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AGRAWAL, Ramesh Chandra, 1936-  
APPLICATIONS OF OPERATIONS RESEARCH  
TECHNIQUES IN AGRICULTURE.

Iowa State University, Ph.D., 1967  
Economics, agricultural

University Microfilms, Inc., Ann Arbor, Michigan

APPLICATIONS OF OPERATIONS RESEARCH TECHNIQUES  
IN AGRICULTURE

by

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A Dissertation Submitted to the  
Graduate Faculty in Partial Fulfillment of  
The Requirements for the Degree of  
DOCTOR OF PHILOSOPHY

Major Subject: Agricultural Economics

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1967

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## INTRODUCTION

## Why Operations Research—The Problem of Decision Making

Decision making is an universally applicable phenomenon. Everyone in the world is required to make decisions at every step of his life. Though we may not be particularly conscious of it, we make decisions every day and every hour of our active life. For example, a housewife has to decide not only what to feed the family tonight but also how to prepare the food to be served, what ingredients to use and furthermore, from where to purchase these ingredients, when and at what price, etc. Similarly, from the time he has taken that big decision to pursue farming for a living, a person has to make very many important initial and then day-to-day decisions of the following type:

- a. Where to start farming? What should be the size of the farm? What type of residential buildings are to be preferred or made on the farm?
- b. What type of farming should he have, viz. grain farming, hog farming, dairy farming, beef cattle, or some other type?
- c. What resources should he acquire and in what quantities? Whether should he have one tractor or two, small, medium or large in size. What attachments and special machines he should purchase for his business?

- d. How should he finance his farm? How much should he borrow and how much should be his own investment? Of the borrowed capital, how much should he take for long-term, how much for medium and how much for short-term periods? When should he borrow and at what terms? What should be the source or sources of borrowing?
- e. Once acquired, what allocation of his resources is most efficient?
- f. What should he produce, when and how much of each?
- g. What level of technology should he use? What operations should he perform, at what time and in what manner?
- h. Once the produce has been obtained, he has to decide as to what to sell, and how much to sell, when, where and at what prices?

Some of these decisions are taken after a good deal of time devoted to thinking; whereas others have to be spontaneous. Some decisions are relatively more important than others. Some are long-term in nature implying thereby that the consequences of these decisions are felt for a long time after they have been taken; e.g., decision to expand the size of the farm, construction of a new elevator, an increase in the capital outlay of plant and equipment, etc. Other decisions have consequences that are shorter in duration; e.g., whether to irrigate a crop today or tomorrow.

The basic components of decision-making problem can be broadly grouped into:

1. objectives,
2. a set of alternative courses of actions available, and
3. uncertainty.

Let the set of objectives be

$$\phi = \{o_j\}$$

Let the set of alternative courses of actions available be

$$A = \{a_i\}$$

where  $i = 1 \dots n$ ,  $j = 1 \dots m$ , and  $m$  and  $n$  are finite. Then given the objective of a firm  $O = \{o_j / o_j \in \phi\}$ , the problem is reduced to the choice of suitable subset  $A_k$  of  $A$  such that  $A_k$  optimises the objective function.

When complete information about the future is available and there is no uncertainty involved in decision-making, choice of a suitable  $A_k$  is rather trivial. The problem of decision-making, in fact, arises due to the fact that one's knowledge of the future is not perfect. He has to take decision in the present for the future in the realm of uncertainty. Uncertainty is a subjective phenomenon and the parameters of probability distribution cannot be established empirically. No two managers may visualize the future happenings in the same manner and, therefore, given the same set of

circumstances, the decision of A may be different from B. As he tries to look into the future, a decision-maker forms expectations of the returns or outcomes or consequences of his decision. Therefore, it is important to keep in mind that he does not maximize his utility or payoff, but maximizes the expected value of his utility function or payoff. Thus given an  $r \times s$  matrix  $U = u_{ij}$  ( $i=1, \dots, r, j=1, \dots, s$ ) where  $u_{ij}$  is decision-maker's utility for the consequence associated with his action  $a_i$  when nature is in state  $t_j$ , his problem of taking a decision under uncertainty is finding a subset of actions which are expected to be optimal in some sense (based on his objective or criterion function).

#### The Special Nature of Farming Enterprise

Returns in any enterprise are a function of several variables, the more important being the quantity of output produced from a given set of resources, the cost of production per unit of output and the price at which the output is sold. Let

$q$  = quantity of output produced from a given set of resources,

$c$  = cost of production per unit of output,

$P$  = price per unit of output,

$\pi$  = profits.

Then

$$\pi = f(q, P, c)$$

However,  $q$ ,  $P$  and  $c$  are in themselves dependent on other factors, e.g., location of the farm, climatic conditions, social conditions, nature of output, etc. Since farming is an out-of-door enterprise, carried on a relatively large area, it is affected by variations in climatic conditions. Moreover, the production of crops and livestock is not instantaneous. It is a biological phenomenon with a time lag between the initiation of the process of production and the realization of the output. Similarly, most farm products are comparatively perishable in nature and, therefore, (especially in less advanced economies where they are not processed, canned or frozen in quantities to be available on the shelf), their supply is seasonal which lends them susceptible to price variability. These are some of the peculiarities of farming that make it more liable to the phenomenon of uncertainty than other enterprises. Thus a farmer, like any other entrepreneur, faces uncertainty of several types when he commits his present resources to the production of a commodity forthcoming at a later time. Broadly speaking, these are:

- a. uncertainty of yield,
- b. uncertainty of prices of inputs and outputs,
- c. technological uncertainty,

- d. uncertainty due to social, legal and political factors.

However, unlike the other entrepreneurs, the degree of uncertainty faced by a farmer is greater than that in other enterprises due to the peculiarities of agriculture pointed out elsewhere.

The degree of uncertainty can be gauged by looking at the probability density function (or distribution) of the expectations of the farmer with regard to the yields and prices. The greater the skewness, the smaller the degree of uncertainty and vice-versa. A leptokurtic distribution indicates that the farmer is more confident of the prices or yields being within a short range. On the other hand, a platykurtic distribution represents a greater degree of uncertainty on the part of the decision maker. (Please see Figure 1.)

#### Decision Making Techniques

The question then arises, "Given the objective function (reflecting the goals of the decision-maker), the functional relationships between dependent and the independent (decision) variables, the constraints, etc., what tools are available to the decision-maker to help him in choosing the appropriate values of the controllable variables which maximize the expected value of the objective function?" Simon (93, p. 8) has divided these techniques of decision making into (a) traditional and (b) modern as in the following table.

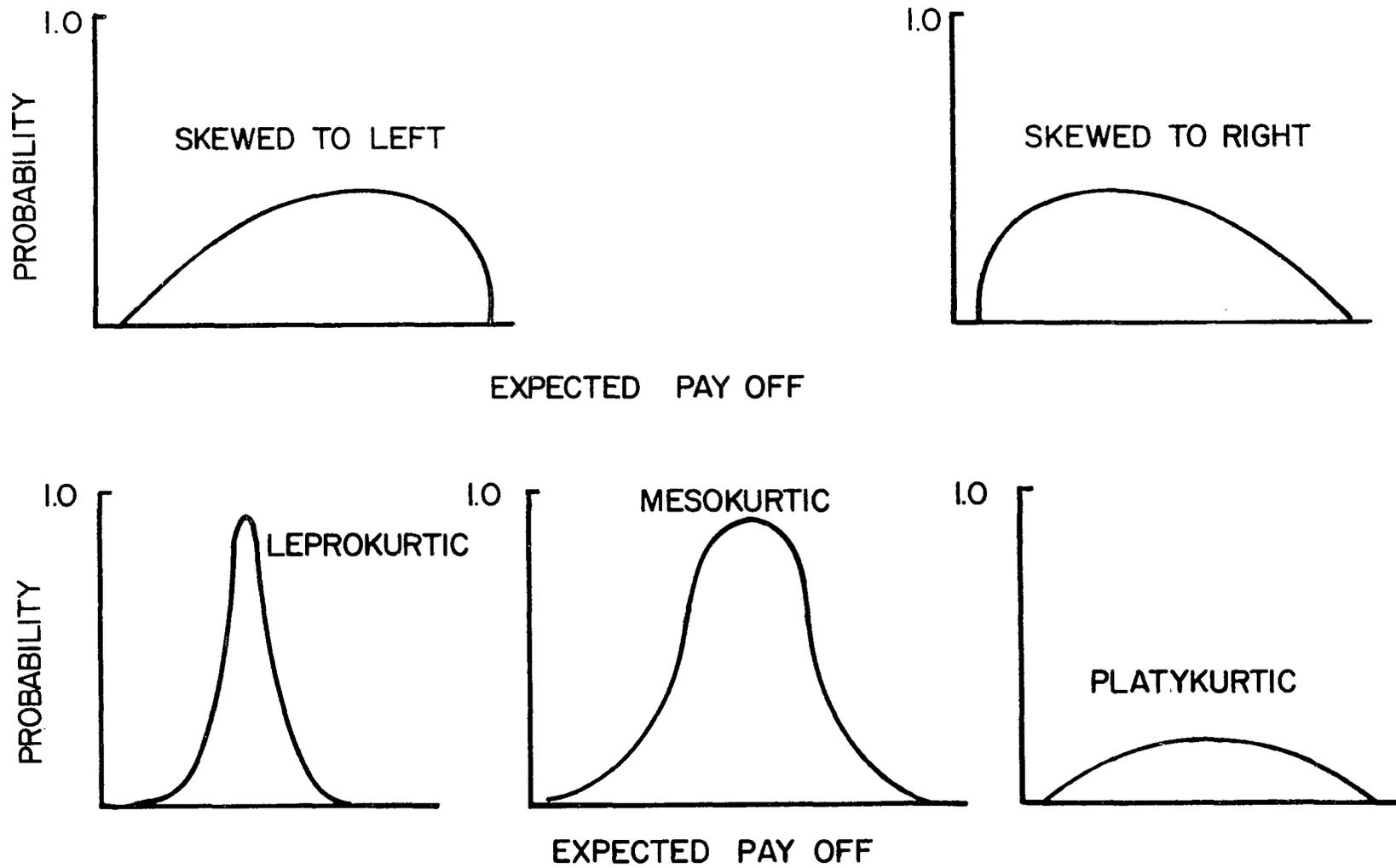


Figure 1. Skewness, kurtosis and degree of uncertainty

Traditional and modern techniques of decision making

Types of decisions	Decision making techniques	
	Traditional	Modern
<p><b>Programmed:</b>                      Routine, repetitive decisions. Organization develops specific processes for handling them.</p>	<ol style="list-style-type: none"> <li>1. Habit</li> <li>2. Clerical routine: Standard operating procedures</li> <li>3. Organization structure: Common expectations, a system of subgoals, well-defined informational channels</li> </ol>	<ol style="list-style-type: none"> <li>1. Operations research: Mathematical analysis, models, computer simulation</li> <li>2. Electronic data processing</li> </ol>
<p><b>Nonprogrammed:</b>                      One-shot, ill-structured novel policy decisions. Handled by general problem-solving processes.</p>	<ol style="list-style-type: none"> <li>1. Judgment, intuition, and creativity</li> <li>2. Rules of thumb</li> <li>3. Selection and training of executives</li> </ol>	<p>Heuristic problem-solving techniques applied to:</p> <ol style="list-style-type: none"> <li>(a) training human decision makers</li> <li>(b) constructing heuristic computer programmes</li> </ol>

The most general and pervasive traditional programmed method, according to Simon, is habit with standard operating procedures and organization structure closely related to it. Proper judgment, creativity and selection and training of the decision-maker still remain significant constituents of success not only in decision making, but in every walk of life. However, these traditional methods have been inadequate and rather naive. Of these traditional methods, Mr. Simon (93, p. 13) writes, "We have not had, in the past, adequate knowledge of the processes that are involved in decision making in complex situations. Human thinking, problem solving, and learning have been mysterious processes which we have labeled but not explained. Lacking an understanding of these processes, we have had to resort to gross techniques for improving nonprogrammed decision making: selection of men who have demonstrated their capacity for it; further development of their powers through professional training and planned experience; protection of nonprogrammed activity from the pressure of repetitive activity by establishing specialized organizational units to carry it on." These traditional methods had, perhaps, limited applicability and scope in complicated situations, could not come to grips with complex logistics problems and required considerably more time and effort to put up the needed facts before the decision maker in the proper form. These difficulties with the traditional

methods coupled with the increasing use of quantitative methods in economics by the mathematicians and economists (econometricians) have been responsible for the emergence of the modern programmed techniques that have served to fill a void in the area of decision making. Of all the decision making techniques—traditional and modern—the methods referred to as 'Operations Research Techniques' constitute, by far, the most powerful tools available to the decision maker in helping and guiding him to take appropriate action through the use of mathematical, logical and scientific means. Operations research, its brief history, and definition and the reasons for undertaking this study of testing some selected operations research techniques to the special conditions of agriculture are the subject matter of the next chapter.

## OPERATIONS RESEARCH—BRIEF HISTORY AND DEFINITION.

## WHY THIS STUDY?

## A Brief History

The techniques of operations research owe their origin to the endeavour of applications of scientific, mathematical and logical principles to the solutions of the military problems. The birth of these methods may be traced to the writings of F. W. Lancaster, who, during the First World War, applied mathematical analysis to military operations. He studied relationships between victory of forces and their superiority in fire power and number. Presentation of a very simplified Lancaster model is interesting, if for nothing else, for its historical significance.

Simplified Lancaster model

Let there be two armies S and T fighting against each other. There are N soldiers in S and M in T with a fire power of n and m respectively. Lancaster assumed such an encounter of the two armies where n fire power of S was directed equally against members of T and vice versa. Then

$$(1) \text{ Rate of loss of the army S} = - \frac{dS}{dt} = cmT \text{ and}$$

$$(2) \text{ Rate of loss of force T} = - \frac{dT}{dt} = cnS$$

where c is a constant and t is time.

- (3) He defines the equality of the strength of forces in this manner. According to him, S and T are equal in

$$\text{strength if } \frac{1}{S} \cdot \frac{dS}{dt} = \frac{1}{T} \cdot \frac{dT}{dt}.$$

By (1), (2) and (3), he got

$$(4) \quad nS^2 = mT^2$$

Therefore, if  $n > m$ , then  $S < T$  and vice versa.

Thus he concluded that the strength of a force was proportional to its firepower multiplied by the square of its elements (i.e., soldiers).

#### Subsequent developments

Lancaster's work, in the nature of 'a priori' investigation, had little use for handling complicated problems and, therefore, no actual effect on operations in World War I. But it certainly was an useful prelude to the subsequently effective applications of operations research in World War II and later. During the Second World War, Professor P. M. S. Blackett (9), a Nobel Laureate, wrote two notes setting out some of the principles of operations research and the methods of analysis. Operations research groups were organized, first in Britain and then in the U.S.A., Canada and Australia. Operations research proved to be a valuable asset to the Allies in bringing them victory and, therefore, the U.S. military establishments continued and supported projects for the development of operations research even after the War was over.

The association of military operations to the growth of

operations research does not mean that it can be or is exclusively applied to military purposes. In fact, Professor Blackett (9), in his celebrated paper on 'Operational Research', clearly mentioned that operational research could "help us to tackle wisely some of the problems of peace". He was very much alive to the fact that in Economics techniques analogous to what he called 'Operational Research' were used and even said that the problems of analyzing war operations were nearer to problems in biology or economics than those in physics. However, it is not clear if at the time of its origin the use of operations research to the science of management was also envisaged. The increasing use of machines in the recent past has led to a rapid growth of enterprises, so much so that the task of management has become more and more complex and difficult. Hence, operations research has played an increasingly important role in decision making in all fields of business activity such as transportation, manufacturing, purchasing and selling. Many big firms have their own cell of operation researchers. Operations research techniques have been used by government and social organizations for several widely varying purposes. The national planning in Puerto Rico is a case in point. "'Operation Bootstrap' has, in fact, transformed that country from an impoverished agricultural island to a thriving, semi-industrialized community with a standard of living which compares favourably with that of

other Latin American countries" (95). By now operations research has established itself as one of the most important sciences in the field of business management. It not only helps in identifying different states and strategies of 'nature', but also in listing alternative courses of action open to the decision maker and the outcomes associated with them, thus suggesting which strategy for him to choose and employ under a given set of circumstances. The question then arises, "What is operations research?"

#### What is Operations Research? (Definitions and Concepts)

Variouly termed as 'Operational Research' (by Blackett and others in Britain), 'Operations Research', 'Operations Analysis' and 'Operations Evaluation', this science has been defined in different ways by different workers. Due to its diverse nature and wide variety of uses, it is difficult to find a definition which is simultaneously simple and satisfactory. One of the first definitions was proposed by Morse and Kimball (76) and was accepted in 1952 at the founding meeting of the Operations Research Society of America. It is as follows:

"Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control" (76, p. 1).

One of the conspicuous weaknesses of this definition is its failure to distinguish operations research from a number

of other disciplines related to business problems. The definition holds equally well even if we replace 'Operations Research' with 'Quality Control' or 'Cost Accounting'.

Another definition of 'Operations Research' enumerates the various techniques like linear programming, nonlinear programming, theory of probability, queuing and inventory theories, PERT and CPM, etc. Some researchers do not agree with this and think that to define the science of operations research in terms of these methods is a mistake similar to that of defining 'medicine' as a collection of drugs used by a doctor to cure the illness.

Still others have defined 'Operations Research' as a "quantitative commonsense", or "Operations Research is what operations research workers do". Needless to say that the latter definition, though technically correct, is rather ridiculous and fails to shed any light on the nature and contents of the science of operations research. Simon (93) thought that a sociological definition of operations research was more understandable and defensible. According to him, "Operations research is a movement that, arising out of military needs of World War II, has brought the decision making problems of management within the range of interests of large numbers of natural scientists and, particularly, of mathematicians and statisticians." Regarding the place of operations research in Economics, he points out that, "The

operations researchers soon joined forces with mathematical economists who had come into the same area to the mutual benefit of both groups." Mr. Simon also found it hard to draw a meaningful line between operations research and management science. In my view, the former is a subset of the latter as the operations research techniques show us the ways but do not include act of final choice by the decision maker. The management science includes both of these.

According to Yates (111), "Operational Research consists of the application of scientific research to the problems arising in administration and planning....By 'methods of scientific research' I mean that combination of observation, experiment and reasoning (both deductive and inductive) which scientists are in the habit of using in their scientific investigations....Experiments form an integral part of operational research...."

Ackoff (1) defines an operation as a "set of acts required for the accomplishment of some desired outcome." He enumerates four components of an organization, viz., communication, content, control and structure. According to him, "Control is a matter of directing the organization toward desired objectives and it is obtained by efficient decision making by those who manage the operations. Assisting managers to control organizations (i.e., improving their decision making) has been an important objective of operations research.

To me, the science of operations research deals with attacking the problems, faced by the decision maker, through identifying the problem or problems in question, defining the alternatives available to him and also the various states of 'nature', apprising him of the payoffs associated with each combination of the elements of these alternatives and strategies and then suggesting, to him, the best course of action obtained through the use of logic, mathematics and other sciences. This decision maker may be a business executive, a farmer, a physicist, an economist, a military officer, national planner or anyone else. This is illustrated in Figure 2.

#### Components of Operations Research

The important components (steps) of an operations research project are:

1. Identification and formulation of the problem
2. Defining the objective function to be optimized
3. Construction of a mathematical model satisfying the constraints on the values of the variables
4. Obtaining the empirical estimates of the parameters
5. Solving the model and finding out the course or courses of action that would optimize the objective function

Formulation of a mathematical model is perhaps the

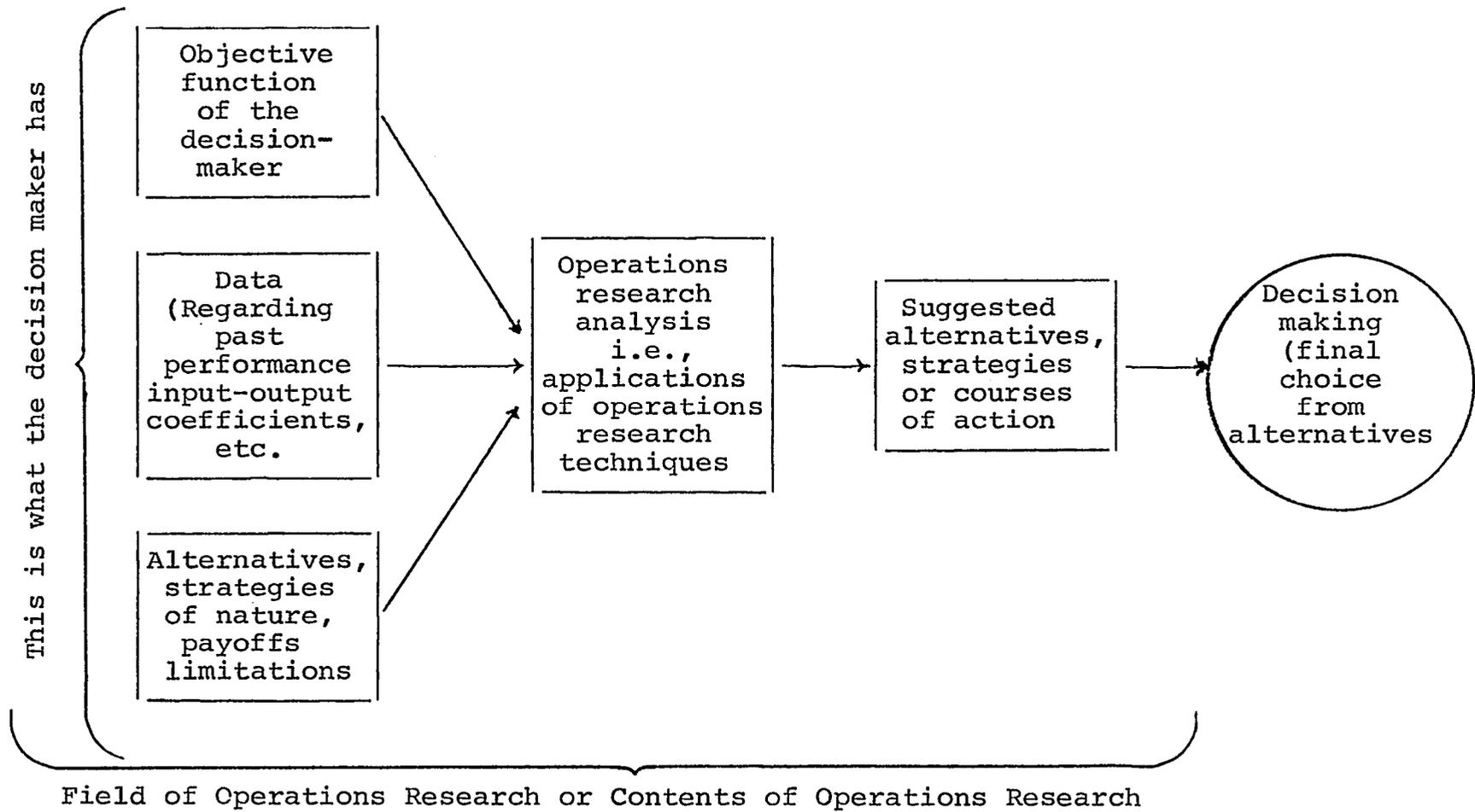


Figure 2. Contents of operations research

distinguishing component of operations research not found in the traditional methods of decision making. The form of a model may vary from a simple graph to a highly complex mathematical relationship. According to Saaty (83), "The fundamental conceptual device which enables one to regard the operation as a whole is a model which is essentially a hypothesis. A model is an objective representation of some aspects of a problem by means of a structure enabling theoretical subjective manipulations aimed at answering certain questions about the problem. This representation which attempts to establish a correspondence between the problem and the rational thought, may be realized by forming a physical model, such as a wind tunnel for testing aircraft, or a theoretical model, such as equations immediately related to an operation."

A model is an abstract formulation and, therefore, makes it easier to brood over and tackle the problem under consideration. Perhaps the greatest utility of a mathematical model lies in its property of lending itself to generalization, i.e., a solution obtained in a given problem may well apply to another set of circumstances. The choice of the model to be used depends on several factors like the nature of the problem, available alternatives, objectives to be achieved, the nature and adequacy of data, types of tools available and the ability and competence of the worker handling these tools.

## Important Operations Research Techniques

The important tools used in operations analysis are:

1. Mathematical programming
  - (a) Linear programming
  - (b) Nonlinear programming
  - (c) Dynamic programming
2. Game theory
3. Probabilistic models
  - (a) Queuing
  - (b) Inventory control
  - (c) Monte Carlo method
4. Transportation models  
(special cases of linear programming)
5. Simulation techniques
6. Time-network analysis
  - (a) PERT
  - (b) CPM
7. Sequential analysis
8. Other methods
  - (a) Input-output analysis
  - (b) Capital budgeting
  - (c) Forecasting
  - (d) Theory of information
  - (e) Searching processes

The Study of Applications of  
Operations Research in Agriculture

The work of Dr. Thornthwaite (100)

One of the first, interesting and challenging applications of operations research in agriculture was by Dr. C. W. Thornthwaite on Seabrook farm during the years 1946-50. In 1946, Seabrook farm had 7,000 acres of peas to be harvested for freezing and canning purposes. Returns from freezing the peas are greater than those from canning them. The best stage for freezing peas is to harvest and freeze them within a couple of days of their maturing. Those not harvested or frozen at the right time have to be canned thus bringing about a substantial reduction in profits of the farmer. The problems faced by the Seabrook farm were:

- (a) There was no scientific way to find out whether the peas had reached the right stage to be harvested for freezing.
- (b) At the time of maturity (i.e., whenever the manager thought that the peas were ready to be picked), even after using all the machine and manpower of the farm and working round the clock, the pickers could not keep up with the ripening peas.
- (c) During peak harvesting period, freezers could not keep pace with the pickers. Therefore, those peas which could not be frozen in time (within a couple of days of picking) had to be canned.

Dr. Thornthwaite tackled the first problem with the help of the evidence that the rate of growth and development of the plant depended on climatic factors. He calculated what he termed 'growth units' for each variety of peas. Further, on the basis of temperature, its intensity and duration, and other climatic factors, he calculated 'growth units' for a particular day, week or month. Thus when 'growth units' required for that variety of pea to mature had accumulated from the time of sowing, the proper stage of maturity of the peas to be picked had been reached. The last two problems were solved by establishing a planting schedule so that all peas on the farm would not mature at the same time. By sowing them in parts, at adequate time intervals, it was possible to harvest all peas at the proper time without undue strain on the labor since the peas matured on different dates. This example illustrates that it is observation, techniques, application and decision that are important for solving a problem of any nature.

#### The work of Yates (111)

In 1949, Yates (111) published a paper regarding the use of operational research made in the field of agriculture in the United Kingdom.

During the Second World War, Great Britain had not only to expand its food supply through increased food production, but also to economize on its imports which included things

like fertilizer needed to increase the agricultural production. These were conflicting situations and, therefore, it was necessary to cut down import to an extent that would not jeopardize agricultural production. Mr. Yates designed and conducted surveys with the objectives of finding out

- (a) the average response of different crops to different amounts of fertilizers and regional and other relevant differences in these responses;
- (b) how farmers actually used the fertilizers made available to them.

Not only the amounts of various kinds of fertilizers to be imported were estimated, but account was taken of their availability, shipping, etc.

The meaning and application of operational research in agriculture as given by Yates is rather narrow, limited to the sampling and surveys only, but it did serve the purpose of demonstrating that we can use methods in agriculture to give a picture of the situation on which sound action could be based.

#### The Need and Scope of This Study

Farming is quite different from the other conventional industries in many respects. Agriculture is an out-of-door occupation, greatly affected by natural factors like soil, climate, etc. The decisions have to be taken in a shorter

time in agriculture as compared to other industries. For example, a farmer is planning to hoe his crop tomorrow. All of a sudden there is a frost warning. He has to take immediate action to irrigate his crop to save it from damage by frost. Likewise, excess rain, severe draught and other natural phenomena may force him not only to take immediate decision, but also action.

Farming is a biological phenomenon. The plants and animals are living and cannot be treated in the same manner as the output of industries like radio, books, tables, etc.

In the United States and other developed nations, farming is commercialized and returns from agriculture may not be much lower than those in other industries. However, it is quite different in the developing nations. Farming is subsistence in nature, is a way of life and returns are lower than those in other industries.

'Time-lag' is a peculiarity of agriculture that distinguishes it from other industries. Production is not instantaneous. After the crops have been sown, time must elapse before they can be harvested. In the case of young dairy cattle, other livestock and orchards, one has to wait, not for months, but for years before he can get any returns on his investment. The turnover is slower.

Agricultural products are bulky and perishable in nature and, therefore, the problems of selling and storage in

agriculture are different from other industries. The produce has to be processed, in many cases, before it can be properly stored. Cold storage is required to keep fruits, potatoes, etc., fresh. Due to bulky nature of agricultural products, the size of storage plant has to be larger.

All these and other peculiarities of agriculture are responsible for the difference in the nature of decision problems faced by a farmer and a company executive. For example, due to its being a biological phenomena and the time lag, there are cycles of over-production and under-production in agriculture. Hog cycles are an example of this. The price and yield variability is greater in farming. As pointed out elsewhere, the nature and extent of uncertainties in farming are quite different.

The various operations research techniques mentioned above have been extensively used in solving widely different types of managerial and executive problems in military and business spheres. Some of these like linear programming have been applied to a considerable extent to farming as well. However, little has been done with regard to several other techniques to see as to how well they lend themselves to the decision-making process in agriculture, where problems are quite different from other industries as described above. The objective of this study is to examine the extent of suitability and applicability of some of the operations research

techniques to the special conditions obtaining in process of decision making in a farming enterprise. As it would be rather stupendous to attempt to test all the techniques, this study is confined to the following:

1. Game Theory
2. Mathematical Programming—  
Linear Fractional Functional Programming
3. Network Analysis—PERT and CPM
4. Queuing Theory

#### Method of Study

It is proposed in this study to give a brief summary of what has already been done in the field of application to agriculture of the technique or techniques under consideration. Then the techniques under discussion would be applied to the data both from the developed and developing countries. In some cases, hypothetical examples with realistic coefficients would be used. These applications would help us in finding the suitability of the different techniques for economies at different levels of development. Efforts will be made to indicate the special areas of farming enterprise where a particular technique of operations research could be applied with advantage.

## APPLICATIONS OF GAME THEORY IN AGRICULTURE

## Introduction

Though the theory of games in its most elementary and rudimentary form can be traced back to the writings of Borel in the 1920's and some earlier papers written by J. von Neumann in 1928 and 1937, it was only in 1943-44 when that monumental work, "Theory of Games & Economic Behaviour", by J. von Neumann and Oskar Morgenstern (104) was published that the 'Game Theory' was put in its proper perspective. The book dealt with the development of a mathematical theory of games and its applications to the economic and sociological problems. Most of the subsequent work in game theory has been an extension to and based on this work of von Neumann and Morgenstern. For example, in his "Statistical Decision Functions Which Minimize the Maximum Risk", Abraham Wald (105) drew heavily on the theory of 'two-person zero-sum game' to develop a theory of statistical decisions. Blackwell and Girshick (10), Drescher (30), Gale (34), Kuhn and Tucker (56), Luce and Raiffa (61), McKinsey (63), Wald (105) and Williams (110), to mention a few, have made significant contributions to the theory of games.

Game theory models deal with the process of decision-making by two or more opponents having conflicting interests. The success of one is the failure of the other. For example, in chess, as in any other game, the aim of each player is to defeat his opponent, to inflict on him the greatest loss or to

let him get away with the minimum of gain. There are a number of strategies available to each player of a game, but the choice of a strategy or strategies depends primarily on the existing conditions which, in part, are the functions of the strategies employed by the opponent. The common examples of games are chess, poker, bridge, matching pennies, checkers, tick-tack-toe, military confrontations, etc.

The elements of a game are the players (having conflicting interests), a code of rules for playing the game, and the pay-offs (gains or losses) associated with different combinations of strategies or moves made by the players.

Let A and B be the two players of a game. If S is the strategy space of player A and T the strategy space of player B and P the pay-off matrix, then game  $G = (S, T, P)$  is a triplet.

$$S = (s_i | i = 1, 2, 3 \dots n)$$

$$T = (t_j | j = 1, 2 \dots m)$$

Then P matrix has elements  $p_{ij}$  ( $i = 1, 2 \dots n$  and  $j = 1, 2 \dots m$ ); P is  $n \times m$  matrix; and  $p_{ij}$  is the pay-off to player A associated with his  $i$ th strategy when B employs his  $j$ th strategy.

### Illustration 1

Two players A and B are playing this game. Each player can show either one or two fingers and both players are required to show them simultaneously. Further, suppose that

if the number of fingers shown by each player is identical, player A wins from B an amount (in pennies) equal to the total of the fingers shown by the two; otherwise, he is required to pay to B pennies equal to the total number of fingers shown by both. Thus the pay-off matrix from the viewpoint of player A is:

		B	
		$t_1$	$t_2$
A	$s_1$	2	-3
	$s_2$	-3	4

where  $s_1$  and  $s_2$  are the two strategies of A such that when he employs  $s_1$  he shows one finger and if he uses  $s_2$  he shows two fingers. Similarly,  $t_1$  and  $t_2$  are the two strategies of B in which he shows one and two fingers, respectively. So the strategy set of player A is  $(s_1, s_2)$  and that of B is  $(t_1, t_2)$ .

If player A chooses to play  $s_1$  and B plays  $t_1$ , A receives two pennies from B. Therefore,  $p_{11} = 2$ . Similarly,  $p_{22} = 4$ ,  $p_{12} = -3$ ,  $p_{21} = -3$ .

#### Two Person Zero-Sum Game

The simplest form of a game is a "two-person zero-sum" game. As the term indicates, the game is played only by two players. One player's loss is equal to the other's gain and

that is why the name 'zero-sum'. If  $P$  (with elements  $p_{ij}$ ) is the pay-off matrix from the viewpoint of player A and  $Q$  (with elements  $q_{ij}$ ) is the pay-off matrix of the same game from player B's viewpoint, then  $p_{ij} = -q_{ij}$ . In our example, the pay-off to B would be of the following form:

		B	
		$t_1$	$t_2$
A	$s_1$	-2	3
	$s_2$	3	-4

Clearly  $p_{ij} = -q_{ij}$ .

The importance of the study of a "zero-sum" game lies in that any game can be reduced to a zero-sum game. As  $p_{ij} = -q_{ij}$ , the study and analysis of game theory, in general, and any game in particular can be done by considering matrix  $P$  only. From now on throughout this work, unless otherwise specified, we shall always be writing a pay-off matrix from the viewpoint of player A, whose strategies are represented by the rows of  $P$ .

#### The Maxi-min and Mini-max Strategies

The pay-off to player A, as a result of his employing his maxi-min strategy is equal to  $\max_i \min_j p_{ij}$ . Similarly, the mini-max strategy for B results in the pay-off equal to  $\min_j \max_i p_{ij}$ .

The logic of why a player employs his maxi-min or mini-max strategy can best be illustrated by an example. We will assume throughout our study of game theory (unless specified otherwise) that each player has full knowledge of all the previous moves of his own and those of his opponent at the time of making a decision for his  $k$ th move. We also assume that a player is rational and would therefore, try to play in manner as to make maximum gain or hold his opponent to a pay-off which represents the minimum loss to himself (i.e., minimum gain to the opponent).

### Illustration 2

Let the pay-off matrix  $P$  be the following for a game between A and B.

		B					$l_i = \min_j p_{ij}$
		$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	
A	$s_1$	5	8	7	9	12	5
	$s_2$	15	12	9	17	18	9
	$s_3$	11	6	8	7	9	6
	$s_4$	8	4	2	5	15	2
	$\max_i p_{ij} = h_j$	15	12	9	17	18	

The goal of A is to maximize his gains at the least possible risk to himself, whereas B's aim is to minimize his own losses, i.e., hold A to a minimum of gain.

In our example, if A chooses to play his first strategy,  $s_1$ , then the worst that B can do to A is to play  $t_1$ , thus limiting the pay-off to 5 to A. In other words,  $\min_j p_{1j} = 5$ .

Similarly, if A plays  $s_2$ , B can minimize his (B's) loss by choosing  $t_3$  and the corresponding pay-off would be 9. Thus, if we denote  $l_i = \min_j p_{ij} = \text{minimum over } j \text{ in the } i\text{th row}$ , we have,

$$l_1 = \min_j p_{1j} = 5$$

$$l_2 = \min_j p_{2j} = 9$$

$$l_3 = \min_j p_{3j} = 6$$

$$l_4 = \min_j p_{4j} = 2$$

Now the question arises as to which strategy should A choose if he has to make the first move. If we look at it carefully, we will notice that A will choose to play  $s_2$  because in this case he is assured of a pay-off of at least 9. If he plays any strategy other than  $s_2$ , his gain can be reduced to as low as 5 (if he plays  $s_1$ ), 6 (if he uses  $s_3$ ) and 2 (if he plays  $s_4$ ). These (5,6,2) are less than 9. Therefore, 9 is the maximum of the minimum (called maxi-min) and  $s_2$  is A's maxi-min strategy.

Now let us look at this game from a different angle. Suppose now that B has to move first. If B plays  $t_1$ , the worst that A can do to him is to play  $s_2$ , thus inflicting a

loss of 15 on B; i.e.,  $\max_i p_{i1} = 15$ . If we let  $h_j = \max_i p_{ij}$  (i.e., maximum in the  $j$ th column), then

$$h_1 = \max_i p_{i1} = 15$$

$$h_2 = \max_i p_{i2} = 12$$

$$h_3 = \max_i p_{i3} = 9$$

$$h_4 = \max_i p_{i4} = 17$$

$$h_5 = \max_i p_{i5} = 18$$

For B,  $t_3$  is the safest (best) strategy because in that case, the worst that A can do to B is to play  $s_2$  and inflict on B a loss of 9. However, the choice of any strategy other than  $t_3$  on the part of B may result in a greater gain to A and, therefore, a greater loss to him. For example, if B plays  $t_1$  and A chooses to play  $s_2$  (which he must be rational), the loss to B would be to the tune of 15 (greater than 9). The same holds true for  $t_2$ ,  $t_4$  and  $t_5$ . Therefore, B tries to minimize  $h_j$  over  $j$  and in our example, the  $\min h_j = h_3 = 9$ . This is the minimum of the maximum and  $t_3$  is B's mini-max strategy. Thus for A, the maxi-min strategy is  $s_2$  and

$$\max_i \min_j p_{ij} = \max_i l_i = 9.$$

Similarly for B, mini-max strategy is  $t_3$  and

$$\min_j \max_i p_{ij} = \min_j h_j = 9.$$

Let

$$\max_i l_i = L \quad \text{and} \quad \min_j h_j = H.$$

In our example,  $L = H = 9$  is called the value of the game. Note that 9 is the minimum in its row (row 2) and maximum in its column (column 3) and, therefore,  $p_{23} = 9$  is a saddle point for the game.

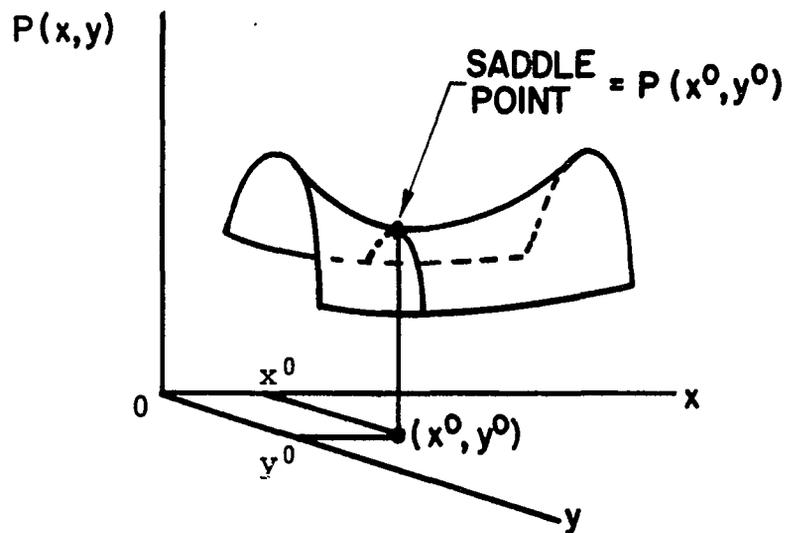


Figure 3. Diagram of a saddle point

The strategies that result in a saddle point are called "optimal" or "equilibrium" strategies. In our example,  $(s_2, t_3)$  is a set of optimal strategies. It is interesting to note that the first subscript of the saddle point  $p_{23}$  is the same as the subscript of the optimal strategy of A and the second subscript 3 is the same as the subscript of the

optimal strategy of B. It should always be kept in mind that  $l_i \leq h_j$  for all  $i$  and  $j$ .

### Finite and Infinite Games

If the number of elements in  $S$  and  $T$  is finite, the game is also finite. If  $S$  and  $T$  contain infinite number of elements, the game is said to be infinite.

Two finger morra, tick-tack-toe, stone paper scissors, etc., are the games of finite type. Missile race, war between two nations, some search type games are the examples of infinite games. In agriculture, for example, the states of nature in terms of climatic conditions can be broadly termed as finite. However, the game can be converted into an infinite game when the state of nature is subdivided into infinite number of categories if very minute differences in temperature prevailing on different days and different time, precipitation, length of the growing period allowed by nature in different years, duration and frequency of heat and cold waves and their intensity, etc., are taken into account to describe different states of nature. Thus, whether a game is finite or infinite also depends on the criteria by which the strategies are distinguished and the extent of (range of) tolerance limits describing a particular state of nature.

## Pure and Mixed Strategies

In Illustration 2 (pp 31-35) we noted that both A and B used only one of the strategies available to each of them. A used  $s_2$  and B  $t_3$ . Also there existed a saddle point  $p_{23}$ . Such games are called 'strictly determined games', 'games of pure strategy', or 'simple games'.

However, it is not always necessary that a saddle point may exist. In such cases, the players do not have optimal pure strategies but rather a mixture of different strategies available to them with some probability attached to each (by some chance mechanism). If we define  $n$ -dimensional simplex  $S^n$  in  $n$ -dimensional Euclidean space  $E^n$  as

$$S^n \equiv X = \left[ \begin{array}{l} (x_1, x_2, \dots, x_n) / x_i \geq 0, i = 1, \dots, n \\ \sum_{i=1}^n x_i = 1 \end{array} \right]$$

and similarly if we define  $m$ -dimensional simplex  $T^m$  in  $E^m$  ( $m$ -dimensional Euclidean space) as

$$T^m \equiv Y = \left[ \begin{array}{l} (y_1, y_2, \dots, y_m) / y_j \geq 0, j = 1, 2, \dots, m \\ \sum_{j=1}^m y_j = 1 \end{array} \right]$$

Then these  $x_i$  and  $y_j$  can be interpreted as probability

distributions for different components of  $S$  and  $T$ . The point  $x_i \in S^n$ ,  $x_i = (0, 0, \dots, 1, 0, 0, 0)$  where  $x_i = 1$  in the  $i$ th place and zero elsewhere is the pure strategy of player A. Similarly, the point  $y_j = (0, 0, \dots, 1, 0, \dots, 0)$  in  $T^m$  is the pure strategy of player B. And  $\theta(G) = (S^n, T^m, P(x, y))$  is called the mixed extension of the game  $G = (S, T, P)$  where

$$P(x, y) = \sum_{i=1}^n \sum_{j=1}^m x_i y_j P_{ij}$$

is called the expected pay-off if A plays his  $i$ th strategy with probability  $x_i$  and B employs his  $j$ th strategy with probability  $y_j$ . Note that

$$P(i, y) = \sum_{j=1}^m y_j P_{ij}$$

$$P(x, j) = \sum_{i=1}^n x_i P_{ij}$$

and, therefore,

$$P(x, y) = \sum_{i=1}^n x_i P(i, y) = \sum_{j=1}^m y_j P(x, j)$$

The pay-off matrix of Illustration 1 is an example of a game without a saddle point.

		B	
		$t_1$	$t_2$
A	$s_1$	2	-3
	$s_2$	-3	4

Suppose that the rules of the game (in Illustration 1) require that one player shows his fingers first and then the other player shows his. Given this, if A is the maximizer, he has to make the first move and plays his first strategy, i.e.,  $s_1$ . Since B has a knowledge of this before he makes his move, it is clear that he will employ his second strategy,  $t_2$ . Likewise, if A uses  $s_2$ , B would gain the maximum by using  $t_1$ . Thus for A,

$$\min_j p_{1j} = p_{12} = -3$$

$$\min_j p_{2j} = p_{21} = -3$$

and the

$$\max_i \min_j p_{ij} \equiv L = p_{12} = p_{21} = -3$$

Similarly for B,

$$\max_i p_{i1} = p_{11} = 2$$

$$\max_i p_{i2} = p_{22} = 4$$

and

$$\min_j \max_i p_{ij} \equiv M = p_{11} = 2$$

Since  $L \neq H$ , the game does not have a saddle point and the players will mix their strategies with some probabilities

rather than using their pure strategies.

The principle in the case of mixed strategies is the same as that in pure strategies. Let the value of the mixed extension of  $G$  [which we termed as  $\theta(G)$ ] be called  $V(\theta)$ . Then  $A$  will try to mix his strategies in a manner that

$$\max_x \min_y p(x,y) \geq V(\theta)$$

Similarly,  $B$  will try to play his strategies with probabilities  $y_j$  such that

$$\min_y \max_x p(x,y) \leq V(\theta)$$

In a finite game,

$$\max_x \min_y p(x,y) = \min_y \max_x p(x,y) = V(\theta)$$

For our example let  $x_1$  be the probability attached to  $s_1$  and  $x_2$  to  $s_2$ . In our example, as there are only two strategies,  $x_1 + x_2 = 1$  or  $x_2 = 1 - x_1$ . If  $y_1$  is the probability attached by  $B$  to  $t_1$  and  $y_2 = 1 - y_1$  to  $t_2$ , then

$$(x_1)(p_{11}) + (x_2)(p_{21}) = (x_1)(p_{12}) + (x_2)(p_{22}) = V(\theta) \quad I$$

and

$$(y_1)(p_{11}) + (y_2)(p_{12}) = (y_1)(p_{21}) + (y_2)(p_{22}) = V(\theta) \quad II$$

Substituting  $(1-x_1)$  for  $x_2$  and values of  $p_{ij}$  in  $I$ , we have

$$2x_1 + (1-x_1)(-3) = (x_1)(-3) + (1-x_1)4$$

or

$$x_1 = \frac{7}{12}, \quad x_2 = \frac{5}{12} \quad \text{and} \quad V(\theta) = -\frac{1}{12}$$

Similarly,

$$y_1 = \frac{7}{12}, \quad y_2 = \frac{5}{12} \quad \text{and} \quad V(\theta) = -\frac{1}{12}$$

In practice very few games have solutions with pure strategies. Generally a mixture of strategies has to be employed to reach a value of the game.

#### Games in the Normal and Extensive Forms

Games of the form  $G = (S, T, P)$  fall into the category of 'normal' games. The components of this form of game, as we have seen, are a given set of players, sets of pure strategies available to each of these players and the pay-offs that result from the particular strategies employed by the players out of their strategy sets.

The games in the 'extensive' form are represented by means of a game tree. A game tree has nodes, branches and terminal pay-offs resulting from a combination of moves by the players. Each node of the tree represents a move and is characterized by the fact that at each node a player has a certain amount of information available to him with regard to what has already happened up to the point this move is to be

made by him. These are, therefore, termed as 'information sets' of a player. A player may have all the information about the happenings just preceding his present move. In this case his move is based on what is termed as 'perfect information'. His information set is called a "perfect information set" and the game where all the information sets available to all the players are 'perfect' is called a "perfect information game". Every perfect information game has a pure value and, therefore, a saddle point.

However, in many cases, the player may not be aware of all the events to the point of making his present move. Then his move is not based on a perfect information. Such an information set is an 'imperfect information set' and the games involving such sets are called "imperfect information" games.

#### Illustration of a perfect information game

The following example, though very arbitrary, would serve the purpose. The game is as follows:

- a. Player F has two rectangular pieces A and B. The initial position of his pieces is such that A is on Square No. 1 and B on Square No. 3.
- b. Player G (his opponent) has two round pieces C and D with C on the 7th and D on the 9th square before the start of the game.
- c. F starts the game and has to make the first move.

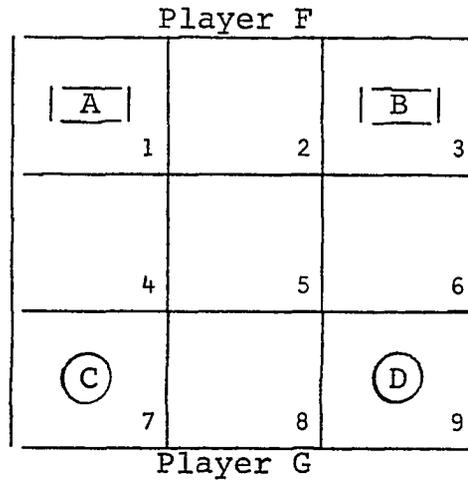


Figure 4. Initial positions of F and G

- d. Both players can move only vertically. They cannot move horizontally or diagonally.
- e. Jumping is allowed. If a piece belonging to a player jumps a piece of his opponent, the latter loses his jumped piece.
- f. The game is over as soon as a piece is jumped and the player whose piece jumps over the opponent's piece wins the game.
- g. The loser pays the winner \$1.
- h. Each player, before he makes his *i*th move, has a full knowledge (perfect information) of all the previous moves that have been made. This can be had by looking at the position of pieces on the board.

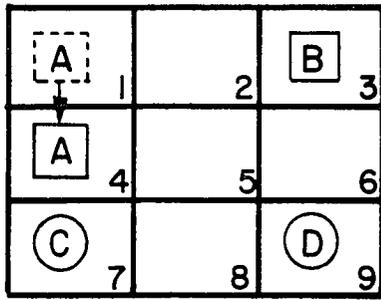
The following are the possible moves:

- I. Player F's first move will be one of the following:
  - I.i. Move A to 4th square.
  - I.ii. Move B to 6th square.
- II. G's first move will depend on the first move of F; i.e., whether F's move was I.i or I.ii.
  1. In case of I.i, G can either,
    - (a) jump C over A to Square No. 1. Call it I.i.a. Then the game is over and won by G.

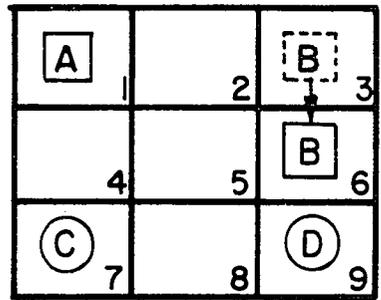
or

    - (b) move D to Square No. 6. Call it move I.i.b.
  2. In the event of move I.ii., G can either,
    - (c) Move C to Square No. 4. Call it move I.ii.c.
    - (d) Jump D over B to Square No. 3. The game is won by G. This is move I.ii.d.
- III. F's second move. It will depend on his initial moves I.i and I.ii and also whether G made his (a), (b), (c) or (d) move.
  - (e) In case of I.i.a. No further move as the game won by G.
  - (f) In case of I.i.b. F jumps B over D to Square No. 9. This is move I.i.b.f. The game won by F.
  - (g) In case of I.ii.c. F jumps A over C to Square No. 7. Call it move I.ii.c.g. The game won by F.
  - (h) In case of I.ii.d. No move by F as he has already lost the game to G.

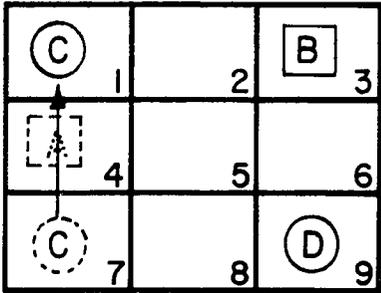
All the moves described above can be represented in the form of a game tree (see Figure 6). The pay-offs are shown



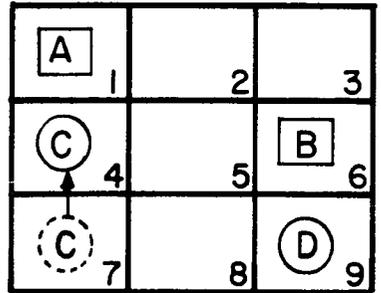
5.a MOVE I.i



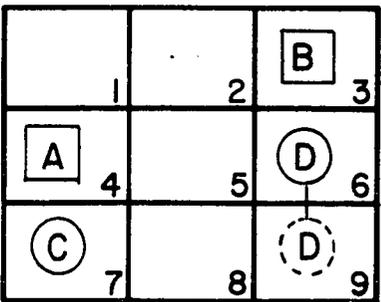
5.b MOVE I.ii



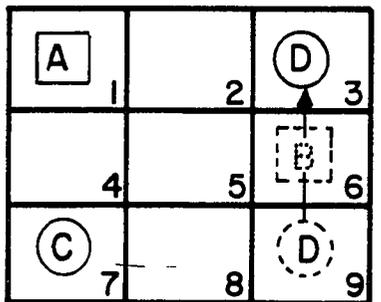
5.c MOVE I.i.a



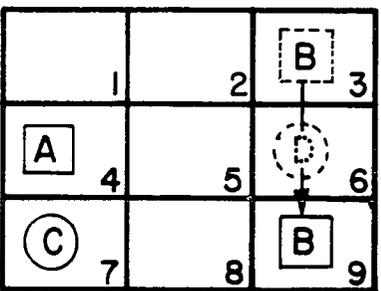
5.d MOVE I.ii.c



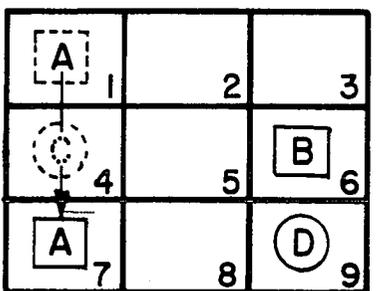
5.e MOVE I.i.b



5.f MOVE I.ii.d



5.g MOVE I.i.b.f



5.h MOVE I.ii.c.g

Figure 5. Possible moves by F and G

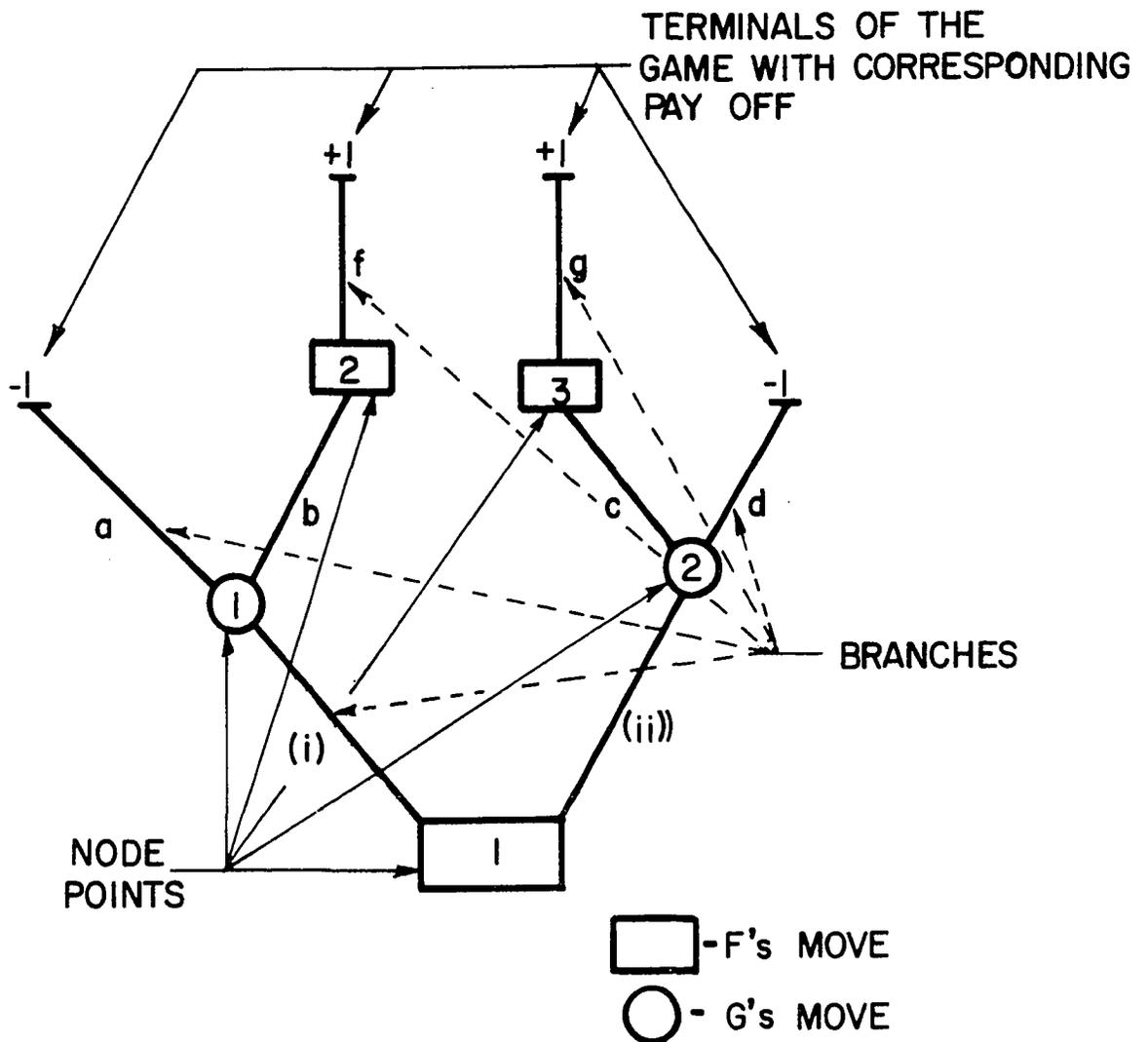


Figure 6. Game tree with perfect information sets

from F's viewpoint. (If we want to show the pay-offs from G's point of view, the tree will remain the same, only the signs of pay-offs will be reversed.)

$\boxed{1}$ ,  $\boxed{2}$ ,  $\boxed{3}$ ,  $\textcircled{1}$ ,  $\textcircled{2}$  are the node points of the tree. The rectangles represent the possible moves that F might make and the circles represent the possible moves of G. Only one player makes a move at a given node point. A 'move' in a game tree consists of going from one node point to another higher node point. The numbers in the node points represent the information sets of the two players as we shall explain later.

Let us study the tree. At  $\boxed{1}$  (his first move), F has two alternatives—(i) or (ii)—available to him. Thus, alternatives open to a player at a given node point are represented by the number of branches that emanate from that node point. If F chooses (i), G has two alternatives, a or b, and that is why two branches from  $\textcircled{1}$ . If he chooses (a), he wins the game and, therefore, the pay-off to F is  $-\$1$ . However, if G chooses (b), then F makes his move (f) and wins  $\$1$ . Therefore, the pay-off is shown to be  $+1$ . Similarly we can analyze the game when F's initial move is (ii). It would be worth noting that (1) if both players are rational and (2) if F makes the first move, player G always wins  $\$1$  and F always loses  $\$1$ .

### Information sets

At (1), the information set available to player G is different from that at (2). At (1), G knows that F has made move (i) and at (2), G knows that F's move was (ii) which is different from (i), and that is why the difference in the information set. Similarly at  $\overline{2}$ , F knows that while his initial move was (i), G's move was b; whereas at  $\overline{3}$ , his information set consists of his knowledge that his initial move was (ii) followed by move c on G's part. Thus each number in the node point denotes a different information set. In a game of perfect information, the total of node points for each player is equal to the total number of his information sets. Since our example is of a perfect information game, we find that there are 3 node points representing 3 possible moves of player F and his information sets are also 3 in number. Likewise, G has 2 node points and 2 information sets.

- As we shall see later, in the case of 'games of imperfect information', the number of information sets of a player is less than the number of node points corresponding to his moves. At some of his node points the same information may be available to a player and then all such node points are assigned the same number denoting the same information set.

### From extensive to normal form

All the games in the extensive form can be converted into those of a normal form. We shall show this for the game tree

drawn in our example.

Since there are 3 information sets of F, his strategy set would consist of moves ( $|\underline{1}|$ ,  $|\underline{2}|$ ,  $|\underline{3}|$ ). At  $|\underline{1}|$ , he has two alternatives, (i) or (ii); but at  $|\underline{2}|$  he has only one move, (f); and at  $|\underline{3}|$ , he has only (g) open to him. Similarly, the strategy set for player G will consist of (①), (②). At ①, he can either make move (a) or (b) and at ② he can choose between (c) and (d).

The strategies of F then are (i, f, ·) and (ii, ·, g) where · represents a void (not available) strategy. It means that if F chooses i at  $|\underline{1}|$ , then  $|\underline{3}|$  and, therefore, alternative (g) is out of the question. If he chooses (ii) at  $|\underline{1}|$ , then move  $|\underline{2}|$  and, therefore, (f) does not come into the picture.

The strategies of G can also be found out in the same fashion. The 4 strategies open to him are: (a,c), (b,d), (a,d) and (b,c).

The strategies and the corresponding pay-offs are shown in the pay-off matrix P shown on the following page.

According to the rules of the game, F always makes the first move and loses if player G is rational; if F uses  $s_1$ , G will always employ  $t_1$  or  $t_2$  and win \$1. If F plays  $s_2$ , G will use either  $t_3$  or  $t_4$ . It may be pointed out that if G started the game, he would always lose.

		G				
		$t_1$	$t_2$	$t_3$	$t_4$	
(   1     2     3   ) ( (1) (2) )		(a,c)	(a,d)	(b,c)	(b,d)	
F	$s_1 = (i, f, \cdot)$	-1	-1	1	1	-1
	$s_2 = (ii, \cdot, g)$	1	1	-1	-1	-1
	$\max_j p_{ij} = h_j$	1	1	1	1	

### Illustration of an imperfect information game

In an imperfect information game, either some information is withheld from the players or a chance move is involved. Most of the games requiring simultaneous moves by the players are games of imperfect information. The two-finger game described earlier is an example of such a game. Since both the players in that example are required to show their fingers simultaneously, they do not know about the move of the other player. The tree of such a game is given in Figure 7.

It is worth noting that since B does not have any knowledge as to whether A will show one or two fingers, his information sets on both node points (representing his move) are identical. As pointed out earlier, if two node points belonging to a player have identical information sets, the game is an 'imperfect information game'.

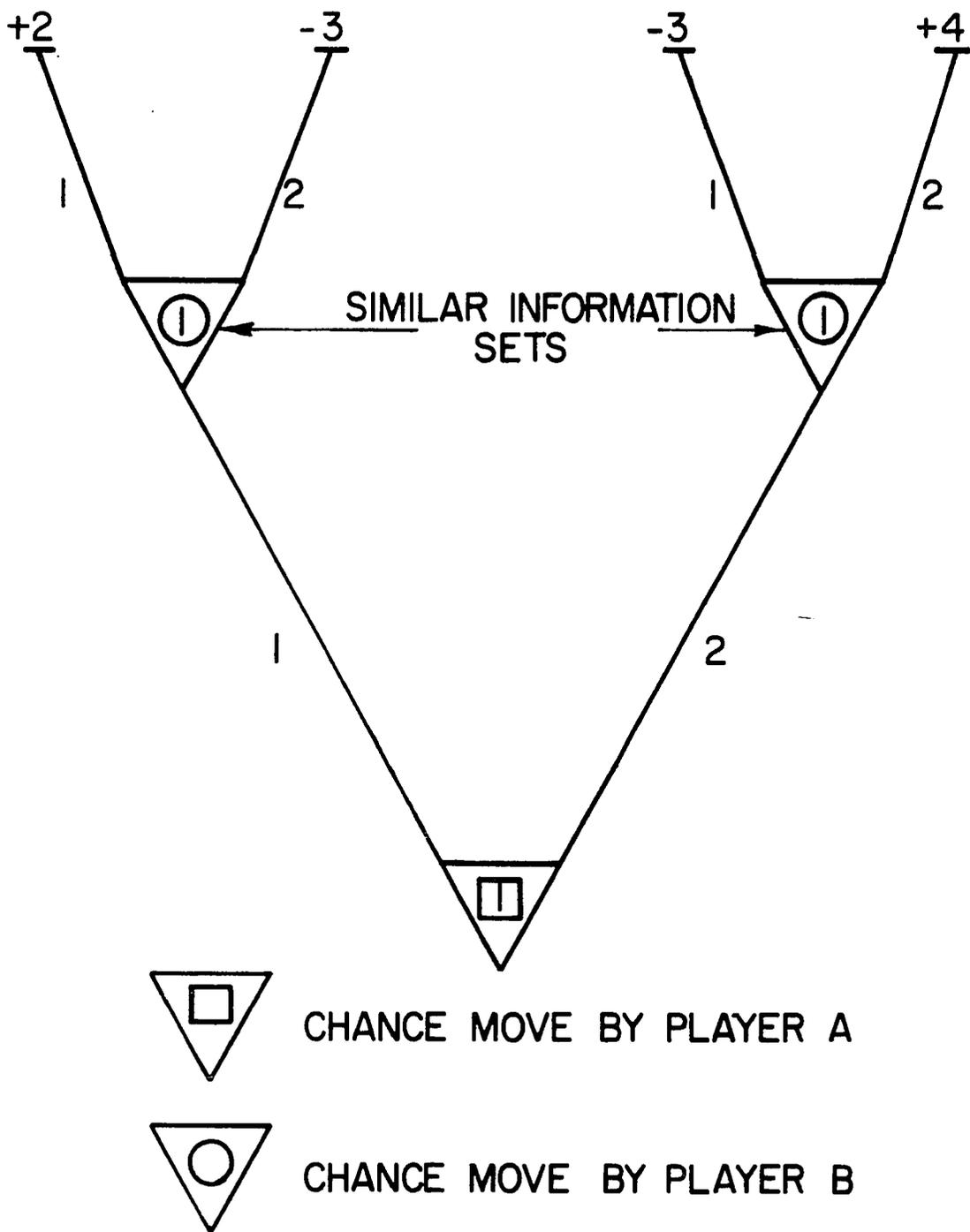


Figure 7. Game tree with imperfect information sets

## Symmetric Games—A Special Form of Matrix Games

The games having the pay-off matrix  $P$  with the property that  $p_{ij} = -p_{ji}$  are called 'symmetric games'. Since such a matrix is known as 'skew-symmetric' matrix, the symmetric games are sometimes also called as 'skew-symmetric' games.

The most important properties of a symmetric game are:

- (a) the value of a symmetric game is zero, and
- (b) the optimal strategy sets for both players of a symmetric game are identical.

Therefore, if we can find out the optimal strategy set  $X^\circ$  for player A, then we have automatically found out the optimal strategy set  $Y^\circ$  for B.

The usefulness of symmetric games lies in the fact that every game can be associated with a symmetric game. One of the most common symmetric games associated with a game having an  $n \times m$  pay-off matrix  $P$  takes the following form.

$$\bar{P} = \begin{array}{c|ccc} & n & m & 1 \\ \hline n & 0 & P & -1 \\ \hline m & -P' & 0 & 1 \\ \hline 1 & 1 & -1 & 0 \end{array}$$

The pay-off matrix  $\bar{P}$  of this symmetric game is of the dimensions  $m+n+1 \times m+n+1$ .

The solution of the symmetric game is used in the following manner to find out the optimal strategy sets and solution

of the original game (with pay-off matrix P):

Let the optimal strategy for symmetric game  $\bar{P}$  be equal to  $(x^0, y^0, \lambda)$  such that

$$x^0 = (x_1, x_2, \dots, x_n)$$

and

$$y^0 = (y_1, y_2, \dots, y_m)$$

if

$$\sum_i x_i = \sum_j y_j = K > 0$$

(because we know that  $x_i \geq 0$  for all  $i$ ,  $y_j \geq 0$  for all  $j$ , and that the set of optimal strategies for both players A and B is the same). Then the game with pay-off matrix P has a solution such that  $\frac{x^0}{K}$  is optimal for A,  $\frac{y^0}{K}$  is optimal for B, and  $\frac{\lambda(x^0, y^0)}{K}$  is the optimal solution for the original game; i.e., the optimal strategy set for game with pay-off matrix P is  $(\frac{x^0}{K}, \frac{y^0}{K}, \frac{\lambda(x^0, y^0)}{K})$ .

### Some Basic Results (Theorems) in Game Theory

The purpose of giving some of the fundamental results in game theory here is to help the reader find the reasons for using certain methods in solving games and to provide him with a greater insight to game theory. No attempt has been made to

provide the proofs, since this step is outside the scope of this study. These results presented are in terms of finite games only. Most of these results also hold for mixed extension of  $G$ .

Let  $G = (S, T, P)$  where  $P$  has elements  $p_{ij}$  with dimensions  $n \times m$ . Let

$$l_i = \min_j p_{ij}$$

$$h_j = \max_i p_{ij}$$

$$H = \min_j \max_i p_{ij} = \min_j h_j$$

$$L = \max_i \min_j p_{ij} = \max_i l_i$$

Then

1.  $l_i \leq h_j$  for all  $i$  and  $j$ .
2. When  $L = H$ , the game has a pure value,  $V$  and  $L=H=V$ .

This is called the mini-max theorem.

3. If  $(s^0, t^0)$  is an optimal solution of  $G$  and  $G$  has a pure value  $V$ , then  $V = P(s^0, t^0)$ .

4.  $(s^0, t^0)$  is a saddle point of  $G$  if, and only if,  
 $P(s, t^0) \leq P(s^0, t^0) \leq P(s^0, t)$  for all  $s \in S$  and  $t \in T$ .

As would appear from the above,

- a. If  $(s^0, t^0)$  is an optimal solution for  $G$ , then it is a saddle point and vice versa.
- b. If  $(s^0, t^0)$  and  $(s_0, t_0)$  are saddle points of  $G$ , then  $(s^0, t_0)$  and  $(s_0, t^0)$  are also saddle

points of  $G$ .

5. Every perfect information game has a pure value.
6. If  $G$  is a finite game such that

$$\sum_{i=1}^n p_{ij} = a \text{ for all } j \quad (j = 1, 2, \dots, m)$$

$$\sum_{j=1}^m p_{ij} = b \text{ for all } i \quad (i = 1, 2, \dots, n)$$

then

$$(a)(m) = (b)(n) \quad \text{and} \quad \frac{b}{m} = \frac{a}{n} = v.$$

7. Two games  $G = (S, T, P)$  and  $G' = (S', T', P')$  are equal if, and only if,
  - a.  $V_G = V_{G'}$  and
  - b. sets of optimal strategies in both games are identical.
8. A perfect information finite game has a saddle point.

#### Different Methods of Solving the Finite Games

##### Method No. 1. Use of maxi-min and mini-max strategies—algebraic solution

Existence of pure mini-max and maxi-min strategies and, therefore, a saddle point Here both players start with the premise that their opponent knows their strategy before making his move and will, therefore, succeed in inflicting on them the greatest loss possible under the circumstances. Therefore, maxi-minimizer looks over the minimum gain or maximum loss

that may be allowed to him by his opponent and then chooses either the strategy that allows him maximum of the minimum gains from different strategies (maxi-minimizer) or (in case he is a loser) the strategy which minimizes the maximum loss. This is done in the following fashion.

Suppose there are two players A and B playing against each other. Let the pay-off matrix of the game from the viewpoint of player A be  $P$  with elements  $p_{ij}$  ( $i = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, m$ ). Further, if  $S = (s_1, s_2, \dots, s_n)$  is the strategy space of A and  $T = (t_1, t_2, \dots, t_m)$  is the strategy space of B, then A will first of all find out the minimum value in each row ( $i$ th row represents the pay-off to A when he employs  $s_i$ ).

Let  $\min_j p_{ij} = l_i$ , then  $(l_1, l_2, \dots, l_i, \dots, l_n)$  are the minimum values of payoffs in rows 1, 2,  $\dots$ ,  $i$ ,  $\dots$ ,  $n$ , respectively. Then A examines the values of  $l_i$  and chooses the strategy whose  $l_i$  is maximum in value. Suppose  $L = \max_i l_i = l_5$ , then under this method A will always play  $s_5$ , his fifth pure strategy. The philosophy behind this sort of approach is that one starts to plan his play with the assumption that his opponent is going to do the worst to him. That is why he looks at the lowest payoff to himself for every strategy. He (A) then picks up the best of the worst, i.e., maximum of the minimum.

The other player (B) tries to minimize his loss to A.

Payoffs to A due to B's  $j$ th strategy  $t_j$  are given in the  $j$ th column of  $P$  and assuming that his opponent A will try to inflict on him the greatest loss, he finds out as to what would be the maximum that he will have to pay to A if he uses his  $j$ th strategy.

Let  $h_j = \max_i p_{ij}$ , then  $h_1, h_2, \dots, h_j, \dots, h_m$  are the maximum of the payoffs (losses to B) in columns 1, 2,  $\dots, j, \dots, m$ , respectively. Since he tries to minimize his payment to A, he chooses the strategy  $j^0$  such that  $\min_j h_j = h_{j^0} = H$ . For example, if  $j^0 = 2$ , then B can always be sure that if he uses  $t_2$ , the maximum loss to him will be no more than the minimum of the maximum loss for each  $t_j$ . Thus  $t_2$  is B's mini-max strategy.

By our assumption of existence of a saddle point,  $H = L =$  value of the game and if  $s_5$  is optimal pure strategy of A and  $t_2$  is optimal pure strategy for B, the value of the game will be the payoff at  $p_{52}$ . Illustration 2 (given earlier) is a numerical example of using this method.

Mixed strategies If  $P$  does not have a saddle point, the players mix their strategies with some probabilities. The principle is the same as for pure strategies. In fact, pure strategies are a special case of mixed strategies. In pure strategy case, the probability of using the mini-max and maxi-min strategy is one and is zero for the other strategies of both the players. When strategies are mixed, positive

probabilities are assigned by each player to more than one of his own strategies. (Please see page 37.) The algebraic method of calculating these probabilities has already been illustrated on pages 39 and 40.

Dominated and dominant strategies      The process of finding mini-max and maxi-min strategies, especially if the game does not have a pure value, becomes very cumbersome if the size of the pay-off matrix is very large. In most cases it is possible to reduce the size of the matrix by using the method of deleting the strictly dominated rows and strictly dominating columns from P.

Definition: Row  $i'$  is strictly dominated by row  $i$  iff for all  $j$ ,  $p_{ij} > p_{i'j}$ . Column  $j$  strictly dominates column  $j'$  iff for all  $i$ ,  $p_{ij'} < p_{ij}$ .

The reason for deleting the strictly dominated rows and strictly dominating columns may be illustrated by a simple example given below.

Let A and B play a game with the following pay-off matrix from the viewpoint of A.

		B		
		$t_1$	$t_2$	$t_3$
A	$s_1$	15	16	10
	$s_2$	22	19	5
	$s_3$	12	3	9

By using the conventional mini-max and maxi-min principles, we see that  $s_1$  is the optimal pure strategy of A,  $t_3$  for B, and the value of the game is 10.

However, on looking at the pay-off matrix more closely, we note that A will always prefer to play  $s_1$  over  $s_3$  since  $p_{1j} > p_{3j}$  for all  $j$ . By the same token, B will always prefer  $t_3$  over  $t_1$  as for all  $i$ ,  $p_{i1} > p_{i3}$ ; i.e., his payment to A is always less in  $t_3$  than that in  $t_1$  for all  $i$ . Therefore,  $s_3$  is obsolete for A and  $t_1$  is obsolete for B and both can be deleted from S and T without changing the outcome of the game. Thus the original matrix can be reduced to the following form:

		B			
		$t_2$	$t_3$		
A	$s_1$	16	10	≡	$P_1$
	$s_2$	19	5		

As would appear, the value of the game again is 10. In fact, we could further delete  $t_2$  as  $t_2$  in the above reduced matrix  $P_1$  dominates  $t_3$ . Then the pay-off matrix will be reduced to

		B			
		$t_3$			
A	$s_1$	10	≡	$P_2$	
	$s_2$	5			

$P_2$  can again be reduced by throwing out  $s_2$  as it is dominated by  $s_1$  and the value of the game equal to 10 can thus be found out, in many cases, by successive deletion of dominating columns and dominated rows.

The question then arises whether 'weakly' dominated rows and 'weakly' dominating columns could be dealt with in the same manner and whether this would deplete  $X^0$  and  $Y^0$ , the optimal mixed strategy sets of A and B. The answer to both questions is "yes". However, before giving any illustration of the deletion of weakly dominated rows and weakly dominating columns, it seems in order that they are defined here.

Definition: Row  $i$  weakly dominates row  $i'$  if, and only if,  $p_{ij} \geq p_{i'j}$  for all  $j$  and  $p_{ij} > p_{i'j}$  for at least one  $j$  for  $i$  and  $i'$  belonging to  $P$ . Similarly, column  $j$  of  $P$  is weakly dominated by column  $j'$  if  $p_{ij} \leq p_{ij'}$  for all  $i$  and  $p_{ij} < p_{ij'}$  for at least one  $i$ .

We shall use the principle of deleting weakly dominated rows and weakly dominating columns for a  $4 \times 5$  pay-off matrix  $P$  given on the next page.

Steps in reducing the matrix  $P$ :

1. Delete  $s_1$  as it is weakly dominated by  $s_2$  and  $s_4$ , and get  $P_1$ .
2. We get  $P_2$  by deleting  $t_2$  and  $t_5$  and both of them dominate  $t_1$ .

$$P = \begin{array}{c|ccccc} & t_1 & t_2 & t_3 & t_4 & t_5 \\ \hline s_1 & 11 & 18 & 15 & 14 & 19 \\ s_2 & 15 & 19 & 15 & 14 & 25 \\ s_3 & 5 & 11 & 16 & 9 & 8 \\ s_4 & 18 & 18 & 15 & 19 & 26 \end{array}$$

3.  $P_3$  is obtained by deleting  $s_2$  from  $P_2$  as  $s_2$  is dominated by  $s_4$  in  $P_2$ .

4. Delete  $t_4$  from  $P_3$  as it dominates  $t_1$ .

By doing the above four operations on  $P$  we finally get  $P_4$

which is very easily solved with the following optimal

solution;  $X^0 = (0, 0, \frac{3}{14}, \frac{11}{14})$ ,  $Y^0 = (\frac{1}{14}, 0, \frac{13}{14}, 0, 0)$ , and the value of the game is  $\frac{213}{14}$ .

$$\begin{array}{c} P_1 \\ \begin{array}{c|ccccc} & t_1 & t_2 & t_3 & t_4 & t_5 \\ \hline s_2 & 15 & 19 & 15 & 14 & 25 \\ s_3 & 5 & 11 & 16 & 9 & 8 \\ s_4 & 18 & 18 & 15 & 19 & 26 \end{array} \end{array} \quad \begin{array}{c} P_2 \\ \begin{array}{c|ccc} & t_1 & t_3 & t_4 \\ \hline s_2 & 15 & 15 & 14 \\ s_3 & 5 & 16 & 9 \\ s_4 & 18 & 15 & 19 \end{array} \end{array} \quad \begin{array}{c} P_3 \\ \begin{array}{c|ccc} & t_1 & t_3 & t_4 \\ \hline s_3 & 5 & 16 & 9 \\ s_4 & 18 & 15 & 19 \end{array} \end{array} \quad \begin{array}{c} P_4 \\ \begin{array}{c|cc} & t_1 & t_3 \\ \hline s_3 & 5 & 16 \\ s_4 & 18 & 15 \end{array} \end{array}$$

Still more useful application of the 'principle of dominance' lies in the property that if a matrix  $P$  can be partitioned in the following manner

$$P = \begin{bmatrix} P_1 & P_3 \\ P_2 & P_4 \end{bmatrix}$$

where  $P_1, P_2, P_3, P_4$  are the submatrices of  $P$ . Further, if every row of  $P_2$  is strictly dominated by some rows of  $P_1$  or some convex combinations of the rows of  $P_1$  and if columns or some convex combination of columns of  $P_1$  is strictly dominated by every column of  $P_3$ , then  $P_2, P_3, P_4$  may be deleted and the solution for  $P_1$  will also be the solution for the game with pay-off matrix  $P$ . For example, the pay-off matrix  $P$  does not have a saddle point and, therefore, mixed strategies will be used. Moreover, there are no dominated rows or dominating

$$P = \begin{vmatrix} 10 & 4 & 9 & 13 \\ 3 & 7 & 8 & 7 \\ 6 & 5 & 2 & 5 \\ 5 & 4 & 18 & 1 \end{vmatrix}$$

columns. Solving such a game is rather complicated. However, if  $P$  is subdivided into  $P_1, P_2, P_3$  and  $P_4$  such that

$$P_1 = \begin{vmatrix} 10 & 4 \\ 3 & 7 \end{vmatrix} \quad P_2 = \begin{vmatrix} 6 & 5 \\ 5 & 4 \end{vmatrix} \quad P_3 = \begin{vmatrix} 9 & 13 \\ 8 & 7 \end{vmatrix} \quad P_4 = \begin{vmatrix} 2 & 5 \\ 18 & 1 \end{vmatrix}$$

Let  $\gamma_{ij}^{(k)}$  represent the element in the  $i$ th row and  $j$ th column of the  $k$ th submatrix. Then, for example,

$$\gamma_{11}^{(1)} = 10,$$

$$\gamma_{12}^{(3)} = 13 ,$$

$$\gamma_{21}^{(4)} = 18 , \text{ and}$$

$$\gamma_{22}^{(2)} = 4 .$$

If we calculate, we find that

$$\left(\frac{1}{2}\right) (\gamma_{11}^{(1)}) + \left(\frac{1}{2}\right) (\gamma_{21}^{(1)}) > \gamma_{11}^{(2)} \text{ and also } > \gamma_{21}^{(2)}$$

$$\left(\frac{1}{2}\right) (\gamma_{12}^{(1)}) + \left(\frac{1}{2}\right) (\gamma_{22}^{(1)}) > \gamma_{12}^{(2)} \text{ and also } > \gamma_{22}^{(2)}$$

Thus some convex combination of rows of  $P_1$  dominates every row of  $P_2$ . Further,

$$\left(\frac{1}{2}\right) (\gamma_{11}^{(3)}) + \left(\frac{1}{2}\right) (\gamma_{12}^{(3)}) > \gamma_{11}^{(1)} \text{ and also } > \gamma_{21}^{(1)}$$

and

$$\left(\frac{1}{2}\right) (\gamma_{21}^{(3)}) + \left(\frac{1}{2}\right) (\gamma_{22}^{(3)}) > \gamma_{12}^{(1)} \text{ and also } > \gamma_{22}^{(1)}$$

Therefore, some convex combination of columns of  $P_3$  dominates every column of  $P_1$ . We can, therefore, delete  $P_2$ ,  $P_3$ ,  $P_4$  and just solve  $P_1$ . This is much simpler than solving  $P$ . The optimal solution for  $P_1$  is

$$x = \left(\frac{4}{10}, \frac{6}{10}\right) , \quad y = \left(\frac{3}{10}, \frac{7}{10}\right) \quad \text{and} \quad \text{Value} = \frac{58}{10} .$$

The solution in terms of the game with pay-off matrix  $P$  can, then, be written in the following manner:

$$x^0 = \left(\frac{4}{10}, \frac{6}{10}, 0, 0\right)$$

$$y^0 = \left(\frac{3}{10}, \frac{7}{10}, 0, 0\right)$$

$$\text{Value} = \frac{58}{10}$$

Method No. 2. Solution of games in the extensive form

If we have a tree set up for a game, the solution is obtained rather easily. Again the pay-offs are given from A's viewpoint. The algorithm for solving the game then can be stated as follows.

Start from the terminal pay-offs and proceed down the tree in the following fashion:

- (a) Maximize at the node point belonging to the maxi-minimizer, i.e., A.
- (b) Minimize at mini-maximizer's (B's) node points.

Let us solve the problem whose tree is sketched on page 45. We start from the top of this tree and first come across node points  $\overline{2}$  and  $\overline{3}$  belonging to A. As the pay-offs are shown for A (and he is the maxi-minimizer), we maximize at his node points and minimize at B's. Therefore, we maximize at  $\overline{1}$ ,  $\overline{2}$  and  $\overline{3}$ . From  $\overline{2}$  there emerges only one branch (f) which terminates in a pay-off of +\$1. So the maximum value at node point  $\overline{2}$  is +\$1. Similarly at  $\overline{3}$ , we have a value of +\$1.

Now comes B's turn, whose node points are (1) and (2).

He minimizes at his node point. There are two branches from (1) and two from (2). At (1), he has a choice between pay-off of  $-\$1$  and  $+\$1$  to A. Naturally he chooses the minimum of the two, i.e.,  $-\$1$ , which he does when he makes move (a). For (2) he chooses (d) with a pay-off of  $-\$1$  to A.

Again it is A's turn at (1) from which emanate two branches—i and ii. He chooses one with the maximum, but since in our example the pay-offs at (1) and (2) node points (which are connected to  $\underline{1}$  by i and ii) are the same ( $-\$1$  in each case), A is indifferent between the two and the value at the lowest node point (root of the tree) is  $-\$1$ . This value at the root of the tree is the value of the game. For our example the value of the game is  $-\$1$ .

If we denote the branches showing the strategies chosen by each player by, say, thick line (just for the sake of distinction), the sequence in this thick line—which is continuous from the root of the tree to the terminal pay-off—shows the set of strategies followed by the two players. For example, in our case, it is either (i,a) or (ii,d); i.e., if A uses i, then B uses a and if A employs ii, B will use d.

In case of chance moves, we multiply the pay-offs by their respective probabilities and the values so obtained are then used for maximization at A's and minimization at B's node points in the manner described above.

Figure 8 is an illustration of the game tree (page 45)

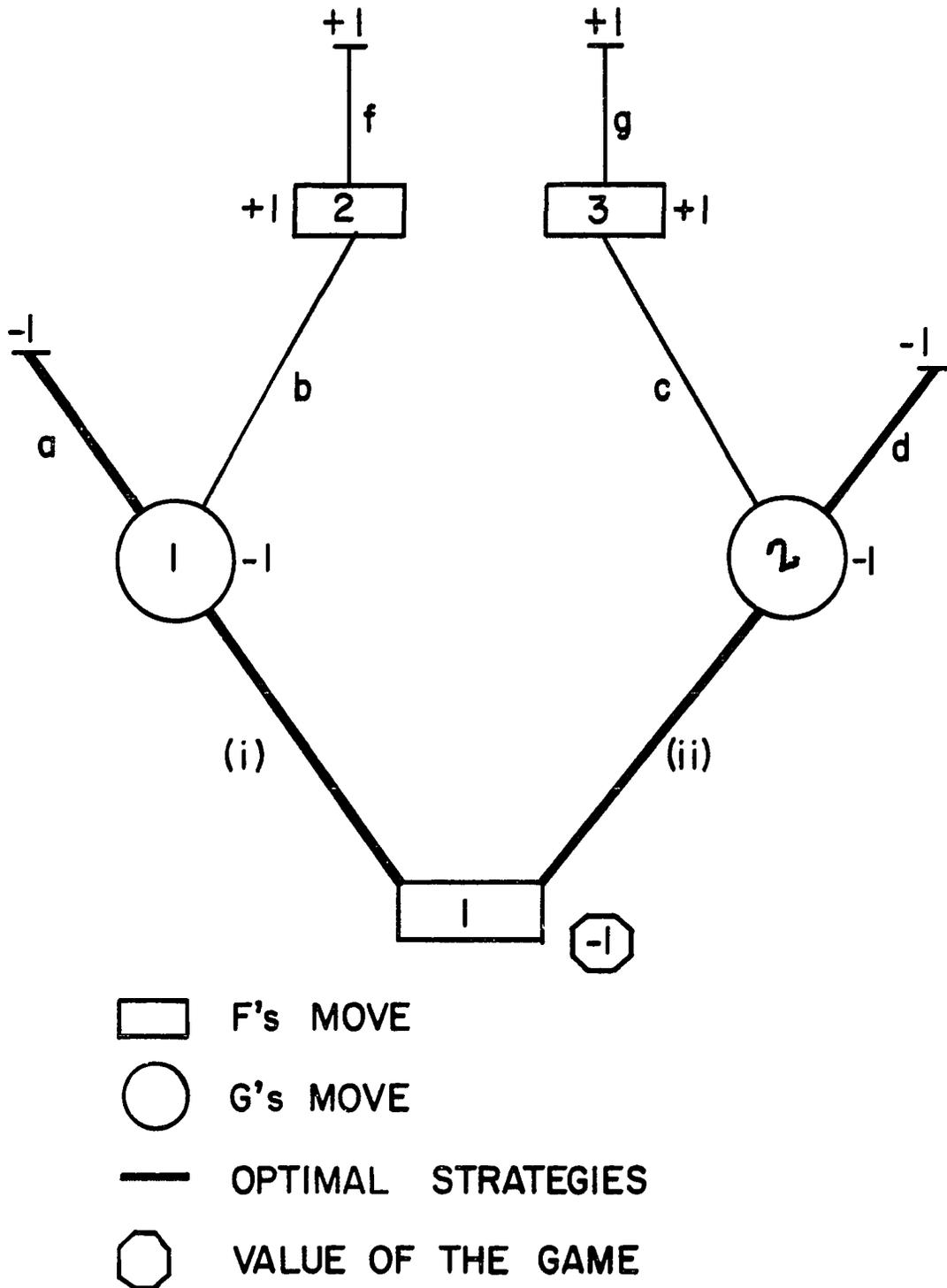


Figure 8. Optimal strategies and value of a game in extensive form

with the maximum and minimum values at different node points and the optimal solution (value) of the game.

### Method No. 3. Geometric methods

If the pay-off matrix is  $n \times 2$  or  $2 \times m$ , the game can be rather easily solved with the help of geometry. Geometric methods are most conducive to a matrix whose vectors (either rows or columns) belong to the  $E^2$  space, though they can also be used for  $n \times 3$  and  $3 \times m$  matrix games.

The two most commonly used geometric methods are:

- a. Line-envelop construction and
- b. Sliding wedge construction.

The former is easier to picture and, therefore, is the more common of the two.

Line-envelop construction Let us explain it by taking the illustration given on page 29. The pay-off matrix to A was:

		B	
		$t_1$	$t_2$
A	$s_1$	2	-3
	$s_2$	-3	4

Case (i). We will plot the lines from A's viewpoint. Let us assume that A attaches a probability  $x_1$  to  $s_1$  and  $1-x_1$  to  $s_2$ .

Define  $l_j = p_{1j}x_1 + p_{2j}x_2$  where  $x_2 = 1-x_1$ , then

$$l_1 = p_{11}x_1 + p_{21}(1-x_1) = (2)(x_1) + (-3)(1-x_1)$$

$$l_2 = p_{12}x_1 + p_{22}(1-x_1) = (-3)(x_1) + (4)(1-x_1).$$

Then we take an envelop of the lowest lines and choose the highest point (equivalent to a maxi-min). The envelop is NQR and the maximum is attained at Q. Draw a perpendicular PQ from Q on the x axis (see Figure 9). The height of PQ is the value of the game and  $\frac{FP}{FG} = x_1$  and  $\frac{PG}{FG} = 1-x_1 = x_2$ . By measurement,  $PQ = -\frac{1}{12}$ ,  $x_1 = \frac{7}{12}$ , and  $x_2 = \frac{5}{12}$ ; i.e.,  $x^0 = (\frac{7}{12}, \frac{5}{12})$  which is the same as obtained by algebraic solution.

Case (ii). We can also plot these lines from B's point of view. In that case  $l_i = p_{i1}y_1 + p_{i2}y_2$  where  $y_2 = 1-y_1$  and  $y_1$  and  $y_2$  are the probabilities of B's employing  $t_1$  and  $t_2$ . The only difference in this (from that described above for A) is that here we take an envelop of the highest contour lines and choose the lowest point in that envelop (equivalent to mini-max). It can be confirmed that  $V_0$  (G) obtained by this method is the same as that obtained by the diagram on page 68.

The steps in the line-envelop construction method can be summarized as follows. Let

$$P = \begin{bmatrix} p_{11}, p_{12}, p_{13}, \dots, p_{1m} \\ p_{21}, p_{22}, \dots, p_{2m} \end{bmatrix}$$

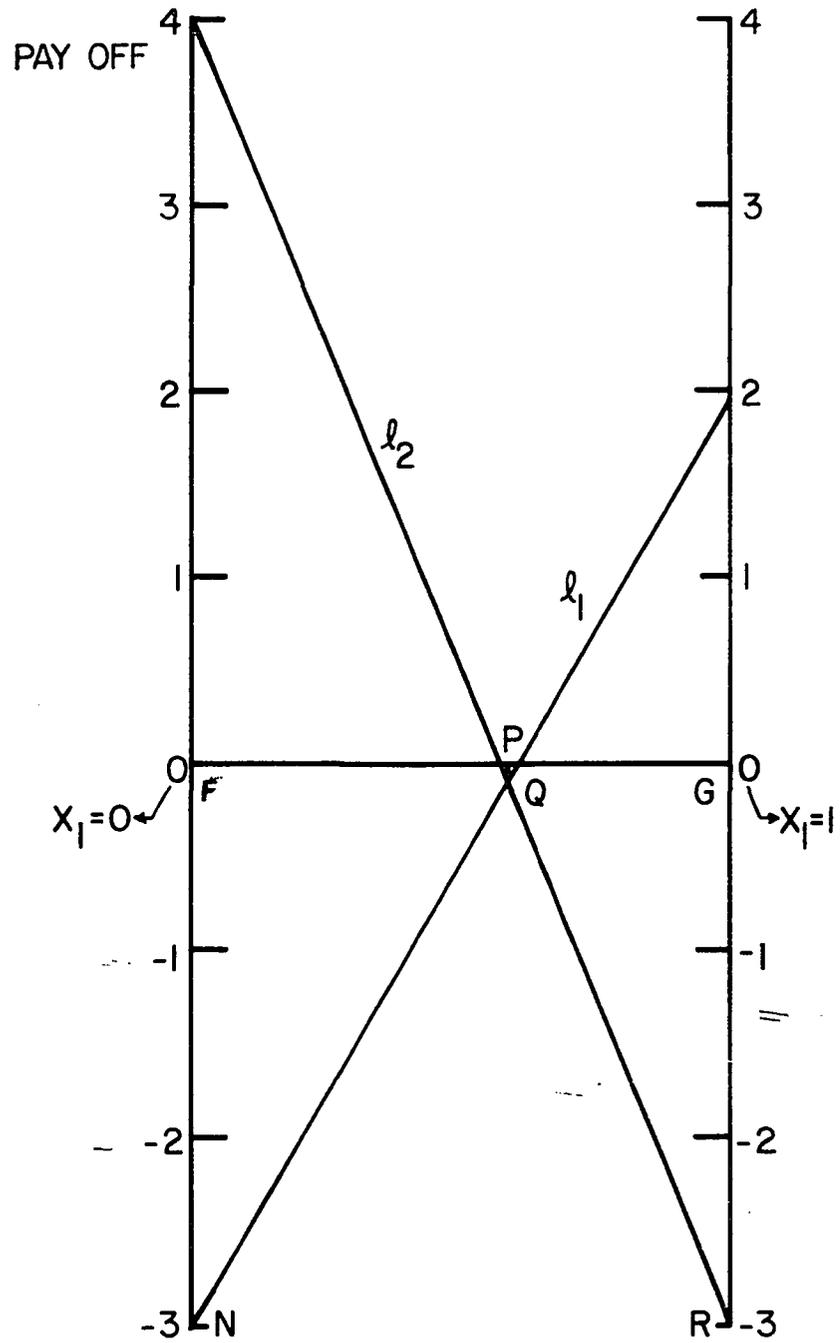


Figure 9. Line-envelop construction

Step 1: Construct  $m$  lines  $l_j(x_1) = x_1 p_{1j} + (1-x_1)p_{2j}$   
 $0 \leq x_1 \leq 1, j = 1, 2, \dots, m$

Step 2: Define an envelop function

$$(i) \quad \phi(x_1) = \min_j l_j(x_1) \text{ for case (i) above.}$$

$$(ii) \quad \phi(y_1) = \max_i l_i(y_1) \text{ for case (ii).}$$

Step 3: Find point

$$(a) \quad x_1^0 \text{ such that } \phi(x_1^0) = \max_{x_1} \phi(x_1) \text{ for case (i) and}$$

$$(b) \quad y_1^0 \text{ such that } \phi(y_1^0) = \min_{y_1} \phi(y_1) \text{ for case (ii).}$$

Step 4: Find  $z$  such that

$$z \cdot l_1(x_1) + (1-z) \cdot l_2(x_1) = \phi(x_1^0)$$

for case (i), or for case (ii) find  $h$  such that

$$h \cdot l_1(y_1) + (1-h) \cdot l_2(y_1) = \phi(y_1^0).$$

As pointed out, this method is quite handy for problems with pay-off matrix of  $2 \times m$  or  $n \times 2$  dimensions.

Sliding-wedge construction To explain this method

also we shall use a  $2 \times m$  matrix. Let  $P$  = the same as given in line envelop construction example. Then

Step 1: Plot  $m$  points in 2-space. The  $j$ th point is

$$= (p_{1j}, p_{2j}), j = 1, 2, \dots, m.$$

Step 2: Construct a convex hull of these points. A convex hull of a set  $A$  is the smallest convex set that contains set  $A$ . If  $CH(A) = \text{convex hull of } A \subseteq E^n$ , then

$$\text{CH}(A) = \left[ \begin{array}{l} \sum_{i=1}^r \lambda_i x^i \mid \sum_{i=1}^r \lambda_i = 1, \lambda_i \geq 0 \\ \\ x^i \in A \text{ for all } i = 1 \dots r \text{ and} \\ r \text{ can be any integer such} \\ \text{that } r \leq n \end{array} \right]$$

Step 3: Construct a translation of the negative quadrant  $O_{\lambda_0}$  along the line  $a_1 = a_2$  where

$$O_{\lambda} = (a_1, a_2 \mid a_1 \leq \lambda, a_2 \leq \lambda) \quad \text{and}$$

$$\lambda_0 = \min \lambda \text{ such that } O_{\lambda} \cap \text{CH}(A) \neq \phi$$

where  $\phi$  is a null set.

Step 4: Construct a separating line L such that

$$L = \left[ \begin{array}{l} (a_1, a_2) \mid x_1^0 a_1 + x_2^0 a_2 = \lambda_0, (x_1^0, x_2^0) \in A^2 \\ \\ \text{and } x_1^0 + x_2^0 = 1 \end{array} \right]$$

and where

$$x_1^0 a_1 + x_2^0 a_2 \leq \lambda_0 \text{ for all } a_1, a_2 \in O_{\lambda_0}$$

and

$$x_1^0 a_1 + x_2^0 a_2 \geq \lambda_0 \text{ for all } a_1, a_2 \in \text{CH}(A)$$

Step 5: Find a point  $y^0 \in T$  (strategy space of B) such that

$\left[ (p_1 y^0, p_2 y^0) \in O_{\lambda} \cap \text{CH}(A) \right]$ . Then game  $\theta(G)$  has a value equal to  $V_{\theta(G)}$ ;  $x^0$  is optimal for A and  $y^0$  is optimal for B.

Figure 10 explains the above steps.

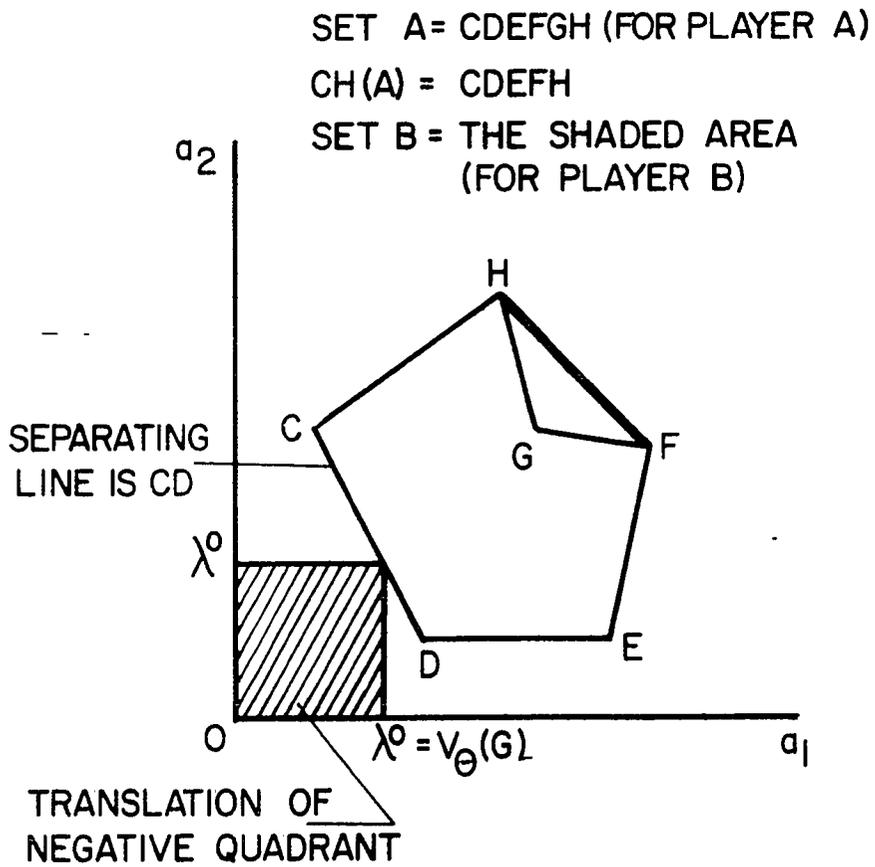


Figure 10. Sliding-wedge construction

Case I.

- (a) Separating line CD is unique  $\rightarrow X^0$  is unique
- (b) Only one point K is common to A and B  $\rightarrow Y^0$  is unique

Case II.

- (a) Separating lines not unique  $\rightarrow X^0$  is not unique
- (b) Only one point common to both A and B  $\rightarrow Y^0$  is unique

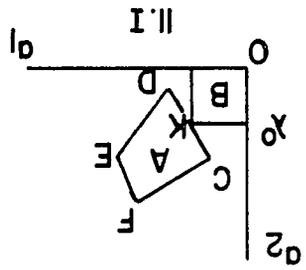
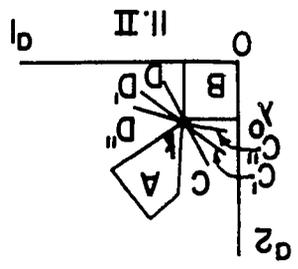
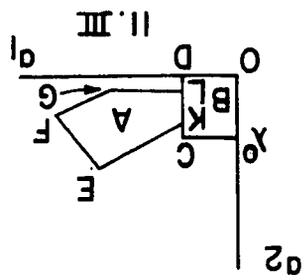
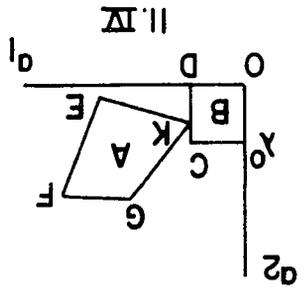
Case III.

- (a) Separating line CD is unique  $\rightarrow X^0$  is unique
- (b) There is a whole set of common points KL  $\rightarrow Y^0$  is not unique

Case IV.

- (a) Separating line CD is unique  $\rightarrow X^0$  is unique
- (b) Only one common point K  $\rightarrow Y^0$  is unique

Figure 11. Different types of sets available for Player A and the choice of strategy sets available to A and B for these different types



## Other methods of solving games

Method of determinants There may be cases (even in a  $3 \times 3$  matrix) where neither there exists any saddle point nor one can see any dominated rows and dominating columns to be deleted to make the job of solving a game easier. In such a case, we can calculate the probabilities by the use of determinants in the following manner.

Step 1: Let  $P$  be  $n \times m$  where  $n \leq m$ . If  $n < m$ , delete any  $m-n$  number of columns from  $P$  and convert it into a square matrix. Call this  $P_R$ . If  $n=m$ , we have the required square matrix for succeeding operations. If  $n \geq m$ , then  $n-m$  number of rows had to be deleted from  $P$  to make it square.  $P_R$  is  $n \times n$ .

Step 2: Subtract the  $i$ th row from the  $(i-1)$ th row of  $P_R$ . Call the resulting matrix  $\bar{P}$ .  $\bar{P}$  has elements  $p_{i',j'}$ , where  $i' = 1, 2, \dots, n-1$  and  $j = 1, 2, \dots, n$ .

Step 3: Leave the first column and take the determinant of the reduced  $n-1 \times n-1$  matrix. The figure thus obtained may be called  $D_1$ . Similarly calculate  $D_2$  by leaving the second column of  $\bar{P}$  and taking the determinant.  $D_3 \dots D_n$  are calculated in the same manner. Then  $Y^0 = (y_1^0, y_2^0, \dots, y_n^0)$ , the optimal strategy set for  $B$ , can be found out in the following manner ( $B$  is mini-maximizer and  $A$  is maxi-minimizer):

$$y_1^0 = \frac{|D_1|}{\sum_{j=1}^n |D_j|}, \quad y_2^0 = \frac{|D_2|}{\sum_{j=1}^n |D_j|}, \quad \dots \quad y_n^0 = \frac{|D_n|}{\sum_{j=1}^n |D_j|}$$

and

$$y^0 = \left[ \frac{|D_1|}{\sum_{j=1}^n |D_j|}, \quad \dots, \quad \frac{|D_n|}{\sum_{j=1}^n |D_j|} \right]$$

The optimum mixed strategy set for A can be found out in the same manner. In this case we take columns instead of rows. The matrix obtained by subtracting the  $j$ th column of  $P_R$  from its  $(j-1)$ th column is  $n \times n-1$ . Call this matrix  $\tilde{P}$ . Its elements are  $p_{ij}$ ,  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, n-1$ .

Again we find out determinants  $C_i$  by leaving out the  $i$ th row,  $i = 1, 2, \dots, n$  and the optimal strategy set  $x^0$  for A is calculated like this

$$x^0 = \left[ \frac{|C_1|}{\sum_{i=1}^n |C_i|}, \quad \dots, \quad \frac{|C_n|}{\sum_{i=1}^n |C_i|} \right]$$

Illustration

Let

$$P = \begin{bmatrix} 3 & -1 & 3 \\ 4 & -2 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

P neither has a saddle point nor any such strategies which can be eliminated according to the dominance criterion. In such a case we resort to the method just explained (method of determinants).

(1) For B

Calculate  $\bar{P}$

$$\bar{P} = \begin{bmatrix} 1 & -1 & -2 \\ -2 & 6 & 2 \end{bmatrix}$$

From this we calculate the values of  $D_1, D_2, D_3$ .

$$D_1 = 10 \quad , \quad D_2 = -2 \quad \text{and} \quad D_3 = 4$$

Therefore,

$$|D_1| = 10 \quad , \quad |D_2| = 2 \quad \text{and} \quad |D_3| = 4$$

The optimal strategy set for B

$$Y^0 = \left( \frac{10}{10+2+4}, \frac{2}{10+2+4}, \frac{4}{10+2+4} \right) = \left( \frac{10}{16}, \frac{2}{16}, \frac{4}{16} \right)$$

$$\text{Value of the game} = V_{\theta(G)_B} = 2.5$$

(2) For A

Calculate  $\tilde{P}$

$$\tilde{P} = \begin{bmatrix} -4 & 4 \\ -6 & 3 \\ 2 & 1 \end{bmatrix}$$

From  $\tilde{P}$  we find the following values of  $C_i$ .

$$C_1 = 0 \quad , \quad C_2 = -4 \quad , \quad C_3 = 12$$

therefore,

$$|C_1| = 0 \quad , \quad |C_2| = 4 \quad , \quad |C_3| = 12$$

and

$$x^0 = \left( \frac{0}{0+4+12}, \frac{4}{0+4+12}, \frac{12}{0+4+12} \right) = \left( 0, \frac{4}{16}, \frac{12}{16} \right)$$

Again  $V_{\theta(G)_A} = 2.5$ .

Thus we find that this method gives us consistent results; i.e.,  $V_{\theta(G)_A} = V_{\theta(G)_B}$ .

In our example  $P$  was a square matrix. If  $P$  is  $n \times m$  where  $n < m$ , say  $m = n+1$ . Then we delete one column arbitrarily (say first column of  $P$  to find  $P_R$ ) and solve in the above described manner. It is possible that we may not get a consistent solution; i.e., if  $V_{\theta(G)_A} \neq V_{\theta(G)_B}$  for some  $x^0, y^0$ , then we do not delete first column from  $P$  to get  $P_R$ , but delete the second column. We go on trying in this manner till we finally reach a consistent solution.

### Applications of Game Theory Models in Agriculture

Though von Neumann and Oskar Morgenstern published their book in 1943-44, it was more than a decade later that the game theory models were applied to agriculture. All the studies

relating to applications of game theory in agriculture have examined the usefulness of a few or all of the following approaches (excluding e) in the process of decision making by a farm firm:

- a. Wald's Maxi-min and Mini-max Criterion
- b. Laplace's Criterion of 'Insufficient Reason' - Naive Theory
- c. Hurwicz Optimism-Pessimism Criterion
- d. Savage's Minimum Regret Approach
- e. Savage's Subjective Probability Criterion
- f. The Theory of the Satisficer as given by Simon
- g. The Shackle Theory of Potential Surprise

These models can be broadly classified into:

- (i) Probabilistic Models, and
- (ii) Strategic Models

Whether a model is probabilistic or strategic depends on the assumptions made. If we know (or assume) that the parameters are not constants but random variables with a known probability distribution, we call it a 'probabilistic model'. In a 'strategic model', we know for granted that the parameter will take only one of a given set of values. These values are acquired from previous experiments or experience. As we shall see, in the former (probabilistic models), we arrive at a pay-off matrix (by using Bayesian methods) by the application of these probabilities and then apply to the pay-off matrix

thus obtained, the maxi-min, mini-max, minimum regret or other criteria to take a decision whose expected value (profit) is the highest. Thus the value of the criterion function depends on the values that the parameter takes as a random variable and on its probability distribution. In a 'strategic model' (also sometimes called as 'game theoretic model' in the sense of being strictly based on Neumann-Morgenstern approach), the value of a game depends on the strategy that a player adopts and the set of values that the parameter can take.

The studies by Dillon and Heady (27) and Walker, Heady, Tweeten and Pesek (106) were of the first few to be in the direction of testing the usefulness of 'game theoretic models' in farmers' decision problems, especially with production decisions under uncertainty. Kelsey and Janssen (53) applied game theory approach in studying bilateral monopoly in farm leasing. The theory lends itself to policy matters in agriculture as has been shown by Langham (58) in his paper on 'Game Theory Applied to a Policy Problem of Rice Farmers'.

The game theory models can be applied to most of the types of decision making problems that a farmer has to face. Right from the time when he makes that big decision of choosing farming as a career, the agriculturist is required to take initial and day-to-day, big and small decisions—all of them important and many of them crucial in one way or the other—in the nature of, for example, where to start the farm? what

should be the size of the farm? should it be purchased or rented? how much investment must he make and in what forms? what should be sources of financing his business and what should be the contribution of each of these sources? what type of resources should he have and in what quantity? what enterprises should he choose and how should he allocate his resources among these selected enterprises? what techniques of production should he use and at what time should he perform a given operation (like sowing, fertilizing, irrigating, feeding, etc.) and to what extent? out of the farm produce, how much should he keep for his own consumption and for his farm and how much should he sell? when should he sell and at what price?

Here it is proposed to describe in brief the important game theory criteria (mentioned on page 78) that are used in decision making and then demonstrate their application (of the first five criteria) to certain practical situations that a farmer faces. Since the last two criteria (i.e., Simon's and Shackel's) are highly subjective and their pay-off matrix rather imprecise in the sense that it is very difficult to write it accurately, we shall not apply them. In the light of the results of the application of the first five criteria [(a) to (e) on page 78], it shall be endeavoured to suggest the suitability of these criteria under different sets of circumstances, because these criteria do not necessarily suggest the same strategy for a given pay-off matrix. As we

shall see, there are some cases when every criterion gives a different result. Finally, the author will suggest a decision making model of his own 'Excess Benefit Criterion', which seems to combine the properties of the Wald's and Regret principles. I should hasten to point out that the applications of these criteria shall be made to only some of the farmer decision situations suggested above as it would be extremely difficult to anticipate and examine every possible situation.

In these game theory models, the other player (playing against the farmer who is to make decisions) can be weather conditions, social and political situations, market condition, insect, pests and diseases of plants and animals, persons or groups that he has to face in making his decisions, etc. This other player is commonly referred to as 'nature' and includes not only climatic conditions and pests and diseases, but also all other things, situations, groups and persons that affect the outcome of the decision taken by the farmer. This nomenclature is adopted for the sake of convenience of analysis. However, sometimes it may be rather confusing. Therefore, the term 'nature' shall be used to denote opponent of a farmer under all situations except when a farmer is directly in conflict with another farmer. For example, if there are two farmers vying with each other to purchase a piece of land from a third party and bid the price of that piece up, this we

shall call as a game against a 'person'. However, the general price level of land in the market is a 'state of nature'.

This distinction between nature and person, as an opponent of the farmer, seems to be necessary from the viewpoint that when two persons are trying to compete against each other (their interests are conflicting), each tries to do the worst to his competitor. However, in most other cases (which we have termed as 'states of nature'), this is not true. For example, the nature (in the sense of climate, etc.) does not always try to do the worst to the farmer and, therefore, cannot be treated in the same manner as the competing farmer, in our example, bidding up prices. In our applications and examples, the farmer will be playing games against nature (in the sense described in our definition), unless otherwise specified.

One of the distinguishing features of game theory models with 'nature' as an opponent is that it is only the farmer who gains or loses and the outcome of the game does not make the 'nature' poorer or richer. 'Nature' is passive in the sense of gains and losses to itself. Therefore, in these models, generally speaking, it is meaningless to operate on columns and find out the resulting pay-off to nature. The pay-off matrix in our examples will always be from the viewpoint of the farmer and, therefore, we shall be doing operations on rows only and not on columns unless required by the algorithm in the model or otherwise necessary.

Different criteria of choice

Wald's model (maxi-min criterion) It is the conventional criterion of solving a given pay-off matrix of a game directly by taking the minimum in each row and then choosing the strategy which provides the maximum pay-offs of these row-minimums.

Again, if  $P$  equals pay-off matrix of a game with  $n \times m$  dimensions and  $G$  equals  $(S, T, P)$  with  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, m$ , then  $\min_j p_{ij} \equiv l_i$  for all  $i$ . Take  $\max_i l_i \equiv l_{i^0}$ . Here the farmer chooses his  $i^0$ th strategy. Playing in this manner, the farmer assures himself of a certain minimum under the worst circumstances. This  $i^0$  is his pure strategy which is equivalent to choosing a single crop, raising only one kind of cattle, or feeding a single type of ration to his animals. However, as it is not always necessary for a pay-off matrix to have a saddle point, the farmer may be required to mix his strategies in order to arrive at a solution. Mixing strategies in farming will amount to choosing a combination of crops or livestock or both and is, therefore, completely relevant and meaningful in the context of decision-making in agriculture. The optimum solution of mixed strategies can be found by the geometric method (if there are either only two states of nature or if the farmer has only two strategies), or by the use of algebraic method described earlier.

As would appear, this criterion is rather conservative,

but perhaps this is jumping the gun. We shall compare the pros and cons of all the criteria after we have discussed them and examined their applicability.

Laplace's criterion—the application of the law of averages One of the most naive and widely used models of selecting a strategy or strategies is called the Laplace's criterion, based on the 'principle of insufficient reason'. The argument for the application of this criterion is that since the decision maker does not have any knowledge of the 'state of nature' that is going to prevail in the period for which he is to make a decision, it should be based on the assumption that each state of nature was equally likely to occur. In this criterion, equal probabilities are assigned to each state of the nature appearing in the payoff matrix and then the strategy with the maximum expected pay-off is chosen. In other words, this is the application of the law of averages and the strategy with the best average gets selected. As there are  $m$  possible 'states of nature', the weight assigned to each column ('state of nature') is  $1/m$ .

Let the expected pay-off (calculated by Laplace's criterion) to the farmer for his  $i$ th strategy be  $k_i$ , then

$$k_i = \sum_{j=1}^m p_{ij} m^{-1} = m^{-1} \sum_{j=1}^m p_{ij}$$

Let  $\max_i k_i = k_{i^0}$ , then the farmer will choose  $i^0$ th strategy.

As pointed out, here the principle of arithmetic average is used. This has been the conventional method with the researchers and the extension workers in arriving at the conclusions of their experiments and making recommendations. The statistical tests of significant difference between treatments are also based on the averages of these treatment values.

Hurwicz "Optimism-Pessimism" criterion This criterion takes into account only the highest and the lowest pay-offs in a given row and ignores the others.

Let a constant  $r$ ,  $0 \leq r \leq 1$ , represent the optimism index of the farmer. This implies that the farmer attaches a probability of  $r$  to the highest pay-off in any row of the matrix;  $1-r$  is the probability attached by the farmer to the getting of the lowest pay-off in any row. This  $1-r$  is termed as the 'pessimism index'. The expected pay-off for each strategy (row) is then calculated in the following manner and the strategy with the highest expected pay-off is chosen.

Let  $C_i$  be the expected pay-off to the farmer for his  $i$ th strategy (row). Then

$$C_i = r \cdot \max_j p_{ij} + (1-r) \cdot \min_j p_{ij}$$

$\max_i C_i = C_{i^0}$  and the  $i^0$ th strategy is selected by the farmer under this criterion.

It is clear that at  $r = 0$ , Hurwicz' model is equivalent to that of Wald's.

This method reminds one of 'Range' as a measure of dispersion. Range takes into consideration only the highest and the lowest values and is not affected at all by the values other than these. Hurwicz' method does exactly the same. It takes into account only the extreme values and ignores other data completely. It should be pointed out that to get reliable and consistent results in this method, the coefficient of correlation between  $r$  and the 'state of nature' should be zero. However, the choice of  $r$  and, therefore, that of the strategy would depend on the degree of optimism or pessimism of the decision-maker.

Minimum regret criterion of Savage      The philosophy underlying the use of this model is that a decision-maker always tries to minimize his regret. The regret to a farmer, if he uses his  $i$ th strategy and if  $j^0$ th 'state of nature' prevails, is defined as

$$P_{ij^0} - \max_i P_{ij^0}$$

Let  $R$  be the regret matrix with elements  $r_{ij}$  and  $\max_i P_{ij}$  be equal to  $P_{i^0j}$  for a given  $j$ , then

$$r_{ij^0} = P_{ij^0} - P_{i^0j^0} .$$

In other words,  $r_{ij}$  is the magnitude of difference between the actual pay-off to a farmer for following his  $i$ th strategy and the (maximum) pay-off that could have been obtained if the

farmer had known the true 'state of nature' that actually prevailed. So  $R$  is derived from the original pay-off matrix. Clearly,  $r_{ij} \leq 0$  for all  $i$  and all  $j$ .

For selecting the most desirable strategy, Wald's criterion is applied to matrix  $R$ . Let  $l_i^R = \min_j r_{ij}$  and  $\max_i l_i^R = l_{i^0}^R$ . The farmer selects  $i^0$ th strategy.

This criterion allows for the use of both pure and mixed strategies and the interpretation for these is the same as in the case of Wald's criterion.

We can also apply Hurwicz' and Laplace's principles to the regret matrix  $R$ .

Illustration of the use of these models      Let

		States of nature		
		$t_1$	$t_2$	$t_3$
$P \equiv$	$s_1$	11	3	7
	$s_2$	5	5	14
	$s_3$	21	1	1
	$s_4$	10	13	4

(1) Wald's criterion

$$\begin{aligned}
 l_1 &= 3 \\
 l_2 &= 5 \\
 l_3 &= 1 \\
 l_4 &= 4 \\
 \max_i l_i &= l_2
 \end{aligned}$$

Result: choose  $s_2$

(2) Laplace's criterion

$$E(s_1) = k_1 = \left(\frac{1}{3}\right)(11) + \left(\frac{1}{3}\right)(3) + \left(\frac{1}{3}\right)(7) = \frac{21}{3}$$

$$E(s_2) = k_2 = \left(\frac{1}{3}\right)(5) + \left(\frac{1}{3}\right)(5) + \left(\frac{1}{3}\right)(14) = \frac{24}{3}$$

$$E(s_3) = k_3 = \left(\frac{1}{3}\right)(21) + \left(\frac{1}{3}\right)(1) + \left(\frac{1}{3}\right)(1) = \frac{23}{3}$$

$$E(s_4) = k_4 = \left(\frac{1}{3}\right)(10) + \left(\frac{1}{3}\right)(13) + \left(\frac{1}{3}\right)(4) = \frac{27}{3}$$

$$\max_i k_i = k_4$$

Suggested strategy:  $s_4$

(3) Hurwicz' criterion

Let  $r = .7$  = optimism-index of the farmer. Then  $1-r = .3$  is his pessimism-index. Expected pay-off for  $s_1$  equals

$$\begin{aligned} C_1 &= (r) \cdot \max_j p_{ij} + (1-r) \min_j p_{ij} \\ &= (.7)(11) + (.3)(3) = 8.6. \end{aligned}$$

Similarly,

$$C_2 = (.7)(14) + (.3)(5) = 11.3$$

$$C_3 = (.7)(21) + (.3)(1) = 15.0$$

$$C_4 = (.7)(13) + (.3)(4) = 10.3$$

$$\max_i C_i = C_3 = 15.0$$

Recommendation: select  $s_3$

## (4) Savage's minimum regret criterion

First of all we have to get R from P,

$$r_{ij^0} = p_{ij^0} - \max_i p_{ij^0},$$

for a given 'state of nature'  $j^0$ .

$$r_{i1} = p_{i1} - \max_i p_{i1}$$

$$\max_i p_{i1} = p_{31} = 21$$

therefore,

$$r_{i1} = p_{i1} - 21$$

$$r_{11} = p_{11} - p_{31} = 11 - 21 = -10$$

$$r_{21} = p_{21} - p_{31} = 5 - 21 = -16$$

$$r_{31} = p_{31} - p_{31} = 21 - 21 = 0$$

$$r_{41} = p_{41} - p_{31} = 10 - 21 = -11.$$

Similarly,

$$\max_i p_{i2} = 13 \quad \text{and} \quad \max_i p_{i3} = 14$$

and

$$r_{12} = -10 \quad r_{22} = -8 \quad r_{32} = -12 \quad r_{42} = 0$$

$$r_{13} = -7 \quad r_{23} = 0 \quad r_{33} = -13 \quad r_{43} = -10.$$

The R matrix then is

		States of nature		
		$t_1$	$t_2$	$t_3$
R =	$s_1$	-10	-10	-7
	$s_2$	-16	-8	0
	$s_3$	0	-12	-13
	$s_4$	-11	0	-10

Note that  $r_{ij} \leq 0$

$$l_i^r = \min_j r_{ij}$$

$$l_1^r = \min_j r_{1j} = -10$$

$$l_2^r = -16$$

$$l_3^r = -13$$

$$l_4^r = -11$$

The maximum of the minimum (the least regret) is  $l_1^r = -10$  which is obtained if the farmer chooses  $s_1$ .

Result: farmer should select  $\underline{\underline{s_1}}$

Summary of the results:

Criterion/model	Suggested strategy
Wald	S <sub>2</sub>
Laplace	S <sub>4</sub>
Hurwicz	S <sub>3</sub>
Savage's Regret	S <sub>1</sub>

It would be worthwhile here to recall a suggestion made earlier that it was not necessary that all criteria would suggest the use of the same strategy. One of the purposes of this illustration was to show this divergence in optimal strategies suggested by these four criteria. However, from this one must not rush to conclude that all these models will necessarily suggest different strategies for the same pay-off matrix. In general, this may not be true. As we shall later see in the applications to the problems of agriculture, in many cases, these criteria may come up with the same optimal strategies and in some other situations, completely different choices may be given by these different models.

Savage's subjective probability theory One of the common features of the above four criteria is the underlying assumption that the decision maker is acting under 'complete ignorance', whereas in practice, this is not necessarily true. He, through past experience of his own or other sources or through just his belief, might have some vague idea of the expected state of affairs. For example, a farmer is planning to take a crop in the coming season. The region is wet, but once in a while drought also strikes. In such cases it seems

impossible to assign objective probabilities to these events (of rain and drought), but subjectively he may assign some probabilities to the events of 'rain' and 'no rain'.

Suppose a farmer is faced with the decision as to whether he should take crop 'C' or crop 'D'. Further, assume that if it rains, taking crop 'C' is the best proposition; but if he takes 'C' and it does not rain, the crop fails entirely with no returns to him. The other crop, 'D', does not give as good returns as 'C' when it rains, but does give some returns even when there is 'no rain'. Suppose in utility terms his problem is reduced to the following form:

		State of nature		
		$S_1$	$S_2$	
		(rain)	(no rain)	
$A_1$ (take crop 'C')	1	0		
$A_2$ (take crop 'D')	Y	Y		(where $0 < Y < 1$ )

Here everything hinges on the way he appraises the relative possibilities of 'rain' and 'no rain'. If he assigns a subjective probability  $\geq Y$  to the event 'rain', he would choose crop 'C'. In other words, from his behavior we could deduce whether the 'a priori' probability assigned by him to 'rain' was  $\geq Y$  or  $< Y$ .

Thus by processing one's information (whatever he has), one can generate an 'a priori' probability distribution over the state of nature, which is appropriate for making decisions.

This reduces the decision problem from one of uncertainty to that of risk. The a priori distribution obtained in this manner is called a 'subjective probability distribution'.

To Savage goes much of the credit for developing the above view and, therefore, in the following lines shall be outlined, in brief, the postulates, premise, definitions and theorems as given by him for a personalist theory of decisions. It is important to note that he, without assuming any objective probability, has been able to develop a subjective probability measure.

#### Savage's Theory:

1. Let  $S$  be the set of states of the world with an infinite number of elements  $s, s', \dots$  ( $s, s' \dots$  are exhaustive and exclusive).
2. Events  $E, E', \dots$  are the sets of states or events ( $E \subset S, E' \subset S, \dots$ ).
3. Let the set of consequences be  $C$  whose elements are  $c, c', c'', \dots$ .
4. Let  $\alpha$  be a set of acts whose elements are  $A, A', \dots$ . The acts are arbitrary functions from  $S$  to  $C$ .
5. Each act assigns a consequence to each state of the world.  $(A, s)$  are act-state pairs and a consequence  $A(s)$  from set  $C$  is assigned to each one of these pairs. Thus  $A(s) = c$  means that consequence 'c' is assigned to state 's' by act 'A'.

6. The binary relation  $\succeq$  between a pair of consequences.

$\succeq$  means ' is preferred or indifferent to '.

With these, the postulates and definitions are given below:

POSTULATE 1:  $\succeq$  is a weak ordering of acts. In other words, it can be said that every pair of acts is comparable and transitive. Thus, initially  $\succeq$  is applied to acts and not consequences.

Definition 1:  $A \succeq A'$  if and only if  $B \succeq B'$  for every  $B$  and  $B'$  that agree with  $A$  and  $A'$  respectively on  $E$  and with each other on  $\sim E$ .  $A$  agrees with  $B$  on  $E$  if and only if  $A(s) = B(s)$  for each  $s \in E$  and  $B$  and  $B'$  agree with each other on  $\sim E$  if and only if  $B(s) = B'(s)$  for each  $s \in \sim E$ .

POSTULATE 2: For every  $A, A'$  and  $E$ ,  $A \succeq A'$  given  $E$  or  $A' \succeq A$  given  $E$ , i.e., conditional preference is well defined.

Definition 2:  $c \succeq c'$  if and only if  $A \succeq A'$  when  $A(s) = c$  and  $A'(s) = c'$  for every  $s \in S$ . Thus, preference or indifference between consequences is defined in terms of the preference or indifference between acts.

Definition 3: An event  $\phi$  is null if and only if for every  $A$  and  $A'$ ,  $A \succeq A'$  given  $\phi$  and  $A' \succeq A$  given  $\phi$ . In other words, an event  $\phi$  is null if an individual considers it impossible.

POSTULATE 3: If  $A(s) = c$  and  $A'(s) = c'$  for every  $s \in E$  and  $E$  is non-null, then  $A \succeq A'$  given  $E$  if and only if  $c \succeq c'$ .

This asserts that conditional preferences do not affect preference between consequences.

Definition 4: Event  $E$  is said to be not more probable than  $E'$  if and only if

- (i)  $c$  and  $c'$  are two consequences such that  $c \geq c'$ ,
  - (ii)  $A(s) = c$  for  $s \in E$  and  $A(s) = c'$  for  $s$  in  $\sim E$ , and
  - (iii)  $A'(s) = c$  for  $s$  in  $E'$  and  $A'(s) = c'$  for  $s \in \sim E'$ ,
- then  $A' \geq A$ .

POSTULATE 4: Probability-wise any two events are comparable, i.e., for every  $E$  and  $E'$ ,  $E \geq E'$  or  $E' \geq E$ .

POSTULATE 5: There is at least one pair of acts and, therefore, at least one pair of consequences, which are not indifferent; i.e.,  $A$  and  $A'$ ,  $A \geq A'$  or  $A' \geq A$ , and  $A(s) \geq A'(s)$  or  $A'(s) \geq A(s)$ .

POSTULATE 6: If  $A > A'$ , then for every consequence  $c$  (no matter how desirable or undesirable it may be) there exists a partition of  $S$  (into a finite number of exhaustive and exclusive events) such that if either  $A$  or  $A'$  is so modified on any one element of the partition as to take value  $c$ , other values remaining undisturbed, then  $>$  still holds between modified  $A$  and  $A'$  or between  $A$  and modified  $A'$  as the case may require.

—This postulate, together with others, has two significant implications:

- (a) It implies existence of a probability measure  $P$

such that for every  $E, E', P(E) \leq P(E')$

if and only if  $E \leq E'$ , and

- (b) It implies the existence of a unique utility function  $u$ .

POSTULATE 7: Let  $A'$  be an act and let  $A'_s$  be the constant which agrees with  $A'$  for the state  $s$ . Then:

- (a)  $A \geq A'_s$  given  $E$  for all  $s \in E \rightarrow A \geq A'$  given  $E$ , and  
 (b)  $A'_s \geq A$  given  $E$  for all  $s \in E \rightarrow A' \geq A$  given  $E$ .

Below are given the two important theorems which have been proved by Savage in the framework of the above postulates.

THEOREM 1: There exists a unique real-valued function  $P$  defined for the set of events (subsets of  $S$ ) such that,

- (a)  $P(E) \geq 0$  for all  $E$ .  
 (b)  $P(S) = 1$ .  
 (c) If  $E$  and  $E'$  are disjoint, then  $P(E \cup E') = P(E) + P(E')$ .  
 (d)  $E$  is not more probable than  $E'$  if and only if  $P(E) \leq P(E')$ .

This  $P$  is termed a 'personalistic probability measure'. It reflects the individual's reported feelings as to which of a pair of events is more likely to occur.

THEOREM 2: There exists a real-valued function  $u$  defined over the set of consequences having the following property.

If  $E_i$  ( $i = 1, 2, \dots, n$ ) is a partition of  $S$  and  $A$  is an act

with the consequences  $c_i$  on  $E_i$ , and if  $E'_i$  (where  $i = 1, 2, \dots, m$ ) is another partition of  $S$  and  $A'$  is an act with the consequence  $c'_i$  on  $E'_i$ , then  $A \succeq A'$  if and only if

$$\sum_{i=1}^n u(c_i) \cdot P(E_i) \geq \sum_{i=1}^n u(c'_i) \cdot P(E'_i).$$

The  $u$  function is a utility function.

It has thus been shown that no objective probabilities have been assumed and yet expected utilities reflect the preferences accurately. As pointed out by Luce and Raiffa (61), "A subjective probability measure arises as a consequence of his (Savage's) axioms. This is in turn used to calibrate utilities, and it is established that it can be done in such a way that expected utilities correctly reflect preferences. Thus Savage's contribution—a major one in the foundations of decision making—is a synthesis of the von Neumann-Morgenstern utility approach to decision making and de Finetti's calculus of subjective probability."

Application of Bayesian theorem—maximization (or minimization) of expected values      With the application of Bayes' theorem to the 'a priori' subjective probabilities, one can arrive at a model (pay-off matrix) on which to base his decisions. In this procedure, the expectation is taken into account and the expected value of the gain (or loss) is maximized (or minimized). Suppose we begin with initial probability and gather new evidence and experience about the

world, we can obtain 'a posteriori' subjective probabilities by combining the 'a priori' probabilities with conditional (subjective) probabilities pertaining to that evidence. We then deal only with this 'a posteriori' model and form it as a basis of our decision. This is illustrated by the following hypothetical (but by no means impractical) situation faced by a farmer.

A poultry farmer has four egg grading machines and supplies only A grade eggs to the market. Each machine can grade 100 packs of A grade eggs (each pack containing 30 dozen) in a work day of 12 hours. These machines are such that normally eggs graded A may have .5% of B grade eggs.

The farmer has a contract for supplying 100 packs of A grade eggs a day to a food store and if the supply contains more than .5% grade B eggs, \$100 are deducted by the store from payment to the farmer for each 100 packs supplied. He is also penalised to the tune of \$100 if he fails to fulfill his contract to supply 100 packs a day. The rest of the graded eggs (300 packs) are sold by him in the open market daily as he has no storage facilities. Thus, at the beginning of every day he has no stock of eggs.

At the end of a certain working day, after the eggs have been graded and four separate consignments of eggs graded by the four different machines have been made, it is discovered that one of the machines was adjusted such that it turned out

lots with 98% A and 2% B grade eggs instead of the usual 99.5% A and .5% B. Though the lots of 100 packs each turned out by each machine are kept separately, unfortunately, it is not known which lot was turned out by the defective machine. Due to the nature of his contract, the farmer has to immediately decide as to which lot to send to the food store. In such a situation he can apply the Bayesian theorems and use the subjective probability method in the following manner.

There are four consignments (lots) and one of them has 2% B grade eggs.

Penalty for supplying a defective lot = \$100.

Therefore, expected penalty =  $\$100/4 = \$25$ .

Suppose he decides to take a sample of two packs from one lot and runs it over a non-defective machine to test whether the pack has .5% or 2% B grade eggs. There are three possibilities. He observes that

(a) both samples have .5% B grade eggs— $Q_1$ ,

(b) both packs have 2% defective— $Q_2$ ,

(c) one pack has .5% B grade eggs and the other pack has 2% (this situation can happen because, though the good machine turns out lots having only .5% B grade eggs, overall, individual packs may have a greater or smaller percentage of B grade eggs)— $Q_3$ .

Let

$t_1$  = consignment sampled comes from the lot having .5% B grade eggs

$t_2$  = sampled consignment comes from the lot having 2% B grade eggs,

$s_1$  = the decision of the farmer to supply the lot sampled,

$s_2$  = the decision of the farmer to supply any lot other than that sampled.

Let

$y_1$  = the probability of  $t_1$  'state of nature',

$y_2$  = the probability of  $t_2$  'state of nature'.

Then

$$y_1 = \frac{3}{4} \quad \text{and} \quad y_2 = \frac{1}{4}.$$

$$\text{Probability of } \left( \frac{t_j}{Q_i} \right) = \frac{y_j \cdot \text{Probability of } \left( \frac{Q_i}{t_j} \right)}{\sum_{j=1} y_j \cdot \text{Probability of } \left( \frac{Q_i}{t_j} \right)}$$

$$j = 1, 2$$

$$i = 1, 2, 3$$

$$(a.1) \quad P\left(\frac{Q_1}{t_1}\right) = (.995)^2 (.005)^0 = .990025$$

$$(a.2) \quad P\left(\frac{Q_1}{t_2}\right) = (.98)^2 (.02)^0 = .9604$$

$$(b.1) \quad P\left(\frac{Q_2}{t_1}\right) = (.995)^0 (.005)^2 = .000025$$

$$(b.2) \quad P\left(\frac{Q_2}{t_2}\right) = (.98)^0 (.02)^2 = .00004$$

$$(c.1) \quad P\left(\frac{Q_3}{t_1}\right) = (.995)^1 (.005)^1 \cdot 2 = (.004975) 2$$

$$(c.2) \quad P\left(\frac{Q_3}{t_2}\right) = (.98)^1 (.02)^1 \cdot 2 = (.0196) 2$$

Therefore,

$$P\left(\frac{t_1}{Q_1}\right) = \frac{\left(\frac{3}{4}\right)(.990025)}{\left(\frac{3}{4}\right)(.990025) + \left(\frac{1}{4}\right)(.9604)} = .756$$

$$P\left(\frac{t_2}{Q_1}\right) = \frac{\left(\frac{1}{4}\right)(.9604)}{\left(\frac{3}{4}\right)(.990025) + \left(\frac{1}{4}\right)(.9604)} = .244$$

$$P\left(\frac{t_1}{Q_2}\right) = \frac{\left(\frac{3}{4}\right)(.000025)}{\left(\frac{3}{4}\right)(.000025) + \left(\frac{1}{4}\right)(.0004)} = .16$$

$$P\left(\frac{t_2}{Q_2}\right) = \frac{\left(\frac{1}{4}\right)(.0004)}{\left(\frac{3}{4}\right)(.000025) + \left(\frac{1}{4}\right)(.0004)} = .84$$

$$P\left(\frac{t_1}{Q_3}\right) = \frac{2\left(\frac{3}{4}\right)(.004975)}{2\left(\frac{3}{4}\right)(.004975) + \left(\frac{1}{4}\right)(.0196)2} = .43$$

$$P\left(\frac{t_2}{Q_3}\right) = \frac{2\left(\frac{1}{4}\right)(.0196)}{2\left(\frac{3}{4}\right)(.004975) + \left(\frac{1}{4}\right)(.0196)2} = .57$$

The pay-off matrix is given below:

	$t_1$	$t_2$
$s_1$	0	-100
$s_2$	$-\frac{100}{3}$	0

For  $(s_2, t_1)$  we get a value of  $-100/3$  because, if we decide to supply any of the remaining three (non-sampled) lots and if  $t_1$  (the consignment sampled comes from .5% B grade) actually exists, the defective lot then is in the consignments not sampled and for  $s_2$  the probability of its being supplied is  $1/3$ . Thus the expected pay-off is  $-\$100/3$ .

(a) If  $Q_1$  is observed, then

$$E(s_1) = (0)(.756) + (-100)(.244) = -24.4 \dots \text{a.1}$$

$$E(s_2) = (-100/3)(.756) + (0)(.244) = -25.2 \dots \text{a.2}$$

Decision: As  $E(s_1) \geq E(s_2)$ , select  $s_1$ ; i.e., supply the sampled lot.

(b) If  $Q_2$  is observed, then

$$E(s_1) = (0)(.16) + (-100)(.84) = -84 \dots \text{b.1}$$

$$E(s_2) = (-100/3) + (0)(.84) = -5.33 \dots \text{b.2}$$

Recommendation: Use  $s_2$ ; i.e., supply any lot other than that sampled.

(c) If  $Q_3$  is observed, then

$$E(s_1) = (0)(.43) + (.57)(-100) = -57 \dots \text{c.1}$$

$$E(s_2) = (-100/3)(.43) + (0)(.57) = -14.3 \dots \text{c.2}$$

Suggestion: Play  $s_2$ ; i.e., supply any lot other than that sampled.

Thus it has been shown as to how the concept of subjective probability can be used in taking decisions. Subjective probabilities, when transformed in the manner suggested above

(by Bayes' theorem), through experience, tend to approach relative frequencies. Thus, 'reasonable men' could be expected to have similar subjective probabilities for events for which they had similar experience (or belief). Some people raise an objection and say that an individual may not be consistent and may provide two different judgments concerning probabilities at two different times prior to implementation of a strategy and these two judgments may be found to be incompatible. But they should not forget that the individual acquires new information and experience in the period between two judgments. There may be still other reasons for this inconsistency such as the individual may not be careful in his judgment, he may be careful but vague about his true preference or his value system may have changed, etc. Hurwicz and Hodges and Lehman have given suggestions to partially cope with this problem of inconsistency.

One of the advantages of this approach (of subjective probability) is that, in many cases, decision maker's subjective probabilities will be determined by routine data collection and analysis methods. In that case, the determination of these probabilities can be left to the analyst and manager is freed for more difficult decisions.

In several instances, the manager may be inexperienced. In such a case he can hire an expert having experience of the matter and associate him with the decision process.

One of the drawbacks of subjective probability, as pointed out by the objectivists, is that 'a priori' probabilities raise empirical difficulties in the sense that it is not clear, and rather impossible, to estimate the postulated probabilities. The subjectivists' answer to this is that 'relative frequencies' (sometimes called the objective probabilities) are, in themselves, to a great extent, a result of considerable subjective judgment by the analyst.

An objection raised by Roegen (80) against the theory of subjective probability is that, "If all events could be expressed as Boolean polynomials of some elementary events that need only to be mutually exclusive, the structure of the beliefs of any individual would be completely characterized by the manner in which he would distribute probabilities to these elementary events. This probability distribution is otherwise arbitrary and does not have to reflect any stochastic aspect of this material world." Mr. Roegen is not completely wrong in this criticism of the subjective probability, but he neglects the aspect of learning from experience. And once the initial belief is known, by Bayes' theorem, the future ones are all determined automatically.

A summary statement showing the type of strategies (pure or mixed) suggested by the different criteria

Assumption: The game does have a solution.

Decision model	Type of strategy suggested	Remarks
1) Wald's	a. Pure	When pay-off matrix has a saddle point.
	b. Mixed	When pay-off matrix does not have a saddle point.
2) Laplace's	Pure only	
3) Hurwicz	Pure only	
4) Regret	a. Pure	Only if one single row (strategy) dominates all other strategies in the original pay-off matrix so that maximum regret for the dominating strategy will be zero for all states of nature. However, in such a case, whatever decision model we use, the suggested strategy will be the same. Thus, when Regret criterion suggests a pure strategy, there is no need to apply other criteria.
	b. Mixed	Whenever no single row dominates over all other rows in a pay-off matrix $P$ , as is generally the case, regret models suggest a mixture of strategies.
5) Subjective probability	Pure only	

Applications to actual data

On the following pages are given the applications of these five models. All these examples have been taken from actual situations and contain actual data obtained on the field unless specified otherwise. These are only a few of the examples representing only a few sets of situations as it may be too unwieldy to cover all the decision making problems that a farmer faces.

The method of application would be the same to all situations. For the use of Savage's criterion of subjective probability, we have already shown as to how we can calculate these probabilities. Therefore, in these applications that follow we shall just be taking the probabilities assigned by the farmer to the different 'states of nature' as given.

Application 1. Choosing an enterprise A young person has made up his mind to take up farming for a living in Illinois. For reasons of economy and efficiency he decides to have a farm of at least 180 acres; but because of being a novice in farming, he does not want to take undue risks and also because of managerial limitations, he has put an upper bound of 500 acres on the size of his farm at least for the first five years. However, he is not sure as to which farm (of the farms for sale) he and his family would like to live on, so he does not know the size of his farm in advance. His goal is to maximize 'earning on capital and management per

Table 1. Average earnings on capital and management in dollars per acre (77)

		Size of farm in acres		
		t <sub>1</sub> 180-259	t <sub>2</sub> 260-339	t <sub>3</sub> 340-499
s <sub>1</sub> Grain farming	North Illinois	33.57	37.76	38.87
	South Illinois	17.72	25.95	26.57
s <sub>2</sub> Hog farming	North Illinois	23.41	23.53	25.57
	South Illinois	26.87	25.18	25.54
s <sub>3</sub> Beef cattle	North Illinois	7.39	19.62	15.44
	South Illinois	22.16	16.25	-
s <sub>4</sub> Dairy cattle	North Illinois	25.82	24.79	29.36
	South Illinois	31.90	30.57	28.50

acre'. Given the data in Table 1, he has to choose a type of farming out of grain farming, hog farming, dairy farming and raising beef cattle.

The example has a rather peculiar characteristic in the sense that one single row dominates all other rows and, therefore, every criterion gives the same results. Grain farming in the Northern Illinois is the most paying proposition. Therefore, he must try to purchase a farm in Northern Illinois and select grain farming.

In practice, however, it may either not always be possible for the person to be able to purchase a farm of his choice in the North or his family may have a preference for living in

Southern Illinois. As illustrated in Table 1, the average earnings on capital and management in Southern Illinois for each type of farming were:

Type of farming	180-259 acres	260-339 acres	340-500 acres
Grain	\$17.72	\$25.95	\$26.57
Hog	26.87	25.18	25.54
Dairy	31.90	30.57	28.50
Beef cattle	22.16	16.25	-

Naturally he will choose dairy farming if he is to farm in Southern Illinois. Again a single row (dairy farming) dominates all other rows and, therefore, all criteria will suggest the same strategy.

Once the farmer has decided on the nature of enterprises that he would follow, he has to take decisions regarding the technology to be used in raising the crops or feeding the livestock. These will consist of, for example, the number of ploughings to be given to prepare the field; the type and quantities of manures and fertilizers to be applied; the choice of varieties of crops; seed rate and the time of sowing; number of irrigations to be given, if any; cultural practices to be adopted; etc.

Here we shall apply these criteria to the following three situations:

- (i) Selection of level of fertilizer or manure to crops
- (ii) Optimum sowing time
- (iii) Decision on when to sell

Application 2. Which level of  $P_2O_5$  will be most desirable for wheat The most common form of application of phosphorus to a crop is  $P_2O_5$ . A researcher in India has tried three levels of  $P_2O_5$  (0 lbs, 20 lbs and 40 lbs per acre) to wheat on an experimental basis on which he will base his recommendations. Table 2 shows the yield of wheat obtained for these three levels of  $P_2O_5$ .

Table 2. Average response of wheat to combinations of the dozes of N and  $P_2O_5$  at Kotah (Rajasthan)—yield in mds/acre (66)

	Application of $P_2O_5$ in lbs per acre	Nitrogen levels		
		$t_1$	$t_2$	$t_3$
$s_1$	0	13.60	15.47	16.90
$s_2$	20	13.17	14.37	17.73
$s_3$	40	14.80	15.93	15.93

In order to apply the different decision criteria we calculate  $l_i$ ,  $k_i$ ,  $C_i$ , the different values for different sets of subjective probabilities (we have chosen three sets, viz. [.2, .5, .3], [.3, .2, .5] and [.1, .4, .5]) for the three states of nature, R (regret) matrix and  $l_i^r$ . These are given on the following page.

	$l_i$	$k_i$	$C_i$ $r=.3$	$r=.7$	Subjective probability values (.2,.5,.3) (.3,.2,.5) (.1,.4,.5)		
$s_1$	13.60	15.32	14.59	15.91	15.525	15.624	15.998
$s_2$	13.17	15.09	14.54	16.36	15.138	15.690	15.930
$s_3$	14.80	15.55	15.14	15.59	15.704	15.591	15.817

## Regret matrix

	$t_1$	$t_2$	$t_3$	$l_i^r$
$s_1$	-1.20	-.46	-.83	-1.20
$s_2$	-1.63	-1.56	0	-1.63
$s_3$	0	0	-1.80	-1.80

Individual judgment (subjectivity) and farmer's attitudes do not affect the results given by Wald's, Laplace's and Regret criteria. In the case of the remaining two criteria, viz Hurwicz and subjective probability, the choice is affected by the optimism-pessimism index of the farmer and the variations in the subjective probabilities assigned to the occurrence of the different states of nature by the individual decision makers. Therefore, in this and most of the succeeding examples we have taken two values of  $r$  ( $r=.3$ ,  $r=.7$ ) and calculated the pay-offs for different  $s_i$  for three farmers who assign different sets of subjective probabilities to the states of nature. As the calculation of subjective probabilities has been demonstrated earlier, we have only applied the criteria of subjective

probability to this example for illustrative purposes.

The strategies suggested by the different criteria are given below:

Criterion	Suggested strategy
Wald's . . . . .	$s_3$
Laplace's . . . . .	$s_3$
Hurwicz' - $r=.3$ . . . . .	$s_3$
$r=.7$ . . . . .	$s_2$
Regret criterion . . . . .	$s_1$
Subjective probability	
$(.2, .5, .3)$ . . . . .	$s_3$
$(.3, .2, .5)$ . . . . .	$s_2$
$(.1, .4, .5)$ . . . . .	$s_1$

Wald's, Laplace's, Hurwicz' ( $r=.3$ ), and subjective probability criteria with probabilities of .2 assigned to  $t_1$ , .5 to  $t_2$ , and .3 to  $t_3$  all suggest  $s_3$ .

Hurwicz' model ( $r=.7$ ) and subjective probability set  $(.3, .2, .5)$  recommend  $s_2$ .

On the basis of regret model and subjective probability set  $(.1, .4, .5)$ ,  $s_1$  will be selected.

Application 3. Optimum level of farm yard manure for Bajra (Pennisetum typhoideum)      Farm yard manure is the single most important source of organic matter to the soil in Indian farming. In the experiments conducted at the Indian Agricultural Research Institute, it was endeavoured to determine the effects of five levels of application of farm yard

manure on the yields of 'bajra' crop. The five levels were 0, 2.5, 5, 10 and 20 tons of farm yard manure per acre. The results are reported for five years, 1952-1956.

The pay-off matrix is given in Table 3.

Table 3. Bajra (Pennisetum typhoideum) yields in mds/acre at five levels of farm yard manure (67)

	Farm yard manure tons/acre.	t <sub>1</sub> 1952	t <sub>2</sub> 1953	t <sub>3</sub> 1954	t <sub>4</sub> 1955	t <sub>5</sub> 1956
s <sub>1</sub>	0	15.4	16.1	15.1	16.1	6.4
s <sub>2</sub>	2.5	15.7	17.0	15.9	16.6	5.0
s <sub>3</sub>	5	16.1	15.9	15.8	16.8	6.3
s <sub>4</sub>	10	16.6	17.1	16.7	17.5	6.2
s <sub>5</sub>	20	15.7	21.7	18.9	21.0	5.5

Strategies suggested by different models are shown below:

Strategy suggested when:

Criterion	data for 1956 included	data for 1956 excluded
Wald's	s <sub>1</sub>	s <sub>4</sub>
Laplace's	s <sub>5</sub>	s <sub>5</sub>
Hurwicz'		
r = .3	s <sub>5</sub>	r ≤ .175 s <sub>4</sub>
r = .5	s <sub>5</sub>	r > .175 s <sub>5</sub>
Regret	s <sub>5</sub>	s <sub>5</sub>

Application 4. Choice of optimum time of sowing of gram (Cicer arietinum L.) The farmer must keep his fields ready to be able to sow the crop at the most appropriate time. The importance of this is enhanced by the very fact that sowing at the proper time will add to the returns to the farmer without any extra costs to him.

The data in Table 4 were obtained as a result of trials conducted in West Bengal, India, to determine the effects of time of sowing on the yield of 'gram'. Gram is a winter crop. Six sowing dates, viz. October 13, October 28, November 12, November 27, December 12 and December 27 (interval of 15 days), were tried for a period of three years.

Table 4. Optimum time of sowing gram (Cicer arietinum L.) in West Bengal—grain yield in quintals per hectare (96)

		$t_1$	$t_2$	$t_3$
Sowing date		1961-1962	1962-1963	1963-1964
$s_1$	October 13	30.95	19.98	15.54
$s_2$	October 28	36.81	24.10	17.77
$s_3$	November 12	37.11	16.13	15.04
$s_4$	November 27	17.87	19.29	10.85
$s_5$	December 12	12.96	10.80	2.74
$s_6$	December 27	5.88	3.96	2.59

Suggested optimum time of sowing gram (Cicer arietinum):

Criterion	Optimal date of sowing	
Wald's	October 28	S <sub>2</sub>
Laplace's	October 28	S <sub>2</sub>
Hurwicz'		
$r > .878$	November 12	S <sub>3</sub>
$r \leq .878$	October 28	S <sub>2</sub>
Regret	October 28	S <sub>2</sub>

But for Hurwicz' criterion at  $r > .878$ , all the criteria suggest October 28 as the optimum date of sowing 'gram' in West Bengal.

Application 5. Optimum time of sowing wheat      Wheat is a winter crop in India and in most parts of the United States as well. Experiments were conducted in the canal irrigated area of Rajasthan state (India) to find out the difference in yields of wheat due to the differences in the time of sowing. Seven different dates of sowing, ten days apart, were tried. The pay-off matrix for these sowing dates (in terms of yield in maunds per acre) is given in Table 5.

Optimum time of sowing wheat as suggested by different models

Model	Dates suggested	
Wald's	November 4	S <sub>3</sub>
Laplace's	November 4	S <sub>3</sub>
Hurwicz'		
$r > .8$	November 14	S <sub>4</sub>
$r \leq .8$	November 4	S <sub>3</sub>
Regret	November 4	S <sub>3</sub>

The pattern of suggested optimal strategy is similar to that obtained for 'gram' in application No. 4 on page 112.

Table 5. Effect of time of sowing (and seed rate) on yield of wheat under canal irrigation in Rajasthan (102)

	Sowing date	Yield in quintals per hectare		
		t <sub>1</sub> 1960-1961	t <sub>2</sub> 1961-1962	t <sub>3</sub> 1962-1963
s <sub>1</sub>	October 15	26.65	28.15	26.40
s <sub>2</sub>	October 25	28.25	28.80	30.25
s <sub>3</sub>	November 4	28.35	30.40	33.85
s <sub>4</sub>	November 14	26.20	29.45	34.45
s <sub>5</sub>	November 24	25.30	31.35	29.45
s <sub>6</sub>	December 4	21.95	25.40	29.90
s <sub>7</sub>	December 14	23.50	22.55	28.25
s <sub>8</sub>	December 24	16.70	19.95	10.90

At  $r > .8$ , Hurwicz' criteria suggests s<sub>4</sub> as optimal. According to all the other criteria, s<sub>3</sub> is the best.

Applications 6-12. When to sell? The most common goal of farm firms is profit maximization. It is precisely for this reason that after a person has decided to enter farming business, he makes decisions regarding the most profitable enterprises and the technology to be adopted in carrying out these enterprises in order to maximize the yields from a given set of resources and alternatively to minimize costs. However, profits are a function of not only costs, but also the price at which the farmer sells his produce. Decision models under study can be equally well applied to this problem.

In Tables 6-12 are given the prices of corn (in dollars per bushel), soybeans (in dollars per bushel), hay (in dollars per ton), oats (in dollars per bushel), hogs (in dollars per cwt.), wheat (in dollars per bushel), and wool (in dollars per pound) for 1964, 1965 and 1966. They are the average prices that prevailed during different months of these three years. For the sake of simplicity, let us assume that the farmer has his own storage facilities and costs of storage are insignificant. Further, we also assume that there are no differentials in rates of interest for the receipts from sales for different months. To sum up, we assume that the greater the farm prices, the larger the profit to the farmer. Given these, the farmer has to decide as to in what month or months he must sell his crop to get maximum returns. The following statement shows the best time of selling for different farm products. Let  $s_1, s_2, s_3, \dots, s_{12}$  be the action of selling a crop in January, February, March,  $\dots$ , December, respectively.

Criterion	When to sell (pure strategy suggested)						
	Corn	Soybeans	Hay	Oats	Hogs	Wheat	Wool
Wald's	$s_5$	$s_1$	$s_1$	$s_1, s_2$ $s_4, s_5$	$s_9$	$s_1, s_2$	$s_6$
Laplace's	$s_9$	$s_7$	$s_2$	$s_5$	$s_8$	$s_1$	$s_5$
Hurwicz' $r=.3$	$s_9$	$s_8$	$s_1$	$s_5$	$s_8$	$s_1$	$s_6$
$r=.7$	$s_9$	$s_8$	$s_2$	$s_{12}$	$s_1$	$s_1$	$s_6$
Regret	$s_7$	$s_7, s_8$	$s_1, s_2, s_3$	$s_1, s_2$ $s_3, s_4$ $s_5, s_6$	$s_8$	$s_2, s_5$	$s_5$

Table 6. Prices of Iowa farm products—corn (107)

		Dollars per bushel		
		t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>
		1964	1965	1966
S <sub>1</sub>	January	1.02	1.09	1.14
S <sub>2</sub>	February	1.02	1.10	1.13
S <sub>3</sub>	March	1.06	1.10	1.09
S <sub>4</sub>	April	1.11	1.14	1.14
S <sub>5</sub>	May	1.12	1.16	1.16
S <sub>6</sub>	June	1.10	1.18	1.16
S <sub>7</sub>	July	1.06	1.15	1.22
S <sub>8</sub>	August	1.07	1.11	1.27
S <sub>9</sub>	September	1.10	1.11	1.28
S <sub>10</sub>	October	1.05	1.05	1.23
S <sub>11</sub>	November	1.00	.95	1.20
S <sub>12</sub>	December	1.09	1.02	1.25

Table 7. Prices of Iowa farm products—soybeans (107)

		Dollars per bushel		
		t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>
		1964	1965	1966
S <sub>1</sub>	January	2.60	2.70	2.62
S <sub>2</sub>	February	2.54	2.75	2.71
S <sub>3</sub>	March	2.51	2.81	2.66
S <sub>4</sub>	April	2.42	2.79	2.74
S <sub>5</sub>	May	2.31	2.65	2.86
S <sub>6</sub>	June	2.32	2.68	3.02
S <sub>7</sub>	July	2.31	2.65	3.38
S <sub>8</sub>	August	2.31	2.49	3.53
S <sub>9</sub>	September	2.45	2.43	2.90
S <sub>10</sub>	October	2.49	2.28	2.77
S <sub>11</sub>	November	2.54	2.34	2.82
S <sub>12</sub>	December	2.68	2.47	2.85

Table 8. Prices of Iowa farm products—hay (107)

		Dollars per ton		
		t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>
		1964	1965	1966
S <sub>1</sub>	January	19.50	19.50	23.50
S <sub>2</sub>	February	19.00	19.50	24.50
S <sub>3</sub>	March	18.00	19.50	22.50
S <sub>4</sub>	April	17.00	19.90	20.90
S <sub>5</sub>	May	17.50	21.50	21.40
S <sub>6</sub>	June	17.00	21.80	19.90
S <sub>7</sub>	July	16.00	21.40	19.10
S <sub>8</sub>	August	16.50	20.40	18.70
S <sub>9</sub>	September	18.00	21.40	19.90
S <sub>10</sub>	October	18.30	21.20	19.20
S <sub>11</sub>	November	18.50	21.10	19.90
S <sub>12</sub>	December	19.30	21.90	20.10

Table 9. Prices of Iowa farm products—oats (107)

		Dollars per bushel		
		t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>
		1964	1965	1966
S <sub>1</sub>	January	.68	.67	.67
S <sub>2</sub>	February	.67	.68	.67
S <sub>3</sub>	March	.66	.68	.67
S <sub>4</sub>	April	.67	.68	.67
S <sub>5</sub>	May	.67	.69	.67
S <sub>6</sub>	June	.65	.68	.67
S <sub>7</sub>	July	.62	.65	.66
S <sub>8</sub>	August	.62	.63	.66
S <sub>9</sub>	September	.62	.63	.68
S <sub>10</sub>	October	.62	.63	.68
S <sub>11</sub>	November	.64	.64	.70
S <sub>12</sub>	December	.65	.65	.70

Table 10. Prices of Iowa farm products—hogs (107)

		Dollars per cwt.		
		t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>
		1964	1965	1966
S <sub>1</sub>	January	14.30	15.30	27.30
S <sub>2</sub>	February	14.30	16.30	27.20
S <sub>3</sub>	March	14.00	16.30	24.20
S <sub>4</sub>	April	13.80	16.70	21.90
S <sub>5</sub>	May	14.20	19.60	21.90
S <sub>6</sub>	June	14.70	22.40	22.90
S <sub>7</sub>	July	16.00	23.20	23.00
S <sub>8</sub>	August	15.70	23.60	24.70
S <sub>9</sub>	September	16.10	22.20	22.20
S <sub>10</sub>	October	15.00	22.80	21.20
S <sub>11</sub>	November	13.90	23.50	19.00
S <sub>12</sub>	December	14.50	27.10	18.50

Table 11. Prices of Iowa farm products—wheat (107)

		Dollars per bushel		
		t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>
		1964	1965	1966
S <sub>1</sub>	January	1.95	1.44	1.43
S <sub>2</sub>	February	1.93	1.43	1.44
S <sub>3</sub>	March	1.85	1.40	1.40
S <sub>4</sub>	April	1.92	1.38	1.40
S <sub>5</sub>	May	1.92	1.37	1.44
S <sub>6</sub>	June	1.56	1.34	1.63
S <sub>7</sub>	July	1.37	1.33	1.78
S <sub>8</sub>	August	1.37	1.39	1.80
S <sub>9</sub>	September	1.42	1.40	1.80
S <sub>10</sub>	October	1.44	1.40	1.67
S <sub>11</sub>	November	1.46	1.40	1.67
S <sub>12</sub>	December	1.44	1.42	1.69

Table 12. Prices of Iowa farm products—wool (108)

		Dollars per pound		
		t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>
		1964	1965	1966
s <sub>1</sub>	January	.43	.47	.47
s <sub>2</sub>	February	.48	.46	.48
s <sub>3</sub>	March	.51	.47	.51
s <sub>4</sub>	April	.53	.48	.53
s <sub>5</sub>	May	.54	.49	.55
s <sub>6</sub>	June	.55	.50	.52
s <sub>7</sub>	July	.54	.46	.49
s <sub>8</sub>	August	.52	.46	.48
s <sub>9</sub>	September	.48	.46	.48
s <sub>10</sub>	October	.47	.48	.48
s <sub>11</sub>	November	.48	.49	.47
s <sub>12</sub>	December	.47	.49	.46

It is noted here that though Wald's and Regret criteria have suggested several strategies for oats, the best would be one which allows the farmer to sell his crop earliest. Therefore, if the oats are harvested in April, the farmer would do well to adopt s<sub>5</sub> and s<sub>6</sub>, the earliest that he can sell after harvest.

### Discussion of results

Here we shall be discussing the five decision models in the light of the results suggested by these for different sets of circumstances.

Wald's criterion      The assumption underlying the use of this criterion is that the farmer is a rank pessimist. He

always thinks in terms of the worst that can happen to him and, therefore, finds out the minimum return (security level) from different possible strategies and then chooses the one affording him the maximum security level. Such a decision maker has a short-time horizon and wants to best protect himself under the worst circumstances. This may be due to several reasons. For example, he may have just started to farm and, therefore, will not like to take risks. Maybe, as is the case in India, the farmer is able to grow hardly enough to feed himself and his family at the subsistence level and, though yields affording him to live at a higher level than subsistence would be welcome and extremely helpful, yields lower than a certain level would spell disaster. Therefore, in Application 2 he would prefer using 40 lbs.  $P_2O_5$  per acre, though if the state of nature would be  $t_3$ , he could have got as much as 17.73 maunds per acre with the use of only 20 lbs. per acre of  $P_2O_5$  instead of 40 lbs. Thus it is apparent that, in our example, a farmer employing Wald's criterion for decision-making can never achieve the maximum possible yield of 17.73 maunds per acre. Nevertheless, by choosing  $s_3$  (by Wald's model), he has protected himself against the possibility of obtaining low yields of 13.17 which he could very well get by using  $s_2$ . It is worth noting that  $s_2$  has the peculiarity of having both the minimum (13.17) and the maximum (17.73) of the values in the matrix and, therefore, may be paying as well as hazardous.

A beginner, subsistence or a non-enterprising farmer, would be quite skeptical and even evasive in choosing  $s_2$ . Wald's criterion is appropriate for such a situation.

Application 3 brings out the conservative nature of Wald's approach rather prominently. From the data, it appears that 1956 was an unusually bad year and the chances of such a poor state of nature occurring in the future (on the basis of our data) are only one in five. Therefore, if we ignore for a moment the figures for 1956 and consider the results for only the remaining four years, we find that  $s_1$  (no farm yard manure) is strictly dominated by not one but three strategies (viz.  $s_2$ ,  $s_4$  and  $s_5$ ) and would, therefore, be counted off under Wald's criteria. But as soon as we take into account the observations for that abnormal year 1956, this strictly dominated strategy  $s_1$  is suggested by Wald's model. Thus Wald's approach gives all weight to small values in a row and no weight to large values. This could perhaps be best illustrated by the following hypothetical pay-off matrix.

	$t_1$	$t_2$
$s_1$	2	4
$s_2$	0	500

Here  $s_1$  would always be preferred to  $s_2$  by Wald's model, though intuitively this seems rather unreasonable as long as the chances of occurrence of  $t_2$  are more than  $1/249$ .

Moreover, it has been pointed out earlier that, as a player against farmer, nature is passive in the sense that it does not always try to do the worst to the farmer. In such cases of a single state of nature being entirely different from the other states of nature which are more like each other among themselves, it may be better to ignore that extreme state of nature in case the probability of its occurrence is small (say less than 20 percent).

Another problem with this criterion is that addition of a constant to the pay-off for each strategy for a given state of nature  $t_{j_0}$  may result in the suggestion of a different strategy. This implies absence of the property of 'column linearity'. For example, suppose we add ten maunds to each figure for 1956 (to make it look like more of a normal year) in Table 3, the pay-off matrix would then be:

	1952	1953	1954	1955	1956	$l_i$
$s_1$	15.4	16.1	15.1	16.1	16.4	15.1
$s_2$	15.7	17.0	15.9	16.6	15.0	15.0
$s_3$	16.1	15.9	15.8	16.8	16.3	15.8
$s_4$	16.6	17.1	16.7	17.5	16.2	16.2
$s_5$	15.7	21.7	18.9	21.0	15.5	15.5

For this pay-off matrix the choice of Wald's criteria is  $s_4$  which is different from  $s_1$ —the strategy suggested in the

original matrix.

This approach is suitable for a farmer who is either a novice in farming, or has little equity, or is a subsistence farmer or a risk-averter by temperament, or has some other heavy responsibilities (like a large family to support), or has no other source of income to fall back upon in rainy days. This would also adequately suit a farmer who has been constantly undergoing a loss for the last few years. It is desirable, when the frequency of the unfavorable states of nature is more than that of favorable states.

Laplace's criterion      This criterion has long been used by researchers and extension workers to make recommendations in agriculture. They have done this by taking the arithmetic averages of the values for different strategies and then testing whether the difference between these means is statistically significant or not. The difference in the two approaches (Laplace model as such and statistically testing the significance of the difference in means) is that in applying the Laplace criterion we choose the strategy with the highest average expected pay-off; we do not care whether this expected average pay-off is significantly different from other expected pay-offs or not. This model gives smaller weight to extreme values and, therefore, has a tendency to iron out the variations due to abnormal states of nature. The expected pay-off given by this criterion is always greater than or equal to the

expected average pay-off given by any other criterion. This is simple to apply and has the property of 'column linearity'.

A characteristic of this criterion is that it suggests the same strategy as optimal no matter on which matrix (whether P or R) it is applied.

Of the 12 applications made in this study, Laplace's principle agrees with Wald's approach in six instances and with regret model in nine cases.

For the most appropriate use of the criterion, one of the requisites is to list all the possible pertinent states of nature. Clearly this is not a finite set and creates problems in proper generalizations. Chernoff (17) has given an axiomatic treatment in defense of this principle.

Another objection to this model may be regarding assumption of complete ignorance itself. Does a farmer act under complete ignorance? Does he not have even the vaguest idea as to what state of nature is more likely?

For a farmer who has been in farming business at least for sometime now and wants to stay for a sufficiently long time (say 20 years or more), Laplace principle may be most appropriate because the longer the period, the better the operation of the 'law of averages'. This model always suggests that pure strategy which has the highest expected pay-off of all other possible strategies available to him.

This approach would also be appropriate to apply to a

farmer who, though he has enough capital and experience of farming elsewhere, decides to farm in a new region about which he does not have much practical knowledge and experience.

Regret criterion Savage's regret criterion is the least conservative of all the approaches. However, it should not be interpreted to mean that this model would always suggest a strategy different from the conservative Wald's criterion. Of the twelve examples given, the two criteria suggested the same strategies in six cases. For a pay-off matrix given on page 121, regret criterion gives more sensible results than Wald's criterion.

$$P = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{cc} t_1 & t_2 \\ \hline s_1 & \begin{array}{|c|c|} \hline 2 & 4 \\ \hline \end{array} \\ \hline s_2 & \begin{array}{|c|c|} \hline 0 & 500 \\ \hline \end{array} \end{array}$$

The R for the above P is

$$R = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{cc} t_1 & t_2 \\ \hline s_1 & \begin{array}{|c|c|} \hline 0 & -496 \\ \hline \end{array} \\ \hline s_2 & \begin{array}{|c|c|} \hline -2 & 0 \\ \hline \end{array} \end{array}$$

Suggestion: Choose  $s_2$ .

The assumption underlying this criterion, as the name suggests, is that the farmer tries to minimize his regret. The minimum pay-off under this criterion is always equal to or

lower than that suggested by Wald's model (as the latter recommends the maximum of the minimum), but the former may also (though not necessarily) provide the farmer with the opportunity to take advantage of the highest prices or maximum yields. In Table 7 Wald's model recommends to sell soybeans in January, whereas regret approach suggests that the best time to sell it is July or August. If the farmer sells soybeans in January, he can expect to get at least \$2.60 per bushel; whereas, if he sells in July or August, he might end up getting only \$2.31—a loss of \$.29 per bushel. But if he goes by the suggestion of the regret criterion and sells soybeans in August, he can get as much as \$3.53 per bushel—\$.83 more than the price of \$2.70 which was the maximum that he could expect to get in January.

In case of wheat (Table 11), Savage regret approach gives inappropriate (and inferior to Wald's) suggestion of selling it in May. Not only the minimum price in May of \$1.37 per bushel, which is less than the minimum of \$1.43 in January as suggested by Wald's principle, but the maximum in May of \$1.92 is also less than the maximum of \$1.95 in January. In this case, therefore, it would be unwise to use the regret principle.

As pointed out, the basic assumption underlying this model is that the farmer minimizes his regret. In practice, there is no evidence to support this argument. In fact,

Chernoff (17) has seriously questioned the very definition of regret. According to him, "It has never been clearly demonstrated that differences in utility do in fact measure what one may call regret (risk). In other words, it is not clear that the regret of going from a state of utility 5 to a state of utility 3 is equivalent in some sense to that of going from a state of utility 11 to one of utility 9."

This criterion may give such a large weight to a small advantage in one state of nature that it may be preferred over a considerable advantage in another state. Another objection to this approach is that the presence of undesirable strategies may influence the choice among the remaining strategies.

This criterion may be especially suitable for the young enterprising farmers who would always like to cash in on the opportunity of greater returns. It could be called a 'non-conservative' criteria because it is based on the assumption of occurrence of largest possible regret.

Hurwicz' criterion This principle will not satisfy those looking for complete objectivity as the recommendations of the criterion would vary with the degree of optimism or pessimism of the decision-maker. The more optimistic the farmer, the greater the weightage given to the maximum value of the strategy and vice versa. For example, in Table 2 at  $r=.3$  the suggested strategy is the application of 40 lbs. per acre of  $P_2O_5$ , but if a farmer is more optimistic and has  $r=.7$ ,

say, he would end up choosing the application of 20 lbs. In the example of optimum sowing time for wheat, a very optimistic farmer (with  $r > .8$ ) would choose to sow his crop on around the 14th of November. If he were not so optimistic, the choice would fall on the 4th of November. As the value of 'r' decreases, the suggested strategy conforms more and more to that given by Wald's criterion. At  $r = .3$ , Hurwicz' model gives the same result as Wald's criterion in eight out of 12 applications. At  $r = 0$ , Hurwicz' criterion is the same as Wald's.

However, the basic and rather difficult question to answer is the determination of 'r'. One of the methods to derive an appropriate 'r' has been suggested by Luce and Raiffa. The basic idea behind the use of Hurwicz' method was to strike a compromise between the extreme liberal and conservative criteria. Therefore, extreme optimism and pessimism suggest strategies similar to the most liberal and most conservative criteria, respectively, and the very purpose of this criterion (to examine a weighted combination of the best and the worst states) will be defeated.

One of the big drawbacks of this criterion is that it takes into consideration only the extreme values and completely ignores the rest of the data. For instance, given to choose between  $s_7$  and  $s_8$  as an optimum selling time of oats (Table 9), any rational decision maker would prefer  $s_7$  to  $s_8$ . But whatever the value of  $r$ , Hurwicz' criterion is indifferent

between the two strategies.

Another objection to this model would be that any randomization over optimal acts done on the basis of Hurwicz' approach is not necessarily optimal for this model. Further, if a decision maker is easily vulnerable to outside influences and quickly changes his mind, he is likely to have a tendency to revise his 'r' rather hastily and may then impart wrong weights to the best and the worst outcomes.

Subjective probability model      The four decision criteria discussed above assume that the farmer is acting under complete uncertainty. However, this is rarely true. From his own experience or from the experience of others, or from his education or just his belief, he might form some expectations in his own mind regarding the state of affairs and assign the 'a priori' probabilities to their occurrence, assuming that these states of nature were random. Then through observations and experience and by applying Bayesian methods, 'a priori' probabilities can be transformed into 'a posteriori' probabilities. The process of transforming these subjective (a priori) probabilities of the various states of nature into conditional (a posteriori) probabilities has already been explained on pages 91-104. The relative merits and demerits of this approach have also been discussed earlier on page 105 and, therefore, shall not be discussed here any more. It would, however, be worth noting as to how

the assignment of different sets of probabilities to different states of nature would result in entirely different recommendations. In Table 2 (showing pay-off matrix with respect to response of wheat yield to the different levels of  $P_2O_5$ ), we have applied three different sets of subjective probability. The set  $a = (.2, .5, .3)$  assigns a probability of .2 to  $t_1$ , .5 to  $t_2$  and .3 to  $t_3$ . The other two sets are  $b = (.3, .2, .5)$  and  $c = (.1, .4, .5)$ .

Let set 'a' represent the subjective probability set of farmer A, 'b' of farmer B and 'c' of farmer C. Given the pay-off matrix in Table 2, A will choose  $s_3$  and apply 40 lbs. of fertilizer, B will select  $s_2$  and use 20 lbs., whereas C will go for  $s_1$  and would not apply any fertilizer at all. The three sets, 'a', 'b' and 'c', suggest three different strategies as optimum. Due care, therefore, has to be exercised in the calculation of 'a posteriori' probabilities, otherwise the criterion would lose all its utility and lead to inconsistent results.

As we have seen, Wald's criterion is pessimistic, whereas the regret principle assumes optimism. The nature never tries to do its worst to the farmer. In practice, especially in the developing economy, the farmer looks at the worst that could happen for a given state of nature and feels that he is better off to the extent of getting more than the worst pay-off. For this reason, the author suggests the following 'benefit

criterion' which seems to blend the properties of the regret and maxi-min principles.

The criterion of excess (surplus) benefit

The criterion—what is it?      The underlying assumptions regarding the behavior of the decision maker under this criterion are as follows:

a. He has full knowledge of the strategies available to him and also of the possible states of nature that can prevail.

b. He knows the pay-off matrix  $P$ .

c. He looks at the pay-offs for a given state of nature and finds out his strategy (say  $\bar{i}$ ) which gives him the lowest pay-off for that state of nature. If that state of nature prevails, the worst decision that he could make would be to play his  $\bar{i}$ th strategy ( $s_{\bar{i}}$ ). If he chose any strategy  $i$  other than  $\bar{i}$ , he has definitely done better than the worst that he could do for that state of nature. How much better has he done? We assume that he can find the answer to this question by deducting the lowest pay-off for that state of nature from the pay-off that he would get (for that state of nature) if he follows his  $i$ th strategy. This difference can be variously termed as the 'benefit', 'excess benefit', 'surplus' or 'surfeit' resulting out of his choosing a strategy other than the worst for a given state of nature. (The matrix thus obtained is his 'benefit' matrix).

d. He tries to maximize the minimum benefit, i.e., he

applies the maxi-min principle to this 'benefit' matrix.

Let  $P = p_{ij}$  be the pay-off matrix of the decision maker.

Let  $\min_i p_{i\bar{j}} = p_{\bar{i}\bar{j}}$  for a given state of nature  $t_{\bar{j}}$ .

Then if the prevalent state of nature is  $\bar{j}$ , the worst that a decision maker could do was to play strategy  $s_{\bar{i}}$ . So if he chose to play any strategy  $i$ ,  $i \neq \bar{i}$ , and if the true state of nature was  $\bar{j}$ , the pay-off to him would be  $p_{i\bar{j}}$ ,  $i \neq \bar{i}$  and  $p_{i\bar{j}} \geq p_{\bar{i}\bar{j}}$  as  $p_{\bar{i}\bar{j}} = \min_i p_{i\bar{j}}$ . The difference between  $p_{i\bar{j}}$  and  $p_{\bar{i}\bar{j}}$  is the benefit to the player due to his employing  $i$ th strategy rather than  $\bar{i}$ th strategy.

Let  $B$  with elements  $b_{ij}$  ( $i = 1, 2, \dots, n$ ), ( $j = 1, 2, \dots, m$ ) be the benefit matrix, then

$$b_{i\bar{j}} = p_{i\bar{j}} - \min_i p_{i\bar{j}} = p_{i\bar{j}} - p_{\bar{i}\bar{j}}$$

After the excess benefit matrix has been calculated in the above manner, the maxi-min criterion is applied to  $B$  matrix to find out the optimal strategy.

The following example will illustrate the mechanism:

Example: Let the pay-off matrix  $P$  be the following

		$t_1$	$t_2$	$t_3$
$P =$	$s_1$	7	13	5
	$s_2$	10	9	6
	$s_3$	8	7	8.5

It may be noted that no single row is dominated by another row or any convex combination of other rows. Similarly, no column is dominated either by a single column or a convex combination of other columns.

The 'benefit' matrix is calculated as follows:

$$\min_i p_{i1} = p_{11} = 7.$$

Therefore,

$$b_{11} = p_{11} - p_{11} = 7 - 7 = 0,$$

$$b_{21} = p_{21} - p_{11} = 10 - 7 = 3,$$

$$b_{31} = p_{31} - p_{11} = 8 - 7 = 1.$$

Similarly,

$$b_{12} = p_{12} - p_{32} = 13 - 7 = 6,$$

$$b_{22} = 9 - 7 = 2,$$

$$b_{32} = 0;$$

$$b_{13} = 0,$$

$$b_{23} = 1,$$

and

$$b_{33} = 3.5.$$

The B matrix then is

$$B = \begin{array}{c|ccc} & t_1 & t_2 & t_3 \\ \hline s_1 & 0 & 6 & 0 \\ \hline s_2 & 3 & 2 & 1 \\ \hline s_3 & 1 & 0 & 3.5 \end{array}$$

$$\min_j b_{1j} = b_{11} = b_{13} = 0$$

$$\min_j b_{2j} = b_{23} = 1$$

$$\min_j b_{3j} = b_{32} = 0$$

Using the maxi-min criterion, we find that

$$\max_i \min_j = b_{23} = 1$$

and the decision is to choose  $s_2$ .

In case of a tie (the maxi-min criterion suggests more than one strategy), it is suggested that the tie may be broken by trying the next higher (than the minimum) pay-off for each row and then again applying the maxi-min criterion. For example, the minimum for both  $s_1$  and  $s_3$  is 0 in B and if we have to make choice between these two, we should look for the next higher value (than the lowest, which is zero in this case). For  $s_1$  it is 6, whereas for  $s_3$  it is 1. As  $6 > 1$ , we choose  $s_1$ .

If we use Wald's criterion for P in the example, the decision maker chooses  $s_3$ . The regret matrix calculated from

this pay-off matrix  $P$  is:

	$t_1$	$t_2$	$t_3$
$s_1$	-3	0	-3.5
$s_2$	0	-4	-2.5
$s_3$	-2	-6	0

Applying the maxi-min criterion, we find that the choice falls on  $s_1$ . To sum up for the  $P$  in the example, the three criteria suggest the following strategies:

Criterion	Optimal strategy suggested
Wald's	$s_3$
Regret	$s_1$
Benefit	$s_2$

In our hypothetical example each principle suggests a different strategy.

The question then arises, "Why in some cases do these three criterion give different results?" The answer to this question lies in examining 'what happens when the original pay-off matrix  $P$  is converted to a regret or benefit matrix?' When  $P$  is reduced to a regret or benefit matrix, the position (in order of being highest, next highest, ...lowest) of the different strategies  $s_1, s_2, \dots, s_n$  in relation to each other remains the same for a given state of nature  $t_j$ . This is clearly brought out by Figure 12 drawn for the pay-off matrices  $P, R$  and  $B$  in the example.

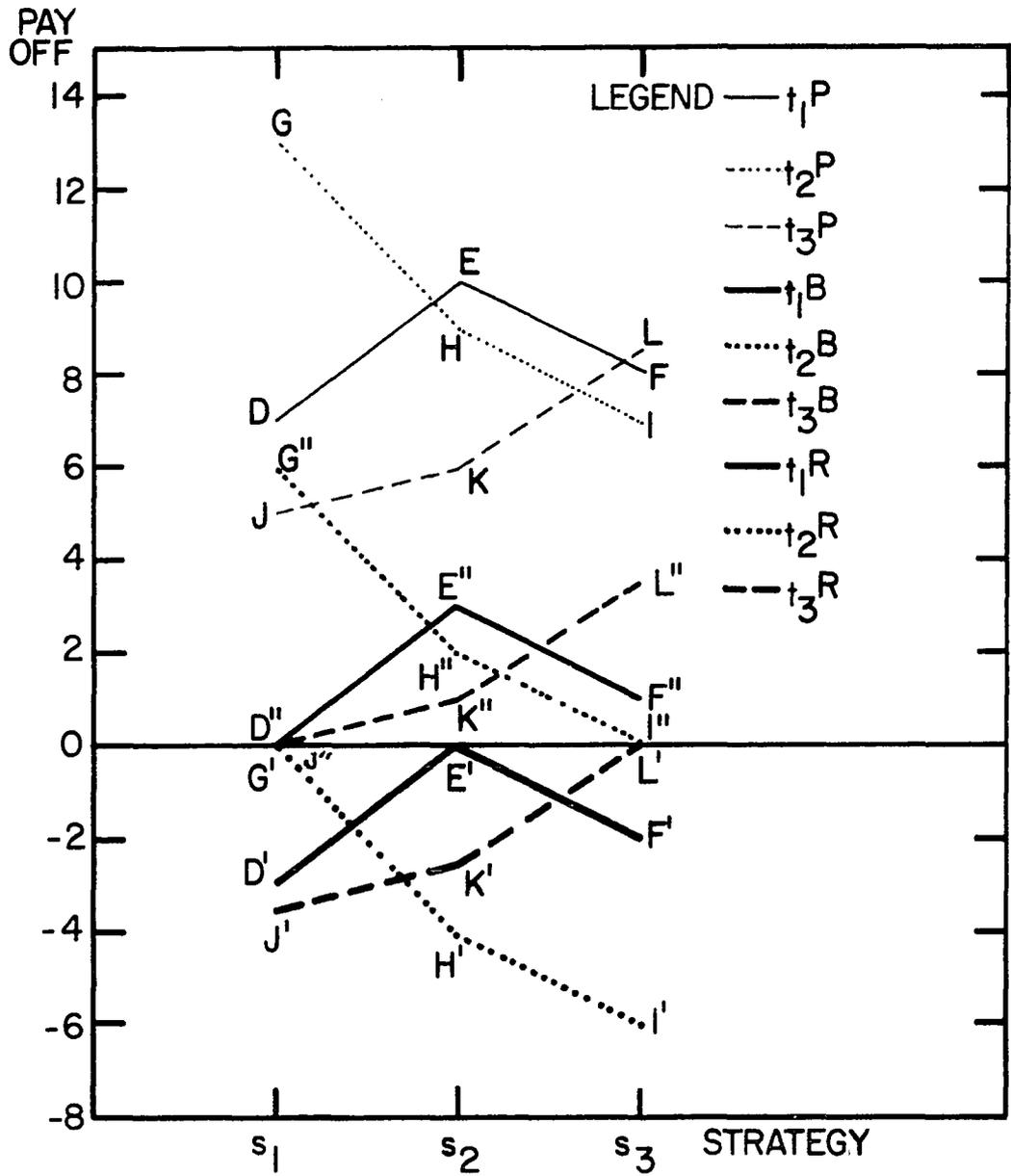


Figure 12. Pay-offs for different strategies of nature

Let

$t_1^P$  be the graph (curve) for  $t_1$  of the pay-offs (of the original matrix P) associated with  $s_1$ ,  $s_2$  and  $s_3$ .

This is DFE in the picture.

$t_2^P$  and  $t_3^P$  be graphs for  $t_2$  and  $t_3$ , respectively, of the pay-offs of P associated with  $s_1$ ,  $s_2$  and  $s_3$ . These are GHI and JKL, respectively.

Similarly,

$t_1^R$ ,  $t_2^R$ , and  $t_3^R$  are the graphs for  $t_1$ ,  $t_2$ , and  $t_3$ , respectively, of the pay-offs of R associated with  $s_1$ ,  $s_2$ , and  $s_3$ . They are D'E'F', G'H'I', and J'K'L', respectively.

Likewise,

$t_1^B$ ,  $t_2^B$ , and  $t_3^B$  are the graphs for  $t_1$ ,  $t_2$ , and  $t_3$ , respectively, of the pay-offs of B associated with  $s_1$ ,  $s_2$ , and  $s_3$ . They are labelled as D"E"F", G"H"I", and J"K"L", respectively.

It can be seen that the graphs for  $t_1^P$ ,  $t_1^R$ , and  $t_1^B$ , i.e., DEF, D'E'F', and D"E"F", are parallel to each other. The same holds true for  $t_2$  and  $t_3$ . That is to say that GHI, G'H'I' and G"H"I" are parallel to each other. Similarly, JKL, J'K'L' and J"K"L" are also parallel to each other. This

is indicative of the fact that relative position of the  $s_i$  within a given  $t$  is unchanged under all the types of matrices. What happens is that in benefit matrix when we deduct the minimum pay-off in a given  $t_j$  from other pay-offs in the same column, we are only displacing the curve for that  $t_j$ , pushing it down to the (tune) extent of minimum pay-off in that  $t_j$ . Thus, DEF has been displaced to D"E"F" and every point on DEF is seven units farther (towards the higher side) from the corresponding point on D"E"F" relating to a given  $s_i$ . The minimum over  $i$  in  $t_2$  is also 7 and, therefore, GHI and G"H"I" are parallel to each other at a distance of 7 units (whatever be the unit of measurement of pay-off, e.g., bushels, dollars, tons, etc.).

Likewise, in the case of  $t_3$ , the minimum in the column is 5 and the two graphs JKL and J"K"L" are 5 units apart from each other at any given point on the  $s_i$  scale.

In the case of a regret matrix, the original curve for a given  $t_j$  is displaced in the downward direction by a distance equal to the maximum in that column ( $t_j$ ). Thus for  $t_1$  the maximum is 10 and DEF is displaced to D'E'F' by 10 units. Similarly, GHI is shifted to G'H'I', the distance being 13 units, and JKL moves to J'K'L', 8 units below its original position.

The displacement in case of a regret matrix is always more than that in the benefit matrix because the maximum in a

column is always  $\geq$  minimum in that column. If the minimum in a column is zero, then  $t_j^P$  and  $t_j^B$  remain the same. For example, if the minimum in  $t_1^P$  would have been zero, then DEF and D"E"F" would have coincided with each other.

The reason for getting different results from different criteria (pay-off matrices) should be evident by now. As we just noted, the displacement of a given  $t_j$  is not uniform. For instance, DEF is displaced by 7 units to D"E"F", whereas the displacement in JKL to J"K"L" is of the order of only 5. The result is that, no doubt, the relative positions of pay-offs for  $s_i$  in a given  $t_j$  remain the same, the positions of pay-offs for a given  $s_i$  (in a row) change in relation to each other due to this difference in the degree of displacement. This point can be well illustrated by referring to Figures 12 and 13.

Let  $s_1^P$ ,  $s_2^P$ , and  $s_3^P$  be the sets of pay-offs associated with  $s_1$ ,  $s_2$  and  $s_3$ , respectively, for the pay-off matrix P.

Let  $s_1^R$ ,  $s_2^R$ , and  $s_3^R$  be the sets of pay-offs of the regret matrix associated with  $s_1$ ,  $s_2$  and  $s_3$ , respectively.

Let  $s_1^B$ ,  $s_2^B$ , and  $s_3^B$  be the sets of pay-offs of the benefit matrix associated with  $s_1$ ,  $s_2$  and  $s_3$ , respectively.

First let us consider  $t_1^P$ (DEF),  $t_2^P$ (GHI), and  $t_3^P$ (JKL) in Figure 13. The set  $s_1^P$  consists of points (G,D,J). Since more is preferred to less and the height of the point directly represents the pay-off, for the set  $s_1^P$  it is true that  $G > D > J$ .

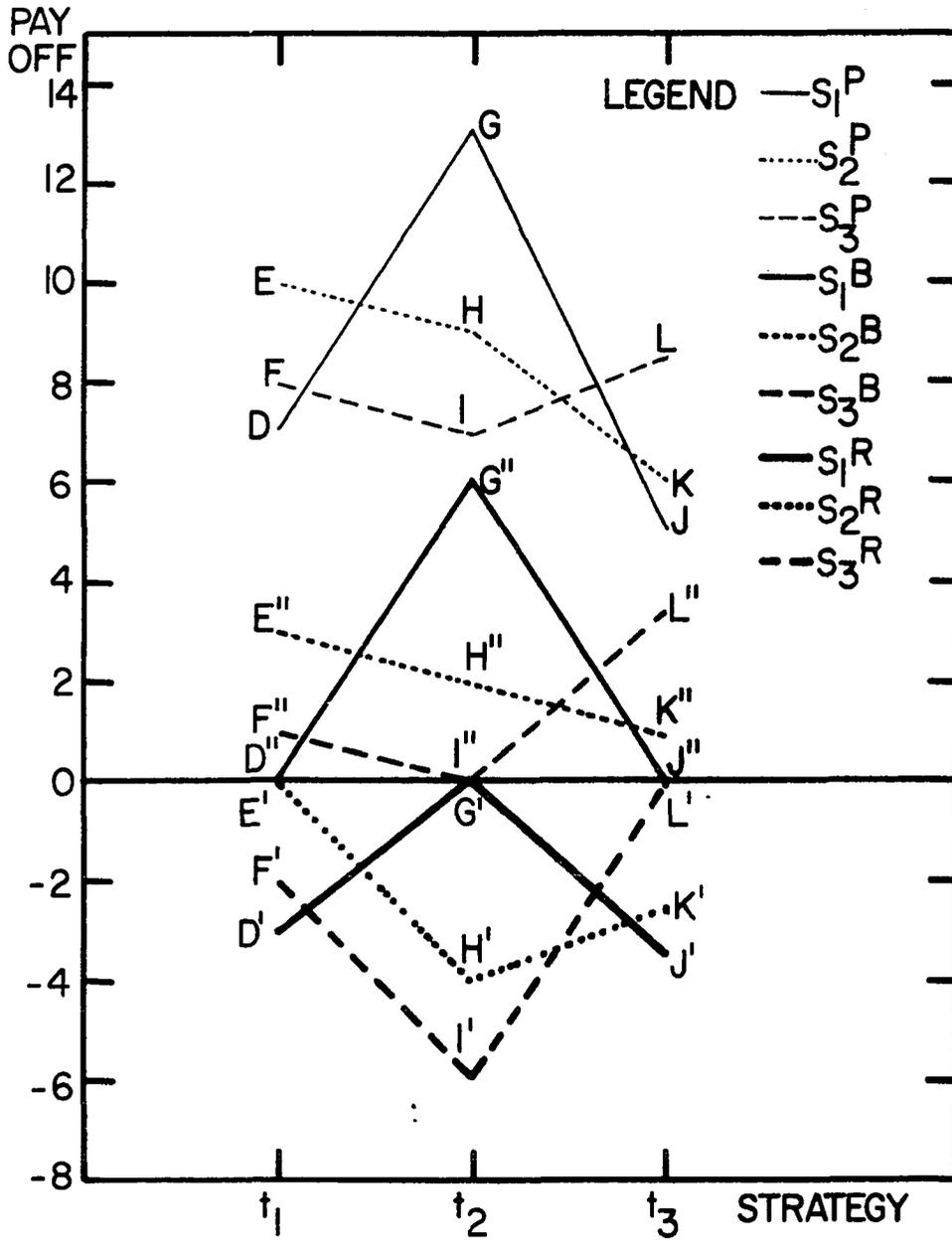


Figure 13. Pay-offs for different strategies of player

Similarly, for  $s_2^P$  and  $s_3^P$  we have  $E>H>K$  and  $L>F>I$ , respectively. In  $P$ , for  $s_1$  the lowest point is  $J$ , for  $s_2$  it is  $K$  and for  $s_3$  it is  $I$ . Since  $I$  is the highest of these,  $I>K>J$ , using the usual maxi-min principle,  $I$  represents the optimal pay-off and  $s_3$  is the optimal strategy.

Now consider  $t_1^R = (D'E'F')$ ,  $t_2^R = (G'H'I')$  and  $t_3^R = (J'K'L')$ . The set  $s_1^R$  consists of  $(G'D'J')$ , which are displaced  $G$ ,  $D$  and  $J$  of  $s_1^P$ . However, note that the order of these elements in  $s_1^R$  is  $(G'>D'>J')$ , which is the same as  $(G>D>J)$  obtained for  $s_1^P$ .

For  $s_2^R$ , the order is  $(E'>K'>H')$ , which differs from  $(E>H>K)$  in  $s_2^P$ . It is this change in the order of displaced elements which is responsible for the choice of a different strategy in the regret criterion as compared to the Wald's criterion. The minimum element of  $s_1^R$  is  $J'$ ,  $s_2^R$  is  $H'$ , and for  $s_3^R$  it is  $I'$ . From Figure 12 we find that  $(J'>H'>I')$ . Hence we choose  $s_1$ .

Likewise, for  $s_1^B$ ,  $s_2^B$ , and  $s_3^B$  the elements in order of preference are  $(G''>D''\approx J'')$ ,  $(E''>H''>K'')$  and  $(L''>F''>I'')$  and the minimum elements of each set are  $D''\approx J''$ ,  $K''$  and  $I''$ , respectively. The order of these points in accordance with their relative heights is  $K''>I''\approx D''\approx J''$  and the choice naturally falls on  $K''$ , an element of  $s_2$ .

All that has been described above can also be verified from Figure 13. This figure has graphs of different  $s_i$  for

$t_1$ ,  $t_2$  and  $t_3$ . The same nomenclature for two points in the two figures denotes that the two points are the same. For example, point J in Figure 12 represents  $(t_3, s_1)$ , which is  $p_{13} = 5$ . In Figure 13 J also represents  $(s_1, t_3)$ , i.e.,  $p_{13}$  and the corresponding pay-off is 5.

Characteristics of the benefit criterion and its comparison with Wald's and regret principles In the Wald's criterion we make use of the original pay-off matrix, whereas in the other two methods, P is transformed into R and B. The method of deriving R from P is different from that of calculating B.

In all these cases (to all these matrices—P, R and B), the maxi-min principle is applied to find out the optimal strategy.

In calculating the regret matrix it is the maximum pay-off in a column which is considered. It is, therefore, rather an optimistic approach. In arriving at the 'benefit' matrix, it is the worst consequence for a given state of nature which is taken into account. The decision maker starts looking for what could have been the worst strategy that he could have chosen given a state of nature and then treats all pay-offs in excess of the lowest pay-off as the benefits accruing from his choosing that particular strategy rather than the worst. This is, of course, more pessimistic than the 'regret' principle.

The benefit matrix is similar to 'regret' matrix in that

as is the case with the latter, it is arrived at by deducting a constant from other pay-offs for a given state of nature. However, it resembles Wald's criterion in the sense that the minimum of a row of  $P$  in Wald's criterion has to be in the same column as the minimum of that row in the  $B$  matrix obtained from  $P$ . That is, if

$$\min_j p_{ij} = p_{ij},$$

then it is necessary that

$$\min_j b_{ij} = b_{ij}.$$

In our example, for row 1

$$\min_j p_{1j} = p_{13}$$

and we see that

$$\min_j b_{1j} = b_{13} \text{ (also } b_{11}\text{)}.$$

Similarly, in row 2

$$\min_j p_{2j} = p_{23} \quad \text{and} \quad \min_j b_{2j} = b_{23}.$$

For row 3

$$\min_j p_{3j} = p_{32} \quad \text{and} \quad \min_j b_{3j} = b_{32}.$$

This is not necessarily true for regret matrix. Again in our example, in row 2

$$\min_j p_{2j} = p_{23} \quad \text{and} \quad \min_j r_{2j} = r_{22}.$$

It must be re-emphasized that in each case (Wald's, Regret and Excess Benefit) we apply the same maxi-min principle to all the matrices, viz. P, R and B. The difference is only in the matrix derived from the original pay-off matrix under different assumptions.

If  $\min_i p_{ij} = 0$  for all  $j$ , then  $P = B$  and Wald's and benefit principles give the same results.

Laplace's criterion suggests the same strategy as optimal no matter on which matrix (P, R or B) it is applied. It is simply because of the fact that when we consider the averages, the variations in displacements are ironed out.

Applications of benefit criterion in agriculture      The following table gives the strategies suggested by this criterion for the twelve applications done elsewhere. The strategies suggested by Wald, Laplace, Hurwicz and Savage's regret principles are also given here for the sake of convenience in comparison of the strategies suggested by these criteria.

Benefit and Wald criteria suggest the same strategy in five of the applications shown in Table 13. Regret and benefit principles agree on four choices. Three of these cases, viz. 1, 4 and 5, are such where the same strategy is suggested by Wald, regret and benefit models. In Application 3 benefit criterion chooses  $s_4$ , application of 10 tons of farm yard manure per acre. This has not been suggested by any

Table 13. Strategies chosen by different criteria

Application No.	Strategies suggested by criterion					
	Benefit	Wald	Laplace	Hurwicz (.3,.7)	Hurwicz (.7,.3)	Regret
1	S <sub>1</sub>	S <sub>1</sub>	S <sub>1</sub>	S <sub>1</sub>	S <sub>1</sub>	S <sub>1</sub>
2	S <sub>1</sub>	S <sub>3</sub>	S <sub>3</sub>	S <sub>3</sub>	S <sub>2</sub>	S <sub>1</sub>
3	S <sub>4</sub>	S <sub>1</sub>	S <sub>5</sub>	S <sub>5</sub>	S <sub>5</sub>	S <sub>5</sub>
4	S <sub>2</sub>	S <sub>2</sub>	S <sub>2</sub>	S <sub>2</sub>	S <sub>2</sub>	S <sub>2</sub>
5	S <sub>3</sub>	S <sub>3</sub>	S <sub>3</sub>	S <sub>3</sub>	S <sub>3</sub>	S <sub>3</sub>
6	S <sub>9</sub>	S <sub>5</sub>	S <sub>9</sub>	S <sub>9</sub>	S <sub>9</sub>	S <sub>7</sub>
7	S <sub>12</sub>	S <sub>1</sub>	S <sub>7</sub>	S <sub>8</sub>	S <sub>8</sub>	S <sub>7</sub> , S <sub>8</sub>
8	S <sub>5</sub>	S <sub>1</sub>	S <sub>2</sub>	S <sub>1</sub>	S <sub>2</sub>	S <sub>1</sub> , S <sub>2</sub> , S <sub>3</sub>
9	S <sub>12</sub>	S <sub>1</sub> , S <sub>2</sub> , S <sub>4</sub> , S <sub>5</sub>	S <sub>5</sub>	S <sub>5</sub>	S <sub>12</sub>	S <sub>1</sub> , S <sub>2</sub> , S <sub>3</sub> , S <sub>4</sub> , S <sub>5</sub> , S <sub>6</sub>
10	S <sub>9</sub>	S <sub>9</sub>	S <sub>8</sub>	S <sub>8</sub>	S <sub>1</sub>	S <sub>8</sub>
11	S <sub>10</sub> , S <sub>11</sub> , S <sub>12</sub>	S <sub>1</sub> , S <sub>2</sub>	S <sub>1</sub>	S <sub>1</sub>	S <sub>1</sub>	S <sub>2</sub> , S <sub>5</sub>
12	S <sub>6</sub>	S <sub>6</sub>	S <sub>5</sub>	S <sub>6</sub>	S <sub>6</sub>	S <sub>5</sub>

other criterion. If the state of nature is  $t_5$ , the worst that a farmer could do was to apply 2.5 tons of manure to an acre. If he chose  $s_4$ , as suggested by benefit principle, he is better off than when he would have used  $s_5$  as suggested by regret criterion. However, under  $t_5$ , it is worse than the pay-off for  $s_1$ , which is chosen by Wald's principle.

For Applications 3, 8, 9 and 11, the choice as given by the benefit principle is different from those by other of the four criteria.

This criterion of 'excess benefit' combines Wald's criterion (for original pay-off matrix  $P$ ) and regret principle in many ways. For example, look at  $s_2^P$ ,  $s_2^R$ , and  $s_2^B$  in Figure 13. Whereas  $s_2^P$  (EHK) has a kink on the upper side at H, for the regret criterion,  $s_2^R$  (E'H'K') has a kink on the lower side of the curve at H'. The graph of  $s_2$  for the 'benefit criterion', i.e.,  $s_2^B$ , is a straight line (E"H"K") showing a compromise between the two criteria—Wald's (rank pessimist criterion) and Savage's regret (rank optimist) criterion—and, therefore, can be termed as neither too optimistic nor too pessimistic. Under the most favorable circumstances, it may be better than Wald's but poorer as compared to the regret principle. However, under unfavorable conditions, it may be inferior to Wald's but superior to the regret criterion. (By inferior and superior, we mean affording lower or higher pay-offs, respectively.) Of course, it might be desirable to

further test its suitability to different situations, but the very fact that it is a hybrid of Wald's and Savage's criteria should render it more vigorous and productive than the above two for decision making in agriculture.

### Summary

Efforts were made in the foregoing applications and discussion of the various principles of choice to suggest the suitability of different approaches to the farmer (decision maker) under different sets of circumstances. The desirability and, therefore, the choice of a decision model would be largely governed by the:

- a. Goals of the farmer—whether he wants to maximize profits or satisfaction or family happiness, etc.
- b. His psychology and outlook.
- c. His financial status in terms of his equity, total investments in the farm and outside, the size of the farm, etc.
- d. The nature of the enterprises, with respect to their diversity and variability, or of returns.
- e. The variability in the states of nature.
- f. Farmer's resources.
- g. His education and age.
- h. His knowledge of technology, of states of nature, his standing in the farming business in terms of years, his experience.
- i. His family situation.

The set of permutations and combinations of these and

other factors is infinite. Therefore, it is rather difficult to prescribe a rule of thumb, but the suggestions and observations of this study made in discussing the pros and cons of various criterion in the light of their applications to these examples would hopefully serve to provide a guideline to the farmer. Milnor (70), in his paper entitled "Games Against Nature", has laid down a set of ten properties of an acceptable decision criterion (a criterion yielding a complete ordering for all acts is defined by him as acceptable) and has demonstrated that of the four approaches (Wald's, Laplace's, Hurwicz' and regret), no single criterion possesses all the properties. It would be interesting to test the criterion of benefit and see how many of the required properties are present in this principle. However, it appears to be more realistic and useful than any other criteria.

## APPLICATIONS OF MATHEMATICAL PROGRAMMING IN AGRICULTURE

If the structure of a system and the objective function can be expressed in terms of a mathematical model, the desired solution can be computed by means of the techniques grouped under a general heading of 'Mathematical Programming'. They include tools like linear programming (integer and non-integer, variable price and variable resource, perturbation techniques, etc.) and non-linear programming (e.g., quadratic programming, concave and convex programming) and dynamic programming. For special situations, techniques of recursive and parametric programming, etc. have been developed. Dantzig (23) has given the classification of the programming problems as shown on page 149.

As the name suggests, like any other mathematical tool, mathematical programming is a mathematical technique without any economic content. Its sole purpose is to indicate the optimum solution to a problem for a given set of circumstances.

It would be rather ambitious to attempt to deal with all classes of mathematical programming. Therefore, we shall confine ourselves to the study of only the following aspects of linear programming:

- a. Characteristics of linear programming problems and the general results (theorems) of linear programming
- b. Connection between game theory and linear programming
- c. Linear fractional functional programming—a brief

## Classification of Programming Problems

Discrete or Continuous Multistage or Non-multistage (Dynamic or Non-Dynamic)	Special cases of these classifications
<b>DETERMINISTIC</b>	
<b>Linear</b>	
General structures	Linear inequality theory
Special structures	Dynamic systems, Leontief models, Networks
<b>Non-linear</b>	
Convex	Decreasing pay-off, Chemical equilibrium, Convex programs
Non-convex	Increasing returns to scale, Many local maxima
<b>PROBABILISTIC</b>	
<b>No Opponents</b>	
Known probability distribution	Inventory control, Markov chains
Unknown probability distribution	Sequential decisions
<b>Against Opponents</b>	
Two person games	Zero-sum games
Multi-person games	Coalition theory

description of a new algorithm to solve problems of linear fractional functional programming and its application in agriculture

### Linear Programming

One of the simple but perhaps the most important and widely used techniques developed in the field of operations research is linear programming. Its applicability to the practical applications in the fields of management economics has been largely responsible for its development to the present level. The objective is to optimize (maximize or minimize as the case may be) the function  $f(x)$  where  $f(x) = FX + k$  is a vector function. It is linear.

$F$  is a functional,

$k$  is a constant, and

$X$  ranges over a convex polyhedral set of points.

The maximization (or minimization) of the objective function is subject to certain linear constraints. The usual way of writing a maximization problem in a matrix form is:

Maximize  $z = c'x$ , subject to

$$Ax \leq b,$$

$$x \geq 0$$

where

$A$  is  $m \times n$  matrix

$c$  is  $n \times 1$  vector

$x$  is  $n \times 1$  vector

$b$  is  $m \times 1$  vector

and

$c'x = Z$  is the objective function.

This problem can be written as

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{subject to } a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$x_m \geq 0.$$

In a compact form the problem is

$$\text{Maximize } Z = \sum_{j=1}^n c_j x_j$$

$$\text{subject to } \sum_{j=1}^n a_{ij} x_j \leq b_i$$

$$x_j \geq 0.$$

where

$$i = (1, 2, \dots, m) \quad \text{and} \quad j = (1, 2, \dots, n).$$

The problem in the preceding form (maximization problem) is referred to as the 'primal problem' or 'primal form' of linear programming problem. A maximization problem can be quickly converted into an equivalent minimization problem in the following manner.

Write down the objective function as

$$- (- \text{maximize } c'x)$$

subject to

$$Ax \leq b,$$

$$x \geq 0.$$

The statements - (minimize - c'x) and - (- maximize c'x) are equivalent. Therefore, the problem can be written as

$$- (\text{minimize } - c'x)$$

subject to

$$Ax \leq b$$

$$x \geq 0.$$

First the problem is solved for the objective function (minimize - c'x). Then the negative of the solution gives the value of the objective function in the original problem. A minimization problem can be converted to a maximization problem in exactly the same manner.

A linear programming problem arises primarily due to the following two reasons:

- (a) The number of constraints is not equal to the number of variables (i.e.,  $m \neq n$ ) and, therefore, matrix A does not have a full rank.
- (b) The inequalities of the constraints.

The basic assumptions underlying a linear programming model are those of

1. Additivity of the resources and activities
2. Linearity of the objective function
3. Non-negativity of the decision variables
4. Divisibility of activities and resources
5.  $m$  and  $n$  being finite
6. Resource supplies, input-output coefficients, prices of resources and activities, etc., known with certainty

One of the essential parts of a linear programming problem is building up the model. Dantzig (23) has given the following five steps of model building in linear programming.

Step 1: Define the 'activity set'. This involves a decomposition of the system into its basic components called activities and choosing a unit for each activity in terms of which its quantity or level can be measured.

Step 2: Define the 'item set'. The items that are either produced or consumed (negative production) by the activities are determined and a suitable unit of measurement chosen for each of these items. One item is

selected such that the net quantity of it produced by the system as a whole measures the 'cost' (or such that its negative measures the 'profit') of the entire system.

Step 3: Determine the 'input-output coefficients'. These coefficients are the proportion of activities and the item flows. Commonly speaking, they refer to the quantity of each item produced or consumed by the operation of each activity at its unit level.

Step 4: Determine the 'exogenous flows'. This step requires determination of the inputs or outputs in 'net' terms of the items between the system, taken as a whole, and the outside.

Step 5: Determine the material balance equations. Unknown (non-negative) activity levels  $x_1, x_2, \dots$  are assigned to all activities in the system and then 'material balance equation' for each item is written down. The equation asserts that the algebraic sum of the flows of that item into each activity (given as the product of the activity level by the appropriate input-output coefficient) is equal to the exogenous flow of the item.

All these steps result in a set of mathematical relationships characterizing all the feasible programs of the system. This set is termed as the 'linear programming model'.

Primal and dual

Every linear programming problem comes in a package of two, i.e., 'primal' and its 'dual'. For every primal problem there exists an equivalent dual problem.

<u>Primal problem</u>	<u>Dual problem</u>
Maximize $c'x$	Minimize $b'y$
subject to $Ax \leq b$	subject to $A'y \geq c$
$x \geq 0$	$y \geq 0$

where  $A$ ,  $b$ ,  $c$  and  $x$  have the same dimensions as given on pages 150-151. In this illustration  $y$  is a vector with dimensions  $m \times 1$ , i.e.,  $y = (y_1, y_2, \dots, y_m)$ . The dual variable is  $y$  (dual of  $x$ ). If the primal is a maximization problem, its dual would always be a minimization problem and vice versa. A 'primal' is the 'dual' to its own 'dual', i.e., the dual of a dual of a primal problem is the primal problem itself. Further, if there are  $n$  variables and  $m$  constraints in the primal problem, in the dual we shall have  $m$  variables and  $n$  constraints. Solving one (either primal or dual) would give a solution to both.

The utility of duality lies not only in the fact that it yields a number of powerful theorems in linear programming, but also because it has very important economic interpretations in terms of shadow prices, etc. Further, if  $m$  is much larger than  $n$ , it would be rather easy to solve the dual rather than

the primal problem.

Applications of linear programming in agriculture

Linear programming has been used in agriculture almost since its very inception. In 1951 Waugh (109) applied this technique to the problem of minimization of costs of feed for dairy cows. Koopmans' "Activity Analysis of Production and Resource Use" has a paper entitled, "On the Choice of a Crop Rotation Plan" by Hildreth and Reiter (42). "Linear Programming Methods" by Heady and Candler (39) deals exclusively with applications in the field of agriculture. Boles (11) has written on "Linear Programming and Farm Management Analysis". Perhaps the most extensive use of linear programming in agriculture has been in the field of feed-mixing with the object of minimization of cost of feed. Even stochastic programming models have been used and in some states services are available to the farmer to advise him (on an individual basis) on the least-cost-feed-mix on his farm with the use of computer. For example, Doan Agricultural Service has used linear programming as a management service to the farmer in the United States. Several land grant colleges have undertaken programs for farm planning, on a limited scale, as a part of their extension activity. Use of linear programming for individual farmers is commonly referred to as 'program planning' and has been widely used in Europe and Japan and to a limited extent in the U.S.A. Barker (4)

conducted a study on the use of linear programming in making farm management decisions and came to the conclusion that, "linear programming can be of value in farmer decision-making by providing quantitative estimates of returns for specified alternatives and levels of resource use" and the larger the size of the farm, the larger the number of alternatives and the greater the likelihood of benefits from linear programming exceeding its costs.

In addition to their use at the micro-level, i.e., cost minimization and profit maximization on an individual farm, linear programming techniques have been applied with advantage at the macro-level for solving the problems of agricultural marketing and spatial analysis. Studies in inter-regional production and adjustments for major crops have been made through the use of 'spatial linear programming' technique. Transportation models are the simplest of linear programming models applied in agriculture.

Since simple linear programming techniques have been extensively used in agriculture and it is virtually impossible to touch on all types and phases of linear programming applications in farming, it is proposed to limit the treatment of linear programming in this study to

- (a) general characteristics of the feasible set of a linear programming problem,
- (b) important theorems and results in linear programming,

- (c) relationship of game theory to linear programming,  
and
- (d) application of a relatively new technique of linear  
fractional functional programming in agriculture.

We shall be dealing here only with the problems of maximization. Problems of minimization can be handled in a similar manner by converting them to those of maximization in the manner given on page 152.

### General features

Once again the general format of a maximization problem is

$$\begin{aligned} &\text{Maximize } Z = c'x \\ &\text{subject to } Ax \leq b, \\ &\quad \quad \quad x \geq 0 \end{aligned}$$

where

A is  $m \times n$  matrix,  
 x is  $n \times 1$  vector,  
 b is  $m \times 1$  vector,  
 c is  $n \times 1$  vector.

Clearly, we shall be operating in the positive orthant with dimension of  $n-1$ .

As we have  $m$  rows and  $n$  decision variables, the total number of half spaces is  $(m + n)$ . These half spaces are closed. Thus the power set of a linear programming problem is

a closed convex polyhedron since it is an intersection of  $m + n$  half spaces and because a half-space (whether it is open or closed) is always convex.

If  $F$  is a function on Euclidean space  $X^n$  (the members of  $X^n$  being  $x_1, x_2, \dots, x_n$ ) and  $b = (b_1, b_2, \dots, b_m)$  is a vector of numbers, then

$$FX \leq b \quad \text{and} \quad FX \geq b$$

are closed half spaces in Euclidean space  $X^n$  and

$$FX < b \quad \text{and} \quad FX > b$$

are open half spaces in  $X^n$ .

For a two-dimensional case in our example, say  $a_{11}x_1 + a_{12}x_2 \leq b$ , the diagram will be something like Figure 14. In the diagram line  $PQ$  is a set with the property that

$$PQ = \left[ \begin{array}{l} (x_1) \\ (x_2) \end{array} \middle| a_{11}x_1 + a_{12}x_2 = b_1 \right]$$

Note that  $PQ$  divides the positive quadrant (whole space) into two parts.

$N$  and  $M$  are half spaces. If  $M$  contains  $PQ$ , then  $M$  is closed half space, otherwise open. Similarly, if  $N$  contains  $PQ$ ,  $N$  is then closed. If not, it is open.

a. If  $M$  contains  $PQ$ , then set  $M$  is defined as

$$M = \left[ \begin{array}{l} (x_1) \\ (x_2) \end{array} \middle| a_{11}x_1 + a_{12}x_2 \geq b_1 \right]$$

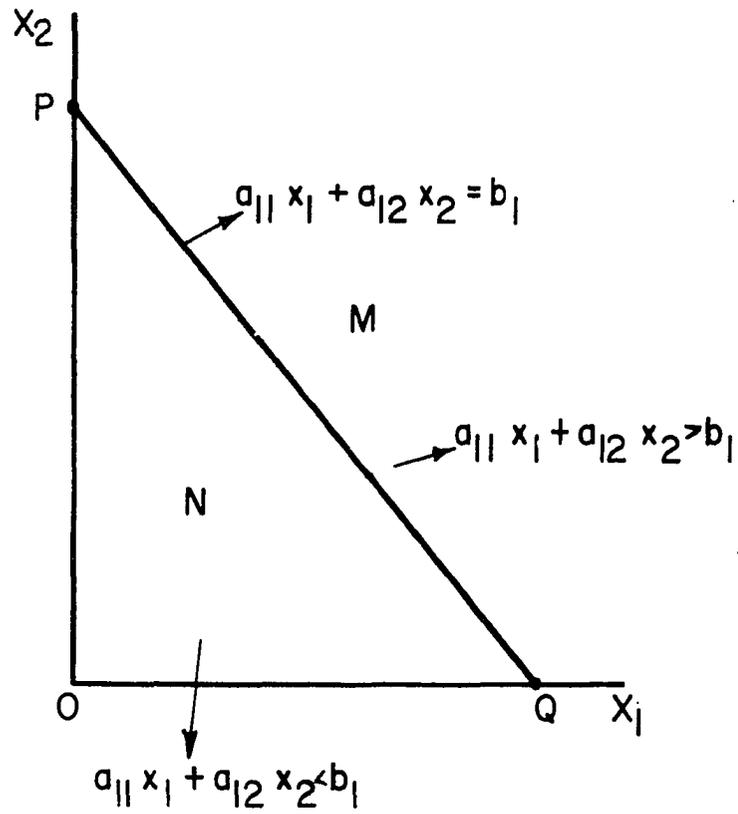


Figure 14. Open and closed half-spaces in a two-dimensional space

b. Without PQ being in M,

$$M = \left[ \begin{array}{l} (x_1) \\ (x_2) \end{array} \middle| a_{11}x_1 + a_{12}x_2 > b_1 \right]$$

Likewise,

a. If N contains PQ, then

$$N = \left[ \begin{array}{l} (x_1) \\ (x_2) \end{array} \middle| a_{11}x_1 + a_{12}x_2 \leq b_1 \right]$$

b. If PQ is not contained in N,

$$N = \left[ \begin{array}{l} (x_1) \\ (x_2) \end{array} \middle| a_{11}x_1 + a_{12}x_2 < b_1 \right]$$

The problem in solving a linear programming problem is to find efficient set (determined by the objective) of extreme points of this closed convex polyhedral set, as every extreme point is a basic feasible solution to the set of constraints and also that every basic feasible solution is an extreme point of the convex set of feasible solutions. If K is this power set defined by  $Ax \leq b$ , then  $e_k$  is an (kth) extreme point of K if

a.  $e_k$  belongs to K, and

b.  $e_k$  is the intersection of the bounding hyperplanes.

The property of the extreme points  $e_k$  is that every point in K can be expressed as a convex combination of the  $e_k$ 's. For an

$n$ -simplex, the number of extreme points is  $n + 1$ , i.e., if  $j = 1, 2, \dots, n$ , then  $k = 1, 2, \dots, n+1$ . As every extreme point is a basic feasible solution to the set of constraints and also as every basic feasible solution is an extreme point of the convex set of feasible solutions, the optimum (maximum or minimum as the case may be) value of  $c'x$ , as  $x$  varies over  $K$ , will be at one or more of the extreme points of  $K$ . Therefore, the answer to the linear programming problem lies in

- a. finding values of the objective function at the extreme points, and
- b. choosing the optimum.

The simplex method is a technique of finding out these corner positions (extreme values). In this method, the slack variables,  $x_{n+1}, \dots, x_{n+m}$  are introduced to convert the inequalities to equalities and the coefficients of these slack variables in  $c$  vector are zero. The original problem is then transformed into the following new problem:

$$\text{Maximize } \bar{Z} = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0x_{n+1} + \dots + 0x_{n+m}$$

subject to

$$\begin{array}{cccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + 1x_{n+1} & = & b_1 \\ \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + 1x_{n+m} & = & b_m \end{array}$$

$$\begin{aligned}
 x_1 &\geq 0 \\
 x_2 &\geq 0 \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 x_{n+m} &\geq 0
 \end{aligned}$$

Let us call

$$L \equiv \left[ \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \\ \vdots \\ x_{n+m} \end{array} \right) \middle| \left[ A \quad I_m \right] \left( \begin{array}{c} x_1 \\ \vdots \\ \vdots \\ x_{n+m} \end{array} \right) = b, \begin{array}{c} x_1 \geq 0 \\ \vdots \\ \vdots \\ x_{n+m} \geq 0 \end{array} \right] \subseteq E^{m+n}$$

where  $I_m$  is  $m \times m$  identity matrix and  $E^{n+m}$  is euclidean space of dimension  $n + m$ .

Then there is 1 - 1 correspondence between extreme points of  $K$  and  $L$ . As the value of the objective function  $Z$  would be equal to that of  $\bar{Z}$ , the two problems—original and transformed—are equivalent and solving the new problem would automatically give the solution to the original problem. Simplex method takes advantage of this property of 1 - 1 correspondence and deals with the transformed problem. The method enables us to go from one extreme point to another directly and then to another until an optimum is reached. The problem of finding a starting extreme point (initial basic feasible solution) is handled in the simplest manner by starting with all the activities at the zero level. In a two-dimensional case,

Figure 15, this starting point is 'E'.

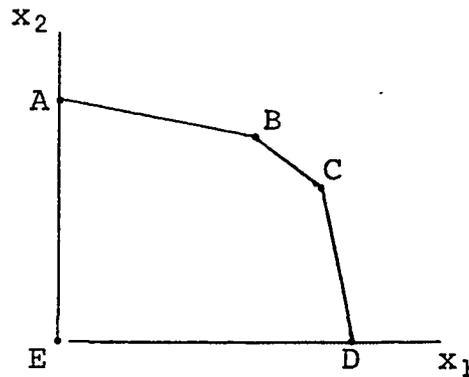


Figure 15. Initial basic feasible solution

At E,  $x_1 = 0$  and  $x_2 = 0$ . Starting from E, we go to A, B, C or D (as may be necessary) in steps and testing at every step whether we have reached the optimal solution.

Some basic results in linear programming

<u>Primal form (problem)</u>	<u>Dual problem</u>
Maximize $c'x$	Minimize $b'y$
subject to $Ax \leq b,$	subject to $A'y \geq c,$
$x \geq 0$	$y \geq 0$
$y$ is the dual variable	

A problem in the dual form is converted to that in a primal form by simply changing the signs of all parameters and vice versa.

Before some results are stated, it would be helpful to define 'lagrangian form' and its 'saddle point'.

Definition:  $f(x,y) = c'x + y(b-Ax)$  is called the lagrangian form of the primal problem.

$g(y,x) = - (y'b) + (yA-c)x$  is called the lagrangian form of the dual.

It should be noted that  $f(x,y) = -g(y,x)$ .

Definition:  $x^*, y^*$ , a pair of vectors, is a saddle point of  $f(x,y)$ , if

(a)  $x^* \geq 0$ ,  $y^* \geq 0$ , and

(b) for all  $x \geq 0$ ,  $y \geq 0$ ,  $f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y)$ .

Result 1: If  $x^*$  is feasible for primal and  $y^*$  is feasible for the dual, then  $y^*b \geq cx^*$ .

Result 2:  $x^*$  is a solution to the primal if there exists a  $y^*$  such that  $(x^*, y^*)$  is a saddle point of  $f(x,y)$ .

Result 3: If the primal and dual problems have feasible vectors, then both problems have solutions and all pairs of solutions  $x^*$  and  $y^*$  are such that  $y^*b = cx^*$ . This is called the 'existence theorem'.

Result 4: If  $x^*$  is optimal for the primal and  $y^*$  is optimal for the dual, then  $y^*b = cx^*$ . This is sometimes referred to as the 'fundamental theorem of linear programming'.

Result 5: If

(a)  $x^*$  is feasible for primal,

(b)  $y^*$  is feasible for the dual, and if

(c)  $y^*b = cx^*$ ,

then

- (a)  $x^*$  is a solution to the primal,
- (b)  $y^*$  is a solution to the dual, and
- (c)  $(x^*, y^*)$  is a saddle point of  $f(x, y)$ .

Result 6: If the primal problem has a solution  $x^*$ , then

- (a) the dual problem has a solution  $y^*$ , and
- (b)  $y^*b = cx^*$ .

This is called the 'duality theorem'.

Result 7: If  $x^*$  and  $y^*$  are solutions to the primal and the dual respectively, then

$$(Ax^*)_i < b_i \longrightarrow y_i^* = 0, \text{ and}$$

$$(y^*A)_j > c_j \longrightarrow x_j^* = 0.$$

The property of  $y^*b = cx^*$  can be used with advantage to derive the following results with the help of calculus under the assumption that the first and second order derivatives exist.

- (a) If we attempt to evaluate  $\frac{\partial f(x, y)}{\partial b_i}$  at  $(x^*, y^*)$ , we find that

$$\left. \frac{\partial f(x, y)}{\partial b_i} \right|_{\substack{x=x^* \\ y=y^*}} = y^*.$$

- (b) Similarly,

$$\left. \frac{\partial f(x, y)}{\partial c_j} \right|_{\substack{x=x^* \\ y=y^*}} = x^*$$

(c) In the comparative static analysis sense:

(i)  $\frac{\partial f(x,y)}{\partial b_i} > 0 \longrightarrow f(x,y)$  is a concave

non-decreasing function in  $b$ , i.e., if we hold others constant and vary  $b$  only, the objective function takes a concave non-decreasing shape as shown in Figure 16.i.

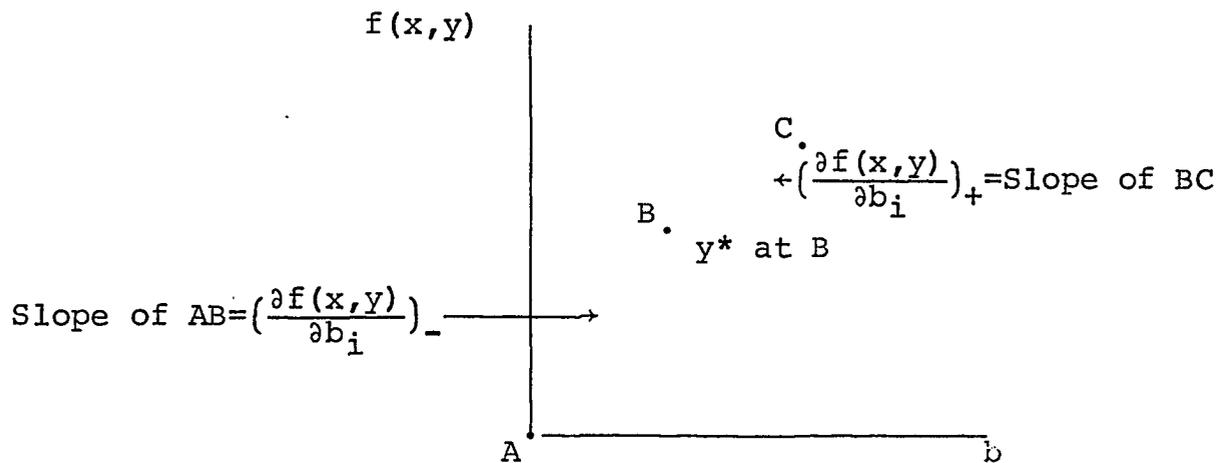


Figure 16.i. Concave non-decreasing function

$$0 \leq \left(\frac{\partial f(x,y)}{\partial b_i}\right)_+ \leq y^* \leq \left(\frac{\partial f(x,y)}{\partial b_i}\right)_-$$

(ii) Likewise,  $\frac{\partial f(x,y)}{\partial c_j} > 0 \longrightarrow f(x,y)$  is a convex

non-decreasing function in the  $c$  space as can be seen in Figure 16.ii.

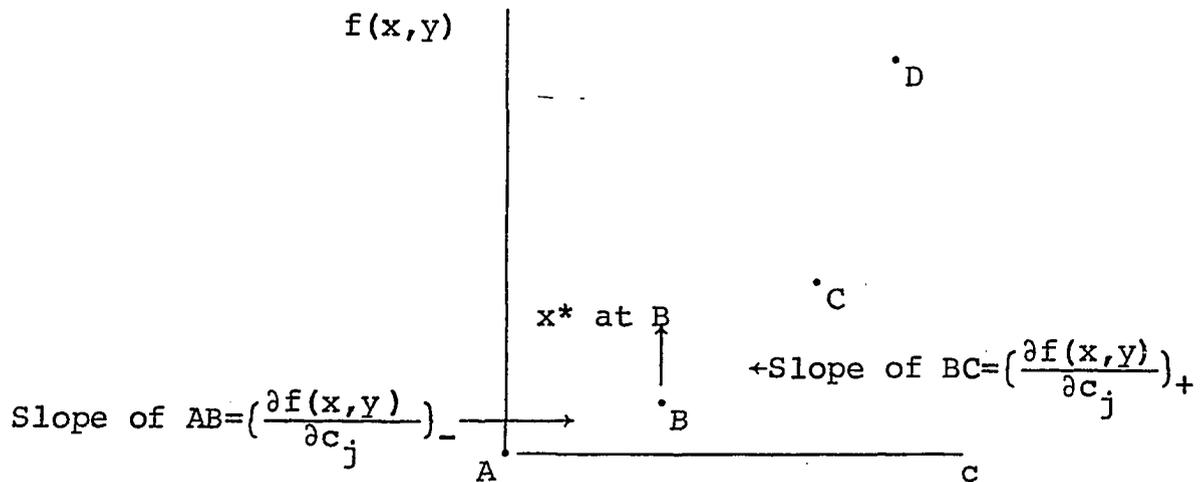


Figure 16.ii. Convex non-decreasing function

$$\text{Here } 0 \leq \left(\frac{\partial f(x, y)}{\partial c_j}\right)_- \leq x_j^* \leq \left(\frac{\partial f(x, y)}{\partial c_j}\right)_+$$

### Relationship of game theory to linear programming

It would be interesting and useful to note that the game theory and linear programming are closely related. Here the equivalence of and connection between game theory and linear programming shall be demonstrated in brief.

1. The method used below for illustrating the connection between game theory and linear programming has been given by Karlin (50).

Let  $A$  be the coefficient matrix for the usual primal linear programming problem and let its size be  $m \times n$ . Matrix  $A$  is then enlarged in the following manner and let this enlarged matrix be called  $P$ .

$$P = \begin{array}{ccc|c} & n & m & 1 \\ \hline & 0 & -A' & -c \\ & A & 0 & -b \\ \hline & c' & b' & 0 \end{array} \begin{array}{l} n \\ m \\ 1 \end{array}$$

where

$m$ ,  $n$  and  $1$  denote the size of the columns or rows,

$c$  is  $n \times 1$  vector (same as in primal problem),

$b$  is  $m \times 1$  vector (same as in linear programming problem),

$0$  in the top left corner is an  $n \times n$  matrix with all its elements being zeros,

$0$  in the middle is  $m \times m$  matrix with all elements being zeros,

$0$  in the bottom right corner is a scalar,

$P$  is a skew-symmetric matrix with dimensions  $(m+n+1) \times (m+n+1)$ .

Therefore, the value of the game represented by  $P$  is zero and the optimal strategy sets  $X^0$  and  $Y^0$  are identical for the two players.

Let

$$x^* = (x_1^*, x_2^*, \dots, x_n^*)$$

be the optimal probabilities attached to first ' $n$ ' rows/columns

$$y^* = (y_1^*, y_2^*, \dots, y_m^*)$$

be the optimal probabilities attached to the next 'm' rows/columns, and where  $x^*$  is a column vector and  $y^*$  is a row vector. Then

$$x^0 = y^0 = (x^*, y^*, \lambda)$$

where

$\lambda$  is the probability attached to the last row/column.

It may be recalled here that  $P$  is skew-symmetric and, therefore, the game with pay-off matrix  $P$  is a symmetric game.

The connection between game theory and linear programming is then given by the following result.

If symmetric game with pay-off matrix  $P$  has a solution with  $\lambda > 0$ , then both the primal and the dual linear programming problems have solutions.

The solution for the primal problem is  $\frac{c'x^*}{\lambda}$ .

The solution to the dual problem is  $\frac{y^*b}{\lambda}$ , and  $\lambda = 1 - \sum x_j^* - \sum y_i^*$ .

Since optimal solution to a primal is equal to optimal solution for its dual, we have

$$\frac{c'x^*}{\lambda} = \frac{y^*b}{\lambda}$$

and

$$c'x^* = y^*b.$$

This demonstrates how game theory is related to linear programming and how can we get a solution to one from the solution to the other. Unfortunately, solving for  $(x^*, y^*, \lambda)$  is no easy task in itself if the size of  $P$  is large. Therefore, it might not be out of place to point out that a game problem can be converted into a linear programming form with great ease in the following manner.

2. Let there be two players  $P$  and  $Q$  with the pay-off matrix  $A$ , the dimensions of which are  $m \times n$ . Further, let  $P$  be the maximinimizer and  $Q$  be the minimaximizer. Also let

$$y^* = (y_1^*, y_2^*, \dots, y_m^*)$$

be the optimal strategy set for  $P$ , and

$$x^* = (x_1^*, x_2^*, \dots, x_n^*)$$

be the optimal strategy set for  $Q$ . Further, let  $V$  be the value of the game.

Then if  $P$  uses his strategies with optimal probabilities  $y^*$ , his expected pay-off will be at least  $V$  no matter what  $Q$  does. Similarly, if  $Q$  uses his optimal strategy set  $x^*$ , his expected loss would, at most, be  $V$  regardless of  $P$ 's play.

Let

$$w_i = \frac{y_i^*}{V} \quad \text{and} \quad z_j = \frac{x_j^*}{V}.$$

As  $x_j^*$  and  $y_i^*$  are probabilities,

$$\sum x_j^* = 1 \quad \text{and} \quad \sum y_i^* = 1.$$

Therefore,

$$\sum z_j = \frac{1}{V} \quad \text{and} \quad \sum w_i = \frac{1}{V}.$$

Since P is a maxi-minimizer, he would try to maximize V or minimize  $\sum w_i$ . Thus his problem can be written as follows:

$$\begin{aligned} & \text{Minimize} \quad \sum w_i \\ & \text{subject to} \quad y_1^* a_{11} + y_2^* a_{21} + \dots + y_m^* a_{m1} \geq V \\ & \quad \quad \quad \cdot \\ & \quad \quad \quad \cdot \\ & \quad \quad \quad \cdot \\ & \quad \quad \quad y_1^* a_{1n} + y_2^* a_{2n} + \dots + y_m^* a_{mn} \geq V \\ & \quad \quad \quad y_i^* \geq 0 \\ & \quad \quad \quad \sum y_i^* = 1 \end{aligned}$$

Dividing both sides of the constraint inequalities by V we have

$$\begin{aligned} & \text{Minimize} \quad \sum w_i \\ & \text{subject to} \quad w_1 a_{11} + w_2 a_{21} + \dots + w_m a_{m1} \geq 1 \\ & \quad \quad \quad \cdot \\ & \quad \quad \quad \cdot \\ & \quad \quad \quad \cdot \\ & \quad \quad \quad w_1 a_{1n} + w_2 a_{2n} + \dots + w_m a_{mn} \geq 1 \\ & \quad \quad \quad w_i \geq 0 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \\ \\ \\ \\ \text{call it} \\ G \end{array}$$

This is just in the form of the usual dual problem.

Similarly for player Q, the problem is to

$$\begin{array}{rcl}
 \text{Maximize} & \Sigma z_j & \\
 \text{subject to} & a_{11}x_1^* + a_{12}x_2^* + \dots + a_{1n}x_n^* & \leq V \\
 & \cdot & \cdot \\
 & \cdot & \cdot \\
 & \cdot & \cdot \\
 & a_{m1}x_1^* + a_{m2}x_2^* + \dots + a_{mn}x_n^* & \leq V \\
 & & \\
 & x_i^* & \geq 0 \\
 & \Sigma x_i^* & = 1
 \end{array}$$

Dividing both sides of the constraint inequalities by V we have

$$\begin{array}{rcl}
 \text{Maximize} & \Sigma z_j & \\
 \text{subject to} & a_{11}z_1 + a_{12}z_2 + \dots + a_{1n}z_n & \leq 1 \\
 & \cdot & \cdot \\
 & \cdot & \cdot \\
 & \cdot & \cdot \\
 & a_{m1}z_1 + a_{m2}z_2 + \dots + a_{mn}z_n & \leq 1 \\
 & & \\
 & z_i & \geq 0
 \end{array}
 \left. \vphantom{\begin{array}{rcl} \text{Maximize} & \Sigma z_j & \\ \text{subject to} & a_{11}z_1 + a_{12}z_2 + \dots + a_{1n}z_n & \leq 1 \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & a_{m1}z_1 + a_{m2}z_2 + \dots + a_{mn}z_n & \leq 1 \\ & & \\ & z_i & \geq 0 \end{array}} \right\} \begin{array}{l} \text{Call it} \\ H \end{array}$$

This can be easily verified that H is 'dual' to G and vice versa.

### Linear Fractional Functionals Programming and Its Possibilities of Being Applied to Agriculture

The problem here is to optimize the objective function which is a ratio of two linear functions of the decision variable. Of course, the constraints are linear.

If  $x$  is the decision variable, the format of the problem would be

$$\begin{aligned} \text{Optimize (maximize or minimize)} \quad \phi(x) &= \frac{c'x + r}{d'x + q} \\ \text{subject to} \quad Ax &= b \\ x &\geq 0 \end{aligned}$$

where

$c$ ,  $x$ ,  $b$  and  $A$  are the same as in the ordinary linear programming problem,

$r$  and  $q$  are scalars, and

$d$  is  $n \times 1$  vector.

Charnes and Cooper (16), in their paper on "Programming with Linear Fractional Functionals", solved the problem by resolving it into two linear programming problems in the following manner.

Let us assume that the problem given above was solved.

Let

$$K = \left[ x \mid Ax = b, \quad x \geq 0 \right]$$

be non-empty and bounded.

Then they transformed the problem (broke it up) into two linear programming problems as shown below and solved them.

- (a) Maximize  $c'Y + rt$   
 subject to  $AY - bt = 0$   
 $d'Y + qt = 1$   
 $Y \geq 0$   
 $t \geq 0$
- (b) Maximize  $-c'Y - rt$   
 subject to  $AY - bt = 0$   
 $-d'Y - qt = 1$   
 $Y \geq 0$   
 $t \geq 0$

As would appear, the transformation used is  $Y = tx$  in such a way that the denominator of the objective function is reduced to unity.

Kanti Swaroop (98) has proposed a more direct approach in his paper on "Linear Fractional Functionals Programming". He has established conditions for the optimality criterion, starting from the basic feasible solution and demonstrating conditions under which the solution can be improved. The utility of the method lies in its similarity to the 'simplex' method. First of all, algorithm given by him shall be described in brief and then applied to a hypothetical example in agriculture.

He has assumed that the denominator is always positive for all feasible solutions (i.e.,  $d'x + q \geq 0$ ). In his paper

he proves that the maximum of  $\phi(x)$  will be at the basic feasible solution. He then proceeds to enunciate his algorithm. The problem is

$$\begin{aligned} \text{Maximize } \phi(x) &= \frac{c'x + r}{d'x + q} \\ \text{subject to } Ax &\leq b \\ x &\geq 0. \end{aligned}$$

First of all, slack vectors are added in the usual fashion and  $A$  is enlarged to  $[A \quad I_m] = P$  with columns  $p_j$  ( $j = 1, 2, \dots, n+m$ ).

If  $X_B$  is the initial basic feasible solution such that

- (1)  $BX_B = b$ , where  $B = (b_1, \dots, b_m)$  is  $m \times m$  matrix as each  $b$  is  $m \times 1$  vector, then
- (2)  $X_B = B^{-1}b$  and  $X_B \geq 0$ .

Let  $c'_B$  be those components of vector  $c$  that are coefficients associated with  $X_B$  and let  $d'_B$  be the components of vector  $d'$  that are coefficients associated with  $X_B$ .

- (3) Let  $f = c'_B X_B + r$ .
- (4) Let  $g = d'_B X_B + q$ .
- (5) Let  $h = f/g$ .

It is also assumed that for this  $X_B$

- (6)  $u_j = B^{-1}p_j$ ,
- (7)  $f_j = c'_B u_j$ , and
- (8)  $g_j = d'_B u_j$

are known for every column  $p_j$  of  $P$  not in  $B$ .

Let the new basic feasible solution be called  $X_B^1$ . Also let us represent basis and other things associated with this new solution by a superscript of 1, except for B for which we shall use the subscript 1 rather than superscript for convenience. Thus,  $B_1$  (rather than  $B^1$ ) is associated with  $X_B^1$ . Then

$$(9) \quad X_B^1 = B_1^{-1} b$$

where  $B_1$  differs from B only in that the kth element of  $B_1$  is  $p_j$  and not  $b_j$ ; i.e., if

$$(10) \quad B = (b_1, \dots, b_k, \dots, b_m), \text{ then}$$

$$(11) \quad B_1 = (b_1, \dots, b_{k-1}, p_j, b_{k+1}, \dots, b_m).$$

Let  $x_{Bij}$  refer to that element of  $X_B$  which is in the ith row and the jth column of  $X_B$ , then

$$(12) \quad x_{Bij}^1 = x_{Bij} - x_{Bkj} (u_{ij}/u_{kj}) \text{ for } i \neq k, \text{ and}$$

$$(13) \quad x_{Bkj}^1 = x_{Bkj}/u_{kj} \text{ when } i = k.$$

Let us call  $x_{Bkj}^1$  as  $\theta$ .

$$(14) \quad p_j = \sum_i u_{ij} b_i.$$

A cursory inspection of (12) and (13) immediately brings out the fact that in the simplex method we find out the new basic feasible solution from the old basic feasible solution in exactly the same manner as above.

Let the value of the new objective function be  $h^1 = \frac{f^1}{g^1}$ .

$$(15) \quad f^1 = f + x_{Bkj}^1 (c_j - f_j) = f + \theta (c_j - f_j)$$

$$(16) \quad g^1 = g + \theta(d_j - g_j)$$

$X_B^1$  preferred to  $X_B$  if

$$(A) \quad h^1 > h; \text{ i.e.,}$$

$$(B) \quad \frac{f^1}{g^1} > \frac{f}{g}; \text{ i.e.,}$$

$$(C) \quad gf^1 > g^1f; \text{ i.e.,}$$

$$(D) \quad gf^1 - fg^1 > 0; \text{ i.e.,}$$

$$(E) \quad g[f + \theta(c_j - f_j)] - f[g + \theta(d_j - g_j)] > 0; \text{ i.e.,}$$

$$(F) \quad g(c_j - f_j) - f(d_j - g_j) > 0; \text{ i.e.,}$$

$$(G) \quad \delta_j > 0 \quad \text{where} \quad \delta_j \equiv g(c_j - f_j) - f(d_j - g_j).$$

It is then shown that if we start with a basic feasible solution and if there is a vector  $a_j$  (in  $A$ ) not in the basis and for which  $\delta_j > 0$ , then there exists another basic feasible solution such that  $h^1 > h$ . In the absence of degeneracy, the number of steps involved in moving from initial feasible basis to the optimal basis is finite as long as  $m$  and  $n$  are finite because no basis shall be repeated in these iterations and it is impossible for the same basis to have different values of the objective function.

How do we know that we have reached the optimum? The optimal solution to this maximization problem is obtained when for the columns of  $P$  not in the basis, the corresponding  $\delta_j \leq 0$ .

At the optimal point, for the columns of  $P$  in the basis,

$\delta_j = 0$  as at that point

$$(17) \quad f_j^* = c_B' u_j^* = c_B' B^{-1} p_j^* = c_B' B^{-1} b_i = c_j, \text{ and}$$

$$(18) \quad g_j^* = d_B' u_j^* = d_B' B^{-1} p_j^* = d_B' B^{-1} b_i = d_j, \text{ and,}$$

therefore,

$$(19) \quad g(c_j - f_j^*) - f(d_j - g_j^*) = 0.$$

### Applications of Linear Fractional Functionals Programming in Agriculture

The linear fractional functionals programming technique has applicability wherever the objective function is a ratio. For example, at the macro-level, one of the objectives of economic planning may be to maximize the rate of growth of income in agricultural sector, to increase per capita income of farm families in a country, etc. Both of these examples are ratio concepts.

Let

$Y_t$  = national income of an economy from  
agriculture at time  $t$ ,

$Y_{t+1}$  = national income of that economy (in  
agricultural sector) at time  $t+1$ .

The rate of growth of this income in terms of current income is then equal to

$$\frac{Y_{t+1} - Y_t}{Y_t}.$$

Clearly this is a ratio.

Likewise, if  $N_t$  is that population of the country that is dependent on agriculture for a living at time  $t$ , then

$$\frac{Y_t}{N_t}$$

is the per capita income of that economy in agriculture at time  $t$ .

There are many situations in agriculture where the objective is the optimization of a ratio. A farmer where there is a shortage of labor may be interested in maximizing returns per hour of man labor used on the farm. A beginner farmer or one who is financially in a tight situation may be interested in maximizing profits per dollar of investment. Several measures of farm profits such as farm labor income, family labor income, farm business income, net returns, etc., can be converted into a ratio by an appropriate denominator. In all these situations the techniques of linear fractional functionals programming can be used with advantage. Given below is a very simple example of the application of linear fractional functional programming in agriculture.

A farmer proposes to put 100 acres of land under cultivation this year. He has a choice of raising corn and wheat. Given below is the basic data for these two crops on his farm (based on his records from the previous years).

Name of the crop	Corn	Wheat
Yields per acre (bushels)	40	20
Total labor requirements (per acre)	2.5 hours	3.0 hours
(a) Labor for preparation of bed and sowing	.8 hours	.8 hours
(b) Cultivation	.9 hours	-
(c) Harvesting, etc.	.8 hours	2.2 hours
Expected returns per acre	\$40.00	\$30.00

We assume that all costs (except for labor requirement) are identical for both crops. We further assume that whether the farmer raises any crop or not, he has to devote four hours during the crop season to the maintenance and upkeep of the equipment such as tractor, combine, etc. The farmer has 300 hours of labor available for the crop season and he is interested in maximizing returns per hour of labor.

Let  $x_1$  be the acreage under corn,

$x_2$  be the acreage under wheat.

The problem is to find out as to how much area he should have under corn and how much under wheat. In other words, the decision variables are  $x_1$  and  $x_2$  and we have to determine their desired values.

As the return from one acre of corn is \$40.00 and that from wheat is \$30.00, the total return from cropping, i.e., the denominator of the  $\phi(x)$ , is  $40x_1 + 30x_2$ .

Since 2.5 hours are required for growing an acre of corn and three hours of labor are required for producing an acre of

wheat and four hours of labor are required for the maintenance of machinery irrespective of the crops, the total number of hours of labor required to be used up on the farm in the process of taking these crops is  $2.5x_1 + 3x_2 + 4$ . The farmer wants to maximize returns per hour of labor used, the denominator of the objective function  $\phi(x)$  is  $2.5x_1 + 3x_2 + 4$ , and the objective function is

$$\phi(x) = \frac{40x_1 + 30x_2}{2.5x_1 + 3x_2 + 4}.$$

The problem is to

$$\text{Maximize } \frac{40x_1 + 30x_2}{2.5x_1 + 3x_2 + 4}$$

$$\text{subject to } x_1 + x_2 \leq 100$$

$$2.5x_1 + 3x_2 + 4 \leq 300$$

$$\text{or } 2.5x_1 + 3x_2 \leq 296$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

By adding the slacks in the usual fashion, the problem is transformed as

$$\text{Maximize } \frac{40x_1 + 30x_2 + 0x_3 + 0x_4}{2.5x_1 + 3x_2 + 0x_3 + 0x_4 + 4}$$

$$\text{subject to } x_1 + x_2 + x_3 + 0x_4 = 100$$

$$2.5x_1 + 3x_2 + 0x_3 + x_4 = 296$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0$$

$$x_4 \geq 0$$

The initial tableau is set up on page 184.

The initial basic feasible solution has  $x_3$  and  $x_4$ . The coefficients of these, i.e.,  $x_3$  and  $x_4$ , in  $c$  and  $d$  vectors are zeros. Therefore,

$$c_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$d_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_B = \begin{bmatrix} 100 \\ 296 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2.5 & 3 & 0 & 1 \end{bmatrix}$$

$$B = [P_3, P_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let  $x_{Bi0}$  denote the element in the  $i$ th row and the 0th (zeroth) column for the basic vector. For this initial tableau,

$$f = 0x_3 + 0x_4 = 0$$

The initial tableau:

				$c_j$	40	30	0	0		
				$d_j$	2.5	3	0	0		
	$d'_B$	$c'_B$	Activity at non-zero level	Level ( $p_0$ )	$x_1$ ( $p_1$ )	$x_2$ ( $p_2$ )	$x_3$ ( $p_3$ )	$x_4$ ( $p_4$ )	$\frac{x_{B_i}}{u_{ij}}$	
←	0	0	$x_3$ (land)	100	1	1	1	0	100/1	$\theta=100$
	0	0	$x_4$ (labor)	296	2.5	3	0	1	296/2.5	
	$g=4$	$f=0$	and $h=\frac{0}{4}=0$							
			$c_j - f_j$		40	30	0	0		
			$d_j - g_j$		2.5	3	0	0		
			$\delta_j$		160	120	-	-		

$$g = 0x_3 + 0x_4 + 4 = 4, \text{ and}$$

$$h = \frac{f}{g} = \frac{0}{4} = 0$$

$$\delta_j = g(c_j - f_j) - f(d_j - g_j), \quad j = 0, 1, 2, \dots, n.$$

$$f_j = 0 \quad \text{for all } j$$

$$g_j = 0 \quad \text{for all } j$$

Therefore,

$$(c_1 - f_1) = (40 - 0) = 40$$

$$(c_2 - f_2) = (30 - 0) = 30$$

$$(d_1 - g_1) = (2.5 - 0) = 2.5$$

$$(d_2 - g_2) = (3.0 - 0) = 3.0$$

$$\delta_1 = 4(40 - 0) - 0(2.5 - 0) = 160$$

$$\delta_2 = 4(30 - 0) - 0(3.0 - 0) = 120$$

Similarly,

$$\delta_3 = 0$$

$$\delta_4 = 0$$

To go to the second tableau (from one basic feasible solution to another better basic feasible solution), we choose the largest value of  $\delta_j$  which  $160 = \delta_1$ . Therefore,  $j = 1$  and  $p_j = p_1$  is the activity that comes in the basis. Now the question as to which activity goes out,  $x_3$  or  $x_4$ .

$$\frac{x_{B10}}{u_{11}} = \frac{x_3}{u_{11}} = \frac{100}{1} = 100$$

$$\frac{x_{B20}}{u_{21}} = \frac{x_4}{u_{21}} = \frac{296}{2.5} = 118.4$$

$$\theta = \min \left[ \frac{x_{B10}}{u_{11}}, \frac{x_{B20}}{u_{21}} \right] = \min(100, 118.4) = 100 = \frac{x_{B10}}{u_{11}}.$$

As  $\theta = \frac{x_3}{u_{11}}$ , the outgoing activity is  $x_3$  which is in the first row of the initial tableau. Therefore,  $k = 1$ . So we know  $j = 1, k = 1$ .

Now the values in the next tableau can be calculated in the following manner. Let the superscript 1 refer to the items belonging to the new (next to the initial) tableau.

$$\text{vector } x_{Bkj}^1 = \frac{x_{Bkj}}{u_{kj}} = \frac{x_{B1j}}{u_{11}}$$

Now  $x_{B1}$ , i.e., the first row in the old basis (initial tableau) is  $x_3$ , i.e.,  $k = 1 \longrightarrow x_{Bkj}^1 = x_{B1j}^1$  for the new tableau.

$$x_{10}^1 = \frac{x_{30}}{u_{11}} = \frac{100}{1} = 100$$

$$x_{11}^1 = \frac{x_{31}}{u_{11}} = \frac{1}{1} = 1$$

$$x_{12}^1 = \frac{x_{32}}{u_{11}} = \frac{1}{1} = 1$$

$$x_{13}^1 = \frac{x_{33}}{u_{11}} = \frac{1}{1} = 1$$

$$x_{14}^1 = \frac{x_{34}}{u_{11}} = \frac{0}{1} = 0$$

These are the values for the 1st row in the new tableau, i.e., for  $i = k = 1$ .

For  $i \neq k$ , we have only one more row, i.e.,  $i = 2$ . We know that for  $i \neq k$ ,  $x_{Bij}^1 = x_{Bij} - x_{Bkj} \left( \frac{u_{ij}}{u_{kj}} \right)$ . In the second row of the initial tableau we have  $x_4$ . So  $x_{B2j}^1 = x_{4j}^1$  as the second row in the new tableau belongs to  $x_4$  and, therefore,  $x_4$  is in the new basis also.

Therefore,

$$x_{4j}^1 = x_{4j} - x_{1j}^1 \left( \frac{u_{21}}{u_{11}} \right)$$

$$x_{40}^1 = x_{40} - x_{10}^1 \left( \frac{u_{21}}{u_{11}} \right) = 296 - 100(2.5) = 46$$

$$x_{41}^1 = 2.5 - 1(2.5) = 0 = u_{21}^1$$

$$x_{42}^1 = u_{22}^1 = 3 - 1(2.5) = .5$$

$$x_{43}^1 = u_{23}^1 = 0 - 1(2.5) = -2.5, \text{ and}$$

$$x_{44}^1 = u_{24}^1 = 1 - 0(2.5) = 1.$$

So the next tableau is as shown on page 188.

$$f^1 = (40)(100) + (30)(0) + (0)(0) + (0)(46) = 4000$$

$$g^1 = (2.5)(100) + (3)(0) + (0)(0) + (0)(46) + 4 = 254$$

$$h^1 = \frac{4000}{254}$$

$c_j^1 - f_j^1$  and  $d_j^1 - g_j^1$  have been calculated in the same manner as the  $u_{ij}^1$ .

The final tableau:

				$c_j$	40	30	0	0
				$d_j$	2.5	3	0	0
	$d'_B$	$c'_B$	Activity at non-zero level	Level ( $p_0$ )	$x_1$ ( $p_1$ )	$x_2$ ( $p_2$ )	$x_3$ ( $p_3$ )	$x_4$ ( $p_4$ )
→	2.5	40	$x_1$ (corn)	100	1	1	1	0
	0	0	$x_4$ (labor)	46	0	.5	-2.5	1
g=254    f=4000    and $h=\frac{4000}{254}$								
			$c_j - f_j$		0	-10	-40	0
			$d_j - g_j$		0	.5	-2.5	0
			$\delta_j$		-	-4540	-160	-

$$\begin{aligned}
 c_1^1 - f_1^1 &= 40 - (40)(1) = 0 - \\
 c_2^1 - f_2^1 &= 30 - (40)(1) = -10 \\
 c_3^1 - f_3^1 &= 0 - (40)(1) = -40 \\
 c_4^1 - f_4^1 &= 0 - (40)(0) = 0
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 d_1^1 - g_1^1 &= (2.5) - (2.5)(1) = 0 \\
 d_2^1 - g_2^1 &= 3 - (2.5)(1) = .5 \\
 d_3^1 - g_3^1 &= 0 - (2.5)(1) = -2.5 \\
 d_4^1 - g_4^1 &= 0 - (2.5)(0) = 0
 \end{aligned}$$

and

$$\delta_2^1 = (254)(-10) - (4000)(.5) = -2540 - 2000 = -4540$$

$$\delta_3^1 = (254)(-40) - (4000)(-2.5) = (-10160 + 10000) = -160$$

$\delta_1^1$  and  $\delta_4^1$  need not be calculated as we calculate  $\delta_j$  only for those  $j$ 's that are not in the basis.

$\delta_2^1 < 0$ ,  $\delta_3^1 < 0 \longrightarrow$  we have reached an optimal solution.

The optimal solution as given by the last tableau is:

$$x_1 = 100$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_4 = 46$$

This means the farmer should raise corn on all the 100 acres of land. He should not grow any wheat.

$x_3$  is land disposal activity and, as all the land has been suggested to be taken under corn,  $x_3$  is zero in the final solution.

$x_4$  is labor disposal. As one acre of corn requires 2.5 hours of labor, 100 acres will need 250 hours of labor for growing corn. In addition, the farmer would spend four hours for maintenance. Thus the total labor requirement for this program comes to 256 hours. In all, 300 hours of labor are available and, therefore, unused labor ( $x_4$ ) is to the tune of 46 hours in the final tableau. The value of the objective function is  $h = \frac{4000}{254}$ .

Complete solution is obtained by putting the two tableaus together as given on page 191.

One of the salient features of this algorithm is its close resemblance to the simplex algorithm. Calculation of  $\theta$ , determination of the outgoing row and the incoming column are all similar to those in the simplex technique. In addition to the slack variables, which were used in the above example, the artificial variables can be taken care of in this in the manner similar to the one in simplex method. If the problem is not degenerate and  $m$  and  $n$  are finite, the solution to the problem can be obtained in a finite number of steps.

An important drawback of this algorithm is that it can not be used in case the values of  $r$  and  $q$ , the scalars in the objective function, are zero at the same time. We start with an initial basic feasible solution where  $c'_B$  and  $d'_B$  are zero vectors; and if  $r = q = 0$ , then  $f = 0$ ,  $g = 0$  and the values of  $\delta_j = g(c_j - f_j) - f(d_j - g_j) = 0$  irrespective of the values of

Combined tableaus:

		$c_j$	40	30	0	0				
		$d_j$	2.5	3	0	0				
		$d'_B$	$c'_B$	Activity at non-zero level	Level ( $p_0$ )	$x_1$ ( $p_1$ )	$x_2$ ( $p_2$ )	$x_3$ ( $p_3$ )	$x_4$ ( $p_4$ )	$\frac{x_{B_i}}{u_{ij}}$
Initial tableau	←	0	0	$x_3$ (land)	100	1	1	1	0	100/1 $\theta=100$
		0	0	$x_4$ (labor)	296	2.5	3	0	1	296/2.5
		$g=4$	$f=0$	and $h=\frac{0}{4}=0$						
				$c_j - f_j$		40	30	0	0	
			$d_j - g_j$		2.5	3	0	0		
			$\delta_j$		160	120	-	-		
Final tableau	→	2.5	40	$x_1$ (corn)	100	1	1	1	0	
		0	0	$x_4$ (labor)	46	0	.5	-2.5	1	
		$g=254$	$f=4000$	and $h=\frac{4000}{254}$						
				$c_j - f_j$		0	-10	-40	0	
			$d_j - g_j$		0	.5	-2.5	0		
			$\delta_j$		-	-4540	-160	-		

$(c_j - f_j)$  and  $(d_j - g_j)$ . This would imply optimality of the solution in the very initial tableau which suggests that the optimal solution lies in having all the real activities at the zero level (i.e., producing nothing) and this may be far from truth. However, this drawback may not be so serious (or even relevant) for agriculture as there are some costs involved in most cases even when the farmer does not produce anything and, therefore,  $q$  is positive. As we noted in the above example, whether the farmer grew any crop or not, he would have some labor for upkeep of machines. Similarly, if the farming is carried out with the help of draft animals, whether the animals work or not, farmer has to spend some time to look after them. The same holds good, for example, when the farmer is interested in maximizing returns per dollar of capital (fixed and variable) invested in the farming business. Again, the machines like tractors, combine, cotton picker and cob-sheller will depreciate at least a little with the passage of time even when they lie completely idle for the whole year. Similarly, an Indian farmer will have to spend money to feed his bullocks even when they are not in use. These are in the nature of fixed costs and would, therefore, not allow  $q$  to go to zero;  $q$  would remain strictly positive.

Though the example given above is rather simple, it serves to bring out the utility of the linear fractional functionals programming. The size of  $A$  matrix was  $2 \times 2$  in

our example. When  $m$  and  $n$  are quite large and there are a number of decision variables involved, it would be much easier to find an answer to the problem by this method. The linear fractional functionals programming technique holds great promise in its application to agriculture when the objective is to maximize returns per hour of family labor or hired labor or both, or the farmer is interested in optimizing things like returns per dollar of capital invested.

## APPLICATIONS OF NETWORK ANALYSIS IN AGRICULTURE

The last few years have witnessed a growth and improvement of the old and emergence of the new management planning and control tools. Many of these tools are rather complex and demand a high degree of sophistication in mathematics on the part of the technician. However, during the period 1956-58, powerful yet relatively simple techniques have been developed and are being widely used by the industry, government, businessmen, etc. for planning, scheduling and also controlling their projects. Grouped under the head of 'network' or 'flow' plans, these were independently and concurrently developed by the military and the industry and include PERT (Program Evaluation and Review Technique), CPM (Critical Path Method), CPS (Critical Path Scheduling), etc. The 'network' is composed of a series of related events and activities and their relationship is sequential. PERT was developed by the U.S. Navy Special Projects Office in connection with the Fleet Ballistic Missile Program. CPM and CPS were developed under the auspices of the industry in the United States.

Before proceeding any further, it would perhaps be helpful to define some terms that would be frequently encountered in connection with the 'network analysis'.

Activity: It is a time-consuming element and occurs between two events. In other words, an activity represents a 'thing' that is required to be done in order to go from one event to another. The activities are represented in a network diagram by an arrow.

Events: They are meaningful accomplishments, goals or milestones. As two events are connected by an activity, they are at the beginning as well as at the end of it. In a network diagram they are denoted by a node or a box.

Positive Slack: It is the amount of additional time available to the firm to perform the series of activities in a given slack path and still allows the activity to be completed within the required time.

Negative Slack: It is the amount of time which is not available to the firm to perform the series of activities in a given slack path and still allow the activity to be completed within the required time.

Critical Path: Critical Path, in a network, is a path having the largest amount of negative or the smallest amount of positive slack.

Fragnet: It is a part of a project network.

Interface: It is the relationship between activities and events that constrain their completion between two or more fragnets.

A network consists of a series of related activities and events. It is a flow plan, a graphic or a pictorial representation of the activities and the events which lead to the attainment of the ultimate goal and also depicts a plan with activities and events arranged in order of precedence. The plan also includes the time scheduling, i.e., time taken by one activity in going from one event to another. Thus 'network' helps the decision maker in defining the critical path and then, in making decisions consistent with his resources and requirements. It not only enables the management to plan ahead, but also to take stock of the situation at every stage of planning and production and may even alert him in time to obviate future sources of trouble. With proper modifications, the technique can be used for the purposes of determining the least cost schedules. Thus with the use of this rather simple tool, given the objective, the management may be in a position to plan the best possible use of resources in a way that the milestone is reached within the appointed time and given costs. See Figure 17.

#### A Brief History

CPM technique was a result of the efforts made by the IEC (Integrated Engineering Control) group of E. I. Du Pont de Nemours and Company. The company faced a problem of increasing time and costs involved in bringing new products from the

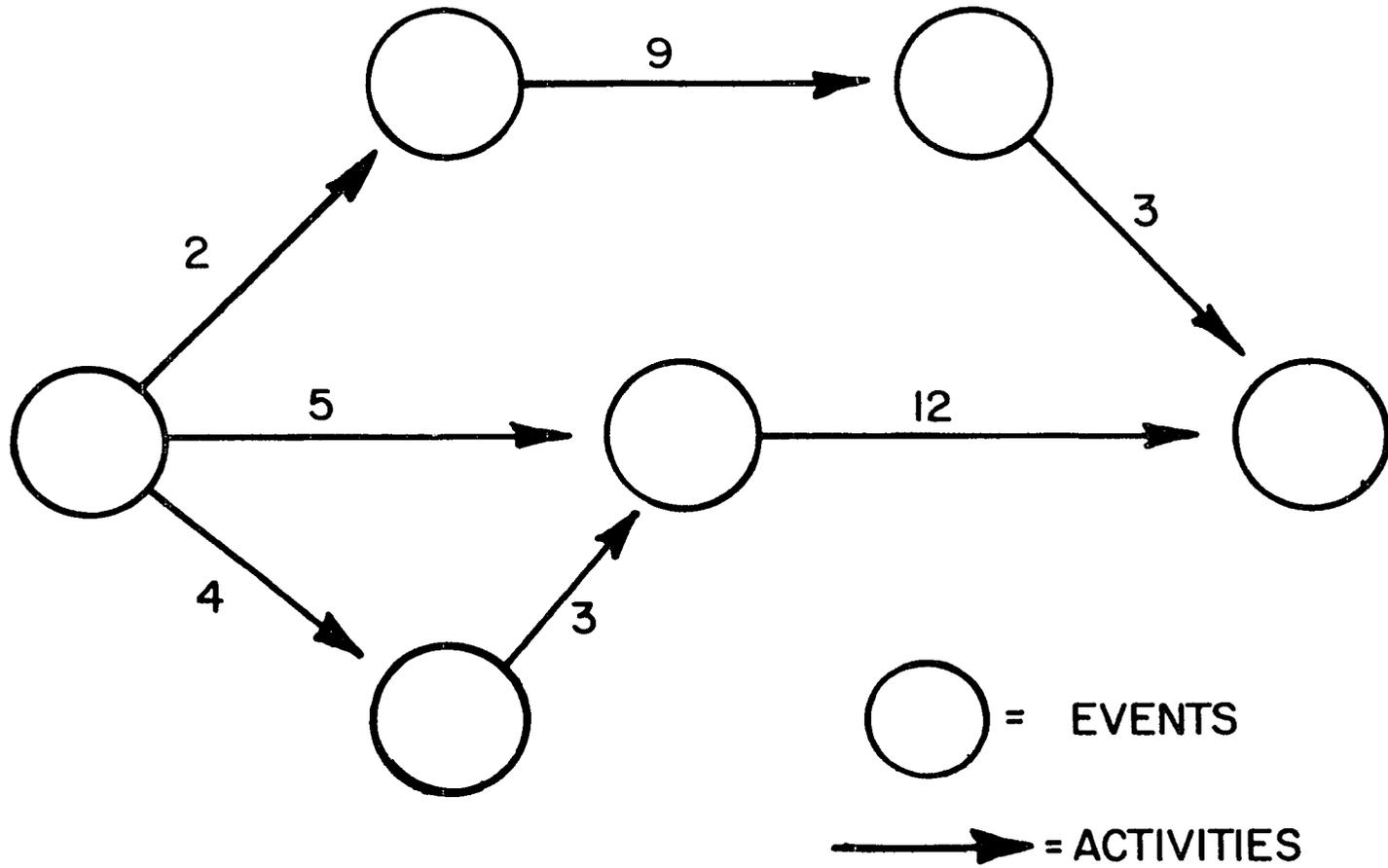


Figure 17. A network diagram

research laboratory to the production line. The management realized that, by proper planning and scheduling, a level of co-ordination could be obtained and a shift from research cell to the production department of the company would be less time and money consuming. To find out improved methods for planning and scheduling, IEC decided on exploring the scope of using electronic computers. As a consequence of the joint efforts of Mr. Morgan R. Walker of Du Pont and Mr. James E. Kelly, Jr., of Univac Applications Research Center, CPM and MCX (Minimum-cost Expediting) techniques came into existence. The tests of these methods conducted by Du Pont in collaboration with Remington Rand Corporation proved successful in 1957.

At about the same time, PERT was developed as a result of a research project of the U.S. Navy relating to the Polaris Fleet Ballistic Missile program. The objective of the Project PERT (Program Evaluation Research Task and later rechristened as Program Evaluation and Review Technique) was to develop, test and implement a methodology for providing management with integrated and quantitative evaluation of:

- (a) progress to date and the outlook for accomplishing the objectives of the Fleet Ballistic Missile Program,
- (b) validity of established plans and schedules for accomplishing the program objectives, and
- (c) effects of changes proposed in established plans.

The essential features of PERT were developed by July 1958

and its application to the Fleet Ballistic Missile program is reported to have resulted in the completion of the program two years ahead of schedule.

The applications of these techniques has been extended to several areas other than time. To mention just a few, these are areas of costs, manpower and capital requirements. CPM is widely used in the construction industry. A relatively new technique called 'inter-project scheduling' will make it possible to allocate resources and also assign priorities for a number of fragnets of the principal project with regard to the variables like costs, time, labor, machinery and equipment. There are still other techniques like MCX (Minimum-cost Expediting) and RPSM (Resource Planning and Scheduling Method) which are either a part or an extension of CPM and PERT techniques. Of these, PERT is perhaps the most widely used. It is proposed to briefly describe PERT technique and then compare it with CPM. Finally, the applicability of the 'network' analysis in agriculture shall be examined.

#### PERT System

Though the PERT technique was originally conceived to cope with the problems related to time (i.e., planning and scheduling of the project), its applications in the areas of costs and performance are also proving useful. The team of Malcolm, Roseboom, Clark and Fazar (65) was largely

responsible for the development of this technique. The group laid down the following requirements for the evaluation purposes. These may be considered, in a way, to be the requirements for PERT.

1. To obtain a reliable estimate of time needed for each activity in the project. This estimate was to be provided by a competent person who had a thorough knowledge of the nature and type of work involved. For best results, the probability distribution of the time estimated to be required by each activity was to be found out.
2. To have an exact idea of the sequence of activities and events. There will be some activities that can go on concurrently, whereas, in other cases, it would be necessary to complete an activity before the other could be undertaken.

After the required information has been obtained, it is translated into a graph in the form of a 'network' of a sequence of activities and events to be achieved in a certain order. This is called a 'flow plan' and its format is shown in Figure 18.  $E_1, E_2, \dots, E_7$  are the events and  $A_1, A_2, \dots, A_8$  are the activities joining them in a specified manner. Circles represent the events and the arrows indicate the activities. Initially, a flow plan is made up of activities and events, but it may also show the time required to complete

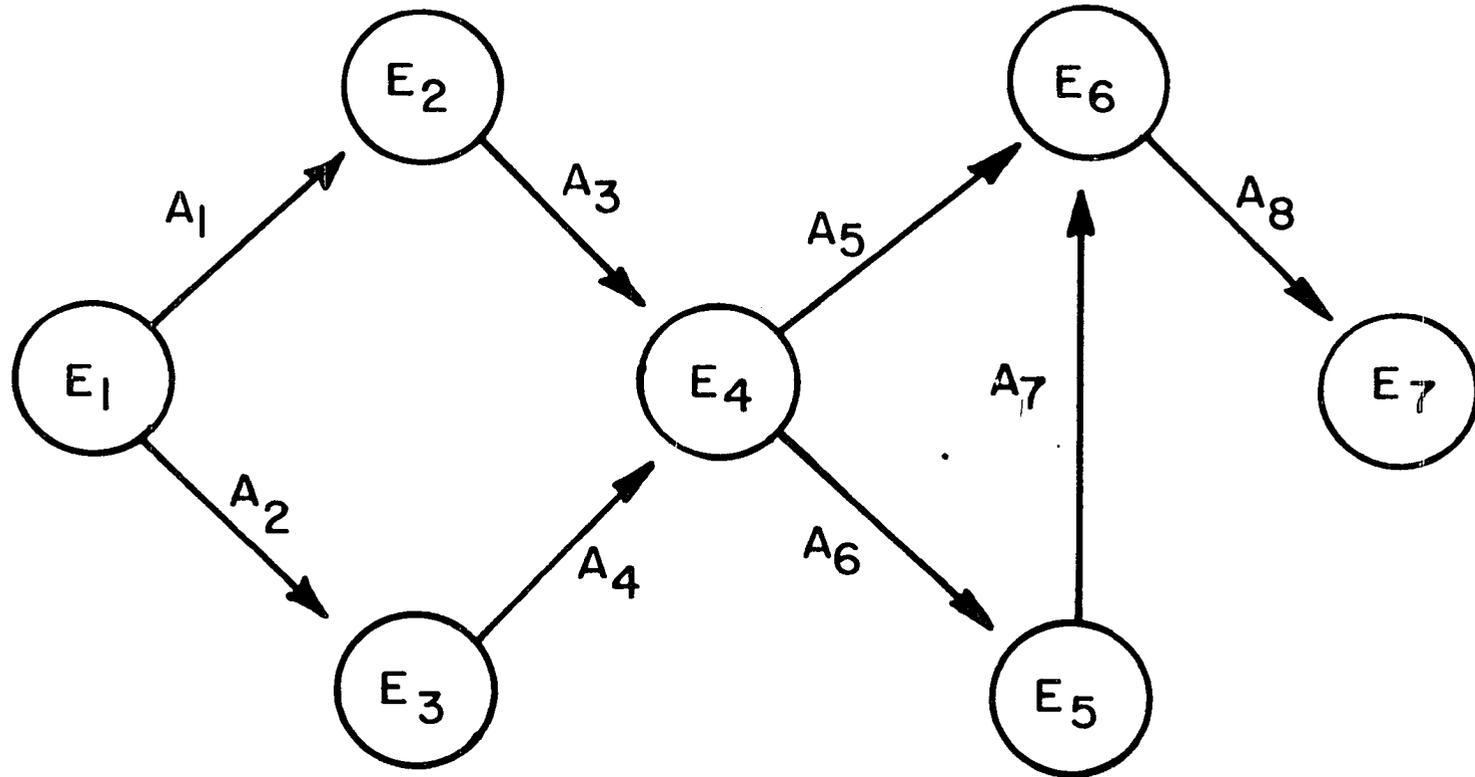


Figure 18. A flow plan

an activity, i.e., to go from one event to the other.

Suppose a new machine is to be designed and put in the market and the company wants to know as to how much time will it take? The problem may be tackled by making a diagram in the manner of Figure 18.  $E_1$  may be taken to be the starting point and  $E_3$ , for example, may be the procurement of raw material like steel.  $E_4$  may represent the testing of the model of the machine and  $E_7$ , being last, is the milestone of the project, say the new machine reaching the market. Similarly,  $A_2$  may represent the activity of procuring the raw material. These activities must be performed in the order shown in the network. For example,  $A_3$  has to succeed  $A_1$ , i.e.,  $A_3$  cannot be taken up until  $A_1$  has been completed. Similarly,  $A_4$  cannot precede  $A_2$ . Of course,  $A_1$  and  $A_2$ , according to the diagram, can be carried out simultaneously as they are not directly related to each other. Looping in the network is not permitted, i.e., the circularity between the preceding and succeeding events is not allowed. In drawing this network, it is assumed that resource needs for each activity are known with certainty. The status of a development program at a given point of time is a function of resources, technical performance and the period of time. Given the resources and technical performance, PERT approach deals with the time variable only.

The question then arises that 'once the sequence of

activities and events is laid out, how is the expected elapsed time for each activity found out?' The problem was tackled by the PERT team in the ensuing manner. In order to formulate a probability distribution of expected time required to complete an activity, each engineer responsible for that activity was asked to provide his three estimates of elapsed time—'optimistic', 'pessimistic' and 'most likely'. This furnishing of three estimates by every engineer helped in taking into account the difficulties that may be encountered in carrying out an activity and variability within the activity, thereby allowing a more precise formulation of expectations.

Optimistic estimate was based on the assumption that there were no hold-ups in the project plan at any stage and that each activity was completed in the shortest possible time, i.e., everything went right the very first time.

Pessimistic estimate of elapsed time was taken under the assumption of maximal potential difficulties faced in the completion of an activity, i.e., when nothing seems to go right.

Most likely estimate was based on the assumption of everything going normally, i.e., in a manner that would be usual.

Given these estimates, the formulae for calculating the expected elapsed time and the uncertainty involved in these expectations were developed on the following assumptions:

- (a) The extremes, i.e., optimistic and pessimistic expectations are realized less often as compared to the 'most likely'.
- (b) The probability distribution will have its peak at the 'most likely' estimate.
- (c) The peak of the distribution can be anywhere between the optimistic and the pessimistic estimates.
- (d) This probability distribution is a Beta-distribution.

Given these assumptions, the following equations were developed to calculate the expected elapsed time and the uncertainty involved.

$$\begin{aligned}
 (1) \quad t_e &= \frac{1}{3} \left[ 2m + \frac{1}{2}(a + b) \right] \\
 &= \frac{a + 4m + b}{6}
 \end{aligned}$$

$$(2) \quad \sigma_{t_e}^2 = \left[ \frac{1}{6}(b - a) \right]^2$$

where

- $t_e$  = expected elapsed time,
- $\sigma_{t_e}^2$  = variance of the expected elapsed time—this variance is a measure of uncertainty involved in  $t_e$ ,
- $a$  = optimistic estimate,
- $b$  = pessimistic estimate, and
- $m$  = most likely estimate of elapsed time.

The distribution is given in Figure 19.

If we define mid range as  $\frac{a + b}{2}$ , and if the distance between the mid range and most likely estimate is  $d$ , then  $t_e$  will be at a distance of  $\frac{d}{3}$  from  $m$  and at a distance of  $\frac{2d}{3}$  from the mid range. The line  $k t_e$  divides the area under the curve into two equal parts.

If

$$a = 24 \text{ days,}$$

$$b = 42 \text{ days, and}$$

$$m = 30 \text{ days,}$$

then

$$t_e = \frac{1}{3} \left[ 2(30) + \frac{1}{2}(24 + 42) \right] = \frac{60 + 33}{3} = 31 \text{ days}$$

and

$$\sigma_{t_e}^2 = \left[ \frac{1}{6}(b - a) \right]^2 = \left[ \frac{42 - 24}{6} \right]^2 = 9.$$

Let  $t_{e_i}$  be the estimate of the elapsed time for the  $i$ th activity where  $i = 1, 2, \dots, 8$ . Suppose the following values of  $t_{e_i}$  are obtained.

$$t_{e_1} = 5 \text{ weeks}$$

$$t_{e_2} = 4 \text{ weeks}$$

$$t_{e_3} = 3 \text{ weeks}$$

$$t_{e_4} = 7 \text{ weeks}$$

$$t_{e_5} = 12 \text{ weeks}$$

$$t_{e_6} = 3 \text{ weeks}$$

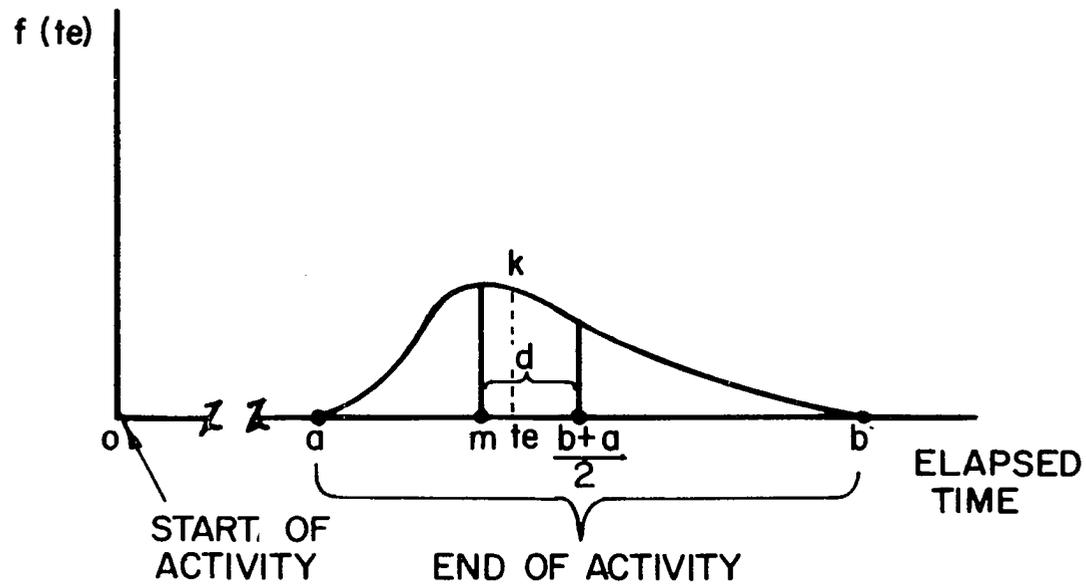


Figure 19. Distribution of elapsed time

$$t_{e_7} = 6 \text{ weeks, and}$$

$$t_{e_8} = 1 \text{ week}$$

On including these in the flow plan of Figure 18, we get the diagram shown in Figure 20.

Once these estimates of time have been calculated, the data are organized for convenience in analysis. A table is prepared with the events being listed in some sequence. Then the analysis is done to determine the 'critical path', the path taking the longest time to go from the start to the finish. If the sequence is small and simple, this can be done manually. Through experience it has been found that plans involving up to a hundred activities can be handled in this fashion without much trouble. If the sequence is too big and complex to be handled manually, the analysis of the data and determination of the critical path is done with help of the computers. For the network represented on page 197 (Figure 17), the data can be organized and the network compressed as shown in Figure 21. The events are placed on the list called the 'list of sequenced events'. The ordering is backward, starting first with the milestone. The preparation of this list shall be demonstrated in the example of the application to agriculture.

The critical path is the one that takes the longest time. For the network on pages 197 and 208, it is very simple to determine the critical path which is shown in Figure 22 with

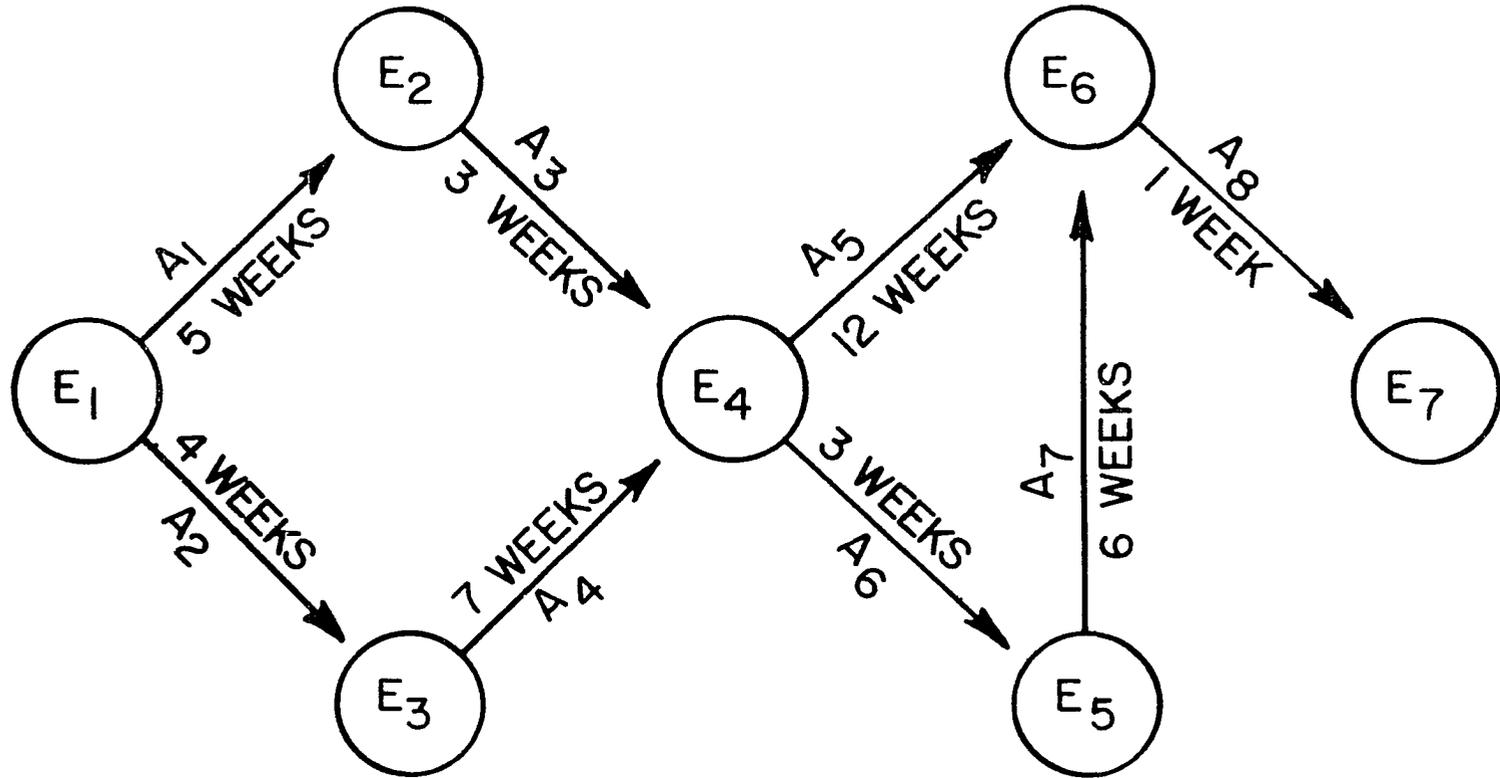


Figure 20. Flow plan with estimates of elapsed time

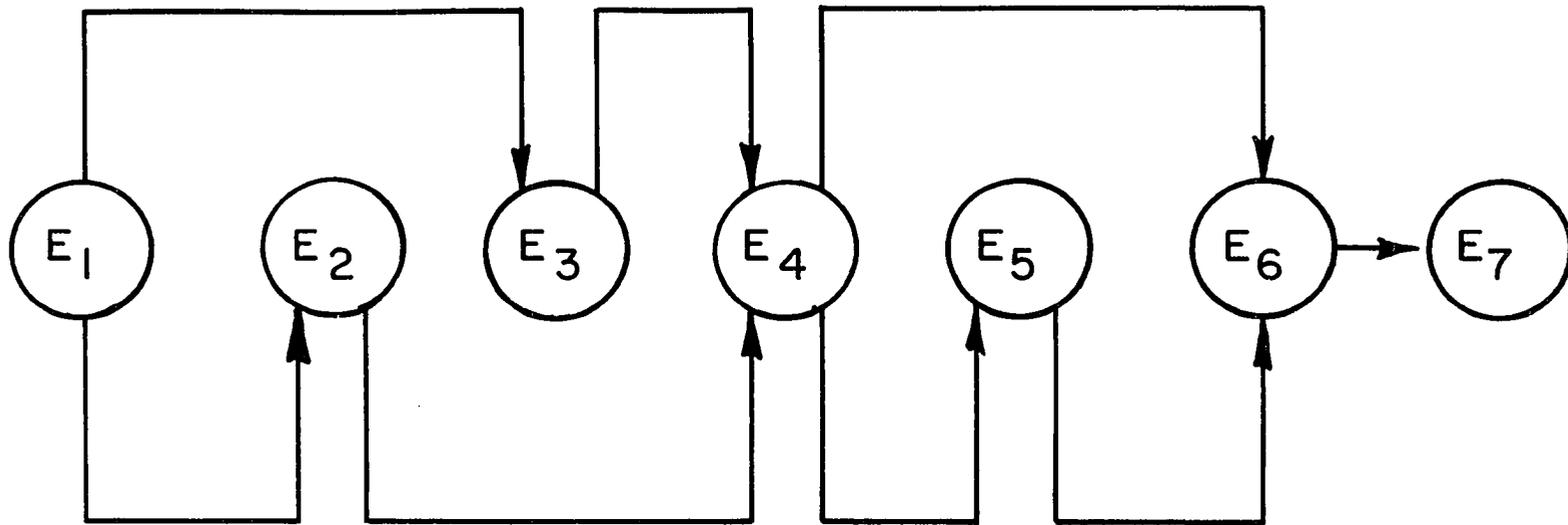


Figure 21. Diagram depicting sequenced events

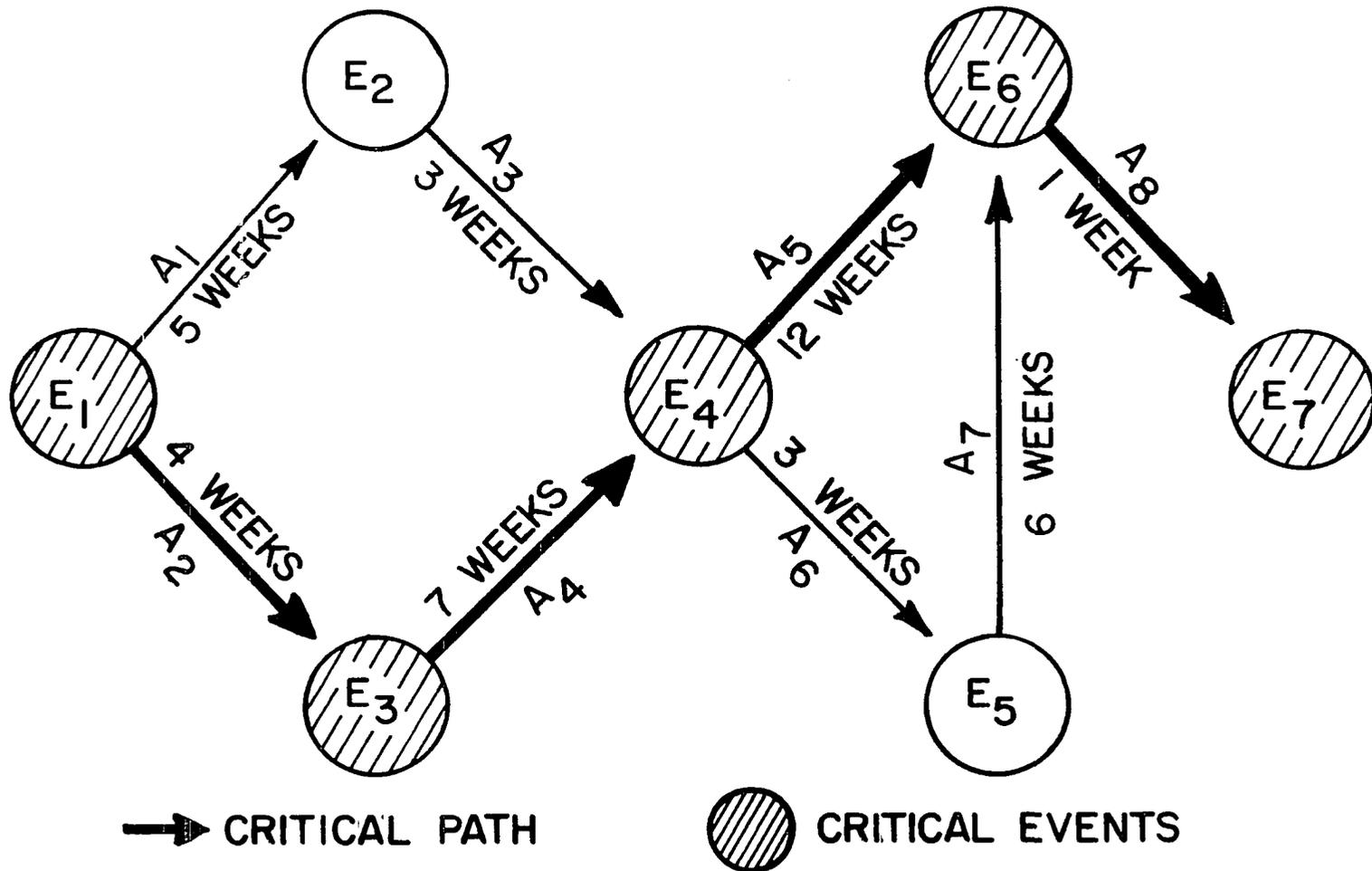


Figure 22. Critical path and critical events

heavy lines.

In our example, the duration of the critical path is 24 weeks. The events lying on the critical path are the critical events and are shaded in the diagram. Here they are  $E_1$ ,  $E_3$ ,  $E_4$ ,  $E_6$ , and  $E_7$ . If any of the critical events is delayed beyond its expected date of accomplishment, the delay in achieving the ultimate goal may be expected to be of the same magnitude. There may be events in the system which may be completed before they are needed. Slack is the difference between the time when they are actually needed and the time of their completion. Clearly, activities on the critical path have zero slacks. It is always useful to examine the path with the largest slack for bringing about improvement in the present plan. The uncertainty factor of the critical path is found by adding the variances for each activity along this path.

The probability of meeting an existent schedule is approximated with the normal distribution, utilizing the property of the 'Central Limit Theorem'.

Let

$T_{os}$  be the existing scheduled time,

$T_{oe}$  be  $= \sum_i t_{ei}$  where  $t_{ei}$  relates to the  $i$ th activity,

and

$$\sigma_{T_{oe}}^2 = \sum_i (\sigma_{t_{ei}}^2).$$

Then the probability of meeting an existent schedule is equal to  $\frac{\text{the shaded area in Figure 23}}{\text{total area under the curve}}$ . As we assume this distribution to be normal, we

(a) calculate  $\frac{T_{os} - T_{oe}}{\sigma_{t_e}}$ , and then

(b) use the table of the area under the normal curve to find out the probability.

It has been mentioned elsewhere that military and the industry simultaneously developed similar techniques. The one developed by the former is called the PERT system and the one developed by the latter is referred to as the CPM (critical path method). Both these methods are similar to each other. Like PERT, CPM is also a network analysis in which the network depicts the relationship of the activities in the system. Lately, extensions of these two techniques have gone to the extent of making them look alike. For example, PERT is being applied to the problems of cost and manpower. However, there seem to be some basic differences between the two as given below:

- (1) CPM is activity-oriented, whereas PERT is event-oriented.
- (2) In CPM, only one time estimate is used, in PERT we use three.
- (3) CPM and its extensions like MCX and RPSM were concerned with resources-manpower, equipment,

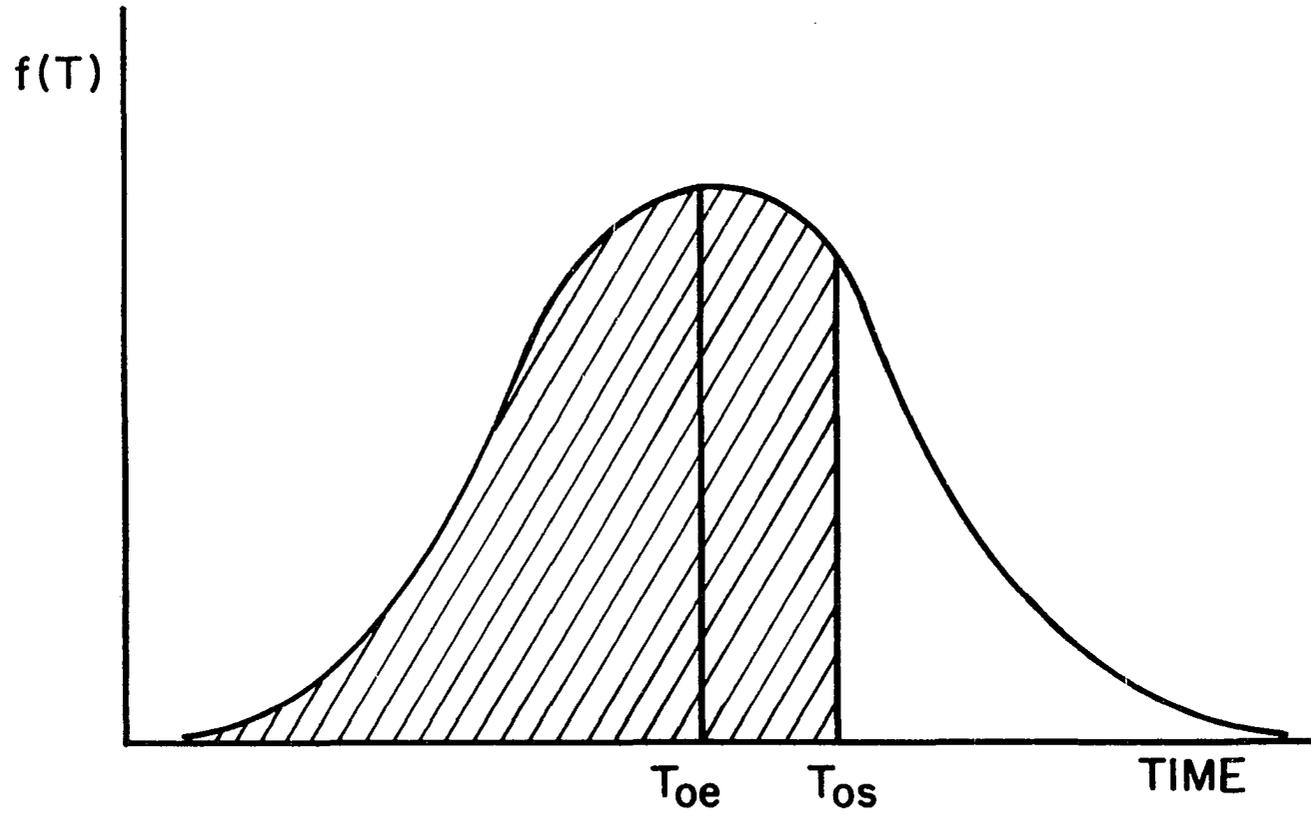


Figure 23. Approximate probability of meeting an existent schedule

working space, etc., whereas PERT took only the time variable into account.

Because of these differences, researchers and administrators have found CPM to be more satisfactory for controlling projects (due to its being activity oriented) and PERT has proved more useful for the purposes of project evaluation due to its being event oriented. Resource Planning and Scheduling Method (RPSM) is a special extension of CPM through which resource limitations can be considered and availability and limitations of capital, working space, material, etc., may be specified. MCX (Minimum-Cost Expediting), as the name suggests, is a technique that helps management in choosing the least costs operations.

#### Applications in Agriculture

Though quite young, 'network analysis' has been extensively used in industries for planning, scheduling and evaluation of the projects. Very few applications of this analysis have been made in agriculture. Perhaps the most detailed application of this has been done by Heiland, Jaendl and Kastner (40) who applied this to the problem of labor. In 1964 Morris and Nygaard (73) used modified critical path algorithm to find out the solution to the problem of selecting a machine for corn production and also in making the comparisons in the costs of different systems of hog production. In

his paper, "PERT: a prospective aid to better management", Schroeder (89) applied the technique to a rural community development program.

An illustration .

A farmer in India decides to follow a one-year rotation of green manure—wheat. Leguminous crops like sunnhemp are grown in the 'KHARIF' season and ploughed in the soil at their appropriate stage of growth to serve as a green manure for the ensuing wheat crop. Usually, this has to be done from the middle of August to the first week of September. At this time the green manure plants are succulent enough to quickly release nutrients in the soil and also there is enough time for soil to absorb these nutrients. The farmer can plough in the green manure either with the help of a tractor, which he can get on rent from the co-operative society of which he is a member, or by using bullocks. Ploughing in by the tractor takes two days, whereas ploughing in by the bullocks is expected to take six days. However, due to the heavy demand for the tractor, it may not be available immediately at the required time but is expected to take eight days to be available after the demand for the same has been put before the co-operative society. He cannot prepare his land properly till he has ploughed the green manure in. Further, sowing can be done only at least after six weeks of ploughing in the green manure and at the latest by the 23rd of October. He can

either sow with the bullock-drawn seed drill or in rows behind the plough, but again he does not own the seed drill but has to obtain it from the co-operative, which may take up to three days. Suppose his starting point is the 21st of August when his green manure crop is standing in the field and needs to be ploughed in the next 10-day period. It means that between now (August 21) and the completion of sowing (October 23) he has, in all, 62 days in which to complete all these activities. The events, activities and the expected time required by each of the activities are given in Figure 24. His activities are:

- A<sub>1</sub> = procuring the tractor
- A<sub>2</sub> = ploughing in the green manure with the help of bullocks
- A<sub>3</sub> = ploughing in the green manure crop by tractor
- A<sub>4</sub> = preparation of the field for sowing by bullocks
- A<sub>5</sub> = procuring the drill for sowing
- A<sub>6</sub> = sowing behind the plough
- A<sub>7</sub> = sowing by the seed drill

The respective events are:

- E<sub>1</sub> = start (August 21)
- E<sub>2</sub> = tractor procured
- E<sub>3</sub> = field green manured
- E<sub>4</sub> = field prepared and ready for sowing
- E<sub>5</sub> = drill procured
- E<sub>6</sub> = sowing completed (October 23)

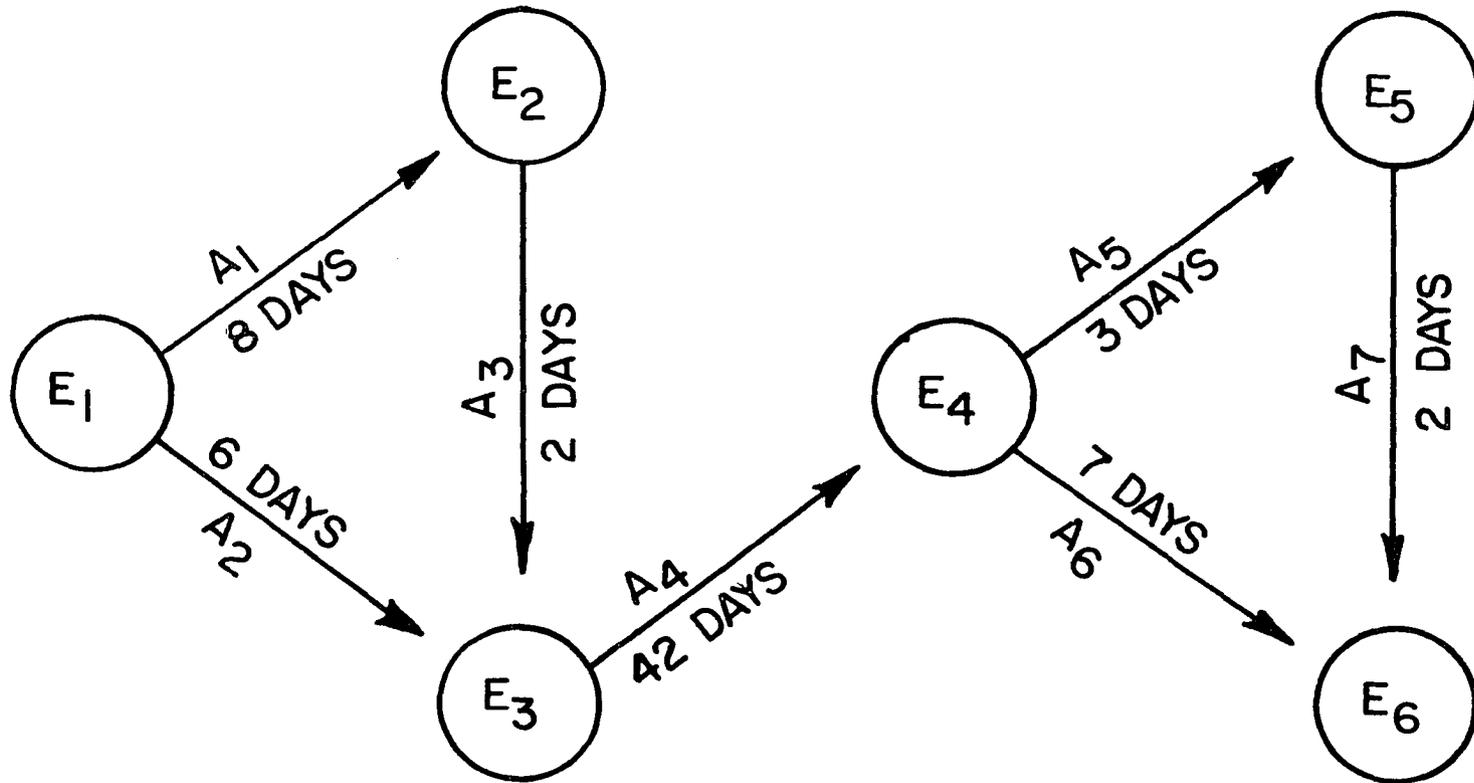


Figure 24. Flow plan of the farmer with expected time required for each activity

By organizing and compressing the network we get the diagram of the sequence of events as shown in Figure 25. The sequencing of these events is listed in Table 14.

Table 14. Sequencing of events

Event	Immediately preceding event		Immediately following event	
	Event No.	Mean $t_e$ (days)	Event No.	Mean $t_e$ (days)
6	5	2	-	-
	4	7		
5	4	3	6	2
4	3	42	5	3
			6	7
3	2	2	4	42
	1	6		
2	1	8	3	2
1	-	-	2	8
			3	6

From this list of sequenced events we compute the expected time for events by starting from the bottom (starting point) and examining the activities emanating from here. We then choose the longest time and prepare a table of 'outputs from

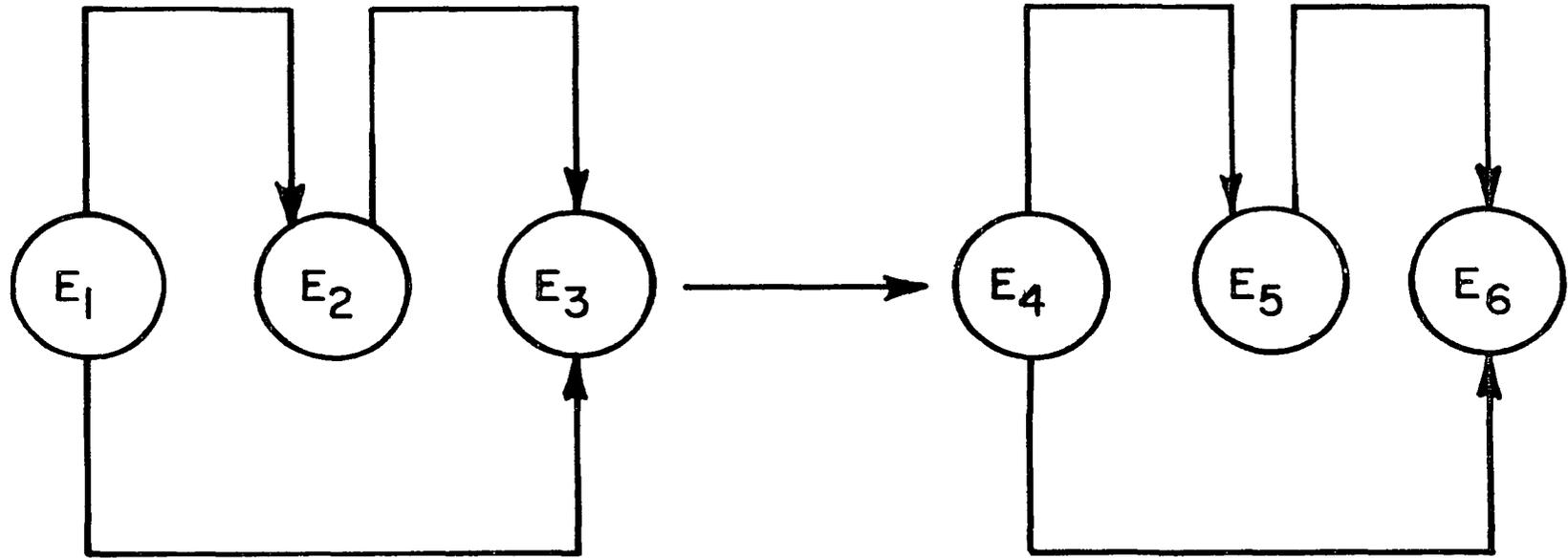


Figure 25. Sequenced events for Figure 24

Table 15. Outputs from analysis

Event	Expected time ( $T_e$ ) (days)	Latest time $T_{oL}$ (days)	Slack $T_{oL} - T_e$ (days)
$E_6$	59	62	3
$E_5$	55	60	5
$E_4$	52	55	3
$E_3$	10	10	0
$E_2$	8	8	0
$E_1$	-	-	-

analysis'. Table 15 is prepared in this manner. For example, from  $E_1$  sprout two activities,  $A_1$  and  $A_2$ . Only  $A_1$  goes to  $E_2$  with  $t_e = 8$  days. Therefore, we enter 8 against  $E_2$  in the 'outputs from analysis' table under column  $T_e$ . From  $E_1$  to  $E_3$  we can either go through a combination of  $A_1$  and  $A_2$  or by  $A_3$  alone. The former involves  $8 + 2 = 10$  days and the latter 6 days. Since we choose one with the longest expected time, this value is 10 and we write 10 against  $E_3$  under  $T_e$  and so on. As already mentioned earlier,

$$T_e = \sum_i t_{e_i}$$

Therefore, the number in the first row against  $E_6$  under  $T_e$  indicates the time of the critical path.

A look at the output from analysis table will show that the farmer could take three days more in preparing the field and would still be in time for sowing the crop. Further, the delivery of the drill could take five days longer without delaying the sowing. If the farmer is sowing behind the plough, he can afford to take another three days and still be able to finish sowing by the 23rd of October. Figure 26 shows the critical path for this example.

#### Discussion and Suggestions

This is just one example of how 'network analysis' can help the farmer in scheduling his operations. There are several areas in agriculture where it can be used with advantage. There is an acute shortage of labor in agriculture in many parts of the world. Even in regions where labor is in surplus, its demand far exceeds the supply during certain periods of the year, especially during the sowing and harvesting seasons. This is largely due to the nature of agriculture. Farming is an out-of-door occupation and greatly depends on weather conditions over which man has practically no control. The optimum time for given operations may be of a very short duration. This is especially true for sowing and harvesting. For the best germination to start with and proper growth later, the ideal sowing conditions for a crop may be just for seven to eight days and the farmer must complete the job during this

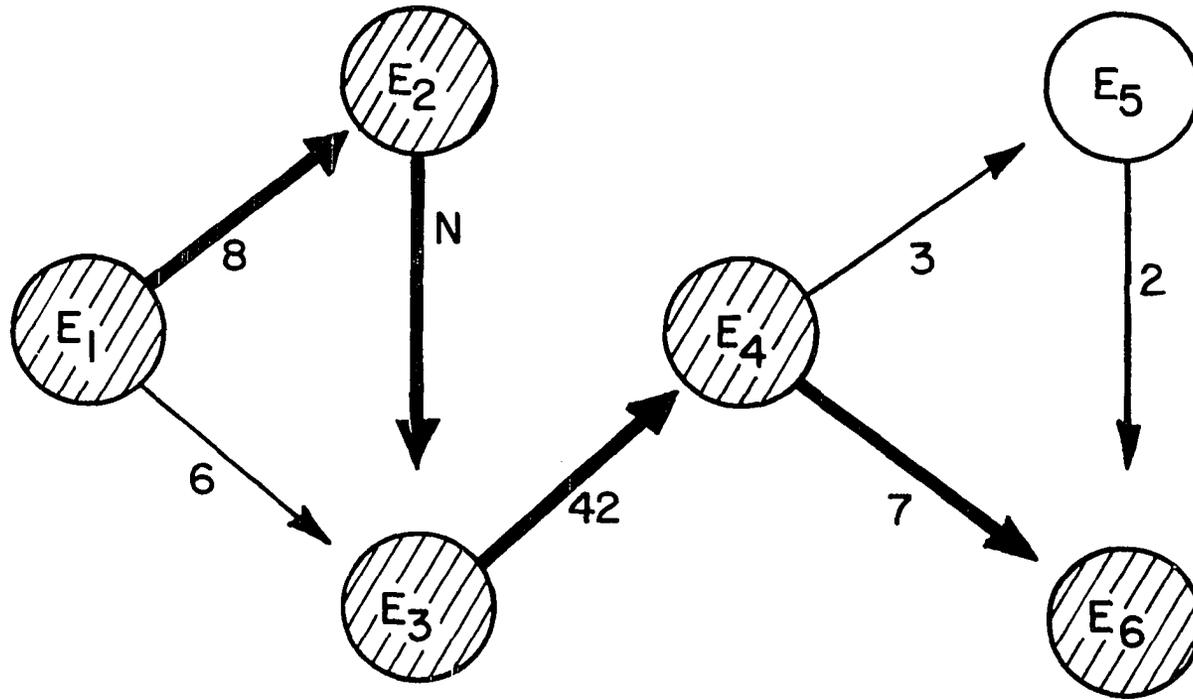


Figure 26. Critical path and critical events for the farmer

time of the existence of the optimum conditions. Thus these seven to eight days are critical. The same holds good for harvesting. If, for example, wheat is harvested too early, there may be excessive moisture in the grain resulting in poor quality product. If the harvesting is done late, the ears may wither with a loss in yield. Therefore, for a good yield, it is necessary that the harvesting be done within a certain time of the crop reaching the critical stage of maturity.

Due to the simultaneous need of labor on farms in the area, the demand for labor at these times is great. Planning and scheduling can help the farmer a great deal in this respect. In fact, crop raising is only one area. Feeding and care of the livestock can be planned in a similar fashion. There exists a great potential for the use of network analysis in the fields of processing and marketing of agricultural products. These industries operate on a very large scale with billions of dollars of investment and sales, and have problems of the same nature and dimensions as other industries. The problem of harvesting, freezing and canning of peas on the Seabrooks farm was solved largely by proper planning and scheduling of sowing. Time and motion studies in agriculture which have existed for a long time resemble these techniques. CPM has proved quite helpful in the construction industry and, therefore, can be used with advantage for construction of huge elevators for storage of agricultural commodities. In

places where farmers depend on cooperative organizations for the supply of fertilizers, and rent machinery and equipment from these for cultivation, a plan can be drawn up to allow the most efficient use of the machinery by larger number of members in a given period of time. There are regions where the farmer has to wait for days and even weeks for his turn to get water from canals or tube wells to irrigate his crops where this tool could be used to improve the situation. One of the assets of network analysis techniques, when not used on a very large scale, is their simplicity of being conceptualized. No special training is required on the part of the farmer for letting him involve himself in the use of this analysis. In fact, with his practical knowledge, if the number of activities is 50-60 or even 100, a farmer, with some hours of effort, can set up the network of schedule for his farm.

As mentioned elsewhere, this analysis is not limited to time only, but has been extended to the areas of resource allocation and cost minimization. If an equipment can be used for several activities, those activities cannot be scheduled to be performed at the same time. Tractor is a good example of this kind of equipment in agriculture. It can be used for ploughing, fertilizing, sowing, cultivating, irrigating and even transporting the produce. If a farmer has only one tractor, he cannot use it for ploughing the field and fetching

fertilizer from the market at the same time. There can be only one path connecting all those activities that are to be performed by the tractor. Again network analysis would be extremely helpful in drawing up the schedule providing for the most efficient use of the tractor.

Analogous to the time minimization is the problem of cost minimization. Replace 'time' with 'cost' and the tool called PERT/time technique becomes PERT/cost technique. A time network has the time listed on the arrows indicating the time taken by that activity. In a cost network, instead of time, the cost of that activity is indicated. However, it is not necessary that the least time consuming path is also least costly and vice versa. In most cases, the reverse is true. Some extensions of PERT analysis have been done in the direction of time-cost to determine an optimum mix of these two variables. Thus the analysis allows taking decisions with respect to both planning and control. However, these extensions of cost and time-cost have not been so successful as the original PERT/time analysis because the calculations are quite involved and iterations are so large that it becomes necessary to seek the refuge of the computer. Moreover, time-cost relationships are discrete and non-linear.

Perhaps the best use of this analysis at the macro-level could be made by the states and the nations for the purposes of economic planning. Time is important in national planning

both for the developed and the developing nations, but for the latter it is rather critical. Let us take the case of India. The efforts of the Indian government are directed towards a long range planning and all round development of the country at a reasonable cost and within a reasonable amount of time. However, its immediate goal is to increase production in the agricultural sector, especially food production, in the shortest possible time to combat the acute food shortage faced by the nation. The problem is being attacked from all directions. For example, emphasis is being given to the increased use of fertilizers. However, before a fertilizer can be increasingly used, its availability in required quantities is necessary which can be achieved through opening of new factories and increasing imports. Likewise, more area is being brought under improved seeds for which new seed multiplication farms are being started. Network analysis can help not only in the planning and scheduling of the projects needed for increasing agricultural production and assigning priorities to these projects, but also to take stock of the situation after the program has been in existence for sometime. As pointed out earlier, it can help in pinpointing weaknesses and difficulties in the program well in time so that they can be taken care of before it is too late.

Since 'human element' plays such an important role in planning, scheduling, estimating time, and evaluating the

project, PERT, CPM and other network techniques may suffer from weaknesses of this element. Some participants may be too conservative while others may be too liberal. Critical premises are based on personal estimates and, therefore, are susceptible to human error, whether the error is in judgment or outlook. As mentioned elsewhere, time-cost analysis has not been successful so far and the present system fails to correlate expenditure and progress. Some researchers have questioned the very assumption of the Beta distribution and, therefore, the formulae derived to calculate expected elapsed time and its variance. According to Hartley and Wortham (36), "Although some currently used PERT computations take account of variation in the completion times of individual operations, the methods used are approximate and are known to lead to (a) optimistic project completion times and (b) misidentification of 'critical paths'."

In spite of drawbacks, these techniques have been enjoying wide applications in industrial, governmental and military activities and have made significant contributions to the improved planning and control of the projects. Efforts are being made to modify these in a manner that would allow meaningful coordination of time, cost and function. The analysis is simple to comprehend and the operator is very much a part of the system. These virtues have led to its wide acceptance by scientists, administrators and technicians. It not only

presents the project as an entity in its entirety in a diagram laying bare the relationships of activities and events and identifies the critical path, but also enables the administrator to control the activities and evaluate the results of the action in a comparatively short time. Its simplicity makes it all the more amenable to be used by farmers and small industrial organizations.

## APPLICATIONS OF WAITING LINE PROBLEMS AND THE THEORY OF QUEUES

Everyone is familiar with the problem of having to wait. It may be for purchasing tickets to a baseball game, making payments at the counter for goods purchased in a supermarket, paying toll at the booths on the highways, getting machines serviced at a repair facility, depositing or withdrawing money from the bank in person, travelling on a road with signal lights or landing at a busy airport. While many times one has to wait at these facilities, at other times these facilities remain idle. The common features of the waiting problems are:

- a. Someone or something which requires the use of some facility or service. This someone or something is commonly referred to as the 'customer'.
- b. The service or the facility.
- c. The idleness of (or waiting by)
  - (i) the customer to be serviced.
  - (ii) the service or the facility.
- d. The actual act of providing the facility or service to the customer.

Irrespective of whether waiting is on the part of the customer or the service facility, there is always a cost associated with waiting, because waiting involves consumption of time. While waiting at the traffic lights or toll, the automobiles consume fuel. If there are long queues at the bank, the customer may get annoyed and perhaps switch to

another bank. However, if there are too many windows at the bank, most of the time they may remain idle and involve service cost. The problem is to strike a balance as to minimize the waiting costs both to the customer and the service agency.

The problems of this nature are attacked by what may be grouped under the broad term of 'Queuing Theory' or 'Waiting Line Models'. The origin of this theory can be traced to the pioneer work of Erlang (14) for the Copenhagen Telephone Company. The use of queuing has not been confined to the problems of operations of telephone systems. It has been widely used for solving problems such as determination of optimum number of clerks that should be placed in a company's factory tool cribs, flow of scheduled air and train traffic, traffic delays at toll booths, determining optimum worker-supervisor ratio, problems of loading and unloading, repairs and maintenance of equipment, etc. These problems are also termed as 'servicing problems' because the queuing is for getting the service or facility.

The queues may be single-channel or multi-channel. In the former only one customer can be serviced at one time. In the latter, more than one can be taken care of at the same time. In a simple servicing system we have the following:

- a. Arrival of the customer into the system.
- b. Waiting by the customer before getting service.

c. Actual servicing.

d. Departure of the customer after being serviced.

These have been illustrated in Figure 27.

First of all the customer arrives at the service station. The system cannot provide facilities or services to an arbitrarily large number of customers. This implies waiting if the service is being rendered to others. In some cases the departure may be immediately after the service has been rendered. In others, some time may have to elapse between actual service and departure (e.g., some stations would like to check over again after they have serviced). After it has entered the system, how long will it take for the customer to get service and then leave? The answer to the question depends on several factors. The important of these are:

A. Rate of arrival of customers into the system.

B. Rate of servicing and departure.

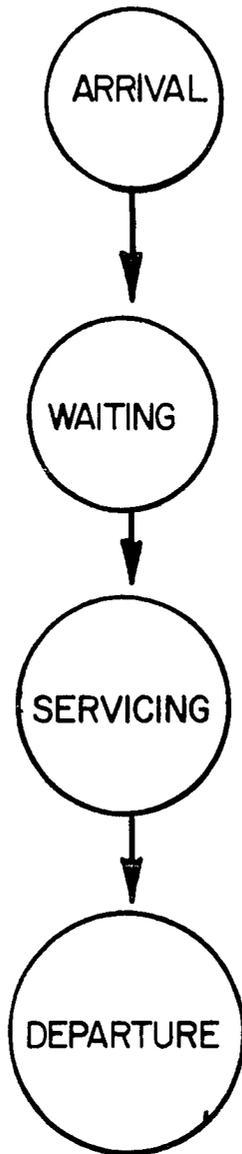
The latter depends on,

(i) Nature of service required.

(ii) Types of facilities available, e.g., whether they are modern or old, personnel are quick or slow, etc.

(iii) Number of facilities in the system.

A and B can be called as the elements of a queuing model. These determine the (expected) average number of customers in a queue waiting for service, average time required by them

SINGLE-CHANNEL  
SYSTEM

## MULTIPLE-CHANNEL SYSTEM

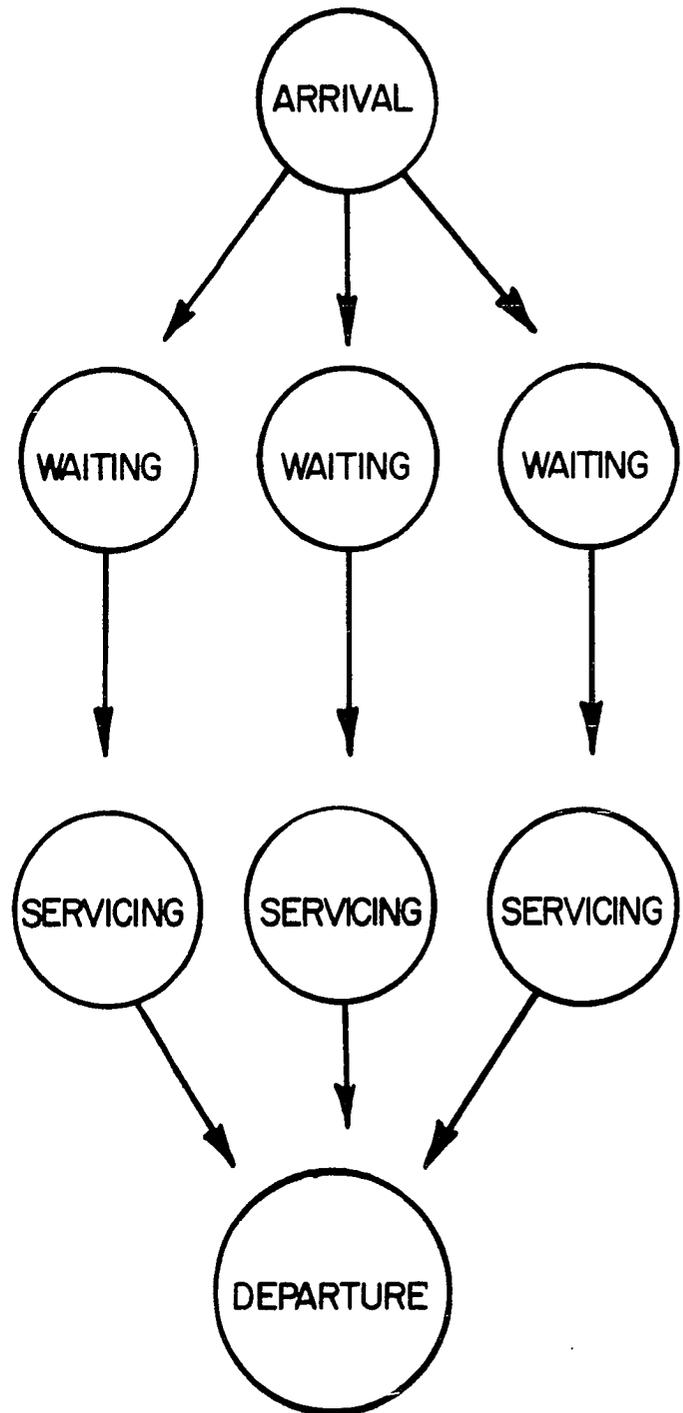


Figure 27. Components of a queuing system

before they actually start getting the service, the time taken in getting the service proper, and finally leaving the scene.

Let

$W$  be the waiting time—time interval between the arrival of the customer at the station and actually start getting the service,

$A$  be the arrival rate of customers, and

$S$  be the servicing rate,

then

$$W = f(A, S).$$

$A$  and  $S$ , in themselves, depend on:

1. probability distributions of arrivals,
2. number of servicing facilities and servicing policy (i.e., whether first come first served or served in order of magnitude of work or some other criterion),
3. queue discipline (for example, in a bank, a customer may switch from one line to another if the latter is shorter), and
4. probability distributions for servicing time.

Let

$S_i$  be the time taken by the service station, only to render the service to the  $i$ th customer (it does not include any waiting time on part of  $i$ ),

$t_i$  be the time when the  $i$ th customer arrives at the station, and

$W_i$  be the waiting time for the  $i$ th customer.

Let us assume that the variables  $S_i$ ,  $t_i$  and  $W_i$  are randomly and independently distributed with finite means (because if the expected values of servicing and waiting time are indefinite, the problem would be unrealistic and meaningless). Let us further assume that the first customer arrives exactly when the service station opens and, therefore, has not to wait. Then

$$W_1 = 0$$

$$W_2 = -(t_2 - t_1) + s_1.$$

$$W_3 = W_2 - (t_3 - t_2) + s_2$$

In general,

$$W_i \begin{cases} = W_{i-1} - (t_i - t_{i-1}) + S_{i-1} & \text{for } W_{i-1} - (t_i - t_{i-1}) + S_{i-1} > 0 \\ = 0 & \text{for } W_{i-1} - (t_i - t_{i-1}) + S_{i-1} \leq 0. \end{cases}$$

Three cases of this could be:

$$(i) \quad -(t_i - t_{i-1}) + S_{i-1} = 0 \text{ for all } i.$$

Then we have a balanced situation and neither the customer nor the servicing facility shall have to wait. This is because  $S_{i-1} = (t_i - t_{i-1})$  for all  $i$ . It means that the servicing time for the  $(i-1)$ st customer is equal to the difference in the times of arrivals of the  $i$ th and the  $(i-1)$ st customer.

Thus  $W_i = 0$  for all  $i$  and no customer has to wait.

$$(ii) \quad -(t_i - t_{i-1}) + S_{i-1} < 0 \text{ for all } i \text{ or } S_{i-1} < (t_i - t_{i-1}).$$

There is no waiting on the part of the customer. It is the service that remains idle.

$$(iii) \quad -(t_i - t_{i-1}) + S_{i-1} > 0 \text{ for all } i, \text{ i.e., } S_{i-1} > (t_i - t_{i-1}).$$

As servicing time of  $(i-1)$ st customer is greater than the difference in the time of arrivals, the  $i$ th customer has to invariably wait. This is just the reverse of (ii).

In practice these three cases do not exist in isolation (i.e., no single of them is true for all  $i$ ). We observe a combination of these. At times the facility is idle. At another time the customer waits, while in some cases a customer arrives just when his predecessor is leaving the station.

If we know the probability distribution

$$f \left[ -(t_i - t_{i-1}) + S_{i-1} \right],$$

the most likely value of  $(S_{i-1} - t_i + t_{i-1})$  can be found out. From this we can draw inferences about the behavior of the waiting lines.

Let  $F_i(z)$  be the probability that  $W_i \leq z$ , i.e.,

$$\begin{aligned}
P(W_i \leq z) &= P\left[\{W_{i-1} - (t_i - t_{i-1}) + S_{i-1}\} \leq z\right] \\
&= \int P\left[W_{i-1} \leq z + (t_i - t_{i-1}) - S_{i-1}\right] \cdot P\left[S_{i-1} - (t_i - t_{i-1})\right] \\
&= \int F_{i-1}\left[z + (t_i - t_{i-1}) - S_{i-1}\right] df\left[S_{i-1} - (t_i - t_{i-1})\right].
\end{aligned}$$

This is a monotonically increasing function.

The most common frequency distributions observed for arrivals and servicing time are:

- (a) gamma distribution—also called Erlang distribution,
- (b) Poisson distribution, and
- (c) negative exponential distribution.

(a) gamma distribution and its properties:

$$f(z; i, \beta) \begin{cases} = \frac{1}{\Gamma(i) \beta^i} z^{i-1} e^{-\frac{z}{\beta}} & \text{for } 0 \leq z < \infty \\ = 0 & \text{for } z < 0 \end{cases}$$

$$\Gamma(i) = (i-1)!$$

Therefore,

$$f(z) = \frac{1}{(i-1)! \beta^i} z^{i-1} e^{-\frac{z}{\beta}} \quad \text{for } 0 \leq z \leq \infty$$

Let  $\theta = \frac{1}{\beta}$ , then

$$f(z) = \frac{\theta^i}{(i-1)!} z^{i-1} e^{-\theta z} \quad \text{for } 0 \leq z \leq \infty$$

$F(z) = I_z(i)$  is called incomplete gamma function.

$$F(z) = \frac{1}{\Gamma(i)} \int_0^z z^{i-1} e^{-z} dz \quad \text{for } 0 < z < \infty$$

$$= \frac{\Gamma_z(i)}{\Gamma(i)} = \frac{\Gamma_z(i)}{(i-1)!}$$

$$\text{The mean of gamma distribution} = \frac{i}{\theta}$$

$$\text{Variance of gamma distribution} = \frac{i}{\theta^2}$$

When  $i \rightarrow \infty$ , gamma distribution tends to a constant servicing time distribution.

(b) Poisson distribution:

$$f(z) = \frac{e^{-\theta} \theta^z}{z!} ; \quad 0 < z < \infty$$

$$\text{Mean of the distribution} = \theta$$

$$\text{Variance of the distribution} = \theta$$

Poisson distribution has only one parameter,  $\theta$ .

(c) negative exponential distribution:

$$F(z) = \begin{cases} 1 - e^{-\theta z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

The probability density function is of the form

$$f(z) = \begin{cases} \theta e^{-\theta z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

$$\text{The mean of the distribution} = \frac{1}{\theta}$$

$$\text{Variance} = \frac{1}{\theta^2}$$

Negative exponential distribution is a special case of gamma distribution with  $i = 1$ .

As we shall see later, in all the above distributions,  $\theta$  can be interpreted as the arrival rate.

### A Simple Single-Channel Queue Model

#### Assumptions

- (1) Only one customer can be serviced at a time.
- (2) The customers are serviced in the order in which they arrive.
- (3) The arrivals and servicing are randomly and independently distributed.

Let

$$\theta = \text{average rate of arrivals,}$$

- $\gamma$  = average rate of servicing,  
 $A_i(t)$  = the probability that  $i$  customers arrive  
in time period  $t$ ,  
 $S_i(t)$  = the probability that  $i$  customers are  
serviced in time ' $t$ '.

If, for example, arrivals follow a poisson distribution, then

$$A_i(t) = \frac{e^{-\theta t} (\theta t)^i}{i!}$$

Similarly, if servicing follows a negative exponential distribution, then

$$S_i(t) = \gamma e^{-\gamma t}$$

- (4)  $\theta > 0, \gamma > 0$ .
- (5)  $\gamma > \theta$ , i.e., servicing rate is bigger than arrival rate.  
 $\tau = \frac{\theta}{\gamma}$  is termed as 'traffic density' or 'traffic intensity'.

The assumption implies  $0 < \tau < 1$ .

If  $\tau \rightarrow 1$ , as we shall see, the length of the waiting time approaches infinity.

- (6) There is no possibility of two or more arrivals or servicings at the same moment of time.
- (7) During a small period of time  $\Delta t$ , the probabilities of arrival of more than one unit or its being serviced are insignificant.

Let  $P_i(t)$  ( $i = 0, 1, 2, \dots$ ) be the probability that  $i$

customers are waiting in the queue at time  $t$ . Then  $P_n(t)$  is the probability of  $n$  customers waiting at time  $t$ .

$P_i(t)$  are called 'state' probabilities. Let  $P_i$  be the 'steady state' probabilities.

$$P_i = \lim_{t \rightarrow \infty} P_i(t)$$

This means that after sufficient time there will be little or no fluctuation in the probability of the system having 'i' individuals in the queue, i.e.,

$$\lim_{t \rightarrow \infty} \frac{d}{dt} P_i(t) = 0$$

Thus  $P_i(t)$  is a function of time, whereas  $P_i$  is not.

Steady state probabilities are derived from state probabilities in the following manner.

Consider the time interval between  $t$  and  $t + \Delta t$ . The waiting line would not increase during  $\Delta t$  under one of the following conditions.

<u>Case No.</u>	<u>Condition or events occurring during <math>t</math> and <math>t+\Delta t</math></u>
1	no arrivals
2	no arrivals, one departure
3	one arrival, two departures
4	two arrivals, three departures
.	.
.	.
.	.

Let  $\theta \Delta t$  be the probability of one new customer arriving at the station between  $t$  and  $t + \Delta t$ .

Let  $\gamma\Delta t$  be the probability of one customer being serviced between  $t$  and  $t + \Delta t$ . Then the probability that no customer arrives during this time interval is  $1 - \theta\Delta t$ , and the probability that no customer is serviced during this period is  $1 - \gamma\Delta t$ .

Let  $P_0(t + \Delta t)$  be the probability that there are no customers waiting at time  $t + \Delta t$ .

$$P_0(t+\Delta t) = \left[ P_0(t) \cdot \text{Probability of Case 1} \right. \\ \left. + P_1(t) \cdot \text{Probability of Case 2} + \Sigma 0_i(\Delta t) \right]$$

where  $0_i(\Delta t)$  are terms of higher order in  $\Delta t$  and have been assumed to vanish when  $\Delta t \rightarrow 0$ . We shall, therefore, drop  $\Sigma 0_i(\Delta t)$  from subsequent equations. Ignoring  $\Sigma 0_i(\Delta t)$ , we get

$$P_0(t+\Delta t) = P_0(t)(1-\theta\Delta t) + P_1(t) \left[ \gamma\Delta t(1-\theta\Delta t) \right]$$

In general,  $P_i(t+\Delta t)$  is the probability that  $i$  customers are waiting in queue at time  $t + \Delta t$ . For  $i > 0$

$$P_i(t+\Delta t) = \left[ P_i(t)(1-\theta\Delta t-\gamma\Delta t) + P_{i-1}(t)\theta\Delta t + P_{i+1}(t)\gamma\Delta t \right]$$

$$\text{or } \frac{P_i(t+\Delta t) - P_i(t)}{\Delta t} = P_{i-1}(t) \cdot \theta + P_{i+1}(t) \cdot \gamma - (\theta + \gamma)P_i(t)$$

By taking the limit as  $\Delta t \rightarrow 0$ , the following differential equation is obtained:

$$\frac{d}{dt}P_i(t) = P_{i-1}(t)\theta - (\theta+\gamma)P_i(t) + P_{i+1}(t)\cdot\gamma$$

(a) For  $i > 0$ , the steady state probability equation has the form

$$\frac{d}{dt}P_i = 0 = P_{i-1}\cdot\theta - (\theta+\gamma)P_i + P_{i+1}\cdot\gamma$$

because  $P_i$  does not change with time. It remains steady and, therefore, rate of change, i.e.,  $\frac{d}{dt}P_i = 0$ .

(b) For  $i = 0$ , we have

$$0 = \gamma P_1 - \theta P_0$$

or

$$P_0 = \frac{\gamma}{\theta}P_1 = \frac{1}{\tau}P_1$$

and

$$P_1 = \tau P_0.$$

Similarly,

$$P_2 = \tau P_1 = \tau^2 P_0$$

$$P_3 = \tau P_2 = \tau^2 P_1 = \tau^3 P_0$$

⋮

⋮

⋮

$$P_i = \tau^i P_0$$

⋮

⋮

$$P_n = \tau^n P_0.$$

Knowing that  $\sum_{i=0}^{\infty} P_i$  must equal to 1, we have

$$\sum_{i=0}^{\infty} P_i = 1 = P_0 \sum \tau^i$$

$$\sum \tau^i = \frac{1}{1-\tau}$$

and

$$P_0 \sum \tau^i = \frac{P_0}{1-\tau} = 1$$

or

$$P_0 = 1 - \tau$$

and

$$P_i = \tau^i (1-\tau).$$

Thus the probability that  $i$  customers would be waiting in the queue is  $\tau^i (1-\tau)$ .

It is important to note that  $P_i$  does not depend on absolute values of  $\theta$  and  $\gamma$  as such, but solely on their ratios, i.e.,  $\tau$ .

The mean length of waiting line is equal to  $E(i)$ . Call it  $\bar{i}$ .

$$\begin{aligned} E(i) &= \sum_{i=0}^{\infty} iP_i \\ &= \sum_{i=0}^{\infty} i\tau^i(1-\tau) \\ &= \tau(1-\tau) \frac{1}{(1-\tau)^2} = \frac{\tau}{1-\tau} \end{aligned}$$

If we change an assumption and consider  $i$  to be the total number of persons in the system, i.e., waiting and being serviced, then  $\bar{i}$  becomes the average number of persons in the system.

Let

$\bar{w}$  be the average number of customers waiting in the line,

$\bar{s}$  be the average number of customers being serviced such that

$\bar{s} \leq 1$  for a single-channel queue,

then

$$\bar{i} = \bar{w} + \bar{s}$$

or

$$\bar{w} = \bar{i} - \bar{s}.$$

The rate of service =  $\tau$ .

$$\text{As } P_0 = (1-\tau) \longrightarrow P_{\bar{s}} = 1 - (1-\tau) = \tau$$

In a single channel queue only one customer can be serviced at a time. Therefore,

$$\begin{aligned} \bar{s} &= 0 P_0 + 1 P_{\bar{s}} \\ &= 0(1-\tau) + 1(\tau) = \tau \end{aligned}$$

and

$$\bar{w} = \bar{i} - \bar{s} = \frac{\tau}{1-\tau} - \tau = \frac{\tau^2}{1-\tau}$$

## Multiple-Channel Queues

So far, we have dealt with the problems of single-channel queues where there was provision for servicing only one customer at a time. In many cases, facilities exist to provide service to a number of customers at the same time, e.g., several check-out lanes in a super market, ticket windows at railway stations, or tellers at a bank. These are multi-channel phenomena.

Let

$\theta$  = the arrival rate in the system,

$\gamma$  = the rate of servicing per channel, and

$c$  = the number of channels or servicing facilities.

Then  $\tau = \frac{\theta}{\gamma}$  is the 'traffic intensity' per channel and the traffic intensity for the whole system is equal to  $\frac{\tau}{c}$ . Call it  $\bar{\tau}$ .

Let  $i$  be the number of customers in the system, i.e., those being serviced plus those who are waiting.

Under the assumptions 2, 3, 4 and 7 of single-channel queues and the assumption that  $\bar{\tau} < 1$  (this is a practical and realistic assumption because it means that the system can serve more customers than are expected to arrive), it can be shown that

$$P_i = \begin{cases} \frac{\tau^i}{c^{i-c} c!} \cdot P_0 & \text{for } i \geq c \\ \frac{\tau^i}{i!} \cdot P_0 & \text{for } i \leq c \end{cases}$$

or

$$\begin{aligned}
 P_i &= \frac{\tau^{i-c} \cdot \tau^c}{c^{i-c} c!} \quad \text{for } i \geq c \\
 &= (\bar{\tau})^{i-c} \cdot \frac{\tau^c}{c!} \quad \text{for } i \geq c
 \end{aligned}$$

Again using the property that the sum of probabilities adds up to 1, we find that

$$\begin{aligned}
 P_0 &= \frac{1}{\sum_{i=0}^c \frac{\tau^i}{i!} + \frac{1}{c!} \sum_{i=c+1}^{\infty} \frac{\tau^i}{c^{i-c}}} \\
 &= \frac{1}{\sum_{i=0}^c \frac{\tau^i}{i!} + \frac{1}{c!} \sum_{i=c+1}^{\infty} \frac{\tau^{i-c}}{c^{i-c}} \tau^c} \\
 &= \frac{1}{\sum_{i=0}^c \frac{\tau^i}{i!} + \frac{\tau^c}{c!} \sum_{i=c+1}^{\infty} \bar{\tau}^{i-c}}
 \end{aligned}$$

This calculation of  $P_0$  is rather complex. But in our model,  $\bar{\tau} < 1$  and, therefore, the series  $\sum_{i=c+1}^{\infty} \bar{\tau}^{i-c}$  converges. This property renders the determination of  $P_0$  easier. Therefore, for practical purposes,

$$P_0 = \left[ \sum_{i=0}^c \frac{\tau^i}{i!} \right]^{-1}$$

It is interesting to note that if the number of servicing facilities is infinite, i.e.,  $c = \infty$ , then

$$P_0 = \frac{1}{\sum_{i=0}^{\infty} \frac{\tau^i}{i!}} = \frac{1}{e^\tau} = e^{-\tau}$$

Since  $c = \infty$ , it necessarily means that  $i \leq c$  and in this case

$$P_i = \frac{\tau^i}{i!} e^{-\tau} \quad \text{for } 0 \leq i \leq c$$

and this is a Poisson distribution.

For calculation purposes the following are more convenient:

$$(a) \quad P_i = \frac{\tau}{i} P_{i-1} \quad \text{for } i \leq c, \text{ and}$$

$$(b) \quad P_i = \tau \cdot P_{i-1} \quad \text{for } i \geq c.$$

#### Applications in Agriculture

Few applications of queuing theory in agriculture have been reported. Cox, Glickstein and Greene (22) applied the theory in determining livestock unloading facilities. They simulated conditions for a ninety-minute period by Monte Carlo method at a facility for various livestock volumes and calculated the probabilities for different  $i$ 's for three and four dock unloading facilities. According to them, the management could assign conditions as it wished to meet and determine the cost. This was to be done by assigning costs of extra dock

facilities and of manpower requirements to unload or service in certain time intervals at each level the management was interested in. It would have been helpful if an actual calculation of how it is to be done was given by them. Lu (60) applied the queuing theory to determine the optimum checkout facilities at a nationally known chain food store in Detroit under given conditions. His study is based on the assumption that the  $A_n(t)$  followed a poisson distribution and that the service was exponentially distributed.

Simmons (91) attempted to determine appropriate plant loading facilities for fleet milk distribution trucks through the use of this theory under the conditions of poisson probability distribution of arrivals, variable average arrival rates, constant service times and a first-come-first-served discipline. He estimated the total waiting time as:

$$E(w) = \frac{\lambda}{2\mu(\mu-\lambda)}$$

where

$E(w)$  = estimated waiting time,

$\lambda$  = average arrival rate in trucks per minute,

and

$\mu$  = average loading time in trucks per minute.

He found that the estimates of waiting time were different under the two assumptions of constant arrival rate and different arrival rates for each hour.

Through the use of two illustrations, we shall attempt to demonstrate the practical applications of single and multi-channel queues in the agricultural industry. Later we shall mention some other areas where the theory could be applied.

Illustration of the application of a single-channel model in agriculture

The following example, though simple, will serve to indicate the situations where we could use this model.

An Agricultural University in India has a fleet of fifty tractors which, during use, break down and require repairs.

Let us assume the following:

- a. The average rate of breakdown is one tractor per day.
- b. The University has only one service station with a squad of machinists and repairmen. The station can service only one tractor at a time and with the given size of the squad, can repair, on an average, two tractors a day.
- c. The University loses 55 rupees a day in case a tractor stands idle for one day.
- d. All the assumptions made in the single-channel queuing model on pages 238 and 239 hold good in this case.
- e.  $\gamma(C)$  is the level of servicing coefficient such that  $\gamma(C) = .5 + .01C$  for  $C > 50$  rupees where  $C$  is the weekly cost of maintaining the service facilities to keep  $\gamma$  at  $\gamma(C)$  level.

f.  $\theta < \gamma$ . If we do not make this assumption, the system will be congested to infinity.

The problem is to find whether the university should maintain its servicing facilities at the present level or change its present capacity of repairing two tractors per day without increasing the number of channels. If the capacity has to be changed, what should be the extent of this change? In other words, the problem is to find that value of  $\gamma$  which would minimize the sum of costs of repairs in terms of waiting costs (where tractors have to wait to get repair) and the cost of actually repairing the tractors.

We can proceed in the following manner.

$$\begin{aligned} \theta &= 1 = \text{average number of breakdowns per day} \\ \gamma &= \text{average number of tractors that can be} \\ &\quad \text{repaired in a day} \\ &= 2 \\ \tau &= \frac{\theta}{\gamma} = \frac{1}{2} \\ E(i) &= \bar{i} = \text{expected number of tractors waiting to} \\ &\quad \text{be repaired} \\ &= \frac{\tau}{1-\tau} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \end{aligned}$$

It is apparent that as  $\tau$ , the ratio of arrivals to the servicing facilities, increases, the value of  $E(i)$  increases. As  $\tau$  approaches 1,  $E(i)$ , the number of tractors in waiting, would approach infinity.

The probabilities of 0, 1, 2, 3, 4, ... tractors waiting in the line are:

$$\begin{aligned}
 P_0 &= \frac{1}{2} \\
 P_1 &= \frac{1}{4} \\
 P_2 &= \frac{1}{8} \\
 P_3 &= \frac{1}{16} \\
 &\vdots \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

Under our assumptions, only one would be serviced at a time and the rest would be waiting. Therefore, the probabilities of 1, 2, 3, 4, ... tractors being in the system (those being serviced plus those waiting to get service) are given below:

Number of tractors in the system	Probability
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$
⋮	⋮
⋮	⋮
⋮	⋮

In our example,  $\theta = 1$ . We have assumed, for practicality, that  $\theta < \gamma$ . This would mean that  $\gamma(C) > 1$  and that the optimum solution should satisfy this condition.

Substituting the equation of  $\gamma(C)$  in the above, we have

$$.5 + .01C > 1, \text{ or } C > 50 \text{ rupees.}$$

The average loss (costs) to the university per day due to non-repair of tractors =  $E(i) \cdot \text{costs per day for each } i$   
 $= 1 \cdot \text{rupees } 55 = \text{Rupees } 55.$

The daily cost of maintaining the service facilities =  $\frac{C}{7}.$

Therefore, the total costs per day to the university =  $55 + \frac{C}{7}.$

Let us call these costs  $K.$   $K = 55 + \frac{C}{7}.$

Under the existing circumstances,

$$\gamma(C) = 2 = .5 + .01C \text{ or } C = \text{Rupees } 150.$$

Therefore, the value of  $K$  under the present service facilities is  $= 55 + \frac{150}{7} = \text{Rupees } 76.5$  approximately.

Our object is to find a value of  $\gamma(C)$  that would minimize the expected value of  $K.$

$$E(K) = E(i) \cdot \text{costs per day for each } i + \frac{C}{7}$$

$$= \left[ \frac{\frac{\theta}{\gamma(C)}}{1 - \frac{\theta}{\gamma(C)}} \right] 55 + \frac{C}{7}$$

$$= \left[ \frac{\frac{1}{.5 + .01C}}{1 - \frac{1}{.5 + .01C}} \right] 55 + \frac{C}{7}$$

$$E(K) = \frac{55}{.01C - .5} + \frac{C}{7}$$

Our objective is to minimize  $E(K)$  with respect to  $C$ . Therefore, we take the derivative of  $E(K)$  with respect to  $C$  and set it equal to zero.

$$\frac{dE(K)}{dC} = \frac{(-55)(.01)}{(.01C - .5)^2} + \frac{1}{7} = 0$$

$$\text{or } (.01C - .5)^2 = 3.85$$

$$\text{or } .0001C^2 - .01C + .25 = 3.85$$

$$\text{or } C^2 - 100C - 3600 = 0$$

Solving this quadratic equation, we get

$$C = \text{Rupees } 246 \text{ approximately.}$$

This means that  $\gamma(C)$  should be such that

$$C = \text{Rs } 246,$$

$$\gamma(C) = .5 + (.01)(246) = 2.96 \approx 3.$$

This suggests that  $\gamma$  should be increased from 2 to 3. For  $C = 246$  and  $\gamma = 3$ ,

$$K = \left[ \frac{\frac{1}{3}}{1 - \frac{1}{3}} \right] 55 + \frac{246}{7}$$

or

$$K = \text{Rs } 62.7.$$

For  $\gamma = 3$ ,  $K = 62.7$  rupees.

For  $\gamma = 2$ ,  $K = 76.5$  rupees.

Therefore, the total costs per day to the university would be less if the repair squad and facilities are increased to the extent that three tractors could be repaired per day.

It is important to keep in mind that optimum size of repair facilities depends not only on the costs of servicing, but also on the loss due to the tractor standing idle. For example, if the loss due to one tractor standing idle was only Rupees 15 per day to the university, then

$$E(K) = \frac{15}{.01C - .5} + \frac{C}{7}.$$

Taking  $\frac{dE(K)}{dC}$  and setting it to equal to zero, we have:

$$\frac{-.15}{(.01C - .5)^2} + \frac{1}{7} = 0$$

or

$$C^2 - 100C - 8000 = 0$$

or

$$C = 152 \text{ rupees approximately}$$

and

$$\begin{aligned} \text{optimal value of } \gamma(C) &= .5 + .01C \\ &= .5 + 1.52 \\ &= 2.02. \end{aligned}$$

This suggests that if loss due to a tractor standing idle were 15 rupees, the present capacity of servicing facility was

optimum and, therefore, should be maintained at the existing level.

Illustration of application of a multiple-channel model in agriculture

Sugar cane is a cash crop. A large part of sugar cane grown in India is converted into sugar in the factories. There it is the responsibility of the producer to deliver the cane at one of the collection centers of the factory. Usually there are a few weighing machines and a large number of carts and trucks at the station. As a consequence, the farmer has to wait for a long time to get his produce weighed and get the weight slip which is brought to the cashier to receive payment for the delivered cane.

Let us assume that the sugar factory has five weighing stations. Let the rate of arrivals be 100 carts and/or trucks per hour. Let the rate of servicing be 25 carts and/or trucks per hour. Then

$$\begin{aligned}c &= 5 \\ \theta &= 100 \\ \gamma &= 25 \\ \tau &= \frac{100}{25} = 4 \\ \bar{\tau} &= \frac{4}{5}\end{aligned}$$

The management of the sugar factory wants to take a decision regarding installation of additional weighing facilities. Under the assumptions that we made previously for the

multi-channel queuing model,

$$\begin{aligned}
 P_0 &= \left[ \sum_{i=0}^5 \frac{\tau^i}{i!} \right]^{-1} \\
 &= \frac{1}{1 + \frac{(4)^2}{2!} + \frac{(4)^3}{3!} + \frac{(4)^4}{4!} + \frac{(4)^5}{5!}} \\
 &= .0257
 \end{aligned}$$

$$P_1 = (4) (P_0) = (4) (.0257) = .1028$$

$$P_2 = \left(\frac{4}{5}\right) (P_1) = .2058$$

•  
•

$$P_5 = \left(\frac{4}{5}\right) (P_4) = .21952$$

$$P_6 = \left(\frac{4}{5}\right) (P_5) = .175616$$

$$P_7 = \left(\frac{4}{5}\right) (P_6) = .141$$

•  
•  
•

$$\begin{aligned}
 \text{Average number of} \\
 \text{arrivals in the system} &= \sum_{i=0}^{\infty} iP_i \\
 &= (0)P_0 + (1)P_1 + (2)P_2 + \dots \\
 &= 13 \text{ approximately.}
 \end{aligned}$$

$$\begin{aligned}
 &\text{Average number} \\
 &\text{of those waiting} \\
 &\text{at any given time} = 1 \cdot \text{Probability of one waiting} \\
 &\quad + 2 \cdot \text{Probability of two waiting} \\
 &\quad + \dots \\
 &= (1 \times P_6) + (2)(P_7) + \dots \\
 &= 5 \text{ approximately.}
 \end{aligned}$$

$$\begin{aligned}
 &\text{Average number of} \\
 &\text{arrivals being serviced} = 13 - 5 = 8
 \end{aligned}$$

$$\begin{aligned}
 &\text{Average waiting} \\
 &\text{time for each farmer} = \left(\frac{1}{25}\right)(P_5) + \left(\frac{2}{25}\right)(P_6) + \left(\frac{3}{25}\right)(P_7) + \dots \\
 &= .23 \text{ hours approximately} \\
 &= 14 \text{ minutes.}
 \end{aligned}$$

We can make similar calculations for  $c = 6, 7, 8, \dots$ .

Depending on the value that the management attaches to the reduction in waiting time of the farmer and cost of installing and operating the additional weighing machines, optimal value of  $c$  can be calculated in a manner similar to that used in the single-channel example.

### Summary

Arrival and servicing rates are the backbones of queuing models. These rates can be measured and probabilities of a

given number of arrivals and probability that a given time will be required to service can be calculated. Mathematical analysis used in finding out the characteristics of waiting lines is rather complex. Monte Carlo methods can be used with advantage to these problems of queuing.

Queuing problems are a special case of Inventory Control. However, there is a basic difference in the analysis of the two because the latter are not waiting line problems and various states do not occur randomly. "In any case where something requires some kind of service from one of a limited number of facilities and where there is a cost associated with any delay caused by the something's having to wait for a facility we will have to use waiting line analysis rather than the other inventory models" (69, p. 397).

The two applications endeavour to demonstrate the way in which the problems can be solved with the help of queuing theory. It can be applied to several types of problems in agriculture. Repairs and maintenance analysis, problems of loading and unloading, production and shipment of seed, and manufacture of feed are some of the areas. In industries akin to agriculture, e.g., tractor and implements manufacture and fertilizer production, theory of queues can be applied exactly in the same manner as in other industries. Due to its very nature of waiting, it is likely that the theory may not interest small farmers.

A great variety of models to handle diversified situations have been developed and a decision maker may find one to suit his requirements. Wherever the problem involves a flow of persons or things through various stages or points, queuing theory can be applied.

## SUMMARY AND CONCLUSIONS (Epilogue)

From an humble beginning during the First World War, the operations research today has grown into a highly sophisticated science. Its use now requires a knowledge of advanced mathematical analysis and operation. Advancements in computer science have rendered operations research more operational and useful.

Though increasingly used in business management and other areas, operations research is still very much defense-oriented. It should not be forgotten, however, that along with defense, bread and butter, shelter and clothing are also essential for survival. Wars are not won or lost just with military hardware. The power of endurance of people to the war and its after-effects are equally important.

It is, therefore, high time that these tools of operations research, primarily developed for defense, were applied to management and decision-making in other fields directly affecting welfare of mankind. Mr. A. M. Mood (72), in his presidential address to the Operations Research Society of America, remarked, "I believe our talents are too much monopolized by the defense business. That business is showing some signs of diminishing so that mere survival demands that we diversify. More importantly, we wish our Society to be a growing viable activity and hence to find itself useful in whatever area of human affairs it can make a significant

contribution.... There are important areas in which we are making little or no contribution and in which operations analysis has the potential to make quite salient contributions." These important areas, as pointed out by Mood were education, health, welfare, urban affairs and agriculture. Applications of operations research techniques in the field of agriculture has been the theme of this study.

Application of operations research to problems of decision making in agriculture is hardly twenty years old. One does come across some works where study of problems allied to agriculture is a by-product of research for defense purposes. The objective of "Nuclear War and Soil Erosion: Some Problems and Prospects" by Katz (51), for example, highlights some of what was presently understood about soil erosion in order to provide a backdrop for further substantive investigations and considerations that were more directly applicable to postulated post-attack situation. The primary objective, thus, was the effect of nuclear war as related to the problems of soil erosion. The work had nothing to do with the problems of decision making in agriculture. As mentioned earlier, our study concerns itself with the sole objective of examining the suitability of operations research techniques for decision making in agriculture industry.

Perhaps the first and the most widely used techniques in agriculture is linear programming. A few applications of

non-linear programming have been done. Game theory, simulation, time-network analysis, queuing theory, inventory control and other techniques have been sparingly applied. "Evaluation of alternative operations research techniques is an endless task far beyond the scope of a single paper or a single person" (47, p. 1417). If not endless, it is, to say the least, ambitious and stupendous. This study was, therefore, confined to applications of game theory, linear fractional functionals programming, time-network analysis and queuing theory.

The most common applications of game theory in agriculture are of games against 'nature'. The author found that the four principles of choice used—viz., Wald's criterion, Savage's regret principle, Hurwicz' optimism-pessimism method, and Laplace's criterion—were rather unsatisfactory for these types of games. Hurwicz' criterion involved subjectivity and Laplace's principle assumes complete ignorance on the part of the farmer. Wald's method implied extreme pessimism, whereas regret principle was based on high optimism. The last two criteria represent the two extremes. In light of these drawbacks, the author has devised and suggested the criterion of 'benefit'. It is a blend of Wald's and regret criteria, being neither too optimistic nor too pessimistic. From the applications carried out it seems to hold promise.

The underlying assumption in game theory is that every

player tries to do the worst to his opponents. In the models of 'games against nature', therefore, we base our conclusions on this premise. In reality 'nature' is not an opponent of the farmer and does not try to do worst to him. The assumption, therefore, is unrealistic, the conclusions may be invalid and the suitability of models to make production decisions seems questionable. The model could, however, be applied with advantage by the farmer in making decisions regarding purchasing his requirements, marketing his products, size of the warehouse to be constructed, etc.

The applicability of mathematical programming in agriculture cannot be overemphasized. Linear, dynamic and recursive programming have been used to widely varying problems. Range X analysis can help the farmer in deciding upon the best farm plan for a given set of resources, within what limits can an activity be varied without affecting the optimality of the original solution, to determine the grain mix that can meet all nutrient requirements of the animal at least cost, and similar problems. A newer technique of linear fractional functionals programming holds great promise in its application to agriculture when the objective is to maximize returns per hour of family labor or hired labor or both, or the farmer is interested in optimizing returns per dollar of investment. The technique can also be used in planning at the state or national level to determine the optimum plan to maximize per

capita income in agriculture or for the economy as a whole.

Given the out-of-door nature of farm work, proper scheduling of operations is important. Time-network analysis can help a farmer not only in planning ahead, but also in making evaluations at every stage of planning and production and obviating future sources of trouble. When fully developed, cost and time-cost analysis applications of PERT would be useful in agriculture. Their primary asset is simplicity and low cost.

Theory of queues has been applied to problems of waiting lines and inventory controls in industries. In agriculture, its use may be rather limited to large firms. In addition to problems of waiting for loading and unloading, repairs and maintenance analysis, we can use it for firms that manufacture tractors and other agricultural machinery. The manufacture and shipment of these machines may follow a certain probability distribution which can be determined through studies of frequencies. Likewise, the arrival of orders for the machines will also be according to a certain distribution. The manufacturers can cut the time for which the farmer has to wait to get their product because they might lose their market if he has to wait too long. Within the factory, we can use this theory to cut the costs if, during its manufacture, the machine has to wait for a process. Similar applications can be made in seed and fertilizer industry. Let us take the seed

industry, for example. Seeds for corn, wheat and other crops are produced during the season. However, the orders for them are huge at their sowing time and almost nil after that. Many of these orders will arrive simultaneously. Here again the problem of receiving the orders for seeds and shipping them may be formulated as a queuing problem. For a small farmer, as a decision maker, the theory may not be very useful.

There are several techniques of operations research other than those mentioned above. Here we shall mention them and point out some of the areas allied to agriculture where they can be or have been applied. Burt (15) has suggested how income tax aspects of farm investment and growth can be analyzed through dynamic programming. Applications of this tool would be helpful in finding out factors significantly responsible for acceleration or retardation of the growth of a farm firm. For example, to test the extent of effect of risk on the expansion of a farm, a stochastic dynamic programming model can be used. Integer programming has realistic applications in determination of optimum number of hogs to be raised or cattle to be fed because the solutions of these in terms of fractions of hogs or cattle are simply meaningless. Decomposition principle has potential in the areas of regional and inter-regional competition. For example, it can be used to find such optimal solutions at the farm level that are consistent with the optimum at the district, county, state or

the national level.

Replacement models, as the name itself suggests, can be applied to determine an optimal replacement policy for a tractor or machine, livestock and other assets of this nature. Some interesting and useful applications of inventory control models have been done in agriculture. One such example is "Price and Productive Uncertainties in Dynamic Planning" by Hurter and Moses (44). Their model assumes a single farmer decision maker and incorporates inter-temporal and inter-spatial considerations. First, production and sale of one commodity in a single market are considered. Then the authors introduce production and price uncertainties and use chance constrained method in solving it. The analysis then is extended to several commodities and several markets.

Sensitivity analysis is a good tool to partly overcome the problem of uncertainty in farming. Before some irrigation or multi-purpose river project is undertaken, it needs to be evaluated in terms of its expected costs and pay-offs. Changes in pay-offs due to cost variations can be found out and different ranges in which a given cost structure is optimal can be calculated for appropriate decision.

Zusman and Amiad (112) have demonstrated how simulation can be used for farm planning under conditions of weather uncertainty. Simulation techniques provide a trial and error method and have an advantage that "...once a simulation program

has been developed for one farm, it may be adapted to other farms with only slight modifications" (112, p. 594). It is especially useful for areas where farms are more or less homogeneous.

Forecasting techniques can be used for several purposes. Projections of demand for and supply of agricultural commodities would be an important application. These projections can be used for extrapolation of agricultural prices. Use of forecasting techniques for weather conditions is helpful to farmers. Three recent studies by Rand Corporation have attempted to study economic gains from storm warning (26), the utility of weather forecasting for the raisin industry (54), and how economic decisions are related to weather information (78).

Markovian analysis can be applied to the study of dynamic changes in the structure of agriculture, e.g., farm size, production functions, land ownership patterns, etc., to help the policy maker in evaluating the effects of their policies on agriculture and modify them in the light of results. Levy (59) has shown how uncertainty of agricultural production function can be incorporated into a model of economic growth and then he describes an investment decision in agriculture using dynamic programming in Markov chains.

The utility and applicability of operations research techniques in agriculture or in any other sector of the

economy depends on its structure and level of development. For example, agriculture in India is entirely different from that in the United States both in structure and technology. The problems of Indian agriculture are low productivity, small size of holdings getting still smaller. American agriculture is highly mechanized, the size of the farm is increasing, the percentage of population engaged in agriculture is decreasing and smaller numbers of farmers are producing increasingly larger quantities of food and other agricultural commodities. Therefore, one technique may be useful to Indian agriculture, but may need some modifications before it can be applied to American agriculture and vice versa. In the United States and other developed nations there are business firms that use computers to solve the problems of individual farmers. This facility does not exist in most of the developing countries and, therefore, it is possible that some problems requiring the use of computers may not be solved. But this should not be taken as a serious drawback. For example, in India, where an average farmer carries on subsistence farming on a small size of holding with meager resources, a computer may neither be required nor be feasible for him (because of high costs) for preparing farm plans. This could be taken care of by calculators. His farm plan would be rather simple because of the features of Indian agriculture mentioned above.

The progress of operations research has been phenomenal.

New tools are being developed at a fast pace and the applications of the existing tools are being tried under different sets of assumptions. Several of these assumptions may be realistic in agriculture. As this study indicates, most of these tools can be used with advantage in all the four fields of agricultural economic activities, viz., production, consumption, exchange and distribution.

Of course, the final decision still lies with the man and, therefore, human element is still the most significant part of the decision making process. It is necessary that these powerful tools are used judiciously. Indiscriminate use may lead to disaster. A tool, at best, is only as good as its user.

But there is no doubt that the future would see increasing and diversified applications of operations research techniques in agriculture and other industries. One cannot help agreeing with Burt in that "Potential applications of operations research in farm management are probably far greater than we realize. If growth of linear programming uses over the past decade is any indication of what to expect in the future for other techniques, operations research is still in its infancy as far as farm management is concerned" (15, p. 1426). I would go one step further and broaden the scope of his statement by substituting 'agricultural industry' for 'farm management'.

## REFERENCES

1. Ackoff, Russell L., ed. Progress in operations research. New York, New York, John Wiley and Sons. 1961.
2. Afzal M., McCoy, J. E. and Orazem, F. Development of inventory models to determine feed reserves for beef-cattle production under unstable climatic conditions. Journal of Farm Economics 47: 948-963. 1965.
3. Babbar, M. M., Tintner, G. and Heady, E. O. Programming with considerations of variations in input coefficients. Journal of Farm Economics 35: 333-341. 1955.
4. Barker, Randolph. Use of linear programming in making farm management decisions. Cornell University Agricultural Experiment Station Bulletin 993. 1964.
5. Baumol, William J. Economic theory and operations analysis. 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall Inc. 1965.
6. Bellman, Richard and Dreyfus, S. F. Applied dynamic programming. Princeton, New Jersey, Princeton University Press. 1963.
7. Berman, F. B. Resource allocation in a PERT network under continuous time cost-functions. Management Science 10: 734-745. 1964.
8. Bishop, C. F. Programming farm-nonfarm allocation of farm family resources. Journal of Farm Economics 38: 396-407. 1956.
9. Blackett, P. M. S. Operational research. Advancement of Science 5: 26-38. 1948.
10. Blackwell, David H. and Girshick, M. A. Theory of games and statistical decisions. New York, New York, John Wiley and Sons. 1954.
11. Boles, J. N. Linear programming and farm management analysis. Journal of Farm Economics 37: 1-25. 1955.
12. Bolton, B. Variable resource programming for appraising farm adjustments opportunities. Agricultural Economics Research 16: 12-23. 1964.

13. Bostwick, D. Yield probabilities as a Markov process. *Agricultural Economics Research* 14: 49-56. 1962.
14. Brockmeyer, E., Halztrom, H. L. and Jensen, A. The life and works of A. K. Erlang. Copenhagen, Denmark, Copenhagen Telephone Company. 1948.
15. Burt, Oscar R. Operations research techniques in farm management: potential contribution. *Journal of Farm Economics* 47: 1418-1426. 1965.
16. Charnes, A. and Cooper, W. W. Programming with linear fractional functionals. *Naval Research Logistics Quarterly* 9: 181-186. 1962.
17. Chernoff, H. Rational selection decision functions. *Econometrica* 22: 422-443. 1954.
18. Chernoff, H. and Moses, L. E. Elementary decision theory. New York, New York, John Wiley and Sons. 1959.
19. Chorafas, Dimitris N. Operations research for industrial management. New York, New York, Reinhold Publishing Corporation. 1958.
20. Churchman, C. W., Ackoff, R. L. and Arnoff, F. L. Introduction to operations research. New York, New York, John Wiley and Sons. 1957.
21. Cooke-Yarborough, R. F. Critical path planning and scheduling. *Review of Marketing and Agricultural Economics* 40: 36-48. 1964.
22. Cox, B., Glickstein, A. and Greene, J. H. Application of queuing theory in determining livestock unloading facilities. *Journal of Farm Economics* 40: 104-117. 1958.
23. Dantzig, G. B. Linear programming and extensions. Princeton, New Jersey, Princeton University Press. 1963.
24. DeCoster, Don T. PERT/Cost—the challenge. *Management Services* May-June 1964: 13-18. 1964.
25. de Finetti, Bruno. Foresight: its logical laws, its subjective sources. In Kyburg, Henry E., Jr. and Smokler, Howard E., eds. *Studies in subjective probability*. pp. 93-158. New York, New York, John Wiley and Sons. 1964.

26. Demsetz, Harold. Economic gains from storm warnings: two Florida case studies. Rand Corporation Research Memorandum RM-3168-NASA. 1962.
27. Dillon, J. L. and Heady, E. O. Theories of choice in relation to farmer decisions. Iowa Agricultural Experiment Station Research Bulletin 485. 1960.
28. Dorfman, Robert. Operations research. American Economic Review 50: 575-623. 1960.
29. Dorfman, Robert, Samuelson, P. and Solow, R. M. Linear programming and economic analysis. New York, New York, McGraw-Hill Book Company, Inc. 1958.
30. Dresher, Melvin. Games of strategy: theory and applications. Englewood Cliffs, New Jersey, Prentice-Hall, Inc. 1961.
31. Enrick, N. L. Management operations research. New York, New York, Holt, Rinehart and Winston. 1965.
32. Fishburn, Peter C. Decision and value theory. New York, New York, John Wiley and Sons. 1964.
33. Ford, L. R. and Fulkerson, D. B. Flows in networks. Princeton, New Jersey, Princeton University Press. 1962.
34. Gale, David. The theory of matrix games and linear economic models. Providence, Rhode Island, Department of Mathematics, Brown University. 1957.
35. Gomory, Ralph E. An algorithm for the mixed integer problem. Rand Corporation Research Project P-1885. 1960.
36. Hartley, H. O. and Wortham, A. W. A statistical theory for PERT critical path analysis. Management Science 12: B469-B481. 1966.
37. Heady, Earl O. Agricultural policy under economic development. Ames, Iowa, Iowa State University Press. 1962.
38. Heady, Earl O. Economic of agricultural production and resource use. Englewood Cliffs, New Jersey, Prentice-Hall, Inc. 1952.
39. Heady, Earl O. and Candler, W. Linear programming methods. Ames, Iowa, Iowa State University Press. 1958.

40. Heiland, H., Jaendl, A. and Kastner, W. Zur Anwendung der Netzplantechnik in der landwirtschaftlichen Unternehmensforschung. *Agrarwirtschaft* 15: 89-101. 1966.
41. Hildreth, Clifford. Problem of uncertainty in farm planning. *Journal of Farm Economics* 39: 1430-1441. 1957.
42. Hildreth, Clifford and Reiter, Stanley. On the choice of a crop rotation plan. In Koopmans, Tjalling C., ed. *Activity analysis of production and resource allocation*. pp. 177-188. New York, New York, John Wiley and Sons. 1951.
43. Hill, Lawrence S. The application of queuing theory to the span of control. *Academy of Management Journal* 6: 58-69. 1963.
44. Hurter, Arthur P., Jr. and Moses, Michael A. Price and productive uncertainties in dynamic programming. Rand Corporation Research Project P-3434. 1966.
45. Hurwicz, L. Optimality criteria for decision making under ignorance. Mimeo. Cowles Commission Discussion Paper 350. 1951.
46. Hutton, Robert F. Operations research techniques in farm management: survey and appraisal. *Journal of Farm Economics* 47: 1400-1414. 1965.
47. Inhen, I. A. Discussion: operations research techniques in farm management: survey and appraisal. *Journal of Farm Economics* 47: 1415-1417. 1965.
48. Jeffreys, Harold. *Theory of probability*. 3rd ed. Oxford, England, Clarendon Press. 1961.
49. Judge G. and Swanson, R. Markov chains: basic concepts and suggested uses in agricultural economics. *Australian Journal of Agricultural Economics* 6: 49-61. 1962.
50. Karlin, Samuel. *Mathematical methods and theory in games, programming and economics*. Vol. I. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc. 1959.
51. Katz, Y. H. Nuclear war and soil erosion: some problems and prospects. Rand Corporation Research Memorandum RM-5302-TAB. 1966.
52. Kelley, J. F. Critical path planning and scheduling: mathematical basis. *Operations Research* 9: 296-320. 1961.

53. Kelsey, M. P. and Janssen, M. R. Bilateral monopoly in farm leasing. *Journal of Farm Economics* 40: 1828-1831. 1958.
54. Kolb, L. L. and Rapp, R. R. The utility of weather forecast to raisin industry. Rand Corporation Research Memorandum RM-2748-NASA. 1961.
55. Koopmans, Tjalling C. Activity analysis of production and resource allocation. New York, New York, John Wiley and Sons. 1951.
56. Kuhn, H. W. and Tucker, A. W., eds. Linear inequalities and related systems. Princeton, New Jersey, Princeton University Press. 1956.
57. Lancaster, F. W. Aircraft in warfare. London, England, Constable. 1916.
58. Langham, M. R. Game theory applied to a policy problem of rice farmers. *Journal of Farm Economics* 45: 151-163. 1963.
59. Levy, Frank S. Markovian agriculture—some first thoughts. Rand Corporation Research Project P-2950. 1964.
60. Lu, John Y. Optimum supermarket check-out facilities: an application of queuing theory. *Journal of Farm Economics*. 43: 27-44. 1961.
61. Luce, R. D. and Raiffa, H. Games and decisions. New York, New York, John Wiley and Sons. 1957.
62. McCloskey, J. F. and Trefethen, F. N., eds. Operations research for management. Vol. I. Baltimore, Maryland, The John Hopkins Press. 1954.
63. McKinsey, J. C. C. Introduction to the theory of games. 1st ed. New York, New York, McGraw-Hill. 1952.
64. Malcolm, Donald G. Integrated research and development management systems. In Dean, Burton V., ed. Operations research in research and development. pp. 124-148. New York, New York, John Wiley and Sons. 1963.
65. Malcolm, Donald G., Roseboom, J. H., Clark, C. F. and Fazar, W. Application of a technique for research and development program evaluation. *Operations Research* 7: 646-669. 1959.

66. Mann, H. S. and Purnapraghnachar, H. Further inferences from the data on response of wheat and paddy to phosphate as affected by nitrogen. *Indian Journal of Agronomy* 8: 384-392. 1963.
67. Mann, H. S., Raheja, P. C. and Dillon, H. S. Studies in crop rotation. 7. long term double cropping of wheat and millets (*Pennisetum typhoideum*). *Indian Journal of Agronomy* 7: 354-364. 1963.
68. Merrill, W. C. Alternative programming models involving uncertainty. *Journal of Farm Economics* 47: 595-609. 1965.
69. Miller, David W. and Starr, Martin K. Executive decisions and operations research. p. 397. Englewood Cliffs, New Jersey, Prentice-Hall, Inc. 1960.
70. Milnor, John. Games against nature. Rand Corporation Research Memorandum RM-679. 1951.
71. Moder, J. C. and Phillips, C. R. Project management with CPM and PERT. New York, New York, Reinhold Publishing Company. 1964.
72. Mood, Alex, M. Diversification of operations research. *Operations Research* 13: 169-178. 1965.
73. Morris, W. H. and Nygaard, A. Application of an optimizing path algorithm in the comparison of farm work methods. *Journal of Farm Economics* 46: 410-417. 1964.
74. Morris, W. T. The analysis of management decisions. Homewood, Illinois, Richard D. Irwin. Rev. ed. 1964.
75. Morse, P. M. Queues, inventories and maintenance. New York, New York, John Wiley and Sons. 1958.
76. Morse, P. M. and Kimball, G. E. Methods of operations research. New York, New York, John Wiley and Sons. 1951.
77. Mueller, A. G. and Wilken, D. F. Summary of Illinois farm business records. University of Illinois College of Agriculture Circular 891. 1964.
78. Nelson, R. R. and Winter, S. G. A case study in the economics of information and coordination: the weather forecasting system. Rand Corporation Research Project P-2788. 1963.

79. Nelson, R. R. and Winter, S. G. Weather information and economic decisions: a preliminary report. Rand Corporation Research Memorandum RM-2620-NASA. 1960.
80. Roegen, Nicholas G. The nature of expectations and uncertainty. In Bowman, M., ed. Expectations, uncertainty and business behavior. pp. 20-21. New York, New York, John Wiley and Sons. 1958.
81. Roman, Daniel D. The PERT system: an appraisal of program evaluation review technique. Journal of Academy of Management 5: 57-65. 1962.
82. Saaty, Thomas L. Elements of queuing theory with applications. New York, New York, McGraw-Hill Book Company, Inc. 1961.
83. Saaty, Thomas L. Mathematical methods of operations research. New York, New York, McGraw-Hill Book Company, Inc. 1959.
84. Sadowski, Wieslaw. The theory of decision making. English edition; translated from Polish. New York, New York, Pergamon Press. 1965.
85. Sasieni, M., Yaspan, A. and Friedman, L. Operations research: methods and problems. New York, New York, John Wiley and Sons, Inc. 1959.
86. Savage, L. J. The foundations of statistical inference. New York, New York, John Wiley and Sons, Inc. 1962.
87. Savage, L. J. The foundations of statistics reconsidered. In Kyburg, Henry E., Jr., and Smokler, Howard E., eds. Studies in subjective probability. pp. 173-188. New York, New York, John Wiley and Sons. 1964.
88. Savage, L. J. The theory of statistical decision. American Statistical Association Journal 46: 55-67. 1951.
89. Schroeder, W. P. PERT: a perspective aid to better management. Canadian Journal of Agricultural Economics 9: 1-11. 1963.
90. Shackle, G. L. S. Expectations in economics. Cambridge, England, Cambridge University Press. 1949.
91. Simmons, K. A queuing theory application with time dependent parameters. Journal of Farm Economics 43: 1408-1409. 1961.

92. Simon, H. A. Models of man. New York, New York, John Wiley and Sons, Inc. 1957.
93. Simon, H. A. The new science of management decisions. New York, New York, Harper and Brothers. 1960.
94. Stoller, David D. Operations research: process and strategy. Berkeley, California. University of California Press. 1964.
95. Stuart, Chase. Operations bootstrap in Puerto Rico: report of progress. Planning pamphlet 75. 1951. Original not available; cited in Trefethen, F. N. A history of operations research. In McCloskey, J. F. and Trefethen, F. N., eds. Operations research for management. Vol. 1. p. 33. Baltimore, Maryland, The John Hopkins Press. 1954.
96. Sur, S. C., Sen Gupta, K. and Sen, S. Optimum time of sowing gram (Cicer Arietinum L.) in West Bengal. Indian Journal of Agricultural Science 36: 173-179. 1966.
97. Swanson, E. R. Application of programming analysis to corn belt farms. Journal of Farm Economics 38: 408-420. 1956.
98. Swaroop, Kanti. Linear fractional functions programming. Operations Research 13: 1029-1036. 1965.
99. Theil, Henri, Boot, John C. G. and Klock, Teun. Operations research and quantitative economics. New York, New York, McGraw-Hill Book Company. 1965.
100. Thornthwaite, Charles W. Operations research in agriculture. In McCloskey, J. F. and Trefethen, F. N., eds. Operations research for management. Vol. 1. pp. 3-35. Baltimore, Maryland, The John Hopkins Press. 1954.
101. Thrall, R. M., Coombs, C. H. and Davis, R. L., eds. Decision processes. New York, New York, John Wiley and Sons, Inc. 1957.
102. Tomar, P. S. and Mathur, O. P. Effect of time of sowing (and seed rate) on yield of wheat under canal irrigation in Rajasthan. Indian Journal of Agronomy 11: 66-69. 1966.

103. Vajda, S. Mathematical programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc. 1961.
104. von Neumann, John and Morgenstern, Oskar. Theory of games and economic behavior. 3rd ed. Princeton, New Jersey, Princeton University Press. 1953.
105. Wald, Abraham. Statistical decision functions. New York, New York, John Wiley and Sons. 1950.
106. Walker, O. L., Heady, Earl O., Tweeten, L. G. and Pesek, J. T. Application of game theory models to decision of farm practices and resource use. Iowa Agricultural Experiment Station Research Bulletin 488. 1960.
107. Wallize, John A., ed. Prices of Iowa farm products, 1930-1965. Iowa Farm Science 20, No. 8: 20. 1966.
108. Wallize, John A., ed. Prices of Iowa farm products, 1930-1966. Iowa Farm Science 21, No. 8: 20. 1967.
109. Waugh, F. V. Discussion: linear programming approach to the solution of practical problems in farm management and micro-agricultural economics. Journal of Farm Economics 43: 404-405. 1961.
110. Williams, J. D. The complete strategyst. New York, New York, John Wiley and Sons. 1966.
111. Yates, F. Agriculture, sampling and operational research. Institut International de Statistique Bulletin 32: 220-227. 1949.
112. Zusman, P. and Amiad, A. Simulation: a tool for farm planning under conditions of uncertainty. Journal of Farm Economics 47: 574-595. 1965.

## ACKNOWLEDGMENTS

I am deeply indebted and grateful to Dr. Earl O. Heady under whose supervision this work was done. But for his great interest, encouragement and guidance, this would not have been possible. Throughout my stay at Iowa State University he has been extremely kind and sympathetic and helped me in my academic and professional problems. Words fail me to express my deep debt of gratitude to him.

I am also grateful to Drs. David Huntsberger, William G. Murray, Frank Riecken, Erik Thorbecke, Eugene Brady and A. Gordon Ball for kindly acting on my Graduate Committee and offering helpful comments and criticism of the manuscript.

Dr. D. K. Hotchkiss taught me the elements of statistical theory and the first formal course in operations research was given to me by Dr. David Thomas. I am grateful to both of them.

I am extremely grateful to the Rockefeller Foundation for the fellowship and to the U. P. Agricultural University for granting me study leave.

It is impossible for me to express my debt to my parents.

To my wife Pushpa, and my children, Bhunesh, Renu and Rahul, I am indeed grateful for their forbearance, understanding and encouragement during the period of my study at Iowa State University.