THE IMPEDANCE OF A LOOP NEAR A CONDUCTING HALF-SPACE

Afroz J. M. Zaman, Stuart A. Long and C. Gerald Gardner
Department of Electrical Engineering
University of Houston
Houston, Texas 77004

ABSTRACT

The change in complex impedance between an ideal one-turn circular coil located above and parallel to a conducting half-space with respect to a similar isolated coil has been calculated. From this result a series expansion of the integrand allows the solution to be approximated by terms expressed as complete elliptic integrals. Results have been calculated for the change in impedance as a function of the lift-off distance and the conductivity of the half-space for a coil of representative radius.

INTRODUCTION

The eddy current method of nondestructive evaluation entails the induction of eddy currents in a conductive test object by a time-varying field produced by a suitable distribution of impressed currents (via an excitation or primary coil), and the detection of the resultant field, usually by an inductive search coil which may be either a separate secondary coil or the primary coil itself. (See Fig. 1.) The method is ordinarily used at frequencies sufficiently low to neglect effects due to displacement current; hence a theoretical analysis entails calculating either a transfer impedance for a primary coil and secondary coil in the presence of the test object, or the calculation of the self impedance of a primary coil in the presence of the test object. In practice one often needs only the change in impedance produced by the test object or by changes in the nominal properties of the test object (e.g. changes in its geometry or position with respect to the test coil or coils, or distributed or localized changes in the resistivity of the test object). The most general case, allowing arbitrary configurations of primary and secondary coils and arbitrary test objects can be handled only by numerical methods. Certain idealized arrangements can be treated analytically either exactly or in useful approximation. In virtually all cases of practical interest, the analysis eventually reduces to the evaluation of certain integrals which cannot be expressed in closed form in terms of standard transcendental functions.

In this paper we discuss the case of a one-turn circular coil located above and parallel to the surface of a homogeneous conductive half-space. From the standard boundary value problem approach we obtain the general expression for the change in coil impedance, ΔZ, produced by the half space; ΔZ is given in terms of an integral over a separation parameter. A series expansion of one term in the integrand permits the integral to be expressed as a series of terms each of which is expressible in terms of complete elliptic integrals. The leading terms of this series approximate ΔZ asymptotically for sufficiently small values of skin depth of the halfspace.

The problem addressed here has previously been treated by Cheng [1] who evaluated ΔZ by numerical methods for various choices of the relevant parameters. Similarly, Dodd and Deeds [2] have devised a digital computer program capable of handling circular test coils in the presence of layered planar and coaxial cylindrical test objects. Such brute force numerical procedures are valuable for design purposes, but have the disadvantage of somewhat concealing the essentially simple manner in which the final result depends upon the parameters of the problem. The approach taken here, while less universal than the purely numerical approach; results in relatively simple, though approximate and restricted, formulas for ΔZ in terms of the basic parameters of the problem.

For illustrative and comparative purposes, some selected numerical examples are also given.
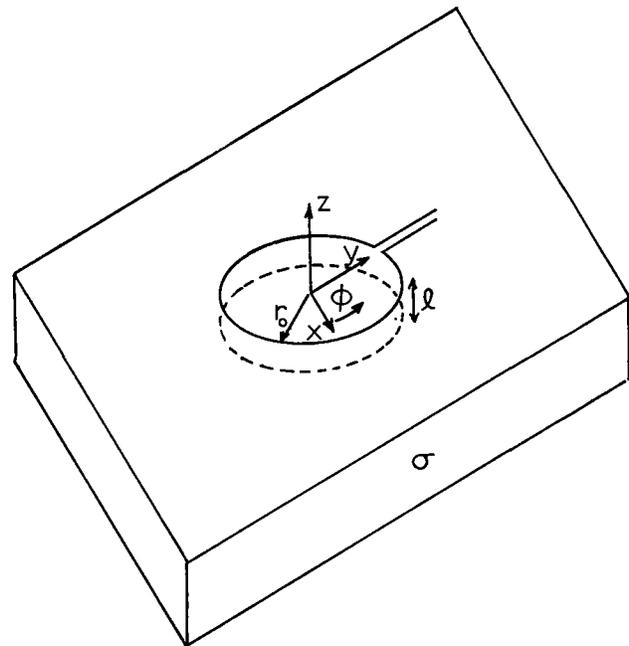

Fig. 1 Geometrical configuration of loop near a conductor.

THEORETICAL ANALYSIS

The basic geometry of the problem is shown in Fig. 1 and consists of a loop radius r_0 oriented parallel to and at a distance ℓ above homogeneous half-space of conductivity σ. Beginning with the basic equation for the vector potential

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$$\nabla^2\vec{A} + k^2\vec{A} = -\mu_0 I(t)\frac{\delta(r_0-r)}{r}\delta(\ell-z) \tag{1}$$

and noting the symmetry of the problem, it is seen that the only component of the vector potential present is the circumferential component, A_ϕ, and that A_ϕ is a function of r and z only. Making the usual low-frequency, quasi-static approximation that the $k^2\vec{A}$ term is negligible for z>0, we have:

$$\nabla^2\vec{A} = \frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial A_\phi}{\partial r} + \frac{\partial^2 A_\phi}{\partial z^2} - \frac{A_\phi}{r^2} = 0 \text{ for } z>0 \tag{2}$$

and, with $k^2 \doteq -j\omega\mu\sigma$ for z<0:

$$\frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial A_\phi}{\partial r} + \frac{\partial^2 A_\phi}{\partial z^2} - \frac{A_\phi}{r^2} - j\omega\mu\sigma A_\phi = 0 \tag{3}$$

for z<0

Solving by the separation of variables technique and using the limiting behavior at z→∞ and r→0 yield the following expressions for the solution to Equations (2) and (3).

$$A_{\phi 1}(r,z) = \int_0^\infty B_1 e^{-\alpha z} J_1(\alpha r) d\alpha, z>\ell>0 \tag{4}$$

$$A_{\phi 2}(r,z) = \int_0^\infty [C_2 e^{\alpha z}+B_2 e^{-\alpha z}] J_1(\alpha r) d\alpha, \tag{5}$$

$\ell>z>0$

$$A_{\phi 3}(r,z) = \int_0^\infty C_3 e^{\alpha_1 z} J_1(\alpha r) d\alpha \quad z<0 \tag{6}$$

where α is the separation constant and $\alpha_1^2 = \alpha^2+j\omega\mu\sigma$.

Since the electric field is proportional to A_ϕ, the boundary conditions for the tangential electric field can be satisfied by equating the values of A_ϕ at the z = ℓ plane.

$$\int_0^\infty B_1 e^{-\alpha\ell} J_1(\alpha r) d\alpha = \int_0^\infty (C_2 e^{\alpha\ell}+B_2 e^{-\alpha\ell}) J_1(\alpha r) d\alpha \tag{7}$$

Multiplying both sides by the integral operator $\int_0^\infty \{\ldots\} J_1(\alpha'r) r dr$ and using the Fourier-Bessel identity [3] give an algebraic equation for the unknown coefficients. The radial component of the magnetic field can also be found from the vector potential; $H_r = -\frac{\partial}{\partial z} A_\phi$. H_r is discontinuous at the position of the loop $(r=r_0, z=\ell)$ by an amount equal to the surface current density there.

$$\left[-\frac{\partial}{\partial z} A_{\phi 1}+\frac{\partial}{\partial z} A_{\phi 2}\right]_{z=\ell} = \mu I \delta(r-r_0) \tag{8}$$

which yields another equation for the coefficients. The boundary conditions may also be similarly applied

at z = 0 where both E_ϕ and H_r are continuous, yielding two more expressions for the constants B_1, C_2, B_2 and C_3. These four equations can then be solved for the constants and used in Equations (4), (5), and (6) to evaluate the vector potential.

Since our principal interest lies in evaluating the vector potential at the location of the loop the most direct route is to evaluate the constant B_1:

$$B_1 = \frac{\mu I r_0 J_1(\alpha r_0)}{2} \left[e^{\alpha\ell}+e^{-\alpha\ell}\frac{(1-\frac{\alpha_1}{\alpha})}{(1+\frac{\alpha_1}{\alpha})}\right] \tag{9}$$

Thus

$$A_{\phi 1}(r,z) = \frac{\mu I r_0}{2} \int_0^\infty J_1(\alpha r_0) J_1(\alpha r) e^{(-\alpha z-\alpha\ell)} \left[e^{+2\alpha\ell}+(\frac{\alpha-\alpha_1}{\alpha+\alpha_1})\right] d\alpha \tag{10}$$

The two terms in the square brackets represent respectively the vector potential due to the loop itself and that due to the currents induced in the conducting plane. This second term due to the conductive half-space, will produce the change in impedance from the case of the isolated loop to the case of the loop near the plane. This change in vector potential is thus given by this second term.

$$\Delta A_{\phi 1}(r,z) = \frac{\mu I r_0}{2} \int_0^\infty J_1(\alpha r_0) J_1(\alpha r) e^{-\alpha(z+\ell)}(\frac{\alpha-\alpha_1}{\alpha+\alpha_1}) d\alpha \tag{11}$$

This change in vector potential can be used to calculate the change in impedance due to the presence of the conductor by integrating the tangential electric field around the position of the loop:

$$\Delta Z = \pi\omega r_0^2 j \int_0^\infty J_1^2(\alpha r_0) e^{-2\alpha\ell}(\frac{\alpha-\alpha_1}{\alpha+\alpha_1}) d\alpha \tag{12}$$

The integrand factor $(\alpha-\alpha_1)/(\alpha+\alpha_1)$, essentially a reflection factor, has modulus equal to or less than unity, the extreme value being assumed for $\alpha=0$ and $\alpha=\infty$. The integrand factor $J_1^2(\alpha r_0)$ guarantees that the value of the integral is negligibly affected by values of α greater than about, $10/r_0$. Practical values of r_0 are usually of the order of 10^{-2}m. For such values of r_0 the important range for α is $0\leq\alpha\leq 10^3 \text{m}^{-1}$, while the quantity $\omega\mu_0\sigma[=2/(\text{skin depth})^2]$ is, in many practical cases, of the order of 10^7 (e. g., for aluminum at 50 KHz, $\omega\mu_0\sigma = 1.5\times 10^7$). For such cases, $\alpha^2/\omega\mu_0\sigma\leq 0.1$, and $(\alpha-\alpha_1)/(\alpha+\alpha_1)$ may be expanded as a power series in $\alpha/\sqrt{\omega\mu_0\sigma}$:

$$\frac{\alpha-\alpha_1}{\alpha+\alpha_1} = -1 + \frac{2}{\sqrt{j}}\frac{\alpha}{\kappa} - \frac{2}{j}\frac{\alpha^2}{\kappa^2} + \ldots = -1 + (1-j)(\alpha\delta) + j(\alpha\delta)^2 + \ldots \tag{13}$$

where $\delta = \sqrt{2/\omega\mu_0\sigma}$, and $\kappa = \sqrt{\omega\mu_0\sigma}$.

We expect the series above to converge rapidly

provided $\alpha\delta<<1$. As we shall presently show, it is convenient to adopt r_0 as a characteristic length. Since the value of ΔZ is determined almost entirely by values of α for which $\alpha r_0 \leq 10$, we have rapid convergence of the integrated series if $\delta/r_0<<1/10$. Separating $\Delta Z = \Delta R + j\Delta X$ into real and imaginary parts we have:

$$\Delta X = -\mu_0\omega\pi r_0^2\{\int_0^\infty J_1^2(\alpha r_0)e^{-2\alpha\ell}d\alpha -$$

$$\int_0^\infty \delta J_1^2(\alpha r_0)e^{-2\alpha\ell}\alpha d\alpha\} \qquad (14)$$

$$\Delta R = \mu_0\omega\pi r_0^2 \int_0^\infty \delta J_1^2(\alpha r_0)e^{-2\alpha\ell}\alpha d\alpha -$$

$$\int_0^\infty \delta^2 J_1^2(\alpha r_0)e^{-2\alpha\ell}\alpha^2 d\alpha\} \qquad (15)$$

These changes in resistance and reactance can be represented by three integrals:

$$\Delta X = -\pi\omega\mu_0 r_0(I_1(\beta) - (\frac{\delta}{r_0})I_2(\beta)) \qquad (16)$$

$$\Delta R = \pi\omega\mu_0 r_0((\frac{\delta}{r_0})I_2(\beta) - (\frac{\delta}{r_0})^2 I_3(\beta)) \qquad (17)$$

where $\beta = 2\ell/r_0$ and

$$I_1(\beta) \doteq \int_0^\infty J_1^2(x)e^{-\beta x}dx \qquad (18)$$

$$I_2(\beta) \doteq -\frac{d}{d\beta} I_1(\beta) \qquad (19)$$

$$I_3(\beta) \doteq \frac{d^2}{d\beta^2} I_1(\beta) \qquad (20)$$

$I_1(\beta)$ is just the Laplace transform of $J_1^2(x)$ [4]:

$$I_1(\beta) = \frac{1}{\pi} Q_{1/2}(1+ \frac{1}{2} \beta^2) \qquad (21)$$

where $Q_{1/2}$ is the Legendre function of the second kind of order 1/2.

$I_2(\beta)$ is therefore given by

$$I_2(\beta) = -\frac{\beta}{\pi} Q'_{1/2}(1+\frac{1}{2} \beta^2) \qquad (22)$$

where the prime indicates differentiation with respect to the argument. The required derivative may be found from the recursion relation [5]

$$(x^2-1)Q'_{1/2}(x) = \frac{x}{2} Q_{1/2}(x) - \frac{1}{2}Q_{-1/2}(x) \qquad (23)$$

For convenience in evaluation, both $Q_{1/2}$ and $Q_{-1/2}$ may be expressed in terms of complete elliptic integrals [5]:

$$Q_{1/2}(x) = x(\frac{2}{x+1})^{1/2}K[(\frac{2}{x+1})^{1/2}] -$$

$$[2(x+1)]^{1/2}E[(\frac{2}{x+1})^{1/2}] \qquad (24)$$

$$Q_{-1/2}(x) = (\frac{2}{x+1})^{1/2}K[(\frac{2}{x+1})^{1/2}] \qquad (25)$$

where $K(k)$ and $E(k)$ are respectively the complete elliptical integrals of the first and second kind of modulus k:

$$K(k) = \int_0^{\pi/2}(1-k^2\sin^2 t)^{-1/2}dt \qquad (26)$$

$$E(k) = \int_0^{\pi/2}(1-k^2\sin^2 t)^{1/2}dt \qquad (27)$$

Values of $K(k)$ and $E(k)$ may be obtained from standard tables or from readily available computer software.

$I_3(\beta)$ may likewise be reduced to an expression involving $K(k)$ and $E(k)$. However, for most practical cases, the factor $(\delta/r_0)^2$ by which $I_3(\beta)$ is multiplied is so small that the contribution to ΔR from the term proportional to $I_3(\beta)$ is negligible.

RESULTS

To illustrate the changes in impedance as a function of the lift-off distance ℓ and the conductivity σ, calculations were made for a loop of radius $r_0 = 1.27$ cm (diameter of one inch) at distances ℓ from .05 to 1.5 cm, and for conductivities from 0.1 to 4 times that of aluminum ($\sigma_0 = 3.8\times10^7$ mho/m). These results are shown in Figs. 2 and 3 as a function of ℓ for various constant conductivities. The normalized dimensionless changes in impedance $\Delta X/\omega\mu r_0$ and $\Delta R/\omega\mu r_0$ are chosen as the quantities to be plotted. For all values of conductivity the value of $\Delta X/\omega\mu r_0$ is seen to approach a large negative value as ℓ decreases showing the known decrease in total inductance as the loop approaches the plane. As ℓ becomes large $\Delta X/\omega\mu r_0$ approaches zero as required. Similarly in Fig. 2 $\Delta R/\omega\mu r_0$ is seen to give a large positive contribution for small ℓ and approaches zero as ℓ becomes large.

To illustrate the effects of the conductivity on the changes in impedance for several constant values of lift-off, the results for the same loop are shown in Figs. 4 and 5. The change in reactance $\Delta X/\omega\mu r_0$ is seen to be very nearly independent of conductivity over the range considered. The value of $\Delta R/\omega\mu r_0$, however, is seen to increase for lower values of σ. This resistance term, of course, approaches zero as the conductivity approaches that of a perfect conductor.

Both the variations in resistance and reactance can be combined into the one graph shown in Fig. 6 by plotting ΔX versus ΔR. The solid lines thus show the change in impedance as the lift-off is changed, while the dashed lines show the variation with changing conductivity for constant lift off ℓ.

The limiting values of $\Delta X/\omega\mu r_0$ for large values of σ can be checked by comparing the calculated values with that of the case of a loop above a perfectly conducting plane. Using image theory the mutual inductance between two identical loops located a distance 2ℓ apart can be found to be $M = 2.54$ Nr_0 [6] where N is a tabulated function of r_0 and ℓ. The

values of M and ΔL at 50 KHz were compared for values of ℓ between 2.5 and 15 cm and quite good agreement was found (within 10^{-4} μH).

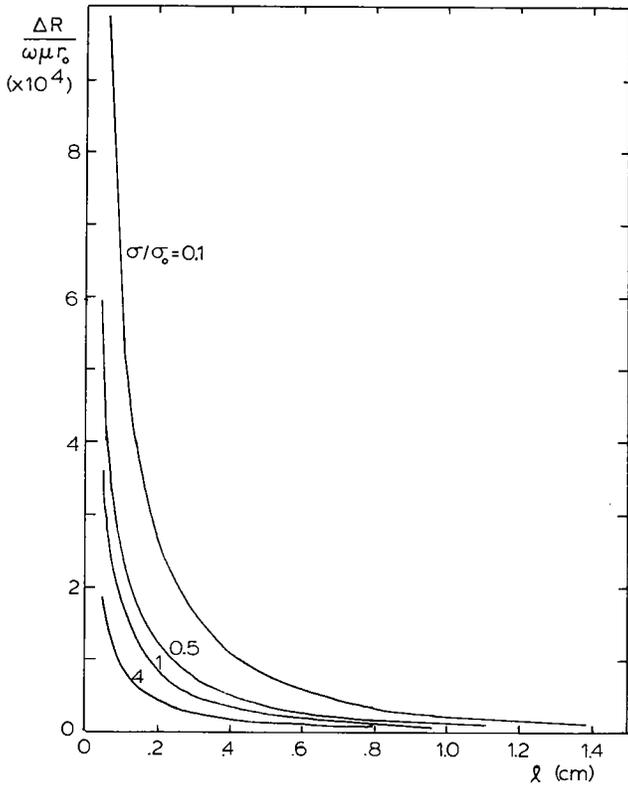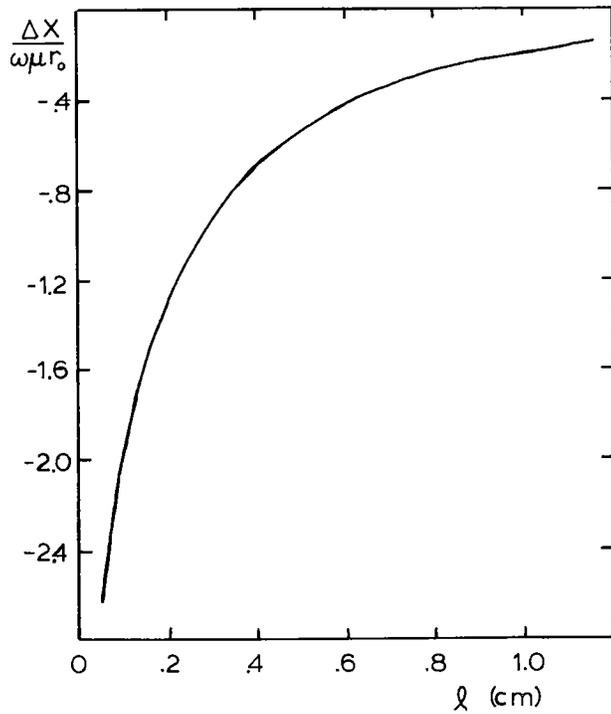Fig. 2   Change in normalized resistance versus lift-off distance.

Fig. 4   Change in normalized resistance versus conductivity.
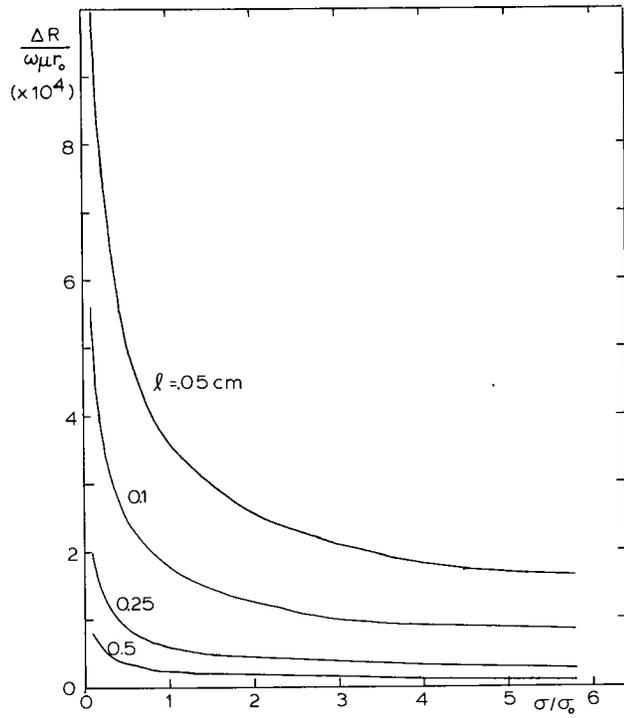

Fig. 3   Change in normalized reactance versus lift-off distance.

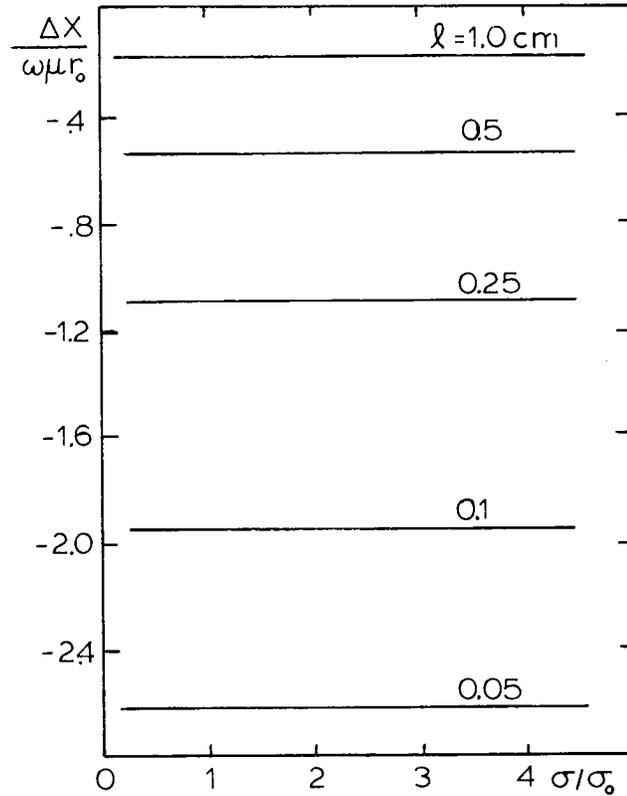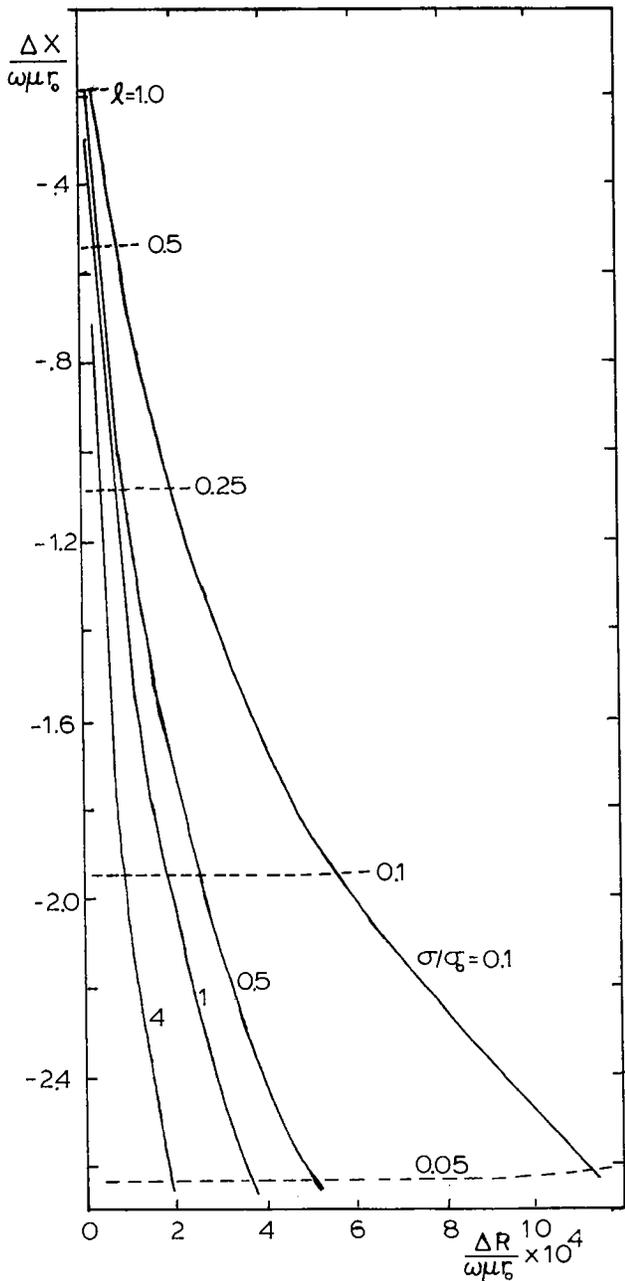Fig. 5   Change in normalized reactance versus conductivity.

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Fig. 6 Change in reactance versus change in resistance.

off and conductivity.

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CONCLUSIONS

For the commonly occurring case where $\delta << 0.1 r_0$, the change in coil inductance is essentially the value that would occur if the substrate were perfectly conductive; ΔL is thus dominated by its dependence on lift-off. The change in resistance is, for constant lift-off, proportional in first order to skin depth (or, for constant frequency, proportional to the square root of substrate conductivity); however, ΔR is also strongly dependent upon lift-off. Second-order changes in ΔL and ΔR, due to small variations in ℓ and σ about nominal values, are well approximated by linear functions of $\Delta\ell$ and $\Delta\sigma$; hence variations in ΔL and ΔR may readily be interpreted in terms of corresponding variations in lift-

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Jim Martin, Chairman (Rockwell Science Center): Thank you. That was an excellent presentation. We
        have time, I think, for one question. Please remember to identify yourself.

Mr. Lincoln (USC): Did you compare your theory with the case of a single straight wire over a
        conducting plane by letting the radius of your loop become large and calculating the induction
        for the loop?

Stuart Long (University of Houston): No, I did not.

Mr. Lincoln: That should also work.

Stuart Long: Would you still have to assume a perfect conductor?

Mr. Lincoln: No, that has already been done.

Stuart Long: Okay, that would be a good test, then.

Jim Martin, Chairman: We will defer the remainder of the presentations until after the break.