Asymptotic behavior of the solutions to a family of PDE's arising from the chemotaxis equations of Keller and Segal

by

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Dedication

I dedicate this work to my friends and family, who have always supported me in every endeavor that I have ever undertaken. In particular, I thank my parents who have supported me and believed in me even when I had doubts myself. I could not have succeeded in my goals without their support.

I would also like to thank the faculty and staff at Iowa State University for their expert guidance and support during my tenure at my beloved alma mater.

Table Of Contents

List Of Figures

Abstract

The system $u_t = u_{xx} - (uv_x)_x$, $v_t = u - Av$ is considered where A is a non-negative, self-adjoint operator which commutes with the Laplacian. The operator is considered to have eigenvalues $\lambda_n = n^\rho \lambda_1$, and the system is considered on $[0,1] \times [0,T]$ with homogeneous Neumann boundary conditions. The operators which lead to global solutions and those that lead to solutions which blow up in finite time are considered as a function of ρ, using an application of the methods of Hillen and Potapov [*Math. Methods Appl. Sci.*, 27 (2004), pp. $1783-1801$] to analyze the global case and those of Halverson, Levine, and Renclawowicz [*Siam J. Appl. Math.*, 65 (2004), pp. $336-360$; 66 (2005), pp. $361-364$] to analyze the finite time blowup case. Some numerical results are provided to back up the analysis. Some questions and directions for future study are posed.

Chapter 1. Introduction

It is well understood that organisms respond to chemical signals. Much of the behavior at all levels of life can be explained in its simplest aspect as response to a chemical signal. The insect kingdom provides a classical example in the behavior of ants. It is well known that these insects lay down chemical trails for the rest of the colony to follow. This allows them to direct others to food sources, away from dangers, and ultimately back to the hive. Simply put, chemotaxis is how organisms respond to such signals. As the benefits and applications of an understanding of these phenomena are considerable, there is much interest in mathematically modeling them.

One often considered organism for modeling is a soil bacterium known as myxobacteria [14, 18]. These bacteria exhibit gliding behavior on suitable surfaces and produce a slime trail, on which they prefer to glide. When starved, they come closer together and form fruiting bodies wherein they survive in a dormant state [18]. One theory of how this aggregation occurs is that it arises as a result of the bacteria following these slime trails. Stevens points out that modeling the trail following behavior by means of a stochastic cellular automaton does lead to the bacteria moving in the direction of higher bacteria concentration, as is desirable in such a model, but does not lead to the stable aggregation centers that one would need to see if this behavior alone described the aggregation patterns.

Both Stevens [18] as well as Othmer and Stevens [14] analyzed the behavior of the myxobacteria by means of cellular automaton and random walks taking into account chemotactic behavior as well as the slime trails, which does lead to the aggregation behavior that one would wish to see in the model when it has captured the relative mechanism.

Othmer and Stevens, in particular, question why the slime trail following alone does not suffice to form the stable aggregation centers. They point out that the bacteria do begin to aggregate in these models, but the aggregation is not stable. They observe that some of the data suggests that the trail following alone may suffice, but that the type of models that lead to this behavior strictly from the slime trail following have not been found.

In analyzing this model, Othmer and Stevens [14] provide both analytical and numerical evidence that solutions to this sort of model can be unstable. Levine and Sleeman [12] consider certain limiting cases of these models and provide further evidence that such behavior is possible. In some of the cases considered, they were able to provide exact solutions to the model to support the conclusion, and were able to establish that solutions can blow up in finite time.

We will now look at a well-known model of chemotaxis to which much research has been devoted and derive it in a classical manner.

1.1 The Keller-Segal model of chemotaxis

Since its introduction [6], the Keller-Segal model has been a considerable focus of attention. Here we will focus on one particular manifestation of it. Consider the system

$$\begin{aligned} u_t &= u_{xx} - (uv_x)_x \\ v_t &= u + v_{xx} - av \end{aligned} \tag{1.1}$$

where $u(x,t)$ is the population of an organism under study at a point x and time t, and v is the concentration of a chemotactic agent to which the organism responds.

1.1.1 Keller and Segal's derivation of the model

Keller and Segal [6] present the following derivation for the chemotaxis equations in two dimensions for amoebae populations. Let $a(x,y,t)$ denote the concentration of amoebae at the point (x,y) and at time t. Amoebae respond to a chemical known as acrasin whose concentration we denote by $p(x,y,t)$. We let $n(x,y,t)$ denote the concentration of acrasinase, which degrades acrasin. Keller and Segal note that the amoebae produce both chemicals, at a rate $f(p)$ and $g(p,n)$ respectively, and that the two chemicals react to form a complex C, whose concentration we will denote with $c(x, y, t)$. They give the following chemical equation:

$$p + n \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} C \overset{k_2}{\rightarrow} n + \text{product}. \tag{1.2}$$

Focusing on a region A where the amoebae are located and letting $Q_a(x, y, t)$ denote the concentration of amoebae produced at the point (x, y) at time t per unit time, and letting $J_a(x, y, t)$ denote the flux of amoebae mass, we have

$$\frac{\partial}{\partial t} \iint_A a(x, y, t)\ dx\ dy = \iint_A Q_a(x, y, t)\ dx\ dy - \int_{\partial A} J_a(x, y, t) \cdot \eta_A\ ds \tag{1.3}$$

where ∂A is the boundary of A and η_A is the unit exterior normal to ∂A. Notice that the first integral on the left is the total amoebae population and thus the left hand side of the equation is simply the rate of change in the amoeba population. The first integral on the right is the total created population at time t and the second integral can be interpreted as the population leaving the region A at time t.

From this, Keller and Segal, after an application of the divergence theorem to the last term in (1.3) and considering (1.3) over arbitrary A, derive that

$$\frac{\partial}{\partial t} a(x, y, t) = Q_a(x, y, t) - \nabla \cdot J_a(x, y, t).$$

From the same approach, similar equations for $p(x, y, t),\ n(x, y, t),$ and $c(x, y, t)$ can be derived. They take the flux terms to be of diffusive type. That is, with constants $D_p, D_c,$ and D_n along with $D_1(p, a), D_a(p, a)$, these are of the form

$$J_a = -D_a \nabla a + D_1 \nabla p,$$

$$J_p = -D_p \nabla p,$$

$$J_c = -D_c \nabla c,$$

$$J_n = -D_n \nabla n$$

where the first equation reflects how the amoebae respond to the chemotaxtic agent as well as their own diffusive movements. They discuss briefly possible forms of the term D_1.

Keller and Segal choose to ignore reproduction and take $Q_a \equiv 0$. The remaining production terms rely on the chemical equation (1.2). Thus we have

$$Q_p = -k_1 pn + k_{-1}c + af(p),$$

$$Q_n = -k_1 pn + (k_{-1} + k_2)c + ag(p,n),$$

$$Q_c = k_1 pn - (k_{-1} + k_2)c.$$

With these values for the production and flux terms, Keller and Segal present the system of PDE's:

$$\frac{\partial a}{\partial t} = -\nabla \cdot (D_1 \nabla p) + \nabla \cdot (D_a \nabla a), \tag{1.4}$$

$$\frac{\partial p}{\partial t} = -k_1 pn + k_{-1}c + af(p) + D_p \nabla^2 p, \tag{1.5}$$

$$\frac{\partial n}{\partial t} = -k_1 pn + (k_{-1} + k_2)c + ag(p,n) + D_n \nabla^2 n, \tag{1.6}$$

$$\frac{\partial c}{\partial t} = k_1 pn - (k_{-1} + k_2)c + D_c \nabla^2 c. \tag{1.7}$$

They simplify this system by making the assumptions $k_1 pn - (k_{-1} + k_2)\, c = 0$ and $n + c = n_0$ where n_0 is a constant. To reach these assumptions, assume that $\frac{\partial c}{\partial t} \equiv 0$ and that $D_c \equiv 0$. This will give the first assumption. To reach the second assumption, look at $\frac{\partial}{\partial t}(n + c)$ and assume $D_n = D_c = g(p,n) \equiv 0$. After some manipulation of the equations, they obtain a new system

$$\begin{aligned} \frac{\partial a}{\partial t} &= -\nabla \cdot (D_1 \nabla p) + \nabla \cdot (D_a \nabla a), \\ \frac{\partial p}{\partial t} &= -\kappa(p)p + af(p) + D_p \nabla^2 p \end{aligned} \tag{1.8}$$

from (1.4) and (1.5) where

$$\kappa(p) = \frac{n_0 k_2 K}{1 + Kp} \quad \text{and} \quad K = \frac{k_1}{k_{-1} + k_2}.$$

Equations (1.6) and (1.7) are no longer needed in the analysis after this simplification as the (a,p) pair no longer depends on them.

Taking the system (1.8) as a general form (letting κ be any function and not necessarily the specific one from the derivation), we can obtain (1.1) by

$$D_a(a,p) \equiv 1, \qquad D_1(a,p) = a, \qquad D_p = 1, \qquad f(p) \equiv 1, \qquad \kappa(p) \equiv \tilde{a}$$

where the function pair (a, p) serves as the pair (u, v). $\tilde{a}$ is the same as the constant a in (1.1) and the $\tilde{a}$ notation is used to advoid confusing it with the function a in the above derivation.

1.2 The Fujita model

Now, for motivation, we consider the Fujita Model [3, 9]

$$u_t = \triangle u + u^p \tag{1.9}$$

on $\mathbb{R}^N$ with u equal to a given $L^1(\mathbb{R}^N)$ function at the initial time which is non-negative and positive on a positive measure subset and $p > 1$.

Fujita was able to show the following result [9]:

Theorem 1. *Consider the Fujita problem* (1.9)*.*

- *If* $1 < p < 1 + \frac{2}{N}$ *then all non-trivial non-negative solutions blow up in finite time.*
- *If* $p > 1 + \frac{2}{N}$ *then there are non-trivial non-negative global solutions when the initial values are sufficiently small.*

Levine [9] points out that when the solution fails to exist globally, it blows up pointwise. He states that for a global solution to exist, the initial value function can not satisfy

$$\left(\frac{1}{p+1}\right) \int_{\mathbb{R}^N} u_0^{p+1} \ dx > \left(\frac{1}{2}\right) \int_{\mathbb{R}^N} \mid \nabla u_0 \mid^2 \ dx,$$

as well as pointing out that it has been shown that the value $1 + \frac{2}{N}$ is in the blowup case.

The meaning of *sufficiently small* is that there is some $\delta(t_0)$ such that

$$0 \le u_0(x) < \delta(4\pi t_0)^{-N/2} \exp(-\frac{\mid x \mid^2}{4t_0})$$

for some t_0 [10], that is, the data lies under a small Gaussian.

Levine further goes on to look at the same problem on a set D (instead of $\mathbb{R}^N$) with the added condition that the function vanishes at the boundary of D (or at infinity when D is not bounded). When D is a domain with a bounded complement, the results of Theorem 1 hold [2].

He states an analogous result for the case that D is an *orthant*, a set

$$\mathbb{D}_k = \{x \in \mathbb{R}^N \mid x_1 > 0,\ \ldots,\ x_k > 0\}$$

where $\mathbb{D}_0 = \mathbb{R}^N$. In this case theorem 1 hold with the critical value $1 + \frac{2}{k+N}$ in place of $1 + \frac{2}{N}$.

Similar results are known for various domains where the Green's function for the heat equation can be found [9].

1.3 The main problem

We will now consider a generalized version of model (1.1). Consider a non-negative self-adjoint operator A which commutes with the Laplacian and which has eigenvalues of the form $\lambda_n = \lambda_1 n^\rho$ for some real number $\rho \geq 0$. Now consider the system

$$\begin{aligned} u_t &= u_{xx} - (uv_x)_x \\ v_t &= u - Av \end{aligned} \tag{1.10}$$

on the interval $[0, 1]$ with homogenous Neumann boundary conditions,

$$u_x(0, t) = u_x(1, t) = v_x(0, t) = v_x(1, t) = 0. \tag{1.11}$$

There are several possibilities for such an operator A, for example A can be viewed as the negative Laplacian with Neumann boundary conditions raised to the $\frac{\rho}{2}$ power. As a more general example, we could let A act on a function v by $Av = \sum_{n=0}^{\infty} \lambda_n < v, \phi_n > \phi_n$ where $\{\phi_n\}_{n=0}^{\infty}$ are the eigenfunctions of the Neumann Laplacian and $\lambda_n \sim cn^\rho$ as $n \to \infty$.

We conjecture that there is a rough analog of the Fujita theorem for this problem. Namely that there is some value of ρ above which all solutions remain global and below which some solutions can blow up. We further conjecture that this value of ρ is 1. Establishing this conjecture is the main focus of this research and the remainder of this work will be devoted to partially establishing this conjecture.

We will approach this problem as follows. For $\rho \leq 1$, we use the Fourier cosine series in order to write the system in terms of an infinite system of ODE's. We will then derive our results based on these. For $\rho > 1$, we will consider a semigroup formed from A and the mappings of this semigroup between Sobelev spaces. In one illustration of this approach, we will treat A as the operator defined by $[\hat{Au}](\varsigma) = | \varsigma |^{\rho} \hat{u}(\varsigma)$ as would be suggested by the Fourier transform if we viewed A as defined on the entire real line.

We will first look at the case of $\rho > 1$ and apply the methods of Hillen and Potapov [5] to show that solutions remain global in this case under certain additional conditions on A. We will then consider the case $\rho < 1$ and use the methods of Halverson, Levine, and Renclawowicz [4,11] to partially show that there are solutions which blow up in finite time. The case of $\rho = 1$ will be handled in chapter 5, and we will show that it belongs to the blowup case.

1.4 The spaces under consideration

We now define the various spaces that we will need to consider.

As in [11] we define a space of sequences of real valued functions, $\ell^i_\beta([0,T])$, by

$$\{g_n\}_{n=1}^{\infty} \in \ell^i_\beta([0,T]) \text{ if and only if } \sum_{n=1}^{\infty} n^{\beta} \mid g_n(t) \mid^i < \infty$$

for all $t \in [0,T]$, where β is a non-negative real number. For non-negative real numbers α and β, we say that $\{g_n(t)\}_{n=1}^{\infty}$ is in $\ell^i_\beta([0,T]) \times \ell^j_\alpha([0,T])$ if $\{g_n(t)\}_{n=1}^{\infty} \in \ell^i_\beta([0,T])$ and $\{g'_n(t)\}_{n=1}^{\infty} \in \ell^j_\alpha([0,T])$. We will omit the subscript if it is 0 and omit the superscript if it is 1.

We will consider the typical $L^p([0,T])$ spaces which are the spaces of functions such that $\int_0^T \mid f(t) \mid^p \ dt < \infty$, and the Sobelev spaces $W^{p,q}([0,T])$ which are the spaces of functions such that $\frac{d^r}{dt^r} f(t) \in L^q([0,T])$ for $r \leq p$. As usual $L^\infty([a,b])$ is defined to be the set of functions, $f(t)$, which are bounded almost everywhere on (a,b).

We use $|| \cdot ||_{L^p[a,b]}$ to denote the $L^p[a,b]$ norm and will simply write $|| \cdot ||_{L^p}$ when the interval is clearly understood. To keep notation readable, we will sometimes, when necessary, only write $|| \cdot ||_p$ which will be understood to mean the same. In all cases $|| \cdot ||_{p,q}$ will denote the norm of the Sobelev space.

Chapter 2. Global Existence and the approach of Hillen and Potapov

We turn to proving that the solutions of (1.10) are global for $\rho > 1$ when Lemma 2 presented below holds for the operator A. We will use the approach of Hillen and Potapov [5] to prove that solutions are bounded above (in some norm) on any finite interval $[0, T]$ when the operator, A, satisfies Lemma 2. It follows from this that the solutions are global in this norm. We let $u_0(x)$ and $v_0(x)$ denote the initial values of $u(x,t)$ and $v(x,t)$.

2.1 Hillen and Potapov's approach to global existence

Hillen and Potapov consider the problem to which this paper concerns itself in the case that $A = aI - \triangle$, where I denotes the identity operator and $\triangle$ denotes the second derivative operator. In establishing global existence in this case, they consider the following lemma [5]:

Lemma 1. *Let M be a bounded n-dimensional C^∞ manifold without boundary. Let $T_\triangle(t) = e^{\triangle t}$ denote the solution semigroup of the heat equation on M. Assume $0 < t \leq 1,\ p \geq q,\ s \geq r$ then*

$$T_\triangle(t) : W^{r,q}(M) \to W^{s,p}(M)$$

with norm $Ct^{-\alpha}$, where

$$\alpha = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{1}{2}(s - r).$$

This lemma allows them to derive various bounds on the norms of the solutions which allow them to show that the solutions are bounded on finite time intervals.

Hillen and Potapov point out that, with periodic boundary conditions, the heat equation on a rectangle meets the assumptions on M [5]. Additionally, the conditions can be met with homogenous Neumann boundary conditions by considering reflected periodic extensions of the original rectangle [5].

2.2 The Main Lemma

The major mechanics of the approach are handled by two lemmas. In Hillen and Potapov, only Lemma 1 is needed. We will need a second lemma similar to this one that will serve the same purpose for the operator A. When our operator is one such that the lemma holds in the correct manner (to be discussed in section 2.3), we will be able to derive global existence. We use the following variation on Lemma 1:

Lemma 2. *Let M be a bounded n-dimensional C^∞ manifold without boundary. Let $T_A(t) = e^{-At}$. Assume $0 < t \leq 1,\ p \geq q,\ s \geq r$ then*

$$T_A(t) : W^{r,q}(M) \to W^{s,p}(M)$$

with norm bounded by $Ct^{-\alpha}$, where $\alpha = \alpha(r, q, s, p, n)$.

Notice that this is the same as lemma 1 except for the power of t involved. As we will only be interested in the one-dimensional case, we will drop the dependence on dimension.

Hillen and Potapov give [19, page 274] as a source for Lemma 1, however a proof of the lemma is not provided in this source, and one was not found through a search of the literature. We had hoped to be able to generalize a proof of Lemma 1 in order to prove Lemma 2.

To see how such a result may be derived, we use the following approach. Note that this approach, however, takes place on $\mathbb{R}$ which does not meet the requirements of the manifold in Lemma 2, so only provides an outline of an approach to the proof and not a fully rigorous proof.

Consider the problem $u_t + Au = 0$, where $u(x, 0) = u_0(x)$. Thus $u(x, t) = e^{-At}u_0(x)$.

Taking the Fourier transform of the equation, we have $\hat{u}_t(\varsigma) = - \mid \varsigma \mid^\rho \hat{u}$. Thus

$$\hat{u}(\varsigma) = e^{-|\varsigma|^\rho t}\hat{u}_0(\varsigma).$$

Then, we see that

$$u(x, t) = C \int_{-\infty}^{\infty} e^{\varsigma x i - |\varsigma|^\rho t}\hat{u}_0(\varsigma)\ d\varsigma,$$

and thus

$$u(x, t) = D \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x-y)\varsigma i - |\varsigma|^\rho t} u_0(y)\ d\varsigma\ dy.$$

Finally, we can see that operation with e^{-At} is equivalent to convolution with the kernel

$$K_\rho(x,t) = D \int_{-\infty}^{\infty} e^{x\varsigma i - |\varsigma|^\rho t} \, d\varsigma.$$

Also, we can show (by Fourier transforms) that

$$\frac{d^\alpha}{dx^\alpha}(u \star v) = \left(\frac{d^\beta}{dx^\beta} u\right) \star \left(\frac{d^{\alpha-\beta}}{dx^{\alpha-\beta}} v\right).$$

From the above and an inequality of convolutions, we have

$$|| \frac{d^\alpha}{dx^\alpha} K_\rho \star u ||_p \ \leq \ || \frac{d^{\alpha-\beta}}{dx^{\alpha-\beta}} K_\rho ||_r \ \ || \frac{d^\beta}{dx^\beta} u ||_q$$

where $\frac{1}{q} + \frac{1}{r} - \frac{1}{p} = 1$ [1, page 104].

Notice, again by Fourier transforms, that

$$\frac{d^c}{dx^c} K_\rho = D(-i)^c \int_{-\infty}^{\infty} \varsigma^c e^{x\varsigma i - |\varsigma|^\rho t} \, d\varsigma.$$

Then we have that

$$| \frac{d^c}{dx^c} K_\rho | \leq \ D \int_{-\infty}^{\infty} | \varsigma |^c \, e^{-|\varsigma|^\rho t} \, d\varsigma = 2D \int_0^{\infty} | \varsigma |^c \, e^{-|\varsigma|^\rho t} \, d\varsigma.$$

Using the transformation $u = t^{\frac{1}{\rho}} \varsigma$, we see that the integral on the right hand side is bounded above by $2Dt^{\frac{-(c+1)}{\rho}} \int_0^\infty | u |^c \, e^{-u^\rho} \, du$, which is in turn bounded above by $C(c)t^{\frac{-(c+1)}{k}}$ where the constant C depends on the order of the derivative c. Then

$$|| \frac{d^\alpha}{dx^\alpha} K_\rho \star u ||_p \leq C(\alpha - \beta, r) t^{\frac{-(\alpha-\beta+1)}{\rho}} \ || \frac{d^\beta}{dx^\beta} u ||_q \ .$$

Note that the lemma will hold with $\alpha(r, q, s, p) = \frac{1}{\rho}(s - r + 1)$. This is sufficient to establish what we want when $\rho \geq 2$. We will need a different value for α for other values of ρ. We derive one such value of α in section 2.6.

2.3 Admissible Parameters

We need to use Lemma 1 in nine different ways and Lemma 2 in three different ways in order to prove global existence and uniqueness of the solutions. Hillen and Potapov provide a criteria on parameters that they call *admissible parameters*. We need a similar concept. Using these two lemmas together, we will obtain integrals of the form $\int_0^t (t-s)^{-k}\, ds$ for various values of k that come from the lemmas. In order for these to converge, we must have $k < 1$. This is where the concept of admissible parameters enters.

We consider parameters $\sigma,\ p,\ r,\ P$, and Q in addition to ρ. Examining the various instances where Lemma 1 will be applied we will see values of $k = \frac{3}{4}$ and $k = \frac{1}{2}$ will be needed in the local existence and uniqueness proof and values of

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{p} - P\right), \qquad \frac{1}{p} - \frac{Q}{r}, \qquad 1 - \frac{1}{2p}, \qquad \frac{2-\sigma}{2} - \frac{1}{2}\left(\frac{1}{p} - \frac{2}{r}\right), \qquad \frac{\sigma}{2}, \qquad \frac{\sigma-1}{2} + \frac{1}{2p}, \qquad \frac{1}{2} + \frac{1}{2p}$$

in the global existence proof.

Doing the same for Lemma 2, the value $\alpha(0, p, \sigma, p)$ will appear in the local and global existence proofs and the value $\alpha(0, 1, \sigma, r)$ will appear in the global existence proof.

The instances of Lemma 1 will be the same as those in [5] that led Hillen and Potapov to consider the following constraints:

$$\begin{aligned} &1 < \sigma < 2, && \frac{1}{\sigma-1} < p < \infty, && \frac{2p}{\sigma p+1} < r < \frac{1}{\sigma-1}, \\ &1 < P < 1 + \frac{1}{p}, && \frac{1}{P} + \frac{1}{Q} = 1, && \frac{1}{p} < \frac{Q}{r} < \frac{1}{p} + 2 \end{aligned} \tag{2.1}$$

We need to add to this the constraints that $\alpha(0, p, \sigma, p) < 1$ and $\alpha(0, 1, \sigma, r) < 1$. This is all we need in order for the global existence and uniqueness arguments to apply to our problem. Notice that when $\rho \geq 2$ these are satisfied as they are both simply $\frac{\sigma}{\rho}$ and these are taken care of by the first constraint. When our estimates in Section 2.6 hold for $\rho < 2$, these constraints can be satsified by modifiing the first constraint of (2.1) above to $1 < \sigma < \rho$ and adding the constraint $r < \frac{1}{2-\rho}$.

Hillen and Potapov provide an algorithm for finding admissible parameters. We provide a slight variation to their algorithm that will work in our case in Appendix A.

Before proving global existence of solutions we first establish local existence and uniqueness by an application of the contraction mapping principle. The proof is similar to that used by

Hillen and Potapov. The major distinguishing feature is the replacement of $T_\triangle$ by T_A at the appropriate places in the argument.

2.4 Local Existence and Uniqueness

Let $z \in L^\infty([0,1]\times[0,T])$. Then, solving $v_t = u - Av$ with $v(0) = v_0$ where $u = z$, we have

$$v(t) = T_A(t)v_0 + \int_0^t T_A(t-s)z(s)\ ds.$$

With $0 \le t \le 1$, we have $T_A : L^p \to W^{\sigma,p}$ with norm $C_1 t^{-\alpha_1}$ where $\alpha_1 = \alpha(0,p,\sigma,p)$ and $T_A : W^{\sigma,p} \to W^{\sigma,p}$ with norm 1. Thus

$$\begin{aligned}
|| \, v(t) \, ||_{\sigma,p} &\le || \, v_0 \, ||_{\sigma,p} + \int_0^t || \, T_A(t-s)z(s) \, ||_{\sigma,p}\ \ ds \\
&\le || \, v_0 \, ||_{\sigma,p} + \int_0^t C_1(t-s)^{-\alpha_1} \, || \, z(s) \, ||_p\ \ ds \\
&\le || \, v_0 \, ||_{\sigma,p} + \int_0^t C_1(t-s)^{-\alpha_1} \sup_{[0,T]} || \, z(t) \, ||_\infty\ \ ds \\
&\le || \, v_0 \, ||_{\sigma,p} + C_1 t^{1-\alpha_1} \sup_{[0,T]} || \, z(t) \, ||_\infty \ .
\end{aligned}$$

Now, we look at the second equation, $u_t = u_{xx} - u_x v_x - u v_x$ with $u(0) = u_0$. This gives us

$$u(t) = T_\triangle(t)u_0 - \int_0^t T_\triangle(t-s)u_x v_x\ ds - \int_0^t T_\triangle(t-s)u v_{xx}\ ds.$$

We have the Sobelev embedding $W^{\sigma-1,p} \hookrightarrow C^0$ for $p > \frac{1}{\sigma-1}$. Also $T_\triangle : W^{-1,q} \to W^{\frac{1}{2},q}$ with norm $C_2 t^{-\frac{3}{4}}$. Let $q > 2$ so that $W^{\frac{1}{2},q} \hookrightarrow C^0$. Note that $|| \, u \, ||_p \le || \, u \, ||_\infty$ as we are interested in functions on $[0,1]$. Then

$$\begin{aligned}
|| \int_0^t T_\triangle(t-s)u_x v_x\ ds \, ||_\infty &\le \sup_{[0,T]} || \, v_x \, ||_\infty \int_0^t || \, T_\triangle(t-s)u_x \, ||_\infty\ \ ds \\
&\le C_2 t^{\frac{1}{4}} \sup_{[0,T]} || \, v \, ||_{1,\infty} \sup_{[0,T]} || \, u \, ||_\infty \ .
\end{aligned}$$

With our choice of p, $W^{\sigma,p} \hookrightarrow C^1$. We see that

$$|| \int_0^t T_\triangle(t-s) u_x v_x \; ds ||_\infty \leq C_2 t^{\frac{1}{4}} \sup_{[0,T]} || v ||_{\sigma,p} \sup_{[0,T]} || u ||_\infty .$$

Using that $T_\triangle : W^{\sigma-2,p} \to W^{\sigma-1,p} \hookrightarrow C^0$ with norm $C_3 t^{-\frac{1}{2}}$, we have that

$$|| \int_0^t T_\triangle(t-s) u v_{xx} \; ds ||_\infty \leq C_3 t^{\frac{1}{2}} \sup_{[0,T]} || v ||_{\sigma,p} \sup_{[0,T]} || u ||_\infty .$$

Combining all results, we have that

$$|| u(t) ||_\infty \leq || u_0 ||_\infty + C_0 \left(t^{\frac{1}{4}} + t^{\frac{1}{2}} \right) \left(|| v_0 ||_{\sigma,p} + C_1 t^{1-\alpha_1} \sup_{[0,T]} || z(t) ||_\infty \right) \sup_{[0,T]} || u ||_\infty$$

or

$$|| u ||_{L^\infty([0,1]\times[0,T])} \leq \frac{|| u_0 ||_\infty}{1 - C_0 \left(T^{\frac{1}{4}} + T^{\frac{1}{2}} \right) \left(|| v_0 ||_{\sigma,p} + C_1 T^{1-\alpha_1} \; || z(t) ||_{L^\infty([0,1]\times[0,T])} \right)}$$

where $C_0 = \max\{C_2, C_3\}$.

Define a mapping $H : L^\infty([0,1] \times [0,T]) \to L^\infty([0,1] \times [0,T])$ by $Hz = u$, i.e. given a function z let u be the function obtained by first solving the v equation above and then solving the u equation. Let $m > || u_0 ||_\infty$. Then for small T, H takes $B_m(0)$ into itself where

$$B_m(0) = \{ \phi(t) \in L^\infty([0,1] \times [0,T]) \mid \; || \phi ||_{L^\infty([0,1]\times[0,T])} < m, \phi(0) = u_0 \}.$$

Let $z, Z \in B_m(0)$ and let $u = Hz$, $U = HZ$. Let v, V be the corresponding solutions of the v equations. Then, as the v equation is linear,

$$|| v(t) - V(t) ||_{\sigma,p} \leq C_1 t^{1-\alpha_1} \sup_{[0,T]} || z(t) - Z(t) ||_\infty .$$

Also, we know that

$$\begin{aligned} u(t) - U(t) = & - \int_0^t T_\triangle(t-s) \left[(u_x - U_x) \, v_x + U_x \, (v_x - V_x) \right] \; ds \\ & - \int_0^t T_\triangle(t-s) \left[(u - U) \, v_{xx} + U \, (v_{xx} - V_{xx}) \right] \; ds. \end{aligned}$$

Then, by our estimates above,

$$|| u(t) - U(t) ||_\infty \leq C_0 \left(t^{\frac{1}{4}} + t^{\frac{1}{2}} \right) \left(\sup || u - U ||_\infty \sup || v ||_{\sigma,p} + \sup || U ||_\infty \sup || v - V ||_{\sigma,p} \right)$$

with $C_0 = \max\{C_2, C_3\}$ again (the supremum terms are taken over $[0,T]$).

There is some constant $\hat{C} = \hat{C}(|| u_0 ||_\infty,\ || v_0 ||_{\sigma,p},\ T)$ so that

$$\max\{\sup_{[0,T]} || v ||_{\sigma,p},\ \sup_{[0,T]} || U ||_\infty\} \leq \hat{C}$$

by the estimates above. Thus

$$|| Hz - HZ ||_{L^\infty([0,1]\times[0,T])} \leq \left(\frac{C_1 \hat{C} T^{1-\alpha_1} \left(T^{\frac{1}{2}} + T^{\frac{1}{4}}\right)}{1 - \hat{C}\left(T^{\frac{1}{2}} + T^{\frac{1}{4}}\right)} \right) || z - Z ||_{L^\infty([0,1]\times[0,T])} .$$

Then, we can see that if T is small enough, H is a contraction on $B_m(0)$ and there is a unique local solution of the initial value problem in the sense that

$$\max\{\sup_{[0,T]} || u(t) ||_\infty, \sup_{[0,T]} || v(t) ||_{\sigma,p}\} < \infty.$$

2.5 Global Existence

We prove the following theorem:

Theorem 2. *Let* $u_0 \in L^1([0,1]) \bigcap L^\infty([0,1])$ *and* $v_0 \in W^{\sigma,p}([0,1])$. *Let* (σ, p, r, P, Q) *be admissible. Then for* $T > 0$, *there is* $K(T) = K(T, || u_0 ||_1, || u_0 ||_\infty, || v_0 ||_{\sigma,p}, \sigma, p, r, P, Q)$ *such that*

$$\max\{\sup_{0\leq t\leq T} || u(t) ||_\infty,\ \sup_{0\leq t\leq T} || v(t) ||_{\sigma,p}\} \leq K(T).$$

We follow Hillen and Potapov's approach. We will establish each of their lemmas in turn using the two Lemmas 1 and 2 in place of their single Lemma 1. We first establish an upper bound on $|| v(t) ||_{\sigma,r}$ where σ and r come from the admissible parameters. We then establish an upper bound on $|| u(t) ||_p$ using this bound on $v(t)$. Note that as $u(t)$ is in $L^\infty[0,1]$ for all small positive t, it must be in $L^p[0,1]$ for such t as well. In fact, it is worth noting again that $|| u(t) ||_p \leq || u(t) ||_\infty$ in the spaces we are interested in. With this we will be able to obtain a bound on $|| u(t) ||_\infty$. The existence of these bounds will tell us that the pair $(u(t), v(t))$ gives a global solution in the sense that $u(t) \in L^\infty[0,1]$ and $v(t) \in W^{\sigma,p}[0,1]$ for all $t > 0$.

The key to Hillen and Potapov's approach is the application of Lemma 1, hence proofs of these lemmas will be quite similar to their's. The most significant difference will be the appearance of both operators $T_{\triangle}(t)$ and $T_A(t)$ instead of just $T_{\triangle}$. We need to apply both Lemma 1 and Lemma 2 to handle these two operators.

We first prove the bound on $v(t)$. We need the observation that the u equation is a conservation equation which preserves positivity [5]. Thus $\| u(t) \|_{L^1} = \| u_0 \|_{L^1} \equiv M$ for all t where the solution exists.

Lemma 3. *Let $0 < t \leq 1$. Then $\| v(t) \|_{\sigma,r} \leq \| v_0 \|_{\sigma,r} + C_4 t^{1-\alpha_4} M$ where M, C_4, and α_4 will be defined below in the proof. Also for $T > 0$, there is some $K_1(T) = K_1(T, \| v_0 \|_{\sigma,r}, C_4, M)$ such that*

$$\sup_{0 \leq t \leq T} \| v(t) \|_{\sigma,r} \leq K_1(T).$$

Proof. As $v_t = u - Av$, we have that

$$v(t) = T_A(t) v_0 + \int_0^t T_A(t-s) u(s)\, ds.$$

Using lemma 2, we see that $T_A(t) : L^1 \to W^{\sigma,r}$ with norm $C_4 t^{-\alpha_4}$ where $a_4 = \alpha(0, 1, \sigma, r)$. Therefore, if $0 \leq a_4 < 1$, we have

$$\| v(t) \|_{\sigma,r} \leq \| v_0 \|_{\sigma,r} + C_4 t^{1-\alpha_4} \sup_{[0,T]} \| u(s) \|_{L^1} \leq \| v_0 \|_{\sigma,r} + C_4 t^{1-\alpha_4} M.$$

If we take the solution at time $t = 1$ to be the initial condition of a new problem, we can solve this on $[1, 2]$ and get a bound for the solution on $[0, 2]$ by using the larger bound. Repeating this, we obtain the bound $K_1(T)$. □

We can now use our bound on $v(t)$ in order to establish the $L^p[0, 1]$ bound on $u(t)$.

Lemma 4. *With (σ, p, r, P, Q) admissible, there is some $K_2(T) = K_2(T, \| u_0 \|_p, M, K_1(T))$ so that*

$$\sup_{0 \leq t \leq T} \| u(t) \|_p \leq K_2(T).$$

Proof. As $u_t = u_{xx} - u_x v_x - u v_{xx}$, we have that

$$u(t) = T_{\triangle}(t) u_0 - \int_0^t T_{\triangle}(t-s) u_x v_x\, ds - \int_0^t T_{\triangle}(t-s) u v_{xx}\, ds.$$

Using lemma 1 with $\alpha_5 = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{p} - P\right)$ where $1 + \frac{1}{p} > P$ and $\frac{1}{P} + \frac{1}{Q} = 1$, we have, by Young's inequality, that

$$\| \int_0^t T_\triangle(t-s) u_x v_x \; ds \|_p \leq \| \int_0^t T_\triangle(t-s) \frac{| u_x |^P}{P} \; ds \|_p + \| \int_0^t T_\triangle(t-s) \frac{| v_x |^Q}{Q} \; ds \|_p$$

and thus, with $t \in (0, 1]$,

$$\| \int_0^t T_\triangle(t-s) \frac{| u_x |^P}{P} \; ds \|_p \leq C_5 t^{1-\alpha_5} \frac{M^P}{P} \leq C_5 \frac{M^P}{P}.$$

Using the same lemma with $\alpha_6 = (\frac{1}{p} - \frac{Q}{r})$ and noting that $\frac{1}{p} < \frac{Q}{r} < 2 + \frac{1}{p}$, we have that

$$\| \int_0^t T_\triangle(t-s) \frac{| v_x |^Q}{Q} \; ds \|_p \leq C_6 t^{1-\alpha_6} \sup_t \| v \|_{\sigma,r}^Q \leq C_6 K_1^Q(1)$$

which relies on the fact that $v_x^Q \in L^{\frac{r}{Q}}$. Now, we note that, also by Young,

$$\| \int_0^t T_\triangle(t-s) u v_{xx} \; ds \|_p \leq \| \int_0^t T_\triangle(t-s) \frac{u^2}{2} \; ds \|_p + \| \int_0^t T_\triangle(t-s) \frac{v_{xx}^2}{2} \; ds \|_p .$$

Using lemma 1 with $\alpha_7 = 1 - \frac{1}{2p}$ for the first term and $\alpha_8 = \frac{2-\sigma}{2} - \frac{1}{2}\left(\frac{1}{p} - \frac{2}{r}\right)$ for the second, we see that this is bounded by $C_7 M^2 + C_8 K_1^2(1)$. Thus,

$$\| u(t) \|_p \leq \| u_0 \|_p + \left(C_5 \frac{M^P}{P} + C_6 K_1^Q(1) + C_7 M^2 + C_8 K_1^2(1) \right).$$

Iterating as we did in lemma 3, we obtain the bound $K_2(T)$. □

With this bound on $u(t)$, we will turn to proving a slightly different bound on $v(t)$ which will depend on fewer parameters. In particular, it will depend on the upper bound of $u(t)$ as well as the initial conditions on v. This will only rely on the existence of an upper bound on $u(t)$, as the last lemma asserts, instead of the specific bound from lemma 3.

Lemma 5. *For $T > 0$, there is $K_3(T) = K_3(\| v_0 \|_{\sigma,p}, K_2(T))$ such that*

$$\sup_{0 \leq t \leq T} \| v(t) \|_{\sigma,p} \leq K_3(T).$$

Proof. Let $\alpha_9 = \alpha(0, p, \sigma, p)$. Then,

$$\| v(t) \|_{\sigma,p} \leq \| v_0 \|_{\sigma,p} + \| \int_0^t T_A(t-s) u(s) \; ds \|_{\sigma,p}$$

$$\leq \| v_0 \|_{\sigma,p} + \int_0^t \| T_A(t-s)u(s) \|_{\sigma,p} \; ds$$

$$\leq \| v_0 \|_{\sigma,p} + \left(C_9 \int_0^t (t-s)^{\alpha_9} \; ds \right) \sup_{[0,T]} \| u \|_p$$

$$\leq \| v_0 \|_{\sigma,p} + C_9 t^{1-\alpha_9} K_2(1).$$

Again, by iteration, we obtain $K_3(T)$. □

We now have the desired bound on $v(t)$. Finally, we can turn to establishing the desired bound on $u(t)$.

Lemma 6. *For $T > 0$ and (σ, p, r, P, Q) admissible, there is a $K_4(T) = K_4(T, \| u_0 \|_\infty, K_2, K_3)$ such that*

$$\sup_{0 \leq t \leq T} \| u(t) \|_\infty \leq K_4(T).$$

Proof. Note that

$$\| u(t) \|_\infty \leq \| u_0 \|_\infty + \| \int_0^t T_\triangle(t-s) u_x v_x \; ds \|_\infty + \| \int_0^t T_\triangle(t-s) u v_{xx} \; ds \|_\infty \, .$$

Note now that if $p > \frac{1}{\sigma-1}$, then $W^{\sigma-1,p} \hookrightarrow C^0$. Also, note that $v_x \in W^{\sigma-1,p}$. By lemma 1, $T_\triangle : W^{-1,p} \to W^{\sigma-1,p}$ with norm $C_{10} t^{-\frac{\sigma}{2}}$. Thus

$$\| \int_0^t T_\triangle(t-s) u_x v_x \; ds \|_\infty \leq C_{10} t^{1-\frac{\sigma}{2}} \sup_t \| u \|_p \| v \|_{\sigma,p} \, .$$

Also, we know that

$$\| \int_0^t T_\triangle(t-s) u v_{xx} \; ds \|_\infty \leq \| \int_0^t T_\triangle(t-s) \frac{u^2}{2} \; ds \|_\infty + \| \int_0^t T_\triangle(t-s) \frac{v_{xx}^2}{2} \; ds \|_\infty \, .$$

As $u^2 \in L^{\frac{p}{2}}$ and $T_\triangle(t) : L^{\frac{p}{2}} \to W^{\sigma-1,p} \hookrightarrow C^0$ with norm $C_{11} t^{-\alpha_{11}}$ where $\alpha_{11} = \frac{\sigma-1}{2} + \frac{1}{2p}$,

$$\| \int_0^t T_\triangle(t-s) \frac{u^2}{2} \; ds \|_\infty \leq C_{11} t^{1-\alpha_{11}} \sup_{[0,T]} \| u \|_p^2 \, .$$

Also, $T_\triangle(t) : W^{\sigma-2,\frac{p}{2}} \to W^{\sigma-1,p} \to C^0$ with norm $C_{12}t^{-\alpha_{12}}$ where $\alpha_{12} = \frac{1}{2} + \frac{1}{2p}$. Thus

$$|| \int_0^t T_\triangle(t-s)\frac{v_{xx}^2}{2}\, ds ||_\leq C_{12}t^{1-\alpha_{12}} \sup_{[0,T]} || v ||_{\sigma,p}^2 .$$

Taken all together, and with $0 \leq t \leq 1$, we have

$$|| u ||_\infty \leq || u_0 ||_\infty + \left(C_{10}K_2(1)K_3(1) + C_{11}K_2^2(1) + C_{12}K_3^2(1)\right).$$

Once again, by iteration, we obtain $K_4(T)$. □

Notice that the first lemma only used the fact that $u(t)$ is bounded in the L^1 norm. Thus as long as $u(t)$ exists, $v(t)$ exists and is bounded in norm on compact intervals. The second lemma only used the fact that $v(t)$ exists and is bounded in norm. The third lemma used that $u(t)$ exists and is bounded in the L^p norm. Thus it follows that $u(t)$ is global in the L^p sense and that $v(t)$ is global in the $W^{\sigma,p}$ sense. From this we then have that $u(t)$ is essentially bounded on all compact intervals.

Looking at how these lemmas built on each other, we only needed that the u equation was a conservation equation and that the initial functions were in the right spaces. Although we had that $|| u(t) ||_1 = || u_0(t) ||_1$, we only needed that $|| u(t) ||_1 \leq || u_0(t) ||_1$ for the first lemma. This was the only place that we used this fact. Theorem 2 follows immediately from these lemmas. In particular, it is just a combined statement of Lemmas 5 and 6.

2.6 Lemma 2 when $1 < \rho < 2$

We consider Lemma 2 when $\rho < 2$. We use an approximation to the value of $|| \frac{d^{\alpha-\beta}}{dx^{\alpha-\beta}}K_\rho ||_z$ from section 2.2 for $1 < \rho < 2$. Note that

$$\frac{d^c}{dx^c}K_\rho = D\int_{-\infty}^{\infty} (i\varsigma)^c e^{ix\varsigma - |\varsigma|^\rho t}\, d\varsigma.$$

First consider the integral above along $[0, \infty)$. We denote this as $K_{\rho,c}^+$. Looking at the power of the exponential, let $F^+(x,t,\varsigma) = ix\varsigma - \varsigma^\rho t,$. Then $\frac{\partial}{\partial\varsigma}F^+(x,t,\varsigma) = ix - t\rho\varsigma^{\rho-1}$. This is zero at the value $\varsigma_0^+ = \left(\frac{ix}{\rho t}\right)^{\frac{1}{\rho-1}}$. Now let $G^+(x,t) = F^+(x,t,\varsigma_0^+)$, and let $H^+(x,t,\varsigma) = F^+(x,t,\varsigma) - G^+(x,t)$. Then

$$G^+(x,t) = (ix)^{\frac{\rho}{\rho-1}}t^{-\frac{1}{\rho-1}}\rho^{-\frac{1}{\rho-1}}\left(\frac{\rho-1}{\rho}\right).$$

Now, we approximate $H^+(x,t,\varsigma)$ around ς_0^+. We have

$$H^+(x,t,\varsigma) = -C(ix)^{-\frac{2-\rho}{\rho-1}} t^{\frac{1}{\rho-1}} \left(\varsigma - \varsigma_0^+\right)^2 + \text{higher order terms}.$$

Then

$$K_{\rho,c}^+ = e^{G^+(x,t)} \int_0^\infty (i\varsigma)^c \exp\left(-C(ix)^{-\frac{2-\rho}{\rho-1}} t^{\frac{1}{\rho-1}} \left(\varsigma - \varsigma_0^+\right)^2 + \text{higher order terms}\right) \, d\varsigma.$$

If the higher order terms are such that they can be ignored in the approximation, then with the correct change of variables, we have

$$K_{\rho,c}^+ = \frac{e^{G^+(x,t)} i^c}{(A(x,t))^{1+c}} \int_{-\varsigma_0^+ A(x,t)}^{\infty - \varsigma_0^+ A(x,t)} (A(x,t)\varsigma_0^+ + \varsigma)^c e^{-\varsigma^2} \, d\varsigma$$

where $A(x,t) = \sqrt{C}(ix)^{-\frac{2-\rho}{2(\rho-1)}} t^{\frac{1}{2(\rho-1)}}$.

A search of the literature was unable to turn up any detailed explanation on when the higher order terms could be ignored. The only source found with a detailed treatment of such methods was [17], however, the asymptotic behavior derived was not the type needed for this treatment of the problem. This is discussed more in Sections 9.1 and 9.3.

We now do the same for $K_{\rho,c}^-$. Then, we see that $\varsigma_0^- = (-1)^{\frac{\rho}{\rho-1}} \varsigma_0^+$. Following the same steps as in the previous case, similar formulas hold except for the appearance of a factor of $(-1)^{\frac{\rho}{\rho-1}}$. Let ρ be such that $(-1)^{\frac{1}{\rho-1}} = 1$. We will discuss this requirement in section 2.7.1

When this is true

$$K_{\rho,c}^- = \frac{e^{G^+(x,t)} i^c}{(A(x,t))^{1+c}} \int_{-\infty - \varsigma_0^+ A(x,t)}^{-\varsigma_0^+ A(x,t)} (A(x,t)\varsigma_0^+ + \varsigma)^c e^{-\varsigma^2} \, d\varsigma.$$

Then, we see that

$$\frac{d^c}{dx^c} K_\rho \approx \frac{e^{G^+(x,t)} i^c}{(A(x,t))^{1+c}} \int_{-\infty - \varsigma_0^+ A(x,t)}^{\infty - \varsigma_0^+ A(x,t)} (A(x,t)\varsigma_0^+ + \varsigma)^c e^{-\varsigma^2} \, d\varsigma. \tag{2.2}$$

By considering the correct paths in the complex plane, we see that the the absolute value of the integral involved above has a bound dependent upon $A(x,t)$. Thus, we have,

$$\left| \frac{d^c}{dx^c} K_\rho(x,t) \right| \leq C(c) \frac{\left(\max\{1, | A(x,t)\varsigma_0^+ |^c\}\right)}{| A(x,t) |^{1+c}} e^{\Re G^+(x,t)} \tag{2.3}$$

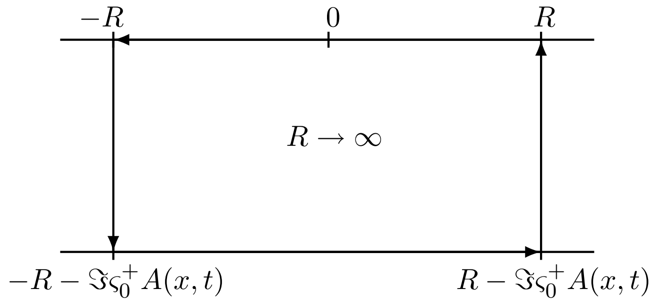

Figure 2.1 The contour used for evaluating integral (2.2)

where $C(c) = \int_{-\infty}^{\infty} (1+ \mid u \mid^c)\, e^{-u^2}\ du,\ \Re$ indicates the real-value function, and $\Im$ indicates the imaginary-value function.

In order to ensure that the exponential term is integrable, we need that $\Re\ i^{\frac{\rho}{\rho-1}} < 0$. This condition will be discussed in section 2.7.2.

We need to compute the L^z norm of the right hand side of (2.3). By carefully keeping track of the powers of t and by making an appropriate change of variables to remove the t term from the power of the exponential, we obtain

$$|| \frac{d^c}{dx^c} K_\rho(x,t) ||_z \leq K t^{\left(\frac{1}{\rho}\right)\left(\frac{1}{z}-1\right)-\frac{c}{\rho}}$$

for some computable constant K. Then, noting that $c = s - r$ and $\frac{1}{z} = 1 + \frac{1}{p} - \frac{1}{q}$, we see that when the above approximations hold, lemma 2 holds with

$$\alpha(r,q,s,p) = \frac{s-r}{\rho} + \frac{1}{\rho}\left(\frac{1}{q} - \frac{1}{p}\right).$$

Notice that this value of α is identical to the one which appears in lemma 1, except for the multiple of $\frac{1}{2}$ has been replaced by the multiple $\frac{1}{\rho}$. This does not seem surprising. If $\rho = 2$, we are in the case of Lemma 1 and thus it should reduce to the power in the Lemma and it does as expected.

It is worth noting that the approximation method above does obtain true equality with the heat kernel when applied to it's integral form, which strongly justifies it's use.

2.7 Restrictions required on ρ for the case $1 < \rho < 2$.

Here we discuss the special requirements that ρ must satisfy for the approximations used in establishing lemma 2. Recall that these requirements came from the attempt to prove lemma 2 for the case $1 < \rho < 2$. Therefore these requirements are not needed when $\rho > 2$ and the lemma is established for all such values of ρ in this case. We first turn to the requirement that ρ is a rational number of a certain form. We discuss another complication of this approximation in appendix A.

2.7.1 Rational form of ρ

Recall that we needed ρ to be such that $(-1)^{\frac{1}{\rho-1}} = 1$. We look at what values of ρ satisfy this condition. If we write $-1 = \exp((1+2k)\pi i)$ then $1 + 2k = 2m(\rho - 1)$ for integers k and m. Therefore

$$\rho = 1 + \frac{k}{m} + \frac{1}{2m}.$$

For numbers of this form, we can state the following:

Theorem 3. *Let* $\mathbb{P} = \{x \in \mathbb{Q} \bigcap (1,2) \mid x = 1 + \frac{k}{m} + \frac{1}{2m}$ *for some* $k, m \in \mathbb{Z}\}$*. Then* $\mathbb{P}$ *is dense in* $(1,2)$.

Proof. Let $x \in (1,2)$ be rational. Let $\epsilon > 0$ be given.

Choose $l > 0$ an integer so that $10^{-l} < \epsilon$.

Choose $a > 0$ and $b > 0$ to be integers so that $x = 1 + \frac{a}{b}$. Let $k = 10^l a$ and $m = 10^l b$. Let $y = 1 + \frac{k}{m} + \frac{1}{2m}$. Then

$$0 < y - x = \frac{1}{(2)(10^l b)} \leq 10^{-l} < \epsilon.$$

Thus there is a y in $\mathbb{P}$ within ϵ of x, and then $\mathbb{P}$ is dense in $\mathbb{Q} \bigcap (1,2)$.

Therefore $\mathbb{P}$ is dense in $(1,2)$. □

In view of Theorem 3, we believe that this restriction on ρ is not actually required and is an artificial condition that arises because of the method of proof. This would need to be shown, but it seems likely that if the result is true on a dense set in the interval $(1,2)$ that it would be true on the whole interval.

2.7.2 Interval condition on ρ

Recall, that we additionally needed $\Re\, i^{\frac{\rho}{\rho-1}} < 0$. Again, by examining the polar form of the expression, we can see in this case that

$$1 + 4n < (1 + 4k)\rho^\star < 3 + 4n$$

where $\rho^\star = \frac{\rho}{\rho-1}$ or $\rho > \frac{3}{2}$.

From the above section $\rho^\star$ must be rational. Then we can choose k an integer so that $4k\rho^\star \in \mathbb{Z}$.

Then $1 + 4n - 4k\rho^\star < \rho^\star < 3 + 4n - 4k\rho^\star$. Letting $m = n - k\rho^\star$, we obtain

$$1 + 4m < \rho^\star < 3 + 4m.$$

Now, if we solve the two directions of inequality for ρ, we obtain

$$\frac{3 + 4m}{2 + 4m} < \rho < \frac{1 + 4m}{4m}$$

or

$$1 + \frac{1}{2 + 4m} < \rho < 1 + \frac{1}{4m}.$$

Then, we see that ρ must be in

$$\left(\frac{3}{2}, \infty\right) \bigcup \left(\frac{7}{6}, \frac{5}{4}\right) \bigcup \left(\frac{11}{10}, \frac{9}{8}\right) \bigcup \cdots.$$

Note that this leaves 'gaps' in the values of ρ that we can consider. It is important to note that the last such gap is $[\frac{5}{4}, \frac{3}{2}]$.

Notice that these gaps tend to *cluster* around 1, ie that there is a gap in any interval of the form $(1, k)$ for $k > 1$ and that the maximum size of the gaps in such an interval become smaller as $k \to 1$.

Noting that these gaps have length $\frac{1}{4}\left(\frac{1}{(1+m)(1+2m)}\right)$, we see that the total length of the gaps is given by

$$\frac{1}{4} + \frac{1}{4}\sum_{m=1}^{\infty} \frac{1}{(1 + m)(1 + 2m)}.$$

Noting that $\frac{1}{1+2m} \leq \frac{1}{1+m}$ we can bound this number above by $\frac{\pi^2}{24} \approx 0.41$. Thus, just under half the interval $(1,2)$ falls outside our acceptable values of ρ (a better estimate of 0.35 can be obtained by using the 100th partial sum and the upper bound on the tail from the integral test).

Chapter 3. Derivation of the ODE problem

3.1 Derivation of the Full Problem

Following the methods of [11], we seek a solution of the form

$$u(x,t) = U(t) + \tilde{u}(x,t) \tag{3.1}$$
$$v(x,t) = V(t) + \Psi(x,t)$$

where $(U(t), V(t))$ is the spatially homogenous solution of (1.10).

Now, noting that $U(t) = \mu$ is constant, from the first equation in 1.10 we can see that

$$\tilde{u}_t = \tilde{u}_{xx} - \left((\mu + \tilde{u})\,\Psi_x\right)_x = \tilde{u}_{xx} - \mu\Psi_{xx} - \tilde{u}_x\Psi_x - \tilde{u}\Psi_{xx}. \tag{3.2}$$

From the second equation, we can see that $V' + \Psi_t = \mu + \tilde{u} - AV - A\Psi$ and thus, as $V' = \mu - AV$,

$$\tilde{u} = \Psi_t + A\Psi, \tag{3.3}$$
$$\tilde{u}_t = \Psi_{tt} + A\Psi_t,$$
$$\tilde{u}_x = \Psi_{tx} + (A\Psi)_x\,,$$
$$\tilde{u}_{xx} = \Psi_{txx} + A\Psi_{xx}.$$

The fourth line uses the hypothesis that A commutes with the Laplacian. Using these with (3.2), we see that $\Psi(x,t)$ satisfies the following:

$$\Psi_{tt} + (\mu - A)\Psi_{xx} - \Psi_{txx} + A\Psi_t = -(\Psi_t\Psi_x)_x - ((A\Psi)\Psi_x)_x. \tag{3.4}$$

Proceeding in the same manner as [11], we seek a solution of the form

$$\Psi(x,t) = \sum_{n=1}^{\infty} g_n(t)\cos(Cnx) \tag{3.5}$$

where C is an integer multiple, M, of 2π so that the conservation equations

$$\int_0^1 \Psi_t \; dx = \int_0^1 A\Psi \; dx = 0$$

hold in order to ensure that the mass $\int_0^1 u \; dx$ is conserved, where

$$u(x,t) = \mu_0 + \Psi_t(x,t) + A\Psi(x,t).$$

Looking at the left hand side of (3.4), we have

$$\begin{aligned} \Psi_{tt} &= \sum_{n=1}^{\infty} g_n''(t)\cos(Cnx)\,, \\ \Psi_{xx} &= -C^2\sum_{n=1}^{\infty} n^2 g_n(t)\cos(Cnx)\,, \\ \Psi_{txx} &= -C^2\sum_{n=1}^{\infty} n^2 g_n'(t)\cos(Cnx)\,, \\ A\Psi_{xx} &= -C^2\sum_{n=1}^{\infty} n^2\lambda_n g_n(t)\cos(Cnx)\,, \\ A\Psi_t &= \sum_{n=1}^{\infty} \lambda_n g_n'(t)\cos(Cnx)\,. \end{aligned} \tag{3.6}$$

Then we see that the left hand side of (3.4) becomes

$$\sum_{n=1}^{\infty}\left[g_n'' + \left(C^2n^2+\lambda_n\right)g_n' - \left(\mu-\lambda_n\right)C^2n^2 g_n\right]\cos(Cnx)\,. \tag{3.7}$$

Now, looking at the right hand side, we find

$$\Psi_t = \sum_{n=1}^{\infty} g_n'(t)\cos(Cnx)\,, \tag{3.8}$$

$$\Psi_x = -C\sum_{n=1}^{\infty} n g_n(t)\sin(Cnx)\,,$$

$$A\Psi = \sum_{n=1}^{\infty} \lambda_n g_n(t)\cos(Cnx)\,.$$

Then the first term on the right hand side becomes

$$(\Psi_t \Psi_x)_x = -C\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n g_n g_k' \sin(Cnx)\cos(Ckx)\right]_x$$

$$= -C^2 \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[n^2 g_n g_k' \cos(Cnx)\cos(Ckx) - nk g_n g_k' \sin(Cnx)\sin(Ckx)\right]$$

$$= -\frac{C^2}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[n^2 g_n g_k' \left(\cos(C(n+k)x) + \cos(C(n-k)x)\right)\right]$$
$$+ \frac{C^2}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[nk g_n g_k' \left(\cos(C(n-k)x) - \cos(C(n+k)x)\right)\right]$$

$$= -\frac{C^2}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[n g_n g_k' \left((n+k)\cos(C(n+k)x) + (n-k)\cos(C(n-k)x)\right)\right]$$

$$= -\frac{C^2}{2}\sum_{n=2}^{\infty}\left[n\left(\sum_{k=1}^{n-1} k g_k g_{n-k}'\right)\cos(Cnx)\right] + \frac{C^2}{2}\sum_{n=1}^{\infty}\left[n\left(\sum_{k=1}^{\infty} k g_k g_{n+k}'\right)\cos(Cnx)\right]$$
$$- \frac{C^2}{2}\sum_{n=1}^{\infty}\left[n\left(\sum_{k=1}^{\infty} k g_k g_{k-n}'\right)\cos(Cnx)\right].$$

The second term can be written as

$$((A\Psi)\,\Psi_x)_x = -C\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n\lambda_k g_n g_k \cos(Ckx)\sin(Cnx)\right]_x$$

$$= -C^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[n^2\lambda_k g_n g_k \cos(Cnx)\cos(Ckx) - nk\lambda_k g_n g_k \sin(Ckx)\sin(Cnx)\right]$$

$$= -\frac{C^2}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[n^2\lambda_k g_n g_k \left(\cos(C(n+k)x) + \cos(C(n-k)x)\right)\right]$$
$$+ \frac{C^2}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[nk\lambda_k g_n g_k \left(\cos(C(n-k)x) - \cos(C(n+k)x)\right)\right]$$

$$= -\frac{C^2}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n\lambda_k g_n g_k \left[(n+k)\cos(C(n+k)x) + (n-k)\cos(C(n-k)x)\right]$$

$$= -\frac{C^2}{4}\sum_{n=2}^{\infty} n\left(\sum_{k=1}^{\infty}\left(k\lambda_{n-k} + (n-k)\lambda_k\right) g_k g_{n-k}\right)\cos(Cnx)$$
$$-\frac{C^2}{2}\sum_{n=1}^{\infty} n\left(\sum_{k=1}^{\infty}\left((n+k)\lambda_k - k\lambda_{n+k}\right) g_k g_{n+k}\right)\cos(Cnx).$$

The first sum here is found by taking the $\cos(C(n+k)x)$ term in the previous line and considering the values of $m = n+k$ and summing over m and n. Then the role of m and n in the sum is interchanged. These two sums are then added and the result is divided by two. The last sum is found from the $\cos(C(n-k)x)$ term by looking at the part of the sum where $(n-k)$ is positive and where it is negative.

We now see that the g_n's satisfy the following infinite system of ordinary differential equations:

(3.9)
$$g_n'' + (C^2n^2 + \lambda_n)g_n' - (\mu - \lambda_n)C^2n^2 g_n$$
$$= \frac{C^2}{2}n\left[\sum_{k=1}^{n-1}\left(kg_kg_{n-k}' + \left(\frac{k\lambda_{n-k} + (n-k)\lambda_k}{2}\right)g_kg_{n-k}\right)\right.$$
$$\left. + \sum_{k=1}^{\infty}\left(\left((k+n)g_k'g_{n+k} - kg_kg_{n+k}'\right) + \left((n+k)\lambda_k - k(\lambda_{n+k})\right)g_kg_{n+k}\right)\right].$$

3.2 The Approximate Problem

We define the approximate problem to be the system (3.9) without the infinite sum terms but with the same initial conditions, ie

(3.10)
$$g_n'' + (C^2n^2 + \lambda_n)g_n' - (\mu - \lambda_n)C^2n^2 g_n = \frac{C^2}{2}n\left[\sum_{k=1}^{n-1}\left(kg_kg_{n-k}' + \left(\frac{k\lambda_{n-k} + (n-k)\lambda_k}{2}\right)g_kg_{n-k}\right)\right].$$

3.3 Notation

To simplify the expressions of these systems, we introduce the following notation:

$$(g \star h)_n = \sum_{k=1}^{n-1} g_k h_{n-k}, \qquad\qquad (g, h) = \sum_{k=1}^{\infty} g_k h_k.$$

We let $T_n g$ indicate the g sequence shifted by n, Mg indicate the g sequence multiplied by it's index, and λg indicate the g sequence multiplied the eigenvalues, i.e.

$$\begin{aligned} T_n g &= g_{1+n},\ g_{2+n},\ g_{3+n}, \dots, \\ Mg &= g_1,\ 2g_2,\ 3g_3, \dots, \\ \lambda g &= \lambda_1 g_1,\ \lambda_2 g_2,\ \lambda_2 g_3, \dots \end{aligned}$$

Then (3.9) becomes

$$\begin{aligned} L_n g_n &= g_n'' + \left(C^2 n^2 + \lambda_n\right) g_n' - (\mu - \lambda_n) C^2 n^2 g_n \\ &= \frac{1}{2} C^2 n \Big[(Mg \star g')_n + \frac{1}{2}\left((Mg \star \lambda g)_n + (\lambda g \star Mg)_n\right) \\ &\qquad + (T_n Mg, g') - (Mg, T_n g') + (\lambda g, T_n Mg) - (Mg, T_n \lambda g) \Big] \end{aligned} \tag{3.11}$$

and (3.10) becomes

$$\begin{aligned} L_n g_n &= g_n'' + \left(C^2 n^2 + \lambda_n\right) g_n' - (\mu - \lambda_n) C^2 n^2 g_n \\ &= \frac{1}{2} C^2 n \left[(Mg \star g')_n + \frac{1}{2}\left((Mg \star \lambda g)_n + (\lambda g \star Mg)_n\right) \right]. \end{aligned} \tag{3.12}$$

3.4 Spaces for the sequences of functions

We define precisely which sequence spaces we need. We will later see that we need to consider a norm which is the sum of the $\ell([0,T])$ norms of g, Mg, λg, and g'. Let $m = \max\{1, \rho\}$. Then, for each of these pieces of the norm to be finite, we must have that $g \in \ell^m([0,T])$ and $g' \in \ell([0,T])$.

Taking this into consideration, we will work in the space $\ell^m([0,T]) \times \ell([0,T])$.

Chapter 4. Existence and Finite Time Blow up of some solutions of the Approximate Problem

We consider (3.10) with the initial conditions $g_n(0) = a_n$ and $g_n' = n\sigma a_n$ where the values a_n and σ are chosen so that $\{g_n(t) = a_n e^{n\sigma t}\}_{n=1}^{\infty}$ is a solution to the system which blows up in finite time.

Then we see that for $n \geq 2$, we have

$$\left[n\sigma^2 + (4\pi^2 M^2 n^2 + \lambda_n)\sigma - (\mu - \lambda_n)4\pi^2 M^2 n\right] a_n$$
$$= 2\pi^2 M^2 \sum_{k=1}^{n-1} \left[k(n-k)\sigma + \frac{k\lambda_{n-k} + (n-k)\lambda_k}{2}\right] a_k a_{n-k}.$$

Notice that for $n = 1$ we must have $a_1 \left(\sigma^2 + (4\pi^2 M^2 + \lambda_1)\sigma - (\mu - \lambda_1)4\pi^2 M^2\right) = 0$. In order for our solution to blow up, we must have $\sigma > 0$. If $a_1 = 0$, then $a_n = 0$ for all n, and thus $a_1 \neq 0$. In order to have $\sigma > 0$, we must have $\lambda_1 \leq \mu$. Then

$$\left[n\sigma^2 + (C^2 n^2 + \lambda_n)\sigma - (\mu - \lambda_n)C^2 n\right] = \left[n\sigma^2 + (C^2 n^2 + \lambda_1 n^\rho)\sigma - (\mu - \lambda_1 n^\rho)C^2 n\right]$$
$$= \left[C^2(n^2 - n)\sigma + \lambda_1(n^\rho - n)\sigma + \lambda_1 C^2(n^{\rho+1} - n)\right].$$

Therefore

$$\left[(n^2 - n)\sigma + \frac{\lambda_1}{C^2}(n^\rho - n)\sigma + \lambda_1(n^{\rho+1} - n)\right] a_n = \frac{1}{2}\sum_{k=1}^{n-1} \left((n-k)^{1-\rho}\sigma + \lambda_1\right) k(n-k)^\rho a_k a_{n-k}.$$

Let $b_n = \frac{n^2-n}{n-n^\rho}$ and let $d_n = \frac{n^{\rho+1}-n}{n-n^\rho}$. Then

$$2\sigma\left[b_n - \frac{\lambda_1}{C^2} + d_n\frac{\lambda_1}{\sigma}\right] a_n = \frac{1}{n-n^\rho}\sum_{k=1}^{n-1}\left(\frac{(n-k)^{1-\rho}\sigma + \lambda_1}{\sigma}\right) k(n-k)^\rho a_k a_{n-k}. \tag{4.1}$$

Notice that $n \geq k+1$ and thus $n - k \geq 1$. Therefore $(n-k)^\rho \leq (n-k)$. Thus

$$2\sigma\left[b_n - \frac{\lambda_1}{C^2} + d_n\frac{\lambda_1}{\sigma}\right] a_n \leq \frac{1}{n-n^\rho}\sum_{k=1}^{n-1}\left(\frac{\sigma + \lambda_1}{\sigma}\right) k(n-k) a_k a_{n-k}.$$

Denote $\sigma^\star = \frac{\lambda_1+\sigma}{\sigma}$. Let $\tilde{a}_n$ be such that

$$2\sigma\left[b_n - \frac{\lambda_1}{C^2} + d_n\frac{\lambda_1}{\sigma}\right]\tilde{a}_n = \frac{\sigma^\star}{n-n^\rho}\sum_{k=1}^{n-1} k(n-k)\tilde{a}_k\tilde{a}_{n-k}.$$

Then if $\tilde{a}_1 \geq a_1$, we have that $a_n \leq \tilde{a}_n$ for all n.

We now note that $b_n = n(\frac{n-1}{n-n^\rho}) = n(1+\frac{n^\rho-1}{n-n^\rho}) \geq n$. We choose M large enough so that $\frac{\lambda_1}{C^2} < 1$ and therefore it is less than $\frac{n}{2}$ for $n \geq 2$. Thus

$$2(b_n - \frac{\lambda_1}{C^2} + d_n\frac{\lambda_1}{\sigma}) \geq 2(n - \frac{n}{2}) = n.$$

We now let $\hat{a}_1 \geq \tilde{a}_1$ and

$$n\hat{a}_n = \frac{\sigma^\star}{n-n^\rho}\sum_{k=1}^{n-1} k(n-k)\hat{a}_k\hat{a}_{n-k}.$$

Then, we have $\hat{a}_n \geq \tilde{a}_n \geq a_n$.

For the moment, consider $f(x) = x^\rho$. Then, with $\rho < 1$, $f''(x) < 0$ for positive x, and we have that $x^\rho \leq 1 + \rho(x-1)$. Therefore $n - n^\rho \geq (1-\rho)(n-1)$.

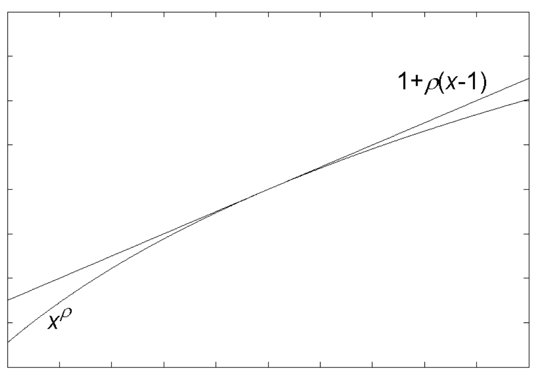

Figure 4.1 The upper bound on x^ρ. The expression is bounded above by it's tangent line at $(1,1)$.

If we let $a'_1 \geq \hat{a}_1$ and define a'_n by

$$(1-\rho)(n-1)na'_n = \sigma^\star\sum_{k=1}^{n-1} k(n-k)a'_k a'_{n-k}$$

we have that $a'_n \geq \hat{a}_n \geq \tilde{a}_n \geq a_n$.

If we choose $\delta = \frac{\sigma^\star}{1-\rho} a'_1$, we have $a'_n = \frac{1}{n}\left(\frac{1-\rho}{\sigma^\star}\right)\delta^n$.

This leads to a lower bound on the blowup time for the solution of $\frac{-\ln\delta}{\sigma}$.

We now return to 4.1. Let m_n satisfy

$$2(b_n + d_n\frac{\lambda_1}{\sigma} - \frac{\lambda_1}{C^2})m_n = \frac{1}{n-n^\rho}\sum_{k=1}^{n-1} k(n-k)m_n m_{n-k}.$$

Then $a_n \geq m_n$. Let m'_n satisfy

$$2(b_n + d_n\frac{\lambda_1}{\sigma})(n-n^\rho)m'_n = \sum_{k=1}^{n-1} k(n-k)m'_k m'_{n-k}.$$

Then $m'_n \leq m_n \leq a_n$.

Finally, we note that

$$2(b_n + d_n\frac{\lambda_1}{\sigma}) = 2n\left(\frac{(n-1)+\frac{\lambda_1}{\sigma}(n^\rho - 1)}{n-n^\rho}\right) \leq 2n(1+\frac{\lambda_1}{\sigma})\frac{n-1}{n-n^\rho}.$$

Therefore, if m''_n satisfies

$$2\sigma^\star n(n-1)m''_n = \sum_{k=1}^{n-1} k(n-k)m''_k m''_{n-k}$$

we will have $m''_n \leq m'_n \leq m_n \leq a_n$.

Again, we can solve the final recurrence by $m''_n = \frac{1}{n}\left[\frac{1}{2\sigma^\star}\right]^{n-1}(m''_1)^n$.

If we choose $\epsilon = \frac{m''_1}{2\sigma^\star}$, then $m''_n = \frac{2\sigma^\star}{n}\epsilon^n$.

As $g_n(t) = a_n e^{n\sigma t}$, we have that $\mid g'_n(t) \mid = \mid n\sigma a_n e^{n\sigma t} \mid \geq \mid 2(\sigma+\lambda_1)\epsilon^n e^{n\sigma t} \mid$. We know that $\sigma > 0$ and therefore for large enough time t, we have that the sequence of derivatives of $g_n(t)$ is greater in absolute value than a geometric series with ratio greater than one. As our norm is greater than the ℓ^1 norm of g', the sequence must leave the space $\ell^m([0,T]) \times \ell([0,T])$ at a time no later than $\frac{-\ln\epsilon}{\sigma}$.

Recall from 3.3 that $u(x,t) = \mu + \Psi_t + A\Psi$. Then, by Parseval, we have that

$$\| u(\cdot,t) - \mu \|_{L^2} = \sum_{n=1}^{\infty} \mid g'_n(t) + \lambda_n g_n(t) \mid = \sum_{n=1}^{\infty} \mid \left(1 + n^{\rho-1}\sigma^{-1}\right) g'_n(t) \mid \geq \sum_{n=1}^{\infty} \mid g'_n(t) \mid^2 .$$

A similar argument to that used on $g'_n(t)$, shows that the right hand side of this inequality also blows up in finite time, and thus $u(x,t)$ leaves $L^2([0,1])$ in finite time.

Chapter 5. A Solution of the Full Problem obtained when $\rho = 1$

If we consider the special case $\lambda_n = n\lambda_1$, we can solve the full problem (3.9). As in chapter 4, we let $g_n(t) = A_n e^{n\sigma t}$. Then we see that the two terms involving infinite sums vanish. Letting $B_n = nA_n$, we can further simplify the resulting form of (3.9) to

$$B_n \left(\sigma^2 + (C^2 n + \lambda_1)\sigma - (\mu - n\lambda_1)C^2\right) = \frac{C^2}{2}(\sigma + \lambda_1) \sum_{k=1}^{n-1} B_k B_{n-k}. \tag{5.1}$$

We will again seek a solution which blows up. If we let $n = 1$, (5.1) becomes

$$B_1 \left(\sigma^2 + (C^2 + \lambda_1)\sigma - (\mu - \lambda_1)C^2\right) = 0.$$

If $\sigma < 0$, the solution of this form is global. Otherwise, we must have $\lambda_1 \leq \mu$ as before. We now assume that $B_1 \neq 0$. We will later see that it needs to be between 0 and 2. Then $\sigma^2 = -(C^2 + \lambda_1)\sigma + (\mu - \lambda_1)C^2$.

Using this , we can reduce (5.1) to the following for $n > 1$:

$$B_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} B_k B_{n-k}. \tag{5.2}$$

We can show, by induction, that this has solution $B_n = 2^{1-n} B_1^n$. Then we have

$$g_n(t) = \frac{2^{1-n} B_1^n}{n} e^{n\sigma t},$$

which is our solution to (3.9). Then, our corresponding solution to (3.4) is

$$\Psi(x,t) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{2}\right)^{n-1} B_1^n e^{n\sigma t} \cos(2\pi M n x). \tag{5.3}$$

By writing the cos term as a sum of two complex exponential functions, and taking a derivative with respect to t, we arrive at a sum of two geometric series. Summing these and then taking an antiderivative, (5.3) can be written as

$$\Psi(x,t) = \ln\left(1 - B_1 e^{\sigma t}\cos(2\pi M x) + \frac{1}{4}B_1^2 e^{2\sigma t}\right) + \ln\left(\frac{2}{2-B_1}\right).$$

Let x be an integer. Then $\Psi(x,t) = 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{B_1 e^{\sigma t}}{2}\right)^n$. This diverges if $\frac{B_1 e^{\sigma t}}{2} \geq 1$ or if $t \geq -\frac{1}{\sigma}\ln(\frac{B_1}{2})$. It can easily be seen, from Parseval, that this implies that $\Psi(x,t)$ must leave $L^2([0,1])$ in finite time.

Then we see that in the case $\rho = 1$ that there are solutions which must blow up in finite time. In this case, it was possible to solve the equations in closed form. Unfortunately, it seems that this case is unique in this respect. We derive bounds on the solutions and use these to prove blowup in the remaining cases.

Chapter 6. Local Existence and Uniqueness of Solutions to the Full Problem

We now wish to establish existence and uniqueness of the solutions to the initial value problem for (3.11). Again, we consider the methods of [11]. We first look at uniqueness.

6.1 Uniqueness of solutions

Consider (3.11) and let $g_n(0) = h_n(0)$ and $g_n'(0) = h_n'(0)$ where g and h are solutions. Let $w = g - h$. Then w satisfies

$$\begin{aligned} L_n w_n = \frac{1}{2}C^2 n \big[&(Mw \star g')_n + (Mh \star w')_n + (Mg \star \lambda w)_n + (Mw \star \lambda h)_n \\ &+ (\lambda g \star Mw)_n + (\lambda w \star Mh)_n + (T_n Mw, g') + (T_n Mh, w') \\ &+ (Mw, T_n g') + (Mh, T_n w') + (\lambda w, T_n Mg) + (\lambda h, T_n Mw) \\ &+ (Mw, T_n \lambda g) + (Mh, T_n \lambda w)\big] = F_n(w, w') = F_n(t). \end{aligned} \tag{6.1}$$

Now let r_n^+ and r_n^- be the roots of $r^2 + (C^2n^2 + \lambda_n)r - (\mu - \lambda_n)C^2n^2$. Then

$$w_n(t) = \frac{1}{\Omega_n} \int_0^t \left(e^{r_n^+(t-s)} - e^{r_n^-(t-s)} \right) F_n(s)\ ds \tag{6.2}$$

where

$$\Omega_n = r_n^+ - r_n^- = \sqrt{(C^2n^2 + \lambda_n)^2 + 4(\mu - \lambda_n)C^2n^2}.$$

Let $\lambda_n = n^\rho \lambda_1$. Assume that $\rho < 2$. Then $\Omega_n \approx dn^2$ for large n and some $d > 0$. Also,

$$r_n^- = \frac{1}{2}\left[-(C^2n^2 + \lambda_1 n^\rho) - \sqrt{(C^2n^2 + \lambda_1 n^\rho)^2 + 4(\mu - \lambda_1 n^\rho)C^2n^2} \right] \approx -dn^2.$$

We now look at

$$r_n^+ = \frac{1}{2}\left[-(C^2n^2 + \lambda_1 n^\rho) + \sqrt{(C^2n^2 + \lambda_1 n^\rho)^2 + 4(\mu - \lambda_1 n^\rho)C^2n^2}\right].$$

It is apparent that for n sufficiently large, $r_n^+ < 0$. By multiplying both numerator and denominator by $(C^2n^2 + \lambda_1 n^\rho) + \sqrt{(C^2n^2 + \lambda_1 n^\rho)^2 + 4(\mu - \lambda_1 n^\rho)C^2n^2}$ we can see that

$$r_n^+ = 2\left[\frac{(\mu - \lambda_1 n^\rho)C^2n^2}{(C^2n^2 + \lambda_1 n^\rho) + \sqrt{(C^2 + \lambda_1 n^\rho)^2 + 4(\mu - \lambda_1 n^\rho)C^2n^2}}\right] \approx -dn^\rho$$

for large n and some $d > 0$.

Now, notice that in the definition of $F_n(t)$ there are 14 terms of which we must estimate the norms of the sequences. We will need various forms of these estimates later, so we will estimate $n^{\beta-\alpha}$ times each of these, where we will choose the various values of α and β later needed.

The first one we need to estimate will be $\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mw \star g)_n \mid$. Then

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mw \star g)_n \mid \leq \sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \sum_{k=1}^{n-1} k \mid w_k \mid\mid g'_{n-k} \mid$$

$$\leq 2^{\beta-\alpha} \sum_{n=1}^{\infty}\sum_{k=1}^{n-1} \left((n-k)^{\beta-\alpha}k + k^{\beta-\alpha+1}\right) \mid w_k \mid\mid g'_{n-k} \mid$$

$$= 2^{\beta-\alpha}\left(\mid\mid M^{\beta-\alpha}g' \star Mw \mid\mid + \mid\mid M^{\beta-\alpha+1}w \star g' \mid\mid\right)$$

$$\leq 2^{\beta-\alpha}\left(\mid\mid M^{\beta-\alpha}g' \mid\mid\mid\mid Mw \mid\mid + \mid\mid M^{\beta-\alpha+1}w \mid\mid\mid\mid g' \mid\mid\right).$$

For the next few terms, we see that

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mh \star w')_n \mid \leq 2^{\beta-\alpha}\left(\mid\mid M^{\beta-\alpha}w' \mid\mid\mid\mid Mh \mid\mid + \mid\mid M^{\beta-\alpha+1}h \mid\mid\mid\mid w' \mid\mid\right),$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mg \star \lambda w)_n \mid \leq 2^{\beta-\alpha}\left(\mid\mid M^{\beta-\alpha+1}g \mid\mid\mid\mid \lambda w \mid\mid + \mid\mid M^{\beta-\alpha+\rho}w \mid\mid\mid\mid Mg \mid\mid\right),$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mw \star \lambda h)_n \mid \leq 2^{\beta-\alpha} \left(|| M^{\beta-\alpha+1} w |||| \lambda h || + || M^{\beta-\alpha+\rho} h |||| Mw || \right),$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (\lambda g \star Mw)_n \mid \leq 2^{\beta-\alpha} \left(|| M^{\beta-\alpha+1} w |||| \lambda g || + || M^{\beta-\alpha+\rho} g |||| Mw || \right),$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (\lambda w \star Mh)_n \mid \leq 2^{\beta-\alpha} \left(|| M^{\beta-\alpha+1} h |||| \lambda w || + || M^{\beta-\alpha+\rho} w |||| Mh || \right).$$

We make the assumption that $\alpha \geq \beta$. We will see this is true when we make use of these estimates later. For the next term, since $n^{\beta-\alpha} \leq 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (T_n Mw, g') \mid &\leq \sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \sum_{k=1}^{\infty} (k+n) \mid w_{k+n} || g'_k \mid \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (k+n) \mid w_{k+n} || g'_k \mid \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (k+n) \mid w_{k+n} || g'_k \mid \\
&= \sum_{k=1}^{\infty} \mid g'_k \mid \sum_{n=1}^{\infty} (k+n) \mid w_{k+n} \mid \\
&\leq || Mw || \sum_{k=1}^{\infty} \mid g'_k \mid \\
&= || Mw |||| g' || .
\end{aligned}$$

In the same manner, we see that

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (T_n Mh, w') \mid \leq \| Mh \| \| w' \|,$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mw, T_n g') \mid \leq \| Mw \| \| g' \|,$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mh, T_n w') \mid \leq \| Mh \| \| w' \| .$$

Again, in a similar manner,

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (\lambda w, T_n Mg) \mid \leq \| \lambda w \| \| Mg \|,$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (\lambda h, T_n Mw) \mid \leq \| \lambda h \| \| Mw \|,$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mw, T_n \lambda g) \mid \leq \| Mw \| \| \lambda g \|,$$

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid (Mh, T_n \lambda w) \mid \leq \| Mh \| \| \lambda w \| .$$

Let $||| w ||| = \| w \| + \| \lambda w \| + \| Mw \| + \| w' \|$ and let $G_n = \frac{2F_n}{C^2 n}$. Then, for $\beta \leq \alpha$,

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^\alpha} \mid G_n(t) \mid \leq ||| w(t) ||| \, (||| g(t) ||| + ||| h(t) |||) \,. \tag{6.3}$$

Now, we need to estimate the terms in $||| w |||$. First,

$$\| Mw(t) \| \leq 2 \int_0^t \sum_{n=1}^{\infty} e^{r_1^+(t-s)} \frac{\mid F_n(s) \mid}{\Omega n} \, ds \leq C_1 e^{r_1^+ t} \int_0^t ||| w(s) ||| \, (||| g(s) ||| + ||| h(s) |||) \; ds.$$

Here we use the above estimates with $\beta = 2$ and $\alpha = 2$, where C_1 is some computable constant.

For the remaining terms, we examine terms with factors like $n^\epsilon e^{-dn^\delta t}$. These factors emerge from having to redistribute the powers of n to ensure that the necessary relationship between β and α holds. These factors are bounded above by $Ct^{-\frac{\epsilon}{\delta}}$ where $C = \left(\frac{\epsilon}{d\delta e}\right)^{\frac{\epsilon}{\delta}}$.

Then

$$\| w'(t) \| \leq C_2 \int_0^t \sum_{n=1}^{\infty} \left(\frac{r_n^+}{\Omega_n} e^{r_n^+(t-s)} + \frac{r_n^-}{\Omega_n} e^{r_n^-(t-s)} \right) | F_n(s) | \ ds$$

$$= C_3 \int_0^t \sum_{n=1}^{\infty} \left(n^{\rho-1} e^{r_n^+(t-s)} | G_n(s) | + n e^{r_n^-(t-s)} | G_n(s) | \right) \ ds$$

$$\leq C_4 \int_0^t \sum_{n=1}^{\infty} \left(e^{r_1^+ t} + \frac{1}{\sqrt{t-s}} \right) | G_n(s) | \ ds$$

$$\leq C_4 e^{r_1^+ t} \int_0^t ||| w(s) ||| \, (||| g(s) ||| + ||| h(s) |||) \ ds \\ + C_4 \int_0^t \frac{||| w(s) ||| \, (||| g(s) ||| + ||| h(s) |||)}{\sqrt{t-s}} \ ds.$$

If $\rho > 1$, then we need to estimate $\| \lambda w \|$, otherwise it is bounded above by $\| Mw \|$. In the former case, it is bounded by $\int_0^t t^{\frac{1-\rho}{\rho}} (||| w(s) ||| \, (||| g(s) ||| + ||| h(s) |||)) \ ds$.

For some $m < 1$, we have

$$||| w(t) ||| \leq c e^{r_1^+ t} \int_0^t \frac{||| w(s) ||| \, (||| g(s) ||| + ||| h(s) |||)}{(t-s)^m} \ ds.$$

Let $\ell^*([0,T]) = \{ f$ such that $\max_K ||| f ||| < \infty$ for all compact $K \subset [0,T] \}$. Then for $g, h \in \ell^*$ and finite T, we have $||| w(t) ||| \leq c \int_0^t \frac{||| w(s) |||}{(t-s)^m} \ ds$.

Choose $1 < p < \frac{1}{m}$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$||| w(t) |||^q \leq c t^{\frac{q}{p} - qm} \int_0^t ||| w(s) |||^q \ ds.$$

Letting $\Phi(t) = \int_0^t ||| w(s) |||^q \ ds$ and $\tilde{c}(t) = c t^{\frac{q}{p} - qm}$, we see that

$$\frac{d}{dt} \left(\Phi(t) \exp(-\int_0^t \tilde{c}(s) \ ds) \right) \leq 0.$$

Therefore $0 \leq \Phi(t) \leq \Phi(0)\exp(\int_0^t \tilde{c}(s)\, ds)$. We know that $\Phi(0) = 0$ and thus Φ is identically 0 on $[0,T]$ which shows that $g = h$ on the interval.

6.2 Local Existence of solutions

Now let $Lg = F(g, g'), g(0) = g_0, g'(0) = g_0'$ indicate the inital value problem for (3.11) where $g(t) = \{g_n(t)\}$. Let G satisfy $LG = 0$ with the above inital conditions and let $H = g - G$.

Then $LH = F(G + H, G' + H'), H(0) = 0, H'(0) = 0$.

Thus $H = L^{-1}F(G + H, G' + H')$, where this operator is interpreted in the sense of the variational formula that appears in the existence argument. Let $H^1(t) = \{H_n^1(t) = 0\}_{n=1}^{\infty}$. Let $H^{k+1} = L^{-1}F(G + H^k, G' + H^{k'})$.

Note that $\sup_{[0,T]} ||| G(s)) ||| \leq G_0(T)$ for some constant G_0 depending on T.

Then, it follows, from estimates like those in the previous section, that

$$||| H^{k+1} ||| \leq C \int_0^t \frac{||| G + H^k |||^2}{(t-s)^m}\, ds.$$

Then $||| H^{k+1} ||| \leq Ct^{1-m}(G_0^2 + \sup_{[0,T]} ||| H^k |||^2)$.

Now let $Z_k = \sup_{[0,T]} ||| H^k |||$. Then $Z_{k+1} \leq Ct^{1-m}(G_0^2 + Z_k^2)$. Let $G_1 > Z_1$. Then $G_1 > Z_n$ for all n if $T^{1-m} \leq \frac{G_1}{C(G_0^2+G_1^2)}$. Thus the H^k's are bounded on $[0,T]$.

Now let $W^k = H^k - H^{k-1}$. Then

$$L(W^k) = F(G + H^k, G' + H^{k'}) - F(G + H^{k-1}, G' + H^{(k-1)'}) = F(W^k, W^{k'}).$$

Using the estimates from the uniqueness result, we see that

$$||| W^{k+1} ||| \leq C \int_0^t \frac{||| W^k |||}{(t-s)^m}\, ds.$$

With the same p and q as in the uniqueness result, we see that

$$||| W^{k+1} |||^q \leq Ct^{\frac{q}{p} - qm} \int_0^t ||| W^k(s) |||^q \;\; ds.$$

Then $\sup_{[0,T]} ||| W^{k+1} ||| \leq Ct^{\frac{1}{p} - m + 1} \sup_{[0,T]} ||| W^k |||$.

Thus, if T is small enough that $T^{1-m} \leq 1 - \epsilon$ and $T^{\frac{1}{p} - m + 1} \leq 1 - \epsilon$ for some small ϵ, there is a unique fixed point for L^{-1} by the contraction mapping principle. Thus we have local existence of a solution in $\ell^*([0,T])$.

Chapter 7. Blowup of Solutions to the Full Problem

We follow the methods of [4] in order to attempt to establish blowup. This will allow us to arrive at the same final inequality and derive a contradiction in the same manner as was done in [4].

Near the end of this argument, a certain function $W(t)$ will be introduced. The argument relies on the assumption that this $W(t)$ is in a certain L^p space. In particular, for the case $\rho = 0$ discussed in [4,5], we must have $W(t) \in L^2$ at least, but it can be shown that this is not the case. This leaves this argument potentially incomplete. The particular L^p space needed and why the assumption is needed will be discussed at the end of the argument after $W(t)$ has been explicitly defined. Further details of the potential flaw in the argument will be discussed in Section 9.2.

Define L_n by $L_n h_n = h_n'' + (C^2 n^2 + \lambda_n) h_n' - (\mu - \lambda_n) C^2 n^2$.

Let

$$G_n(z, z') = \frac{1}{2} C^2 n \left[(Mz \star z')_n + \frac{1}{2} \left((Mz \star \lambda z)_n + (\lambda z \star Mz)_n \right) \right]$$

and

$$H_n(z, z') = \frac{1}{2} C^2 n \left[(T_n Mz, z') - (Mz, T_n z') + (\lambda z, T_n Mz) - (Mz, T_n \lambda z) \right].$$

Now, let g and h satisfy

$$L_n g_n = G_n(g, g')$$
$$L_n h_n = G_n(h, h') + H_n(h, h').$$

For the moment, let's focus on g_n. If $a_1 > 0$, σ is the positive root of

$$\sigma^2 + (4\pi^2 M^2 + \lambda_1)\sigma - (\mu - \lambda_1) 4\pi^2 M^2 = 0$$

and

$$2\sigma\left[b_n - \frac{\lambda_1}{C^2} + d_n\frac{\lambda_1}{\sigma}\right]a_n = \frac{1}{n-n^\rho}\sum_{k=1}^{n-1}\left(\frac{(n-k)^{1-\rho}\sigma+\lambda_1}{\sigma}\right)k(n-k)^\rho a_k a_{n-k}$$

then we have that $\{g_n(t) = a_n e^{n\sigma t}\}$ is a solution to $L_n g_n = G_n(g, g')$ which blows up in finite time.

Now, in Chapter 4, we established that there are positive constants a, b, ϵ, δ such that we have $a\epsilon^n \le na_n \le b\delta^n$. Then, it follows, that $\liminf \frac{-\ln na_n}{n\sigma}$ and $\limsup \frac{-\ln na_n}{n\sigma}$ are both finite. We denote these by $\underline{T_b}$ and $\overline{T_b}$ respectively.

We let a_{n_k} be a subsequence such that the limit of $\frac{-\ln na_n}{n\sigma}$ is $\underline{T_b}$. Then

$$\lim_{k\to\infty} n_k a_{n_k} exp(n_k\sigma\underline{T_b}) = 1.$$

We define $A_n = na_n e^{n\sigma\underline{T_b}}$. Then $\lim_{k\to\infty} A_{n_k} = 1$. Now, we see that, for $\tau > 0$,

$$\lim_{t\to\underline{T_b}^-}\sum_{k=1}^{\infty} A_{n_k} e^{-n_k\sigma(\underline{T_b}-t)} = \infty, \tag{7.1}$$

$$\lim_{t\to\underline{T_b}^-}\sum_{k=1}^{\infty} \frac{A_{n_k} e^{-n_k\sigma(\underline{T_b}-t)}}{n_k^{1+\tau}} < \infty. \tag{7.2}$$

Now, if we note that $(Mg)_n(t) = na_n e^{n\sigma t} = \left(na_n e^{n\sigma\underline{T_b}}\right)e^{-n\sigma(\underline{T_b}-t)} = A_n e^{-n\sigma(\underline{T_b}-t)}$, it becomes immediately apparent from (7.1) that the blowup time of g must be no more than $\underline{T_b}$.

Next, note that for any τ and sufficiently large n, we have $\underline{T_b} - \tau \le \frac{-\ln na_n}{n\sigma}$. Then, we must have that $a_n \le \frac{1}{n}e^{-n\sigma(\underline{T_b}-\tau)}$. Consider $|||\ g(t)\ |||$ for $t < \underline{T_b}$. Choose τ so that $t < \underline{T_b} - \tau$. Then, we have, from the tail of $|||\ g\ |||$,

$$\sum_{n=N}^{\infty} na_n e^{n\sigma t} \le \sum_{n=N}^{\infty} e^{-n\sigma(\underline{T_b}-\tau-t)} < \infty.$$

Thus, the blowup time of g can be no earlier than $\underline{T_b}$, and thus it must be exactly $\underline{T_b}$.

Let $w = h - g$. Then

$$\begin{aligned} L_n w_n &= G_n(h,h') - G_n(g,g') + H_n(h,h') \\ &= G_n(h,h') - G_n(g,h') + G_n(g,h') - G_n(g,g') + H_n(h,h') \\ &= G_n(h-g,h') + G_n(g,h'-g') + H_n(h,h') \end{aligned}$$

$$= G_n(w, h') + G_n(g, w') + H_n(h, h') = F_n(s).$$

From the estimates in the existence and uniqueness theorem, we know that

$$||| L_n^{-1} H_n(h, h') ||| \leq C \int_0^t \frac{||| h(s) |||^2}{(t-s)^m} \, ds.$$

Therefore

$$(7.3) \qquad ||| w_n(t) ||| \leq I(t) + J(t) + B \int_0^t \frac{||| h(s) |||^2 + ||| h(s) |||||| w(s) |||}{(t-s)^m} \, ds$$

where

$$I(t) = \int_0^t \sum_{n=1}^{\infty} \left(e^{-dn^p(t-s)} + n e^{-dn^2(t-s)} \right) | (Mw \star g')_n | \; ds = I_1(t) + I_2(t)$$

and

$$J(t) = \int_0^t \sum_{n=1}^{\infty} \left(e^{-dn^p(t-s)} + n e^{-dn^2(t-s)} \right) | (Mw \star \lambda g)_n | \; ds = J_1(t) + J_2(t).$$

We now compute,

$$I_1(t) = \int_0^t \sum_{n=1}^{\infty} e^{-dn^p(t-s)} \; | (Mw \star g')_n | \; ds$$

$$= \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} (n-k) \, | g'_k || w_{n-k} | \, e^{-dn^p(t-s)} \, ds$$

$$= \int_0^t \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} (n-k) \, | g'_k || w_{n-k} | \, e^{-dn^p(t-s)} \, ds$$

$$= \int_0^t \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n \, | g'_k || w_n | \, e^{-d(n+k)^p(t-s)} \, ds$$

$$= \int_0^t \sum_{k=1}^{\infty} | g'_k | \, e^{\frac{-d(n+k)^p(t-s)}{2}} \sum_{n=1}^{\infty} n \, | w_n | \, e^{\frac{-d(n+k)^p(t-s)}{2}} \, ds$$

$$\leq C \int_0^t \sum_{k=1}^{\infty} \mid g_k' \mid e^{\frac{-dk^p(t-s)}{2}} \frac{||| \, w \, |||}{(t-s)^m} \; ds$$

$$\leq C \int_0^t \sum_{k=1}^{\infty} A_k e^{\frac{-dk^p(t-s)}{2} - k\sigma(t-s)} \frac{||| \, w \, |||}{(t-s)^m} \; ds.$$

Also,

$$J_1(t) = \int_0^t \sum_{n=1}^{\infty} e^{-dn^p(t-s)} \mid (Mw \star \lambda g)_n \mid \; ds$$

$$= \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} k^\rho (n-k) \mid g_k \mid\mid w_{n-k} \mid e^{-dn^p(t-s)} \; ds$$

$$= \int_0^t \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} k^\rho n \mid g_k \mid\mid w_n \mid e^{-d(n+k)^p(t-s)} \; ds$$

$$= \int_0^t \sum_{k=1}^{\infty} \mid \lambda_k g_k \mid e^{\frac{-d(n+k)^p(t-s)}{2}} \sum_{n=1}^{\infty} n \mid w_n \mid e^{\frac{-d(n+k)^p(t-s)}{2}} \; ds$$

$$\leq C \int_0^t \sum_{k=1}^{\infty} A_k e^{\frac{-dk^p(t-s)}{2} - k\sigma(t-s)} \frac{||| \, w \, |||}{(t-s)^m} \; ds.$$

Continuing,

$$I_2(t) = \int_0^t \sum_{n=1}^{\infty} n e^{-dn^2(t-s)} \mid (Mw \star g')_n \mid \; ds$$

$$= \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} n(n-k) \mid g_k' \mid\mid w_{n-k} \mid e^{-dn^2(t-s)} \; ds$$

$$= \int_0^t \sum_{k=1}^{\infty} \mid g_k' \mid e^{\frac{-d(n+k)^2(t-s)}{2}} \sum_{n=1}^{\infty} (n+k) e^{\frac{-d(n+k)^2(t-s)}{2}} n \mid w_n \mid \; ds$$

$$\leq \int_0^t \sum_{k=1}^{\infty} A_k e^{\frac{-d(n+k)^2(t-s)}{2} - k\sigma(t-s)} \frac{c \,|||\, w \,|||}{(t-s)^m}\ ds,$$

$$J_2(t) \leq \int_0^t \sum_{k=1}^{\infty} A_k e^{\frac{-d(n+k)^2(t-s)}{2} - k\sigma(t-s)} \frac{c \,|||\, w \,|||}{(t-s)^m}\ ds.$$

Now, let $W(t) = \sum_{k=1}^{\infty} A_k \left(e^{-\frac{d}{2}k^{\rho}(t-s)} + e^{-\frac{d}{2}k^2(t-s)} \right) e^{-k\sigma(t-s)}$.

Then (7.3) becomes

$$|||\, w(t) \,||| \leq C \int_0^t \left(\frac{W(t-s) + |||\, h(s) \,|||}{(t-s)^m} \right) |||\, w(s) \,|||\ \ ds + C \int_0^t \frac{|||\, h(s) \,|||^2}{(t-s)^m}\ ds. \tag{7.4}$$

Now, assume that h is global. $W(t) \in L^1([0,T])$. Assume that $W \in L^{1+\epsilon}([0,T])$. This condition will be discussed in more detail in Section 9.2. Let $p > 1$, $q > 1$, and $r > 1$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Then, we have

$$|||\, w(t) \,||| \leq K_1(h,T) \left(\int_0^t |||\, w(s) \,|||^q\ \ ds \right)^{\frac{1}{q}} + K_2(h,T)$$

by an application of Hölder's inequality where the choice of p, q and r depends on m and ϵ. In particular $p < \frac{1}{m}$ and $r < 1 + \epsilon$. We assume that ϵ is such that this necessary relationship can hold. Note that for $m = \frac{1}{2}$, we will need $\epsilon > 1$ for this to work.

By Gronwall, we have that $|||\, w \,|||$ is bounded on $(0,T)$ [11]. But, as $w = h - g$, we have that

$$|||\, g(t) \,||| \leq |||\, h(t) \,||| + |||\, w(t) \,|||$$

and thus g is bounded, but this is impossible. Thus h can not be bounded in norm, and must blow up in finite time. Therefore there are solutions to (3.11) which blow up in finite time. There could be a problem in the necessary relationship between ϵ and m in this argument. This will be discussed in detail in section 9.2.

Chapter 8. Numerical Results

In this chapter, we will discuss attempts to solve the system (3.9) numerically. This will grant more support to the blowup claim as well as show visually what is happening in the global solutions case. The graphs that have been generated will be analyzed in this chapter to see what they tell us about our solutions. Several issues that the graphs raise will be brought up again in section 9.3. We will first look at the case where solutions are believed to be able to blow up in finite time and then turn our attention to the case where solutions are believed to remain global.

In order to solve the equations with numerical methods, it is necessary to replace the infinite system with a finite one. In order to do so, we choose a cutoff value, N, and set all functions $g_n(t)$ to be zero for $n > N$. This results in a finite system, where the infinite sums in the equations are replaced with finite sums.

For the various constants needed, we choose $\lambda_1 = 1,\ C = 2\pi$, and $\mu = 2$ (the results of Chapters 4 and 5 suggest that μ should be larger than λ_1).

8.1 Blowup solutions

We compute $|||\ g(t)\ |||$ for several values of ρ. Figure 8.1 shows the values computed for four different values of ρ. The initial conditions used are similar to those in Chapter 4 with $a_n = \frac{(.5)^{n-1}}{n}$. Notice that in all cases the solutions appear to blow up in finite time.

The right hand values of t on the graphs are $0.50, 0.50, 0.50,$ and 0.70, so it appears from the figure that as ρ increases the blowup time also increases. This is not entirely surprising if there actually is a value of the parameter which tends to separate the two behaviors. One would expect the solution to take longer and longer to blow up as the parameter value gets closer and closer to this cutoff value. This would mean that solutions get better behaved close to the cutoff value. Notice that we may not be able to say that the blowup time must tend to infinity as we approach the cutoff, however, as if $\rho = 1$ is the cutoff, we know that it may lie in the blowup case, so the blowup time can remain finite as we approach the cutoff provided that it does increase with ρ.

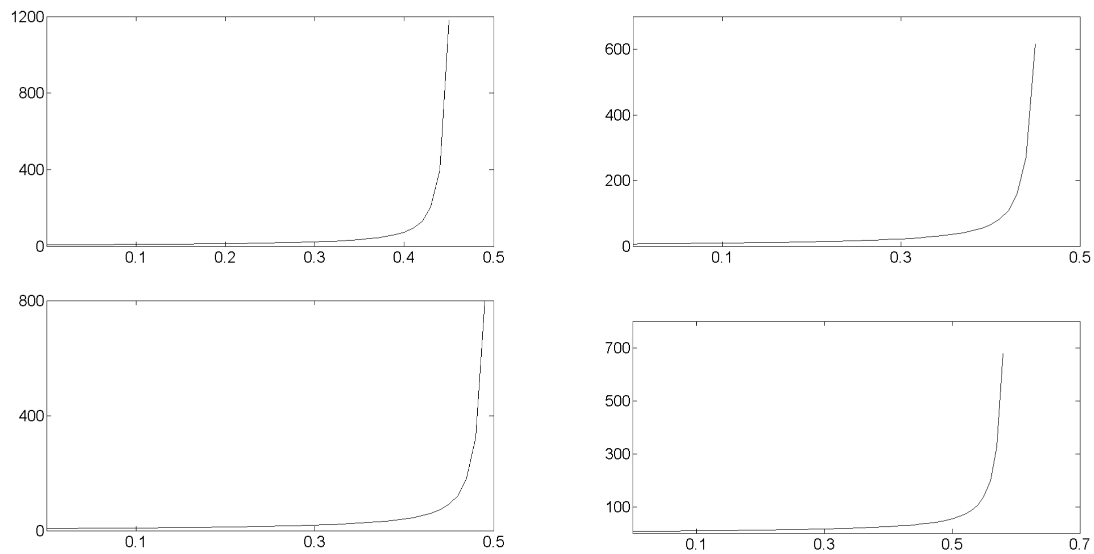

Figure 8.1 Shown are the numerically computed values of $||| \, g(t) \, |||$. The values of ρ are $0.2, 0.25, 0.5,$ and 0.79. The horizontal axis is the time axis.

8.2 Global solutions

Figure 8.2 shows several graphs of $||| \, g(t) \, |||$ on the interval $(0, 2)$. The same initial conditions as in figure 8.1 are used. All of these seem to reach some sort of horizontal asymptote and thus remain global in time. Figure 8.3 shows the same solutions on the larger interval $(0, 10)$. Here the result is even more striking.

Notice as well that in figures 8.2 and 8.3 that it appears that not only are the solutions global, but that they are also bounded. This is much clearer in the second of the figures, where the solutions seem to become nearly horizontal and flat after a certain time value. There is

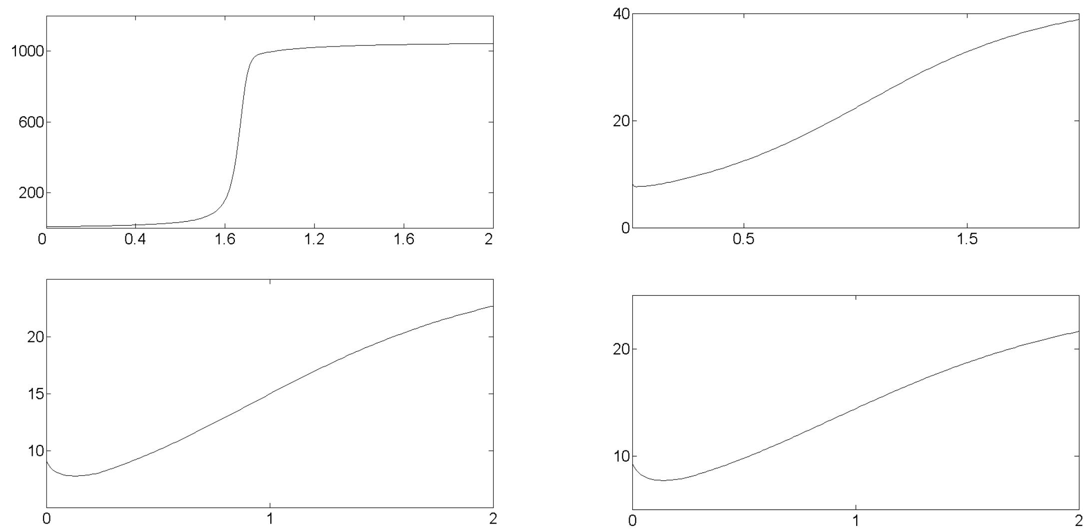

Figure 8.2 Shown are the numerically computed values of $|||\ g(t)\ |||$. The values of ρ are $1.1, 1.6, 1.95,$ and 2.0. These are shown on $(0, 2)$. The horizontal axis is the time axis.

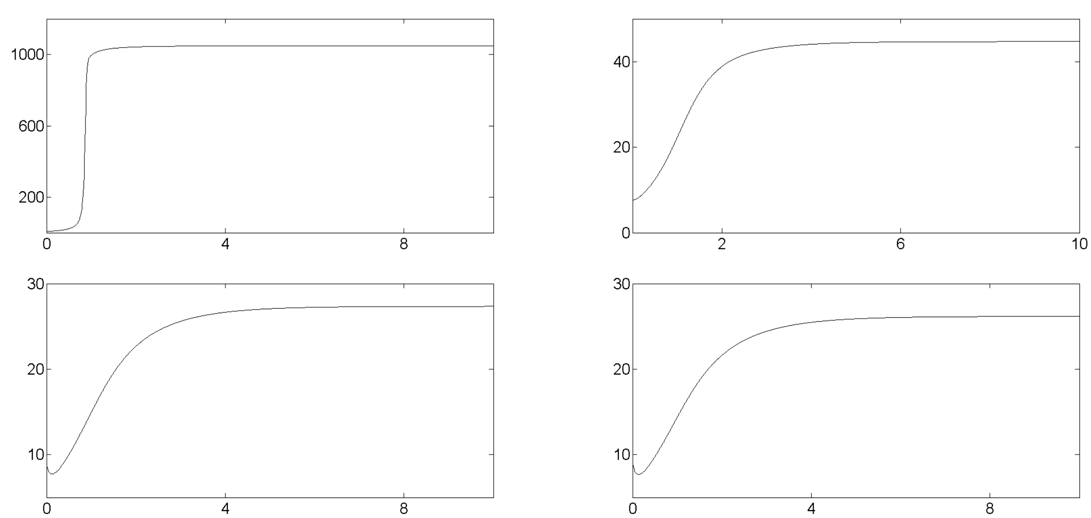

Figure 8.3 Shown are the numerically computed values of $|||\ g(t)\ |||$. The values of ρ are $1.1, 1.6, 1.95,$ and 2.0. These are shown on $(0, 10)$. The horizontal axis is the time axis.

very little difference between the last two values of ρ (and we wouldn't expect much), but notice from the first three values, that it appears that the time where the solutions flatten out increases with ρ.

Notice the much steeper incline on the first image in the figure. This may give insight on what is happening near the cutoff value of ρ. Likely all figures will look like this on an appropriate time scale, however, the much more rapid growth suggests that there is some explosive growth that occurs before smoothing out. It appears that the solutions try to blow up. For the larger values of ρ it seems that the operator A eventually contains this growth. In the smaller values shown in figure 8.1, it may be that A is not powerful enough to check this explosive growth and thus the solutions are able to blow up.

Chapter 9. Conclusions

9.1 The global result

The results for the global condition rely heavily on Lemma 1 and the modified version of it, Lemma 2. If the operator in question is one such that Lemma 2 holds, then the result holds - specifically that solutions both exist and are global for (1.10).

We note that the estimates used in establishing lemma 2 for arbitrary $\rho > 1$ did not depend on the approximation when $\rho > 2$, and thus the result there is well established in those cases. For $\rho < 2$, the approximation was needed, and thus the result can only be said to hold if the approximation is valid. Even when the approximation was valid, the method of finding it gave us some restrictions on the values of ρ for which it was applicable. Taking these into account, we can only make statements about $\rho > \frac{3}{2}$ because of the effects of the gaps (this will be discussed more in Appendix A as well as in section 9.3).

9.2 The blowup result

There is a caveat that must now be mentioned regarding the proof of the blowup condition. We made the assumption that $W \in L^{1+\epsilon}(0,T)$. A similar assumption is needed in [4], however at this time we are uncertain if such a result can hold. Specifically, it can be shown that the function W used in [4] can not be in L^2, but it must be so for the remaining inequalities to hold. Thus the blowup result is potentially flawed. We used $L^{1+\epsilon}$ here where ϵ will ultimately rely on the order of the power of $(t-s)$ in the inequalities. It is possible, with small enough power, that we do not need to go all the way to L^2 for estimating the next set of inequalities. It can easily be shown by comparing the W functions here to those in [4] that this one also can not be in L^2. It can also be shown, however, that in both cases W is in L^1. A tighter estimate giving a smaller power may allow us to work around this problem. At this time, no tighter estimate has been found.

Although the proof for the blow up of solutions of equation (3.11) is incomplete, there is strong reason to believe that the desired result is true. In section 5, we showed that there are solutions which blow up in finite time for this equation when $\rho = 1$. In [4, 11], strong numerical evidence suggests that blowup does occur in the case $\rho = 0$. Taken together, it seems likely that blowup should occur in between these values. In chapter 8, we did provide numerical evidence supporting this prediction. Additionally, although a different (but related) system, the work of Othmer and Stevens [14] as well as Levine and Sleeman [12] both provide numerical evidence arguing this case.

Under the right conditions, we showed that the solutions are global when $\rho > 1$. As the intervals needed tend to 'cluster' around 1 (see section 2.7.2), it seems likely that the cut-off point between the two possible behaviors is actually at 1.

Thus, we have strong evidence of the statement that non-trivial solutions of (3.11) can blow up in finite time for $\rho \leq 1$ and that the solutions are all global for $\rho > 1$ (of course we know for certain that the later is true for $\rho > 2$).

9.3 Future work

There is further work that is needed on this problem. Looking at the numerical results again, it appears that in the cases that the solutions are global that they may even be bounded. Notice that the results of the global proof does not state this. There is a constant, K, which bounds the solutions, but it may be the case that $\lim_{T\to\infty} K(T) = \infty$. The proof does not deny this possibility. Whether there are global solutions which grow without bound, or whether all such solutions are bounded is a question worth further study.

Additionally, it may be worth looking at the global solutions which exist in the cases where solutions are able to blow up (the conclusion was only that there are solutions which blow up, not that all solutions blow up - in fact some trivial solutions certainly can not). In these cases, are global solutions necessarily bounded? Or is this maybe another area where the two cases are distinct? Perhaps all solutions are global and bounded in the global case, but in the blowup case there are also non-bounded global solutions.

One may want to ask about the behavior of blowup times associated with a given set of initial conditions. What happens to these as the parameter approaches the cutoff. For instance where the numerical results suggest that blowup time may increase with ρ, is this actually true? This would give insight into what effect A is having on the solutions. If the cutoff leads to a finite time blowup solution for a given set of initial conditions, this would imply that the limit of the blowup times as the parameter approaches the cutoff is finite. What happens if the cutoff is not part of the blowup case? If the cutoff value leads to a global solution, then is this limit infinite?

Additionally, is it even possible to have a set of initial conditions which leads to blowup solutions below the cutoff point, and does not lead to blowup solutions at the cutoff? If we find a set of initial conditions, S, which leads to a finite time blowup solution at some value of ρ, is it possible that when we let ρ be the cutoff value, these same conditions lead to a global solution? Or must this value always lead to a finite time blowup solution as well? Can we do even better? Some attempts to solve the system numerically gave figures which looked much like the global solution figures for some values of $\rho < 1$ even when they resulted in blowup for even smaller values of ρ, much like in figure 9.1.

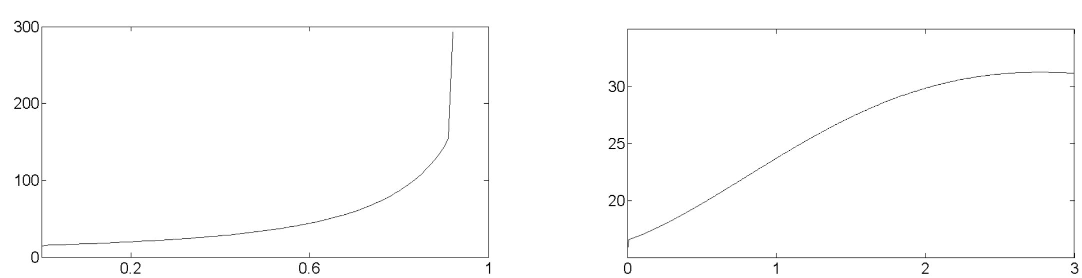

Figure 9.1 Numerical solutions with different initial conditons. The initial conditions are $g_{12}(0) = 1, g'_{12}(0) = 0.1, g_i(0) = g'_i(0) = 0$ for $i \neq 12$. Here $\rho = 0.2$ and 0.35. The horizontal axis is time.

This behavior could be simply the result of replacing the infinite system with a finite system and may go away with a higher cutoff value as in figure 9.2. However, cutoff values where the solution would start blowing up could not be found in all cases where the behavior was observed. The numerical results hint that not all initial conditions which lead to blowup for values of $\rho < 1$ may do so for all such values, but also provide evidence against it at the same time. However, such a result would not be entirely surprising and may be worth further study.

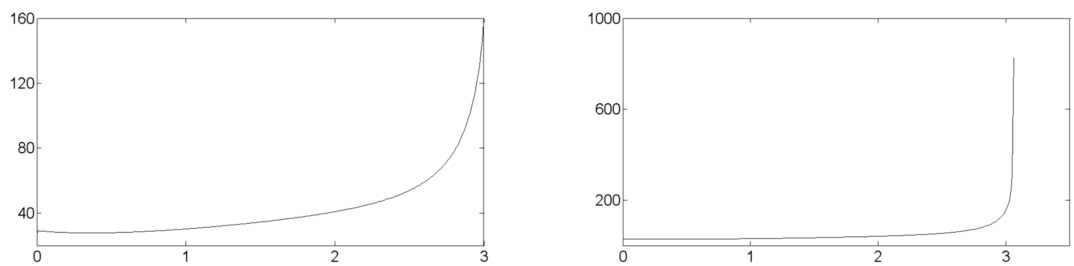

Figure 9.2 The right graph from figure 9.1 with $\rho = 0.35$ is shown with a cutoff value of 200. The results of trying to solve the equations on $[0, 3]$ and on $[0, 4]$ are shown. The horizontal axis is time.

Obviously another area of further study would be to expand the areas where the proofs apply. We can only say with certainty that solutions are global when $\rho \geq 2$. Even when the approximations hold, an observation of appendix A will show that at best we can hope for this to work when $\rho > \frac{1}{2}\left(\sqrt{33} - 3\right)$ due to the admissible parameter criteria, and when we must satisfy the gap criteria of section 2.7.2, only when $\rho > \frac{3}{2}$. A better approximation needs to be found for the cases where $\rho < 2$, and a solid proof that it holds needs to be found. It may be that a different method than that of Hillen and Potapov needs to be used for these cases.

The sometimes invalid assumption of the blowup proof needs to be eliminated. Numerical evidence does strongly suggest that blowup can occur in the cases stated, however numerical evidence alone is not sufficient - global solutions can easily involve values beyond any computer's ability to work with, and still be global solutions. In such cases the numerical results would suggest blowup even if it was not actually occurring. Although this is unlikely, it is a possibility. Additionally, numerical results could be hiding blowup where it was occurring, because of replacing the infinite system with a finite system. Even if the finite systems can eventually blow up in finite time, it could require a very large cut off on the system for this behavior to appear, which could be outside the realm of what would be possible to model. Further work needs done to fix and complete this proof in order to definitively answer these questions.

Appendix A. Admissible Parameter Algorithm

Hillen and Potapov [5] provide an algorithm for finding admissible criteria of the type that they needed in their approach to the problem. We present here a modification of their approach that works for the conditions that we used in section 2.3.

Step 1: Choose σ where

$$\frac{6-\rho}{\rho+2} < \sigma < \rho,$$

Step 2: Choose p where

$$\frac{1}{\sigma-1} < p < \min\{\frac{1}{2(2-\rho)}\left(\rho+\sqrt{\rho^2-4\rho+8}\right),\ \frac{1}{2(\sigma-1)}\left(3-\sigma+\sqrt{4+(\sigma-1)^2}\right)\},$$

Step 3: Choose r where

$$\frac{p^2+p}{1+2p} < r < \min\{\frac{1}{2-\rho}, \frac{1}{\sigma-1}\},$$

Step 4: Choose P where

$$\frac{r(1+2p)}{r(1+2p)-p} < P < 1+\frac{1}{p},$$

Step 5: Let $Q = \frac{P}{P-1}$.

In Hillen and Potapov's algorithm, steps 2 and 3 only have the second term in the minimized set. In step 1, they require that $1 < \sigma < 2$. In the cases $\rho \geq 2$ their algorithm is the correct one to use. In the remaining cases, we require the above algorithm. Hillen and Potapov carefully tested their algorithm to be sure that it provided admissible parameters. The above algorithm, being only a slight modification of their's, works as it only further restricts the choices of their's. A set of parameters acceptable in this algorithm is acceptable in the algorithm of Hillen and

Potapov. The extra restrictions ensure that the parameters chosen meet the extra requirements that we need.

Looking at step 1, we see that in order for this choice to be possible, we must have $\rho > \frac{1}{2}\left(\sqrt{33}-3\right) \approx 1.3723$. Notice that this falls in the last gap in section 2.7.2. Therefore, if this is an absolute requirement to have admissible parameters, then the proof works only when $\rho > \frac{3}{2}$ using the approximation that was found.

This does not seem to be necessarily a fault of the algorithm. If ρ is close to 1, then σ must also be close to 1. Thus p must be large. Then $\frac{2p}{\sigma p+1}$ is close to 2. Thus r must be less than 1 and greater than 2, which is not possible. As long as we have the extra criteria on r, we can not allow ρ all the way down to 1. This means that the estimate in section 2.6 is not quite sufficient to prove that the division between blowup and global solutions occurs at 1.

Appendix B. Matlab Code

The following matlab programs are used to generate the data needed to draw the figures in chapter 8. The program solve.m sets up and solves the equation. The program F.m computes the differential equation.

solve.m

```
% finds the solution of the system


clc;
clear
format long
global N
global lambda
global mu
global C
global sigma



% eigenvalue growth rate
rho = 2.0;

% constants for equation
C = 2*pi;
```

```
mu = 2;
lambda1 = 1;

% cut-off point (must be at least 3)
N = 100 ;

% solve on [0,max_time]
max_time = 2;
delta = 0.01;

% initial conditions
sigma=(-(C^2+lambda1)+sqrt((C^2 + lambda1)^2 + 4*(mu - lambda1)*C^2))/2;
for i=1:N
    g0(i,1) = .5^(i-1)/i;
    g0(N+i,1) = sigma*i*g0(i,1) ;
end
% prepare lambda vector

for i=1:N
lambda(i) = lambda1 * i^rho;
end

% solve IVP

tspan = 0:delta:max_time;
clear s
[s,G] = ode15s(@F,tspan,g0);
```

```
t=s';
g=G';
```

F.m

```
function x = F(t,g)

% F(t,g,m) finds the derivative vector for the system with
% a cut-off after the m-th component
%
% t is the time variable
% g is the vector at time g
% m is the cut-off

global lambda
global N
global mu
global C

global z
global L
global S
global s

m=N;

L = g;
```

```
% find derivatives of the g's (which are the h's)

for n=1:m
z(n) = L(n+m);
end

% find the derivative of the first h

for k=1:m-1
y(k) = (k+1) * L(k+m) * L(k+1) - k * L(k) * L(m+k+1)
     + ((k+1) * lambda(k) - k * lambda(k+1)) * L(k) * L(k+1);
end

s = sum(y);
T = (mu - lambda(1)) * C^2 * L(1) - (C^2 + lambda(1)) * L(1+m);
z(m+1) = (1/2)*C^2*s + T;
clear y


% find the derviative of the last h

for k=1:m-1
y(k) = k * L(k) * L(2*m-k) + (1/2)
     * (k * lambda(m-k) + (m-k) * lambda(k)) * L(k) * L(m-k);
end

s = sum(y);
T = (mu - lambda(m))*C^2 * m^2 * L(m) - (C^2 * m^2 + lambda(m)) * L(2*m);
```

```
z(2*m) = (1/2)*C^2*m*s + T;
clear y

% find the derivatives of the remaining h's

for n = 2:m-1
for k=1:n-1
y1(k) = k * L(k) * L(n-k+m) + (1/2)
      * (k * lambda(n-k) + (n-k) * lambda(k)) * L(k) * L(n-k);
end
for k=1:m-n
y2(k) = (k+n) * L(k+m)*L(n+k) - k*L(k)*L(n+k+m)
      + ((n+k)*lambda(k) - k*lambda(n+k))*L(k)*L(n+k);
end
s1=sum(y1);
s2=sum(y2);
T = (mu-lambda(n))*C^2*n^2*L(n) - (C^2*n^2+lambda(n))*L(n+m);
z(n+m) = (1/2)*C^2*n*(s1 + s2) + T;
clear y1
clear y2
end

% set output variable as a column vector

x = z';
```

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