfrak{M}}$ with domain $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ such that for some set $|\mathfrak{M}|$, each of the following hold.

- For every predicate symbol P, $P^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(P)} \to [0, 1]$.
- For every function symbol f, $f^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(f)} \to |\mathfrak{M}|$.
- For every constant symbol c, $c^{\mathfrak{M}} \in |\mathfrak{M}|$.

[2] An argument can be made for more general *topological* spaces.

$\cdot^{\mathfrak{M}}$ is a *continuous interpretation* if, moreover, each of the following hold.

- $\underline{d}^{\mathfrak{M}} := d$ is a pseudometric.
- For every predicate symbol P, $\Delta(P)$ is a modulus of continuity for P.[3]
- For every function symbol f, $\Delta(f)$ is a modulus of continuity for f.

When $\cdot^{\mathfrak{M}}$ is an interpretation, the quintuple

$$\mathfrak{M} = \big(|\mathfrak{M}|, d, \{P^{\mathfrak{M}} : P \in \mathcal{P} \setminus \{\underline{d}\}\}, \{f^{\mathfrak{M}} : f \in \mathcal{F}\}, \{c^{\mathfrak{M}} : c \in \mathcal{C}\}\big)$$

is an *L-pre-structure.* Moreover, if $\cdot^{\mathfrak{M}}$ is a continuous interpretation, $\mathfrak{M}$ is a *continuous* L-pre-structure. Lastly, if $\cdot^{\mathfrak{M}}$ is a continuous interpretation and $\big(|\mathfrak{M}|, d\big)$ is a complete metric space, then $\mathfrak{M}$ is an *L-structure.* If $|\mathfrak{M}|$ is countable, $\mathfrak{M}$ is called *weak.*

When $\mathfrak{M}$ is an L-pre-structure, $\mathcal{P}^{\mathfrak{M}}$ is the set of *predicates* of $\mathfrak{M}$, $\mathcal{F}^{\mathfrak{M}}$ the set of *functions* of $\mathfrak{M}$, and $\mathcal{C}^{\mathfrak{M}}$ the set of *distinguished points* of $\mathfrak{M}$.

At times, the language of *non*-continuous pre-structures is dropped, and every pre-structure is assumed to be continuous. We find it useful for certain examples, however, to retain access to pre-structures which can *interpret* the symbols of a signature without satisfying the requirements placed by the moduli of continuity. Also what were given here as "L-structures" are often designated as "metric L-structures". In this manuscript, however, we will assume every structure is interpreting a continuous signature, so we drop the prefix "metric".

Example 1. The following are examples of structures (for some related signature).

- A complete metric space of diameter 1 with no additional structure.
- The natural numbers with the discrete metric and addition and multiplication as binary functions.

[3]Here the domain of $P^{\mathfrak{M}}$ is considered as the standard product pseudometric space $\big(|\mathfrak{M}|^{\eta(P)}, (\underline{d}^{\mathfrak{M}})^{\eta(P)}\big)$ and the range the metric space $\big([0,1], |\cdot|\big)$.

- The unit ball of a Banach space over $\mathbb{R}$ or $\mathbb{C}$, the metric induced by the norm of that Banach space, as functions all binary maps of the form

$$f_{\alpha,\beta}(x,y) = \alpha x + \beta y$$

where $|\alpha| + |\beta| \leq 1$ as scalars, and the additive identity 0 is a distinguished point.

- The unit ball of a C^*-algebra with the standard norm as the metric, and multiplication and the $*$-map included as functions.

- When $(\Omega, \mathcal{B}, \mu)$ is a probability space, let M be its measure algebra and d the measure of symmetric difference. Then (M,d) along with μ as a predicate, $\cap$, $\cup$, and $\cdot^c$ as functions, and 0 and 1 as distinguished points is a structure.

We now define when a structure satisfies a given wff. But to do so, we must first define variable assignments.

Definition 11. Let $\mathfrak{M}$ be an L-pre-structure. An *assignment* (on $\mathfrak{M}$) is a map $\sigma : \mathcal{V} \to |\mathfrak{M}|$. When σ is an assignment, $x \in \mathcal{V}$, and $a \in |\mathfrak{M}|$, the assignment $\sigma(x \mapsto a)$ is defined as follows.

$$\sigma(x \mapsto a;\ y) := \begin{cases} a & \text{if } y = x, \\ \sigma(y) & \text{otherwise.} \end{cases}$$

Given a term t, the *interpretation* of t in $\mathfrak{M}$ with σ ($t^{\mathfrak{M},\sigma}$) is defined recursively as follows.

- If $t \in \mathcal{C}$, then $t^{\mathfrak{M},\sigma} := t^{\mathfrak{M}}$.

- If $t \in \mathcal{V}$, then $t^{\mathfrak{M},\sigma} := \sigma(t)$.

- If $t = f(t_0, ..., t_n)$, then $t^{\mathfrak{M},\sigma} := f^{\mathfrak{M}}\big(t_0^{\mathfrak{M},\sigma}, ..., t_n^{\mathfrak{M},\sigma}\big)$.

Definition 12. For every L-pre-structure $\mathfrak{M}$, assignment σ, and wff φ, the *value* (or *truth value*) of φ in $\mathfrak{M}$ with σ ($\varphi^{\mathfrak{M},\sigma}$) is defined recursively as follows.

- $\big(P(t_0, ..., t_n)\big)^{\mathfrak{M},\sigma} := P^{\mathfrak{M}}\big(t_0^{\mathfrak{M},\sigma}, ..., t_n^{\mathfrak{M},\sigma}\big)$.

- $(\neg\varphi)^{\mathfrak{M},\sigma} := 1 - \varphi^{\mathfrak{M},\sigma}$.
- $\left(\frac{1}{2}\varphi\right)^{\mathfrak{M},\sigma} := \frac{1}{2} \cdot \varphi^{\mathfrak{M},\sigma}$.
- $(\varphi \dot{-} \psi)^{\mathfrak{M},\sigma} := \max\left\{\varphi^{\mathfrak{M},\sigma} - \psi^{\mathfrak{M},\sigma}, 0\right\}$.
- $\left(\sup_x \varphi\right)^{\mathfrak{M},\sigma} := \sup_{a\in|\mathfrak{M}|} \varphi^{\mathfrak{M},\sigma(x\mapsto a)}$.
- $\left(\inf_x \varphi\right)^{\mathfrak{M},\sigma} := \inf_{a\in|\mathfrak{M}|} \varphi^{\mathfrak{M},\sigma(x\mapsto a)}$.

When $\varphi^{\mathfrak{M},\sigma} = 0$, $\mathfrak{M}$ with σ *satisfies* φ ($\mathfrak{M}, \sigma \vDash \varphi$). If, moreover, $\varphi^{\mathfrak{M},\sigma} = 0$ for every assignment σ, then $\mathfrak{M}$ *satisfies* φ ($\mathfrak{M} \vDash \varphi$). Notably, when φ is a sentence, $\mathfrak{M} \vDash \varphi$ if and only if there exists some assignment σ such that $\mathfrak{M}, \sigma \vDash \varphi$. If every continuous L-pre-structure satisfies φ, it is *valid*. The set of all valid wffs is denoted $\vDash$. We write $\vDash \varphi$ when $\varphi \in \vDash$ and $\nvDash \varphi$ when $\varphi \notin \vDash$.

It is clear, now, that the space of truth values is $[0, 1]$ and is, itself, a complete metric space of diameter 1. This provides even the most trivial structures with nontrivial complexity, as will be shown in Chapter 5.

Heuristic 2. When φ is a wff with free variables $\vec{x}$ and $\mathfrak{M}$ an L-structure, $\varphi^{\mathfrak{M}}(\vec{a})$ means $\varphi^{\mathfrak{M},\sigma(\vec{x}\mapsto\vec{a})}$ for any assignment σ.

Definition 13. Let Γ be a set of wffs. An L-structure $\mathfrak{M}$ *models* Γ ($\mathfrak{M} \vDash \Gamma$) if for every $\varphi \in \Gamma$, $\mathfrak{M} \vDash \varphi$. A wff φ *follows* from Γ ($\Gamma \vDash \varphi$) if for every continuous L-pre-structure $\mathfrak{M}$, if $\mathfrak{M} \vDash \Gamma$, then $\mathfrak{M} \vDash \varphi$.

The clever reader may already notice subtle differences between the syntactic consequence relation and the semantic following relation. These subtleties will become apparent when we discuss completeness.

The reader may also now be wondering what signatures the structures in Example 1 are actually structures *over*. We present the basic idea for how to generate a signature from a structure.

Example 2. Consider a metric structure

$$\mathfrak{M} = \big(|\mathfrak{M}|, d, \{P_i : i \in I\}, \{f_j : j \in J\}, \{c_k : k \in K\}\big).$$

Let $L_{\mathfrak{M}}$ contain the predicate symbol $\underline{d}$ and predicate symbols $\underline{P_i}$, function symbols $\underline{f_j}$, and constant symbols $\underline{c_k}$, and define η to agree with the arities of each P_i and f_j, and Δ to agree with their uniform continuity. The result is that $\mathfrak{M}$ is an $L_{\mathfrak{M}}$-structure under the natural interpretation.

Exercise 1. Let $\mathfrak{M}$ be an L-pre-structure, φ and ψ be sentences, and $\ell, k \in \mathbb{N}$. Prove the following.

(a) $(\varphi \vee \psi)^{\mathfrak{M}} = \max\big\{\varphi^{\mathfrak{M}}, \psi^{\mathfrak{M}}\big\}$.

(b) $(\varphi \wedge \psi)^{\mathfrak{M}} = \min\big\{\varphi^{\mathfrak{M}}, \psi^{\mathfrak{M}}\big\}$.

(c) $(\varphi \leftrightarrow \psi)^{\mathfrak{M}} = \big|\varphi^{\mathfrak{M}} - \psi^{\mathfrak{M}}\big|$.

(d) $\underline{0}^{\mathfrak{M}} = 0$.

(e) $\underline{1}^{\mathfrak{M}} = 1$.

(f) $\varphi \dot{+} \psi = \min\big\{\varphi^{\mathfrak{M}} + \psi^{\mathfrak{M}}, 1\big\}$.

(g) $m\varphi = \min\big\{m \cdot \varphi^{\mathfrak{M}}, 1\big\}$.

(h) $\big(\underline{\tfrac{\ell}{2^k}}\big)^{\mathfrak{M}} = \frac{\ell}{2^k}$.

Note that all of the above hold for wffs when interpreted along with an assignment.

Exercise 2 (Soundness of I – X)**.** Let $\mathfrak{M}$ be an L-pre-structure. Prove that $\mathfrak{M}$ satisfies axiom schemata I through X.

Exercise 3 (Canonicity of CL for continuous L-pre-structures)**.** Let $\mathfrak{M}$ be an L-pre-structure. Prove the following.

(a) $\mathfrak{M}$ satisfies axiom schemata XI, XII, and XIII if and only if $\underline{d}^{\mathfrak{M}}$ is a pseudometric.

(b) $\mathfrak{M}$ satisfies axiom schema XIV if and only if for every function $f^{\mathfrak{M}}$, $\Delta(f)$ is a modulus of continuity for $f^{\mathfrak{M}}$.

(c) $\mathfrak{M}$ satisfies axiom schema XV if and only if for every predicate $P^{\mathfrak{M}}$, $\Delta(P)$ is a modulus of continuity for $P^{\mathfrak{M}}$.

Deduce that an L-pre-structure satisfies CL if and only if it is a continuous L-pre-structure.

It is well-known that some (pseudo)metric spaces embed into others. There is a similar notion for embeddings of pre-structures.

Definition 14. Let $\mathfrak{M}$ and $\mathfrak{N}$ be continuous L-pre-structures. $E : |\mathfrak{M}| \to |\mathfrak{N}|$ is an L-*morphism* if each of the following hold.

- For every $f \in \mathcal{F}$ and $a_0, ..., a_{\eta(f)-1} \in |\mathfrak{M}|$,

$$E\big(f^{\mathfrak{M}}(a_0, ..., a_{\eta(f)-1})\big) = f^{\mathfrak{N}}\big(E(a_0), ..., E(a_{\eta(f)-1})\big).$$

- For every $P \in \mathcal{P}$ and $a_0, ..., a_{\eta(P)-1} \in |\mathfrak{M}|$,

$$E\big(P^{\mathfrak{M}}(a_0, ..., a_{\eta(P)-1})\big) = P^{\mathfrak{N}}\big(E(a_0), ..., E(a_{\eta(P)-1})\big).$$

E is an *elementary* L-morphism if, moreover, for every $\mathfrak{M}$-assignment σ, $\varphi^{\mathfrak{M},\sigma} = \varphi^{\mathfrak{N},E\circ\sigma}$.

It is important to note that any classical structure can be considered as a metric structure. The following definitions and exercise illustrate this consideration.

Definition 15. A *classical structure* is a quadruple

$$\mathfrak{A} = (|\mathfrak{A}|, \mathcal{P}^{\mathfrak{A}}, \mathcal{F}^{\mathfrak{A}}, \mathcal{C}^{\mathfrak{A}})$$

with the following properties.

- For every $R \in \mathcal{P}^{\mathfrak{A}}$, $R \subseteq |\mathfrak{A}|^{\eta(R)}$.

- For every $f \in \mathcal{F}^{\mathfrak{A}}$, $f : |\mathfrak{A}|^{\eta(f)} \to |\mathfrak{A}|$.

- For every $c \in \mathcal{C}^{\mathfrak{A}}$, $c \in |\mathfrak{A}|$.

Similarly to the continuous signature constructed in Example 2, $\mathfrak{A}$ interprets some first-order language $L_{\mathfrak{A}} = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta)$, with the relevant symbols denoted $\underline{R} \in \mathcal{P}$, $\underline{f} \in \mathcal{F}$, and $\underline{c} \in \mathcal{C}$.

Heuristic 3. If $\mathcal{P}^{\mathfrak{A}}$, $\mathcal{F}^{\mathfrak{A}}$, or $\mathcal{C}^{\mathfrak{A}}$ is empty or a singleton, it is common to simplify the notation of the structure. *E.g.*, $(|\mathfrak{A}|, <_{\mathfrak{A}})$ denotes the structure $(|\mathfrak{A}|, \{<_{\mathfrak{A}}\}, \emptyset, \emptyset)$.

Definition 16. Let $\mathfrak{A}$ be a classical structure and $L_{\mathfrak{A}}$ the language of that structure. The *terms* of $L_{\mathfrak{A}}$ are defined as in Definition 3. The *well-formed formulas* (wffs) of $L_{\mathfrak{A}}$ are generated in the classical manner, with connectives $\to$ and $\neg$ and quantifiers $\exists$ and $\forall$. A wff with no variable symbols is a *sentence*. A wff is *atomic* (or *quantifier-free*) if it contains no quantifiers. *Satisfaction* ($\vDash_{\mathrm{FOL}}$) is defined classically.

Definition 17. Fix a classical $L_{\mathfrak{A}}$-structure

$$\mathfrak{A} = \big(|\mathfrak{A}|, \mathcal{P}^{\mathfrak{A}}, \mathcal{F}^{\mathfrak{A}}, \mathcal{C}^{\mathfrak{A}}\big).$$

Define a signature $L_{\mathrm{MT}(\mathfrak{A})}$ in the following manner.

- The predicate symbols consist of $\mathcal{P}$ along with a new predicate symbol $\underline{d}$.

- The function symbols are just $\mathcal{F}$.

- The constant symbols are just $\mathcal{C}$.

- $\Delta : \mathcal{P} \cup \mathcal{F} \to \mathbb{N}^{\mathbb{N}}$, where $\Delta(\underline{d}) := \mathrm{id}_{\mathbb{N}}$ and for every other predicate and function symbol F and $n \in \mathbb{N}$, $\Delta(F; n) := 1$.

- $\eta : \mathcal{P} \cup \mathcal{F} \to \mathbb{N}$, where $\eta(\underline{d}) := 2$ and for every other predicate and function symbol, the arity is the same as in L_{FOL}.

Then define the *metric transformation* of $\mathfrak{A}$ ($\mathrm{MT}(\mathfrak{A})$) as follows.

$$\mathrm{MT}(\mathfrak{A}) = \big(|\mathfrak{A}|, d, \mathcal{P}^{\mathrm{MT}(\mathfrak{A})}, \mathcal{F}^{\mathrm{MT}(\mathfrak{A})}, \mathcal{C}^{\mathrm{MT}(\mathfrak{A})}\big),$$

where

$$d(a,b) := \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{otherwise,} \end{cases}$$

for every predicate symbol $P \in \mathcal{P}$,

$$P^{\mathrm{MT}(\mathfrak{A})}(t_0, ..., t_{\eta(P)-1}) := 1 - \chi_{P^{\mathfrak{A}}}(t_0, ..., t_{\eta(P)-1}),$$

where χ is the standard *indicator* map, for every function symbol $f \in \mathcal{F}$, $f^{\mathrm{MT}(\mathfrak{A})} := f^{\mathfrak{A}}$, and for every constant symbol $c \in \mathcal{C}$, $c^{\mathrm{MT}(\mathfrak{A})} := c^{\mathfrak{A}}$.

Definition 18. Consider all classical wffs written using only the connectives $\neg$ and $\rightarrow$ and only the quantifier $\forall$ (by well-known results any classical wff is provably equivalent in first-order logic to a wff of this form). Define the *continuous translation* (CT) map from this set into the set of wffs of continuous logic recursively as follows.

- If $\varphi = \neg\psi$, then $\mathrm{CT}(\varphi) := \neg\mathrm{CT}(\psi)$.
- If $\varphi = \psi \rightarrow \theta$, then $\mathrm{CT}(\varphi) := \mathrm{CT}(\theta) \dot{-} \mathrm{CT}(\psi)$.
- If $\varphi = \forall x\ \psi$, then $\mathrm{CT}(\varphi) := \sup_x \mathrm{CT}(\psi)$.

When considered together, the metric transformation of a classical structure and continuous translation of classical wffs show the conservativity of continuous logic over classical logic.

Exercise 4. Let $\mathfrak{A}$ be a classical $L_{\mathfrak{A}}$-structure. Prove that

$$\mathfrak{A} \vDash_{\mathrm{FOL}} \varphi \iff \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\varphi).$$

Hint: Applying the Completeness Theorem (Theorem 1) may simplify the proof.

2.3 Completeness Results

We now recall many results related to completeness which were proven by Ben Yaacov and Pedersen in (5). We have altered some of the notation in order to make these results more applicable to our work, but the results proven remain the same. Important to this work is the introduction of the formal notion of dyadic numerals.

Definition 19. The *dyadic numerals* (Dyad) are all sentences of the form $\underline{\frac{\ell}{2^k}}$ for $\ell, k \in \mathbb{N}$. When $\underline{p} \in \text{Dyad}$, by p we mean the real number such that for every L-pre-structure $\mathfrak{M}$, $\underline{p}^{\mathfrak{M}} = p$.

Maximal consistency is defined similarly to the classical case, but with an extra condition concerning limiting behavior.

Definition 20. A set of wffs Γ is *maximally consistent* if for every pair of wffs φ and ψ, the following hold.

(i) If $\Gamma \vdash \varphi \dot{-} \underline{2^{-k}}$ for every $k \in \mathbb{N}$, then $\varphi \in \Gamma$.

(ii) Either $\varphi \dot{-} \psi \in \Gamma$ or $\psi \dot{-} \varphi \in \Gamma$.

Notably, without the limiting behavior condition, we would not gain the intuitive property that if Γ is maximally consistent, then for every $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is inconsistent. Ben Yaacov and Pedersen implement a continuous version of a Henkin construction to prove their completeness theorem. To accomplish this, Henkin witnesses must be added to the signature.

Definition 21. Given a signature L, the *Henkin extended signature of* L (L^+) is the smallest signature that extends L and that, for every combination of L^+-wff φ, variable symbol x, and $\underline{p}, \underline{q} \in \text{Dyad}$, contains a unique constant symbol $c_{\varphi,x,\underline{p},\underline{q}}$.

When Γ is a set of L^+-wffs, we say it is *Henkin complete* if for every L^+-wff φ, every variable symbol x, and every $\underline{p}, \underline{q} \in \text{Dyad}$,

$$\left(\sup_x \varphi \dot{-} \underline{q}\right) \wedge \left(\underline{p} \dot{-} \varphi[c_{\varphi,x,\underline{p},\underline{q}}/x]\right) \in \Gamma.$$

We now note a relevant lemma and theorem from Ben Yaacov and Pedersen. We also assume that we have a fixed signature L and Henkin extended signature L^+ for the remainder of this section.

Lemma 1 ((ii) of Lemma 8.5, Ben Yaacov and Pedersen, 2010)**.** *Let T be an L-theory. Then for every pair of L-wffs φ and ψ, either $T \cup \{\varphi \dot{-} \psi\}$ or $T \cup \{\psi \dot{-} \varphi\}$ is consistent.*

Theorem 1 (From Theorem 8.10 and Proposition 9.2, Ben Yaacov and Pedersen, 2010)**.** *Let T be an L-theory. Then there exists a maximally consistent, Henkin complete set of L^+-wffs which extends T.*

In what follows, the original Henkin model created will be a continuous L^+-pre-structure. To make the move to a genuine L^+-structure, the following theorem is needed.

Theorem 2 (Theorem 6.9, Ben Yaacov and Pedersen, 2010)**.** *Let $\mathfrak{M}'$ be a continuous L-pre-structure. Then there is an L-structure $\mathfrak{M}$ and an elementary L-morphism of $\mathfrak{M}'$ into $\mathfrak{M}$.*

We now summarize the construction of the Henkin model in (5). Completeness follows.

Definition 22. Let Γ be a maximally consistent, Henkin complete set of L^+-wffs. Define the *Henkin continuous L^+-pre-structure over Γ* ($\mathfrak{M}'_\Gamma$) as follows.

- $|\mathfrak{M}'_\Gamma|$ is the set of all terms of L^+.
- For every constant symbol c of L^+, $c^{\mathfrak{M}'_\Gamma} := c$.
- For every function symbol f of L^+, define $f^{\mathfrak{M}'_\Gamma}$ for each $t_0, ..., t_{\eta(f)-1} \in |\mathfrak{M}'_\Gamma|$ as

$$f^{\mathfrak{M}'_\Gamma}\big(t_0, ..., t_{\eta(f)-1}\big) := f(t_0, ..., t_{\eta(f)-1}).$$

- For every predicate symbol P of L^+, define $P^{\mathfrak{M}'_\Gamma}$ for each $t_0, ..., t_{\eta(P)-1} \in |\mathfrak{M}'_\Gamma|$ as

$$P^{\mathfrak{M}'_\Gamma}\big(t_0, ..., t_{\eta(P)-1}\big) := \sup\big\{p \in [0,1] : \underline{p} \in \text{Dyad and } \underline{p} \dot{-} P(t_0, ..., t_{\eta(P)-1}) \in \Gamma\big\}.$$

The *basic assignment* on $\mathfrak{M}'_\Gamma$ is defined as $\sigma(x) := x$ for every variable symbol x of L^+. By a slight abuse of notation, when $\mathfrak{M}'_\Gamma$ is a Henkin continuous L^+-pre-structure, by $\varphi^{\mathfrak{M}'_\Gamma}$ we mean $\varphi^{\mathfrak{M}'_\Gamma,\sigma}$, and by $\mathfrak{M}'_\Gamma \vDash \varphi$ we mean $\mathfrak{M}'_\Gamma, \sigma \vDash \varphi$, where σ is the basic assignment.

The *Henkin L^+-structure over Γ* ($\mathfrak{M}_\Gamma$) is the structure induced by the metric completion of $\big(|\mathfrak{M}'_\Gamma|, \underline{d}^{\mathfrak{M}'_\Gamma}\big)$ and the elementary morphism given in Theorem 2.

Theorem 3 (Theorem 9.4, Ben Yaacov and Pedersen, 2010)**.** *Let* Γ *be a maximally consistent, Henkin complete set of* L^+*-wffs. Then* $\mathfrak{M}_\Gamma \models \Gamma$.

Corollary 1 (Completeness of Continuous Logic, Theorem 9.5, Ben Yaacov and Pedersen, 2010)**.** *A set of* L*-wffs is consistent if and only if it is (completely) satisfiable.*

Ben Yaacov and Pedersen then introduce important maps from sets of L-wffs into $[0,1]$. These maps serve as upper bounds on relative provability and interpretation of sentences following from those sets of L-wffs.

Definition 23. Let Γ be a set of L-wffs. The *degree of truth with respect to* Γ $(\,\cdot\,^{\circ}_{\Gamma})$ is a map from wffs to $[0,1]$, defined as

$$\varphi^{\circ}_{\Gamma} := \sup\left\{\varphi^{\mathfrak{M},\sigma} : \mathfrak{M},\sigma \models \Gamma\right\}.$$

The *degree of provability with respect to* Γ $(\,\cdot\,^{\circledcirc}_{\Gamma})$ is a similar map, defined as

$$\varphi^{\circledcirc}_{\Gamma} := \inf\left\{p \in [0,1] : \underline{p} \in \text{Dyad and } \Gamma \vdash \varphi \dot{-} \underline{p}\right\}.$$

The Completeness Theorem implies that these maps are the same.

Corollary 2 (Corollary 9.8, Ben Yaacov and Pedersen, 2010)**.** *For any* L^+*-wff* φ *and set of* L*-wffs* Γ, $\varphi^{\circ}_{\Gamma} = \varphi^{\circledcirc}_{\Gamma}$.

A theory is then considered complete if and only if its related degree of truth map is precisely witnessed by interpretation in a structure.

Definition 24. A set of L-wffs Γ is *complete* if there is a structure $\mathfrak{M}$ and assignment σ such that for every L-wff φ,

$$\varphi^{\circ}_{T} = \varphi^{\mathfrak{M},\sigma}.$$

Γ is *incomplete* if it is not complete.

In contrast to the classical case, even if a theory is complete, its set of consequences may not be maximally consistent. This is due to the limiting behavior condition discussed in Definition 20. The Deduction Theorem for continuous logic encounters a similar issue.

Theorem 4 (Deduction Theorem, Theorem 8.1, Ben Yaacov and Pedersen, 2010)**.** *Let Γ be a set of L-wffs. Then for every L-wff ψ, $\Gamma \cup \{\psi\} \vdash \varphi$ if and only if $\Gamma \vdash \varphi \dot{-} m\psi$, for some $m \in \mathbb{N}$.*

Lastly, we note the Generalization Theorem, which will be useful in future work.

Lemma 2 (Generalization Theorem, Lemma 8.2, Ben Yaacov and Pedersen, 2010)**.** *Let Γ be a set of L^+-wffs and φ an L^+-wffs. If x does not appear freely in Γ and $\Gamma \vdash \varphi$, then $\Gamma \vdash \sup_x \varphi$.*

2.4 Infinitary Continuous Logic

Infinitary continuous logic was first developed by Ben Yaacov and Iovino (4) using the language $L^C_{\omega_1\omega}$ which extends a signature L.[4] To define the infinitary well-formed formulas, moduli of continuity are extended to all terms and finitary wffs, and allowed to take a variable symbol as a second argument.

Definition 25. For every variable symbol x, let $\Delta(\cdot, x; 0) := 0$. The following definitions hold for $k \in \mathbb{N} \setminus \{0\}$.

- When y is a constant or variable symbol, define $\Delta(y, x; k) := k$ when $y = x$ and $\Delta(y, x; k) := 0$ when $y \neq x$.

- For every term of the form $t = f(t_0, ..., t_{\eta(f)-1})$, define

$$\Delta(t, x; k) := \max\big\{\Delta\big(t_i, x; (\Delta(f; k)\big) \ : \ 0 \leq i < \eta(f)\big\}.$$

- For every wff of the form $\varphi = P(t_0, ..., t_{\eta(P)-1})$, define

$$\Delta(\varphi, x; k) := \max\big\{\Delta\big(t_i, x; \Delta(P; k)\big) \ : \ 0 \leq i < \eta(P)\big\}.$$

  Then also define

$$\Delta(\neg\varphi, x; k) := \Delta(\varphi, x; k),$$

$$\Delta\Big(\frac{1}{2}\varphi, x; k\Big) := \Delta(\varphi, x; k+1),$$

[4] The language $L^C_{\omega_1\omega}$ was previously denoted $L_{\omega_1\omega}$ but was distinguished by Eagle (14) in order to highlight its preservation of a modulus of continuity.

and

$$\Delta(\varphi \dot{-} \psi, x; k) := \max\left\{\Delta(\varphi, x; k), \Delta(\psi, x; k)\right\}.$$

- For every wff of the form $\varphi = \inf_y \psi$ or $\varphi = \sup_y \psi$, define $\Delta(\varphi, x; k) := 0$ when $y = x$ and $\Delta(\varphi, x; k) := \Delta(\psi, x; k)$ when $y \neq x$.

$\Delta(\varphi, x)$ is a *modulus of continuity of* φ *in* x.

Exercise 5. Let φ be a finitary wff with a single free variable x. Prove that for every L-structure $\mathfrak{M}$, if $d(a, b) < 2^{-\Delta(\varphi,x;k)}$, then $|\varphi^{\mathfrak{M}}(a) - \varphi^{\mathfrak{M}}(b)| \leq 2^{-k}$.

As noted in Footnote 4, one may consider the infinitary extension of a signature L without enforcing the retention of moduli of continuity. This extension was given by Eagle (14) as $L_{\omega_1\omega}$, and we consider its "formulas" to be infinitary pseudowffs.

Definition 26. The *infinitary pseudowffs* are defined recursively as follows.

- Every finitary wff is an infinitary pseudowff.
- If $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of infinitary pseudowffs which share free variables from a finite tuple $\vec{x}$, both

$$\bigwedge_{n\in\mathbb{N}} \varphi_n \qquad \text{and} \qquad \bigvee_{n\in\mathbb{N}} \varphi_n$$

  are infinitary pseudowffs.[5]
- If φ and ψ are infintary pseudowffs and x a variable symbol, each of $\neg\varphi$, $\frac{1}{2}\varphi$, $\varphi \dot{-} \psi$, $\sup_x \varphi$, and $\inf_x \varphi$ are infinitary pseudowffs.

Pseudosentences generalize similarly.

We now introduce the language of equicontinuity to distinguish genuine infinitary wffs from pseudowffs.

[5]The overlapping connectives $\bigvee$ and $\bigwedge$ are used to indicate the distinction of these as *infinite* disjunctions and conjunctions.

Definition 27. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of infinitary pseudowffs in a finite tuple of free variables $\vec{x}$. If for every $k \in \mathbb{N}$ and $x \in \vec{x}$, $\sup_{n\in\mathbb{N}} \Delta(\varphi_n, x; k)$ exists (as a natural number, *i.e.* is bounded), the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is called *uniformly equicontinuous.*

Definition 28. The *infinitary wffs* are defined recursively as follows.

- Every finitary wff is an infinitary wff.
- When $(\varphi_n)_{n\in\mathbb{N}}$ is a uniformly equicontinuous sequence of infinitary pseudowffs, both

$$\bigwedge_{n\in\mathbb{N}} \varphi_n \qquad \text{and} \qquad \bigvee_{n\in\mathbb{N}} \varphi_n$$

  are infinitary wffs, with

$$\Delta\Big(\bigvee_{n\in\mathbb{N}} \varphi_n, x; k\Big) = \Delta\Big(\bigwedge_{n\in\mathbb{N}} \varphi_n, x; k\Big) := \sup_{n\in\mathbb{N}} \Delta(\varphi_n, x; k).$$

- The wffs are also closed under the continuous connectives and quantifiers, with their respective moduli of continuity in each variable symbol given in Definition 25.

The infinitary *sentences* relativize similarly.

Exercise 6. Show that every infinitary pseudosentence is an infinitary sentence.

The interpretation of infinitary formulas is defined similarly to the classical case, but with $\bigvee$ corresponding to infinite conjunction and $\bigwedge$ to infinite disjunction.

Definition 29. The *value* of a wff in a structure is defined as in the finitary case, along with

$$\Big(\bigwedge_{n\in\mathbb{N}} \varphi_n\Big)^{\mathfrak{M}} := \inf_{n\in\mathbb{N}} \varphi_n^{\mathfrak{M}} \qquad \text{and} \qquad \Big(\bigvee_{n\in\mathbb{N}} \varphi_n\Big)^{\mathfrak{M}} := \sup_{n\in\mathbb{N}} \varphi_n^{\mathfrak{M}}.$$

Because of the continuous nature of L-structures, infinitary conjunction and disjunction do not perform as in the classical case. This is illustrated by the following exercise.

Exercise 7. Give an example of an L-structure $\mathfrak{M}$ and an infinitary wff $\bigwedge_{n\in\mathbb{N}} \varphi_n$ such that $\mathfrak{M} \not\models \varphi_n$ for every $n \in \mathbb{N}$, but $\mathfrak{M} \models \bigwedge_{n\in\mathbb{N}} \varphi_n$.

CHAPTER 3. PRELIMINARIES ON COMPUTABILITY THEORY

3.1 Effective Procedures

There is no concept more central to computability theory than that of an effective procedure.

Heuristic 4 (Church-Turing Thesis)**.** A procedure is *effective* (also *computable* or *recursive*) if it is an algorithm of finite length (in a finite language) such that for any input, if it produces an output, then it will do so within a finite number of steps.

Example 3. There is an effective procedure which determines if a given natural number is prime.

Given $n \in \mathbb{N}$, check if any of $\{\frac{n}{m} : m = 2, ..., n-1\}$ is a whole number. If none are, n is prime.

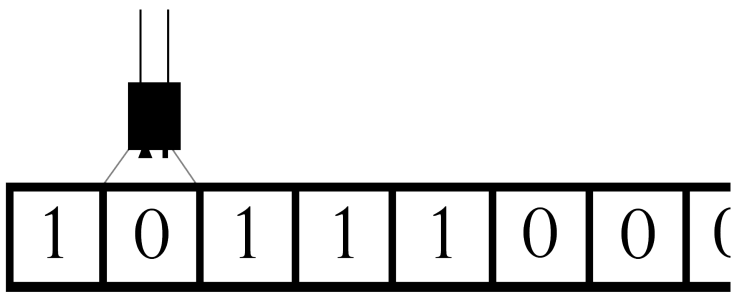
Heuristic 5. For any $m \in \mathbb{N}$, $\mathbb{N}^m$ is effectively the same as $\mathbb{N}$. Hence in the following heuristics and definitions, $A \subseteq \mathbb{N}$ means $A \subseteq \mathbb{N}^m$, for some $m \in \mathbb{N}$.

Example 4 (Gödel Numbering)**.** For every $(n_0, ..., n_{m-1}) \in \mathbb{N}^m$, define $\langle n_0, ..., n_{m-1}\rangle := 2^{n_0+1} \cdot 3^{n_1+1} \cdot \cdots \cdot p_{m-1}^{n_{m-1}+1}$, where p_i is the ith prime number. This is an effective encoding of every $\mathbb{N}^m$ into $\mathbb{N}$.

Turing machines are special sorts of effective procedures, often defined on programs given via tuples, and considered to work on long tapes of 0s and 1s. While such machines (like the figure below the heuristic) can be useful for visualization purposes, we prefer to think of Turing machines in the following sense.

Heuristic 6. A *Turing machine* is an effective procedure whose inputs and outputs are natural numbers. When Φ is a Turing machine, we write $\Phi(n)\downarrow$ if $n \in \mathrm{dom}(\Phi)$ and say Φ *halts* on n.

READ-WRITE TURING MACHINE



Heuristic 7. Any Turing machine may be coded by a natural number, called an *index* of that Turing machine. The Turing machine with index $e \in \mathbb{N}$ is denoted Φ_e and its domain denoted W_e.

Example 5. Suppose the instructions for a Turing machine are given via a finite tuple of natural numbers. Use the encoding from Example 4 to assign those instructions to a unique natural number. This natural number is an index of that machine.

We've presented effective procedures and Turing machines as algorithms, which are just a special sort of well-defined map. Effectivity is then extended to sets if their respective characteristic (indicator) functions are given by effective procedures.

Definition 30. Let $A \subseteq \mathbb{N}$. A is *computable* if there is an Turing machine which, on input $k \in \mathbb{N}$, outputs a 1 if $k \in A$ and a 0 if $k \notin A$.

A is *computably enumerable* (c.e.) if there is a Turing machine whose range is exactly A.

A is *co-computably enumerable* (co-c.e.) if A^c is computably enumerable.

Exercise 8. Prove that a set of natural numbers is computable if and only if it is c.e. and co-c.e.

Exercise 9. Prove that the *halting set*

$$K := \{k \in \mathbb{N} \ : \ k \in W_k\}$$

is c.e. but not computable.

Certain subsets of the natural numbers look effectively identical to other subsets of the naturals. Consider $\{2k : k \in \mathbb{N}\}$ and $\{4k : k \in \mathbb{N}\}$. These sets carry with them the same amount of effective information, since we can simply multiply the first by two to get the second, or divide the second by two to get the first. The ordering of many-one reducibility on subsets of the natural numbers formalizes this notion.

Definition 31. Let $A, B \subseteq \mathbb{N}$. A is *many-one reducible* to B ($A \leq_{\mathrm{m}} B$) if there is a Turing machine Φ with domain $\mathbb{N}$ such that

$$k \in A \iff \Phi(k) \in B.$$

A and B are *many-one equivalent* ($A \equiv_{\mathrm{m}} B$) if $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} A$.

Let $\mathcal{S} \subseteq \mathscr{P}(\mathbb{N})$. A set $A \in \mathcal{S}$ is $\mathcal{S}$*-complete* if for every $B \in \mathcal{S}$, $B \leq_{\mathrm{m}} A$.

In this sense, a single $\mathcal{S}$-complete set carries within it all the effective information of *every* set in $\mathcal{S}$.

As was briefly discussed in the introduction, computers (and humans) are able to query the external world to attempt to answer their internal questions. The act of querying an external object is formalized by relative computation.

Heuristic 8 (Relativized Church-Turing Thesis)**.** Let X be a countable set. A procedure is X*-effective* (also X*-computable* or X*-recursive*) if it is an algorithm of finite length (in a finite language) such that for any input, the procedure can check finitely many elements of X and if it produces an output, then it will do so within a finite number of steps. In such cases, X is called an *oracle*.

There is a similar notion for oracle Turing machines.

Heuristic 9. An *oracle Turing machine with access to* X is an X-effective procedure, where $X \subseteq \mathbb{N}$, whose inputs and outputs are natural numbers.

Any oracle Turing machine with access to X may be coded by a natural number, again called an *index* of that Turing machine. The Turing machine with access to X with index $e \in \mathbb{N}$ is denoted Φ_e^X and its domain denoted W_e^X.

Certain sets can be computed by an oracle Turing machine with access to another set. This is the notion of Turing reducibility.

Definition 32. Let $A, B \subseteq \mathbb{N}$. A is *Turing reducible* to B ($A \leq_{\mathrm{T}} B$) if there is an oracle Turing machine with access to B which, on input $k \in \mathbb{N}$, outputs a 1 if $k \in A$ and a 0 if $k \notin A$. In this case, A is also called *B-computable*, or *computable in B*.

A is *computably enumerable in B* (c.e. in B) if there is an oracle Turing machine with access to B whose range is exactly A.

A is *co-computably enumerable in B* (co-c.e. in B) if A^c is B-computably enumerable.

A and B are *Turing equivalent* ($A \equiv_{\mathrm{T}} B$) if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$.

It may seem at first glance that Turing reducibility and many-one reducibility are identical orderings, but a simple exercise reveals this not to be the case.

Exercise 10. Prove that $K^c \leq_{\mathrm{T}} K$ but $K^c \not\leq_{\mathrm{m}} K$.

This is because K does not contain the effective information stored in K^c, but rather the complement of that effective information. It follows that Turing reducibility is more general than many-one reducibility.

Exercise 11. Prove that for any $A, B \subseteq \mathbb{N}$, if $A \leq_{\mathrm{m}} B$, then $A \leq_{\mathrm{T}} B$.

We can then consider the set of all subsets which are Turing equivalent to a given subset of the naturals.

Definition 33. Let $A \subseteq \mathbb{N}$. The *Turing degree* of A is

$$\deg(A) := \{X \subseteq \mathbb{N} \ : \ X \equiv_{\mathrm{T}} A\}.$$

The set of all Turing degrees is denoted $\mathcal{D}$ and is ordered as

$$\deg(B) \leq \deg(A) \iff B \leq_{\mathrm{T}} A.$$

The Turing degrees have an incredibly rich structure, and many surprising properties.

Lastly, it is sometimes important to discuss effectivity on uncountable sets. The two most useful such sets are Cantor space and Baire space.

Definition 34. *Cantor space* ($2^{\mathbb{N}}$) is the set of all characteristic (indicator) functions on subsets of natural numbers. It can heuristically be considered as the powerset of the natural numbers (see Exercise 12).

Baire space ($\mathbb{N}^{\mathbb{N}}$) is the set of all functions from the natural numbers into themselves. It can heuristically be considered as the set of all sequences of natural numbers. Notably, every Turing machine is a point in Baire space.

Exercise 12. Construct a bijection between $2^{\mathbb{N}}$ and $\mathscr{P}(\mathbb{N})$.

Exercise 13. Construct an injection from $\mathbb{N}^{\mathbb{N}}$ into $\mathscr{P}(\mathbb{N})$.

We consider the Turing degrees of these sets through use of the following.

Definition 35. $\mathrm{Graph} : \mathbb{N}^{\mathbb{N}} \to \mathscr{P}(\mathbb{N})$ is the map from the solution to Exercise 13. Notably, $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$. When $X \in \mathbb{N}^{\mathbb{N}}$, by a slight abuse of notation, the *Turing degree* of X is

$$\deg(X) := \deg(\mathrm{Graph}(X)).$$

3.2 The Arithmetical Hierarchy

We saw in the previous section how many-one reducibility and the oracle computation can classify the computational complexity of subsets of natural numbers. Another way to classify this complexity is the arithmetical hierarchy. Moreover, there is a deep relationship between this hierarchy and oracle computation.

Definition 36. The Σ^0_n, Π^0_n, and Δ^0_n sets are defined recursively for every $n \in \mathbb{N} \setminus \{0\}$. A set $A \subseteq \mathbb{N}$ is

- Σ^0_1 if there is some computable binary relation $R \subseteq \mathbb{N}^2$ such that

$$k \in A \iff \exists s \in \mathbb{N}\ R(s,k);$$

- Π^0_1 if there is some computable binary relation $R \subseteq \mathbb{N}^2$ such that

$$k \in A \iff \forall s \in \mathbb{N}\ R(s,k);$$

- Σ^0_n if there is some Π^0_{n-1} binary relation $R \subseteq \mathbb{N}^2$ such that

$$k \in A \iff \exists s \in \mathbb{N}\ R(s,k);$$

- Π^0_n if there is some Σ^0_{n-1} binary relation $R \subseteq \mathbb{N}^2$ such that

$$k \in A \iff \forall s \in \mathbb{N}\ R(s,k);$$

- Δ^0_n if it is both Σ^0_n and Π^0_n.

A set $A \subseteq \mathbb{N}$ is *arithmetical* if it is Σ^0_n for some $n \in \mathbb{N}$.

In Exercise 9, we introduced one of the simplest non-computable sets. The following generalizes that construction.

Definition 37. The *jump* of a set $X \subseteq \mathbb{N}$ is defined as

$$X' := \{k \in \mathbb{N} \ :\ k \in W^X_k\}.$$

For any $n \in \mathbb{N} \setminus \{0\}$, the *nth-jump* is recursively defined as

$$X^{(n)} := \big(X^{(n-1)}\big)'.$$

The *halting set* is $\emptyset'$.

We can now state the deep relationship between the arithmetical hierarchy and oracle computation, given by Post in (27).

Theorem 5 (Post, 1941)**.** *For every $n \in \mathbb{N} \setminus \{0\}$, a set $A \subseteq \mathbb{N}$ is*

(i) Σ^0_n if and only if it is c.e. in $\emptyset^{(n-1)}$;

(ii) Π^0_n if and only if it is co-c.e. in $\emptyset^{(n-1)}$;

(iii) Δ^0_n if and only if it is computable in $\emptyset^{(n-1)}$.

Corollary 3 (Post, 1941)**.** *For every $n \in \mathbb{N}$, there are sets $A, B \subseteq \mathbb{N}$ such that $A \in \Sigma^0_n \setminus \Delta^0_n$ and $B \in \Pi^0_n \setminus \Delta^0_n$.*

Exercise 14. Prove Corollary 3.

There is also a relation to many-one reducibility.

Corollary 4 (Post, 1941)**.** *For every* $n \in \mathbb{N} \setminus \{0\}$*,* $\emptyset^{(n)}$ *is* Σ^0_n*-complete.*

From Corollary 3, we form the *arithmetical hierarchy*, which forms the following lattice.

$$
\begin{tikzcd}
\vdots & & \vdots \\
& \Delta^0_{n+1} \arrow[ul, dash] \arrow[ur, dash] & \\
\Sigma^0_n \arrow[ur, dash] & & \Pi^0_n \arrow[ul, dash] \\
& \Delta^0_n \arrow[ul, dash] \arrow[ur, dash] & \\
\vdots & & \vdots \\
& \Delta^0_3 \arrow[ul, dash] \arrow[ur, dash] & \\
\Sigma^0_2 \arrow[ur, dash] & & \Pi^0_2 \arrow[ul, dash] \\
& \Delta^0_2 \arrow[ul, dash] \arrow[ur, dash] & \\
\Sigma^0_1 \arrow[ur, dash] & & \Pi^0_1 \arrow[ul, dash] \\
& \Delta^0_1 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
$$

3.3 Classical Computable Structures

Recall that classical structures have associated languages they interpret. To execute effective constructions on these structures, we require that they have countable universes and effectively numbered languages.

Definition 38. A language is *effectively numbered* if there is an effective procedure which maps the natural numbers onto every symbol of the language.

Exercise 15. Prove that if L is effectively numbered then the atomic sentences of L can be effectively numbered.

We then discuss the computability of a given structure relative to computability on the subset of natural numbers which code the (quantifier-free) sentences it satisfies.

Definition 39. Let $\mathfrak{A}$ be a classical structure with an effectively numbered language $L_{\mathfrak{A}}$ such that $\mathcal{C}^{\mathfrak{A}} = |\mathfrak{A}|$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be an effective numbering of the atomic sentences of $L_{\mathfrak{A}}$. The *atomic diagram* of $\mathfrak{A}$ is

$$\left\{n \in \mathbb{N} \ : \ \mathfrak{A} \vDash_{\mathrm{FOL}} \varphi_n\right\}.$$

A classical structure $\mathfrak{A}$ is *computable* (c.e., co-c.e., Δ^0_n, *etc.*) if its atomic diagram is.

3.4 The Hyperarithmetical Hierarchy

The clever reader may have noticed that the arithmetical sets do not include all subsets of the natural numbers. Indeed there are only countably-many arithmetical sets. There are ways of extending the arithmetical hierarchy, however. To construct the *hyper*arithmetical hierarchy, we need to first discuss computable ordinals.

Definition 40. A *computable well-ordering* is a computable structure $\mathfrak{A} = (|\mathfrak{A}|, <_{\mathfrak{A}})$ where $<_{\mathfrak{A}}$ is a well-ordering on $|\mathfrak{A}|$. A *computable ordinal* is an ordinal which is order-isomorphic to some computable well-ordering.

Exercise 16. Prove that the set of all computable ordinals form an initial segment of the ordinals. Deduce that this set is the least non-computable ordinal.

Definition 41. The set of all computable ordinals is denoted ω_1^{CK} (where CK stands for "Church-Kleene").

Exercise 17. Prove that ω_1^{CK} is countable. Deduce that $\omega_1^{\mathrm{CK}} < \omega_1$.

As is standard for effective constructions, we need to associate each computable ordinal with some set of natural numbers.

Definition 42. *Kleene's* O, the standard encoding of computable ordinals, is defined by transfinite recursion as

$$\begin{aligned} \mathrm{O} \ := \{2^a \ : \ a \in \mathrm{O}\} \cup \\ \{3 \cdot 5^e \ : \ \Phi_e \text{ maps } \mathbb{N} \text{ into O and for every } n \in \mathbb{N},\ \Phi_e(n) <_{\mathrm{O}} \Phi_e(n+1)\}, \end{aligned}$$

where $a <_{\mathrm{O}} b$ means $|a|_{\mathrm{O}} < |b|_{\mathrm{O}}$, determined via the function $|\cdot|_{\mathrm{O}} : \mathrm{O} \to \omega_1^{\mathrm{CK}}$, which is also defined via transfinite recursion as

- $|2^a|_{\mathrm{O}} := |a|_{\mathrm{O}} + 1$,
- $|3 \cdot 5^e|_{\mathrm{O}} := \sup\big\{|\Phi_e(n)|_{\mathrm{O}} \ : \ n \in \mathbb{N}\big\}$.

For every ordinal $\alpha \in \omega_1^{\mathrm{CK}}$, define

$$\langle \alpha \rangle := |\alpha|_{\mathrm{O}}^{-1} = \{a \in \mathrm{O} \ : \ |a|_{\mathrm{O}} = \alpha\}.$$

We may now define jumps which go beyond $\emptyset^{(n)}$ for finite n.

Definition 43. For every $a \in \mathrm{O}$, the set $\mathcal{H}(a)$ is defined via transfinite recursion as

- $\mathcal{H}(1) := \emptyset$,
- $\mathcal{H}\big(2^a\big) := \mathcal{H}(a)'$,
- $\mathcal{H}\big(3 \cdot 5^e\big) := \big\{\langle a, b\rangle \ : \ a <_{\mathrm{O}} 3 \cdot 5^e \ \text{ and } \ b \in \mathcal{H}(a)\big\}$.

The hyperarithmetical hierarchy is then defined by a relativization of Post's Theorem.

Definition 44. For every $\alpha \in \omega_1^{\mathrm{CK}}$, a set $A \subseteq \mathbb{N}$ is

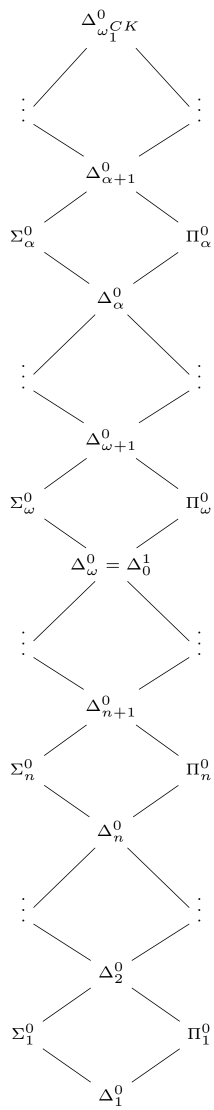
- Σ^0_α if it is c.e. in $\mathcal{H}(a)$ for some $a \in \langle \alpha \rangle$;
- Π^0_α if it is co-c.e. in $\mathcal{H}(a)$ for some $a \in \langle \alpha \rangle$;

- Δ^0_α if it is computable in $\mathcal{H}(a)$ for some $a \in \langle \alpha \rangle$.

A set A is called *hyperarithmetical* if it is Σ^0_α for some $\alpha \in \omega_1^{\mathrm{CK}}$.

An *index* of a Σ^0_α set A is a coded pair $\langle a, e \rangle$ such that $A = W_e^{\mathcal{H}(a)}$. An *index* of a Π^0_α set A is similar but such that $A = \left(W_e^{\mathcal{H}(a)}\right)^c$.

This classifies the hyperarithmetical sets into a lattice of sets which extends the arithmetical sets. There are still, however, only finitely many hyperarithmetical sets.



3.5 Computable Analysis

At the end of Section 3.1, we briefly mentioned how effectivity could be accomplished on uncountable sets. We further formalize that construction here, largely relegated to computation on the real numbers.

Definition 45. A real number r is *computable* if there is an effective procedure which, given $k \in \mathbb{N}$, outputs a rational $q \in \mathbb{Q}$ such that

$$|r - q| < 2^{-k}.$$

A *name* of a real number r is a point in Baire space $\nu(r) \in \mathbb{N}^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $\nu(r; k)$ is a code of a rational q such that

$$|r - q| < 2^{-k}.$$

Exercise 18. Prove that a real number is computable if and only if it is named by a Turing machine.

We can now also consider computable maps on the real numbers.

Definition 46. A map $f : \mathbb{R} \to \mathbb{R}$ is *computable* if there is an effective procedure which, given access to $\nu(r) \in \mathbb{N}^{\mathbb{N}}$ and input $k \in \mathbb{N}$, outputs a rational q such that

$$|f(r) - q| < 2^{-k}.$$

Definition 47. Let A be a countable set. A map $f : A \to \mathbb{R}$ is *computable* if there is an effective procedure which, given $a \in A$ and $k \in \mathbb{N}$, outputs a rational q such that

$$|f(a) - q| < 2^{-k}.$$

Remark 2. All of the above notions of computability relativize for any countable set X.

For more information on computable analysis, see Weihrauch (34).

3.6 Computable Presentations

Since metric structures often have uncountable domains, we need a way of discussing effectivity on such structures. We generally follow the system introduced by Melnikov (23) and recently utilized by Franklin and McNicholl (16).

Similar to the notion of an effectively numbered language for a classical structure, an effectively numbered signature is necessary for effective constructions on structures.

Definition 48. A signature L is *effectively numbered* if there is an effective mapping of the natural numbers onto $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ and, moreover, an effective procedure which, given the code of a predicate or function symbol, outputs that symbol's arity and an index of a Turing machine which serves as a modulus of continuity for that symbol.

Definition 49. Given an L-structure $\mathfrak{M}$ and $A \subseteq |\mathfrak{M}|$, the *algebra generated by* A is the smallest subset of $|\mathfrak{M}|$ containing A that is closed under every function of $\mathfrak{M}$.

A pair $(\mathfrak{M}, g)$ is called a *presentation* of $\mathfrak{M}$ if $g : \mathbb{N} \to |\mathfrak{M}|$ is a map such that the algebra generated by $\operatorname{ran}(g)$ (the range of g) is dense. A presentation of $\mathfrak{M}$ is denoted $\mathfrak{M}^\sharp$. Every point in $\operatorname{ran}(g)$ is called a *distinguished point* of the presentation, and each point in the algebra generated by the distinguished points is called a *rational point* of the presentation ($\mathbb{Q}(\mathfrak{M}^\sharp)$).

Notably, $\operatorname{ran}(g)$ need not be dense, but the *algebra* it generates does.

Definition 50. A presentation $\mathfrak{M}^\sharp$ is *computable* if the predicates of $\mathfrak{M}$ are uniformly computable on the rational points of $\mathfrak{M}^\sharp$.[1]

An *index* of a computable presentation $\mathfrak{M}^\sharp$ is an index of a Turing machine which, given a code of $P \in \mathcal{P}$, codes of $a_0, ..., a_{\eta(P)-1} \in \mathbb{Q}(\mathfrak{M}^\sharp)$, and $k \in \mathbb{N}$, outputs a code of a rational q such that

$$\left| P^{\mathfrak{M}}(a_0, ..., a_{\eta(P)-1}) - q \right| < 2^{-k}.$$

Notions of computability also relativize to any $X \subseteq \mathbb{N}$.

[1] Since the metric is a binary predicate on $\mathfrak{M}$, this entails that the distance between any two rational points is uniformly computable.

Example 6. Consider a finite-dimensional Banach space of $\mathbb{R}$ with basis $\{e_0, ..., e_{N-1}\}$. Let $\mathfrak{X}$ be the metric structure consisting of the unit ball of this Banach space, the metric induced by the norm, as functions all binary maps of the form

$$f_{p,q}(x, y) = px + qy$$

where $p, q \in \mathbb{Q}$ and $|p| + |q| \leq 1$, and 0 as the only distinguished point. Then define $g : \mathbb{N} \to |\mathfrak{X}|$ as $g(n) = e_n$ for every $n \leq N - 1$ and $g(n) = e_{N-1}$ for every $n \geq N$. Clearly the algebra generated by $\operatorname{ran}(g)$ is dense in $|\mathfrak{X}|$. Moreover, with a bit of careful calculation, one may see that this presentation is computable.

Definition 51. Let $\mathfrak{M}^\sharp = (\mathfrak{M}, g)$ be a presentation. A *rational ball* of $\mathfrak{M}^\sharp$ is a ball $B(a, q) \subseteq |\mathfrak{M}|$ such that $a \in \mathbb{Q}(\mathfrak{M}^\sharp)$ and $q \in \mathbb{Q}$.

A tuple of pairs $\big((a_0, q_0), ..., (a_N, q_N)\big) \in \big(\mathbb{Q}(\mathfrak{M}^\sharp) \times \mathbb{Q}\big)^{<\mathbb{N}}$ is a *rational open cover* of $\mathfrak{M}^\sharp$ if

$$|\mathfrak{M}| \subseteq \bigcup_{n=0}^{N} B(a_n, q_n).$$

$\mathfrak{M}^\sharp$ is *computably compact* if the set of all rational open covers of $\mathfrak{M}$ is c.e.

3.7 Previous Effective Completeness Results

To remain as charitable as possible, we want to briefly review two previous results concerning the effective completeness of continuous logic. The first was given in (8) and relates theories to probabilistically decidable structures.

Probabilistic decidability is a form of computation that allows a probability of halting within $[0, 1]$ to be associated with each natural number. First a probability measure is defined on $\mathscr{P}(\mathscr{P}(\mathbb{N}))$.

Definition 52. Let $E : \mathscr{P}(\mathbb{N}) \to [0, 1]$ be such that

$$E(X) = \sum_{n \in \mathbb{N}} \chi_X(n)\ 2^{-(n+1)}.$$

Notably, E is a bijection. The sequence of partial sums which in the limit yield $E(X)$ is sometimes called a *Specker sequence.*

The *Lebesgue probability measure* μ on $\mathscr{P}(\mathscr{P}(\mathbb{N}))$ is defined as

$$\mu(\mathscr{C}) := \mu_{\mathbb{R}}\big(\{E(X) \in [0,1] : X \in \mathscr{C}\}\big),$$

where $\mu_{\mathbb{R}}$ is the standard Lebesgue measure on $\mathbb{R}$.

Definition 53. A *probabilistic Turing machine* is an oracle Turing machine with range $\{0,1\}$ whose oracle is left free as an additional input. A probabilistic Turing machine Φ *accepts* n *with probability* p if $\mu\big(\{X \in \mathscr{P}(\mathbb{N}) : \Phi^X(n) \downarrow = 1\}\big) = p$ and *rejects* n *with probability* p if $\mu\big(\{X \in \mathscr{P}(\mathbb{N}) : \Phi^X(n) \downarrow = 0\}\big) = p$.

Let L be an effectively numbered signature and $(\varphi_n)_{n\in\mathbb{N}}$ an effective enumeration of the L-sentences.

Definition 54. A continuous L-pre-structure $\mathfrak{M}$ is *probabilistically decidable* if there is a probabilistic Turing machine Φ such that $\varphi_n^{\mathfrak{M}} = p$ if and only if Φ rejects n with probability p.

Definition 55. An L-theory T is *decidable* if $\cdot_T^\circ$ is a computable map from $(\varphi_n)_{n\in\mathbb{N}}$ into $[0,1]$.

Calvert then proved the following lemma and theorem.

Lemma 3 (Lemma 4.6, Calvert, 2011)**.** *There is an effective procedure which extends L to its Henkin extended signature L^+.*

Theorem 6 (Theorem 4.5, Calvert, 2011)**.** *Let T be a complete, decidable L-theory. Then there is a probabilistically decidable weak L^+-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash T$.*

The second version of effective completeness was given by Didehvar, Ghasemloo, and Pourmahdian in (13). It was implicitly proven with respect to computable presentations, but restricted the class of computably axiomatizable theories to be linear-complete. (Notably, in (5) it was shown that any computably axiomatizable theory is decidable.)

Definition 56. An L-theory T is *linear-complete* if for every pair of L-wffs φ and ψ, either $T \vdash \varphi \dot{-} \psi$ or $T \vdash \psi \dot{-} \varphi$.

Theorem 7 (Theorem 3.5, Didehvar, Ghasemloo, Pourmahdian, 2010)**.** *Let T be a linear-complete, computably axiomatizable L^+-theory. Then there is a computably presentable*[2] *L^+-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash T$.*

[2] *In (13), they prove the result for a* decidable *structure, which is a structure whose open theory under a given presentation is Σ^0_1. Notably, any such structure is computably presentable.*

CHAPTER 4. GENERALIZED EFFECTIVE COMPLETENESS

4.1 Motivation

For this entire chapter, we assume that we are working in an effectively numbered signature L, with $(\varphi_n)_{n\in\mathbb{N}}$ an effective enumeration of the L-wffs. The main result of this chapter has four major upshots in comparison to the effective completeness theorems found in (8) and (13).

1. The given (name of a) continuous theory need be neither *complete* nor *linear-complete*. Through the process described in Lemma 4, any continuous theory may be effectively extended to a complete theory (though, as a *caveat*, this extension is not unique).

2. The given (name of a) continuous theory need not be *decidable*. The generalized effective completeness theorem given applies to general names of continuous theories, and produces presentations of the same degree.

3. The effective completeness theorem given relates continuous theories to *presentations* of metric structures, which have become the norm for the study of effectivity on metric structures, rather than *probabilistic decidability*.

4. The effective completeness theorem given relates continuous theories to presentations of genuine metric *structures*, rather than *weak structures*, so the assumption of a countable universe may be dropped.

The result is a generalized effective completeness theorem which performs intuitively similarly to effective completeness in the classical setting, only with computable presentations rather than computable copies of structures.

4.2 Model-Theoretic Results

There are four important model-theoretic propositions which extend the results of (5) and are useful for the construction of the effective completeness theorem.

Proposition 1. *Let Γ be a set of L-wffs and B a finite set of L-wffs. Then for every L-wff φ,*

$$\varphi^{\circ}_{\Gamma\cup B} \leq \Big(\varphi \dot{-} \bigvee_{\theta\in B} \theta\Big)^{\circ}_{\Gamma}.$$

Proof. Fix an L-wff φ. If $\varphi^{\circ}_{\Gamma\cup B} = 0$, the result follows trivially. Thus suppose $\varphi^{\circ}_{\Gamma\cup B} > 0$. Notably, this implies that $\Gamma \cup B$ is consistent. Fix $\underline{p} \in$ Dyad such that $p < \varphi^{\circ}_{\Gamma\cup B}$. Then there is some L-structure $\mathfrak{M}$ and assignment σ such that $\mathfrak{M}, \sigma \vDash \Gamma \cup B$ while $\varphi^{\mathfrak{M},\sigma} > p$. But, clearly, since $\mathfrak{M}, \sigma \vDash B$, $\left(\bigvee_{\theta\in B} \theta\right)^{\mathfrak{M},\sigma} = 0$. Hence, $\left(\varphi \dot{-} \bigvee_{\theta\in B} \theta\right)^{\mathfrak{M},\sigma} > p$. Then since $\mathfrak{M}, \sigma \vDash \Gamma$, this implies that $\left(\varphi \dot{-} \bigvee_{\theta\in B} \theta\right)^{\circ}_{\Gamma} > p$. Since this is true for every $p < \varphi^{\circ}_{\Gamma\cup B}$, we have that $\varphi^{\circ}_{\Gamma\cup B} \leq \left(\varphi \dot{-} \bigvee_{\theta\in B} \theta\right)^{\circ}_{\Gamma}$.

□

Proposition 2. *Let Γ be a set of L-wffs and B a finite set of L-wffs such that $\Gamma \cup B$ is consistent. Then there are infinitely many L-wffs φ such that*

$$\Big(\varphi \dot{-} \bigvee_{\theta\in B} \theta\Big)^{\circ}_{\Gamma} = 1.$$

Proof. Recall that $\Gamma \cup B$ is consistent only if there is some L-wff φ such that $\Gamma \cup B \nvdash \varphi$. By Corollary 2, $\varphi^{\circ}_{\Gamma\cup B} > \frac{1}{M}$, for some $M \in \mathbb{N}$. It follows that for every $m \geq M$, $(m\varphi)^{\circ}_{\Gamma\cup B} = 1$. Hence, by Proposition 1, $\left(m\varphi \dot{-} \bigvee_{\theta\in B} \theta\right)^{\circ}_{\Gamma} = 1$, for every $m \geq M$. □

Proposition 3. *Let L be a signature, T an L-theory, and φ an L-wff with free variables $\vec{x}$. Then*

$$\varphi^{\circ}_{T} = \big(\sup_{\vec{x}} \varphi\big)^{\circ}_{T}.$$

Proof. Fix a signature L, an L-theory T, and an L-wff φ with free variables $\vec{x}$. Notice that since T contains only L-sentences, none of $\vec{x}$ appear freely in T. It follows via Corollary 2 and the

Generalization Theorem that

$$\begin{aligned}\varphi^\circ_T &= \inf\left\{p : \underline{p} \in \text{Dyad and } T \vdash \varphi \dot{-} \underline{p}\right\}\\ &= \inf\left\{p : \underline{p} \in \text{Dyad and } T \vdash \sup_{\vec{x}} \varphi \dot{-} \underline{p}\right\}\\ &= \big(\sup_{\vec{x}} \varphi\big)^\circ_T.\end{aligned}$$

□

Proposition 4. *Let T be an L-theory. Then for every L^+-wff θ,*

$$\big(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\big)^\circ_T = \theta^\circ_T,$$

where $\vec{c}$ is the tuple of all constants from L^+, but not in L, appearing in θ.

Proof. Fix an L-theory T and an L^+-wff θ. Recall that no variable from $\vec{x}$ appears freely in T, nor does any constant in $\vec{c}$ appear in T, since it is an L-theory. Hence for every L-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash T$, and every tuple $\vec{a} \in |\mathfrak{M}|$, there is an L^+-structure $\mathfrak{M}^+_{\vec{a}}$ such that $\vec{c}^{\mathfrak{M}^+_{\vec{a}}} = \vec{a}$ and $\mathfrak{M}^+_{\vec{a}} \restriction_L = \mathfrak{M}$. Then

$$\begin{aligned}\big(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\big)^\circ_T &= \sup\left\{\big(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\big)^{\mathfrak{M}} : \mathfrak{M} \vDash T\right\}\\ &= \sup\left\{\sup_{\{\sigma(\vec{x}\mapsto\vec{a}) : \vec{a}\in|\mathfrak{M}|\}} \big(\theta[\vec{x}/\vec{c}]\big)^{\mathfrak{M},\sigma(\vec{x}\mapsto\vec{a})} : \mathfrak{M} \vDash T\right\}\\ &= \sup\left\{\big(\theta[\vec{x}/\vec{c}]\big)^{\mathfrak{M},\sigma(\vec{x}\mapsto\vec{a})} : \mathfrak{M}, \sigma(\vec{x} \mapsto \vec{a}) \vDash T,\ \vec{a} \in |\mathfrak{M}|\right\}\\ &\leq \sup\left\{\big(\theta[\vec{x}/\vec{c}]\big)^{\mathfrak{M}^+_{\vec{a}},\sigma(\vec{x}\mapsto\vec{a})} : \mathfrak{M}^+_{\vec{a}}, \sigma(\vec{x} \mapsto \vec{a}) \vDash T,\ \vec{a} \in |\mathfrak{M}|\right\}\\ &= \sup\left\{\theta^{\mathfrak{M}^+_{\vec{a}},\sigma} : \mathfrak{M}^+_{\vec{a}}, \sigma \vDash T\right\}\\ &\leq \sup\left\{\theta^{\mathfrak{M}^+,\sigma} : \mathfrak{M}^+, \sigma \vDash T\right\}\\ &= \theta^\circ_T.\end{aligned}$$

Now notice that for any L^+-structure $\mathfrak{M}^+$ and assignment σ, there is an assignment $\sigma(\vec{x} \mapsto \vec{c}^{\mathfrak{M}^+}_{\vec{a}})$. Then the L-structure $\mathfrak{M}^+ \restriction_L$ is such that $\theta^{\mathfrak{M}^+,\sigma} = \big(\theta[\vec{x}/\vec{c}]\big)^{\mathfrak{M}^+\restriction_L,\sigma(\vec{x}\mapsto\vec{c}^{\mathfrak{M}^+}_{\vec{a}})}$. It follows that

$$\begin{aligned}
\theta^\circ_T &= \sup\left\{\theta^{\mathfrak{M}^+,\sigma} : \mathfrak{M}^+, \sigma \vDash T\right\} \\
&= \sup\left\{\big(\theta[\vec{x}/\vec{c}]\big)^{\mathfrak{M}^+\restriction_L,\sigma(\vec{x}\mapsto\vec{c}^{\mathfrak{M}^+}_{\vec{a}})} : \mathfrak{M}^+ \restriction_L, \sigma(\vec{x} \mapsto \vec{c}^{\mathfrak{M}^+}_{\vec{a}}) \vDash T\right\} \\
&\leq \sup\left\{\big(\theta[\vec{x}/\vec{c}]\big)^{\mathfrak{M},\sigma} : \mathfrak{M}, \sigma \vDash T\right\} \\
&= \big(\theta[\vec{x}/\vec{c}]\big)^\circ_T.
\end{aligned}$$

But by Proposition 3, $\big(\theta[\vec{x}/\vec{c}]\big)^\circ_T = \big(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\big)^\circ_T$. Therefore, $\theta^\circ_T \leq \big(\sup_{\vec{x}} \theta[\vec{x}/\vec{c}]\big)^\circ_T$. The claim follows. □

4.3 Effective Extensions of Theories

Recall that any L-theory T has an associated degree of truth map $\cdot^\circ_T$. As in Definition 55, then, to analyze the effectiveness of a theory, we will actually consider the effectiveness of the related degree of truth map. Since there are uncountably-many such maps, we introduce a naming system.

Definition 57. Given an L-theory T, we say that $X \in \mathbb{N}^\mathbb{N}$ is a *name* of T if the following hold.

- For every $n, k \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $\langle n, k, m\rangle \in \operatorname{ran}(X)$.
- For every $n, k, m \in \mathbb{N}$, if $\langle n, k, m\rangle \in \operatorname{ran}(X)$, then $q_m \in \big[\ (\varphi_n)^\circ_T - 2^{-k}\ ,\ (\varphi_n)^\circ_T + 2^{-k}\ \big]$.

Proposition 5. *An L-theory is decidable if and only if it has a computable name.*

Proof. For the forward direction, suppose $\cdot^\circ_T$ is computable. Fix a witness of this computability. For every code of a pair, define $X(\langle n, k\rangle) := \langle n, k, m\rangle$ where the witness outputs q_m, given φ_n and precision parameter k. On every natural number which doesn't code a pair, let X be 0. For the reverse direction, suppose X is a computable name. Given φ_n and a precision parameter k, begin computing $\operatorname{ran}(X)$ until a code of a triple of the form $\langle n, k, m\rangle$ is output. It follows that
$q_m \in \big[\ (\varphi_n)^\circ_T - 2^{-k}\ ,\ (\varphi_n)^\circ_T + 2^{-k}\ \big]$. □

Let L^+ be the Henkin extended signature effectively given by Lemma 3, and let $(\theta_n)_{n\in\mathbb{N}}$ be an effective enumeration of the L^+-wffs. The next lemma we present is similar to Lemma 4.7 in (8). However, in our case, the construction is with respect to any *name* of an L-theory, $X \in \mathbb{N}^{\mathbb{N}}$. Moreover, careful consideration is taken with respect to when two L^+-sentences are provably equivalent with respect to a given L^+-theory.

Lemma 4. *There is an effective procedure which given X, a name of an L-theory T, outputs $\Phi(X) \subseteq \mathbb{N}$ such that $T \cup \{\theta_n : n \in \Phi(X)\}$ is consistent, and for every pair of L^+-wffs φ and ψ, either φ and ψ are provably equivalent with respect to $T \cup \{\theta_n : n \in \Phi(X)\}$, or exactly one of $\varphi \dot{-} \psi$ or $\psi \dot{-} \varphi$ is in $\{\theta_n : n \in \Phi(X)\}$.*

Proof. We proceed via partial effective recursion. First define $\Phi_0(X) := \emptyset$, for every $X \in \mathbb{N}^{\mathbb{N}}$. As the recursive assumption, we suppose that at stage s, if X is a name of an L-theory T, then $\Phi_s(X)$ is defined, finite, and $T \cup \{\theta_n : n \in \Phi_s(X)\}$ is consistent. At stage $s+1$, the following procedure attempts to construct $\Phi_{s+1}(X)$.

For every pair of L^+-wffs φ and ψ, define the real number

$$r_{\varphi,\psi,f,s+1} := \Big(\sup_{\vec{x},\vec{y}}\ \sup_{\vec{z}} \Big(\Big((\psi \dot{-} \varphi) \dot{-} \Big(\bigvee_{n\in\Phi_s(X)} \theta_n\Big)\Big)[\vec{z}/\vec{c}]\Big)\Big)^{\circ}_{T},$$

where $\vec{x}$ and $\vec{y}$ are the free variables appearing in $\psi \dot{-} \varphi$ and $\bigvee_{n\in\Phi_s(X)} \theta_n$, respectively, $\vec{c}$ is the (possibly empty) tuple of constants from L^+ and not in L appearing in $(\psi \dot{-} \varphi) \dot{-} \big(\bigvee_{n\in\Phi_s(X)} \theta_n\big)$, and $\vec{z}$ is a $|\vec{c}|$-tuple of variable symbols distinct from $\vec{x}$ and $\vec{y}$. Notably, these real numbers are computable in f, uniformly in φ, ψ, and s. To see this, notice that each recursively defined $\Phi_s(X)$ is finite, each free variable becomes bound by the quantifier, and every constant from L^+ not in L is replaced by a variable and bound. Hence each formula checked above is actually an L-sentence, so X can compute a rational approximation of $r_{\varphi,\psi,X,s+1}$ within $2^{-(s+2)}$. Call such a rational $q_{\varphi,\psi,X,s+1}$. Then search the pairs of L^+-wffs for the first pair φ and ψ such that $\varphi \dot{-} \psi \notin \{\theta_n : n \in \Phi_s(X)\}$ and $q_{\varphi,\psi,X,s+1} \geq 2^{-(s+1)}$. By Proposition 2, there are infinitely many L^+-wffs ψ such that $r_{\underline{0},\psi,X,s+1} = 1$, and hence such that $q_{\underline{0},\psi,X,s+1} \geq 2^{-(s+1)}$. Thus, when X is a name of an L-theory, the procedure will halt. When such a pair φ and ψ is found, search the effective enumeration of the L^+-wffs for the index m of $\varphi \dot{-} \psi$ and define $\Phi_{s+1}(X) := \Phi_s(X) \cup \{m\}$. Clearly, if $\Phi_{s+1}(X)$ is defined, it is also finite, by construction. We now claim that this Φ witnesses the lemma. Fix a name of an L-theory $X \in \mathbb{N}^{\mathbb{N}}$.

To see that $T \cup \{\theta_n : n \in \Phi(X)\}$ is consistent, we show that each $T \cup \{\theta_n : n \in \Phi_{s+1}(X)\}$ is consistent, for every $s \in \mathbb{N}$. We proceed inductively.

Suppose $T \cup \{\theta_n : n \in \Phi_s(X)\}$ is consistent and fix $m \in \Phi_{s+1}(X) \setminus \Phi_s(X)$. By construction this m is the index for $\varphi \dot{-} \psi$ where $q_{\varphi,\psi,X,s+1} \geq 2^{-(s+1)}$. It follows by the definition of $q_{\varphi,\psi,X,s+1}$ and Propositions 3 and 4, Corollary 2, logical equivalence, and the Deduction Theorem, that we have each of the following.

$$\begin{aligned}
&\Big((\psi \dot{-} \varphi) \dot{-} \Big(\bigvee_{n\in\Phi_s(X)} \theta_n\Big)\Big)^\circ_T \geq 2^{-(s+2)} \\
&\qquad \Longrightarrow \quad \Big(\underline{2^{-(s+2)}} \dot{-} \Big((\psi \dot{-} \varphi) \dot{-} \Big(\bigvee_{n\in\Phi_s(X)} \theta_n\Big)\Big)\Big)^\circ_T = 0 \\
&\qquad \Longrightarrow \quad T \vdash \Big(\underline{2^{-(s+2)}} \dot{-} \Big((\psi \dot{-} \varphi) \dot{-} \Big(\bigvee_{n\in\Phi_s(X)} \theta_n\Big)\Big)\Big) \dot{-} \underline{2^{-(s+3)}} \\
&\qquad \Longrightarrow \quad T \cup \{\theta_n : n \in \Phi_s(X)\} \cup \{\psi \dot{-} \varphi\} \vdash \underline{2^{-(s+3)}}.
\end{aligned}$$

Therefore, $T \cup \{\theta_n : n \in \Phi_s(X)\} \cup \{\psi \dot{-} \varphi\}$ is inconsistent. It follows by Lemma 1 that $\varphi \dot{-} \psi$ is consistent with $T \cup \{\theta_n : n \in \Phi_s(X)\}$.

We now need only show that for every pair of L^+-wffs φ and ψ, either φ and ψ are provably equivalent with respect to $T \cup \{\theta_n : n \in \Phi(X)\}$, or exactly one of $\varphi \dot{-} \psi$ or $\psi \dot{-} \varphi$ is in $\{\theta_n : n \in \Phi(X)\}$.

Note that a pair of L^+-wffs φ and ψ is provably equivalent with respect to $T \cup \{\theta_n : n \in \Phi(X)\}$ if and only if for every $s \in \mathbb{N}$, there is some $S \in \mathbb{N}$ such that

$$T \cup \{\theta_n : n \in \Phi_S(X)\} \vdash (\varphi \dot{-} \psi) \dot{-} \underline{2^{-(s+2)}} \qquad \text{and} \qquad T \cup \{\theta_n : n \in \Phi_S(X)\} \vdash (\psi \dot{-} \varphi) \dot{-} \underline{2^{-(s+2)}}.$$

Now fix a pair of L^+-wffs φ and ψ that are not provably equivalent with respect to $T \cup \{\theta_n : n \in \Phi(X)\}$. Then there must be some $s \in \mathbb{N}$ such that for every $S \in \mathbb{N}$, either

$$T \cup \{\theta_n : n \in \Phi_S(X)\} \nvdash (\varphi \dot{-} \psi) \dot{-} \underline{2^{-(s+2)}} \qquad \text{or} \qquad T \cup \{\theta_n : n \in \Phi_S(X)\} \nvdash (\psi \dot{-} \varphi) \dot{-} \underline{2^{-(s+2)}}.$$

At least one of these two cases must hold for infinitely many $S \in \mathbb{N}$. Without loss of generality, since the cases are symmetric, suppose it is the latter. It follows by Corollary 2 that for every $S \in \mathbb{N}$, $(\psi \dot{-} \varphi)^{\circ}_{T \cup \{\theta_n : n \in \Phi_S(X)\}} \geq 2^{-(s+2)}$. Hence by Proposition 1, for every $S \in \mathbb{N}$,

$$\Big((\psi \dot{-} \varphi) \dot{-} \Big(\bigvee_{n \in \Phi_S(X)} \theta_n\Big)\Big)^{\circ}_{T} \geq 2^{-(s+2)}.$$

Then by Propositions 3 and 4, for every $S \in \mathbb{N}$, $r_{\varphi,\psi,X,S+1} \geq 2^{-(s+2)}$. Thus for some $S \geq s+2$, φ and ψ will have to be the first pair such that $\varphi \dot{-} \psi \notin \{\theta_n : n \in \Phi_S(X)\}$ and $q_{\varphi,\psi,X,S+1} \geq 2^{-(s+3)} \geq 2^{-(S+1)}$. It follows that the procedure will place the code for $\varphi \dot{-} \psi$ into $\Phi_{S+1}(X)$, so $\varphi \dot{-} \psi \in \{\theta_n : n \in \Phi(X)\}$. $\square$

It should be noted that, if T is not complete, a name of T does not specify a *unique* consistent extension of T. The above procedure constructs a complete extension, which itself has a unique maximally consistent extension, but the procedure is dependent on the enumeration of the L^+-wffs. When that enumeration changes, if T is not a complete theory, the above extension of T may also change.

4.4 Generalized Effective Completeness

We now come to our main result.

Theorem 8 (Generalized Effective Completeness)**.** *There is an effective procedure which, given a name $X \in \mathbb{N}^{\mathbb{N}}$ of an L-theory T, produces a presentation of an L^+-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash T$.*

Proof. Compute L^+ as in Lemma 3. Given a name of an L-theory $X \in \mathbb{N}^{\mathbb{N}}$, let $\Phi(X)$ be as in Lemma 4. Then, by Theorem 1, extend $T \cup \{\theta_n : n \in \Phi(X)\}$ to a maximally consistent, Henkin complete L^+-theory Γ. By Proposition 3, $\mathfrak{M}_\Gamma \vDash T$.

Since L^+ is effectively numbered, the set of constants of L^+ is also effectively numbered, which we may effectively join to an effective numbering of the variable symbols. Let g' be such an effective numbering. Then, for every $n \in \mathbb{N}$, define $g(n) := [g'(n)]$, the equivalence class of $g'(n)$ in $|\mathfrak{M}_\Gamma|$. By construction, the algebra generated by $\mathrm{ran}(g)$ in $\mathfrak{M}_\Gamma$ is the set of all equivalence classes of terms of L^+, that is, equivalence classes of the elements of $|\mathfrak{M}'_\Gamma|$. It follows that this algebra is dense in $|\mathfrak{M}_\Gamma|$, since by construction $|\mathfrak{M}_\Gamma|$ is the metric completion of $|\mathfrak{M}'_\Gamma|$. Thus $\big(\mathfrak{M}_\Gamma, g\big)$ is a presentation of $\mathfrak{M}_\Gamma$. We further claim that $\big(\mathfrak{M}_\Gamma, g\big)$ is an X-computable presentation.

Fix a code of an N-ary predicate symbol P, codes of rational points $[t_0], ..., [t_{N-1}]$, and a precision parameter $k \in \mathbb{N}$. From these, use g' to decode L^+-terms $t_0, ..., t_{N-1}$ corresponding to $[t_0], ..., [t_{N-1}]$. Then execute the following.

Compute the finite set $D = \big\{\underline{p} \in \mathrm{Dyad} : \text{ the denominator of } p \text{ is less than } 2^{k+2}\big\}$. By the construction of $\Phi(X)$, with access to an oracle that computes X, we may compute the least $M \geq k+2$ such that for all but one $\underline{p} \in D$, exactly one of $\underline{p} \dot{-} P(t_0, ..., t_{N-1})$ or $P(t_0, ..., t_{N-1}) \dot{-} \underline{p}$ is in $\{\theta_n : n \in \Phi_{M+1}(X)\}$.[1] Then compute the finite set
$E = \big\{\underline{p} \in D : \underline{p} \dot{-} P(t_0, ..., t_{N-1}) \in \{\theta_n : n \in \Phi_{M+1}(X)\}\big\}$. Notice, then, that by construction $\underline{\max_{\underline{p} \in E} p} \dot{-} P(t_0, ..., t_{N-1}) \in \Gamma$ and $P(t_0, ..., t_{N-1}) \dot{-} \big(\underline{\min_{\underline{p} \in D \setminus E} p} \dot{+} \underline{2^{-(k+2)}}\big) \in \Gamma$. Therefore,

$$\mathfrak{M}_\Gamma \vDash \underline{\max_{\underline{p} \in E} p} \dot{-} P(t_0, ..., t_{N-1}) \qquad \text{and} \qquad \mathfrak{M}_\Gamma \vDash P(t_0, ..., t_{N-1}) \dot{-} \Big(\underline{\min_{\underline{p} \in D \setminus E} p} \dot{+} \underline{2^{-(k+2)}}\Big).$$

It follows that

$$\max_{\underline{p} \in E} p \;\leq\; \big(P(t_0, ..., t_{N-1})\big)^{\mathfrak{M}_\Gamma} \;\leq\; \Big(\min_{\underline{p} \in D \setminus E} p + 2^{-(k+2)}\Big).$$

This implies that

$$\big(P(t_0, ..., t_{N-1})\big)^{\mathfrak{M}_\Gamma} \;\in\; \Big[\, \Big(\min_{\underline{p} \in D \setminus E} p - 2^{-(k+1)}\Big) \,,\, \Big(\min_{\underline{p} \in D \setminus E} p + 2^{-(k+1)}\Big) \,\Big].$$

Note lastly that this procedure was uniform in $X \in \mathbb{N}^{\mathbb{N}}$. $\square$

Standard effective completeness comes as a corollary.

Corollary 5 (Effective Completeness of Continuous Logic)**.** *There is an effective procedure which, given a computable name of an L-theory T, produces an index of a computable presentation of an L^+-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash T$.*

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[1]It may be that for some $\underline{p} \in D$, for every $M \in \mathbb{N}$, $\big(P(t_0, ..., t_{N-1}) \dot{-} \bigvee_{n \in \Phi_M(X)} \theta_n\big)^\circ_T$ and p differ by less than $2^{-(M+2)}$. This can only occur if $P(t_0, ..., t_{N-1})$ and $\underline{p}$ are provably equivalent with respect to $T \cup \{\theta_n : n \in \Phi(X)\}$, which can happen for at most one $\underline{p} \in D$, since $T \cup \{\theta_n : n \in \Phi(X)\}$ is consistent.

CHAPTER 5. CONTINUOUS NUMERALS

5.1 Finitary Numerals

A *numeral* is a symbolic representation of a number. The common notions that come to mind are the Arabic digits "0" through "9", though also commonly used are the Roman "I", "V", "X", *etc.* In classical logic, the space of truth values includes only two numbers 0 and 1, with 0 normally interpreted as falsity and 1 as truth. As such, in classical logic, we can consider wffs yielding *contradictions* as numerals for 0 and wffs giving *tautologies* as numerals for 1. That is, for every first order structure $\mathfrak{A}$,

$$(\exists x\ x \neq x)^{\mathfrak{A}} = 0 \qquad \text{and} \qquad (\forall x\ x = x)^{\mathfrak{A}} = 1.$$

But in continuous logic, the space of truth values is $[0,1]$. A natural question arises: for which $r \in [0,1]$ is there a wff φ such that for every structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = r$? In other words, which numbers have numerals?

We briefly touched on this topic all the way back in Exercise 1. Namely, (d), (e), and (h) show that every dyadic in $[0,1]$ has a numeral. That is, for any dyadic number in $p \in [0,1]$, there is a finitary sentence $\underline{p}$ such that for every structure $\mathfrak{M}$, $\underline{p}^{\mathfrak{M}} = p$. Indeed, the converse holds, as well.

Proposition 6. *Let $r \in [0,1]$ and suppose there is some wff φ such that for every structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = r$. Then r is a dyadic number.*

Proof. Fix $r \in [0,1]$ and wff φ such that for every structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = r$. Then for the structure $\mathfrak{A} = (\{0\}, d)$, $\varphi^{\mathfrak{A}} = r$. We now proceed via structural induction to show that for every wff ψ, $\psi^{\mathfrak{A}}$ is a dyadic number.

(i) For the base case, suppose $\psi = P(t_0, ..., t_{\eta(P)-1})$. Then ψ is of the form $\underline{d}(x,y)$ for $x, y \in \mathcal{V}$. But notably there is only one variable assignment, which sends every variable to 0. It follows that $\psi^{\mathfrak{A}} = d(0,0) = 0$.

(ii) Suppose ψ is of the form $\neg\theta$, $\frac{1}{2}\theta$, or $\theta \dot{-} \gamma$. By the inductive assumption, we may assume both $\theta^{\mathfrak{A}}$ and $\gamma^{\mathfrak{A}}$ are dyadic numbers. Hence since $\psi^{\mathfrak{A}}$ is one of $1 - \theta^{\mathfrak{A}}$, $\frac{1}{2} \cdot \theta^{\mathfrak{A}}$, or $\max\{\theta^{\mathfrak{A}} - \gamma^{\mathfrak{A}}, 0\}$, $\psi^{\mathfrak{A}}$ is also a dyadic number.

(iii) Suppose ψ is of the form $\sup_x \theta$ or $\inf_x \theta$. Again, by the inductive assumption, we may assume that $\theta^{\mathfrak{A}}$ is a dyadic number. But $(\sup_x \theta)^{\mathfrak{A}} = \sup_{a\in|\mathfrak{A}|} \theta^{\mathfrak{A},\sigma(x\mapsto a)} = \theta^{\mathfrak{A},\sigma(x\mapsto 0)} = \theta^{\mathfrak{A}}$. Similarly for $(\inf_x \theta)^{\mathfrak{A}}$. It follows that $\psi^{\mathfrak{A}} = \theta^{\mathfrak{A}}$, which is a dyadic number.

Therefore $\varphi^{\mathfrak{A}} = r$ is a dyadic number. $\square$

It is also important to note that we may get the codes of such dyadic numerals effectively. The proof of this is simple, but a good exercise for the reader.

Exercise 19. Let L be an effectively numbered signature. Prove that there is an Turing machine which on input $\langle \ell, k\rangle$, produces the code of an L-wff φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = \frac{\ell}{2^{-k}}$.

Remark 3. In the interest of completeness, as was mentioned in Remark 1, some versions of continuous logic include a connective $\underline{u}$ for every continuous mapping on $[0, 1]$. Such extended versions of continuous logic have trivial numerals. Namely, for every $r \in [0, 1]$, there is a connective $\underline{u}_r$ corresponding to the constant map $u_r : [0, 1] \to \{r\}$. In this case, for every structure $\mathfrak{M}$ and sentence φ, $(\underline{u}_r\varphi)^{\mathfrak{M}} = u_r(\varphi^{\mathfrak{M}}) = r$.

5.2 Infinitary Numerals

To discuss numerals encoded by infinitary sentences, we need to review Dedekind cuts.

Definition 58. Fix $r \in \mathbb{R}$. The *open right Dedekind cut* of r is

$$D^{>}(r) := \big\{q \in \mathbb{Q} : q > r\big\}.$$

The *closed right Dedekind cut* of r is

$$D^{\geq}(r) := \big\{q \in \mathbb{Q} : q \geq r\big\}.$$

By a *right Dedekind cut* of r, we mean either $D^{>}(r)$ or $D^{\geq}(r)$.

We then arrive at a simple proposition.

Proposition 7. *For every $r \in [0,1]$, there is an infinitary sentence φ such that for every structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = r$.*

Proof. Actually, there are two very simple such sentences which are straightforward to describe. Fix $r \in [0,1]$ and consider

$$\bigvee\!\!\!\bigvee_{q_m \in (\mathbb{Q}\cap[0,1])\setminus D^{>}(r)} \underline{\left(\lfloor 2^m q_m \rfloor\ 2^{-m}\right)} \qquad \text{and} \qquad \bigwedge\!\!\!\bigwedge_{q_m \in D^{>}(r)} \underline{\left(\lceil 2^m q_m \rceil\ 2^{-m}\right)}.$$

We refer the reader to the proof of Theorem 9 to see how these are both universally interpreted with truth value r. □

This is not very interesting. However, in the next section we introduce the computable fragment of $L^{C}_{\omega_1\omega}$. Investigating which real numbers are representable by such sentences becomes considerably more complicated.

5.3 Computable Infinitary Continuous Logic

Classical computable infinitary logic and its relation to computable structures is covered in (1) and (25). Here we institute a similar program for continuous logic and computably presented structures. We begin with an informal definition, for intuition's sake.

Heuristic 10. The *computable infinitary wffs* are heuristically given as follows, where $\alpha \in \omega_1^{\mathrm{CK}}$.

- The $\Sigma^c_0 = \Pi^c_0$ sets include all quantifier-free, finitary wffs.
- A wff φ is Σ^c_α if it is of the form
$$\varphi = \bigwedge\!\!\!\bigwedge_{i\in I} \inf_{\vec{x}} \psi_i$$
where $I \subseteq \mathbb{N}$ is c.e., each $\psi_i \in \Pi^c_\beta$, for some $\beta < \alpha$, and a modulus of continuity for φ exists and is computable from a code for φ.

- A wff φ is Π^c_α if it is of the form
$$\varphi = \bigvee\!\!\!\bigvee_{i \in I} \sup_{\vec{x}} \psi_i$$
where $I \subseteq \mathbb{N}$ is c.e., each $\psi_i \in \Sigma^c_\beta$, for some $\beta < \alpha$, and a modulus of continuity for φ exists and is computable from a code for φ.

To construct the computable fragment of $L^C_{\omega_1\omega}$ formally, we need to first construct the computable fragment of $L_{\omega_1\omega}$ in the sense of (14).

Definition 59. The *computable infinitary pseudowffs* are formally defined as follows.

Let $\widetilde{S^\Sigma_1} = \widetilde{S^\Pi_1}$ be the set of codes of all quantifier free, finitary wffs, since $\{1\} = \langle 0\rangle$. For every $a \in \mathrm{O} \setminus \{1\}$, let $\widetilde{S^\Sigma_a}$ be the set of codes of all quadruples of the form $(\Sigma, a, \vec{x}, e)$, where $\vec{x}$ is a finite tuple of variable symbols, and $e \in \mathbb{N}$. Define $\widetilde{S^\Pi_a}$ similarly.

For every $a \in \mathrm{O}$ and tuple of variable symbols $\vec{x}$, define $P(\Sigma, a, \vec{x})$ via effective transfinite recursion as the set of all codes for pairs $(j, \vec{z})$, where j is a code for the quadruple $(\Sigma, b, \vec{y}, e')$, for some $b <_{\mathrm{O}} a$, and $\vec{z}$ is a finite sequence of variable symbols of $\vec{y}$ not appearing in $\vec{x}$. Define $P(\Pi, a, \vec{x})$ similarly.

The following effective procedure defines φ_i via effective transfinite recursion, for every $i \in \bigcup_{a \in \mathrm{O}} \widetilde{S^\Sigma_a} \cup \widetilde{S^\Pi_a}$.

- If i is the code of a quantifier-free, finitary wff, then φ_i is just that wff.

- If i is a code for the quadruple $(\Sigma, a, \vec{x}, e)$, with $a \in \langle\alpha\rangle$, then
$$\varphi_i := \bigwedge\!\!\!\bigwedge_{\langle j, \vec{z}\rangle \in W_e \cap P(\Pi, a, \vec{x})} \inf_{\vec{z}} \psi_j.$$
That is, if $(j, \vec{z})$ is one such pair coded in this intersection, then $\inf_{\vec{z}} \psi$ is a disjunct in φ_i, where ψ is the computable infintary pseudowff given by the code j for a quadruple $(\Pi, b, \vec{y}, e')$, for some $b <_{\mathrm{O}} a$ and $\vec{z}$ a finite sequence of variable symbols of $\vec{y}$ not appearing in $\vec{x}$.

- If i is a code for the quadruple $(\Pi, a, \vec{x}, e)$, with $a \in \langle\alpha\rangle$, then
$$\varphi_i = \bigvee\!\!\!\bigvee_{\langle j, \vec{z}\rangle \in W_e \cap P(\Sigma, a, \vec{x})} \sup_{\vec{z}} \psi_j.$$

That is, if $(j, \vec{z})$ is one such pair coded in this intersection, then $\sup_{\vec{z}} \psi$ is a conjunct in φ_i, where ψ is the computable infintary pseudowff given by the code j for a quadruple $(\Sigma, b, \vec{y}, e')$, for some $b <_{\mathrm{O}} a$ and $\vec{z}$ a finite sequence of variable symbols of $\vec{y}$ not appearing in $\vec{x}$.

Then for every computable ordinal α, $\widetilde{\Sigma^c_\alpha}$ denotes the set of all pseudowffs φ_i where $i \in \bigcup_{a \in \langle \alpha \rangle} \widetilde{S^\Sigma_a}$, and $\widetilde{\Pi^c_\alpha}$ denotes the set of all pseudowffs φ_i where $i \in \bigcup_{a \in \langle \alpha \rangle} \widetilde{S^\Pi_a}$. Computable infinitary *pseudosentences* relativize similarly.

Remark 4. Recall that in an effectively numbered signature, every modulus of continuity is a computable function. It follows by Definition 25 that from the code of a finitary wff φ and tuple of variable symbols $\vec{x}$, one may uniformly compute a modulus of continuity for φ in $\vec{x}$. The same is not true, in general, from a code of an infinitary pseudowff. Thus we need to "hard code" computable moduli of continuity into our computable infinitary wffs.

We can now define the computable fragment of $L^C_{\omega_1 \omega}$.

Definition 60. The *computable infinitary wffs* are defined recursively as follows.

- If i' is the code of a quantifier-free, finitary wff and $\Phi_k : \mathbb{N} \to \mathbb{N}$ is a computable modulus of continuity for $\varphi_{i'}$, then $\langle i', k \rangle \in S^\Sigma_1 \cup S^\Pi_1$. When $i = \langle i', k \rangle$, the above wff is φ_i.
- If $\langle \Sigma, a, \vec{x}, e \rangle \in \widetilde{S^\Sigma_a}$, $(\psi_j)_{j \in W_e}$ is a uniformly equicontinuous sequence of computable infinitary wffs, and $\Phi_k : \mathbb{N} \to \mathbb{N}$ is a computable modulus of continuity for
$$\bigwedge_{\langle j, \vec{z} \rangle \in W_e \cap P(\Pi, a, \vec{x})} \inf_{\vec{z}} \psi_j,$$
then $\langle \Sigma, a, \vec{x}, e, k \rangle \in S^\Sigma_a$. When $i = \langle \Sigma, a, \vec{x}, e, k \rangle$, the above infinitary wff is φ_i.
- If $\langle \Pi, a, \vec{x}, e \rangle \in \widetilde{S^\Pi_a}$, $(\psi_j)_{j \in W_e}$ is a uniformly equicontinuous sequence of computable infinitary wffs, and $\Phi_k : \mathbb{N} \to \mathbb{N}$ is a computable modulus of continuity for
$$\bigvee_{\langle j, \vec{z} \rangle \in W_e \cap P(\Sigma, a, \vec{x})} \sup_{\vec{z}} \psi_j,$$
then $\langle \Pi, a, \vec{x}, e, k \rangle \in S^\Sigma_a$. When $i = \langle \Pi, a, \vec{x}, e, k \rangle$, the above infinitary wff is φ_i.

Then, for every computable ordinal α, Σ^c_α denotes the set of all wffs φ_i where $i \in \bigcup_{a\in\langle\alpha\rangle} S^\Sigma_a$ and Π^c_α denotes the set of all wffs φ_i where $i \in \bigcup_{a\in\langle\alpha\rangle} S^\Pi_a$.

It is important to be able to effectively manipulate the codes of the computable infintary wffs to perform all the finitary logical operations at each level. Luckily, this is possible.

Proposition 8. *Fix a computable ordinal α. There is an effective procedure which performs the following.*

(i) *Given a code for a Σ^c_α (or Π^c_α) wff φ, outputs a code for a Π^c_α (or Σ^c_α, respectively) wff which is logically equivalent to $\neg\varphi$.*

(ii) *Given a code for a Σ^c_α (or Π^c_α) wff φ, outputs a code for a Σ^c_α (or Π^c_α, respectively) wff which is logically equivalent to $\frac{1}{2}\varphi$.*

(iii) *Given codes for a pair of Σ^c_α (or Π^c_α) wffs φ and ψ, outputs a code for a $\Pi^c_{\alpha+1}$ (or $\Sigma^c_{\alpha+1}$, respectively) wff which is logically equivalent to $\varphi \vee \psi$.*

(iv) *Given codes for a finite collection of Σ^c_β or Π^c_β wffs, with $\beta < \alpha$, for any possible logical combination of such wffs, outputs a code for a Σ^c_α and Π^c_α wff which is logically equivalent to that combination.*

(v) *Given a code for a Σ^c_α wff φ and tuple of variable symbols $\vec{x}$, outputs a code for a Σ^c_α wff which is logically equivalent to $\inf_{\vec{x}} \varphi$, and a code for a $\Pi^c_{\alpha+1}$ wff which is logically equivalent to $\sup_{\vec{x}} \varphi$. Similarly, given an index for a Π^c_α wff φ and tuple of variable symbols $\vec{x}$, outputs a code for a $\Sigma^c_{\alpha+1}$ wff which is logically equivalent to $\inf_{\vec{x}} \varphi$, and an index for a Π^c_α wff which is logically equivalent to $\sup_{\vec{x}} \varphi$.*

There is a similar effective procedure which does the same for pseudowffs.

Proof. We proceed by effective transfinite recursion on $a \in \mathrm{O}$, *i.e.* we suppose the following claims hold for all $b <_{\mathrm{O}} a$.

(i) If $a = 1$, define $\text{neg}_1(\langle\varphi\rangle) = \langle\neg\varphi\rangle$. For $a \in \mathrm{O} \setminus \{1\}$, given the code $\langle\varphi\rangle \in S_a^\Sigma$, which decodes as $(\Sigma, a, \vec{x}, e, k)$, find a c.e. index e' for the set

$$\big\{\langle\text{neg}_b(j), \vec{u}\rangle \;:\; \exists b <_{\mathrm{O}} a \;\big(j \in S_b^\Pi \;\text{ and }\; \langle j, \vec{u}\rangle \in W_e \cap P(\Pi, a, \vec{x})\big)\big\}.$$

Defining $\text{neg}_a(\langle\varphi\rangle) := \langle\Pi, a, \vec{x}, e', k\rangle$ satisfies the claim. The construction for $\langle\varphi\rangle \in S_a^\Pi$ is similar.

(ii) If $a = 1$, define $\text{half}_1(\langle\varphi\rangle) = \langle\frac{1}{2}\varphi\rangle$. For $a \in \mathrm{O} \setminus \{1\}$, given the code $\langle\varphi\rangle \in S_a^\Sigma$, which decodes as $(\Sigma, a, \vec{x}, e, k)$, find a c.e. index e' for the set

$$\big\{\langle\text{half}_b(j), \vec{u}\rangle \;:\; \exists b <_{\mathrm{O}} a \;\big(j \in S_b^\Pi \;\text{ and }\; \langle j, \vec{u}\rangle \in W_e \cap P(\Pi, a, \vec{x})\big)\big\}.$$

Moreover, find k' such that for every $n \in \mathbb{N}$, $\Phi_{k'}(n) = \Phi_k(n+1)$. Then defining $\text{half}_a(\langle\varphi\rangle) := \langle\Pi, a, \vec{x}, e', k'\rangle$ satisfies the claim. The construction for $\langle\varphi\rangle \in S_a^\Pi$ is similar.

(iii) If $a = 1$, define $\text{and}_1(\langle\varphi\rangle, \langle\psi\rangle) = \langle\varphi \vee \psi\rangle$. For $a \in \mathrm{O} \setminus \{1\}$, given codes $\langle\varphi(\vec{x})\rangle, \langle\psi(\vec{y})\rangle \in S_a^\Pi$, decode them as $(\Pi, a, \vec{x}, e_1, k_1)$ and $(\Pi, a, \vec{y}, e_2, k_2)$, respectively. Force the two wffs to share the same set of variables by c.e. recoding them as $\langle\varphi(\vec{x} \cup \vec{y})\rangle$ and $\langle\psi(\vec{x} \cup \vec{y})\rangle$. Then find a c.e. index e' for the set

$$\begin{aligned}\big\{\langle\text{and}_b(j, \ell), \vec{u} \cup \vec{v}\rangle \;:\; \exists b_1, b_2 <_{\mathrm{O}} a \;&\big(j \in S_{b_1}^\Sigma \;\text{ and }\; \langle j, \vec{u}\rangle \in W_{e_1} \cap P(\Sigma, a, \vec{x} \cup \vec{y})\big)\\ &\text{and }\; \big(\ell \in S_{b_2}^\Sigma \;\text{ and }\; \langle \ell, \vec{v}\rangle \in W_{e_2} \cap P(\Sigma, a, \vec{x} \cup \vec{y})\big)\big\}.\end{aligned}$$

Moreover, find k' such that for every $n \in \mathbb{N}$, $\Phi_{k'}(n) = \max\{\Phi_{k_1}(n), \Phi_{k_2}(n)\}$. Then defining $\text{and}_a(\langle\varphi\rangle, \langle\psi\rangle) := \langle\Pi, a, \vec{x} \cup \vec{y}, e', k'\rangle$ satisfies the claim. The construction for $\langle\varphi\rangle, \langle\psi\rangle \in S_a^\Sigma$ is similar.

(iv) Follows from structural induction with (i), (ii), and (iii), the effective translation of $\varphi \dot{-} \psi \mapsto \neg(\psi \vee (\neg\varphi))$, and the computable embeddings for any $b <_{\mathrm{O}} a$ of S_b^Σ and S_b^Π into S_a^Σ and S_a^Π.

(v) If $a = 1$, define $\inf_{\vec{y},1}(\langle\varphi\rangle) = \langle\inf_{\vec{y}}\varphi\rangle$. For $a \in \mathrm{O} \setminus \{1\}$, given the code $\langle\varphi\rangle \in S_a^\Sigma$, which decodes as $(\Sigma, a, \vec{x}, e, k)$, find a c.e. index e' for the set

$$\big\{\langle\inf_{\vec{y},b}(j), \vec{u}\rangle \;:\; \exists b <_{\mathrm{O}} a j \in S_b^\Pi \;\text{ and }\; \langle j, \vec{u}\rangle \in W_e \cap P(\Pi, a, \vec{x})\big\}.$$

Defining $\inf_{\vec{y},a}(\langle\varphi\rangle) := \langle \Sigma, a, \vec{x} \cup \vec{y}, e', k\rangle$ satisfies the first claim.

Again, if $a = 1$, define $\sup_{\vec{y},1}(\langle\varphi\rangle) = \langle \sup_{\vec{y}} \varphi\rangle$. For $a \in \mathrm{O} \setminus \{1\}$, given the code $\langle\varphi\rangle \in S_a^{\Sigma}$, which decodes as $(\Sigma, a, \vec{x}, e, k)$, we let e' be the index of the c.e. (singleton) set $\{\langle\langle\varphi\rangle, \vec{x}\rangle\} \subseteq P(\Sigma, a, \vec{y} \setminus \vec{x})$. Now let $a' \in \langle |a|_{\mathrm{O}} + 1\rangle$. Then defining $\sup_{\vec{y},a}(\langle\varphi\rangle) := \langle \Pi, a', \vec{y} \setminus \vec{x}, e', k\rangle$ satisfies the second claim. The construction for $\langle\varphi\rangle \in S_a^{\Pi}$ is similar.

Note that the effective procedure for pseudowffs is similar but ignores the moduli of continuity coordinates. □

Moreover, we even gain some infinitary connectives, like bounded infinite summation.

Proposition 9. *For any computable ordinal α and c.e. sequence of Π^c_α wffs $(\gamma_j)_{j\in W_e}$, we may effectively find the code of a Π^c_α wff ψ such that for every L-structure $\mathfrak{M}$ and assignment σ,*

$$\psi^{\mathfrak{M},\sigma} = \min\Big\{ \sum_{j\in W_e} \gamma_j^{\mathfrak{M},\sigma},\ 1 \Big\}.$$

Proof. Let $\big(\Phi_e(n)\big)_{n\in\mathbb{N}}$ be the standard effective enumeration of W_e. By Proposition 8 and the construction of S_a^{Π}, we may compute a code for a Π^c_α wff ψ which is logically equivalent to

$$\bigvee_{n\in\mathbb{N}} (\gamma_{\Phi_e(1)} \dotplus \cdots \dotplus \gamma_{\Phi_e(n)}).$$

The result follows by interpreting this wff in a structure with an assignment. □

5.4 Hyperarithmetical Numerals

We will now investigate which real numbers may be represented by computable infinitary sentences. To do this, we require the following definition, which generalizes the hyperarithmetic degree of right Dedekind cuts of real numbers to sequences of real numbers.

Definition 61. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $g : \mathbb{N} \to \omega_1^{\mathrm{CK}}$. We say $(r_n)_{n\in\mathbb{N}}$ is *weakly uniformly right* Σ^0_g (Π^0_g) if there is a Turing machine $\Phi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, $\Phi(n)$ is a $\Sigma^0_{g(n)}$ ($\Pi^0_{g(n)}$) index of a right Dedekind cut of r_n.

But notably, there are certain Dedekind cuts we don't need to consider in the following constructions.

Proposition 10 (C., McNicholl, 2022+)**.** *There is an effective procedure which, given a Σ^0_α index of a right Dedekind cut of r, produces a Σ^0_α index of $D^{>}(r)$. Similarly, there is an effective procedure which, given a Π^0_α index of a right Dedekind cut of r, produces a Π^0_α index of $D^{\geq}(r)$.*

Proof. Fix $\langle a, e\rangle$, a Σ^0_α index of a right Dedekind cut of r. Then compute e' such that

$$W^{\mathcal{H}(a)}_{e'} = \left\{m \in W^{\mathcal{H}(a)}_{e} : \exists \ell \in W^{\mathcal{H}(a)}_{e}\ q_\ell < q_m\right\}.$$

It follows that $\langle a, e'\rangle$ is a Σ^0_α index of $D^{>}(r)$. Notably, this computation is uniform. The Π^0_α result follows by considering complements. □

The following lemma then does the bulk of the work for our construction.

Lemma 5 (C., McNicholl, 2022+)**.** *Let $\alpha \in \omega_1^{\mathrm{CK}}$ and $s \in \mathbb{R}$. Then the following hold uniformly.*

1. *If $\alpha = \beta + 1$ and a right Dedekind cut of s is Σ^0_α, then there is a sequence of real numbers $(r_n)_{n\in\mathbb{N}}$ such that $s = \inf_{n\in\mathbb{N}} r_n$ and $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Π^0_β.*

2. *If $\alpha = \beta + 1$ and a right Dedekind cut of s is Π^0_α, then there is a sequence of real numbers $(r_n)_{n\in\mathbb{N}}$ such that $s = \sup_{n\in\mathbb{N}} r_n$ and $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Σ^0_β.*

3. *If α is a limit ordinal and a right Dedekind cut of s is Σ^0_α, then there is a computable map $h : \mathbb{N} \to \alpha$ and a sequence of real numbers $(r_n)_{n\in\mathbb{N}}$ such that $s = \inf_{n\in\mathbb{N}} r_n$ and $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Π^0_h.*

4. *If α is a limit ordinal and a right Dedekind cut of s is Π^0_α, then there is a computable map $h : \mathbb{N} \to \alpha$ and a sequence of real numbers $(r_n)_{n\in\mathbb{N}}$ such that $s = \sup_{n\in\mathbb{N}} r_n$ and $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Σ^0_h.*

Proof. Fix $\alpha \in \omega_1^{\mathrm{CK}}$ and $s \in \mathbb{R}$. We proceed by effective transfinite recursion. By Proposition 10, we only need to consider the cases where $D^{>}(s)$ is Σ^0_α and $D^{\geq}(s)$ is Π^0_α.

Case 1. $\alpha = 1$.

(1) Suppose $D^>(s)$ is Σ^0_1. Let $(r_n)_{n\in\mathbb{N}}$ be an effective enumeration of $D^>(s)$. Then $s = \inf_{n\in\mathbb{N}} r_n$.

(2) Suppose $D^\geq(s)$ is Π^0_1. Let $(r_n)_{n\in\mathbb{N}}$ be an effective enumeration of $\mathbb{Q} \setminus D^\geq(s)$. Then $s = \sup_{n\in\mathbb{N}} r_n$.

Case 2. $\alpha = 2$.

(1) Suppose $D^>(s)$ is Σ^0_2. Then we may effectively find a computable $R \subseteq \mathbb{N}^2 \times \mathbb{Q}$ such that for every $q \in \mathbb{Q}$,

$$q > s \iff \exists n \in \mathbb{N}\ \forall k \in \mathbb{N}\ R(n, k, q). \tag{5.1}$$

It follows that

$$q \leq s \iff \forall n \in \mathbb{N}\ \exists k \in \mathbb{N}\ \neg R(n, k, q). \tag{5.2}$$

For every $n \in \mathbb{N}$, define the set

$$S(n) := \big\{q \in \mathbb{Q} : \forall n' \leq n\ \exists k \in \mathbb{N}\ \exists q' \geq q\ \neg R(n', k, q')\big\}.$$

We now claim that each of the following hold.

(i) Each $S(n)$ is nonempty.

(ii) Each $S(n)$ is downward closed (as a subset of the rationals).

(iii) There is some $N \in \mathbb{N}$ such that for every $n \geq N$, $S(n)$ is bounded from above.

(iv) Each $S(n)$ is Σ^0_1, uniformly in $n \in \mathbb{N}$.

(v) $s = \inf_{n\in\mathbb{N}} \sup S(n)$

To see (i), by way of contradiction, suppose some $S(n)$ was empty. Then for some $n' \leq n$, for every $q \in \mathbb{Q}$, $k \in \mathbb{N}$, and $q' \geq q$, $R(n', k, q')$. It follows from 5.1 that for every $q \in \mathbb{Q}$, $q > s$. $\rightarrow\leftarrow$

To see (ii), fix $q \in S(n)$ and $q'' \leq q$. Since $q \in S(n)$, for every $n' \leq n$, there must be some $k \in \mathbb{N}$ and $q' \geq q$ such that $\neg R(n', k, q')$. Hence for every $n' \leq n$, for these k and $q' \geq q \geq q''$ values, $\neg R(n', k, q')$. It follows that $q'' \in S(n)$.

To see (iii), we first show that for some $N \in \mathbb{N}$, $S(N)$ is bounded from above. By way of contradiction, suppose not, *i.e.*, that every $S(N)$ has no upper bound. But by (ii), each $S(N)$ is downward closed. Therefore, every $S(N) = \mathbb{Q}$. It follows that for every $q \in \mathbb{Q}$, for every $n \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $\neg R(n,k,q)$. Hence, by 5.2, for every $q \in \mathbb{Q}$, $q \leq s$. $\rightarrow\leftarrow$

Then notice that for every $n > N$, $S(n) \subseteq S(N)$, by construction. It follows that for every $n \geq N$, $S(n)$ is bounded from above.

To see (iv), notice the universal quantifier in the definition of $S(n)$ is bounded by n, and recall that $\neg R$ is a computable relation.

To see (v), note that by (i), (ii), and (iii), for some $N \in \mathbb{N}$, for every $n \geq N$, $S(n)$ is a left Dedekind cut of $\sup S(n)$. It follows that $\inf_{n\in\mathbb{N}} \sup S(n) = \sup \bigcap_{n\in\mathbb{N}} S(n)$. Then notice that

$$\bigcap_{n\in\mathbb{N}} S(n) = \big\{q \in \mathbb{Q} : \forall n \in \mathbb{N}\ \exists k \in \mathbb{N}\ \exists q' \geq q\ \neg R(n,k,q')\big\}.$$

Hence, by 5.2, $\bigcap_{n\in\mathbb{N}} S(n) = \{q \in \mathbb{Q} : q \leq s\}$. Therefore, $s = \inf_{n\in\mathbb{N}} \sup S(n)$.

By considering complements, we now have the following.

(i) There is some $N \in \mathbb{N}$ such that for every $n \geq N$, $S(n)^c$ is a right Dedekind cut of $\inf S(n)^c$.

(ii) Each $S(n)^c$ is Π^0_1, uniformly in $n \in \mathbb{N}$.

(iii) $s = \inf_{n\in\mathbb{N}} \inf S(n)^c$.

For every $N \in \mathbb{N}$, define $f_N : \mathbb{N} \to \mathbb{N}$ by mapping n to the Π^0_1 index of $S(n+N)^c$. By construction, each of these is a computable function. Moreover, by (i), for some $N \in \mathbb{N}$, the sequence of real numbers $(\inf S(n+N)^c)_{n\in\mathbb{N}}$ is weakly uniformly right Π^0_1, as witnessed by f_N.

(2) Suppose $D^{\geq}(s)$ is Π^0_2. Then we may effectively find a computable $R \subseteq \mathbb{N}^2 \times \mathbb{Q}$ such that for every $q \in \mathbb{Q}$,

$$q \geq s \iff \forall n \in \mathbb{N}\ \exists k \in \mathbb{N}\ R(n,k,q). \tag{5.3}$$

It follows that

$$q < s \iff \exists n \in \mathbb{N}\ \forall k \in \mathbb{N}\ \neg R(n,k,q). \tag{5.4}$$

For every $n \in \mathbb{N}$, define the set

$$S(n) := \{q \in \mathbb{Q} : \forall n' \leq n \; \exists k \in \mathbb{N} \; \exists q' \leq q \; R(n', k, q')\}.$$

We now claim that each of the following hold.

(i) Each $S(n)$ is nonempty.

(ii) Each $S(n)$ is upward closed.

(iii) There is some $N \in \mathbb{N}$ such that for every $n \geq N$, $S(n)$ is bounded from below.

(iv) Each $S(n)$ is Σ^0_1, uniformly in $n \in \mathbb{N}$.

(v) $s = \sup_{n\in\mathbb{N}} \inf S(n)$.

To see (i), by way of contradiction, suppose some $S(n)$ was empty. Then for some $n' \leq n$, for every $q \in \mathbb{Q}$, $k \in \mathbb{N}$, and $q' \leq q$, $\neg R(n', k, q')$. It follows from 5.4 that for every $q \in \mathbb{Q}$, $q < s$. $\rightarrow\leftarrow$

To see (ii), fix $q \in S(n)$ and $q'' \geq q$. Since $q \in S(n)$, for every $n' \leq n$, there must be some $k \in \mathbb{N}$ and $q' \leq q$ such that $R(n', k, q')$. Hence for every $n' \leq n$, for these k and $q' \leq q \leq q''$ values, $R(n', k, q')$. It follows that $q'' \in S(n)$.

To see (iii), we first show that for some $N \in \mathbb{N}$, $S(N)$ is bounded from below. By way of contradiction, suppose not, *i.e.*, that every $S(N)$ has no lower bound. But by (ii), each $S(N)$ is upward closed. Therefore, every $S(N) = \mathbb{Q}$. It follows that for every $q \in \mathbb{Q}$, for every $n \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $R(n, k, q)$. Hence, by 5.3, for every $q \in \mathbb{Q}$, $q \geq s$. $\rightarrow\leftarrow$

Then notice that for every $n > N$, $S(n) \subseteq S(N)$, by construction. It follows that for every $n \geq N$, $S(n)$ is bounded from below.

To see (iv), notice the universal quantifier in the definition of $S(n)$ is bounded by n, and recall that $\neg R$ is a computable relation.

To see (v), notice that by (i), (ii), and (iii), for some $N \in \mathbb{N}$ and every $n \geq N$, $S(n)$ is a right Dedekind cut of $\inf S(n)$. It follows that $\sup_{n\in\mathbb{N}} \inf S(n) = \inf \bigcap_{n\in\mathbb{N}} S(n)$. Then notice that

$$\bigcap_{n\in\mathbb{N}} S(n) = \{q \in \mathbb{Q} : \forall n \in \mathbb{N} \; \exists k \in \mathbb{N} \; \exists q' \leq q \; R(n, k, q)\}.$$

Hence, by 5.3, $\bigcap_{n\in\mathbb{N}} S(n) = \{q \in \mathbb{Q} : q \geq s\}$. Therefore, $s = \sup_{n\in\mathbb{N}} \inf S(n)$.

For every $N \in \mathbb{N}$, define $f_N : \mathbb{N} \to \mathbb{N}$ which maps n to the Σ^0_1 index of $S(n+N)$. By construction, each of these is a computable function. Moreover, by (iii), for some $N \in \mathbb{N}$, the sequence of real numbers $(\inf S(n+N))_{n\in\mathbb{N}}$ is weakly uniformly right Σ^0_1, as witnessed by f_N.

Case 3. $\alpha = k+3$ for some $k \in \mathbb{N}$.

(1) Suppose $D^>(s)$ is Σ^0_{k+3}. From case 2 and relativization, $s = \inf_{n\in\mathbb{N}} r_n$ where $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right $\Pi^0_1(\emptyset^{k+1})$. Hence $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Π^0_{k+2}.

(2) Suppose $D^\geq(s)$ is Π^0_{k+3}. From case 2 and relativization, $s = \sup_{n\in\mathbb{N}} r_n$ where $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right $\Sigma^0_1(\emptyset^{k+1})$. Hence $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Σ^0_{k+2}.

Case 4. $\alpha > 0$ is a limit ordinal. Let $h : \mathbb{N} \to \alpha$ be an effective enumeration of α.

(3) Suppose $D^>(s)$ is Σ^0_α. By well-known techniques we may decompose

$$D^>(s) = \bigcup_{n\in\mathbb{N}} S(n)$$

where each $S(n)$ is Π^0_h, uniformly in $n \in \mathbb{N}$. Moreover, by the same techniques as were used in case 2, we may construct each $S(n)$ to be upwardly closed. Then define $r_n := \inf S(n)$ for every $n \in \mathbb{N}$. By construction, $s = \inf_{n\in\mathbb{N}} r_n$.

(4) Suppose $D^\geq(s)$ is Π^0_α. Again, by well-known techniques, we may decompose

$$D^\geq(s) = \bigcap_{n\in\mathbb{N}} S(n)$$

where each $S(n)$ is Σ^0_h, uniformly in $n \in \mathbb{N}$. And again, by the same techniques as were used in case 2, we may construct each $S(n)$ to be upwardly closed. Then define $r_n := \inf S(n)$ for every $n \in \mathbb{N}$. By construction, $s = \sup_{n\in\mathbb{N}} r_n$.

Case 5. $\alpha = \beta + 1$, for β infinite.

(1) Suppose $D^>(s)$ is Σ^0_α. By definition, this means $D^>(s)$ is c.e. in $\emptyset^{(\beta+1)}$. It follows that $D^>(s)$ is $\Sigma^0_2(\emptyset^{(\beta)})$. Then by relativization and case 2, $s = \inf_{n\in\mathbb{N}} r_n$, where $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right $\Pi^0_1(\emptyset^{(\beta)})$. Hence, $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Π^0_β.

(2) Suppose $D^{\geq}(s)$ is Π^0_α. By definition, this means $D^{\geq}(s)$ is co-c.e. in $\emptyset^{(\beta+1)}$. It follows that $D^{\geq}(s)$ is $\Pi^0_2(\emptyset^{(\beta)})$. Again, by relativization and case 2, $s = \sup_{n\in\mathbb{N}} r_n$ where $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right $\Sigma^0_1(\emptyset^{(\beta)})$. Hence, $(r_n)_{n\in\mathbb{N}}$ is weakly uniformly right Σ^0_β. $\square$

At last we are able to see that every hyperarithmetical real number is representable as a numeral via a computable infinitary sentence of the same hyperarithmetical degree.

Theorem 9 (C., McNicholl, 2022+)**.** *Let L be an effective numbered signature. There are partial computable functions f and g from $(\mathrm{O} \setminus \{1\}) \times \mathbb{N}$ to the set of computable infinitary sentences such that the following hold, for every $1 \leq \alpha < \omega_1^{\mathrm{CK}}$.*

- *If $\langle a, i\rangle$ is a Π^0_α index of a right Dedekind cut of a real number s, then $f(a,i)$ is a code of a Π^c_α sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.*
- *If $\langle a, i\rangle$ is a Σ^0_α index of a right Dedekind cut of a real number s, then $g(a,i)$ is a code of a Σ^c_α sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.*

Proof. Fix an effectively numbered signature L. We proceed to prove the claim via effective transfinite recursion on $\mathrm{O} \setminus \{1\}$. That is, for every notation $a \in \mathrm{O} \setminus \{1\}$, there is an uniform effective procedure which constructs functions f_a and g_a, which halt with the desired properties on inputs (c, i) for $1 <_{\mathrm{O}} c <_{\mathrm{O}} a$ and $i \in \mathbb{N}$.

Case 0. $a = 2$.

Define $f_2 := \emptyset$ and $g_2 := \emptyset$, since there is no $b \in \mathrm{O}$ such that $1 <_{\mathrm{O}} b <_{\mathrm{O}} 2$.

Case 1. $a = 4$.

Given a pair $(2, i)$, define

$$f_4(2,i) := \bigvee\!\!\!\bigvee_{n \in W_i^{\mathcal{H}(2)}} \underline{\left(\lfloor 2^n q_n \rfloor \; 2^{-n}\right)}.$$

Further define

$$g_4(2,i) := \bigwedge\!\!\!\bigwedge_{n \in W_i^{\mathcal{H}(2)}} \underline{\left(\lceil 2^n q_n \rceil \; 2^{-n}\right)}.$$

Notably, $f_4(2,i)$ is a Π^c_1 sentence, while $g_4(2,i)$ is a Σ^c_1 sentence. Then notice that if $(2,i)$ is a Π^0_1 index of a right Dedekind cut of a real number s, then for any L-structure $\mathfrak{M}$,

$$s = \sup_{n \in W_i^{\mathcal{H}(2)}} q_n - 2^{-n} \leq \Big(\bigvee_{n \in W_i^{\mathcal{H}(2)}} \underline{\left(\lfloor 2^n q_n \rfloor \; 2^{-n}\right)} \Big)^{\mathfrak{M}} \leq \sup_{n \in W_i^{\mathcal{H}(2)}} q_n = s.$$

Moreover, if $(2,i)$ is a Σ^0_1 index of a right Dedekind cut of a real number s, then for any L-structure $\mathfrak{M}$,

$$s = \inf_{n \in W_i^{\mathcal{H}(2)}} q_n \leq \Big(\bigwedge_{n \in W_i^{\mathcal{H}(2)}} \underline{\left(\lceil 2^n q_n \rceil \; 2^{-n}\right)} \Big)^{\mathfrak{M}} \leq \inf_{n \in W_i^{\mathcal{H}(2)}} q_n + 2^{-n} = s.$$

For the recursive step, given $a \in \mathrm{O} \setminus \{1\}$, we assume that we can uniformly compute, for any $1 <_{\mathrm{O}} b <_{\mathrm{O}} a$, functions f_b and g_b such that, for any pair (c,i) with $1 <_{\mathrm{O}} c <_{\mathrm{O}} b$, the following hold.

- If (c,i) is an Π^0_γ index of a right Dedekind cut of a real number s, then $f_b(c,i)$ is a code of a Π^c_γ sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.
- If (c,i) is an Σ^0_γ index of a right Dedekind cut of a real number s, then $g_b(c,i)$ is a code of a Σ^c_γ sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.

Case 2. $a = 3 \cdot 5^e$, for some $e \in \mathbb{N}$.

Given input (c,i) such that $1 <_{\mathrm{O}} c <_{\mathrm{O}} a$, notice that $1 <_{\mathrm{O}} c <_{\mathrm{O}} 2^c <_{\mathrm{O}} a$. Then define $f_a(c,i) := f_{2^c}(c,i)$.

Case 3. $a = 2^b$, for $b \in \mathrm{O} \setminus \{1,2\}$.

If input (c,i) is such that $1 <_{\mathrm{O}} c <_{\mathrm{O}} b$, then define $f_a(c,i) := f_b(c,i)$. If input (c,i) is such that $c =_{\mathrm{O}} b$, we consider two subcases.

Subcase 3.1. c is a notation for a successor ordinal.

By Lemma 5, if (c,i) is a Π^0_γ index of a right Dedekind cut of a real number s, we may compute a sequence of indices $\big((\log_2(b), e_n)\big)_{n \in \mathbb{N}}$ for the right Dedekind cuts of $(r_n)_{n \in \mathbb{N}}$, which is weakly uniformly right $\Sigma^0_{\gamma-1}$, and such that

$$s = \sup_{n \in \mathbb{N}} r_n.$$

Then define

$$f_a(c,i) := \bigvee_{n\in\mathbb{N}} g_c(\log_2(c), e_n).$$

By the recursive assumption, for every $n \in \mathbb{N}$, $g_c(\log_2(c), e_n)$ is a $\Sigma^c_{\gamma-1}$ sentence such that for every L-structure $\mathfrak{M}$, $\left(g_c(\log_2(c), e_n)\right)^{\mathfrak{M}} = r_n$. It follows that $f_a(c,i)$ is a Π^c_γ sentence such that for every L-structure $\mathfrak{M}$,

$$\left(f_a(c,i)\right)^{\mathfrak{M}} = \Big(\bigvee_{n\in\mathbb{N}} g_c(\log_2(c), e_n)\Big)^{\mathfrak{M}} = \sup_{n\in\mathbb{N}} \left(g_c(\log_2(c), e_n)\right)^{\mathfrak{M}} = \sup_{n\in\mathbb{N}} r_n = s.$$

Subcase 3.2. c is a notation for a nonzero limit ordinal.

By Lemma 5, if (c,i) is a Π^0_γ index of right Dedekind cut of a real number s, we may compute $h : \mathbb{N} \to \gamma$ and a sequence of indices $\left((c_n, e_n)\right)_{n\in\mathbb{N}}$ for the right Dedekind cuts of $(r_n)_{n\in\mathbb{N}}$ which is weakly uniformly right Σ^0_h, and such that

$$s = \sup_{n\in\mathbb{N}} r_n.$$

Then define

$$f_a(c,i) := \bigvee_{n\in\mathbb{N}} g_{2^{c_n}}(c_n, e_n).$$

By the recursive assumption, for every $n \in \mathbb{N}$, $g_{2^{c_n}}(c_n, e_n)$ is a $\Sigma^c_{h(n)}$ sentence such that for every L-structure $\mathfrak{M}$, $\left(g_{2^{c_n}}(c_n, e_n)\right)^{\mathfrak{M}} = r_n$. It follows that $f_a(c,i)$ is a Π^c_γ sentence such that for every L-structure $\mathfrak{M}$,

$$\left(f_a(c,i)\right)^{\mathfrak{M}} = \Big(\bigvee_{n\in\mathbb{N}} g_{2^{c_n}}(c_n, e_n)\Big)^{\mathfrak{M}} = \sup_{n\in\mathbb{N}} \left(g_{2^{c_n}}(c_n, e_n)\right)^{\mathfrak{M}} = \sup_{n\in\mathbb{N}} r_n = s.$$

$\square$

CHAPTER 6. DIAGRAM COMPLEXITY

6.1 Diagrams of Metric Structures

The following is joint work with McNicholl and Goldbring from (9). Recall the atomic diagrams of classical structures, which are the subsets of natural number codes of quantifier-free sentences they satisfy. A similar definition in the continuous case would leave out information, only considering quantifier-free sentences which are interpreted as zero. Instead, we want to know the truth-value *as a real number* of each quantifier-free sentence. Hence we formally define the open and closed diagrams of a metric structure as follows.

Definition 62. Let $\mathfrak{M}$ be an L-structure. In the following, φ ranges over sentences of L and q ranges over $[0,1] \cap \mathbb{Q}$.

1. The *closed (resp. open) quantifier-free diagram of* $\mathfrak{M}$ is the set of all pairs (φ, q) so that φ is quantifier-free and $\varphi^{\mathfrak{M}} \leq q$ (resp. $\varphi^{\mathfrak{M}} < q$).

2. For every positive integer N, the *closed (resp. open)* Π_N *diagram of* $\mathfrak{M}$ is the set of all pairs (φ, q) so that φ is Π_N and $\varphi^{\mathfrak{M}} \leq q$ (resp. $\varphi^{\mathfrak{M}} < q$). The closed and open Σ_N diagrams are defined similarly.

Since we must restrict ourselves to countable diagrams, the above constructions code for Dedekind cuts rather than the real number truth-value, itself. This leads to further arithmetical complexity which is not found in the classical case.

6.2 Combinatorial Results

We introduce here some results that will support our demonstration of lower bounds. Among these, our main combinatorial result (Theorem 10) is a principle for representing Σ^0_N and Π^0_N sets as solutions of inequalities involving infinite series. We also provide some relational notation which will facilitate the statements of many of our results and their proofs.

Definition 63. Let $N \in \mathbb{N}$, and suppose $R \subseteq \mathbb{N}^{N+1}$.

1. $\neg R = \mathbb{N}^{N+1} - R$.
2. $\vec{\exists} R = \{n \in \mathbb{N} \ : \ \exists x_1 \forall x_2 \ldots Q x_N \ R(n, x_1, \ldots, x_N)\}$.
3. $\vec{\forall} R = \{n \in \mathbb{N} \ : \ \forall x_1 \exists x_2 \ldots Q x_N \ R(n, x_1, \ldots, x_N)\}$.

In Definition 63.2, Q denotes the quantifier $\forall$ if N is even and $\exists$ if N is odd. Similarly, in Definition 63.3, Q denotes the quantifier $\forall$ if N is odd and $\exists$ if N is even. We will also use these conventions in what follows.

Given $R \subseteq \mathbb{N}^{N+1}$, also set

$$R^* = \{(n, x_1, \ldots, x_N) \ \in \mathbb{N}^{N+1} \ : \ \forall x'_1 \leq x_1 \exists x'_2 \leq x_2 \ldots Q x'_N \leq x_N \ R(n, x'_1, \ldots, x'_N)\}.$$

Notably, $R \equiv_{\mathrm{T}} R^*$.

Then fix a uniformly computable family $(R_N)_{N \in \mathbb{N}}$ of relations so that for each $N \in \mathbb{N}$, $R_{2N} \cup R_{2N+1} \subseteq \mathbb{N}^{N+2}$, $\vec{\forall} R_{2N}$ is Π^0_{N+1}-complete, and $\vec{\exists} R_{2N+1}$ is Σ^0_{N+1}-complete.

The following lemma is easily verified by simultaneous induction on N, where the suprema and infima range over $\mathbb{N}$.

Lemma 6 (C., Goldbring, McNicholl, 2021)**.** *For $R \subseteq \mathbb{N}^{N+1}$ and $n \in \mathbb{N}$, we have:*

1. *$n \in \vec{\forall} R$ if and only if* $\inf_{x_1} \sup_{x_2} \ldots Q_{x_N} \chi_R(n, x_1, \ldots, x_N) = 1$.
2. *$n \in \vec{\exists} R$ if and only if* $\sup_{x_1} \inf_{x_2} \ldots Q_{x_N} \chi_R(n, x_1, \ldots, x_N) = 1$.

We also require the following definition to state our main combinatorial result.

Definition 64. For $K, N \in \mathbb{N}$ and $f : \mathbb{N}^{N+1} \to \mathbb{R}$ a bounded function, set:

$$\begin{aligned} \Gamma_K(f; x_1, \ldots, x_N) &= \sum_{x_0=0}^{K} 2^{-(x_0+1)} f(x_0, \ldots, x_N) \\ \Gamma(f; x_1, \ldots, x_N) &= \sum_{x_0=0}^{\infty} 2^{-(x_0+1)} f(x_0, \ldots, x_N). \end{aligned}$$

Define $\Gamma(f) : \mathbb{N}^N \to \mathbb{R}$ by setting $\Gamma(f)(x_1, \ldots, x_N) = \Gamma(f; x_1, \ldots, x_N)$. Note that $\Gamma(f)$ is computable if f is computable and, moreover, an index of $\Gamma(f)$ can be computed from an index of f and a bound on f.

The following is our main combinatorial result, where elements of $\mathbb{N}^{N+2}$ are viewed in the form $(x_0, x_1, \ldots, x_N, n)$.

Theorem 10 (C., Goldbring, McNicholl, 2021)**.** *Let $R \subseteq \mathbb{N}^{N+2}$, and let $n \in \mathbb{N}$.*

1. *$n \in \vec{\forall} R$ if and only if*

$$\inf_{x_1} \sup_{x_2} \ldots Q_{x_N} \Gamma\Big(1 - \frac{1}{2}\chi_{R^*}; x_1, \ldots, x_N, n\Big) \leq \frac{1}{2}.$$

2. *$n \in \vec{\exists} R$ if and only if*

$$\sup_{x_1} \inf_{x_2} \ldots Q_{x_N} \Gamma\Big(\frac{1}{2}\chi_{(\neg R)^*}; x_1, \ldots, x_N, n\Big) < \frac{1}{2}.$$

The proof of this theorem requires some additional lemmas. For the first, note that if $f : \mathbb{N} \to \mathbb{R}$ is a bounded function, then $\Gamma_K(f)$ is a single real number (*i.e.* a constant).

Lemma 7 (C., Goldbring, McNicholl, 2021)**.** *If $f : \mathbb{N} \to \{\frac{1}{2}, 1\}$, then for every $K \in \mathbb{N}$, $\Gamma_K(f) \leq \frac{1}{2}$ if and only if $f(m) = \frac{1}{2}$ for all $m < K$.*

Proof sketch. Fix $K \in \mathbb{N}$. Consider the given sum in base 2. Any $m < K$ for which $f(m) = 1$ leads to a "carry" operation so that the $\frac{1}{2}$-position becomes 1. Adding $f(K)$ would then force the value to be greater than $\frac{1}{2}$. □

Lemma 8 (C., Goldbring, McNicholl, 2021)**.** *Suppose $R \subseteq \mathbb{N}^{N+1}$. Then $\vec{\forall}(R^*) = \vec{\forall} R$.*

Proof sketch. The proof that $\vec{\forall}(R^*) \subseteq \vec{\forall} R$ is straightforward. The other inclusion is demonstrated via Skolemization. □

Lemma 9 (C., Goldbring, McNicholl, 2021)**.** *Fix $R \subseteq \mathbb{N}^{N+2}$ and $1 \leq J \leq N$. Then for every $x_1, \ldots, x_{J-1}, n \in \mathbb{N}$ and every $K \in \mathbb{N}$, we have:*

1. $\sup_{x_J}\inf_{x_{J+1}}\dots Q_{x_N}\Gamma_K(1-\frac{1}{2}\chi_{R^*};x_1,\dots,x_N,n)\leq\frac{1}{2}$ *if and only if* $\Gamma_K(\sup_{x_J}\inf_{x_{J+1}}\dots Q_{x_N}(1-\frac{1}{2}\chi_{R^*});x_1,\dots,x_{J-1},n)\leq\frac{1}{2}$.

2. $\inf_{x_J}\sup_{x_{J+1}}\dots Q_{x_N}\Gamma_K(1-\frac{1}{2}\chi_{R^*};x_1,\dots,x_N,n)\leq\frac{1}{2}$ *if and only if* $\Gamma_K(\inf_{x_J}\sup_{x_{J+1}}\dots Q_{x_N}(1-\frac{1}{2}\chi_{R^*};x_1,\dots,x_{J-1},n)\leq\frac{1}{2}$.

Proof. Set $G=1-\frac{1}{2}\chi_{R^*}$ and note that $\mathrm{ran}(G)\subseteq\{\frac{1}{2},1\}$. Thus, in what follows, all suprema are maxima and all infima are minima. Also, we may assume $K>0$.

We proceed by induction on $N-J$. We begin with the base case for (1), that is, $J=N-1$. Without loss of generality, we may assume that one of the two quantities in (1) is no larger than $\frac{1}{2}$. Since $\Gamma_K(\sup_{x_N}G;x_1,\dots,x_{N-1},n)\geq\sup_{x_N}\Gamma_K(G;x_1,\dots,x_N,n)$, we may assume $\sup_{x_N}\Gamma_K(G;x_1,\dots,x_N,n)\leq\frac{1}{2}$. By Lemma 7, we have that $G(x_0,x_1,\dots,x_N,n)=\frac{1}{2}$ for all $x_N\in\ N$ and all $x_0<K$. By Lemma 7 again, $\Gamma_K(\sup_{x_N}G;x_1,\dots,x_{N-1},n)\leq\frac{1}{2}$.

We now consider the base case for (2). Again, we may assume one of the two quantities in (2) is no larger than $\frac{1}{2}$. Since $\Gamma_K(\inf_{x_N}G;x_1,\dots,x_{N-1},n)\leq\inf_{x_N}\Gamma_K(G;x_1,\dots,x_N,n)$, we assume $\Gamma_K(\inf_{x_N}G;x_1,\dots,x_{N-1},n)\leq\frac{1}{2}$. By Lemma 7, $\inf_{x_N}G(x_0,\dots,x_N,n)=\frac{1}{2}$ for all $x_0<K$. Consequently, for each $x_0<K$, there exists $\xi_{x_0}\in\mathbb{N}$ so that $G(x_0,\dots,x_{N-1},\xi_{x_0},n)=\frac{1}{2}$. Let

$$\xi=\begin{cases}\max_{x_0<K}\xi_{x_0} & \text{if } N \text{ odd}\\ 0 & \text{otherwise.}\end{cases}$$

By the definition of R^*, it follows that $G(x_0,\dots,x_{N-1},\xi,n)=\frac{1}{2}$ for all $x_0<K$. By Lemma 7 again, $\inf_{x_N}\Gamma_K(G;x_1,\dots,x_N,n)\leq\frac{1}{2}$.

We now perform the inductive step for (1). Suppose that $N-J>1$ and set $H=\inf_{x_{J+1}}\dots Q_{x_N}G$. By the inductive hypothesis, it suffices to show that $\sup_{x_J}\Gamma_K(H;x_1,\dots,x_J,n)\leq\frac{1}{2}$ if and only if $\Gamma_K(\sup_{x_J}H;x_1,\dots,x_{J-1},n)\leq\frac{1}{2}$. Without loss of generality, we assume $\sup_{x_J}\Gamma_K(H;x_1,\dots,x_J,n)\leq\frac{1}{2}$ By Lemma 7, for all $x_J\in\ N$ and all $x_0<K$, $H(x_0,x_1,\dots,x_J,n)=\frac{1}{2}$. By Lemma 7 again, $\Gamma_K(\sup_{x_J}H;x_1,\dots,x_{J-1},n)\leq\frac{1}{2}$.

We now carry out the inductive step for (2). In this case, consider the function

$$H=\sup_{x_{J+1}}\dots Q_{x_N}G(x_0,\dots,x_N,n).$$

It suffices to show that $\inf_{x_J} \Gamma_K(H; x_1, \ldots, x_J, n) \le \frac{1}{2}$ if and only if $\Gamma_K(\inf_{x_J} H; x_1, \ldots, x_{J-1}, n) \le \frac{1}{2}$. Without loss of generality, we assume $\Gamma_K(\inf_{x_J} H; x_1, \ldots, x_{J-1}, n) \le \frac{1}{2}$. By Lemma 7, for every $x_0 < K$, $\inf_{x_J} H(x_0, \ldots, x_J, n) = \frac{1}{2}$, whence, for every $x_0 < K$, there exists $\xi_{x_0} \in \mathbb{N}$ so that $H(x_0, \ldots, x_{J-1}, \xi_{x_0}, n) = \frac{1}{2}$. Let

$$\xi = \begin{cases} \max_{x_0 < K} \xi_{x_0} & J \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of R^*, $H(x_0, x_1, \ldots, x_{J-1}, \xi, n) = \frac{1}{2}$ for all $x_0 < K$. By Lemma 7,

$\Gamma_K(\inf_{x_J} H; x_0, \ldots, x_{J-1}, n) = \frac{1}{2}$. □

Though Lemma 9 is not the primary combinatorial result, it is, nonetheless, somewhat surprising. Notably, one does not usually expect to be able to interchange summation with sup or inf. But here R^* allows us to come to a weaker conclusion but one that is just strong enough complete the main proof.

Proof of Theorem 10. It suffices to prove (1); part (2) follows by considering complements. Once again, set $G = 1 - \frac{1}{2}\chi_{R^*}$.

Suppose $n \in \vec{\forall} R$. It follows from Lemmas 6 and 8 that

$$\sup_{x_0} \inf_{x_1} \ldots Q_{x_N} G(x_0, \ldots, x_N, n) = \frac{1}{2}.$$

Thus, by Lemma 7, $\Gamma_K(\inf_{x_1} \ldots Q_{x_N} G; n) \le \frac{1}{2}$. By Lemma 9, we have that

$$\inf_{x_1} \ldots Q_{x_N} \Gamma_K(G; x_1, \ldots, x_N, n) \le \frac{1}{2}.$$

Since $G \le 1$, it follows that

$$\inf_{x_1} \ldots Q_{x_N} \Gamma(G; x_1, \ldots, x_N, n) \le \frac{1}{2} + 2^{-(K+1)}$$

for all $K \in \mathbb{N}$. Hence, $\inf_{x_1} \ldots Q_{x_N} \Gamma(G; x_1, \ldots, x_N, n) \le \frac{1}{2}$.

Conversely, suppose $\inf_{x_1} \sup_{x_2} \ldots Q_{x_N} \Gamma(G; x_1, \ldots, x_N, n) \le \frac{1}{2}$. Since $G > 0$, for every $K \in \mathbb{N}$, $\inf_{x_1} \sup_{x_2} \ldots Q_{x_N} \Gamma_K(G; x_1, \ldots, x_N, n) \le \frac{1}{2}$. By Lemmas 7 and 9, for every $x_0 < K$,

$\inf_{x_1} \sup_{x_2} \ldots Q_{x_N} G(x_0, \ldots, x_N, n) = \frac{1}{2}$. Thus, $\sup_{x_0} \inf_{x_1} \sup_{x_2} \ldots Q_{x_N} G(x_0, \ldots, x_N, n) = \frac{1}{2}$. It follows from Lemma 6 that $n \in \vec{\forall} R^*$. Thus, by Lemma 8, $n \in \vec{\forall} R$. □

6.3 Fintary Diagram Results

We are now ready to pin down the complexity of the diagrams of a computably presented metric structure. We first consider the quantifier-free diagrams.

Proposition 11 (C., Goldbring, McNicholl, 2021)**.** *If $\mathfrak{M}$ is a computably presentable L-structure, then the closed quantifier-free diagram of $\mathfrak{M}$ is Π^0_1 and the open quantifier-free diagram of $\mathfrak{M}$ is Σ^0_1.*

Proof. The proposition follows from the observation that if $\mathfrak{M}$ is computably presentable, then the map $\varphi \mapsto \varphi^{\mathfrak{M}}$ is computable on the set of quantifier-free sentences of L. □

We note that the proof of Proposition 11 is uniform; that is, from an index of a presentation of $\mathfrak{M}$, it is possible to compute a Π^0_1 index of the closed quantifier-free diagram of $\mathfrak{M}$ and a Σ^0_1 index of the open quantifier-free diagram of $\mathfrak{M}$.

We now consider the higher-level diagrams.

Theorem 11 (C., Goldbring, McNicholl, 2021)**.** *Let $\mathfrak{M}$ be a computably presentable L-structure, and let N be a positive integer.*

1. *The closed Π_N diagram of $\mathfrak{M}$ is Π^0_N, and the open Π_N diagram of $\mathfrak{M}$ is Σ^0_{N+1}.*

2. *The closed Σ_N diagram of $\mathfrak{M}$ is Π^0_{N+1}, and the open Σ_N diagram of $\mathfrak{M}$ is Σ^0_N.*

Moreover, the results of (1) and (2) hold uniformly in the sense that from N and an index for a computable presentation for $\mathfrak{M}$, one can compute an index for any of the above diagrams.

Proof. Throughout this proof, we fix a computable presentation $\mathfrak{M}^\sharp$ of $\mathfrak{M}$. We proceed by induction on N, the base case being true by Proposition 11. We now fix a positive integer N and assume that (1) and (2) hold uniformly for every $M < N$.

Fix a Π_N sentence φ and a rational number q. Note that φ has the form $\sup_{\overline{x}} \psi$, where ψ is a Σ_{N-1} wff of L and $\overline{x}$ is a tuple of variables. Since the rational points of $\mathfrak{M}^\sharp$ are dense, $\sup_{\overline{a} \in \mathbb{Q}(\mathfrak{M}^\sharp)} \psi^{\mathfrak{M}}(\overline{a}) = \sup_{\overline{a} \in |\mathfrak{M}|} \psi^{\mathfrak{M}}(\overline{a})$. Thus,

$$\varphi^{\mathfrak{M}} \leq q \quad \Longleftrightarrow \quad (\forall k \in \mathbb{N})\ (\forall \overline{a} \in \mathbb{Q}(\mathfrak{M}^\sharp))\ \psi^{\mathfrak{M}}(\overline{a}) \leq q + 2^{-k}.$$

If $N = 1$, then by the uniformity of Proposition 11, the statement $\psi^{\mathfrak{M}}(\overline{a}) \leq q + 2^{-k}$ is a Π^0_1 condition on $\varphi, \overline{a}, k$. If $N > 1$, then this statement is a Π^0_N condition since (2) is assumed to hold uniformly for $M < N$. In either case it then follows that $\varphi^{\mathfrak{M}} \leq q$ is a Π^0_N condition on φ, q.

Furthermore,

$$\varphi^{\mathfrak{M}} < q \iff (\exists k \in \mathbb{N})\ (\forall \overline{a} \in \mathbb{Q}(\mathfrak{M}^{\sharp}))\ \psi^{\mathfrak{M}}(\overline{a}) \leq q - 2^{-k}.$$

As before, if $N = 1$, then the statement $\psi^{\mathfrak{M}}(\overline{a}) \leq q - 2^{-k}$ is a Π^0_1 condition on $\varphi, \overline{a}, k$. If $N > 1$, then this statement is a Π^0_N condition since (2) is assumed to hold uniformly for $M < N$. In either case, it follows that $\varphi^{\mathfrak{M}} < q$ is a Σ^0_{N+1} condition on φ, q.

Now fix a Σ_N sentence φ and a rational number q. Then φ has the form $\inf_{\overline{x}} \psi$, where ψ is a Π_{N-1} wff of L and $\overline{x}$ is a tuple of variables. Again, since the rational points of $\mathfrak{M}^{\sharp}$ are dense, $\inf_{\overline{a} \in \mathbb{Q}(\mathfrak{M}^{\sharp})} \psi^{\mathfrak{M}}(\overline{a}) = \inf_{\overline{a} \in |\mathfrak{M}|} \psi^{\mathfrak{M}}(\overline{a})$. Thus,

$$\varphi^{\mathfrak{M}} \leq q \iff (\forall k \in \mathbb{N})\ (\exists \overline{a} \in \mathbb{Q}(\mathfrak{M}^{\sharp}))\ \psi^{\mathfrak{M}}(\overline{a}) < q + 2^{-k}.$$

If $N = 1$, then the statement $\psi^{\mathfrak{M}}(\overline{a}) < q + 2^{-k}$ is a Σ^0_1 condition on $\varphi, \overline{a}, k$. If $N > 1$, then this statement is a Σ^0_N condition since (1) is assumed to hold uniformly for $M < N$. In either case, it then follows that $\varphi^{\mathfrak{M}} \leq q$ is a Π^0_{N+1} condition on φ, q.

Finally,

$$\varphi^{\mathfrak{M}} < q \iff (\exists k \in \mathbb{N})\ (\exists \overline{a} \in \mathbb{Q}(\mathfrak{M}^{\sharp}))\ \psi^{\mathfrak{M}}(\overline{a}) < q - 2^{-k}.$$

If $N = 1$, then the statement $\psi^{\mathfrak{M}}(\overline{a}) < q - 2^{-k}$ is a Σ^0_1 condition on $\varphi, \overline{a}, k$. If $N > 1$, then this statement is a Σ^0_N condition since (1) is assumed to hold uniformly for $M < N$. In either case, it then follows that $\varphi^{\mathfrak{M}} < q$ is a Σ^0_N condition on φ, q.

Finally, we note that these arguments are uniform in the sense described above. □

We will now show that these results are the best possible, which is not as trivial as intuition may grant. Since structures in continuous logic must be bounded, it might seem that the unit interval is a natural setting in which to construct these lower bounds. The following, however, prevents this from being the case.

Proposition 12 (C., Goldbring, McNicholl, 2021)**.** *Let $\mathfrak{M}^\sharp$ be a computably compact computable presentation of an L-structure $\mathfrak{M}$. Then the open diagram of $\mathfrak{M}$ is Σ^0_1 and the closed diagram of $\mathfrak{M}$ is Π^0_1.*

Proof. Since $\mathfrak{M}$ is computably presentable, the map $(a_0, ..., a_N) \mapsto \theta^{\mathfrak{M}}(a_0, ..., a_N)$ is computable, uniformly in a predicate θ. It now follows from a standard result in computable analysis (see (31) and Corollary 6.2.5 of (34)) that since $\mathfrak{M}^\sharp$ is computably compact, the map $\theta \mapsto \left(Q_{x_0}...Q_{x_N}\theta\right)^{\mathfrak{M}}$ is also computable. The result follows. □

On the other hand, one may notice that true arithmetic, translated into continuous logic via the discrete metric, would provide *some* lower bounds. While this is the case, it fails to capture the lower bounds on the open Π_N diagrams and closed Σ_N diagrams. We thus attain our optimal bounds by constructing a nontrivial structure which extends the natural numbers under the discrete metric, applying our combinatorial results.

Theorem 12 (C., Goldbring, McNicholl, 2021)**.** *There is a language L' and a computably presentable L'-structure $\mathfrak{M}$ with the following properties:*

1. *The closed quantifier-free diagram of $\mathfrak{M}$ is Π^0_1-complete, and the open quantifier-free diagram of $\mathfrak{M}$ is Σ^0_1-complete.*

2. *For every positive integer N, the closed Π_N diagram of $\mathfrak{M}$ is Π^0_N-complete, and the open Π_N diagram of $\mathfrak{M}$ is Σ^0_{N+1}-complete.*

3. *For every positive integer N, the closed Σ_N diagram of $\mathfrak{M}$ is Π^0_{N+1}-complete, and the open Σ^0_N diagram of $\mathfrak{M}$ is Σ^0_N-complete.*

Proof. Let L' be the metric language that consists of the following.

1. A constant symbol $\underline{0}$.

2. A family of unary predicate symbols $(C_n)_{n\in\mathbb{N}}$.

3. A family of predicate symbols $(P_{N,n})_{N,n\in\mathbb{N}}$, where $P_{2N,n}$ and $P_{2N+1,n}$ are $(N+1)$-ary.

Here, each predicate symbol is assumed to have modulus of continuity equal to the constant function 1.

We now define our L'-structure $\mathfrak{M}$. The underlying metric space of $\mathfrak{M}$ is the set $\mathbb{N}$ of natural numbers equipped with its discrete metric. We also set $\underline{0}^{\mathfrak{M}} = 0$. In order to define the interpretations of the other symbols, we first set

$$f_N = \begin{cases} \Gamma(1 - \frac{1}{2}\chi_{R_N^*}) & N \text{ even} \\ \Gamma(\frac{1}{2}\chi_{(\neg R_N)^*}) & \text{otherwise.} \end{cases}$$

Also, for every $a \in \mathbb{N}$, set

$$C_n^{\mathfrak{M}}(a) = \begin{cases} f_0(n/2) & n \text{ even} \\ f_1((n-1)/2) & \text{otherwise.} \end{cases}$$

That is, for every $n \in \mathbb{N}$, $C_n^{\mathfrak{M}}$ is a *constant* unary predicate of the given truth value. Finally, set $P_{2N,n}^{\mathfrak{M}}(a_0, \ldots, a_N) = f_{2N+2}(a_0, \ldots, a_N, n)$, and let $P_{2N+1,n}^{\mathfrak{M}}(a_0, \ldots, a_N) = f_{2N+3}(a_0, \ldots, a_N, n)$.

It is clear that $\mathfrak{M}$ has a computable presentation. In fact, one may simply take the n-th distinguished point to be n.

We first note that the closed atomic diagram of $\mathfrak{M}$ is Π_1^0-complete. To see this, let φ_n be the sentence $C_{2n}(\underline{0})$. Then, by Theorem 10, $\varphi_n^{\mathfrak{M}} \leq \frac{1}{2}$ if and only if $n \in \vec{\forall} R_0$.

Similarly, the open atomic diagram of $\mathfrak{M}$ is Σ_1^0-complete. This time, let φ_n be the sentence $C_{2n+1}(\underline{0})$. Then, by Theorem 10, $\varphi_n^{\mathfrak{M}} < \frac{1}{2}$ if and only if $n \in \vec{\exists} R_0$.

Next fix a positive integer N. For each $n \in \mathbb{N}$, let φ_n be the sentence

$$\inf_{x_1} \ldots Q_{x_N} P_{2N,n}(x_1, \ldots, x_N),$$

and let ψ_n be the sentence

$$\sup_{x_1} \ldots Q_{x_N} P_{2N+1,n}(x_1, \ldots, x_N).$$

By Theorem 10, $\varphi_n^{\mathfrak{M}} \leq \frac{1}{2}$ if and only if $n \in \vec{\forall} R_{2N}$. Thus, the closed Σ_N diagram of $\mathfrak{M}$ is Π_{N+1}^0-complete. Also by Theorem 10, $\psi_n^{\mathfrak{M}} < \frac{1}{2}$ if and only if $n \in \vec{\exists} R_{2N+1}$. Thus, the open Π_N diagram of $\mathfrak{M}$ is Σ_{N+1}^0-complete.

Since the open Π_{N-1} diagram of $\mathfrak{M}$ is Σ_N^0-complete, it follows that the open Σ_N diagram of $\mathfrak{M}$ is Σ_N^0-complete. It similarly follows that the closed Π_N diagram of $\mathfrak{M}$ is Π_N^0-complete. $\square$

6.4 Infinitary Diagram Results

When formulating our diagram complexity results for infinitary logic, we actually must give up the terminology of "diagrams". This is because the coding of the computable infinitary wffs would make the actual diagrams capable of computing O, which itself is Π^1_1-complete. In order to avoid this pitfall, we focus on the complexity of the right Dedekind cuts of reals of the form $\varphi^{\mathfrak{M}}$ where φ is infinitary.

We first prove our infinitary upper bound result which generalizes our bounds in the finitary case.

Theorem 13 (C., Goldbring, McNicholl, 2021)**.** *Let $\mathfrak{M}$ be a computably presentable L-structure and let φ be a computable infinitary sentence of L.*

1. *If φ is Π^c_α, then $D^>(\varphi^{\mathfrak{M}})$ is $\Sigma^0_{\alpha+1}$ uniformly in a code of φ, and $D^{\geq}(\varphi^{\mathfrak{M}})$ is Π^0_α uniformly in a code of φ.*

2. *If φ is Σ^c_α, then $D^>(\varphi^{\mathfrak{M}})$ is Σ^0_α uniformly in a code of φ, and $D^{\geq}(\varphi^{\mathfrak{M}})$ is $\Pi^0_{\alpha+1}$ uniformly in a code of φ.*

Proof. Fix a computable presentation $\mathfrak{M}^\sharp$ of $\mathfrak{M}$. Let φ be a computable infinitary sentence of L.

Suppose $\varphi \in \Sigma^c_\alpha \cup \Pi^c_\alpha$. A code for φ yields a notation a for α. In the following, all other ordinals considered are less than α. For ease of exposition, we identify each $\beta \leq \alpha$ with its unique notation in $\{b \ : \ b \leq_O a\}$.

We proceed by effective transfinite recursion. Thus, we assume the following hold uniformly in an index of $\mathfrak{M}^\#$.

1. From a $\beta < \alpha$ and a code of a Π^c_β sentence ψ, it is possible to compute a Π^0_β index of $D^{\geq}(\psi^{\mathfrak{M}})$ and a $\Sigma^0_{\beta+1}$-index of $D^>(\psi^{\mathfrak{M}})$.

2. From a $\beta < \alpha$ and a code of a Σ^c_β sentence ψ, it is possible to compute a $\Pi^0_{\beta+1}$-index of $D^{\geq}(\psi)$ and a Σ^0_β-index of $D^>(\psi_i)$.

First suppose that φ is a Π^c_α sentence. Thus, φ has the form $\bigwedge\!\!\!\bigwedge_{i\in I} \sup_{\vec{x_i}} \psi_i$ where I is c.e. and ψ_i is $\Sigma^c_{\beta_i}$ for some $\beta_i < \alpha$. Furthermore, we may assume $(\beta_i)_{i\in I}$ is computable. For $q \in \mathbb{Q}$, we have

$$q \in D^{\geq}(\varphi^{\mathfrak{M}}) \Leftrightarrow (\forall k \in \mathbb{N})(\forall i \in I)(\forall \vec{r} \in \mathbb{Q}(\mathfrak{M}^{\#}))\ q + 2^{-k} \in D^{>}(\psi_i^{\mathfrak{M}}(\vec{r})).$$

As $\emptyset^{(\alpha)}$ computes $D^{>}(\psi_i^{\mathfrak{M}}(\vec{r}))$ uniformly in i, $D^{\geq}(\varphi^{\mathfrak{M}})$ is co-c.e. in $\emptyset^{(\alpha)}$, that is, $D^{\geq}(\varphi^{\mathfrak{M}})$ is Π^0_α. At the same time,

$$q \in D^{>}(\varphi^{\mathfrak{M}}) \iff (\exists k \in \mathbb{N})(\forall i \in I)(\forall \vec{r} \in \mathbb{Q}(\mathfrak{M}^{\#}))\ q - 2^{-k} \notin D^{>}(\psi_i^{\mathfrak{M}}(\vec{r})).$$

Thus, $D^{>}(\varphi^{\mathfrak{M}})$ is $\Sigma^0_2(\emptyset^{(\alpha)}) = \Sigma^0_{\alpha+1}$.

Now suppose φ is a Σ^c_α sentence. Thus, φ has the form $\bigvee\!\!\!\bigvee_{i\in I} \inf_{\vec{x_i}} \psi_i$ where I is c.e. and ψ_i is $\Pi^c_{\beta_i}$ for some $\beta_i < \alpha$ uniformly in i. Let $q \in \mathbb{Q}$. Then,

$$q \in D^{\geq}(\varphi^{\mathfrak{M}}) \iff (\forall k \in \mathbb{N})(\exists i \in I)(\exists \vec{r} \in \mathbb{Q}(\mathfrak{M}^{\#}))q + 2^{-k} \in D^{>}(\psi_i^{\mathfrak{M}}(\vec{r})).$$

Thus, $D^{\geq}(\varphi^{\mathfrak{M}})$ is $\Sigma^0_2(\emptyset^{(\alpha)}) = \Sigma^0_{\alpha+1}$. In addition,

$$q \in D^{>}(\psi_{i_0}^{\mathfrak{M}}) \iff (\exists k \in \mathbb{N})(\exists i \in I)(\exists \vec{r} \in \mathbb{Q}(\mathfrak{M}^{\#}))\ q - 2^{-k} \notin D^{>}(\psi_i^{\mathfrak{M}}(\vec{r})).$$

Thus, $D^{>}(\varphi^{\mathfrak{M}})$ is $\Sigma^0_1(\emptyset^{(\alpha)}) = \Sigma^0_\alpha$.

As these arguments are all uniform in an index of $\mathfrak{M}^{\#}$ and a code for φ, the theorem is proven. □

We now demonstrate the optimality of Theorem 13 by means of the following.

Theorem 14 (C., Goldbring, McNicholl, 2021)**.** *There is a language L'' and an L''-structure $\mathfrak{M}$ so that the following hold for every computable ordinal α.*

1. *There is a computable sequence $(\psi_i)_{i\in\mathbb{N}}$ of Π^c_α sentences of L'' so that $\{i\ :\ \frac{1}{2} \in D^{\geq}(\psi_i^{\mathfrak{M}})\}$ is Π^0_α-complete.*

2. *There is a computable sequence $(\psi_i)_{i\in\mathbb{N}}$ of Σ^c_α sentences of L'' so that $\{i\ :\ \frac{1}{2} \in D^{>}(\psi_i^{\mathfrak{M}})\}$ is Σ^0_α-complete.*

3. *There is a computable sequence* $(\psi_i)_{i\in\mathbb{N}}$ *of* Π^c_α *sentences of* L'' *so that* $\{i\ :\ \frac{1}{2} \in D^{>}(\psi_i^{\mathfrak{M}})\}$ *is* $\Sigma^0_{\alpha+1}$*-complete.*

4. *There is a computable sequence* $(\psi_i)_{i\in\mathbb{N}}$ *of* Σ^c_α *sentences of* L'' *so that* $\{i\ :\ \frac{1}{2} \in D^{\geq}(\psi_i^{\mathfrak{M}})\}$ *is* $\Pi^0_{\alpha+1}$*-complete.*

The remainder of this section is dedicated to the proof of Theorem 14. We begin with the construction of L'' and M''.

Let L_0 be a language consisting of one constant symbol $\underline{q}$ for every $q \in \mathbb{Q} \cap [0,1]$ and let $\mathfrak{M}_0$ be the L_0-structure whose underlying metric space is $[0,1]$ with its usual metric and which interprets each $\underline{q}$ as q. Let L'' be the expansion of L_0 obtained by adding a family $(c_{N,n,x_1,\ldots,x_{N+1}})_{N,n,x_1,\ldots,x_{N+1}\in\mathbb{N}}$ of constant symbols.

Let $\mathfrak{M}$ be the expansion of $\mathfrak{M}_0$ obtained by setting $c^{\mathfrak{M}}_{N,n,x_1,\ldots,x_{N+1}} = \frac{1}{2}(1 - \chi_{R_{2N+1}}(n, x_1, \ldots, x_{N+1}))$. Since $(R_N)_{N\in\mathbb{N}}$ is computable, it follows that $\mathfrak{M}$ is computably presentable.

We now verify that L'' and $\mathfrak{M}$ satisfy the conclusions of Theorem 14. We will need a little additional terminology and two lemmas.

Suppose $(\psi_i)_{i\in\mathbb{N}}$ is a sequence of Π^c_α sentences of L''. We say that a set S is *encoded* by $(\psi_i)_{i\in\mathbb{N}}$ if $\psi_i^{\mathfrak{M}} = 1 - \frac{1}{2}\chi_S(i)$ for all i.

Similarly, if $(\psi_i)_{i\in\mathbb{N}}$ is a sequence of Σ^c_α sentences of L'', we say that a set S is *encoded* by $(\psi_i)_{i\in\mathbb{N}}$ if $\psi_i^{\mathfrak{M}} = \frac{1}{2}(1 - \chi_S(i))$ for all i.

Lemma 10 (C., Goldbring, McNicholl, 2021)**.** *Let α be a computable ordinal.*

1. *Every* Σ^0_α *set is encoded by a computable sequence of* Σ^c_α *sentences.*

2. *Every* Π^0_α *set is encoded by a computable sequence of* Π^c_α *sentences.*

Proof. We prove (1). Part (2) then follows by considering complements. Suppose S is Σ^0_α.

If $\alpha = 0$, then we let

$$\psi_i = \begin{cases} d(\underline{0}, \underline{0}) & i \in S \\ d(\underline{0}, \underline{\tfrac{1}{2}}) & \text{otherwise.} \end{cases}$$

Next suppose $\alpha = N+1$ where $N \in \mathbb{N}$. Let

$$\psi_n = \bigvee_{x_1} \bigwedge_{x_2} \ldots \mathcal{C}_{x_{N+1}} d(c_{N,n,x_1,\ldots,x_{N+1}}, \underline{0}).$$

Here, $\mathcal{C}$ is $\bigvee$ if N is even and $\bigwedge$ if N is odd.

It follows from Lemma 6 that $(\psi_n)_{n\in\mathbb{N}}$ encodes $\vec{\exists} R_{2N+1}$. Since $\vec{\exists} R_{2N+1}$ is Σ^0_{N+1}-complete, it follows that every Σ^0_{N+1} set is encoded by a sequence of computable Σ^c_{N+1} sentences. Furthermore, the construction of such a sequence from a Σ^0_{N+1} index is uniform.

Suppose $\alpha \geq \omega$. Similar to the proof of Theorem 7.9 of (1), we construct a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of Σ^0_α sentences so that $\varphi_n^{\mathfrak{M}} = 1 - \chi_S(n)$. In particular, we replace $\top$ and $\bot$ with $d(\underline{0}, \underline{0})$ and $d(\underline{0}, \underline{1})$ respectively. Setting $\psi_n = \frac{1}{2}\varphi_n$ yields the desired formulae. □

Lemma 11 (C., Goldbring, McNicholl, 2021)**.** *If $(\psi_n)_{n\in\mathbb{N}}$ is a computable sequence of Π^c_α sentences of L'', then there is a computable Π^c_α sentence φ of L'' so that*

$$\varphi^{\mathfrak{M}} = \sum_{n=0}^{\infty} 2^{-(n+1)} \psi_n^{\mathfrak{M}}.$$

Furthermore, a code of φ can be computed from an index of $(\psi_n)_{n\in\mathbb{N}}$.

Proof. Follows directly from Proposition 9.

□

Proof of Theorem 14. Parts (1) and (2) follow directly from Lemma 10.

Now suppose S is $\Sigma^0_{\alpha+1}$ complete. Take a Π^0_α binary relation R so that $S = \vec{\exists} R$. By Lemma 10, there is a computable family $(\psi_{n,x_1})_{n,x_1\in\mathbb{N}}$ of Π^c_α sentences so that for all $n, x_1 \in \mathbb{N}$, $\psi^{\mathfrak{M}}_{n,x_1} = 1 - \frac{1}{2}\chi_R(n, x_1)$. By Lemma 11, there is a computable sequence $(\varphi_n)_{n\in\mathbb{N}}$ of Π^c_α sentences so that

$$\varphi_n^{\mathfrak{M}} = \sum_{x_1=0}^{\infty} 2^{-(x_1+2)} \psi^{\mathfrak{M}}_{n,x_1}.$$

It then follows that $n \in S$ if and only if $\frac{1}{2} \in D^{>}(\varphi_n^{\mathfrak{M}})$, establishing (3). Part (4) follows by considering complements. □

Returning to an earlier point, we note that the closed and open quantifier-free diagrams of $\mathfrak{M}$ are Π^0_1-complete and Σ^0_1-complete, respectively. To see this, fix a Σ^0_1 complete set C, and let $(c_s)_{s=0}^{\infty}$ be an effective enumeration of C. Since C is infinite, we may assume this enumeration is one-to-one. Let

$$p_n = \begin{cases} \frac{1}{2} - 2^{-s} & \text{if } n = c_s \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

It is fairly straightforward to show that $(p_n)_{n\in\mathbb{N}}$ is computable as a sequence of reals. Furthermore, $p_n < \frac{1}{2}$ if and only if $n \in C$. Since L'' contains a constant symbol for each rational number, it follows that the open quantifier-free diagram of $\mathfrak{M}$ is Σ^0_1-complete. The Π^0_1-completeness of the closed quantifier-free diagram follows by considering complements.

We also note that while computably compact domains are insufficient for demonstrating lower bounds in the finitary case, $[0,1]$ works extremely well in the infinitary case. This is because we can build up all the required infinitary sentences without variables, so the computability of extrema plays no role.

CHAPTER 7. CONCLUSION

7.1 Summary

We briefly recall and summarize our three main results.

Theorem 8. *There is an effective procedure which, given a name $X \in \mathbb{N}^{\mathbb{N}}$ of an L-theory T, produces a presentation of an L^{+}-structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$.*

This theorem generalized and extended the results of (8) and (13), taking in any name of an L-theory and outputting a presentation of an L^{+}-structure of the same degree. It was proven using four model-theoretic propositions and a lemma which allows one to effectively extend a theory to a complete theory.

Theorem 9. (C., McNicholl, 2022+) *There are partial computable functions f and g from $(\mathrm{O} \setminus \{1\}) \times \mathbb{N}$ to the set of computable infinitary sentences such that the following hold, for every $1 \leq \alpha < \omega_1^{\mathrm{CK}}$.*

- *If $\langle a, i\rangle$ is a Π^0_α index of a right Dedekind cut of a real number s, then $f(a,i)$ is a code of a Π^c_α sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.*
- *If $\langle a, i\rangle$ is a Σ^0_α index of a right Dedekind cut of a real number s, then $g(a,i)$ is a code of a Σ^c_α sentence φ such that for every L-structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = s$.*

That is, while every real number is trivially represented by an infinitary sentence, the computable infinitary sentences are still strong enough to represent any hyperarithmetical number as a numeral. This implies that a large amount of complexity is carried in every continuous structure from the connectives and the space of truth values, alone. The result was proven via a lemma applying effective transfinite recursion and properties of Dedekind cuts.

Theorems 11, 12, 13, and 14. (C., Goldbring, McNicholl, 2021) *Let $\mathfrak{M}$ be a computably presentable L-structure, and let N be a positive integer.*

1. The closed Π_N diagram of $\mathfrak{M}$ is Π^0_N, and the open Π_N diagram of $\mathfrak{M}$ is Σ^0_{N+1}.

2. The closed Σ_N diagram of $\mathfrak{M}$ is Π^0_{N+1}, and the open Σ_N diagram of $\mathfrak{M}$ is Σ^0_N.

These results hold uniformly in the sense that from N and an index for a computable presentation for $\mathfrak{M}$, one can compute an index for any of the above diagrams. Moreover, there is an L-structure in which these bounds are optimal, and similar correlates hold in the infinitary case.

The finitary optimal bound cannot be realized on computably compact spaces. Because of this, it was proven through a structure over the natural numbers along with a novel combinatorial result, which allows for the encoding of a quantifier via a series inequality. The infinitary optimal bound is easier to realize, as the infinite connectives allow one to quantify over indexed sentences.

Let us know examine some applications of these results.

7.2 Applications and Future Research

First note that the theory of a metric structure may be thought of not as a set (as is it in the discrete setting), but as a map into $[0, 1]$. We gain two immediate corollaries from the results in Chapter 6.

Corollary 6 (C., Goldbring, McNicholl, 2021)**.** *Let $\mathfrak{M}$ be an L-structure with a computably compact computable presentation. Then the theory of $\mathfrak{M}$ is Δ^0_2.*

Corollary 7 (C., Goldbring, McNicholl, 2021)**.** *Let $\mathfrak{M}$ be an L-structure with a hyperarithmetic presentation. Then the theory of $\mathfrak{M}$ is also hyperarithmetic.*

Corollary 6 follows directly from Proposition 12 and Corollary 7 from Theorem 11. An application of the former may be made to computable Stone spaces. Notably, in (22), it was shown that any computable Stone space has a computably compact presentation. We thus attain the following.

Corollary 8 (C., Goldbring, McNicholl, 2021)**.** *Let $\mathcal{X}$ be a computable Stone space. Then the (continuous) theory of $\mathcal{X}$ is Δ^0_2.*

Meanwhile, Corollary 7 has already been applied in (19) in the proof of the following.

Theorem 15 (Theorem 1.1, Goldbring, Hart, 2020)**.** *The following operator algebras have hyperarithmetic theory.*

1. *The hyperfinite II_1 factor $\mathcal{R}$.*
2. *$L(\Gamma)$ for Γ a finitely generated group with solvable word problem.*
3. *$C^*(\Gamma)$ for Γ a finitely presented group.*
4. *$C^*_\lambda(\Gamma)$ for Γ a finitely generated group with solvable word problem.*

In their manuscript, Goldbring and Hart show, via the results of (17), that each of the above have hyperarithmetic presentations. They then apply Corollary 7 to conclude that each of the above have hyperarithmetic theory. Through a careful analysis of the complexity of model-theoretic forcing in continuous logic, they also show that the Cuntz algebra has an arithmetic presentation and hence a hyperarithmetic presentation.

We see two main avenues for further research. The first concerns the hyperfinite II_1 factor $\mathcal{R}$. As it stands, the structure used in the proof of Theorem 12 was somewhat contrived. But in (20), the authors prove that the open Π_1 diagram of $\mathcal{R}$ is not computable. This lends credence to the conjecture that $\mathcal{R}$ is a natural structure which realizes the optimal bounds of Theorem 12.

The second avenue for further research considers the translation of notion of a predicate (relation) on a metric structure being *relatively intrinsically computably enumerable* (r.i.c.e.) into the continuous setting. Similarly, a notion of being Σ_1^c definable in a structure with parameters could also be translated. The ultimate goal of this project would be to find a continuous correlate of the result of (10) and (2) that a relation is r.i.c.e. if and only if it is Σ_1^c definable with parameters. The issues arising concerning this avenue include the continuous nature of definability and the extra complexity which can be carried with parameters from spaces with uncountably many points.

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APPENDIX. SOLUTIONS TO SELECTED EXERCISES

Solution (Exercise 3)**.** Let $\mathfrak{M}$ be an L-pre-structure and σ any assignment. We only sketch canonicity (the forward direction) and leave soundness to the reader.

(a) Suppose $\mathfrak{M}$ satisfies schemata XI, XII, and XIII. For notational simplicity, let $d := \underline{d}^{\mathfrak{M}}$. That $d : |\mathfrak{M}|^2 \to [0,1]$ is given. Then fix $a, b, c \in |\mathfrak{M}|$.

Since $\mathfrak{M}$ satisfies XI,

$$d(a,a) = \big(\underline{d}(x,x)\big)^{\mathfrak{M},\sigma(x\mapsto a)} = 0.$$

Thus d is reflexive.

Now notice

$$d(a,b) = \big(\underline{d}(x,y)\big)^{\mathfrak{M},\sigma((x,y)\mapsto(a,b))} \qquad \text{and} \qquad d(b,a) = \big(\underline{d}(y,x)\big)^{\mathfrak{M},\sigma((x,y)\mapsto(a,b))}.$$

But since $\mathfrak{M}$ satisfies XII,

$$\big(\underline{d}(x,y) \dot{-} \underline{d}(y,x)\big)^{\mathfrak{M},\sigma((x,y)\mapsto(a,b))} = 0$$

which implies

$$d(a,b) \leq d(b,a).$$

The argument works symmetrically under the assignment $\sigma((x,y) \mapsto (b,a))$, implying

$$d(b,a) \leq d(a,b).$$

It follows that $d(a,b) = d(b,a)$. Thus d is symmetric.

Notice, lastly, that

$$d(a,c) = \big(\underline{d}(x,z)\big)^{\mathfrak{M},\sigma((x,y,z)\mapsto(a,b,c))}, \quad d(a,b) = \big(\underline{d}(x,y)\big)^{\mathfrak{M},\sigma((x,y,z)\mapsto(a,b,c))},$$

$$\text{and} \quad d(b,c) = \big(\underline{d}(y,z)\big)^{\mathfrak{M},\sigma((x,y,z)\mapsto(a,b,c))}.$$

And since $\mathfrak{M}$ satisfies XIII,

$$\left(\left(\underline{d}(x,z) \dot{-} \underline{d}(x,y)\right) \dot{-} \underline{d}(y,z)\right)^{\mathfrak{M},\sigma((x,y,z)\mapsto(a,b,c))} = 0$$

which implies (literally) that

$$\max\left\{d(a,b) - d(b,c), 0\right\} \le d(a,c).$$

But notably, this also implies that

$$d(a,b) - d(b,c) \le d(a,c)$$

and hence by addition and the symmetry of d,

$$d(a,b) \le d(a,c) + d(c,b).$$

Thus d satisfies the triangle inequality. Therefore, d is a pseudometric.

(b) Fix $f \in \mathcal{F}$, $n \in \mathbb{N}$, and $a, b \in |\mathfrak{M}|$ such that $f^{\mathfrak{M}}(a,b) < 2^{-\Delta(f;n)}$. This means that its *not* the case that $2^{-\Delta(f;n)} \le f^{\mathfrak{M}}(a,b)$. Hence

$$\mathfrak{M}, \sigma((x,y) \to (a,b)) \not\models \underline{2^{-\Delta(f;n)}} \dot{-} \underline{d}(x,y).$$

But since $\mathfrak{M}$ satisfies XIV, for every pair of tuples of terms $\vec{t_0}$ and $\vec{t_1}$, we have

$$\mathfrak{M}, \sigma((x,y) \to (a,b)) \models \left(\underline{2^{-\Delta(f;n)}} \dot{-} \underline{d}(x,y)\right) \wedge \left(\underline{d}\big(f(\vec{t_0}, x, \vec{t_1}), f(\vec{t_0}, y, \vec{t_1})\big) \dot{-} \underline{2^{-n}}\right).$$

So it must be that

$$\mathfrak{M}, \sigma((x,y) \to (a,b)) \models \underline{d}\big(f(\vec{t_0}, x, \vec{t_1}), f(\vec{t_0}, y, \vec{t_1})\big) \dot{-} \underline{2^{-n}}.$$

It follows via interpretation that

$$d\big(f^{\mathfrak{M}}(\vec{t_0}^{\mathfrak{M}}, a, \vec{t_1}^{\mathfrak{M}}), f^{\mathfrak{M}}(\vec{t_0}^{\mathfrak{M}}, b, \vec{t_1}^{\mathfrak{M}})\big) \le 2^{-n}.$$

Since this holds for every pair of tuples of terms, and every $n \in \mathbb{N}$, it follows that $\Delta(f)$ is a modulus of continuity for $f^{\mathfrak{M}}$.

(c) Similar to (b), *mutatis mutandis*.

Solution (Exercise 4)**.** Let $\mathfrak{A}$ be a classical structure and $\mathrm{MT}(\mathfrak{A})$ its metric transformation. We proceed by structural induction on classical wffs.

First suppose $\varphi = P(t_0, ..., t_{\eta(P)-1})$. Then

$$\begin{aligned}
\mathfrak{A} \vDash_{\mathrm{FOL}} \varphi &\iff (t_0^{\mathfrak{A}}, ..., t_{\eta(P)-1}^{\mathfrak{A}}) \in P^{\mathfrak{A}} \\
&\iff 1 - \chi_{P^{\mathfrak{A}}}(t_0^{\mathfrak{A}}, ..., t_{\eta(P)-1}^{\mathfrak{A}}) = 0 \\
&\iff \mathrm{MT}(\mathfrak{A}) \vDash P(t_0, ..., t_{\eta(P)-1}) \\
&\iff \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\varphi).
\end{aligned}$$

Now suppose $\varphi = \neg\psi$. Then

$$\begin{aligned}
\mathfrak{A} \vDash_{\mathrm{FOL}} \varphi &\iff \mathfrak{A} \vDash_{\mathrm{FOL}} \neg\psi \\
&\iff \mathfrak{A} \nvDash_{\mathrm{FOL}} \psi \\
&\iff \mathrm{MT}(\mathfrak{A}) \nvDash \mathrm{CT}(\psi) \\
&\iff \mathrm{MT}(\mathfrak{A}) \vDash \neg\mathrm{CT}(\psi) \\
&\iff \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\varphi).
\end{aligned}$$

Then suppose $\varphi = \psi \to \theta$. It follows that

$$\begin{aligned}
\mathfrak{A} \vDash_{\mathrm{FOL}} \varphi &\iff \mathfrak{A} \vDash_{\mathrm{FOL}} \psi \to \theta \\
&\iff \big(\mathfrak{A} \vDash_{\mathrm{FOL}} \psi \implies \mathfrak{A} \vDash_{\mathrm{FOL}} \theta\big) \\
&\iff \big(\mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\psi) \implies \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\theta)\big) \\
{}^{*} &\iff \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\theta) \dot{-} \mathrm{CT}(\psi) \\
&\iff \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\varphi).
\end{aligned}$$

Notably, the reverse direction of the biconditional with an asterisk follows from the Completeness Theorem (Theorem 2).

Lastly, suppose $\varphi = \forall x\ \psi(x)$. Then

$$\begin{aligned}
\mathfrak{A} \vDash_{\mathrm{FOL}} \varphi &\iff \mathfrak{A} \vDash_{\mathrm{FOL}} \forall x\ \psi(x) \\
&\iff \forall a \in |\mathfrak{A}|\ \mathfrak{A} \vDash_{\mathrm{FOL}} \psi(\underline{a}) \\
&\iff \forall a \in |\mathfrak{A}|\ \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\psi(\underline{a})) \\
&\iff \forall a \in |\mathfrak{A}|\ \mathrm{MT}(\mathfrak{A}), \sigma(x \mapsto a) \vDash \mathrm{CT}(\psi(x)) \\
&\iff \mathrm{MT}(\mathfrak{A}) \vDash \sup_x \mathrm{CT}(\psi(x)) \\
&\iff \mathrm{MT}(\mathfrak{A}) \vDash \mathrm{CT}(\varphi).
\end{aligned}$$

The result follows.

Solution (Exercise 8)**.** The forward direction is clear. To see the reverse, let $A \subseteq \mathbb{N}$ be c.e. and co-c.e. Given $k \in \mathbb{N}$, begin enumerating A and A^c. Because $A \cap A^c = \emptyset$ and $A \cup A^c = \mathbb{N}$, at some step, k will be enumerated into exactly one of A or A^c. If it was enumerated into A, output a 1. If it was enumerated into A^c, output a 0. It follows that A is computable.

Solution (Exercise 9)**.** To see that K is c.e., at stage $s \in \mathbb{N}$, run the computations $\Phi_0(0)$, $\Phi_1(1)$, ..., and $\Phi_s(s)$ for s-many steps. For every computation which halts in s-many steps, enumerate the index of that computation into K.

To see that K is not computable, we proceed by way of contradiction, *i.e.*, suppose K is computable. It follows that K^c is computable, and hence c.e. Therefore, there is some Turing machine Φ_e such that $\Phi_e(k)\downarrow$ if and only if $k \in K^c$. Then consider $\Phi_e(e)$. If $\Phi_e(e)\downarrow$, then $e \in K^c$ by the above. But by definition, then, $e \in W_e$, so $e \in K$. $\rightarrow\leftarrow$.

On the other hand, if $\Phi_e(e)$ does not halt, then $e \notin K^c$, by the above. But this implies that $e \in K$, so $e \in W_e$, and hence $\Phi_e(e)\downarrow$! $\rightarrow\leftarrow$

Therefore, K is not computable.

Solution (Exercise 10)**.** To see that $K^c \leq_{\mathrm{T}} K$, notice we may perform the following oracle computation with access to K. Given $k \in \mathbb{N}$, check if $k \in K$. If it is, output a 0. If not, output a 1. This K-effective procedure computes K^c.

To see that $K^c \not\leq_{\mathrm{m}} K$, by way of contradiction, suppose it was. Then there is some Turing machine Φ with domain $\mathbb{N}$ such that

$$k \in K^c \iff \Phi(k) \in K.$$

We proceed to effectively enumerate the members of K^c. At stage $s \in \mathbb{N}$, compute $\Phi(0), ..., \Phi(s)$. Then check if any of these has been enumerated into K after s computation steps. For each one that has, enumerate its argument into K^c. This implies K^c is c.e. But by the solution to Exercise 9, we know that K^c is not c.e. $\rightarrow\leftarrow$

Therefore, $K^c \not\leq_{\mathrm{m}} K$.

Solution (Exercise 13)**.** Define $E : \mathbb{N}^{\mathbb{N}} \to \mathscr{P}(\mathbb{N})$ as

$$E(X) := \left\{2^n \cdot 3^{X(n)} \ : \ n \in \mathbb{N}\right\}.$$

To see that E is injective, suppose $E(X) = E(Y)$. By the Fundemental Theorem of Arithmetic, this implies that for every $n \in \mathbb{N}$, $X(n) = Y(n)$. Therefore, $X = Y$.

Solution (Exercise 14)**.** (Sketch) Relativize the proof of Exercise 9 by proving the following.

(i) $\emptyset^{(n)}$ is c.e. in $\emptyset^{(n-1)}$.

(ii) $\emptyset^{(n)}$ is not computable in $\emptyset^{(n-1)}$.

Solution (Exercise 16)**.** Let α be a computable ordinal. By definition, there is a computable well-ordering $\left(|\mathfrak{A}|, <_{\mathfrak{A}}\right)$ of order type α. Let g be an effective numbering of $|\mathfrak{A}|$. Then, for every $n \in \mathbb{N}$, define $|\mathfrak{A}| \restriction_n = \{a \in |\mathfrak{A}| : a <_{\mathfrak{A}} g(n)\}$ and let $<_{\mathfrak{A}\restriction_n}$ be $<_{\mathfrak{A}}$ restricted to this set. It follows that for every $n \in \mathbb{N}$, $\left(|\mathfrak{A}| \restriction_n, <_{\mathfrak{A}\restriction_n}\right)$ is a computable structure. Now let $\beta < \alpha$. Since the order type of $\left(|\mathfrak{A}|, <_{\mathfrak{A}}\right)$ is α, there is some substructure $\left(|\mathfrak{B}|, <_{\mathfrak{A}\restriction\mathfrak{B}}\right) \subseteq \left(|\mathfrak{A}|, <_{\mathfrak{A}}\right)$ with order type β. Moreover, since $|\mathfrak{A}|$ is well-ordered, there is some least $a \in |\mathfrak{A}|$ such that for every $b \in |\mathfrak{B}|$, $b <_{\mathfrak{A}} a$. And since g is an effective numbering of $|\mathfrak{A}|$, there is some $N \in \mathbb{N}$ such that $g(N) = a$. It follows that $\left(|\mathfrak{A}| \restriction_N, <_{\mathfrak{A}\restriction_N}\right) = \left(|\mathfrak{B}|, <_{\mathfrak{A}\restriction\mathfrak{B}}\right)$, and thus is a computable well-ordering of order type β. Therefore, β is a computable ordinal. The result follows.

Solution (Exercise 17)**.** Since every computably well-ordering is witnessed by an effective procedure, and there are only countably-many effective procedures, there are only countably-many computable well-orderings. It follows that there are only countably-many computable ordinals. The result follows.

Solution (Exercise 18)**.** For the forward direction, suppose r is a computable real number. Then there is an effective procedure such that, given $k \in \mathbb{N}$, outputs a rational q such that

$$|r - q| \leq 2^{-k}.$$

Then for every $k \in \mathbb{N}$, define $\Phi(k)$ to be the least code of that q. It follows that Φ is a Turing machine and name of r.

The reverse direction is trivial.

Solution (Exercise 19)**.** (Sketch) Follows from the strings for $\underline{\frac{\ell}{2^{-k}}}$ given in Heuristic 1.