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NONLINEAR SLOSHING.

Iowa State University of Science and Technology
Ph.D., 1964
Mathematics

NONLINEAR SLOSHING

by

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A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

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Iowa State University
Of Science and Technology
Ames, Iowa

1964

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I. INTRODUCTION

The study of the small oscillations of a free surface, under gravity, appears in Lamb's "Hydrodynamics" (1). However, after this work appeared interest in this problem decreased. With the advent of the missile age a renewed interest in the problem has been stimulated.

The dynamic response of the liquid propellant in the tanks of a space vehicle may effect the stability of the vehicle. The alleviation of this influence can be achieved in many ways, among which are proper choice of tank form, tank location, or introduction of baffles.

These fluid oscillations, resulting from such sources as perturbation of the trajectory, have been shown experimentally to be most critical when the excitation frequency is in the region of a natural frequency of lower mode fluid oscillations.

Eulitz and Glaser (2) have compared experimental results with the previously obtained theoretical solutions, which are obtained from a linear boundary value problem. Within the framework of linear theory, the free surface of the fluid, in a container undergoing transverse harmonic vibrations, should exhibit a steady-state planar harmonic motion at all frequencies except resonance. Eulitz and Glaser claimed thorough agreement between the experimental results and the linearized theory.

Hutton (3) notes that the free surface of a fluid in a container, undergoing transverse harmonic vibrations does not necessarily exhibit a steady-state harmonic motion. In fact, the behavior of such a free surface is as follows. If the container is excited at a frequency well below the lowest natural frequency, p_{11}, of small, free-surface oscillations, the steady-state fluid motion is harmonic with a constant peak wave height and a single nodal diameter perpendicular to the direction of excitation. The wave height increases with an increase of the excitation frequency. When the excitation frequency is close to but smaller than, p_{11}, the smoothly oscillating free-surface changes to a violently splashing condition. As the frequency increases this motion continues until a frequency greater than p_{11} is attained. An additional increase in excitation frequency reduces the wave height until the cycle begins again as the next resonant frequency is approached.

Hutton shows that the "sloshing" motion can be accurately predicted in an inviscid liquid if the analysis includes appropriate non-linear effects.

The present paper is an extension of Hutton's work to include not only transverse harmonic oscillations of the container, but also rotational harmonic oscillations. Again the appropriate non-linear effects are included. A comparison is made between the two solutions. In particular stability of the non-linear motion is studied.

II. DEFINITION OF THE BOUNDARY VALUE PROBLEM

The problem under consideration is that of a tank, partially filled with a non-viscous, incompressible liquid, which is mounted in a system which is moving along a prescribed path. Perturbations of the path of the system cause the liquid to oscillate. There exists two possible types of motion to consider. The first is that of surface waves of large amplitudes, possibly of low frequency, which could actually damage the tank structure. For the most part this can be controlled by suitable baffles in the tank. The second, which will be considered here, is that of surface waves of small amplitudes with a frequency near the natural frequency of the control system on the tank, i.e., the natural frequency of the liquid-tank configuration.

Since the tank is in motion along some path it seems reasonable to refer its motion to an inertial coordinate system, for example the earth. However, if any type of measuring device is attached to the tank then it measures quantities in terms of a tank-fixed reference frame which is moving relative to the inertial system. Thus it is necessary to be able to express the tank-fixed system in terms of the inertial system and vice versa.

Let Y_i be an inertial Cartesian coordinate system with origin O' and coordinates y_i; and let X_i be a Cartesian coordinate system moving relative to Y_i, with origin O and coordinates x_i. Then, instantaneously, the position of a particle moving with the X_i system can be described in the Y_i system by

$$y_i = \bar{Z}_i + a_{ji}\, x_j \tag{2.1}$$

where the summation convention is being used and Latin subscripts take on the values 1, 2, and 3. In Equation 2.1, $\bar{Z}_i(t)$, with components measured in Y_i, give the instantaneous displacement of 0 relative to 0'; and

$$a_{ij}(t) = \cos(x_i, y_j) \tag{2.2}$$

measures the instantaneous rotation of X_i with respect to Y_i. Subsequently the following notation will be used: a barred vector has components measured in Y_i and an unbarred vector has components measured in X_i.

Since a_{ij} are a set of direction cosines, they satisfy, for any t,

$$a_{ik}\, a_{jk} = \delta_{ij} \quad , \tag{2.3}$$

where δ_{ij} is the Kronecker delta. Denote df/dt by $\dot{f}$ and take the derivative of Equation 2.3; then

$$a_{ik}\, \dot{a}_{jk} + \dot{a}_{ik}\, a_{jk} = 0 \ . \tag{2.4}$$

Define

$$\omega_{ij} = a_{ik}\, \dot{a}_{jk} \ . \tag{2.5}$$

Thus, using Equation 2.4 and 2.5, one has

$$\omega_{ji} = a_{jk}\, \dot{a}_{ik} = - \dot{a}_{jk}\, a_{ik} = - a_{ik}\, \dot{a}_{jk} = - \omega_{ij}; \tag{2.6}$$

that is, ω_{ij} is a skew-symmetric second order quantity which can be shown to be a second order tensor.

A dual vector ω_i can then be defined such that

$$\omega_{ij} = - \epsilon_{ijk}\, \omega_k$$

where ϵ_{ijk} is the third order alternating tensor. Thus, from Equation 2.6,

$$a_{ik}\, \dot{a}_{jk} = - \epsilon_{ijk}\, \omega_k \tag{2.7}$$

where ω_k is the angular velocity of X_i with respect to Y_i measured in X_i.

The absolute velocity of a particle, whose position is described by Equation 2.1, can be found by differentiating Equation 2.1 with respect to time. This gives

$$\dot{y}_i = \dot{\bar{Z}}_i + a_{ji}\,\dot{x}_j + \dot{a}_{ij}\,x_j = \bar{q}_i \ , \tag{2.8}$$

where $\bar{q}_i$ is the velocity measured in the Y_i system. The measuring device fixed on the tank measures q_i, where q_i is the velocity measured in the X_i system, and

$$q_i = a_{ij}\,\bar{q}_j \ . \tag{2.9}$$

Using Equation 2.8, q_i can be written as

$$q_i = a_{ij}\,(\dot{\bar{Z}}_j + a_{kj}\,\dot{x}_k + \dot{a}_{kj}\,x_k) \ . \tag{2.10}$$

With Equation 2.3 and 2.7 and the fact that $\dot{Z}_i = a_{ij}\,\dot{\bar{Z}}_j$, Equation 2.10 becomes

$$q_i = \dot{Z}_i + \dot{x}_i - \epsilon_{ijk}\,\omega_j x_k \ , \tag{2.11}$$

or

$$q_i = \dot{Z}_i + \dot{x}_i + \epsilon_{ijk}\,\omega_j\,x_k \ , \tag{2.12}$$

where the skew-symmetric property of the alternating tensor ϵ_{ijk} has been used.

The next quantity of interest is the absolute acceleration

$$\bar{a}_i = \dot{\bar{q}}_i \ . \tag{2.13}$$

From Equation 2.9, Equation 2.3 can be written as

$$\bar{a}_i = \frac{d}{dt}\,(a_{ji}q_j) = a_{ji}\,\frac{dq_j}{dt} + \dot{a}_{ji}\,q_j \ . \tag{2.14}$$

The quantity of interest is $a_i = a_{ij}\,\bar{a}_j$; thus from Equation 2.14,

$$a_k = a_{ki}\,a_{ji}\,\frac{dq_j}{dt} + a_{ki}\,\dot{a}_{ji}\,q_j \ , \tag{2.15}$$

or

$$a_k = \frac{dq_k}{dt} + \epsilon_{ijk}\, \omega_i q_j \quad . \tag{2.16}$$

Here it is noted that the velocity q_i is a function of not only the time, but also of the coordinates x_i which are also functions of time, and thus

$$\frac{dq_k}{dt} = \frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial x_i} \frac{dx_i}{dt} \quad . \tag{2.17}$$

From Equation 2.12

$$\frac{dx_i}{dt} = \dot{x}_i = q_i - \dot{Z}_i - \epsilon_{ijk}\, \omega_j\, x_k \quad . \tag{2.18}$$

Equations 2.18, 2.17, and 2.16 give finally

$$a_k = \frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial x_i} [q_i - \dot{Z}_i - \epsilon_{ijs}\, \omega_j x_s] + \epsilon_{ijk} \omega_i q_j \quad . \tag{2.19}$$

In vector form Equations 2.11 and 2.19 appear as

$$\vec{q} = \vec{q}_o + \vec{\omega} \times \vec{r} + \dot{\vec{r}} \tag{2.20}$$

$$\vec{a} = \frac{\partial \vec{q}}{\partial t} + \vec{\omega} \times \vec{q} + [(\vec{q} - \vec{q}_o - \vec{\omega} \times \vec{r}) \cdot \nabla]\, \vec{q} \quad , \tag{2.21}$$

where $\vec{q}_o$ is the velocity of 0 relative to 0'.

The Eulerian equations of motion for an incompressible inviscid fluid are, in the inertial system,

$$\bar{a}_i = \bar{F}_i - \frac{1}{\rho} \frac{\partial p}{\partial y_i} \quad , \tag{2.22}$$

where $\bar{F}_i$ is the specific body force, ρ is the density and p is the pressure. Since $y_i = y_i\ (x_1, x_2, x_3)$,

$$\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial x_k} \frac{\partial x_k}{\partial y_i} = a_{ki} \frac{\partial p}{\partial x_k} \quad . \tag{2.23}$$

Transform Equation 2.22 to the tank fixed system, using Equation 2.23; then

$$a_j = F_j - \frac{1}{\rho} a_{ji}\, a_{ki} \frac{\partial p}{\partial x_k} \quad , \tag{2.24}$$

or

$$a_j = F_j - \frac{1}{\rho}\frac{\partial p}{\partial x_j} \quad . \tag{2.25}$$

Assuming that the motion is irrotational, there exists a potential ϕ such that

$$\bar{q}_i = - \frac{\partial \phi}{\partial y_i} \quad . \tag{2.26}$$

The incompressibility assumption implies that

$$\frac{\partial \bar{q}_i}{\partial y_i} = 0 \quad . \tag{2.27}$$

Equations 2.26 and 2.27 then lead to

$$\nabla^2 \phi = \frac{\partial}{\partial y_i}\left(\frac{\partial \phi}{\partial y_i}\right) = 0 \quad . \tag{2.28}$$

Transform the above to the tank fixed system, noting that

$$\frac{\partial \phi}{\partial y_i} = \frac{\partial \phi}{\partial x_k}\frac{\partial x_k}{\partial y_i} = a_{ki}\frac{\partial \phi}{\partial x_k} \; ;$$

then

$$q_j = - \frac{\partial \phi}{\partial x_j} \quad , \tag{2.29}$$

$$\frac{\partial q_i}{\partial x_j} = 0 \quad , \tag{2.30}$$

$$\nabla^2 \phi = \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_j}\right) = 0. \tag{2.31}$$

Thus the solution of Laplace's equation furnishes a possible potential function for an incompressible, irrotational flow. In order to determine exactly which potential function is the solution certain boundary conditions need to be prescribed.

Consider a tank of arbitrary shape partially filled with fluid, Figure 1. Assume a constant acceleration is acting along the x_3 axis.

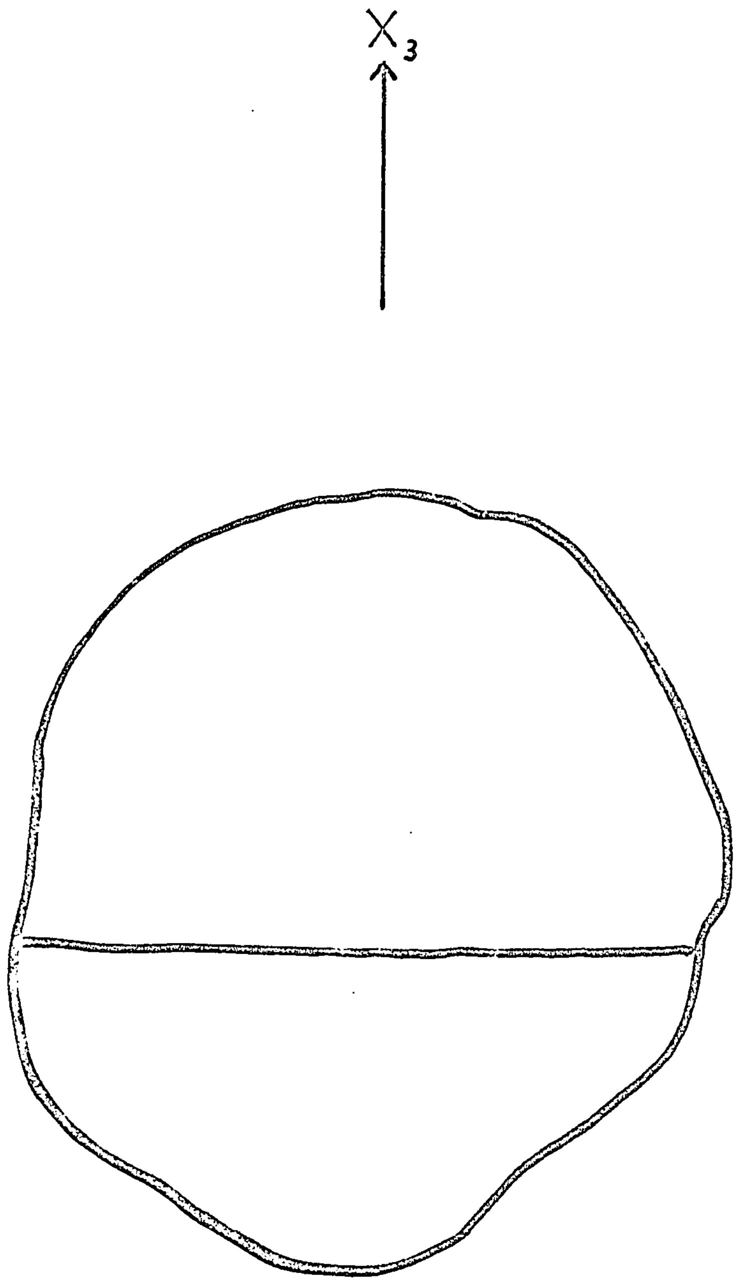

Figure 1. Arbitrarily shaped tank partially filled with liquid.

The surface of the liquid then assumes a planar surface normal to this axis, this surface being called the free surface or quiescent free surface. The origin of the X_i system is taken at the center of gravity of the accelerating fluid system. The motion of the tank-fixed system X_i relative to Y_i, characterized by $\dot{Z}_i$ and ω_i, are oscillatory motions superimposed on the constant-acceleration motion. These will induce perturbations or disturbances of the free surface. The measuring device, traveling with the tank sees only the forcing motions, or perturbations.

In this analysis, it will be assumed that the tank is rigid. With this in mind, the boundary condition on the wetted surface of the tank must be that the velocity of the liquid normal to the tank wall must equal the normal component of velocity of the tank itself. Thus, if υ_i is the unit exterior normal to the tank, and $q_i = -\frac{\partial\phi}{\partial x_i}$,

$$- \upsilon_i \frac{\partial\phi}{\partial x_i} = \upsilon_i \; [\dot{Z}_i + \epsilon_{ijk} \, \omega_j \, x_k] \quad , \tag{2.32}$$

where $\dot{x}_i = 0$ for a rigid tank.

There are two conditions at the free surface. Denoting the disturbed free surface by $\eta \; (x_1, x_2, t)$ and the unit normal to the quiescent free surface by n_i, the kinematic condition that a particle of fluid which travels with the free surface as it moves must have the same velocity as the free surface itself is given as

$$\frac{d}{dt} \, (x_3 - \eta) \Bigg]_{x_3 = \eta} = 0 \quad , \tag{2.33}$$

where x_3 is the displacement of a particle in the x_3 - direction, and, as in Equation 2.17,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial x_k}{\partial t} \frac{\partial}{\partial x_k} \quad . \tag{2.34}$$

Expand Equation 2.33 and use Equations 2.34 and 2.12:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_1} \dot{x}_1 + \frac{\partial \eta}{\partial x_2} \dot{x}_2 = q_3 - \dot{Z}_3 - \epsilon_{3jk} \omega_j x_k , \text{ on } x_3 = n_1 \quad . \tag{2.35}$$

But since $\eta_1 = n_2 = 0$ and $n_3 = 1$, the right hand side of Equation 2.35 can be written as

$$(q_i - \dot{Z}_i - \epsilon_{ijk} \omega_j x_k) n_i \quad .$$

Thus Equation 2.35 becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_i} \dot{x}_i = (q_i - \dot{Z}_i - \epsilon_{ijk} \omega_j x_k) n_i , \text{ on } x_3 = \eta \quad . \tag{2.36}$$

The second condition at the free surface is a dynamic one which states that the pressure at the free surface of the fluid must equal the ambient pressure. To find the form of this boundary condition it is necessary to integrate the equations of motion. Substitute Equation 2.19 into Equation 2.25; the equations of motion become

$$\frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial x_i} [q_i - \dot{Z}_i - \epsilon_{its} \omega_t x_s] + \epsilon_{ijk} \omega_i q_j = F_k - \frac{1}{\rho} \frac{\partial p}{\partial x_k} \quad . \tag{2.37}$$

If the only specific body force is that due to the gravitational field in which the tank system is moving Equation 2.37 can be integrated directly:

$$\frac{p - p_o}{\rho} = \alpha x_3 + \frac{1}{2} (q_i - \dot{Z}_i)^2 - (\epsilon_{ijk} \omega_j x_k) q_i - \frac{\partial \phi}{\partial t} , \tag{2.38}$$

where p_o is the ambient pressure, and α is the magnitude of the acceleration of the tank system. It is assumed here that p_o is a constant. Thus the second boundary condition at the free surface is

$$\frac{\partial \phi}{\partial t} = \alpha\eta + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i} - \dot{Z}_i\right)^2 - \epsilon_{ijk}\omega_j\, x_k \frac{\partial \phi}{\partial x_i}\ , \text{ on } x_3 = \eta\ . \tag{2.39}$$

In summary, the mathematical description of the motion of an incompressible, irrotational fluid confined in a moving, partially filled tank, subject to translational and rotational perturbations, is

$$\nabla^2 \phi = 0\ , \tag{2.40}$$

$$-\upsilon_i \frac{\partial \phi}{\partial x_i} = \upsilon_i\ [\dot{Z}_i + \epsilon_{ijk}\ \omega_j\ x_k]\ \ , \tag{2.41}$$

on the wetted surface, where υ_i is the unit exterior normal to the tank;

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_i}\ \dot{x}_i = -\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i + \epsilon_{ijk}\omega_j x_k\right) n_i, \text{ on } x_3 = \eta, \tag{2.42}$$

where n_i is the normal to the free surface;

$$\frac{\partial \phi}{\partial t} = \alpha\eta + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i} - \dot{Z}_i\right)^2 - \epsilon_{ijk}\ \omega_j\ x_k \frac{\partial \phi}{\partial x_i}, \text{ on } x_3 = \eta\ . \tag{2.43}$$

A. The Linearization of the Problem

If the free surface oscillations are sufficiently small, terms of second order in the velocities can be neglected. Equations 1.40 to 1.43 then become

$$\nabla^2 \phi = 0\ \ , \tag{2.1a}$$

$$-\upsilon_i \frac{\partial \phi}{\partial x_i} = \upsilon_i\ \ [\dot{Z}_i + \epsilon_{ijk}\ \omega_j\ x_k]\ , \text{ on } S_\omega\ , \tag{2.2a}$$

where S_ω is the wetted surface.

$$\frac{\partial \eta}{\partial t} = -\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i + \epsilon_{ijk}\ \omega_j\ \dot{x}_k\right) n_i, \text{ on } x_3 = \eta \tag{2.3a}$$

$$\frac{\partial \phi}{\partial t} = \alpha\eta \qquad\qquad , \text{ on } x_3 = \eta \tag{2.4a}$$

The last two equations can be combined into a single condition

$$\frac{1}{\alpha}\frac{\partial^2 \phi}{\partial t^2} = -\left(\frac{\partial \phi}{\partial x_i} + \dot{Z}_i + \epsilon_{ijk}\ \omega_j x_k\right) n_i, \text{ on } x_3 = \eta\ \ . \tag{2.5a}$$

If it is assumed that $\dot{Z}_i$ and ω_j can furthermore be represented as harmonic oscillations

$$\dot{Z}_i = Z_i^{(0)} e^{i\beta t} \quad , \ \omega_i = \omega_i^{(0)} e^{i\beta t} \ , \tag{2.6a}$$

then it may be assumed that

$$\phi \ (x_i, \ t) = \ \psi \ (x_i) \ e^{i\beta t} \ . \tag{2.7a}$$

The problem then reduces to

$$\nabla^2 \ \psi = 0 \ , \tag{2.8a}$$

$$- \upsilon_i \frac{\partial\psi}{\partial x_i} = \upsilon_i \ [Z_i^{(0)} + \epsilon_{ijk} \ \omega_j^{(0)} \ x_k], \text{ on } S_\omega \tag{2.9a}$$

$$- \frac{\beta^2}{\alpha} \psi - (\frac{\partial\psi}{\partial x_i} + Z_i^{(0)} + \epsilon_{ijk} \ \omega_j^{(0)} \ x_k)n_i, \text{ on } x_3 = 0 \ . \tag{2.10a}$$

The solution to 2.8a through 2.10a can always be obtained if the tank is a prismatic cylinder with x_3 parallel to a generator, and the cross section such that $\nabla^2 \ \psi$ is separable in the appropriate 3-dimensional coordinate system.

III. NON-LINEAR SLOSHING IN A CIRCULAR-CYLINDRICAL TANK

The natural course here is to express the problem in terms of cylindrical polar coordinates (r, θ, z). Before doing this, consider Equation 2.41. There exists two segments of wetted surface: the side of the tank and the bottom of the tank. Consider first the side. Here the normal υ_i has the following components:

$$\upsilon_i = \cos\theta,\ \upsilon_2 = \sin\theta,\ \upsilon_3 = 0\ . \tag{3.1}$$

Thus Equation 2.41 becomes

$$-\upsilon_1 \frac{\partial\phi}{\partial x_1} - \upsilon_2 \frac{\partial\phi}{\partial x_2} = \upsilon_1 \dot{Z}_1 + \upsilon_2 \dot{Z}_2 + \upsilon_1 \epsilon_{1jk}\omega_j x_k + \upsilon_2 \epsilon_{2jk}\omega_j x_k\ . \tag{3.2}$$

On the bottom of the tank the normal υ_i has the components

$$\upsilon_1 = 0\ ,\quad \upsilon_2 = 0,\quad \upsilon_3 = -1\ . \tag{3.3}$$

Thus Equation 2.41 becomes

$$\frac{\partial\phi}{\partial x_3} = -\dot{Z}_3 - \epsilon_{3jk}\omega_j x_k \tag{3.4}$$

Note that $\dot{Z}_3$ is absent in Equation 3.2 and that ω_3 is absent in Equation 3.4. This will in fact be the case for any cylindrical tank whose generators are parallel to the x_3 axis, and is not merely a perculiarity of the circular-cylindrical tank.

In the following the Equations 2.40 through 2.43 will be transformed into cylindrical polar coordinates. So far all the quantities in these equations are measured with respect to the tank-fixed rectangular Cartesian coordinate system. These quantities may also be expressed in a tank fixed cylindrical polar coordinate system. In doing so, it is also convenient to shift the origin from the center of gravity of the fluid to the geometric center of the quiescent free surface. Let $x_1 = x$, $x_2 = y$, $x_3 = z$, and use the usual transformation equations to cylindrical polar coordinates

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z\ ; \tag{3.5}$$

Equation 2.40 becomes

$$\nabla^2 \phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad . \tag{3.6}$$

Let $b_i = \epsilon_{ijk}\,\omega_j x_k$;

$$\begin{aligned} b_1 &= \omega_2\, z - \omega_3\, r\sin\theta\ , \\ b_2 &= \omega_3\, r\cos\theta - \omega_1\, z\ , \\ b_3 &= \omega_1\, r\sin\theta - \omega_2\, r\cos\theta\ . \end{aligned} \tag{3.7}$$

Denote by u_i the quantity $\mathring{Z}_i$; Equations 3.2 and 3.4 become, respectively,

$$\frac{\partial\phi}{\partial r} = -\,u_1\cos\theta - u_2\sin\theta - z\,\omega_2\cos\theta + z\,\omega_1\sin\theta \tag{3.8}$$

$$\frac{\partial\phi}{\partial z} = -\,u_3 - \omega_1\, r\sin\theta + \omega_2\, r\cos\theta \tag{3.9}$$

It may be noted in Equation 3.8 that ω_3 is absent. This is only true for a circular cylinder. To see this, consider the last term, $\upsilon_i\,\epsilon_{ijk}\,\omega_j x_k$ in Equation 2.41. In vector form this is

$$\vec{\upsilon}\cdot(\vec{\omega}\times\vec{\rho}) = \vec{\omega}\cdot(\vec{\rho}\times\vec{\upsilon}) \quad , \tag{3.10}$$

where $\vec{\rho} = (r\cos\theta,\ r\sin\theta,\ z)$. On the side of the circular-cylindrical tank υ_i has components given by Equation 3.1. Thus, the third component of $\vec{\rho}\times\vec{\upsilon}$ is

$$(\vec{\rho}\times\vec{\upsilon})_3 = r\cos\theta\sin\theta - r\cos\theta\sin\theta \equiv 0 \quad . \tag{3.11}$$

Thus from Equation 3.10, ω_3 is absent in the expression $\upsilon_i\epsilon_{ijk}\omega_j x_k$, for a circular cylindrical tank. Since this term in Equation 2.9a is the only place ω_j enters, ω_3 is absent from this boundary condition for this special tank configuration.

On $x_3 = \eta\,(r, \theta, t)$, the unit exterior normal has the following components

$$n_1 = n_2 = 0, \quad n_3 = 1. \tag{3.12}$$

Therefore, from Equations 2.11, 2.34, 3.7, and 3.12, one of the boundary conditions at $z = \eta$ becomes

$$- \frac{\partial\phi}{\partial z} = u_3 - \omega_1 r \sin\theta + \omega_2 r \cos\theta = \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial r}\frac{\partial\phi}{\partial r}$$
$$- \frac{1}{r^2}\frac{\partial\eta}{\partial\theta}\frac{\partial\phi}{\partial\theta} - u_1 \left(\cos\theta \frac{\partial\eta}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\eta}{\partial\theta}\right)$$
$$+ \omega_1 \eta \left(\sin\theta \frac{\partial\eta}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\eta}{\partial\theta}\right) . \tag{3.13}$$

The other boundary condition at $z = \eta$ can be obtained from Equation 2.43 using Equations 3.7 and 3.12:

$$\frac{\partial\phi}{\partial t} = \alpha\eta + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{2\partial\phi}{\partial r}(u_1 \cos\theta + u_2 \sin\theta)\right.$$
$$\left. + \frac{2}{r}\frac{\partial\phi}{\partial\theta}(u_2 \cos\theta - u_1 \sin\theta) + 2\frac{\partial\phi}{\partial z}u_3 + u_1^2 + u_2^2 + u_3^2\right]$$
$$+ \omega_2 \frac{\partial\phi}{\partial r}\eta \cos\theta - \frac{\omega_2}{r}\eta\frac{\partial\phi}{\partial\theta}\sin\theta + \omega_3\frac{\partial\phi}{\partial\theta} . \tag{3.14}$$

Thus if the cylindrical tank has radius a, and the original depth of the fluid is given by $z = -h$, the motion of the contained fluid in a circular cylindrical tank is governed by the following partial differential equation subject to the given boundary conditions:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 , \tag{3.15}$$

$$0 \leqslant r < a, \quad 0 \leqslant \theta \leqslant 2\pi, \; -h < z < \eta \; .$$

On $r = a$, $\frac{\partial\phi}{\partial r} = -u_1 \cos\theta - u_2 \sin\theta - z\,\omega_2 \cos\theta + z\,\omega_1 \sin\theta.$ (3.16)

On $z = -h$, $\frac{\partial\phi}{\partial z} = -u_3 - \omega_1 r \sin\theta + \omega_2 r \cos\theta$. (3.17)

On $z = \eta$, $-\frac{\partial\phi}{\partial z} - u_3 - \omega_1 r \sin\theta + \omega_2 r \cos\theta = \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial r}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial\eta}{\partial\theta}\frac{\partial\phi}{\partial\theta}$

$$-u_1 \quad \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\;\frac{\partial\eta}{\partial\theta} \quad + \omega_1\eta \quad \sin\theta\frac{\partial\eta}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\eta}{\partial\theta} \quad , \tag{3.18}$$

and $\frac{\partial\phi}{\partial t} = \alpha\eta + \frac{1}{2}\left[(\frac{\partial\phi}{\partial r})^2 + (\frac{1}{r}\frac{\partial\phi}{\partial\theta})^2 + (\frac{\partial\phi}{\partial z})^2\right] + \frac{\partial\phi}{\partial r}[u_1 \cos\theta + u_2 \sin\theta]$

$$+ \frac{1}{r}\frac{\partial\phi}{\partial\theta}[u_2 \cos\theta - u_1 \sin\theta] + \frac{\partial\phi}{\partial z}u_3 + \frac{1}{2} \quad u_1^2 + u_2^2 + u_3^2$$

$$+ \omega_2\eta \quad \frac{\partial\phi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\sin\theta \quad + \omega_3\frac{\partial\phi}{\partial\theta} + \frac{\partial\phi}{\partial z}r\,[\omega_1 \sin\theta - {}_2 \cos\theta]. \tag{3.19}$$

To simplify the problem described the following transformation is made:

$$\psi\,(r,\theta,z,t) = \phi\,(r,\theta,z,t) + u_1 r \cos\theta + u_2 r \sin\theta - zr\omega_1 \sin\theta$$

$$+ z\,\omega_2\, r \cos\theta\,. \tag{3.20}$$

Equations 3.15 through 3.19 become:

$$\nabla^2\,\psi = 0. \tag{3.21}$$

On $r = a$, $\frac{\partial\psi}{\partial r} = C.$ (3.22)

On $z = -h$, $\frac{\partial\psi}{\partial z} = -u_3 - 2r\omega_1 \sin\theta + 2r\omega_2 \cos\theta.$ (3.23)

On $z = \eta$, $\frac{\partial\psi}{\partial t} - \mathring{u}_1 r \cos\theta - \dot{u}_2 r \sin\theta + \eta\, r\, \mathring{\omega}_1 \sin\theta - \eta\,\omega_2 r \cos\theta$

$$= \alpha\eta + \frac{1}{2}\left[(\frac{\partial\psi}{\partial r})^2 + (\frac{1}{r}\frac{\partial\psi}{\partial\theta})^2 + (\frac{\partial\psi}{\partial z})^2\right] + \frac{\eta^2}{2}\,(\,\omega_1^2 + \omega_2^2) + 2r^2\,(\omega_1^2 \sin^2\theta$$

$$+ \omega_2^2 \cos^2\theta) + 2\,\frac{\partial\psi}{\partial z}\,(r\omega_1 \sin\theta - r\,\omega_2 \cos\theta + u_3) - 3r^2\,\omega_1\,\omega_2 \sin\theta\cos\theta$$

$$+ 2 u_3 \omega_1 r \sin\theta + 2 u_3\omega_2 r \cos\theta - \eta \omega_2 u_1 + \eta\omega_1 u_2$$

$$+ \eta \omega_3\omega_2 r \sin\theta + u_1\omega_3 r \sin\theta + \omega_3 \frac{\partial\psi}{\partial\theta} + \eta\omega_3\omega_1 r \cos\theta , \qquad (3.24)$$

and $$- \frac{\partial\psi}{\partial z} - u_3 + 2r\omega_2 \cos\theta - 2r\omega_1 \sin\theta = \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial\eta}{\partial\theta}\frac{\partial\psi}{\partial\theta} - \omega_3 \frac{\partial\eta}{\partial\theta} . \qquad (3.25)$$

The free surface η is one of the unknowns; but it may be eliminated between Equations 3.24 and 3.25 as shown in the Appendix. Equations 3.24 and 3.25 may thus be replaced by Equation A.12:

$$a_{oo} + a_{o1} - \frac{a_{11}b_{oo}}{\alpha} + a_{o2} - \frac{a_{11}b_{o1}}{\alpha} + \frac{a_{12}b_{oo}}{\alpha} \pm \frac{a_{11}b_{11}b_{oo}}{\alpha^2} + \frac{a_{22}b_{oo}^2}{21\alpha^2}$$

$$+ B_{11} + B_{12} + B_{13} - a_{11} (\alpha^{-1}\mathring{u}_2 r \sin\theta + \alpha^{-2} [r \mathring{\sigma}_1 \mathring{u}_2 r \sin\theta - b_{11}\mathring{u}_2 r \sin\theta$$

$$+ r \mathring{\sigma}_1 b_{oo}] + \alpha^{-1} [2r^2 \omega_1^2 \sin^2\theta + 2r^2 \omega_2^2 \cos^2\theta + 2 \xi_r r (u_3 - \sigma_1)$$

$$- 3r^2 \omega_1 \omega_2 \sin\theta \cos\theta + 2u_3 r \sigma_1 + u_1 \omega_3 r \sin\theta + \omega_3 \xi_\theta])$$

$$-B_{22} (\alpha^{-1} [b_{oo} + \mathring{u}_2 r \sin\theta] - \alpha^{-2} [b_{11} - r \mathring{\sigma}_1][b_{oo} + \mathring{u}_2 r \sin\theta]$$

$$+ \alpha^{-1} [b_{o1} + 2r^2 \omega_1^2 \sin^2\theta + 2r^2 \omega_2^2 \cos^2\theta + \xi_z r(u_3 - \sigma_1)$$

$$- 3r^2 \omega_1\omega_2 \sin\theta \cos\theta + 2u_3 r\sigma_1 + u_1 {}_3 r \sin\theta + \omega_3 \xi_\theta])$$

$$+ a_{12} \alpha^{-1} \mathring{u}_2 r \sin\theta + B_{23} \alpha^{-1} [b_{oo} + \mathring{u}_2 r \sin\theta] + \frac{a_{22}}{2} \alpha^{-2}(2b_{oo}\mathring{u}_2 r \sin\theta$$

$$+ \mathring{u}_2^2 r^2 \sin^2\theta) + \frac{B_{33}}{2} \alpha^{-2} [b_{oo} + \mathring{u}_2 r \sin\theta]^2 + 0 (\eta^4) = 0, \qquad (A.12)$$

where $\xi_\theta = \frac{\partial\psi}{\partial\theta} (r,\theta,0,t)$; $\xi_z = \frac{\partial\psi}{\partial z} (r,\theta,0,t)$; $\xi_r = \frac{\partial\psi}{\partial r} (r,\theta,0,t)$;

a_{ij}, b_{ij}, and B_{ij} are functions of the potential ψ and its partial derivatives all evaluated at $z = 0$; (see Appendix), and

$$\sigma_1 = \omega_2 \cos \theta - \omega_1 \sin \theta .$$

This leads to the boundary value problem consisting of Equations 3.21, 3.22, 3.23, and A.12, which involves only the potential function ψ and the prescribed tank displacements. The tank displacements are assumed to be

$$x_i (t) = \epsilon_o^i \sin \omega t, \quad \theta_i(t) = \theta_o^i \sin \omega t, \ i = 1, 2, 3, \tag{3.26}$$

with ϵ_o^i and θ_o^i "small" and ω close to or equal to the lowest natural frequency p_{11} is

$$p_{11} = \sqrt{\alpha \lambda_{11} \tanh \lambda_{11} h} \quad , \tag{3.27}$$

where λ_{11} is the first non zero root of

$$J_1' (\lambda_{1n} a) = 0.$$

Here the $x_i(t)$ correspond to translational motion, and the $\theta_i(t)$ correspond to rotational motions. Since ϵ_o^i and θ_o^i are small we may effectively assume each set is the same for all i, say ϵ_o and θ_o, respectively; and furthermore we assume

$$\theta_o = \frac{\epsilon_o}{h} \tag{3.28}$$

The tank velocities $\dot{x}_i(t)$ and $\dot{\theta}_i(t)$ are

$$\dot{x}_i(t) = \epsilon \cos \omega t, \ \dot{\theta}_i(t) = \frac{\epsilon}{h} \cos \omega t, \tag{3.29}$$

where $\epsilon = \omega\epsilon_o$.

A steady-state harmonic solution to this boundary value problem is posed in a perturbation form, in analogy with the Duffing problem (4, 5,6), in terms of the parameter ϵ:

$$\psi = \epsilon^{1/3} [\psi_1 (\vec{r}, t) \cos \omega t + x_1 (\vec{r}, t) \sin \omega t]$$
$$+ \epsilon^{2/3} [\psi_o(\vec{r}) + \psi_2(\vec{r}) \cos 2 \omega t + x_2 (\vec{r}) \sin 2 \omega t]$$

$$+ \epsilon [\psi_3 (\vec{r}) \cos 3\omega t + X_3 (\vec{r}) \sin 3\omega t] \quad , \tag{3.30}$$

where the functions ψ_n and X_n for each value of n, each satisfy

$$\nabla^2 \Phi = 0 ,$$

$$\frac{\partial \Phi}{\partial r} = 0 \quad , \text{ on } r = a,$$

$$\frac{\partial \Phi}{\partial z} = 0, \quad \text{on } z = -h . \tag{3.31}$$

Here $\vec{r}$ means dependence upon r, θ, and z. A set of normal modes of vibration which satisfies Equations 3.31 identically is

$$[A_{mn} (t) \cos m\theta + B_{mn}(t) \sin m\theta] J_m (\lambda_{mn} r) \frac{\cosh \lambda_{mn}(z + h)}{\cosh \lambda_{mn} h} , \tag{3.32}$$

where the J_m are Bessel functions of the first kind of order m, for m a positive integer or zero; and λ_{mn} are an infinite set of numbers for each m obtained from the equation

$$J'_m (\lambda_{mn} a) = 0. \tag{3.33}$$

The functions $A_{mn}(t)$ and $B_{mn}(t)$ will be called the generalized coordinates of the mn'th mode; they depend only on the time t. The natural frequency of small, free-surface oscillations in the mn'th mode is denoted by p_{mn}. When the tank displacements are harmonic motions at a frequency close to or at the lowest natural frequency p_{11}, associated with the J_1 mode, the generalized coordinates A_{11} and B_{11} dominate all other generalized coordinates. Thus it is assumed that the first order terms, ψ_1 and X_1, in Equation 3.9 contain only the J_1 mode; thus ψ_1 and X_1 are chosen as

$$\psi_1 = [f_1 (\tau) \cos \theta + f_3 (\tau) \sin \theta] J_1 (\lambda_{11} r) \frac{\cosh [\lambda_{11} (z+h)]}{\cosh \lambda_{11} h} , \tag{3.34}$$

$$X_1 = [f_2(\tau) \cos \theta + f_4 (\tau) \sin \theta] J_1 (\lambda_{11} r) \frac{\cosh [\lambda_{11} (z+h)]}{\cosh \lambda_{11} h}, \tag{3.35}$$

where the transformations

$$\tau = \tfrac{1}{2}\, \epsilon^{2/3}\, \omega t, \quad p_{11}^2 = \omega^2\, [1 - \upsilon \epsilon^{2/3}] \tag{3.36}$$

have been used. In Equation 3.36, υ is a dimensionless measure of frequency and τ is a dimensionless time parameter. A derivation of the above transformation is given in the Appendix.

As shown in the Appendix Equation 3.30 is then substituted into Equation A.12. Equate to zero the coefficient of $\epsilon^{1/3}$:

$$(\alpha\, \psi_{1z} - p_{11}^2\, \psi_1) \cos \omega t + (\alpha\, X_{1z} - p_{11}^2\, X_1) \sin \omega t = 0, \text{ on } z = 0. \tag{A.22}$$

Equation A.22 is satisfied identically for all time, if ψ_1 and X_1 are chosen as in Equations 3.34 and 3.35. As can be seen, the coefficient of $\epsilon^{1/3}$ involves only the J_1 mode.

The vanishing of the coefficient of $\epsilon^{2/3}$ gives

$$\psi_{oz} = 0\ , \tag{A.27a}$$

$$\alpha\, \psi_{2z} - r\, p_{11}^2\, \psi_2 = 2p_{11}\, \left(X_{1r}\, \psi_{1r} + \frac{X_{1\theta}\psi_{1\theta}}{r^2} + \frac{3\zeta_{11}^2 - 1}{2}\, \lambda_{11}^2\, \psi_1\, X_1\right) \tag{A.27b}$$

$$\alpha\, X_{2z} - r\, p_{11}^2\, X_2 = p_{11}\, \left(X_{1r}^2 - \psi_{1r}^2 + \frac{X_{1\theta}^2}{r^2} - \frac{\psi_{1\theta}^2}{r^2} + \frac{3\zeta_{11}^2 - 1}{2}\, \lambda_{11}^2 (X_1^2 - \psi_1^2)\right). \tag{A.27c}$$

The functions ψ_o, ψ_2, and X_2 are chosen to satisfy Equations A.27 a, b, c. If ψ_o is taken to be constant, Equation A.27a will be satisfied identically. Choose ψ_2 and X_2 to be

$$\psi_2 = \sum_{n=1}^{\infty} \hat{A}_{on}\, J_o\, (\lambda_{on} r)\, \frac{\cosh\, [\lambda_{on}\, (z+h)]}{\cosh \lambda_{on}\, h}$$

$$+ \sum_{n=1}^{\infty} (\hat{A}_{2n} \cos 2\theta + \hat{B}_{2n} \sin 2\theta)\, J_2(\lambda_{2n} r)\, \frac{\cosh\, [\lambda_{2n}\, (z+h)]}{\cosh \lambda_{2n}\, h}\ , \tag{3.37}$$

and

$$X_2 = \sum_{n=1}^{\infty} \hat{C}_{on} J_o (\lambda_{on} r) \frac{\cosh [\lambda_{on} (z+h)]}{\cosh \lambda_{on} h}$$

$$+ \sum_{n=1}^{\infty} (\hat{C}_{2n} \cos 2\theta + \hat{D}_{2n} \sin 2\theta) J_2 (\lambda_{2n} r) \frac{\cosh [\lambda_{2n} (z+h)]}{\cosh \lambda_{2n} h} , \qquad (3.38)$$

where $J_o' (\lambda_{on} a) = J_2' (\lambda_{2n} a) = 0.$

Equations A.27b and A.27c can be satisfied by finding the appropriate generalized coordinates in ψ_2 and X_2. These generalized coordinates can be found by introducing Equations 3.34, 3.35, 3.37, and 3.38 into Equations A.27 b,c and applying a Fourier-Bessel technique using the following orthogonality conditions:

$$\int_0^a r J_o (\lambda_{om} r) J_o (\lambda_{or} r) \, dr = \begin{cases} 0 , & m \neq n \\ \frac{a^2}{2} J_o^2 (\lambda_{on} a), & m = n \end{cases} \qquad (3.39)$$

$$\int_0^a r J_2 (\lambda_{2m} r) J_2 (\lambda_{2n} r) \, dr = \begin{cases} 0 , & m \neq n \\ \frac{\lambda_{2n}^2 a^2 - 4}{2\lambda_{2n}^2} J_2^2 (\lambda_{2n} a), & m = n . \end{cases} \qquad (3.40)$$

These give the generalized coordinates of the J_o and J_2 modes as

$$\hat{A}_{on} = \Omega_{on} (f_1 f_2 + f_3 f_4) , \quad \hat{C}_{on} = \frac{\Omega_{on}}{2} (f_2^2 + f_4^2 - f_1^2 - f_3^2),$$

$$\hat{A}_{2n} = \Omega_{2n} (f_1 f_2 - f_3 f_4) , \quad \hat{C}_{2n} = \frac{\Omega_{2n}}{2} (f_2^2 + f_3^2 - f_1^2 - f_4^2) ,$$

$$\hat{B}_{2n} = \Omega_{2n} (f_1 f_4 + f_2 f_3) , \quad \hat{D}_{2n} = \Omega_{2n} (f_2 f_4 - f_1 f_3), \qquad (3.41)$$

where Ω_{on}, Ω_{2n} are constants defined in the Appendix.

The terms a_{oo}, $a_{o1} - \frac{a_{11}b_{oo}}{\alpha}$, $a_{o2} - a_{o2} - \frac{a_{11}b_{o1}}{\alpha} + \frac{a_{12}b_{oo}}{\alpha}$ $+ \frac{a_{11}b_{oo}b_{11}}{\alpha^2} + \frac{a_{22}b_{oo}^2}{2\alpha^2}$, B_{11} each contribute to the coefficient of ϵ. The coefficient of ϵ contains sin ωt, sin2ωt, sin 3ωt, cos ωt, cos 2ωt, and cos 3ωt. With this type of approximation it is assumed that only the first harmonic terms need vanish. The first harmonic terms from a_{oo} and B_{11} are

$$p_{11}^2 \left\{ \left[\frac{dX_1}{d\tau} - \upsilon\psi_1 - r \cos \theta - r \sin \theta \right] \cos \omega t - \left(\frac{d\psi_1}{d\tau} + \upsilon X_1\right) \sin \omega t \right\}$$

$$- \left(\alpha - \frac{2r}{h} \cos \theta + \frac{2r}{h} \sin \theta\right) \cos \omega t \,, \tag{3.42}$$

where p_{11}^2 r cos θ cos ωt corresponds to the translational motion u_1; p_{11}^2 r sin θ cos ωt corresponds to the translational motion u_2; α cos ωt corresponds to the translational motion u_3; $\frac{2r}{h}$ cos θ cos ωt corresponds to the rotational motion ω_2; $\frac{2r}{h}$ sin θ cos ωt corresponds to the rotational motion ω_1.

The first harmonic terms from $a_{o1} - \frac{a_{11}b_{oo}}{\alpha}$ are

$$p_{11} [\psi_{1r} X_{2r} - X_{1r}\psi_{2r} + \frac{1}{r^2} (\psi_{1\theta} X_{2\theta} - X_{1\theta}\psi_{2\theta}) - \lambda_{11}^2 (\zeta_{11}^2 - 1)(X_1\psi_2 - \psi_1 X_2)$$

$$- \lambda_{11}\zeta_{11} (\psi_1 X_{2z} - X_1\psi_{2z}) + \tfrac{1}{2} (\psi_1 X_{2zz} - X_1\psi_{2zz})] \cos \omega t$$

$$- p_{11} [X_{1r} X_{2r} + \psi_{1r}\psi_{2r} + \frac{1}{r^2} (X_{1\theta}X_{2\theta} + \psi_{1\theta}\psi_{2\theta}) + \lambda_{11}^2 (\zeta_{11}^2 - 1)$$

$$(X_1X_2 + \psi_1\psi_2) - \lambda_{11}\zeta_{11}(X_1X_{2z} + \psi_1\psi_{2z}) + \tfrac{1}{2} (X_1X_{2zz} + \psi_1\psi_{2zz})] \sin \omega t. \tag{3.43}$$

Similarly, there are first harmonic terms from $a_{o2} - \frac{a_{11}b_{o1} - a_{12}b_{oo}}{\alpha}$

$$+ \frac{2a_{11}b_{oo}b_{11} + a_{22}b_{oo}^2}{2\alpha^2}$$ which are lengthy expressions and are not written out here.

The equation obtained by setting the first harmonic terms of the coefficient of ϵ equal to zero is now satisfied in a Rayleigh-Ritz or averaged sense.

This is done by multiplying the equation by

$$J_1\ (\lambda_{11}r) \begin{Bmatrix} \cos\ \theta \\ \sin\ \theta \end{Bmatrix} r dr d\theta\ , \tag{3.44}$$

integrating over the free surface, $0 \leqslant r < a,\ 0 \leqslant \theta \leqslant 2\pi$, and using the known results

$$\int_o^a \int_o^{2\pi} r \begin{Bmatrix} \cos\ \theta \\ \sin\ \theta \end{Bmatrix} J_1(\lambda_{11}r)\ dr\ d\theta = 0, \tag{3.45}$$

$$\int_o^a \int_o^{2\pi} r^2\ \sin\ \theta \cos\ \theta\ J_1\ (\lambda_{11}r)\ dr d\theta = 0, \tag{3.46}$$

$$\int_o^a \int_o^{2\pi} r^2 \begin{Bmatrix} \sin^2\theta \\ \cos^2\theta \end{Bmatrix} J_1\ (\lambda_{11}r)\ dr d\theta = \frac{\pi a}{\lambda_{11}^2} J_1\ (\lambda_{11}a)\quad . \tag{3.47}$$

The contributions from $a_{oo} + B_1$ are

$$\int_o^a \int_o^{2\pi} (a_{oo} + B_{11})\ \cos\ \theta J_1\ (\lambda_{11}r)\ r dr d\theta = \pi p_{11}^2 \left\{ \frac{\lambda_{11}^2\ a^2 - 1}{2\lambda_{11}^2}\ J_1^2\ (\lambda_{11}\ a) \right.$$
$$\left. \left(\frac{df_2}{d\tau} - \upsilon\ f_1\right) - \left[\frac{a}{\lambda_{11}^2}\ J_1\ (\lambda_{11}\ a)\right] (u_1) \right\} \cos\ \omega t\ -\ \pi\ p_{11}^2 \left\{ \frac{\lambda_{11}^2 a^2 - 1}{2\lambda_{11}^2} J_1^2(\lambda_{11}a) \right.$$
$$\left. \left(\frac{df_1}{d\tau} + \upsilon f_2\right) \right\} \sin\ \omega t + \left[\frac{2\pi a}{\lambda_{11}^2 h}\ J_1\ (\lambda_{11}a)\right] (\omega_2)\ \cos\ \omega t\quad , \tag{3.48}$$

$$\int_0^a \int_0^{2\pi} (a_{oo} + B_{11}) \sin \theta J_1 (\lambda_{11} r) r dr d\theta = \pi p_{11}^2 \left\{ \frac{\lambda_{11}^2 a_{11}^2 - 1}{2\lambda_{11}^2} J_1^2 (\lambda_{11} a) \right.$$

$$\left. \left(\frac{df_4}{dt} - \upsilon f_3\right) - \left[\frac{\pi a}{\lambda_{11}^2} J_1(\lambda_{11} a) \right]^{(u_2)} \right\} \cos \omega t - \pi p_{11}^2 \left\{ \frac{\lambda_{11}^2 a^2 - 1}{2\lambda_{11}^2} J_1^2 (\lambda_{11} a) \right.$$

$$\left. \left(\frac{df_3}{d\tau} + \upsilon f_4\right) \right\} \sin \omega t - \left[\frac{2\pi a}{\lambda_{11}^2 h} J_1 (\lambda_{11} a) \right]^{(\omega_1)} \cos \omega t \quad . \qquad (3.49)$$

The contributions from $a_{o1} - \frac{a_{11} b_{oo}}{\alpha} = B_2$ are

$$\int_0^a \int_0^{2\pi} B_2 \cos \theta J_1 (\lambda_{11} r) r dr d\theta = \pi p_{11} [f_1 (f_j f_j) \hat{G}_1 +$$

$$f_4 (f_2 f_3 - f_1 f_4) \hat{G}_2] \cos \omega t + \pi p_{11} [f_2 (f_j f_j) \hat{G}_1 - f_3 (f_2 f_3 - f_1 f_4) \hat{G}_2] \sin \omega t, \qquad (3.50)$$

$$\int_0^a \int_0^{2\pi} B_2 \sin \theta J_1 (\lambda_{11} r) r dr d\theta = \pi p_{11} [f_3 (f_j f_j) \hat{G}_1$$

$$-f_2 (f_2 f_3 - f_1 f_4) \hat{G}_2] \cos \omega t + \pi p_{11} [f_4 (f_j f_j) \hat{G}_1 + f_1 (f_2 f_3 - f_1 f_4) \hat{G}_2] \sin \omega t, \qquad (3.51)$$

where $f_j f_j = f_1^2 + f_2^2 + f_3^2 + f_4^2$, and $\hat{G}_1$, $\hat{G}_2$ are constants defined in the Appendix.

The contributions from $B_3 = a_{o2} - \frac{(a_{11} b_{o1} - a_{12} b_{oo})}{\alpha} + \frac{2a_{11} b_{oo} b_{11} + a_{22} b_{oo}^2}{2\alpha^2}$ are

$$\int_0^a \int_0^{2\pi} B_3 \cos \theta J_1 (\lambda_{11} r) r dr d\theta = - \frac{p_{11}^2}{4} [f_1 (f_j f_j) \hat{H}_1$$

$$+ f_4 \ (f_2 f_3 - f_1 f_4)\hat{H}_2] \cos \omega t - \frac{p_{11}^2}{4} [f_2(f_j f_j)\hat{H}_1 - f_3(f_2 f_3 - f_1 f_4)\hat{H}_2] \sin \omega t \tag{3.52}$$

$$\int_0^a \int_0^{2\pi} B_3 \sin\theta \, J_1 \, (\lambda_{11} r) \, r dr d\theta = - \frac{p_{11}^2}{4} [f_3(f_j f_j) \, \hat{H}_1 - f_2 \, (f_2 f_3$$

$$- f_1 f_4) \, \hat{H}_2] \cos \omega t - \frac{p_{11}^2}{4} [f_4 \, (f_j f_j)\hat{H}_1 + f_1 \, (f_2 f_3 - f_1 f_4)\hat{H}_2] \sin \omega t, \tag{3.53}$$

where $\hat{H}_1$ and $\hat{H}_2$ are constants defined in the Appendix.

From the Rayleigh-Ritz process two ordinary differential equations are obtained. Setting equal to zero the coefficients of sin ωt and cos ωt in each of these equations results in four first order,non-linear, ordinary differential equations.

This system is

$$\frac{df_i}{d\tau} = G_i \ (f_i, \ f_2, \ f_3, \ f_4), \ i = 1, \ 2, \ 3, \ 4 \tag{3.54}$$

where

$$G_1 = - H_{,2} \quad , \qquad G_3 = - H_{,4} \quad ,$$
$$G_2 = H_{,1} \quad , \qquad G_4 = H_{,3} \quad , \tag{3.55}$$

and

$$H = (F_1 + A_2) \ f_1 + (U_2 + A_2) \ f_4 + \tfrac{1}{2} \upsilon \ f_j f_j$$
$$+ \tfrac{1}{4} K_1 \ (f_j f_j)^2 - \tfrac{1}{2} K_2 \ (f_2 f_3 - f_1 f_4)^2 \quad , \tag{3.56}$$

where

$$H_{,i} = \frac{\partial H}{\partial f_i} \quad . \tag{3.57}$$

The constants F_1, K_1, K_2, A_1, A_2, U_2 are defined in the Appendix by Equations A.18, A.19, A.20, and A.21.

A steady-state harmonic solution to the boundary value problem is given by the roots of the four equations

$$G_i\ (f_1, f_2, f_3, f_4) = 0 \quad , i = 1, 2, 3, 4 \,. \tag{3.58}$$

The roots of Equation 3.58 are functions of f_i where the f_i are independent of the time τ. The form of Equations 3.54 is similar to the equations derived by Miles (7) for the undamped spherical pendulum.

There are two solutions to Equations 3.58. The first, called planar motion, is

$$f_1 = \gamma, \quad f_3 = \gamma Q, \quad f_2 = f_4 = 0, \tag{3.59}$$

where γ is a parameter independent of time. The transformed frequency is

$$\upsilon = P\,\gamma^{-1} - K_1\,R\,\gamma^2 \quad , \tag{3.60}$$

where

$$P = -\,(F_1 + A_2), \quad Q = \frac{A_1 + U_2}{F_1 + A_2}\,, \; R = 1 + Q^2 \quad . \tag{3.61}$$

The second solution to Equation 3.58, called non-planar motion, is

$$f_1 = \gamma\ , \quad f_2 = -\,(\gamma^2 - \frac{P\gamma^{-1}}{K_2\,R})^{\frac{1}{2}}Q, \quad f_3 = \gamma Q, \quad f_4^2 = \gamma^2 - \frac{P\gamma^{-1}}{K_2 R} \tag{3.62}$$

with

$$\upsilon = -\,\gamma^{-1}\,(K_3 + \frac{A_2K_1}{K_2}) + RK_4\,\gamma^2 \quad , \tag{3.63}$$

$$K_3 = \frac{K_1}{K_2}\,F_1, \quad K_4 = K_2 - 2K_1 \,. \tag{3.64}$$

It is seen that the non-planar solution is real and hence exists for $\gamma > 0$, when $\gamma^3 - \frac{P}{K_2R} > 0$; and for $\gamma < 0$, when $\gamma^3 - \frac{P}{K_2R} < 0$.

The names planar and non-planar motion are used in analogy with Miles terminology for the spherical pendulum. It is not to be implied that the motion of the free-surface is necessarily described by the names given to the two solutions of Equation 3.58.

IV. STABILITY OF THE STEADY-STATE HARMONIC SOLUTION

To determine the stability of the motion corresponding to a given steady-state solution consider the perturbed solution

$$f_i(\tau) = f_i^{(0)} + c_i \; e^{\lambda\tau} \quad , \quad |c_i| \ll 1, \quad i = 1, 2, 3, 4 \; . \tag{4.1}$$

The $f_i^{(0)}$ are constants corresponding to the steady-state amplitudes of the harmonic solutions of Equation 3.54. The corresponding steady-state solution will be stable if Re $(\lambda) \leqslant 0$ and unstable if Re $(\lambda) > 0$.

Substitute Equation 4.1 into Equation 3.54, neglect products of the c_i's, and use the fact that the $f_i^{(0)}$ are solutions of Equation 3.54;the following set of homogeneous algebraic equations are obtained:

$$\begin{bmatrix} d_{11} + \lambda & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} - \lambda & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} + \lambda & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} - \lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \tag{4.2}$$

where

$$d_{11} = 2K_1 f_1^{(0)} \; f_2^{(0)} + K_2 f_3^{(0)} \; f_4^{(0)} \quad ,$$

$$d_{12} = \upsilon + K_1 \; (f_j^{(0)} \; f_j^{(0)}) + 2K_1 f_2^{(0)^2} - K_2 f_3^{(0)^2} \quad ,$$

$$d_{13} = 2K_1 f_2^{(0)} f_3^{(0)} + K_2 \; [f_1^{(0)} f_4^{(0)} - 2f_2^{(0)} \; f_3^{(0)}] \quad ,$$

$$d_{14} = 2K_1 \; f_2^{(0)} \; f_4^{(0)} + K_2 \; f_1^{(0)} \; f_3^{(0)} \quad ,$$

$$d_{21} = \upsilon + K_1 \; (f_j^{(0)} \; f_j^{(0)}) + 2K_1 f_1^{(0)^2} - K_2 f_4^{(0)^2} \quad ,$$

$$d_{22} = d_{11} \quad ,$$

$$d_{23} = 2K_1 f_1^{(0)} f_3^{(0)} + K_2 f_2^{(0)} f_4^{(0)} ,$$

$$d_{24} = 2K_1 f_1^{(0)} f_4^{(0)} + K_2 [f_2^{(0)} f_3^{(0)} - 2f_1^{(0)} f_4^{(0)}] ,$$

$$d_{31} = d_{24} ,$$

$$d_{32} = d_{14} ,$$

$$d_{33} = 2K_1 f_3^{(0)} f_4^{(0)} + K_2 f_1^{(0)} f_2^{(0)} ,$$

$$d_{34} = \upsilon + K_1 [f_j^{(0)} f_j^{(0)}] + 2K_1 f_4^{(0)2} - K_2 f_1^{(0)2} ,$$

$$d_{41} = d_{23} ,$$

$$d_{42} = d_{13} ,$$

$$d_{43} = \upsilon + K_1 (f_j^{(0)} f_j^{(0)}) + 2K_1 f_3^{(0)2} - K_2 f_2^{(0)2} ,$$

$$d_{44} = d_{33} .$$

Equation 4.2 will have nontrivial solutions only if the determinant of the coefficient matrix is zero. This gives an equation for the allowable values of λ.

A. Stability of Planar Motion

Substituting Equation 3.59 into the expressions for the d_{ij}'s and expanding the determinant of the coefficient matrix in Equation 4.2, one obtains

$$\lambda^4 + \lambda^2 (M_1' + M_2' + 2M_3) + M_1' M_2' + 4K_1^2 \gamma^4 Q^2 M_4 - K_2^2 Q^2 \gamma^4 M_5 + M_3^2 = 0, \tag{4.3}$$

where

$$M_1' = \begin{vmatrix} \upsilon + \gamma^2 (K_1 R - K_2 Q) & 0 \\ 0 & \upsilon + K_1 \gamma^2 (3 + Q^2) \end{vmatrix} ,$$

$$M_2' = \begin{vmatrix} \upsilon + K_1\gamma^2\ (1 + 3Q^2) & 0 \\ 0 & \upsilon + \gamma^2\ (K_1R - K_2) \end{vmatrix},$$

$$M_3 = \begin{vmatrix} K_2\gamma^2Q & 0 \\ 0 & 2K_1\gamma^2Q \end{vmatrix},$$

$$M_4 = \begin{vmatrix} 0 & \upsilon + \gamma^2\ (K_1\ R - K_2) \\ \upsilon + \gamma^2\ (K_1R - K_2Q) & 0 \end{vmatrix},$$

$$M_5 = \begin{vmatrix} \upsilon + K_1\gamma^2\ (3 + Q^2) & 0 \\ 0 & \upsilon + K_1\gamma^2(1 + 3Q^2) \end{vmatrix}.$$

The boundary between stable and unstable planar motion corresponds to $\lambda = 0$. Set $\lambda = 0$ in Equation 4.3 and substitute for υ from Equation 3.60:

$$\gamma^{-4} \quad N_4\ \gamma^{12} - N_3\ P\ \gamma^9 + N_2P^2\gamma^6 + P^3N_1\ \gamma^3 + P^4 \quad = 0, \tag{4.4}$$

where

$$N_1 = -K_4 + 2K_1Q^2 - K_2Q\ ,$$
$$N_2 = K_2K_4Q - K_2Q^2\ (K_2 + 2K_1) - 2K_1\ K_2(1 + Q^3)\ ,$$
$$N_3 = 2K_1K_2^2\ (Q^4 - Q^3 + Q^2 - Q)\ ,$$
$$N_4 = -r\ K_2^2\ K_1^2\ Q^2\ (Q^2 + 1)\ . \tag{4.5}$$

One possible solution to Equation 4.4 is $\gamma = \pm\ \infty$. Since γ is actually an amplitude this would correspond to unstable motion. Letting $\sigma = \gamma^3$, the other possible solution to Equation 4.4 is

$$N_4\sigma^4 - N_3P\sigma^3 + N_2 P^2 \sigma^2 + P^3 N_1\sigma + P^4 = 0. \qquad (4.5)$$

This equation can then be solved for $\sigma = \gamma^3$. The solution of Equation 4.5 for various perturbations is given in Chapter V.

B. Stability of Non-Planar Motion

Substitute Equation 3.62 in the expressions for the d_{ij}'s:

$$d_{11} = \gamma Q K_4 L$$

$$d_{12} = \upsilon + \gamma^2 (2K_1R - Q^2K_4) - \frac{K_1P\gamma^{-1}}{K_2R} (1 + 3Q^2)$$

$$d_{13} = L\gamma (K_4 Q^2 + K_2R)$$

$$d_{14} = Q (K_4\gamma^2 + \frac{2K_1P\gamma^{-1}}{K_2R})$$

$$d_{21} = \upsilon + \gamma^2 (2K_1R - K_4) + P\gamma^{-1} (\frac{1}{R} - \frac{K_1}{K_2})$$

$$d_{22} = d_{11}$$

$$d_{23} = - Q (K_4\gamma^2 - \frac{P\gamma^{-1}}{R})$$

$$d_{24} = -\gamma L (K_4 + K_2R)$$

$$d_{31} = d_{24}$$

$$d_{32} = d_{14}$$

$$d_{33} = -d_{11}$$

$$d_{34} = \upsilon - \gamma^2 (2K_1R - K_4) - \frac{K_1P\gamma^{-1}}{K_2R} (3 + Q^2)$$

$$d_{41} = d_{23}$$

$$d_{42} = d_{13}$$

$$d_{43} = \upsilon + \gamma^2 (2K_1R - K_4Q^2) - P\gamma^{-1} (\frac{K_1}{K_2} + \frac{Q^2}{R})$$

$$d_{44} = d_{33}$$

Substitute the value of υ given by Equation 3.63 into the expressions for d_{12}, d_{21}, d_{34}, d_{43}:

$$d_{12} = \gamma^2 \; (2K_1R + K_4) - \frac{2K_1PQ^2\gamma^{-1}}{K_2R}$$

$$d_{21} = \gamma^2 \; (2K_1R + K_4Q^2) + \frac{P\gamma^{-1}}{R}$$

$$d_{34} = \gamma^2 \; (2K_1R + K_4Q^2) - \frac{2K_1P\;\gamma^{-1}}{K_2R}$$

$$d_{43} = \gamma^2 \; (2K_1R + K_4) + \frac{PQ^2\gamma^{-1}}{R}$$

Substitute the above values of the d_{ij}'s into Equation 4.2 and set the determinant of the coefficient matrix equal to zero to give a fourth degree polynomial for the determination of the parameter λ, such that Equation 4.2 has non-trivial solutions. Stability of the steady-state nonplanar solution is determined by examining the roots of this quartic equation. The regions of stable and unstable motion are given in Chapter V for various perturbations.

V. NUMERICAL EXAMPLE

To compare results obtained here with those obtained by Hutton, the following values of the parameters are used:

$$a = \text{tank radius} = 5.938 \text{ inches},$$
$$h = \text{water depth} = 8.907 \text{ inches},$$
$$\lambda_{11}a = 1.84119 \, ,$$
$$J_1 (\lambda_{11}a) = .581865 \quad . \tag{5.1}$$

Then

$$F_1 = 8.53992, \; \zeta_{11} = 0.99205 \; ,$$
$$p_{11} = 10.897 \text{ rad/sec} = 1.734 \text{ cps},$$
$$A_2 = -.6389 \; ,$$
$$A_1 = - A_2 \; , \; U_2 = F_1 \tag{5.2}$$

To evaluate K_1 and K_2 only the first five terms in the infinite series defined by $\hat{G}_1$ and $\hat{G}_2$ are used to approximate the series. In the calculation of $\hat{G}_1$ and $\hat{G}_2$ the last three terms are about one percent of the first two terms. Thus

$$K_1 = .4853 \times 10^{-5} \; , \; K_3 = -3.0235,$$
$$K_2 = .13707 \times 10^{-4} \; , \; K_4 = 4.0010 \times 10^{-6}. \tag{5.3}$$

A. Planar Motion

Equation 3.60 gives the transformed frequency υ in terms of the parameter γ. The coefficients in this equation depend on the perturbations given to the liquid-tank system. Thus

$$\upsilon = - (F_1 + A_2)\gamma^{-1} - K_1 \left[1 + \left(\frac{A_1 + U_2}{F_1 + A_2}\right)^2 \right] \gamma^2 \quad . \tag{5.4}$$

Equation 4.4 gives the values of γ which separate stable and unstable regions. The coefficients in this equation also depend on the perturbations given to the liquid-tank system.

Case I: Consider $\omega_1 = \omega_2 = u_2 = 0$, $u_1 = \epsilon \cos \omega t$, which is the case considered by Hutton.

Then

$$\upsilon = -8.5399\ \gamma^{-1} - .4853 \times 10^{-5}\gamma^2 \tag{5.5}$$

The motion is unstable for

$$-.1337 < \upsilon < .06459\ , \tag{5.6}$$

where $\gamma = 95.82$ when $\upsilon = -.1337$ and

$\gamma = -85.41$ when $\upsilon = .06459$.

This agrees with the result obtained by Hutton (3).

Case II: $\omega_1 = u_1 = u_2 = 0$, $\omega_2 = \frac{\epsilon}{h} \cos \omega t$;

$$\upsilon = .6389\ \gamma^{-1} - .4853 \times 10^{-5}\ \gamma^2 \tag{5.7}$$

The motion is unstable for

$$-.0237 < \upsilon < .0115 \tag{5.8}$$

where $\gamma = -40.377$ when $\upsilon = -.0237$ and

$\gamma = 35.988$ when $\upsilon = .0115$.

Case III: $u_1 = u_2 = 0$, $\omega_1 = \omega_2 = \frac{\epsilon}{h}$ cos ωt ;

$$\upsilon = .6389\ \gamma^{-1} - .9706 \times 10^{-6}\ \gamma^2\ . \tag{5.9}$$

The motion is unstable for

$$-.1863 < \upsilon < .1934\ , \tag{5.10}$$

where $\gamma = -3.43$ when $\upsilon = -.1863$ and

$\gamma = 3.30$ when $\upsilon = .1934$.

Case IV: $\omega_2 = u_2 = 0$, $\omega_1 = \frac{\epsilon}{h} \cos \omega t$, $u_1 = \epsilon \cos \omega t$;

$$\upsilon = -8.5399\gamma^{-1} - .488 \times 10^{-5}\gamma^2 . \qquad (5.11)$$

The motion is unstable for

$$-5.735 < \upsilon < 5.662 \qquad (5.12)$$

where $\gamma = -1.508$ when $\upsilon = -5.735$ and

$\gamma = 1.489$ when $\upsilon = 5.662$.

Case V: $\omega_1 = u_2 = 0$, $\omega_2 = \frac{\epsilon}{h}\cos\omega t$, $u_1 = \epsilon\cos\omega t$;

$$\upsilon = -7.901\gamma^{-1} - .4853 \times 10^{-5}\gamma^2 . \qquad (5.13)$$

The motion is unstable for

$$-.1269 < \upsilon < .06133, \qquad (5.14)$$

where $\gamma = 93.37$ when $\upsilon = -.1269$ and

$\gamma = -83.23$ when $\upsilon = .06133$.

Case VI: $u_1 = u_2 = \epsilon\cos\omega t$, $\omega_1 = \omega_2 = \frac{\epsilon}{h}\cos\omega t$;

$$\upsilon = -7.9010\,\gamma^{-1} - 1.144 \times 10^{-5}\gamma^2 \qquad (5.15)$$

The motion is unstable for

$$-4.773 < \upsilon < 4.834 \qquad (5.16)$$

where $\gamma = 1.655$ when $\upsilon = -4.773$ and

$\gamma = -1.634$ when $\upsilon = 4.834$.

Case VII: $\omega_2 = u_1 = 0$, $\omega_1 = \frac{\epsilon}{h}\cos\omega t$, $u_2 = \epsilon\cos\omega t$.

The results are the same as in Case V by symmetry.

Case VIII: $\omega_1 = u_1 = 0$, $\omega_2 = \frac{\epsilon}{h}\cos\omega t$, $u_2 = \epsilon\cos\omega t$.

The results are the same as Case IV by symmetry.

Cases I and II, according to (2), should not differ. However, it is seen that the unstable region for Case II is much smaller than the unstable region for Case I, indicating that, at least for stability considerations, rotational oscillations about x_2 are not equivalent to translational oscillations in the x_1 direction. Also note from Case V, that the unstable region is slightly smaller but nearly equivalent to, the unstable region in Case I. Here the rotational motion and translational motion are taking place in the same plane. The rotational motion thus has a much smaller effect on the free-surface motion than the translational motion, even to the extent that the combination is essentially not different than the situation for translation alone.

Cases IV and VIII consider the combination of rotational motion and translational motion in planes perpendicular to one another. From Equation 5.12 the region of unstable motion in these cases is much greater than any of the other cases considered.

B. Non Planar Motion

Case I: $\omega_1 = \omega_2 = u_2 = 0, \quad u_1 = \epsilon \cos \omega t$;

The quartic equation for this case is

$$\lambda^4 + (M_3 + M_4)\,\lambda^2 + M_5\,M_6 = 0 \,, \tag{5.17}$$

where

$$M_3 = \begin{vmatrix} d_{34} & d_{13} \\ d_{24} & d_{12} \end{vmatrix} \,, \qquad M_4 = \begin{vmatrix} d_{12} & d_{13} \\ d_{24} & d_{21} \end{vmatrix} \tag{5.18}$$

$$M_5 = \begin{vmatrix} d_{13} & d_{12} \\ d_{12} & d_{13} \end{vmatrix} \,, \qquad M_6 = \begin{vmatrix} d_{24} & d_{21} \\ d_{34} & d_{24} \end{vmatrix} \,, \tag{5.19}$$

and the d_{ij}'s are defined in Equations 4.6 and 4.7.

The transformed frequency υ is

$$\upsilon = -\ (F_1 + A_2)\ \frac{K_1}{K_2}\ \gamma^{-1} + RK_4\ \gamma^2 \quad . \tag{5.20}$$

For this case $A_2 = 0$, $R = 1$, $F_1 = 8.53992$, and

$$\upsilon = -3.0235\ \gamma^{-1} + 4.001 \times 10^{-6}\ \gamma^2 \quad .$$

The genrralized coordinates f_i for this case are

$$f_1 = \gamma\ ,\quad f_2 = f_4 = 0,\quad f_4^2 = \gamma^2 + \frac{F_1}{K_2}\ \gamma^{-1}\ . \tag{5.22}$$

Since f_4 must be real, $f_4^2 > 0$. Thus γ cannot be in the range

$$(-85.41)^3 = -\frac{F_1}{K_2} < \gamma^3 < 0\ . \tag{5.23}$$

Evaluate $M_3 + M_4$ and $M_5\ M_6$:

$$M_3 + M_4 = 7.5148 \times 10^{-10}\ \gamma\ (\gamma^3 + 3.5699 \times 10^5)\quad , \tag{5.24}$$

$$M_5 M_6 = 2.5677 \times 10^{-15}\ \gamma^{-1}\ (\gamma^3 + 6.2305 \times 10^5)(\gamma^3 + 3.7784 \times 10^5)^1\ , \tag{5.25}$$

and consider Equation 5.17 as a quandratic in λ^2:

$$(\lambda^2)^2 + (M_3 + M_4)(\lambda^2) + M_5 M_6 = 0. \tag{5.26}$$

If $\gamma^3 < 6.2305 \times 10^5$ or $\gamma < -85.41$, $M_5 + M_4 > 0$, and $M_5 M_6 < 0$. Thus by Descarte's rule of signs there is one positive real root of Equation 5.26. This corresponds to a region of unstable motion. The case $-6.2305 \times 10^{-5} < \gamma^3 < 0$ is dealt with in Equation 5.23.

To examine stability for $\gamma > 0$, compute the discriminant of Equation 5.26:

$$\Delta = 52.99 \times 10^{-20} \gamma^9 + 361.4 \times 10^{14}\ \gamma^6 + 5.72.5 \times 10^{-10} \gamma^3 - 241.75 \times 10^{-5}. \tag{5.27}$$

Set $\Delta = 0$, multiply by 10^{20} and let $\sigma = \gamma^3$ to obtain

$$52.99\ \sigma^3 + 36.14 \times 10^5\ \sigma^2 + 572.5 \times 10^{10}\ \sigma - 241.75 \times 10^{15} = 0. \tag{5.28}$$

The only positive real root of Equation 5.28 corresponds to $\gamma = 64.47$. Thus for $0 < \gamma < 64.47$ the roots of Equation 5.26 are complex. If $\gamma > 0$, $M_3 + M_4$ and $M_5 M_6$ are both positive. Thus by Descarte's rule of signs there is no positive real roots of Equation 5.17. Replace λ^2 by $-\ \alpha$ in Equation 5.26 to obtain

$$\alpha^2 - (M_3 + M_4)\alpha + M_5 M_6 = 0. \tag{5.29}$$

Thus by Descarte's rule of sign there exist either two positive real roots or no positive real roots of Equation 5.29. If $\gamma > 64.47$ the roots of Equation 5.29 are real. Hence for $\gamma > 64.47$ Re $(\lambda) \leqslant 0$, and this corresponds to stable motion.

In summary the steady-state nonplanar motion is stable for

$$64.47 < \gamma < \infty \quad , \ -\ 0.03027 < \upsilon < \infty \quad ; \tag{5.30}$$

and unstable for

$$0 < \gamma\ 64.47 \ , \quad -\ \infty \ < \upsilon < -0.03027 \ , \tag{5.31}$$

$$-\ \infty < \gamma < -85.41 \ , \quad 0.06459 < \upsilon < \infty \quad . \tag{5.32}$$

The solution does not exist when

$$-85.41 < \gamma < 0 \quad , \ -\ \infty < \upsilon < 0.06459 \quad ,$$

since then $f_4^2 < 0$.

This motion is stable for a small range of driving frequencies ω which includes the first natural frequency p_{11}. This can be seen from the range of transformed frequencies υ given by Equation 5.30.

The results have agreed with those obtained by Hutton (3).

Case II: $\omega_1 = u_1 = u_2 = 0, \quad \omega_2 = \frac{\epsilon}{h} \cos \omega t$.

Proceeding as in Case I, with $A_1 = U_2 = F_1 = 0$ and $A_2 = -.6389$, the quartic equation to be examined is

$$\lambda^4 + (M_3 + M_4)\lambda^2 + M_5 M_6 = 0, \tag{5.33}$$

where M_3, M_4, M_5, and M_6 are defined by Equations 5.18 and 5.19. Here the d_{ij}'s have different values than in Case I.

The transformed frequency υ is

$$\upsilon = - A_2 \frac{K_1}{K_2} \gamma^{-1} + K_4 \gamma^2 \quad , \tag{5.34}$$

or

$$\upsilon = .2262 \gamma^{-1} + 4.001 \times 10^{-6} \gamma^2 \quad . \tag{5.35}$$

The generalized coordinates for this case are

$$f_1 = \gamma, \; f_2 = f_3 = 0 \; , \; f_4^2 = \gamma^2 + \frac{A_2}{K_2} \gamma^{-1} \tag{5.36}$$

Since f_4 must be real, $f_4^2 > 0$. Thus γ cannot lie in the range

$$0 < \gamma^3 < - \frac{A_2}{K_2} , \tag{5.37}$$

where $A_2 < 0$. This gives

$$0 < \gamma < 35.8 \; , \; .01047 < \upsilon < \infty \quad . \tag{5.38}$$

Regard Equation 5.33 as a quadratic in λ^2 and examine the roots of

$$(\lambda^2)^2 + (M_3 + M_4)(\lambda^2) + M_5 M_6 = 0. \tag{5.39}$$

This gives the following regions of unstable and stable motion respectively:

$$- \infty < \gamma < 0 \qquad - \infty < \upsilon < \infty \quad , \tag{5.40}$$

$$35.8 < \gamma < \infty \qquad .01047 < \upsilon < \infty \quad , \tag{5.41}$$

This motion is not stable about the first natural frequency as opposed to the situation in Case I.

Case III: $\omega_1 = 0,\ u_2 = 0,\ \omega_2 = \frac{\epsilon}{h}\ \cos 2t,\ u_1 = \epsilon \cos \omega t$.

The quartic equation for this case is Equation 5.17 with M_3, M_4, M_5 and M_6 given by Equations 5.18 and 5.19. The d_{ij}'s are evaluated from Equations 4.6 and 4.7 with $A_1 = U_2 = 0$, $Q = 0$, $R = 1$.

The transformed frequency υ is

$$\upsilon = -2.797\ \gamma^{-1} + 4.001 \times 10^{-6}\ \gamma^2\ , \tag{5.42}$$

and the generalized coordinates are

$$f_1 = \gamma,\quad f_2 = f_3 = 0,\quad f_4^2 = \gamma^2 + \frac{(F_1 + A_2)}{K_2}\gamma^{-1}\ . \tag{5.43}$$

Since f_4 is real, $f_4^2 > 0$. Thus γ cannot lie in the range

$$-\left(\frac{F_1 + A_2}{K_2}\right) < \gamma^3 < 0\ . \tag{5.44}$$

Thus the solution does not exist for

$$-75.6 < \gamma < 0\ ,\quad -\infty < \upsilon < .0597 \tag{5.45}$$

The regions of unstable motion are

$$-\infty < \gamma < -8.31\ ,\quad .0613 < \upsilon < \infty\ , \tag{5.46}$$

$$0 < \gamma < 62.58 \qquad -\infty < \upsilon < -0.0399. \tag{5.47}$$

The region of stable motion is

$$62.58 < \gamma < \infty\ ,\quad -0.0399 < \upsilon < \infty\ . \tag{5.48}$$

The similarity between Cases I and III is to be noted as regards the stability of the non planar motion. Thus the rotational motion has a much smaller effect on the stability of the free-surface motion than does the translational motion, even to the extent that the combination is essentially not different than the situation for translation alone.

In all cases considered for the stability of non planar motion there is a region of υ for which it is possible to have both stable planar motion and stable non planar motion.

Case IV: $u_2 = \omega_2 = 0, \quad \omega_1 = \frac{\epsilon}{h} \cos \omega t, \; u_1 = \epsilon \cos \omega t.$

The quartic equation for the determination of the allowable values of λ is

$$\lambda^4 + M\lambda^2 + N = 0 \; , \tag{5.49}$$

where

$$M = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} + \begin{vmatrix} d_{11} & d_{12} \\ d_{32} & d_{33} \end{vmatrix} + \begin{vmatrix} d_{11} & d_{12} \\ d_{43} & d_{44} \end{vmatrix} + \begin{vmatrix} d_{22} & d_{23} \\ d_{32} & d_{33} \end{vmatrix} + \begin{vmatrix} d_{22} & d_{23} \\ d_{43} & d_{44} \end{vmatrix} + \begin{vmatrix} d_{33} & d_{34} \\ d_{43} & d_{44} \end{vmatrix} , \tag{5.50}$$

and N is the determinant of the coefficient matrix in Equation 4.2 with $\lambda = 0$.

The transformed frequency υ is

$$\upsilon = -3.0235 \; \gamma^{-1} + 4.024 \times 10^{-6} \; \gamma^2, \tag{5.51}$$

and the generalized coordinates are

$$f_1 = \gamma \;\; , \; f_2 = -.0748 \; [\gamma^2 + 6.24 \times 10^5 \; \gamma^{-1}]^{\frac{1}{2}}, \tag{5.52}$$

$$f_3 = .0748 \; \gamma \;\; , \; f_4^2 = \gamma^2 + 6.24 \times 10^5 \; \gamma^{-1} \; . \tag{5.53}$$

Since f_4 is real, $f_4^2 > 0$. Thus γ cannot lie in the range

$$-85.41 < \gamma < 0 \;\; , \quad .06459 < \upsilon < \infty \; . \tag{5.54}$$

The motion is unstable for

$$-\infty < \gamma < -85.41 \;\; , \quad .06459 < \upsilon < \infty \tag{5.55}$$

$$59.1 < \gamma < \infty \;\; , \quad -.0371 < \upsilon < \infty \; . \tag{5.56}$$

The motion is stable for

$$31.6 < \gamma < 59.1, \quad -.0915 < \upsilon < -.0371 . \tag{5.57}$$

This case, as in the planar motion, has the largest region of unstable motion.

VI. CONCLUSION

This paper considers the irrotational motion of an incompressible, inviscid fluid contained in a tank which is only partially filled. The tank is subjected to both transverse and rotational vibrations whose frequencies are near the first natural frequency of small free-surface oscillations. The analysis was performed retaining higher-order terms in the free-surface dynamic and kinematic boundary conditions. The theoretical investigation predicts the forcing frequency ranges, for various combinations of rotational and translational motion, over which there are stable steady-state harmonic planar and nonplanar motions. The least stable case occurs when a combination of motions occurs in planes perpendicular to one another. This substantiates the findings of Hutton in that the mechanism that apparently causes sloshing in the unstable regions is a nonlinear coupling of the fluid motions parallel with and perpendicular to the plane in which the translational motion is taking place.

VII. APPENDIX

In the sloshing problem being considered, the free-surface height η is an unknown which may be eliminated by replacing Equations 3.24 and 3.25 by a single equation which does not contain η. Solving Equation 3.24 for η, one obtains

$$\alpha\eta = \Gamma(r,\theta,\eta\ (r,\theta,t),\ t)\ , \tag{A.1}$$

where

$$\Gamma\ (r,\theta,\eta\ (r,\theta,t),\ t) = -\ [\alpha - r\mathring{\omega}_1 \sin\theta + r\ \mathring{\omega}_2 \cos\theta - \omega_2 u_1 + \omega_1 u_2 + \omega_3\omega_2\ r \sin\theta + \omega_3\omega_1\ r\cos\theta]/\ (\omega_1^2 + \omega_2^2) + \quad [\alpha - r\ \mathring{\omega}_1 \sin\theta + r\ \mathring{\omega}_2 \cos\theta - \omega_2 u_1 + \omega_1 u_2 + \omega_3\omega_2\ r\sin\theta + \omega_3\omega_1\ r\cos\theta]^2 - 2(\omega_1^2 + \omega_2^2)[\quad (\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_z^2) - \psi_t + \mathring{u}_1\ r\cos\theta + u_2 r \sin\theta + 2r^2\ (\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta) + 2\ (\psi_z + u_3)(r\omega_1 \sin\theta - r\omega_2\cos\theta) + \omega_3 r\ (u_1 \sin\theta - u_2\cos\theta) - 3\ r^2\ \omega_1\omega_2 \sin\theta\cos\theta + \omega_3\ \psi_\theta]^{\frac{1}{2}}\ /\ (\omega_1^2 + \omega_2^2)\ ,$$

and the positive sign in the quadratic formula is chosen so that $\alpha\eta$ remains finite if ω_1, ω_2, $\mathring{\omega}_1$, $\mathring{\omega}_2$ are all equal to zero. Equation A.1 can now be used to obtain the partial derivatives, η_t, η_r, and η_θ. Thus

$$\begin{aligned}
(\alpha - \Gamma_\eta)\eta_t &= \Gamma_t\ , \\
(\alpha - \Gamma_\eta)\ \eta_r &= \Gamma_r\ , \\
(\alpha - \Gamma_\eta)\ \eta_\theta &= \Gamma_\theta\ ,
\end{aligned} \tag{A.2}$$

Multiply Equation 3.25 by $(\alpha - \Gamma_\eta)$ and use Equation A.2:

$$(-\psi_z - u_3 + 2r\omega_2 \cos\theta - 2r\omega_1 \sin\theta)\ (\alpha - \Gamma_\eta) = \Gamma_t - \Gamma_r\psi_r - (\frac{1}{r^3}\ \psi_\theta + \omega_3)\Gamma_\theta\ ,\ \text{on } z = \eta\ . \tag{A.3}$$

The potential functions in Equation A.3 are evaluated on $z = \eta$, and thus Equation A.3 depends upon η implicitly. Equation 3.24, however, depends upon η both explicitly and implicitly. The wave height η is eliminated between these two equations by first expanding the functions defined by these two equations in Taylor series about $z = 0$.

Introduce the notation

$$\xi_{r^m \theta^n z^p t^s} = \frac{\partial^k \psi}{\partial r^m \, \partial \theta^n \, \partial z^p \, \partial t^s} \Big|_{z=0} \quad , \tag{A.4}$$

where $m + n + p + s = k$.

The Taylor series of the function defined by Equations A.3 and 3.24 can then be written in the form

$$a_o + a_1 \eta + \frac{a_2}{2!} \eta^2 + \frac{a_3}{3!} \eta^3 + \dots \quad = 0 \quad , \tag{A.5}$$

and

$$b_o + b_1 \eta + \frac{b_2}{2!} \eta^2 + \frac{b_3}{3!} \eta^3 + \dots \quad = 0 \quad , \tag{A.6}$$

respectively, where

$$\begin{aligned} b_o = & - b_{oo} - b_{o1} - \mathring{u}_2 r \sin\theta - 2r^2 (\omega_1^2 \cos^2\theta + \omega_2^2 \sin^2\theta) \\ & - 2\xi_z (r\omega_1 \sin\theta - r\omega_2 \cos\theta + u_3) + 3r^2 \omega_1 \omega_2 \sin\theta \cos\theta \\ & - 2 u_3 r \omega_1 \sin\theta + 2u_3 \omega_2 r \cos\theta - u_1 \omega_3 r \sin\theta + \omega_3 u_2 r \cos\theta \\ & - \omega_3 \xi_\theta , \end{aligned}$$

$$\begin{aligned} b_1 = & - b_{11} - b_{12} - \alpha + r \mathring{\omega}_1 \sin\theta - \mathring{\omega}_2 r \cos\theta - 2\xi_{zz} (r\omega_1 \sin\theta \\ & - r\omega_2 \cos\theta + u_3) + \omega_2 u_1 - \omega_3 \omega_2 \sin\theta - \omega_3 \xi_{\theta z} \\ & - \omega_3 \omega_1 r \cos\theta - \omega_1 u_2 \quad , \end{aligned}$$

$$\begin{aligned} b_2 = & - b_{22} - b_{23} - (\omega_1^2 + \omega_2^2) - 2\xi_{zz} (r\omega_1 \sin\theta - r\omega_2 \cos\theta + u_3) \\ & - \omega_3 \xi_{\theta zz} \quad , \end{aligned}$$

$$b_{oo} = -\xi_t + \mathring{u}_1 r\cos\theta \;, \quad b_{o1} = \tfrac{1}{2}\left(\xi_r^2 + \frac{\xi_\theta^2}{r^2} + \xi_t^2\right)$$

$$b_{11} = \frac{\partial b_{oo}}{\partial z} \;, \; b_{12} = \frac{\partial b_{o1}}{\partial z} \;, \; \dot{b}_{oo} = \frac{\partial b_{oo}}{\partial t} \;,$$

$$b_{m+1,n+1} = \frac{\partial b_{mn}}{\partial z} \;, \; m,n = 1, 2, 3 \ldots \qquad (A.7)$$

It is evident that the potential functions are the same order as the wave height. This can be seen by neglecting the products of η and ξ in Equation A.6; then, the first approximation becomes

$$b_{oo} - \alpha\eta = 0 \;,$$

or

$$\eta = -\frac{1}{\alpha}\,\xi_t \qquad (A.8)$$

With this fact in mind the function defined by Equation A.1 is expanded in a binomial expansion and terms of $O(\eta^4)$ are neglected. Here it is assumed that ω_1, ω_2, ω_3, u_1, u_2, u_3 and their time derivatives are the same order as the wave height. This gives;

$$\Gamma(r,\theta,\eta,t) = -[-\psi_t + \mathring{u}_1 r\cos\theta + \tfrac{1}{2}(\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_z^2) + \mathring{u}_2 r\sin\theta$$
$$+ 2r^2(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta) + 2(\psi_z + u_3)(r\omega_1\sin\theta - r\omega_2\cos\theta) + \omega_3 r(u_1\sin\theta$$
$$- u_2\cos\theta) - 3r^2\omega_1\omega_2\sin\theta\cos\theta + \omega_3\psi_\theta] + [-r\,\mathring{\omega}_1\sin\theta + r\mathring{\omega}_2\cos\theta]$$
$$[-\psi_t + \mathring{u}_1 r\cos\theta + \mathring{u}_2 r\sin\theta + \tfrac{1}{2}(\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_2^2) + 2r^2(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta)$$
$$+ 2(\psi_z + u_3)(r\omega_1\sin\theta - r\omega_2\cos\theta) + \omega_3 r(u_1\sin\theta - u_2\cos\theta)$$
$$- 3r^2\omega_1\omega_2\sin\theta\cos\theta + \omega_3\psi_\theta] + \alpha^{-1}[-\omega_2 u_1 + \omega_1 u_2 + \omega_3 r(\omega_2\sin\theta$$
$$+ \omega_1\cos\theta)][-\psi_t + \mathring{u}_1 r\cos\theta + \mathring{u}_2 r\sin\theta] - \alpha^{-2}[r^2\mathring{\omega}^2\sin^2\theta + r^2\omega_2^2\cos^2\theta$$
$$-2r^2\mathring{\omega}_1\mathring{\omega}_2\sin\theta\cos\theta][-\psi_t + \mathring{u}_1 r\cos\theta + \mathring{u}_2 r\sin\theta] + O(\eta^4). \qquad (A.9)$$

Use Equation A.9 to find Γ_r, Γ_θ, Γ_t. Note that on $z = \eta$, $\Gamma_\eta = \Gamma_z$;

then $\alpha - \Gamma_\eta$ can be computed. Substitute these values into Equation A.3 and expand the function defined by the result in a Taylor series about $z = 0$. This leads to Equation A.5 where a_n is defined below.

$$a_k = \frac{\partial^k \ominus}{\partial z^k}\Big|_{z=0} \quad , \; k = 0, 1, 2, 3 \; \ldots$$

where

$$\begin{aligned}
\ominus = & - \alpha \; \psi_z - \psi_{tt} + \ddot{u}_1 \, r \cos\theta + 2\, \psi_r \psi_{rt} + 2 \frac{\psi_\theta \psi_{\theta t}}{r^2} + 2\psi_z \psi_{zt} \\
& - \mathring{u}_1 \left(\psi_r \cos\theta - \psi_\theta \frac{\sin\theta}{1}\right) - \psi_r^2 \psi_{rr} - \frac{\psi_\theta^2 \psi_{\theta\theta}}{r^4} - \psi_z^2 \psi_{zz} + \frac{\psi_z \psi_\theta^2}{r^3} - 2\psi_r \psi_{rz} \psi_z \\
& - 2 \frac{\psi_r \psi_\theta \psi_{r\theta}}{r^2} - 2 \frac{\psi_z \psi_\theta \psi_{\theta z}}{r^2} + (-u_3 + 2r\, [\omega_2 \cos\theta - \omega_1 \sin\theta])[\alpha + \psi_r \psi_{rz} + \psi_\theta \psi_{\theta z} \\
& + \psi_z \psi_{zz} - \psi_{tz} - 2r\psi_{zz} (\omega_2 \cos\theta - \omega_1 \sin\theta) + \omega_3 \psi_{\theta z} - \alpha^{-1} r \, (\mathring{\omega}_2 \cos\theta \\
& - \mathring{\omega}_1 \sin\theta)\left(\psi_r \psi_{rz} + \frac{\psi_\theta \psi_{\theta z}}{r^2} + \psi_z \psi_{zz}\right)] - \psi_z [-2\psi_{zz} (\omega_2 \cos\theta - \omega_1 \sin\theta) + \omega_3 \psi_{\theta z} \\
& - \alpha^{-1} r \,(\mathring{\omega}_2 \cos\theta - \mathring{\omega}_1 \sin\theta)\left(\psi_r \psi_{rz} + \frac{\psi_\theta \psi_{\theta z}}{r^2} + \psi_z \psi_{zz}\right)] - \alpha^{-1} r (\mathring{\omega}_2 \cos\theta \\
& - \mathring{\omega}_1 \sin\theta)\left[\psi_r \psi_{rt} + \frac{\psi_\theta \psi_{\theta t}}{r^2} + \psi_z \psi_{zt} - \psi_{tt} + \ddot{u}_1 \, r \cos\theta + \ddot{u}_2 r \sin\theta \right. \\
& + 4r^2 (\omega_1 \mathring{\omega}_1 \sin^2\theta + \omega_2 \mathring{\omega}_2 \cos^2\theta) + 2\,(\psi_{zt} + \mathring{u}_3) r (\omega_2 \cos\theta - \omega_1 \sin\theta) \\
& + 2r\,(\psi_z + u_3)(\mathring{\omega}_2 \cos\theta - \omega_1 \sin\theta) + \mathring{\omega}_3 r\,(u_1 \sin\theta - u_2 \cos\theta) + \omega_3 r (\mathring{u}_1 \sin\theta \\
& \left. - \mathring{u}_2 \cos\theta) - 3r^2 \sin\theta \cos\theta\, (\mathring{\omega}_1 \omega_2 + \omega_1 \mathring{\omega}_2) + \mathring{\omega}_3 \psi_\theta + \omega_3 \psi_{\theta t}\right] + [\ddot{u}_2 r \sin\theta \\
& + 4r^2 \,(\omega_1 \mathring{\omega}_1 \sin^2\theta + \omega_2 \mathring{\omega}_2 \cos^2\theta) + 2\,(\psi_{zt} + \mathring{u}_3) r (\omega_2 \cos\theta - \omega_1 \sin\theta) \\
& + 2r\,(\psi_z + u_3)(\mathring{\omega}_2 \cos\theta - \mathring{\omega}_1 \sin\theta) + \omega_3 r\,(u_1 \sin\theta - u_2 \cos\theta) + \omega_3 r (\mathring{u}_1 \sin\theta \\
& - \mathring{u}_2 \cos\theta) - 3r^2 \sin\theta \cos\theta (\mathring{\omega}_1 \omega_2 + \omega_1 \mathring{\omega}_2) + \mathring{\omega}_3 \psi_\theta + \omega_3 \psi_{\theta t}] - \alpha^{-1} r (\mathring{\omega}_2 \cos\theta \\
& - \mathring{\mathring{\omega}}_1 \sin\theta)\left[-\psi_t + \mathring{u}_1 \, r \cos\theta + \tfrac{1}{2}\left(\psi_r^2 + \frac{\psi_\theta^2}{r^2} + \psi_z^2\right) + \mathring{u}_2 r \sin\theta + 2r^2 (\omega_1^2 \sin^2\theta \right. \\
& + \omega_2^2 \cos^2\theta) + 2r\,(\psi_z + u_3)\,(\omega_2 \cos\theta - \omega_1 \sin\theta) + \omega_3 r (u_1 \sin\theta - u_2 \cos\theta) \\
& \left. - 3r^2 \omega_1 \omega_2 \sin\theta \cos\theta + \omega_3 \psi_\theta\right] - \alpha^{-1} [-\omega_2 u_1 + \omega_1 u_2 + \omega_3 r (\omega_2 \sin\theta + \omega_1 \cos\theta)]
\end{aligned}$$

$$[-\psi_{tt} + \ddot{u}_1 r\cos\theta + \ddot{u}_2 r\sin\theta] - \alpha^{-1}[-\mathring{\omega}_2 u_1 - \omega_2\mathring{u}_1 + \mathring{\omega}_1 u_2 + \omega_1\mathring{u}_2$$
$$+ \mathring{\omega}_3 r(\omega_2\sin\theta + \omega_1\cos\theta) + \omega_3 r(\mathring{\omega}_2\sin\theta + \mathring{\omega}_1\cos\theta)][-\psi_t + \mathring{u}_1 r\cos\theta$$
$$+ \mathring{u}_2 r\sin\theta] + \alpha^{-2}[2r^2\mathring{\omega}_1\ddot{\omega}_1\sin^2\theta + 2r^2\mathring{\omega}_2\ddot{\omega}_2\cos^2\theta - 2r^2\sin\theta\cos\theta(\ddot{\omega}_1\mathring{\omega}_2$$
$$+ \mathring{\omega}_1\ddot{\omega}_2)][-\psi_t + \mathring{u}_1 r\cos\theta + \mathring{u}_2\sin\theta] + \alpha^{-2}[r^2\mathring{\omega}_1^2\sin^2\theta + r^2\mathring{\omega}_2^2\cos^2\theta$$
$$- 2r^2\mathring{\omega}_1\mathring{\omega}_2\cos\theta\sin\theta][-\psi_{tt} + \ddot{u}_1 r\cos\theta + \ddot{u}_2 r\sin\theta] + \psi_r[-\mathring{u}_2\sin\theta$$
$$- 4r(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta) - 2\psi_{zr}r(\omega_2\cos\theta - \omega_1\sin\theta) - 2(\psi_z + u_3(\omega_2\cos\theta$$
$$- \omega_1\sin\theta) - \omega_3(u_1\sin\theta - u_2\cos\theta) + 6r\,\omega_1\omega_2\sin\theta\cos\theta - \omega_3\psi_{\theta r}$$
$$+ \alpha^{-1}r(\mathring{\omega}_2\cos\theta - \mathring{\omega}_1\sin\theta)(-\psi_{rt} + \mathring{u}_1\cos\theta + \mathring{u}_2\sin\theta) + \alpha^{-1}(\mathring{\omega}_2\cos\theta$$
$$- \mathring{\omega}_1\sin\theta)(-\psi_t + \mathring{u}_1 r\cos\theta + \mathring{u}_2 r\sin\theta)] + \omega_3[\psi_{\theta t} + \mathring{u}_1 r\sin\theta - \mathring{u}_2 r\cos\theta$$
$$- \psi_r\psi_{\theta r} - \frac{\psi_\theta\psi_{\theta\theta}}{r^2} - \psi_z\psi_{\theta z} - 4r^2\omega_1^2\sin\theta\cos\theta + 4r^2\omega_2^2\cos\theta\sin\theta + 2\psi_{z\theta}r$$
$$(\omega_2\cos\theta - \omega_1\sin\theta) - 2r(\psi_z + u_3)(\omega_2\sin\theta + \omega_1\cos\theta) - \omega_3 r(u_1\cos\theta$$
$$+ u_2\sin\theta) + 3r^2\omega_1\omega_2(\cos^2\theta - \sin^2\theta) - \omega_3\psi_{\theta\theta} - \alpha^{-1}r(\mathring{\omega}_2\cos\theta$$
$$- \mathring{\omega}_1\sin\theta)(-\psi_{\theta t} - \mathring{u}_1 r\sin\theta + \mathring{u}_2 r\cos\theta) - \alpha^{-1}r(\mathring{\omega}_2\sin\theta + \mathring{\omega}_1\cos\theta)(-\psi_t$$
$$+ \mathring{u}_1 r\cos\theta + \mathring{u}_2\sin\theta)] + \frac{\psi_\theta}{r^2}[-\mathring{u}_2 r\cos\theta - 4r^2\omega_1^2\sin\theta\cos\theta$$
$$+ 4r^2\omega_2^2\cos\theta\sin\theta - 2\psi_{z\theta}r(\omega_2\cos\theta - \omega_1\sin\theta) - 2r(\psi_z$$
$$+ u_3)(\omega_2\sin\theta + \omega_1\cos\theta) - \omega_3 r(u_1\cos\theta + u_2\sin\theta + 3r^2\omega_1\omega_2(\cos^2\theta$$
$$- \sin^2\theta) - \omega_3\psi_{\theta\theta} - \alpha^{-1}r(\mathring{\omega}_2\cos\theta - \mathring{\omega}_1\sin\theta)(-\psi_{\theta t} - u_1 r\sin\theta + \mathring{u}_2 r\cos\theta)$$
$$- \alpha^{-1}r(\mathring{\omega}_2\sin\theta + \mathring{\omega}_1\cos\theta)(-\psi_t + \mathring{u}_1 r\cos\theta + \mathring{u}_2 r\sin\theta)] + O(\eta^4).$$

Equation A.6 is solved for η:

$$\eta = -\frac{b_o}{b_1} - \left(\frac{b_2}{2b_1}\right)\eta^2 + \ldots = -\frac{b_o}{b_1} + O(\eta^3) \quad , \qquad \text{(A.10)}$$

since $b_2 = O(\eta)$. Substitute Equation A.10 into Equation A.5:

$$a_o + a_1\left[-\frac{b_o}{b_1} + O(\eta^3)\right] + \frac{a_2}{2}\left[-\frac{b_o}{b_1} + O(\eta^3)\right]^2 + O(\eta^4) = 0,$$

or

$$a_o + a_1\left(-\frac{b_o}{b_1}\right) + \frac{a_2}{2}\left(\frac{b_o^2}{b_1^2}\right) + O(\eta^4) \;.. = 0 \;. \tag{A.11}$$

Compute the indicated multiplication; Equation A.11 is then

$$a_{oo} + a_{o1} - \frac{a_{11}b_{oo}}{\alpha} + a_{o2} - \frac{a_{11}b_{o1}}{\alpha} + \frac{a_{12}b_{oo}}{\alpha} + \frac{a_{11}b_{11}b_{oo}}{\alpha^2} + \frac{a_{22}b_{oo}^2}{\alpha^2}$$

$$+ \frac{a_{22}b_{oo}^2}{2!\,\alpha^2} + B_{11} + B_{12} + B_{13} - a_{11}\,(\alpha^{-1}\,\mathring{u}_2 r \sin\theta + \alpha^{-2} r\,\mathring{\sigma}_1\mathring{u}_2 r \sin\theta$$

$$- \alpha^{-2} b_{11}\,\mathring{u}_2 r \sin\theta + \alpha^{-2}\,r\,\mathring{\sigma}_1\,b_{oo} + \alpha^{-1}[2r^2\,\omega_1^2 \sin^2\theta + 2r^2\,\omega_2^2\cos^2\theta$$

$$+ 2\xi_r\,(u_3 - \sigma_1) - 3r^2\,\omega_1\omega_2 \sin\theta\cos\theta + 2u_3 r\sigma_1 + u_1\omega_3\,r\sin\theta + \omega_3\xi_\theta])$$

$$- B_{22}\,(\alpha^{-1}[b_{oo} + \mathring{u}_2 r \sin\theta] - \alpha^{-2}[b_{11} - r\,\mathring{\sigma}_1][b_{oo} + \mathring{u}_2 r\sin\theta] + \alpha^{-1}[b_{o1}$$

$$+ 2r^2\,\omega_1^2\sin^2\theta + 2r^2\,\omega_2^2\cos^2\theta + 2\xi_z r\,(u_3 - \sigma_1) - 3r^2\,\omega_1\omega_2\sin\theta\cos\theta$$

$$+ 2\,u_3 r\,\sigma_1 + u_1\,\omega_3 r\sin\theta + \omega_3\xi_\theta]) + a_{12}\alpha^{-1}\,\mathring{u}_2 r\sin\theta + B_{23}\,\alpha^{-1}[b_{oo}$$

$$+ \mathring{u}_2 r\sin\theta] + \frac{a_{22}}{2!}\,\alpha^{-2}\,(2b_{oo}\,\mathring{u}_2 r\sin\theta + \mathring{u}_2^2\,r^2\sin^2\theta) + \frac{B_{33}}{2!}\,\alpha^{-2}\,[b_{oo}$$

$$+ \mathring{u}_2\,r\sin\theta]^2 + O(\eta^4) = 0, \tag{A.12}$$

where

$$a_{oo} = -\alpha\,\xi_z - \xi_{tt} + \ddot{\mathring{u}}_1\,r\cos\theta,$$

$$a_{o1} = 2\xi_r\xi_{rt} + \frac{2\xi_\theta\xi_{\theta t}}{r^2} + 2\,\xi_z\xi_t - \mathring{u}_1\,\left(\xi_r\cos\theta - \xi_\theta\,\frac{\sin\theta}{r}\right),$$

$$a_{o2} = -\,\xi_r^2\,\xi_{rr} - \frac{\xi_\theta^2\,\xi_{\theta\theta}}{r^4} - \xi_z^2\,\xi_{zz} + \frac{\xi_z\xi_\theta^2}{r^3} - 2\,\xi_r\,\xi_{rz}\,\xi_z$$

$$- \frac{2\xi_r\xi_\theta\xi_{r\theta}}{r^2} - \frac{2\xi_z\xi_\theta\xi_{\theta z}}{r^2},$$

$$a_{11} = -\alpha\, \xi_{zz} - \xi_{ttz} ,$$

$$a_{m+1,\, n+1} = \frac{\partial a_{mn}}{\partial z} ,$$

$$\sigma_1 = \omega_2 \cos\theta - \omega_1 \sin\theta , \qquad \beta_1 = u_1 \sin\theta - u_2 \cos\theta ,$$

$$\sigma_{1,\theta} = \frac{\partial \sigma_1}{\partial \theta} , \qquad \beta_{1,\theta} = \frac{\partial \beta_1}{\partial \theta} ,$$

$$\mathring{\sigma}_1 = \frac{\partial \sigma_1}{\partial t} , \qquad \mathring{\beta}_1 = \frac{\partial \beta_1}{\partial t} ,$$

$$B_{11} = \alpha\, (-u_3 + 2r\sigma_1) + \ddot{\mathring{u}}_2\, r \sin\theta,$$

$$B_{12} = (-u_3 + 2r\sigma_1) b_{11} - \alpha^{-1}\, r\mathring{\sigma}_1\, (\mathring{b}_{oo} + \ddot{\mathring{u}}_2 r \sin\theta) + 4r^2(\omega_1\mathring{\omega}_1 \sin^2\theta$$
$$+ \omega_2\mathring{\omega}_2 \cos^2\theta) + 2\,(\xi_{zt} + \mathring{u}_3) r\sigma_1 + 2(\xi_z + u_3)\, r\mathring{\sigma}_1 + \mathring{\omega}_3\, r\beta_1 + \omega_3 r\mathring{\beta}_1$$
$$- 3r^2 \sin\theta \cos\theta\, (\mathring{\omega}_1\omega_2 + \omega_1\mathring{\omega}_2) + \mathring{\omega}_3\, \xi_\theta + \omega_3\xi_{\theta t} - \alpha^{-1} r\, \ddot{\mathring{\sigma}}_1\, (b_{oo} + \mathring{u}_2 r \sin\theta)$$
$$+ \xi_r\, (-\dot{\mathring{u}}_2 \sin\theta) + \omega_3\, (\xi_{\theta t} + \dot{\mathring{u}}_1\, r \sin\theta - \mathring{u}_2 r \cos\theta) + \frac{\xi_\theta}{r^2}\, (-\dot{\mathring{u}}_2 r \cos\theta) ,$$

$$B_{13} = (-u_3 + 2r\sigma_1)(b_{12} - 2r\xi_{zz}\sigma_1 + \omega_3\, \xi_{\theta z} - \alpha^{-1} r\mathring{\sigma}_1 b_{11}) - \xi_z(-2\xi_{zz}\mathring{\sigma}_1$$
$$+ \omega_3\xi_{\theta z} - \alpha^{-1} r\, \mathring{\sigma}_1 b_{11}) - \alpha^{-1} r\, \mathring{\sigma}_1\, [\mathring{b}_{o1} + 4r^2\, (\omega_1\mathring{\omega}_2 \sin^2\theta + \omega_2\mathring{\omega}_2 \cos^2\theta\,)$$
$$+ 2(\xi_{zt} + \mathring{u}_3)\, r\sigma_1 + 2\,(\xi_z + u_3) r\mathring{\sigma}_1 + \omega_3 r\, \mathring{\beta}_1 + \mathring{\omega}_3 r\, \beta_1 - 3r^2 \sin\theta \cos\theta\, \cdot$$
$$(\mathring{\omega}_1\omega_2 + \omega_1\mathring{\omega}_2) + \omega_3\, \xi_\theta + \omega_3\, \xi_{\theta t}] - \alpha^{-1} r\, \ddot{\mathring{\sigma}}_1 [b_{o1} + 2r^2\, (\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta)$$
$$+ 2\,(\xi_z + u_3) r\sigma_1 + \omega_3 r\beta_1 - 3r^2\, \omega_1\omega_2 \sin\theta \cos\theta + \omega_3\xi_\theta\,] - \alpha^{-1}[-\omega_2 u_1$$
$$+ \omega_1 u_2 - \omega_3 r\sigma_{1,\theta}][\mathring{b}_{oo} + \ddot{\mathring{u}}_2\, r \sin\theta\,] - \alpha^{-1}[-\mathring{\omega}_2 u_1 - \omega_2\mathring{u}_1 + \mathring{\omega}_1 u_2 + \omega_1\mathring{u}_2$$
$$- \mathring{\omega}_3 r\, \sigma_{1,\theta} - \omega_3 r\, \mathring{\sigma}_{1,\theta}\,][b_{oo} + \mathring{u}_2 r \sin\theta] + \alpha^{-2}[2r^2\, \mathring{\omega}_1\ddot{\mathring{\omega}}_1 \sin^2\theta + 2r^2\mathring{\omega}_2\ddot{\mathring{\omega}}_2\cos^2\theta$$
$$-2r^2 \sin\theta \cos\theta\, (\ddot{\mathring{\omega}}_1\mathring{\omega}_2 + \mathring{\omega}_1\ddot{\mathring{\omega}}_2)][b_{oo} + \mathring{u}_2 r \sin\theta\,] + \alpha^{-2}[r^2\, \mathring{\omega}_1^2 \sin^2\theta +$$
$$r^2\, \mathring{\omega}_2^2 \cos^2\theta - 2r^2\, \mathring{\omega}_1\mathring{\omega}_2 \cos\theta \sin\theta\,][\mathring{b}_{oo} + \ddot{\mathring{u}}_2\, r \sin\theta\,] + \xi_r[-4r(\omega_1^2 \sin^2\theta$$

$$+ \omega_2^2 \cos^2\theta) - 2\, \xi_{zr}\, r\, \sigma_1 - 2(\xi_z + u_3)\sigma_1 - \omega_3\beta_1 + 6r\, \omega_1\omega_2 \sin\theta\cos\theta$$

$$- \omega_3\, \xi_{\theta r} + \alpha^{-1} r\, \dot{\sigma}_1\, (-\xi_{rt} + \dot{u}_1 \cos\theta + \dot{u}_2 \sin\theta\,) + \alpha^{-1}\, \dot{\sigma}_1\, (b_{oo} + \dot{u}_2 r\sin\theta\,)]$$

$$+ \omega_3\, [- \xi_r\xi_{\theta\, r} - \frac{\xi_\theta\xi_{\theta\theta}}{r^2} - \xi_z\xi_{\theta z} - 4r^2\omega_1^2 \sin\theta\cos\theta + 4r^2\, \omega_2^2 \cos\theta\sin\theta$$

$$+ 2\xi_z\, r\sigma_1 + (\xi_z + u_3)\, r\sigma_{1,\theta} - \omega_3 r\, \beta_{1,\theta} + 3r^2\, \omega_1\omega_2[\cos^2\theta - \sin^2\theta] - \omega_3\xi_{\theta\theta}$$

$$-\alpha^{-1}\, r\, \dot{\sigma}_1\, (-\xi_{\theta t} - \dot{u}_1 r \sin\theta + \dot{u}_2\, r\cos\theta) + \alpha^{-1}\, \dot{\sigma}_{1,\theta}\, r\, (b_{oo} + \dot{u}_2 r \sin\theta)]$$

$$+ \frac{\xi_\theta}{r^2}\, [-4r^2\omega_1^2 \sin\theta\cos\theta + 4r^2\, \omega_2^2 \sin\theta\cos\theta - \xi_{z\theta}\, r\, \sigma_1 + 2(\xi_z + u_3) r\sigma_{1,\theta}$$

$$- \omega_3 r\, \beta_{1,\theta} + 3r^2\, \omega_1\omega_2\, [\cos^2\theta - \sin^2\theta] - \omega_3\xi_{\theta\theta} - \alpha^{-1}\, r\dot{\sigma}_1(-\xi_{\theta t} - \dot{u}_1 r\sin\theta$$

$$+ \dot{u}_2 r \cos\theta) + \alpha^{-1} r\, \dot{\sigma}_{1,\theta}\, (b_{oo} + \dot{u}_2 r \sin\theta)],$$

$$B_{m+1,n+1} = \frac{\partial B_{mn}}{\partial z}\,. \tag{A.13}$$

In summary, the boundary value problem in terms of the potential function, with the higher order approximation for the free surface condition becomes

$$\nabla^2\, \psi\, (r,\theta,z,t) = 0 \quad \text{in} \quad \begin{matrix} 0 \leqslant r < a \\ 0 \leqslant \theta \leqslant 2\pi \\ -h < z < \eta \end{matrix}\,, \tag{A.14}$$

$$\psi_r = 0\,, \text{ on } r = a,$$

$$\psi_z = -\, u_3 - 2r\omega_1 \sin\theta + 2r\, \omega_2 \cos\theta, \text{ on } z = -h\,,$$

and Equation A.12.

The frequency of the forcing motion is close to or equal to the first natural frequency. The neighborhood of resonance considered is for $|\omega^2 - p_{11}^2| = 0\, (\epsilon^{2/3} p_{11}^2)$ as $\epsilon \rightarrow 0$. Thus

$$p_{11}^2\, (1 + \epsilon^{2/3}\upsilon) = \omega^2\,, \tag{A.15}$$

or

$$p_{11}^2 = \omega^2 \left(1 - \epsilon^{2/3} \upsilon\right) \quad . \tag{A.16}$$

Here p_{11} is the lowest natural frequency of small, free surface oscillations;

$$p_{11} = \sqrt{\alpha \lambda_{11} \tanh \lambda_{11} h} \quad , \tag{A.17}$$

where λ_{11} corresponds to the first non zero root of

$$J_1' (\lambda_{1n} a) = 0 \ .$$

Constants:

$$\Omega_{on} = \frac{I_{o1}^n + I_{o2}^n + K_o I_{o3}^n}{(4p_{11}^2 - p_{on}^2) \frac{a^2}{2p_{11}} J_o^2 (\lambda_{on} a)} \quad ,$$

$$\Omega_{2n} = \frac{I_{21}^n - I_{22}^n + K_o I_{23}^n}{(4p_{11}^2 - p_{2n}^2) \frac{\lambda_{2n}^2 a^2 - 4}{2\lambda_{2n}^2 p_{11}} J_2^2 (\lambda_{2n} a)} \quad ,$$

$$I_{q1}^n = \int_0^{\lambda_{11} a} u J_q \left(\frac{\lambda_{qn}}{\lambda_{11}} u\right) \left[\frac{d}{du} J_1(u)\right]^2 du \quad , \ q = 0, 2,$$

$$I_{q2}^n = \int_0^{\lambda_{11} a} \frac{1}{u} J_q \left(\frac{\lambda_{qn}}{\lambda_{11}} u\right) J_1^2 (u) \, du \ , \quad q = 0, 2 \ ,$$

$$I_{q3}^n = \int_0^{\lambda_{11} a} u J_q \left(\frac{\lambda_{qn}}{\lambda_{11}} u\right) J_1^2 (u) \, du \ , \quad q = 0, 2 \ ,$$

$$I_{q4}^n = \int_0^{\lambda_{11} a} u J_1(u) \frac{d}{du} [J_1(u)] \frac{d}{du} \left[J_q \left(\frac{\lambda_{qn}}{\lambda_{11}} u\right)\right] du, \quad q = 0, 2 \ ,$$

$$K_o = \frac{3\zeta_{11}^2 - 1}{2} \quad ,$$

$$\hat{G}_1 = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \left[\zeta_{on}\zeta_{11} \lambda_{on}\lambda_{11} - \frac{1}{2} \lambda_{on}^2 + \lambda_{11}^2 (1 - \zeta_{11}^2) \right] \Omega_{on} I_{o3}^n \right.$$

$$- \Omega_{on} I_{o4}^n - \Omega_{2n} I_{22}^n - \frac{1}{2} \Omega_{2n} I_{24}^n + \frac{1}{2} \left[\zeta_{11}\zeta_{2n} \lambda_{11}\lambda_{2n} - \frac{1}{2} \lambda_{2n}^2 \right.$$

$$\left. \left. + \lambda_{11}^2 (1 - \zeta_{11}^2) \right] \Omega_{2n} I_{2n}^n \right\} ,$$

$$\hat{G}_2 = \sum_{h=1}^{\infty} \left\{ \left[\zeta_{on}\zeta_{11}\lambda_{on}\lambda_{11} - \frac{1}{2} \lambda_{on}^2 + \lambda_{11}^2 (1 - \zeta_{11}^2) \right] \Omega_{on} I_{o3}^n - \Omega_{on} I_{o4}^n \right.$$

$$\left. + \Omega_{2n} I_{22}^n + \frac{1}{2} \Omega_{2n} I_{24}^n - \frac{1}{2} \left[\zeta_{11}\zeta_{2n} \lambda_{11}\lambda_{2n} - \frac{1}{2} \lambda_{2n}^2 + \lambda_{11}^2 (1 - \zeta_{11}^2) \right] \Omega_{2n} I_{23}^n \right\} ,$$

$$H_1 = 3k_1 + k_2 \quad , \; k_1 = \frac{\pi}{2} \frac{\lambda_{11}^2}{K p_{11}^2} k_{10} \quad ,$$

$$H_2 = 2(k_1 - k_3 + k_4) , \; k_2 = \frac{\pi}{2} \frac{\lambda_{11}}{K\alpha\zeta_{11}} k_{20} \quad ,$$

$$\zeta_{mn} = \tanh (\lambda_{mn} h) , \quad k_3 = \frac{\pi}{2} \frac{\lambda_{11}}{K\alpha\zeta_{11}} k_{30} ,$$

$$p_{mn}^2 = \alpha \lambda_{mn}\zeta_{mn} \quad , \; k_4 = \frac{\pi}{2} \frac{\lambda_{11}}{K\alpha\zeta_{11}} k_{40} \quad ,$$

$$k_{10} = - \frac{1}{4} K \, [6I_1 - I_3 - 2I_4 - 3I_5 + 4I_6 - 2I_7 + \zeta_{11}^2 (3I_2 + 5I_3 + 15I_5) + 3\zeta_{11}^4 I_2] \quad ,$$

$$k_{20} = \frac{1}{2} K \, \zeta_{11}^2 (9I_2 + 4I_3 + 12I_5 + 3\zeta_{11}^2 I_2) \quad ,$$

$$k_{30} = \frac{1}{2} K \, \zeta_{11}^2 (3I_2 - 4I_3 + 4I_5 + \zeta_{11}^2 I_2) \quad ,$$

$$k_{40} = K \, \zeta_{11}^2 (3I_2 + 4I_3 + 4I_5 + \zeta_{11}^2 I_2)$$

$$K = \frac{1}{(\lambda_{11}^2 a^2 - 1) J_1^2 (\lambda_{11} a)} \left(\frac{\lambda_{11}^3}{\alpha\zeta_{11}} \right) \quad ,$$

$$I_1 = \int_0^{\lambda_{11}a} u\, J_1 \left(\frac{dJ_1}{du}\right) \frac{d^2J_1}{du^2}\, du = \frac{1}{\lambda_{11}^2} \int_0^a rJ_1 \left(\frac{dJ_1}{dr}\right)^2 \frac{d^2J_1}{dr^2}\, dr \quad ,$$

$$I_2 = \int_0^{\lambda_{11}a} u\, J_1^4\, du = \lambda_{11}^2 \int_0^a r\, J_1^4\, dr \quad ,$$

$$I_3 = \int_0^{\lambda_{11}a} \frac{1}{u} J_1^4\, du = \int_0^a \frac{1}{r}\, J_1^4\, dr \quad ,$$

$$I_4 = \int_0^{\lambda_{11}a} \frac{1}{u^3} J_1^4\, du = \frac{1}{\lambda_{11}^2} \int_0^a \frac{1}{r^3} J_1^4\, dr \quad ,$$

$$I_5 = \int_0^{\lambda_{11}a} u\, J_1^2 \left(\frac{dJ_1}{du}\right)^2 du = \int_0^a r\, J_1^2 \left(\frac{dJ_1}{dr}\right)^2 dr \quad ,$$

$$I_6 = \int_0^{\lambda_{11}a} \frac{1}{u} J_1^2 \left(\frac{dJ_1}{du}\right)^2 du = \frac{1}{\lambda_{11}^2} \int_0^a \frac{1}{r} J_1^2 \left(\frac{dJ_1}{dr}\right)^2 dr \quad ,$$

$$I_7 = \int_0^{\lambda_{11}a} \frac{1}{u^2} J_1^3 \left(\frac{dJ_1}{du}\right) du = \frac{1}{\lambda_{11}^2} \int_0^a \frac{1}{r^2}\, J_1^3 \left(\frac{dJ_1}{dr}\right) dr \quad ,$$

$$F_1 = \frac{2a}{(\lambda_{11}^2 a^2 - 1) J_1\, (\lambda_{11}a)} \quad , \qquad \text{(A.18)}$$

$$A_2 = - \frac{2F_1}{p_{11}^2 h} \quad , \qquad \text{(A.19)}$$

$$A_1 = \frac{2F_1}{p_{11}^2 h} \quad , \tag{A.20}$$

$$U_2 = F_1 \quad , \tag{A.21}$$

$$K_1 = K_{10} + \Delta K_1$$

$$K_2 = K_{20} + \Delta K_2$$

$$K_{10} = \frac{K}{16} [-18I_1 + 3I_3 + 6I_4 + 9I_5 - 12I_6 + 6I_7 + \zeta_{11}^2(9I_2 - 7I_3 - 21I_5)$$
$$- 3\zeta_{11}^4 I_2] \quad ,$$

$$K_{20} = \frac{K}{8} [-6I_1 + I_3 + 2I_4 + 3I_5 - 4I_6 + 2I_7 + \zeta_{11}^2 (3I_2 + 19I_3 - 7I_5)$$
$$- \zeta_{11}^4 I_2] \quad ,$$

$$\Delta K_1 = -\frac{2p_{11}}{\lambda_{11}^2} K \hat{G}_1 \quad ,$$

$$\Delta K_2 = - \frac{2p_{11}}{\lambda_{11}^2} K\hat{G}_2$$

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IX. ACKNOWLEDGMENTS

I wish to thank Dr. Harry J. Weiss for his interest and valuable guidance during my years of graduate study. His encouragement and assistance in the preparation of this dissertation are also greatly appreciated.

The assistance of General Dynamics Astronautics is also greatly appreciated.