Boundary functions for wavelets and their properties

by

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ABSTRACT

Wavelets and wavelet transforms have been studied extensively since the 1980s. It has been shown that the Discrete Wavelet Transform (DWT) can be applied to an enormous number of applications in virtually every branch of science.

The DWT is designed for decomposing and reconstructing infinitely long signals. In practice, we can only deal with finitely long signals. This raises the very important question of handling boundaries. What should be done near the ends?

Several approaches have been proposed to deal with this problem. In this dissertation, we consider the boundary function approach. The idea is to alter the DWT by constructing appropriate boundary functions at each end so that finite length signals can be analyzed accurately. This dissertation contains two sets of new results. One is about smoothness and approximation order properties of boundary functions. The other is about finding boundary functions. The results have been elaborated upon for some specific wavelets.

CHAPTER 1. Introduction

In most basic terms, wavelets are basis functions (signals) that are local in both time and frequency. Locality in time means that wavelets are zero or nearly zero outside of a finite time interval. Locality in frequency implies that wavelets live in certain frequency bands. They are primarily used for decomposing and reconstructing functions. As a result of their applicability to a variety of research areas such as signal analysis, numerical analysis, physics, music, and many more, the interest in them has grown enormously over the years [11], [12], [18], [27], [29]. Different approaches to wavelet theory in one- or higher-dimensional setting are available in the literature [11], [22], [24], [25].

We start with a refinable function φ, which is the solution to the following two-scale refinement equation

$$\varphi(x) = \sqrt{2} \sum_{k=k_0}^{k_1} h_k \varphi(2x-k). \tag{1.1}$$

Under some mild conditions, φ produces a multiresolution approximation and a wavelet function ψ [22]. Translations and dilations of ψ yield the basis functions. That is,

$$\left\{ \psi_{nk}(x) := 2^{n/2}\psi(2^n x - k), \quad n,\ k \in \mathbb{Z} \right\}$$

forms an orthonormal basis of $L^2(\mathbb{R})$ [10]. Details are given in section 2.4.

Similarly, starting with a refinable function vector $\boldsymbol{\varphi}$, which is the solution to the two-scale matrix refinement equation

$$\boldsymbol{\varphi}(x) = \sqrt{2} \sum_{k=k_0}^{k_1} H_k \boldsymbol{\varphi}(2x-k), \quad k \in \mathbb{Z},$$

one can obtain multiwavelets. Multiwavelets are desirable and useful tools because they can have important properties such as orthogonality, approximation order, smoothness, symmetry or antisymmetry, short support, and continuity simultaneously [15], [23], [26].

Mathematically, analysis on wavelets is represented by the wavelet transform. The Continuous Wavelet Transform maps a function of time f into a function of two arguments (time and frequency) and is defined as

$$W_f(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\psi^*(\frac{t-b}{a})dt,$$

where $a > 0$, $b \in \mathbb{R}$, and ψ is the wavelet function [1]. In words, it is the integral over all time of the function multiplied by the complex conjugate of dilated and shifted wavelets.

The Discrete Wavelet Transform is obtained by discretizing (dyadic sampling) the time-scale parameter in the wavelet transform. This is a remarkable transform in that it is easy to implement – simple algorithms are available, and they do not use the functions φ and ψ explicitly.

The Discrete Wavelet Transform acts on infinitely long signals. For finite signals, a problem occurs near the boundaries. By using some specific data extension approaches, one can extend a finite signal so that DWT can act on the extended signal [6], [32]. However, the usual approaches all suffer from various flaws: loss of orthogonality or approximation order, introducing discontinuities, making the decomposed signal longer, etc.

1.1 Overview of the Dissertation

In this dissertation, we modify the algorithm for the Discrete Wavelet Transform by introducing boundary functions so that it can be applied to finite signals.

In preparation for this, in chapter 3, we follow two approaches to construct boundary functions. The first approach is based on a recursion relation. The second approach focuses on a linear combination of functions that cross boundaries. We first investigate how these two approaches are connected. By this comparison we show that not every refinable boundary function is a combination of boundary-crossing functions or vice versa. We then obtain conditions under which these two approaches are equivalent. Because a certain degree of approximation and continuity are desirable for wavelets, we show under what conditions continuous boundary functions with approximation order 1 can be obtained. In particular, we give an explicit set of coefficients that produces the same continuous boundary function with approximation order 1 for both approaches when there is only a single boundary function. We also demonstrate by an example that this is not the case when we have a boundary function vector – more than one boundary function (see chapter 5). The work contained in this chapter will appear in [2], [3].

In chapter 4, the Madych approach [21] for obtaining boundary functions for scalar wavelets is investigated. We show that under certain extra conditions it also works for multiwavelets. We also construct a more general approach to multiwavelet endpoint modification, which works in all cases. In addition, this construction is unique up to the choice of arbitrary orthogonal matrices. The tradeoff here is that not every orthogonal matrix produces a useful set of coefficients that are used for constructing boundary functions. To overcome this, we search for orthogonal matrices that produce some useful set of coefficients in the sense that they produce boundary functions with certain properties. The choice of orthogonal matrix is based on the results from chapter 3. In fact, the necessity of choosing suitable matrices is what motivated the work in chapter 3. The results obtained in this chapter will appear in [2], [3].

In chapter 5, the results obtained in previous chapters are applied to some specific wavelets. The explicit boundary functions with certain properties for the specific wavelets are obtained as well. We also show by an example that one can construct a regular boundary function that cannot be written as linear combinations of boundary-crossing functions.

In chapter 6, we discuss possible applications and direction for future research.

CHAPTER 2. Wavelet Theory

2.1 Introduction

This chapter begins by stating the basic definitions and theorems in scalar wavelet theory. It then presents basic properties of multiwavelets. The DWT, which will be used throughout the dissertation, for both scalar wavalets and multiwavelets is also introduced.

2.2 Refinable Functions and Scalar Wavelets

Definition 2.1. *A function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ is called a* refinable function *if it satisfies a* two-scale refinement equation *or* recursion relation *of the form*

$$\varphi(x) = \sqrt{2} \sum_{k=k_0}^{k_1} h_k \varphi(2x-k). \tag{2.1}$$

This equation is also referred to as *dilation equation.* The $h_k \in \mathbb{C}$ are called the *recursion coefficients.*

It is possible to consider refinement equations with an infinite number of recursion coefficients, as long as they satisfy suitable decay conditions. However, we shall assume that there are only finitely many recursion coefficients.

Definition 2.2. *The inner product of two functions ϕ and φ is defined by*

$$\langle \phi(x), \varphi(x) \rangle = \int \phi(x) \varphi^*(x) dx,$$

where the * stands for complex conjugation.

Definition 2.3. *φ is called* orthogonal *if*

$$\langle \varphi(x), \varphi(x-k) \rangle = \delta_{0k}\,, \qquad k \in \mathbb{Z}.$$

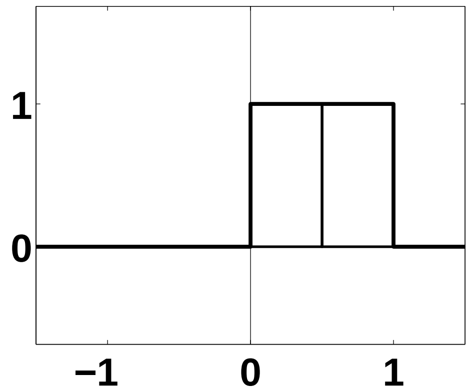

Figure 2.1 Haar scaling function recursion relation

Two refinable functions φ *and* $\tilde{\varphi}$ *are called* biorthogonal *if*

$$\langle \varphi(x), \tilde{\varphi}(x-k) \rangle = \delta_{0k}, \qquad k \in \mathbb{Z}.$$

$\tilde{\varphi}$ is called a *dual* of φ.

Example 2.1. The Haar function is the simple unit-width, unit-height pulse function. It is clear that $\varphi(x)$ can be reconstructed from $\varphi(2x)$ and $\varphi(2x-1)$. That is,

$$\varphi(x) = \varphi(2x) + \varphi(2x-1),$$

(see figure 2.1), which means that the coefficients are

$$h_0 = h_1 = \frac{1}{\sqrt{2}}.$$

The Haar function is orthogonal.

Example 2.2. The hat function is given by

$$\varphi(x) = \begin{cases} 1+x, & \text{if } -1 \le x \le 0; \\ 1-x, & \text{if } 0 < x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

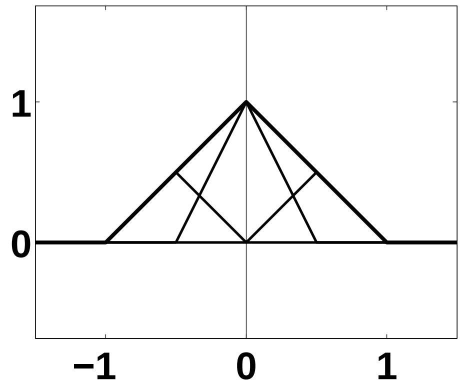

Figure 2.2 Hat function recursion relation

It satisfies

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1)\\ &= \sqrt{2}\big(\frac{1}{2\sqrt{2}}\varphi(2x+1) + \frac{1}{\sqrt{2}}\varphi(2x) + \frac{1}{2\sqrt{2}}\varphi(2x-1)\big),\end{aligned}$$

(see figure 2.2), so the recursion coefficients are

$$h_{-1} = \frac{1}{2\sqrt{2}}, \qquad h_0 = \frac{1}{\sqrt{2}}, \qquad h_1 = \frac{1}{2\sqrt{2}}.$$

Since $\langle \varphi(x), \varphi(x-1) \rangle \neq 0$, it is not orthogonal.

Example 2.3. The scaling function of the Daubechies wavelet D_4 has recursion coefficients

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \qquad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \qquad h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \qquad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

It is orthogonal [11], (see figure 2.3).

Theorem 2.4. *A necessary condition for orthogonality is*

$$\sum_k h_k h^*_{k-2l} = \delta_{0l}. \tag{2.2}$$

This is proved in [20].

As a corollary of this theorem, one can deduce that an orthogonal φ has an even number of recursion coefficients.

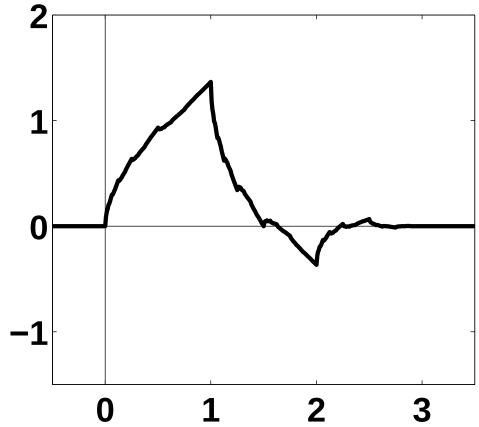

Figure 2.3 D_4 scaling function

Many refinable functions cannot be represented in closed form. However, we are able to compute the point values and analyze other properties, such as smoothness and regularity. But before delving into the details, we first analyze and express the above relations in the frequency domain.

Definition 2.5. *Let f be an integrable function over $\mathbb{R}$. The* Fourier Transform *of f is denoted by $\hat{f}$ and is defined by*

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx.$$

Definition 2.6. *The* symbol *of a refinable function is the trigonometric polynomial*

$$h(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k e^{-ik\xi}.$$

Taking the Fourier transform of the refinement equation (2.1), we obtain

$$\hat{\phi}(\xi) = h(\frac{\xi}{2}) \hat{\phi}(\frac{\xi}{2}). \tag{2.3}$$

Equation (2.3) is the refinement equation in the frequency domain.

Theorem 2.7. *A necessary condition for orthogonality in the frequency domain is*

$$|h(\xi)|^2 + |h(\xi + \pi)|^2 = 1. \tag{2.4}$$

The proof can be found in [11] and [20].

2.3 Cascade Algorithm and Point Values

Definition 2.8. *The* cascade algorithm *is a fixed point iteration applied to the refinement equation.*

The n^{th} *iteration with a chosen initial function* $\varphi^{(0)}$ *is given by*

$$\varphi^{(n+1)}(x) = \sqrt{2}\sum_k h_k \varphi^{(n)}(2x - k).$$

Convergence of the cascade algorithm is part of the sufficient conditions in many settings [11]. The following lemma is an example.

Lemma 2.9. *If the recursion coefficients satisfy (2.2) or equivalently, if the symbol* h *satisfies (2.4), and the cascade algorithm converges, then* φ *is orthogonal and has compact support.*

This is proved in [11].

The cascade algorithm can be used to find approximate point values of $\varphi(x)$. The exact point values of φ can also often be found by solving an eigenvalue problem. While it may not work for some cases, it usually works for continuous φ.

It is easy to show that if φ satisfies the recursion equation (2.1), it has support $[k_0, k_1]$. If we let $\boldsymbol{\varphi}$ be the vector of point values of φ at the integers in $[k_0, k_1]$,

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi(k_0) \\ \vdots \\ \varphi(k_1) \end{pmatrix},$$

the refinement equation admits the form

$$\boldsymbol{\varphi} = T\boldsymbol{\varphi}, \tag{2.5}$$

which is an eigenvalue problem.

Note: For notational convenience, vectors are set in boldface type.

Example 2.4. Consider the scaling function of the Daubechies wavelet D_4.

The equation (2.5) takes the following form

$$T\boldsymbol{\varphi} = \sqrt{2}\begin{pmatrix} h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 \\ 0 & h_3 & h_2 & h_1 \\ 0 & 0 & 0 & h_3 \end{pmatrix}\begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \end{pmatrix} = \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \end{pmatrix}.$$

The solution, normalized to $\sum_k \varphi(k) = 1$, is

$$\varphi(0) = 0, \qquad \varphi(1) = \frac{1+\sqrt{3}}{2}, \qquad \varphi(2) = \frac{1-\sqrt{3}}{2}, \qquad \varphi(3) = 0.$$

Then, one can use the refinement equation to find the values of φ at any dyadic rational.

Remark 2.10. *If we iterate (2.3), we obtain*

$$\hat{\varphi}(\xi) = \left[\prod_k h(\frac{\xi}{2^k})\right]\hat{\varphi}(0).$$

A solution to the refinement equation exists under assumptions that $\hat{\varphi}(0) \neq 0$ and that the infinite product converges. This approach is not very effective to find φ, but it is useful for smoothness estimates and existence proofs.

2.4 Multiresolution Approximations

Multiresolution Approximation

Multiresolution approximation (MRA) is one of the main concepts in wavelet theory. Any multiresolution approximation provides us with not only the wavelet function but also a way for understanding and deriving orthonormal wavelet bases. Basic definitions and results are stated for orthonormal *MRA* throughout this section.

Definition 2.11. *A* multiresolution approximation *of $L^2(\mathbb{R})$ is a chain of closed subspaces $V_n, n \in \mathbb{Z}$,*

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbb{R})$$

satisfying

1. *$V_n \subset V_{n+1}$ for all $n \in \mathbb{Z}$;*

2. *$f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$, for all $n \in \mathbb{Z}$;*

3. *$f(x) \in V_n \Longrightarrow f(x - 2^{-n}k) \in V_n$, for all $k \in \mathbb{Z}$;*

4. *$\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$;*

5. *$\overline{\bigcup_{n\in Z} V_n} = L^2(\mathbb{R})$;*

6. *There exists a function $\varphi \in L^2$ such that*

$$\Big\{\varphi(x-k) : k \in \mathbb{Z}\Big\}$$

   is an orthonormal basis for V_0 [22].

The function φ, whose existence is asserted in (6), is called a *scaling function* of the given MRA.

Condition (2) gives us the main property of an MRA. Each V_n consists of the functions in V_0 compressed by a factor of 2^n. Thus, an orthonormal basis of V_n is given by

$$\Big\{\varphi_{nk}(x) := 2^{n/2}\varphi(2^n x - k), \quad k \in \mathbb{Z}\Big\}.$$

The factor $2^{n/2}$ preserves the L^2-norm.

Since $V_0 \subset V_1$, φ can be written in terms of the basis of V_1 as

$$\varphi(x) = \sum_k h_k \varphi_{1,k}(x) = \sqrt{2}\sum_k h_k \varphi(2x-k), \tag{2.6}$$

for some coefficients h_k. That means that φ is refinable.

Theorem 2.12. *Assume that*

$$h(0) = 1,$$

$$|h(\xi)|^2 + |h(\xi+\pi)|^2 = 1,$$

and that the cascade algorithm converges. Then the solution $\varphi(x)$ for the refinement equation exists and is a scaling function for an MRA.

This is proved in [22].

The orthogonal projection P_n of a function $f \in L^2$ into V_n is defined by

$$P_n f = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{n,k} \rangle \varphi_{n,k}.$$

This is also called an approximation to f at scale 2^{-n}. It is the best approximation of f in V_n. The key point of MRA can be expressed not by using $P_n f$ and increasing n, but by using the difference between approximations to f at successive scales 2^{-n} and 2^{-n-1}.

More precisely, let

$$Q_n = P_{n+1} - P_n.$$

Q_n is also an orthogonal projection onto a closed subspace W_n. W_n is the orthogonal complement of V_n in V_{n+1};

$$V_{n+1} = V_n \oplus W_n.$$

This implies that

$$V_n = \bigoplus_{k=-\infty}^{n-1} W_k.$$

Hence,

$$L^2(\mathbb{R}) = \bigoplus_{n \in \mathbb{Z}} W_n.$$

Mallat [22] showed that an orthonormal basis of W_0 is generated from integer translates of single function $\psi(x) \in L^2(\mathbb{R})$, called *wavelet function* or *mother wavelet,* which is essentially determined by the scaling function. Since $W_0 \subset V_1$, the wavelet function ψ can be represented as

$$\psi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_n \varphi(2t - n) \tag{2.7}$$

for some coefficients g_n, [7].

Since W_n is comprised of functions in W_0 compressed by a factor of 2^n,

$$\left\{ \psi_{nk}(x) := 2^{n/2} \psi(2^n x - k), \quad k \in \mathbb{Z} \right\},$$

forms an orthonormal basis of W_n.

Moreover,

$$\Big\{\psi_{nk},\ n,k \in \mathbb{Z}\Big\}$$

produces an orthonormal basis for $L^2(\mathbb{R})$ [10].

This implies that an arbitrary function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{n,k} \langle f, \psi_{nk} \rangle \psi_{nk}, \tag{2.8}$$

and also as

$$f(x) = \sum_{k} \langle f, \varphi_{jk} \rangle \varphi_{jk} + \sum_{k} \sum_{n \geq j} \langle f, \psi_{nk} \rangle \psi_{nk}. \tag{2.9}$$

Equation (2.8) or (2.9) is called a *wavelet decomposition* of f.

Example 2.5. If we define

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ -1, & \text{if } \frac{1}{2} < x \leq 1; \\ 0, & \text{elsewhere,} \end{cases}$$

then ψ is an orthonormal wavelet for the MRA generated by the Haar scaling function. This is called the *Haar Wavelet*.

For a given orthogonal MRA, the scaling function $\varphi(x)$ and the wavelet function $\psi(x)$ must satisfy the following properties under assumptions that $\varphi(x) \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ and $\psi(x) \in L^1(\mathbb{R})$.

The following statements are all proved in [31].

Theorem 2.13. $\quad |\int_{\mathbb{R}} \varphi(x) dx| = 1.$

Corollary 2.14. $\quad \int_{\mathbb{R}} \psi(x) dx = 0.$

The necessary conditions can be written in terms of recursion coefficients as follows.

1. $\sum_k h_k = \sqrt{2}$;

2. $\sum_k g_k = 0$;

3. $\sum_k h_k h^*_{k-2l} = \sum_k g_k g^*_{k-2l} = \delta_{0l}$;

4. $\sum_k g_k h^*_{k-2l} = 0$ for all $l \in \mathbb{Z}$;

5. $\sum_k h^*_{m-2k} h_{n-2k} + \sum_k g^*_{m-2k} g_{n-2k} = \delta_{nm}$.

2.5 Moments and Approximation Order

Definition 2.15. *The k^{th}* discrete moments *of φ and ψ are defined by*

$$m_k = \frac{1}{\sqrt{2}} \sum_l l^k h_l,$$

$$n_k = \frac{1}{\sqrt{2}} \sum_l l^k g_l.$$

They are related to the symbols by

$$m_k = i^k D^k h(0),$$

$$n_k = i^k D^k g(0),$$

where D stands for differentiation. Discrete moments are uniquely defined [7].

Definition 2.16. *The k^{th}* continuous moments *of φ and ψ are defined by*

$$\mu_k = \int x^k \varphi(x) dx,$$

$$\nu_k = \int x^k \psi(x) dx.$$

They are related to the Fourier transforms of φ and ψ by

$$\mu_k = \sqrt{2\pi} i^k D^k \hat{\varphi}(0),$$

$$\nu_k = \sqrt{2\pi} i^k D^k \hat{\psi}(0).$$

The continuous moment μ_0 is not determined by the refinement equation. It depends on the scaling of φ.

Theorem 2.17. *The continuous and discrete moments are related by*

$$\mu_k = 2^{-k} \sum_{t=0}^{k} \binom{k}{t} m_{k-t} \mu_t,$$

$$\nu_k = 2^{-k} \sum_{t=0}^{k} \binom{k}{t} n_{k-t} \mu_t.$$

Starting with given μ_0, all other continuous moments can be computed from these relations [20].

Definition 2.18. *A function ψ has p* vanishing moments *if*

$$\int_{\mathbb{R}} \psi(x)x^n dx = 0, \quad n = 0, 1, \dots, p-1.$$

Definition 2.19. *A scaling function has* approximation order p *if all polynomials of degree less than p can be expressed as a linear combination of its integer shifts. That means the wavelet transform of any polynomial of degree less than p will have all its wavelet coefficients zero.*

Definition 2.20. *The recursion coefficients h_k satisfy the* sum rules of order p *if*

$$\sum_k (-1)^k k^m h_k = 0, \quad m = 0, 1, \dots, p-1.$$

It is important to know that the sum rules of order p can be equivalently stated as "the symbol h has a zero of order p at $\xi = \pi$." There are actually more equivalent statements, all of which are combined in the following fundamental theorem.

Theorem 2.21. *Assume that the recursion coefficients h_k satisfy the sum rules of order* 1. *Then the following statements are equivalent.*

1. *φ has approximation order p.*
2. *$\{h_k\}$ satisfies the sum rules of order p.*
3. *$h^{(m)}(\pi) = 0, \quad m = 0, 1, \dots, p-1$, and so it can be factored as $h(\xi) = (1 + e^{-i\xi})^p t(\xi)$, where t is some trigonometric polynomial.*
4. *$D^n \hat{\varphi}(2k\pi) = 0, \quad k \in \mathbb{Z} - \{0\}, \quad n = 1, 2, \dots, p-1$.*

This is proved in [28], [19].

Example 2.6. The Haar wavelet has 1 vanishing moment and so only constant functions can be reproduced by the scaling functions. The symbol is

$$h(\xi) = \frac{1}{2}(1 + e^{-i\xi}).$$

2.6 Discrete Wavelet Transform

From now on we concentrate on orthogonal wavelets with compact support. As we defined in section 2.4,

$$\begin{aligned} V_{n+1} &= V_n \oplus W_n \\ &= V_{n-1} \oplus W_{n-1} \oplus W_n \\ &= V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_{n-1}. \end{aligned}$$

The series expansion for a given function $s(x) \in V_n$ is either

$$\begin{aligned} s(x) &= P_n s(x) \\ &= \sum_k \langle s, \varphi_{nk} \rangle \varphi_{nk}(x) \\ &= \sum_k s^*_{nk} \varphi_{nk}(x) \end{aligned}$$

or

$$\begin{aligned} s(x) &= P_{n-1}s(x) + Q_{n-1}s(x) \\ &= \sum_j \langle s, \varphi_{n-1,j} \rangle \varphi_{n-1,j}(x) + \sum_j \langle s, \psi_{n-1,j} \rangle \psi_{n-1,j}(x) \\ &= \sum_j s^*_{n-1,j} \varphi_{n-1,j}(x) + d^*_{n-1,j} \psi_{n-1,j}(x) \end{aligned}$$

where $s^*_{nk} = \langle s, \varphi_{nk} \rangle$, $d^*_{nk} = \langle s, \psi_{nk} \rangle$.

Note that the decomposed signal consists of two pieces $\mathbf{s}_{n-1}$ and $\mathbf{d}_{n-1}$.

Lemma 2.22.

$$\begin{aligned} \langle \varphi_{n-1,j}, \varphi_{nk} \rangle &= h_{k-2j}, \\ \langle \psi_{n-1,j}, \varphi_{nk} \rangle &= g_{k-2j}. \end{aligned}$$

Proof.

$$\begin{aligned}
\langle \varphi_{n-1,j}, \varphi_{nk} \rangle &= \int 2^{\frac{n-1}{2}} \varphi(2^{n-1}x - j) 2^{\frac{n}{2}} \varphi^*(2^n x - k) dx \\
&= \int 2^{\frac{n-1}{2}} \sqrt{2} \sum_l h_l \varphi(2^n x - 2j - l) 2^{\frac{n}{2}} \varphi^*(2^n x - k) dx \\
&= \sum_l h_l \int 2^n \varphi(2^n x - 2j - l) \varphi^*(2^n x - k) dx \\
&= \sum_l h_l \delta_{2j+l,k} \\
&= h_{k-2j}.
\end{aligned}$$

The second identity is proved in a similar manner. □

Algorithm: Discrete Wavelet Transform

Assume we have the signal $\mathbf{s}_n = \{s_{nk}\}$.

Decomposition:

$$\begin{aligned}
s_{n-1,j} &= \sum_k h_{k-2j} s_{nk}, \\
d_{n-1,j} &= \sum_k g_{k-2j} s_{nk}.
\end{aligned}$$

Reconstruction:

$$s_{nk} = \sum_j [h^*_{k-2j} s_{n-1,j} + g^*_{k-2j} d_{n-1,j}].$$

The DWT can also be interpreted as a product of an infinite matrix and a vector. The decomposition step for the *s-coefficients* and the *d-coefficients*, respectively, turns out to be

$$\begin{pmatrix} \vdots \\ s_{n-1,-1} \\ s_{n-1,0} \\ s_{n-1,1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots & & & & & & \\ \dots & h_{-1} & h_0 & h_1 & h_2 & \dots & & & & \\ & & \dots & h_{-1} & h_0 & h_1 & h_2 & \dots & & \\ & & & & \dots & h_{-1} & h_0 & h_1 & h_2 & \dots \\ & & & & & & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \vdots \\ s_{n,-1} \\ s_{n,0} \\ s_{n,1} \\ \vdots \end{pmatrix},$$

and

$$\begin{pmatrix} \vdots \\ d_{n-1,-1} \\ d_{n-1,0} \\ d_{n-1,1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & & & & & & \\ \cdots & g_{-1} & g_0 & g_1 & g_2 & \cdots & & & & \\ & & \cdots & g_{-1} & g_0 & g_1 & g_2 & \cdots & & \\ & & & & \cdots & g_{-1} & g_0 & g_1 & g_2 & \cdots \\ & & & & & & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ s_{n,-1} \\ s_{n,0} \\ s_{n,1} \\ \vdots \end{pmatrix}.$$

The notation becomes easier if we interleave s- and d- coefficients:

$$\underbrace{\begin{pmatrix} \vdots \\ s_{n-1,-1} \\ d_{n-1,-1} \\ s_{n-1,0} \\ d_{n-1,0} \\ s_{n-1,1} \\ d_{n-1,1} \\ \vdots \end{pmatrix}}_{(\mathbf{sd})_{n-1}} = \underbrace{\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & & & & & & \\ \cdots & h_{-1} & h_0 & h_1 & h_2 & \cdots & & & & \\ \cdots & g_{-1} & g_0 & g_1 & g_2 & \cdots & & & & \\ & & \cdots & h_{-1} & h_0 & h_1 & h_2 & \cdots & & \\ & & \cdots & g_{-1} & g_0 & g_1 & g_2 & \cdots & & \\ & & & & \cdots & h_{-1} & h_0 & h_1 & h_2 & \cdots \\ & & & & \cdots & g_{-1} & g_0 & g_1 & g_2 & \cdots \\ & & & & & & \cdots & \cdots & \cdots & \cdots \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} \vdots \\ s_{n,-1} \\ s_{n,0} \\ s_{n,1} \\ \vdots \end{pmatrix}}_{\mathbf{s}_n},$$

where T is an infinite block Toeplitz matrix.

Decomposition:

$$(\mathbf{sd})_{n-1} = T\mathbf{s}_n \tag{2.10}$$

Reconstruction:

$$\mathbf{s}_n = T^*(\mathbf{sd})_{n-1} \tag{2.11}$$

In engineering terms, decomposition and reconstruction amount to the filtering scheme in figure 2.4. Here, the $\uparrow 2$ and $\downarrow 2$ denote the upsampling and the downsampling operators, respectively. $*$ stands for the convolution operator. Digital signal processors use the sequences $\{h_k\}$ and $\{g_n\}$ as filter coefficients, which are also referred to as *lowpass filter* and *highpass filter,* respectively.

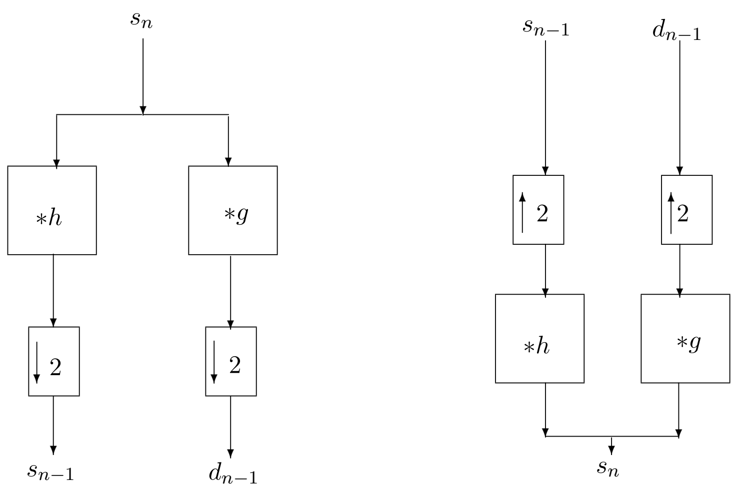

Figure 2.4 Two band filter bank

2.7 Refinable Function Vectors and Multiwavelets

We now consider the case where there are several functions, grouped together into a function vector, which are jointly refinable. The recursion coefficients in this case are no longer scalars. Instead they are matrices, and the symbols are trigonometric matrix polynomials.

Definition 2.23. *A* refinable function vector *is a vector valued function*

$$\boldsymbol{\varphi}(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_r(x) \end{pmatrix},$$

where $\boldsymbol{\varphi}_k : \mathbb{R} \longmapsto \mathbb{C}$*, which satisfies a* two scale matrix refinement equation *of the form*

$$\boldsymbol{\varphi}(x) = \sqrt{2} \sum_{k=k_0}^{k_1} H_k \boldsymbol{\varphi}(2x - k), \quad k \in \mathbb{Z}.$$

r *is called the* multiplicity *of* $\boldsymbol{\varphi}$*. The recursion coefficients* H_k *are* $r \times r$ *matrices.*

The inner product of two function vectors $\boldsymbol{\phi}$ and $\boldsymbol{\varphi}$ is defined by

$$\langle \boldsymbol{\phi}(x), \boldsymbol{\varphi}(x) \rangle = \int \boldsymbol{\phi}(x) \boldsymbol{\varphi}^*(x) dx.$$

For a function vector, $\boldsymbol{\varphi}^*$ denotes the complex conjugate transpose, so this inner product is an $r \times r$ matrix.

The refinable function vector $\boldsymbol{\varphi}$ is called orthogonal if

$$\langle \boldsymbol{\varphi}(x), \boldsymbol{\varphi}(x-k) \rangle = \delta_{0k} I, \quad k \in \mathbb{Z}.$$

I is an $r \times r$ identity matrix.

Two refinable function vectors $\boldsymbol{\varphi}$ and $\tilde{\boldsymbol{\varphi}}$ are called *biorthogonal* if

$$\langle \boldsymbol{\varphi}(x), \tilde{\boldsymbol{\varphi}}(x-k) \rangle = \delta_{0k} I, \quad k \in \mathbb{Z}.$$

Example 2.7. An example with multiplicity 2 is the constant-linear refinable function vector,

$$\boldsymbol{\varphi}(x) = \begin{pmatrix} 1 \\ \sqrt{3}(2x-1) \end{pmatrix},$$

where $x \in [0\,, 1]$.

It satisfies

$$\begin{aligned} \varphi_1(x) &= \varphi_1(2x) + \varphi_1(2x-1), \\ \varphi_2(x) &= [\frac{-\sqrt{3}}{2}\varphi_1(2x) + \frac{1}{2}\varphi_2(2x)] + [\frac{\sqrt{3}}{2}\varphi_1(2x-1) + \frac{1}{2}\varphi_2(2x-1)]. \end{aligned}$$

This function vector is refinable with

$$H_0 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 & 0 \\ -\sqrt{3} & 1 \end{pmatrix}, \quad H_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 & 0 \\ \sqrt{3} & 1 \end{pmatrix}.$$

It is orthogonal.

Theorem 2.24. *A necessary condition for orthogonality is*

$$\sum_k H_k H_{k-2l}^* = \delta_{0l} I. \tag{2.12}$$

This is proved in [20].

Definition 2.25. *The* symbol *of a refinable function vector is the trigonometric matrix polynomial*

$$H(\xi) = \frac{1}{\sqrt{2}} \sum_{k=k_0}^{k_1} H_k e^{-ik\xi}.$$

The *Fourier transform* of the refinement equation is

$$\hat{\boldsymbol{\varphi}}(\xi) = H(\frac{\xi}{2})\hat{\boldsymbol{\varphi}}(\frac{\xi}{2}).$$

Lemma 2.26. *The orthogonality condition* (2.12) *is equivalent to*

$$H(\xi)H(\xi)^* + H(\xi+\pi)H(\xi+\pi)^* = I.$$

A proof can be found in [20].

As in the scalar case, these conditions are sufficient if the cascade algorithm converges.

Definition 2.27. *A matrix A satisfies* Condition E(p) *if it has a nondegenerate $p-$fold eigenvalue* 1*, and all other eigenvalues are smaller than* 1 *in magnitude.*

Condition E means Condition E(1).

Definition 2.28. *A compactly supported refinable function vector $\boldsymbol{\varphi}$ has* linearly independent shifts *if for all sequences of vectors $\{\mathbf{c}_k\}$,*

$$\textstyle\sum_k \mathbf{c}_k^* \boldsymbol{\varphi}(x-k) = 0 \Longrightarrow \mathbf{c}_k = 0$$

for all k.

Definition 2.29. *The refinable function vector $\boldsymbol{\varphi}$ satisfies the* minimal regularity conditions *if $\boldsymbol{\varphi}$ has compact support, $\boldsymbol{\varphi} \in L^2$, $\boldsymbol{\varphi}$ has linearly independent shifts, and $\int \boldsymbol{\varphi}(x)dx \neq 0$.*

Theorem 2.30. *Let φ have minimal regularity. Then the following conditions hold:*

1. *$H(0)$ satisfies condition E.*

2. *There exists a vector $\mathbf{y}_0 \neq 0$ so that*

$$\sum_k \mathbf{y}_0^* \boldsymbol{\varphi}(x-k) = 1.$$

3. *The same vector* $\mathbf{y}_0$ *satisfies*

$$\mathbf{y}_0^* H(\pi k) = \delta_{0k} \mathbf{y}_0^*,$$

$k = 0, 1.$

4. *The same vector* $\mathbf{y}_0$ *satisfies*

$$\mathbf{y}_0^* \sum_l H_{2l+k} = \frac{1}{\sqrt{2}} \mathbf{y}_0^*,$$

$k = 0, 1.$

This is proved in [23].

Definition 2.31. ***Multiresolution Approximation and Multiwavelets***

A Multiresolution Approximation *(MRA) of* L^2 *is a doubly infinite nested sequence of subspaces of* L^2

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$$

with properties

1. $V_n \subset V_{n+1}$ *for all* $n \in \mathbb{Z}$;

2. $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$ *for all* $n \in \mathbb{Z}$;

3. $f(x) \in V_n \Longleftrightarrow f(x - 2^{-n}k) \in V_n$ *for all* $k \in \mathbb{Z}$;

4. $\bigcap_n V_n = \{0\}$;

5. $\overline{\bigcup_n V_n} = L^2(\mathbb{R})$;

6. *There exists a function vector* $\boldsymbol{\varphi} \in L^2$ *so that*

$$\{\varphi_l(x - k) : l = 1, 2, \ldots, r \text{ and } k \in \mathbb{Z}\}$$

forms an orthonormal basis of V_0. $\boldsymbol{\varphi}$ *is called the* multiscaling function.

Condition (2) expresses the main property of an MRA:

Each V_n consists of the function in V_0 compressed by a factor of 2^n. Thus, an orthonormal basis of V_n is given by

$$\{\boldsymbol{\varphi}_{nk}(x) := 2^{n/2} \boldsymbol{\varphi}(2^n x - k), \quad k \in \mathbb{Z}\}.$$

The factor $2^{n/2}$ preserves the L^2-norm.

Since $V_0 \subset V_1$, $\boldsymbol{\varphi}$ can be written in terms of the basis of V_1 as

$$\boldsymbol{\varphi}(x) = \sum_k H_k \boldsymbol{\varphi}_{1k}(x) = \sqrt{2} \sum_k H_k \boldsymbol{\varphi}(2x - k)$$

for some coefficient matrices H_k. It means $\boldsymbol{\varphi}$ is refinable.

The orthogonal projection P_n of a function $f \in L^2$ into V_n is defined by

$$P_n f = \sum_{k \in \mathbb{Z}} \langle f, \boldsymbol{\varphi}_{n,k} \rangle \boldsymbol{\varphi}_{n,k},$$

which is also called an approximation to f at scale 2^{-n}.

As in the scalar case, we define Q_n to be the difference between approximations to f at successive scales 2^{-n} and 2^{-n-1} :

$$Q_n = P_{n+1} - P_n.$$

Q_n is also an orthogonal projection onto a closed subspace W_n. W_n is the orthogonal complement of V_n in V_{n+1};

$$V_{n+1} = V_n \oplus W_n.$$

This implies that

$$V_n = \bigoplus_{k=-\infty}^{n-1} W_k.$$

It is known that there exists a function vector $\boldsymbol{\psi} \in L^2$ so that its integer translates form an orthonormal basis of W_0. Since $W_0 \subset V_1$, the function vector $\boldsymbol{\psi}$ can be represented as

$$\boldsymbol{\psi}(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} G_k \boldsymbol{\varphi}(2x - k)$$

for some coefficients G_k. The function vector $\boldsymbol{\psi}$ is called the *multiwavelet function.* $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ together form a *multiwavelet.*

Example 2.8. The orthogonal scaling function vector of the DGHM wavelet [14], (see figure 2.5), has recursion coefficients

$$H_0 = \begin{pmatrix} \frac{3}{10} & \frac{4}{5\sqrt{2}} \\ \frac{-1}{20\sqrt{2}} & \frac{-3}{20} \end{pmatrix}, \qquad H_1 = \begin{pmatrix} \frac{3}{10} & 0 \\ \frac{9}{20\sqrt{2}} & \frac{1}{2} \end{pmatrix},$$

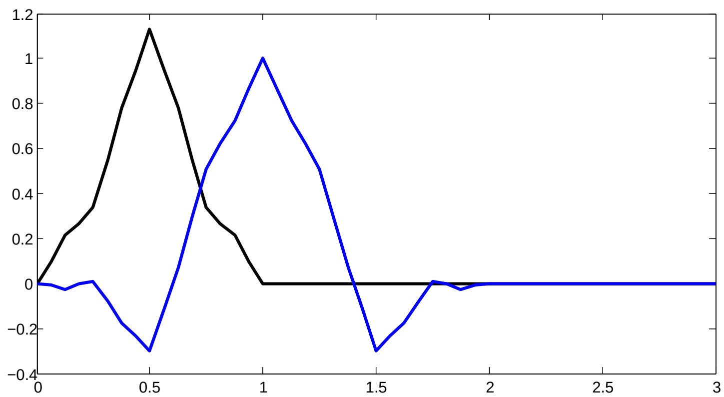

Figure 2.5 DGHM scaling functions

$$H_2 = \begin{pmatrix} 0 & 0 \\ \frac{9}{20\sqrt{2}} & \frac{-3}{20} \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 0 & 0 \\ \frac{-1}{20\sqrt{2}} & 0 \end{pmatrix}.$$

The associated wavelet coefficients are

$$G_0 = \begin{pmatrix} \frac{-1}{20\sqrt{2}} & \frac{-3}{20} \\ \frac{1}{20} & \frac{3}{10\sqrt{2}} \end{pmatrix}, \qquad G_1 = \begin{pmatrix} \frac{9}{20\sqrt{2}} & \frac{-1}{2} \\ \frac{-9}{20} & 0 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} \frac{9}{20\sqrt{2}} & \frac{-3}{20} \\ \frac{9}{20} & \frac{-3}{10\sqrt{2}} \end{pmatrix}, \qquad G_3 = \begin{pmatrix} \frac{-1}{20\sqrt{2}} & 0 \\ \frac{-1}{20} & 0 \end{pmatrix}.$$

From now on, we assume that $\boldsymbol{\varphi}$ satisfies the *minimal regularity conditions.* In particular, $H(0)$ satisfies condition E.

Definition 2.32. *The* k^{th} discrete moments *of* $\boldsymbol{\varphi}$ *and* $\boldsymbol{\psi}$ *are defined by*

$$M_k = \frac{1}{\sqrt{2}} \sum_l l^n H_l,$$

$$N_k = \frac{1}{\sqrt{2}} \sum_l l^n G_l.$$

Discrete moments are $r \times r$ matrices. They are related to the symbols by

$$M_k = i^k D^k H(0),$$

$$N_k = i^k D^k G(0).$$

Discrete moments are uniquely defined.

Definition 2.33. *The* k^{th} continuous moments *of* $\boldsymbol{\varphi}, \boldsymbol{\psi}$ *are*

$$\boldsymbol{\mu}_k = \int x^k \boldsymbol{\varphi}(x)dx,$$

$$\boldsymbol{\nu}_k = \int x^k \boldsymbol{\psi}(x)dx.$$

Continuous moments are $r-$vectors*.*

They are related to the Fourier transforms of $\boldsymbol{\varphi}, \boldsymbol{\psi}$ by

$$\boldsymbol{\mu}_k = \sqrt{2\pi} i^k D^k \hat{\boldsymbol{\varphi}}(0),$$

$$\boldsymbol{\nu}_k = \sqrt{2\pi} i^k D^k \hat{\boldsymbol{\psi}}(0).$$

The continuous moment $\boldsymbol{\mu}_0$ is only defined up to a constant multiple by the refinement equation, depending on the scaling of $\boldsymbol{\varphi}$.

Theorem 2.34. *The continuous and discrete moments are related by*

$$\boldsymbol{\mu}_k = 2^{-k} \sum_{t=0}^{k} \binom{k}{t} M_{k-t} \boldsymbol{\mu}_t,$$

$$\boldsymbol{\nu}_k^{(s)} = 2^{-k} \sum_{t=0}^{k} \binom{k}{t} N_{k-t}^{(s)} \boldsymbol{\mu}_t.$$

In particular,

$$\boldsymbol{\mu}_0 = M_0 \boldsymbol{\mu}_0 = H(0) \boldsymbol{\mu}_0.$$

Once $\boldsymbol{\mu}_0$ *is chosen, all other continuous moments can be computed from these relations* [20]*.*

CHAPTER 3. Refinable Boundary Functions

3.1 Introduction

The motivation in standard wavelet analysis was to construct orthonormal bases for families of functions defined on $\mathbb{R}$. In practice, we often only deal with the functions on a closed subset $[A, B]$ of the real line. In this section we focus on functions on $[A, B]$.

We begin with internal scaling functions supported inside the given interval, and introduce appropriate boundary functions at each end so that this new family of functions obeys the multiresolution hierarchy and gives the accuracy that we want to achieve.

It is shown in [13] that one can form an orthogonal MRA for a closed interval $[A, B]$ by first fixing an orthogonal MRA on the line given by the scaling function φ and the wavelet ψ and then constructing the spaces $V_n[A, B]$. The spaces $V_n[A, B]$ are spanned by scaling functions supported near the left end (left boundary functions), scaling functions supported near the right end (right boundary functions), and interior scaling functions. This construction yields an MRA of $L^2([A, B])$. The wavelet spaces $W_n[A, B]$ are constructed by the relation

$$V_{n+1}[A, B] = V_n[A, B] \oplus W_n[A, B].$$

See [9], [13], [4] for a complete treatment.

In our setup, we consider continuous orthogonal multiwavelets of multiplicity r with recursion coefficients $H_0, H_1, \ldots, H_N, G_0, G_1, \ldots, G_N$, so that the support is a subset of the interval $[0, N]$. We assume that the finite interval is $[0, M]$ with M large enough so that left and right ends don't interfere with each other. We expect that there are M multiscaling functions at level zero (in V_0) or $2^n M$ multiscaling functions at level n (in V_n).

The *interior multiscaling functions* are those whose support fits completely in $[0, M]$. We therefore have $(M-N+1)$ interior multiscaling functions, numbered $\boldsymbol{\varphi}_0$ to $\boldsymbol{\varphi}_{M-N+1}$, at level 0.

Some shifts of $\boldsymbol{\varphi}(x)$ cross the boundaries. These functions are called *boundary-crossing multiscaling functions*. These are $\boldsymbol{\varphi}_{-N+1}$ through $\boldsymbol{\varphi}_{-1}$ at the left end and $\boldsymbol{\varphi}_{M-N+1}$ through $\boldsymbol{\varphi}_{M-1}$ at the right end.

Let L be the number of individual left boundary functions denoted by

$$\varphi_j^L(x),\ j=-L,\dots,-1.$$

Likewise, let R be the number of individual right boundary functions denoted by

$$\varphi_j^R(x),\ j=M-N+1,\dots,M-N+R.$$

Thus, we have

$$(M-N+1)r+L+R=Mr$$

at level 0, which implies that

$$L+R=(N-1)r. \tag{3.1}$$

For orthogonal scalar wavelets, N is odd, so we can always take $L=R$. However, this is not necessarily true for multiwavelets. We will show that the choice of L and R for multiwavelets depends on the recursion coefficients H_k.

From now on, we will consider only the left boundary functions. The analogous analysis can be done at the right end as well.

Let L, not necessarily equal to the multiplicity r, be the number of left boundary functions supported inside $[0,\ N-1]$ at level zero. We stack them into a single vector $\boldsymbol{\varphi}^L$ so that we have the following:

$$\boldsymbol{\varphi}^L(x)=\boldsymbol{\varphi}_0^L(x)=\begin{pmatrix}\varphi_{0,-1}^L(x)\\ \vdots\\ \varphi_{0,-L}^L(x)\end{pmatrix}=\begin{pmatrix}\varphi_{-1}^L(x)\\ \vdots\\ \varphi_{-L}^L(x)\end{pmatrix}$$

We want the boundary functions to be orthogonal to each other and to the interior functions. The interior function vectors are denoted by

$$\boldsymbol{\varphi}_j(x)=\boldsymbol{\varphi}(x-j),\qquad j=0,1,\dots$$

At level n,

$$\begin{aligned} \boldsymbol{\varphi}_n^L(x) &= 2^{\frac{n}{2}}\boldsymbol{\varphi}^L(2^n x), \\ \boldsymbol{\varphi}_{nj}(x) &= 2^{\frac{n}{2}}\boldsymbol{\varphi}(2^n x - j). \end{aligned}$$

To maintain the multiresolution hierarchy, we require the boundary function vector $\boldsymbol{\varphi}^L(x)$ to satisfy the following recursion relation:

$$\boldsymbol{\varphi}_{n-1}^L(x) = A\boldsymbol{\varphi}_n^L(x) + \sum_{l=0}^{N-2} B_l \boldsymbol{\varphi}_{nl}(x), \tag{3.2}$$

where

$$A = \begin{pmatrix} a_{jk} \end{pmatrix} = \begin{pmatrix} a_{-1,-1} & \dots & a_{-1,-L} \\ \vdots & \ddots & \vdots \\ a_{-L,-1} & \dots & a_{-L,-L} \end{pmatrix}_{L\times L}, \qquad \langle \boldsymbol{\varphi}_{n-1,j}^L, \boldsymbol{\varphi}_{n,k}^L \rangle = a_{jk},$$

$$B = \begin{pmatrix} B_0 & B_1 & \dots & B_{N-2} \end{pmatrix}_{L\times(N-1)r}, \qquad \langle \boldsymbol{\varphi}_{n-1}^L, \boldsymbol{\varphi}_{n,k} \rangle = B_k,$$

$$B_j = \begin{pmatrix} \mathbf{b}_{j,1}^* \\ \vdots \\ \mathbf{b}_{j,L}^* \end{pmatrix}_{L\times r}, \qquad j = 0, 1, \dots, N-2.$$

For $n = 1$, the recursion relation yields

$$\boldsymbol{\varphi}^L(x) = \sqrt{2}A\boldsymbol{\varphi}^L(2x) + \sqrt{2}\sum_{l=0}^{N-2} B_l \boldsymbol{\varphi}(2x - l).$$

As we shall show in the following, this representation yields the decomposition and the reconstruction algorithm. Table 3.1 gives the recursion formulas in the interior and at the left end.

Formulas in Interior	Formulas at Left End
$\boldsymbol{\varphi}_{n,j}(x) = 2^{\frac{n}{2}}\boldsymbol{\varphi}(2^n x - j)$	$\varphi^L_{n,j}(x) = 2^{\frac{n}{2}}\varphi^L_j(2^n x) \quad$ (no shift)
$\boldsymbol{\varphi}_{n-1,j} = \sum_l H_{l-2j}\boldsymbol{\varphi}_{n,l}$	$\varphi^L_{n-1,j} = \sum_l a_{j,l}\varphi^L_{n,l} + \sum_l \mathbf{b}^*_{l,j}\boldsymbol{\varphi}_{n,l}$
$\langle \boldsymbol{\varphi}_{n-1,j}, \boldsymbol{\varphi}_{n,k}\rangle = H_{k-2j}$ $\langle \boldsymbol{\varphi}_{n,k}, \boldsymbol{\varphi}_{n-1,j}\rangle = H^*_{k-2j}$ $j,k \geq 0$	$\langle \varphi^L_{n-1,j}, \varphi^L_{n,k}\rangle = a_{jk}$ $\langle \varphi^L_{n,k}, \varphi^L_{n-1,j}\rangle = a^*_{jk}$ $j,k = -L,\ldots,-1.$
	$\langle \boldsymbol{\varphi}^L_{n-1}, \boldsymbol{\varphi}_{n,k}\rangle = B_k$ $k \geq 0.$

Table 3.1 Recursion formulas

Sample Calculations:

$$\begin{aligned}
\langle \varphi^L_{n-1,j}, \varphi^L_{n,k}\rangle &= \langle \sum_l a_{j,l}\varphi^L_{n,l} + \sum_l \mathbf{b}^*_{l,j}\boldsymbol{\varphi}_{n,l}, \varphi^L_{n,k}\rangle \\
&= \sum_l a_{j,l}\delta_{l,k} + 0 \\
&= a_{j,k}.
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{\varphi}_{n,k}, \varphi^L_{n-1,j}\rangle &= \langle \boldsymbol{\varphi}_{n,k}, \sum_l a_{j,l}\varphi^L_{n,l} + \sum_l \mathbf{b}^*_{l,j}\boldsymbol{\varphi}_{n,l}\rangle \\
&= 0 + \sum_l \mathbf{b}^*_{l,j}\delta_{l,k} \\
&= \mathbf{b}^*_{k,j}.
\end{aligned}$$

Decomposition: Let $s \in V_n[0, M]$.

$$\begin{aligned} s &= P_n s = \underbrace{\langle s, \boldsymbol{\varphi}_n^L \rangle}_{(\mathbf{s}_n^L)^*} \boldsymbol{\varphi}_n^L + \sum_k \underbrace{\langle s, \boldsymbol{\varphi}_{n,k} \rangle}_{(\mathbf{s}_{n,k})^*} \boldsymbol{\varphi}_{n,k} + \underbrace{\langle s, \boldsymbol{\varphi}_n^R \rangle}_{(\mathbf{s}_n^R)^*} \boldsymbol{\varphi}_n^R \\ &= P_{n-1} s + Q_{n-1} s \\ &= (\mathbf{s}_{n-1}^L)^* \boldsymbol{\varphi}_{n-1}^L + \sum_j (\mathbf{s}_{n-1,j})^* \boldsymbol{\varphi}_{n-1,j} + (\mathbf{s}_{n-1}^R)^* \boldsymbol{\varphi}_{n-1}^R \\ &\quad + (\mathbf{d}_{n-1}^L)^* \boldsymbol{\psi}_{n-1}^L + \sum_j (\mathbf{d}_{n-1,j})^* \boldsymbol{\psi}_{n-1,j} + (\mathbf{d}_{n-1}^R)^* \boldsymbol{\psi}_{n-1}^R. \end{aligned}$$

Thus,

$$\begin{aligned} (\mathbf{s}_{n-1}^L)^* &= \langle P_n s, \boldsymbol{\varphi}_{n-1}^L \rangle \\ &= (\mathbf{s}_n^L)^* \langle \boldsymbol{\varphi}_n^L, \boldsymbol{\varphi}_{n-1}^L \rangle + \sum_k (\mathbf{s}_{n,k})^* \langle \boldsymbol{\varphi}_{n,k}, \boldsymbol{\varphi}_{n-1}^L \rangle + (\mathbf{s}_{n-1}^R)^* \langle \boldsymbol{\varphi}_n^R, \boldsymbol{\varphi}_{n-1}^L \rangle \\ &= (\mathbf{s}_n^L)^* A^* + \sum_k (\mathbf{s}_{n,k})^* B_k^* + 0. \end{aligned}$$

If we take the transpose of both sides, we wind up with

$$\mathbf{s}_{n-1}^L = A \mathbf{s}_n^L + \sum_k B_k \mathbf{s}_{n,k} \tag{3.3}$$

Likewise, if

$$\boldsymbol{\psi}_{n-1}^L = C \boldsymbol{\varphi}_n^L + \sum_l D_l \boldsymbol{\varphi}_{n,l},$$

then

$$\mathbf{d}_{n-1}^L = C \mathbf{s}_n^L + \sum_k D_k \mathbf{s}_{n,k} \tag{3.4}$$

As we expressed earlier, notation becomes simpler if we interleave $\mathbf{s}-$ and $\mathbf{d}-$ coefficients at level $n-1$:

$$\mathbf{s}_n = \begin{pmatrix} \mathbf{s}_n^L \\ \mathbf{s}_{n,k} \\ \mathbf{s}_n^R \end{pmatrix}, \text{ where } \quad \mathbf{s}_n^L = \begin{pmatrix} s_{n,-L}^L \\ \vdots \\ s_{n,-1}^L \end{pmatrix}.$$

If

$$(\mathbf{sd})_{n-1} = \begin{pmatrix} \mathbf{s}_{n-1}^{L} \\ \mathbf{d}_{n-1}^{L} \\ \mathbf{s}_{n-1,0} \\ \mathbf{d}_{n-1,0} \\ \vdots \\ \mathbf{s}_{n-1}^{R} \\ \mathbf{d}_{n-1}^{R} \end{pmatrix},$$

then

$$(\mathbf{sd})_{n-1} = T_n \mathbf{s}_n, \tag{3.5}$$

$$\mathbf{s}_n = T_n^* (\mathbf{sd})_{n-1}, \tag{3.6}$$

where

$$T_n = \left(\begin{array}{c|cccccccc|c} L_0 & L_1 & \dots & L_K & 0 & \dots & \dots & 0 & 0 \\ \hline 0 & T_0 & T_1 & \dots & T_K & 0 & \dots & 0 & 0 \\ \vdots & 0 & T_0 & T_1 & \dots & T_K & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & T_0 & T_1 & \dots & T_K & 0 \\ \hline 0 & 0 & \dots & \dots & 0 & R_0 & R_1 & \dots & R_K \end{array} \right).$$

Here

$$T_k = \begin{pmatrix} H_{2k} & H_{2k+1} \\ G_{2k} & G_{2k+1} \end{pmatrix}_{2r \times 2r}, \quad L_k = \begin{pmatrix} B_{2k-2} & B_{2k-1} \\ D_{2k-2} & D_{2k-1} \end{pmatrix}_{2L \times 2r}, \quad L_0 = \begin{pmatrix} A \\ C \end{pmatrix}_{2L \times L},$$

where $k = 1, \dots, K$.

In the discrete setting, the inner products in Table 3.1 go directly into the matrix representation of the transformation.

Example 3.1. Consider the standard orthogonal Chui-Lian $CL(2)$ multiwavelet [8].

It has multiplicity 2 and is supported on $[0, 2]$. Recursion coefficients are H_0, H_1, H_2. For simplicity, we consider the interval [0,4]. Then, we expect to have 4 multifunctions (scaling functions) or 8 single functions at level 0. As shown in figure 3.1, there are 3 interior multifunctions or 6 single functions. Thus, we expect 1 single function at each end. At level -1, wavelets come in to play. As shown in figure 3.2, there is 1 interior multifunction, and 1 single function at each end. It adds up to 4 single functions for the scaling functions. Likewise, there are 4 single functions for the wavelet. Then, we have a total of 8 functions as expected. The following is the decomposition of a signal with this transform.

$$\begin{pmatrix} s^L_{-1,-1} \\ d^L_{-1,-1} \\ \mathbf{s}_{-1,0} \\ \mathbf{d}_{-1,0} \\ s^R_{-1,1} \\ d^R_{-1,1} \end{pmatrix} = \left(\begin{array}{c|ccc|c} A & B & 0 & 0 & 0 \\ C & D & 0 & 0 & 0 \\ \hline 0 & H_0 & H_1 & H_2 & 0 \\ 0 & G_0 & G_1 & G_2 & 0 \\ \hline 0 & 0 & 0 & E & F \\ 0 & 0 & 0 & G & H \end{array}\right) \begin{pmatrix} s^L_{0,-1} \\ \mathbf{s}_{0,0} \\ \mathbf{s}_{0,1} \\ \mathbf{s}_{0,2} \\ s^R_{0,3} \end{pmatrix},$$

where

$$L_0 = \begin{pmatrix} A \\ C \end{pmatrix}_{2\times 1}, \quad L_1 = \begin{pmatrix} B \\ D \end{pmatrix}_{2\times 2}, \quad R_0 = \begin{pmatrix} E \\ G \end{pmatrix}_{2\times 2}, \quad R_1 = \begin{pmatrix} F \\ H \end{pmatrix}_{2\times 1}.$$

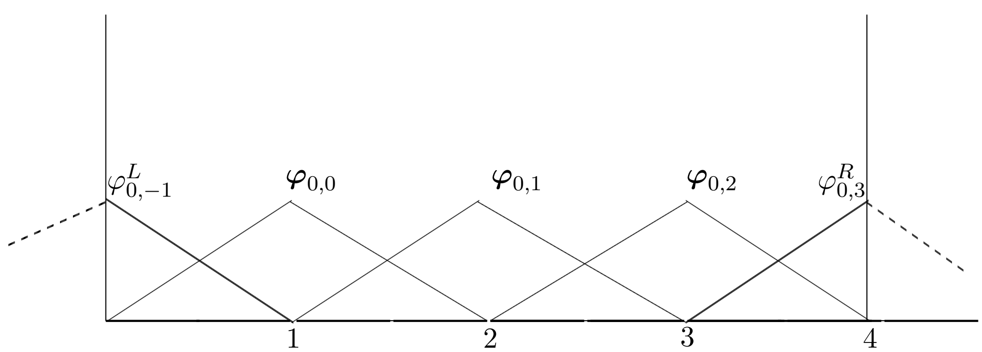

Figure 3.1 CL(2) at level 0

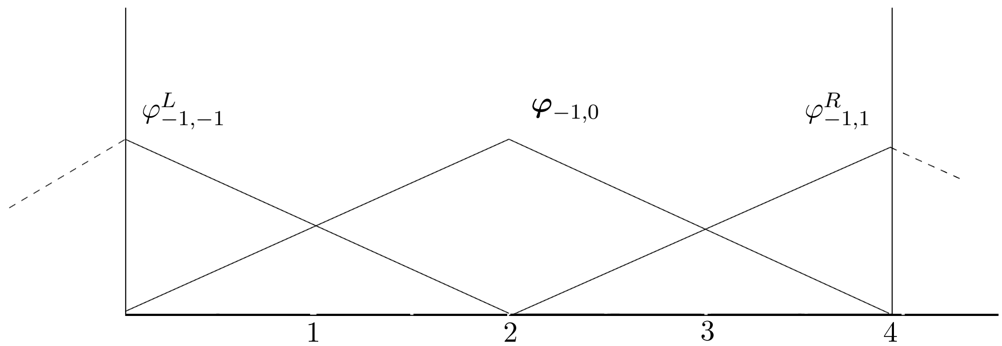

Figure 3.2 CL(2) at level -1

3.2 Endpoint Functions

As we pointed out earlier, it suffices to consider the left boundary functions. Recall that the recursion formula at the left end yields

$$\boldsymbol{\varphi}^L_{n-1}(x) \quad = \quad A\boldsymbol{\varphi}^L_{n,l}(x) + \sum_l B_l \boldsymbol{\varphi}_{n,l}(x). \tag{3.7}$$

A is of size $L \times L$, and each B_l is of size $L \times r$.

Definition 3.1. *We will call* $\boldsymbol{\varphi}^L$ refinable *if it satisfies such a recursion relation. We will call* $\boldsymbol{\varphi}^L$ *a* regular boundary function *if it is refinable, continuous, has approximation order* 1, *and* $\boldsymbol{\varphi}^L(0) \neq \mathbf{0}$.

It is also quite natural to expect that boundary functions are linear combinations of boundary crossing functions. We then have

$$\boldsymbol{\varphi}^L(x) = \sum_k C_k \boldsymbol{\varphi}_k(x). \tag{3.8}$$

Each C_k is of size $L \times r$.

A reasonable direction for analysis is to explore how the equations (3.7) and (3.8) are connected. To be more precise, we would like to know how to find $C = (\ldots, C_k, \ldots)$ if we are given A and $B = (\ldots, B_l, \ldots)$ and vice versa. We also want to know if all C (A and B) lead to useful A and B (C). We note that we are only interested in regular boundary functions, because the A and B coefficients are needed for the wavelet transform algorithms. Since we want to be able to represent functions that are not zero at the endpoints, the condition $\boldsymbol{\varphi}^L(0) \neq \mathbf{0}$ is added.

We are on the point of making our analysis at the left end. We devote this section to investigate the connection between (3.7) and (3.8).

The recursion relation at the left end for $n = 1$ is

$$\boldsymbol{\varphi}^L(x) = \sqrt{2}A\boldsymbol{\varphi}^L(2x) + \sqrt{2}\sum_{l=0}^{N-2} B_l\boldsymbol{\varphi}(2x - l). \tag{3.9}$$

The interior recursion for $n = 1$ is

$$\boldsymbol{\varphi}(x) = \sqrt{2}\sum_{k=0}^{N} H_k\boldsymbol{\varphi}(2x - k). \tag{3.10}$$

Assume that the left endpoint functions are linear combinations of boundary crossing functions. That is,

$$\boldsymbol{\varphi}^L(x) = \sum_{k=-N+1}^{-1} C_k\boldsymbol{\varphi}(x - k), \quad \text{for } x \geq 0. \tag{3.11}$$

Then, the left hand-side of (3.9) can be written as

$$\begin{aligned}
\boldsymbol{\varphi}^L(x) &= \sum_{k=-N+1}^{-1} C_k\boldsymbol{\varphi}(x - k) \\
&= \sum_{k=-N+1}^{-1} C_k[\sqrt{2}\sum_{l=0}^{N} H_l\boldsymbol{\varphi}(2x - 2k - l)] \\
&= \sqrt{2}\sum_{l}\sum_{k} C_kH_l\boldsymbol{\varphi}(2x - 2k - l) \\
&= \sqrt{2}\sum_{s}(\sum_{k} C_kH_{s-2k})\boldsymbol{\varphi}(2x - s) \\
&= \sqrt{2}\sum_{k}(\sum_{l} C_lH_{k-2l})\boldsymbol{\varphi}(2x - k)
\end{aligned} \tag{3.12}$$

and the right-hand side of (3.9) yields

$$\sqrt{2}A\boldsymbol{\varphi}^L(2x) + \sqrt{2}\sum_{l=0}^{N-2} B_l\boldsymbol{\varphi}_{n,l}(x) = \sqrt{2}A\sum_{k=-N+1}^{-1} C_k\boldsymbol{\varphi}(2x - k) + \sqrt{2}\sum_{k=0}^{N-2} B_k\boldsymbol{\varphi}(2x - k) \tag{3.13}$$

Comparing (3.12) and (3.13), we obtain the following identities.

$$\sum_{l} C_lH_{k-2l} \;=\; AC_k, \quad k = -K, \ldots, -1, \quad l = -N+1, \ldots, -1, \tag{3.14}$$

$$\sum_{l} C_lH_{k-2l} \;=\; B_k, \quad , k = 0, \ldots, N-2, \quad l = -N+1, \ldots, -1. \tag{3.15}$$

In matrix notation, (3.14) and (3.15) are equivalent to

$$CY = AC, \tag{3.16}$$

$$CZ = B, \tag{3.17}$$

where

$$(Y||Z) = (H_{l-2j}) = \left(\begin{array}{cccccc||ccccc} H_{N-1} & H_N & 0 & \dots & & & 0 & \dots & & & \\ H_{N-3} & H_{N-2} & H_{N-1} & H_N & 0 & \dots & \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & H_4 & \dots & H_N & \dots & \\ \dots & \dots & \dots & \dots & H_0 & H_1 & H_2 & H_3 & \dots & H_{N-1} & H_N \end{array}\right).$$

The order of both Y and Z is $(N-1)r \times (N-1)r$.

The first question is whether it is always possible to find C given A and B, or vice versa. Recall that the Kronecker product of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}_{mp\times nq},$$

and the vec operator transforms a matrix into a vector by stacking the columns of the matrix one below the other. For example, let D be a matrix of order $m \times n$ with the j^{th} column defined as $\mathbf{d}_j$. Then

$$\text{vec}(D) = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_n \end{pmatrix}_{mn\times 1}.$$

There is an important identity that relates the Kronecker product and the vec operator:

$$\text{vec}(PQR) = (R^T \otimes P)\text{vec}(Q), \tag{3.18}$$

where $P, Q,$ and R are matrices of order $m \times n$, $n \times r$, and $r \times s$, respectively. This is proved in [16].

We now focus on the identity

$$CY = AC.$$

If $\varsigma=$ rowspan(C), then rowspan$(AC) \subset \varsigma$. This implies that ς is a left invariant subspace of Y, so the rows of C must be a linear combinations of left eigenvectors of Y.

If there is a single boundary function, this is actually a straight eigenvalue problem:

$$\mathbf{c}^* Y = a\mathbf{c}^*, \qquad \mathbf{c}^* = C \text{ and } a = A.$$

An alternative approach uses the identity (3.18).

$$\begin{aligned} CY = AC \iff& I_L C Y = A C I_{(N-1)r} \\ \implies& (Y^T \otimes I_L)\mathrm{vec}(C) = (I_{(N-1)r} \otimes A)\mathrm{vec}(C) \\ \implies& [(Y^T \otimes I_L) - (I_{(N-1)r} \otimes A)]\mathbf{c} = \mathbf{0}, \end{aligned}$$

where $\mathbf{c} = \mathrm{vec}(C)$ and I_L is the $L \times L$ identity matrix.

If $(Y^T \otimes I_L) - (I_{(N-1)r} \otimes A)$ is a nonsingular matrix, then $\mathbf{c} = \mathbf{0}$. Therefore, we require $(Y^T \otimes I_L) - (I_{(N-1)r} \otimes A)$ to be a singular matrix and need to determine its nullspace, which is again an eigenvalue problem. Both approaches show that the answer to the question we raised at the beginning is "no". An arbitrary C does not in general lead to A and B, nor do A and B always lead to C.

Example 3.2. Consider the standard orthogonal Chui-Lian $CL(2)$ multiwavelet. In this case,

$$N = 2,\ K = 1,\ Y = H_1,\ Z = H_2,\ A = (a),\ C = \mathbf{c}^*,\ B = \mathbf{b}^*.$$

The identities $CY = AC$ and $CZ = B$ provide two sets of solutions:

$$a = \sqrt{2}/4\,, \quad \mathbf{c}^* = \begin{bmatrix} 0 & 1 \end{bmatrix}\,, \quad \mathbf{b}^* = \begin{bmatrix} \frac{\sqrt{14}}{8} & \frac{-\sqrt{14}}{8} \end{bmatrix}, \tag{3.19}$$

and

$$a = \sqrt{2}/2\,, \quad \mathbf{c}^* = \begin{bmatrix} 1 & 0 \end{bmatrix}\,, \quad \text{and} \quad \mathbf{b}^* = \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} \end{bmatrix}. \tag{3.20}$$

What we have shown here is that given C either as $\begin{bmatrix} 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \end{bmatrix}$ or (a multiple of these), there exist corresponding A and B so that boundary function satisfies the recursion relation

(3.9). Similarly, given A and B, there can be found a corresponding C so that boundary function can be obtained from a linear combination of boundary crossing functions. We shall show that if there is only a single boundary function, the choice $\mathbf{c}^* = (\boldsymbol{\mu}_0^*, \boldsymbol{\mu}_0^*, \ldots)$ always works. However, we note that other choices of C do not lead to refinable functions.

3.3 Continuity for Scalar Boundary Functions

In this section, we will discuss and derive conditions for constructing continuous scalar boundary functions. As a first example, we will investigate the case of scalar wavelet with four recursion coefficients. For this case, we obtain necessary and sufficient conditions. We also show that a similar analysis can be done for the multiwavelet case (see next section).

Let us assume that we have a continuous scalar wavelet with four recursion coefficients and compactly supported on $[0, 3]$. There is one left boundary scaling function. In addition, we also assume that the boundary scaling function has support on $[0, 2]$. Then the recursion relation for the boundary scaling function yields

$$\varphi^L(x) \quad = \quad \sqrt{2}[a_{-1}\varphi^L(2x) + b_0\varphi(2x) + b_1\varphi(2x-1)] \tag{3.21}$$

In what follows, we show that under some conditions on a_{-1}, b_0, and b_1, φ^L is continuous. The cascade algorithm applied to the equation (3.21) will determine the boundary function. Assume $x \in [1, 2]$. Then $\varphi^L(2x) = 0$, so (3.21) becomes

$$\varphi^L(x) = \sqrt{2}[b_0\varphi(2x) + b_1\varphi(2x-1)] \quad \text{on } [1, 2].$$

The cascade algorithm converges in a single step and determines $\varphi^L(x)$ on $[1, 2]$. The continuity of φ carries over to φ^L.

In the next step,

$$\varphi^L(x) = 2a_{-1}[b_0\varphi(4x) + b_1\varphi(4x-1)] + \sqrt{2}[b_0\varphi(2x) + b_1\varphi(2x-1)] \quad \text{on } [\frac{1}{2}, 2].$$

If we keep on applying the cascade algorithm, we determine $\varphi^L(x)$ on $(0, 2]$. At each step, $\varphi^L(x)$ is a linear combination of continuous functions, and therefore continuous.

Furthermore, $\varphi^L(x)$ is uniformly continuous on $[\epsilon, 2]$ for every $\epsilon > 0$ independent of choice of

a_{-1}, b_0, b_1 simply because $\varphi^L(x)$ is continuous on the closed interval $[\epsilon, 2]$ for every $\epsilon > 0$. We therefore only need to explore the behavior of $\varphi^L(x)$ as $x \to 0$.

For $x = 0$, equation (3.21) says

$$\varphi^L(0) = \sqrt{2}a_{-1}\varphi^L(0),$$

which can only possibly be nonzero and continuous at 0 if $a_{-1} = \frac{1}{\sqrt{2}}$.

Otherwise, we will not get any desirable results, since either $\varphi^L(0) = 0$, which is not useful in applications, or it is discontinuous at 0, possibly unbounded. However, we will investigate $\lim_{x\to 0} \varphi^L(x)$ independent of the value of a_{-1}.

Take any $0 < x_0 \leq 1$, so that $x_0 - 1 \leq 0$.

$$\begin{aligned}
\varphi(\frac{x_0}{2}) =& \sqrt{2}[h_0\varphi(x_0) + h_1 \underbrace{\varphi(x_0-1)}_{=0} + h_2 \underbrace{\varphi(x_0-2)}_{=0} + h_3 \underbrace{\varphi(x_0-3)}_{=0}] \\
=& \sqrt{2}h_0\varphi(x_0), \\
\varphi(\frac{x_0}{4}) =& (\sqrt{2}h_0)^2\varphi(x_0), \\
\varphi(\frac{x_0}{8}) =& (\sqrt{2}h_0)^3\varphi(x_0), \\
& \vdots \\
\varphi(2^{-n}x_0) =& (\sqrt{2}h_0)^n\varphi(x_0).
\end{aligned} \tag{3.22}$$

From (3.21),

$$\begin{aligned}
\varphi^L(\frac{x_0}{2}) =& \sqrt{2}[a_{-1}\varphi^L(x_0) + b_0\varphi(x_0) + b_1 \underbrace{\varphi(x_0-1)}_{=0}] \\
=& (\sqrt{2}a_{-1})\varphi^L(x_0) + (\sqrt{2}b_0)\varphi(x_0), \\
\varphi^L(\frac{x_0}{4}) =& (\sqrt{2}a_{-1})\varphi^L(\frac{x_0}{2}) + \sqrt{2}b_0\varphi(\frac{x_0}{2}) \\
=& (\sqrt{2}a_{-1})^2\varphi^L(x_0) + [(\sqrt{2}a_{-1})(\sqrt{2}b_0) + (\sqrt{2}b_0)(\sqrt{2}h_0)]\varphi(x_0), \\
& \vdots \\
\varphi^L(2^{-n}x_0) =& (\sqrt{2}a_{-1})^n\varphi^L(x_0) \\
& + (\sqrt{2}b_0)(\sqrt{2}a_{-1})^{n-1}\varphi(x_0) \\
& + (\sqrt{2}b_0)[(\sqrt{2}a_{-1})^{n-2}(\sqrt{2}h_0) + \ldots + (\sqrt{2}a_{-1})(\sqrt{2}h_0)^{n-2} + (\sqrt{2}h_0)^{n-1}]\varphi(x_0).
\end{aligned} \tag{3.23}$$

We now consider several cases for (3.23).

It follows from the continuity of φ, and the convergence of the cascade algorithm, that $\sqrt{2}h_0 < 1$.

Case 1: $\sqrt{2}a_{-1} > 1$

In this case, it is conceivable that the coefficients of both $\varphi^L(x_0)$ and $\varphi(x_0)$ will cancel each other and lead a useful φ^L for some particular x_0. However, this is unlikely to happen for all x_0. Thus, φ^L will most likely become unbounded in this case. We will not consider this further.

Case 2: $\sqrt{2}a_{-1} = 1$

(3.23) reads

$$\begin{aligned}\varphi^L(2^{-n}x_0) =&\varphi^L(x_0) + (\sqrt{2}b_0)[1 + (\sqrt{2}h_0) + \ldots + (\sqrt{2}h_0)^{n-2} + (\sqrt{2}h_0)^{n-1}]\varphi(x_0)\\ \lim_{n\to\infty} \varphi^L(2^{-n}x_0) =&\varphi_{lb}(x_0) + (\sqrt{2}b_0)\frac{1-(\sqrt{2}h_0)^n}{1-\sqrt{2}h_0}\varphi(x_0)\end{aligned}$$

for every $x_0 \in (0,1]$.

If it is continuous at 0, then

$$\varphi^L(0) = \varphi^L(x) + \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\varphi(x), \quad \forall x \in (0,1].$$

In particular,

$$\varphi^L(0) = \varphi^L(1) + \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\varphi(1).$$

Thus,

$$\varphi^L(x) = \begin{cases} \varphi^L(0) - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\varphi(x) & \text{if } x \in [0,1]; \\ \sqrt{2}[b_0\varphi(2x) + b_1\varphi(2x-1)] & \text{if } x \in [1,2]. \end{cases} \tag{3.24}$$

Case 3: $\sqrt{2}a_{-1} < 1$

This leads to $\varphi^L(0) = 0$, continuous at zero, which is not useful in practice.

We have obtained a necessary condition:

If φ^L is continuous and $\varphi^L(0) \neq 0$, then $a_{-1} = \frac{1}{\sqrt{2}}$.

We now show that $\dfrac{b_0}{b_1} = \dfrac{h_2}{h_3}$ is a sufficient condition. To achieve this, we claim that the function defined by (3.24) satisfies (3.21) if $\dfrac{b_0}{b_1} = \dfrac{h_2}{h_3}$.

If x is in $[1,2]$, then the two identities for $\varphi_{lb}(x)$ are equal.

$$\begin{aligned}
\varphi^L(x) &= \sqrt{2}[b_0\varphi(2x) + b_1\varphi(2x-1)] && \text{(from 3.24)}\\
\varphi^L(x) &= \underbrace{\varphi^L(2x)}_{=0} + \sqrt{2}[b_0\varphi(2x) + b_1\varphi(2x-1)] && \text{(from 3.21)}
\end{aligned}$$

Let x be in $[0, \frac{1}{2}]$. Equation (3.24) leads to

$$\begin{aligned}
\varphi^L(x) &= \varphi^L(0) - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\varphi(x)\\
&= \varphi^L(0) - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_0\varphi(2x)\big]\\
&\quad - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_1\underbrace{\varphi(2x-1)}_{=0} + \sqrt{2}h_2\underbrace{\varphi(2x-2)}_{=0} + \sqrt{2}h_3\underbrace{\varphi(2x-3)}_{=0}\big]\\
&= \varphi^L(0) - 2\frac{b_0h_0}{1-\sqrt{2}h_0}\varphi(2x).
\end{aligned}$$

On the other hand, equation (3.21) leads to

$$\begin{aligned}
\varphi^L(x) &= \varphi^L(2x) + \sqrt{2}b_0\varphi(2x) + \sqrt{2}b_1\underbrace{\varphi(2x-1)}_{=0}\\
&= \varphi^L(0) - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\varphi(2x) + \sqrt{2}b_0\varphi(2x)\\
&= \varphi^L(0) + \big(\sqrt{2}b_0 - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big)\varphi(2x)\\
&= \varphi^L(0) - 2\frac{b_0h_0}{1-\sqrt{2}h_0}\varphi(2x)
\end{aligned}$$

The two identities that we obtained for $\varphi^L(x)$ are the same, so there is no need to impose any constraints.

The important interval is $[\frac{1}{2},1]$. Let x be in $[\frac{1}{2},1]$. From equation (3.24), we get

$$\begin{aligned}
\varphi^L(x) &= \varphi^L(0) - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\varphi(x)\\
&= \varphi^L(0) - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_0\varphi(2x) + \sqrt{2}h_1\varphi(2x-1)\big]\\
&\quad -\frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_2\underbrace{\varphi(2x-2)}_{=0} + \sqrt{2}h_3\underbrace{\varphi(2x-3)}_{=0}\big]\\
&= \varphi^L(0) - \frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_0\big(\sqrt{2}h_0\varphi(4x) + \sqrt{2}h_1\varphi(4x-1)\big)\big]\\
&\quad -\frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_0\big(\sqrt{2}h_2\varphi(4x-2) + \sqrt{2}h_3\varphi(4x-3)\big)\big]\\
&\quad -\frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_1\big(\sqrt{2}h_0\varphi(4x-2) + \sqrt{2}h_1\varphi(4x-3)\big)\big]\\
&\quad -\frac{\sqrt{2}b_0}{1-\sqrt{2}h_0}\big[\sqrt{2}h_1\big(\sqrt{2}h_2\underbrace{\varphi(4x-4)}_{=0} + \sqrt{2}h_3\underbrace{\varphi(4x-5)}_{=0}\big)\big].
\end{aligned}$$

After combining like terms, we have

$$\begin{aligned}
\varphi^L(x) =&\varphi^L(0) - 2\sqrt{2}\frac{b_0(h_0)^2}{1-\sqrt{2}h_0}\varphi(4x) - 2\sqrt{2}\frac{b_0h_0h_1}{1-\sqrt{2}h_0}\varphi(4x-1)\\
&-2\sqrt{2}b_0h_0\frac{h_1+h_2}{1-\sqrt{2}h_0}\varphi(4x-2) - 2\sqrt{2}b_0\frac{(h_1)^2+h_0h_3}{1-\sqrt{2}h_0}\varphi(4x-3).
\end{aligned} \tag{3.25}$$

On the other hand, equation (3.21) leads to

$$\begin{aligned}
\varphi^L(x) =&\varphi^L(2x) + \sqrt{2}b_0\varphi(2x) + \sqrt{2}b_1\varphi(2x-1)\\
=&\sqrt{2}b_0\varphi(4x) + \sqrt{2}b_1\varphi(4x-1) + \sqrt{2}b_0\varphi(2x) + \sqrt{2}b_1\varphi(2x-1).
\end{aligned} \tag{3.26}$$

Use the recursion relation for $\varphi(2x)$ and $\varphi(2x-1)$. That is

$$\varphi(2x) = \sqrt{2}h_0\varphi(4x) + \sqrt{2}h_1\varphi(4x-1) + \sqrt{2}h_2\varphi(4x-2) + \sqrt{2}h_3\varphi(4x-3) \tag{3.27}$$

$$\varphi(2x-1) = \sqrt{2}h_0\varphi(4x-2) + \sqrt{2}h_1\varphi(4x-3) + \sqrt{2}h_2\underbrace{\varphi(4x-4)}_{=0} + \sqrt{2}h_3\underbrace{\varphi(4x-5)}_{=0} \tag{3.28}$$

Substitute (3.27) and (3.28) into (3.26) and combine like terms to get

$$\begin{aligned}\varphi^L(x) =&\sqrt{2}b_0\varphi(4x)+\sqrt{2}b_1\varphi(4x-1)+\sqrt{2}b_0\big[\sqrt{2}h_0\varphi(4x)+\sqrt{2}h_1\varphi(4x-1)+\sqrt{2}h_2\varphi(4x-2)\\ &+\sqrt{2}h_3\varphi(4x-3)\big]+\sqrt{2}b_1\big[\sqrt{2}h_0\varphi(4x-2)+\sqrt{2}h_1\varphi(4x-3)\big]\\ =&\sqrt{2}b_0(1+\sqrt{2}h_0)\varphi(4x)+(\sqrt{2}b_1+\sqrt{2}h_1\sqrt{2}b_0)\varphi(4x-1)\\ &+2(b_0h_2+b_1h_0)\varphi(4x-2)+2(b_0h_3+b_1h_1)\varphi(4x-3).\end{aligned} \tag{3.29}$$

Equate (3.25) and (3.29). This lead us to the following identity.

$$\begin{aligned}\varphi^L(0) \;=\;& \underbrace{\big[\sqrt{2}b_0(1+\sqrt{2}h_0)+2\sqrt{2}\frac{b_0(h_0)^2}{1-\sqrt{2}h_0}\big]}_{a}\varphi(4x)\\ &+\underbrace{\big[2\sqrt{2}\frac{b_0h_0h_1}{1-\sqrt{2}h_0}+\sqrt{2}b_1+(\sqrt{2}h_1)(\sqrt{2}b_0)\big]}_{b}\varphi(4x-1)\\ &+\underbrace{\big[2b_0h_2+2b_1h_0+2\sqrt{2}b_0h_0\frac{h_1+h_2}{1-\sqrt{2}h_0}\big]}_{c}\varphi(4x-2)\\ &+\underbrace{\big[2b_0h_3+2b_1h_1+2\sqrt{2}b_0\frac{(h_1)^2+h_0h_3}{1-\sqrt{2}h_0}\big]}_{d}\varphi(4x-3)\end{aligned}$$

for $x \in [\frac{1}{2}, 1]$.

Notice that the right hand side of above identity is a constant. For scalar scaling functions,

$$\sum_j \varphi(x-j) = \text{constant},$$

hence all coefficients of φ that appears in the identity must be the same.

If $a = b$, then

$$\begin{aligned}\sqrt{2}b_0(1+\sqrt{2}h_0)+2\sqrt{2}\frac{b_0(h_0)^2}{1-\sqrt{2}h_0} =&2\sqrt{2}\frac{b_0h_0h_1}{1-\sqrt{2}h_0}+\sqrt{2}b_1+(\sqrt{2}h_1)(\sqrt{2}b_0)\\ \sqrt{2}b_0(1+\sqrt{2}h_0)(1-\sqrt{2}h_0)+2\sqrt{2}b_0(h_0)^2 =&2\sqrt{2}b_0h_0h_1\\ &+\sqrt{2}b_1(1-\sqrt{2}h_0)+(\sqrt{2}h_1)(\sqrt{2}b_0)(1-\sqrt{2}h_0)\\ b_0 =&b_1(1-\sqrt{2}h_0)+(\sqrt{2}h_1)(b_0)\\ b_0(1-\sqrt{2}h_1) =&b_1(1-\sqrt{2}h_0).\end{aligned} \tag{3.30}$$

We use the fact that

$$\sum_k h_{2k} = \sum_k h_{2k+1} = \frac{1}{\sqrt{2}}.$$

In particular for the case that we are considering this means

$$\begin{aligned} h_0 + h_2 &= \frac{1}{\sqrt{2}}, \\ h_1 + h_3 &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Hence, the equation (3.30) becomes

$$b_0 h_3 = b_1 h_2 \tag{3.31}$$

If $a = c$, then

$$\begin{aligned} \sqrt{2}b_0(1+\sqrt{2}h_0) + 2\sqrt{2}\frac{b_0(h_0)^2}{1-\sqrt{2}h_0} &= 2b_0h_2 + 2b_1h_0 + 2\sqrt{2}b_0h_0\frac{h_1+h_2}{1-\sqrt{2}h_0} \\ \sqrt{2}b_0 &= 2b_1h_0(1-\sqrt{2}h_0) + 2b_0 + h_2 + 2\sqrt{2}b_0h_0h_1 \\ b_0(\sqrt{2}h_0 - 2h_0h_1) &= 2b_1h_0h_2 \\ b_0h_0(1-\sqrt{2}h_1) &= \sqrt{2}b_1h_0h_2 \\ b_0h_0h_3 &= b_1h_0h_2 \end{aligned}$$

If $a = d$, then

$$\begin{aligned} \sqrt{2}b_0(1+\sqrt{2}h_0) + 2\sqrt{2}\frac{b_0(h_0)^2}{1-\sqrt{2}h_0} &= 2b_0h_3 + 2b_1h_1 + 2\sqrt{2}b_0\frac{(h_1)^2+h_0h_3}{1-\sqrt{2}h_0} \\ \sqrt{2}b_0(1-2(h_1)^2) &= 2b_0h_3 + 2b_1h_1(1-\sqrt{2}h_0) \\ b_0(\sqrt{2}h_0 - 2h_0h_1) &= 2b_1h_0h_2 \\ 2b_0h_3(1+\sqrt{2}h_1) &= 2b_0h_3 + 2\sqrt{2}b_1h_1h_2 \\ b_0h_1h_3 &= b_1h_1h_2 \end{aligned}$$

All equalities lead to the same sufficient condition.

Assuming this condition is satisfied, we obtain

$$\varphi^L(x) = \begin{cases} \varphi^L(0) - \frac{b_0}{h_2}\varphi(x) & \text{if } x \in [0,1]; \\ \sqrt{2}[b_0\varphi(2x) + b_1\varphi(2x-1)] & \text{if } x \in [1,2], \end{cases} \tag{3.32}$$

and

$$\begin{aligned}
\varphi^L(0) &= \varphi^L(1) + \frac{b_0}{h_2}\varphi(1) \\
&= [\sqrt{2}b_1 + \frac{b_0}{h_2}]\varphi(1) + \sqrt{2}b_0\varphi(2) \\
&= \sqrt{2}b_1[1 + \frac{1}{\sqrt{2}h_3}]\varphi(1) + \sqrt{2}b_0\varphi(2).
\end{aligned}$$

3.4 Continuity for Multiboundary Functions

We now do the same analysis for multiwavelets as we did for scalar wavelets. Assume that we have a continuous multiwavelet with 4 recursion coefficients and compactly supported on $[0,3]$. In addition, we also assume that the boundary scaling function has support on $[0,2]$. We want the boundary function with compact support $[0,2]$ to satisfy the dilation equation. Hence, the recursion relation has the following form:

$$\boldsymbol{\varphi}^L(x) = \sqrt{2}A\boldsymbol{\varphi}^L(2x) + \sqrt{2}B_0\boldsymbol{\varphi}(2x) + \sqrt{2}B_1\boldsymbol{\varphi}(2x-1). \tag{3.33}$$

The cascade algorithm shows that $\boldsymbol{\varphi}^L$ is continuous on $(0,2]$, as in the scalar case.

$$\begin{aligned}
\boldsymbol{\varphi}^L(x) &= \sqrt{2}[B_0\boldsymbol{\varphi}(2x) + B_1\boldsymbol{\varphi}(2x-1)] \quad \text{on} \quad [1,2], \\
\boldsymbol{\varphi}^L(x) &= 2AB_0\boldsymbol{\varphi}(4x) + 2AB_1\boldsymbol{\varphi}(4x-1) + \sqrt{2}B_0\boldsymbol{\varphi}(2x) \\
&+ \sqrt{2}B_1\boldsymbol{\varphi}(2x-1)] \quad \text{on} \quad [1/2,2].
\end{aligned}$$

If we continue iterating the boundary function, we determine $\boldsymbol{\varphi}^L(x)$ on $(0,2]$. The continuity of boundary function follows from the continuity of interior functions.

As in the scalar case, we focus on the behavior of the boundary function as $x \to 0$.

If $x = 0$,

$$\boldsymbol{\varphi}^L(0) = \sqrt{2}A\boldsymbol{\varphi}^L(0).$$

Since we do not want $\boldsymbol{\varphi}^L(0) = \mathbf{0}$, this implies that $\left(\frac{1}{\sqrt{2}}, \boldsymbol{\varphi}^L(0)\right)$ is an eigen-pair of A.

Consider

$$\boldsymbol{\varphi}(x) = \sqrt{2}\left[H_0\boldsymbol{\varphi}(2x) + H_1\boldsymbol{\varphi}(2x-1) + H_2\boldsymbol{\varphi}(2x-2) + H_3\boldsymbol{\varphi}(2x-3)\right]$$

Let x_0 be fixed in $(0,1]$, then

$$\begin{aligned}
\boldsymbol{\varphi}(\frac{x_0}{2}) =& \sqrt{2}\big[H_0\boldsymbol{\varphi}(x_0) + H_1\boldsymbol{\varphi}(x_0-1) + H_2\boldsymbol{\varphi}(x_0-2) + H_3\boldsymbol{\varphi}(x_0-3)\big] \\
=& [\sqrt{2}H_0]\boldsymbol{\varphi}(x_0), \\
\boldsymbol{\varphi}(\frac{x_0}{4}) =& [\sqrt{2}H_0]^2\boldsymbol{\varphi}(x_0), \\
\boldsymbol{\varphi}(\frac{x_0}{8}) =& [\sqrt{2}H_0]^3\boldsymbol{\varphi}(x_0), \\
& \vdots \\
\boldsymbol{\varphi}(2^{-n}x_0) =& [\sqrt{2}H_0]^n\boldsymbol{\varphi}(x_0),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\varphi}^L(\frac{x_0}{2}) &= \sqrt{2}A\boldsymbol{\varphi}^L(x_0) + \sqrt{2}B_0\boldsymbol{\varphi}(x_0) + \sqrt{2}B_1\boldsymbol{\varphi}(x_0-1) \\
&= \sqrt{2}A\boldsymbol{\varphi}^L(x_0) + \sqrt{2}B_0\boldsymbol{\varphi}(x_0), \\
\boldsymbol{\varphi}^L(\frac{x_0}{4}) &= \sqrt{2}A\boldsymbol{\varphi}^L(\frac{x_0}{2}) + \sqrt{2}B_0\boldsymbol{\varphi}(\frac{x_0}{2}) \\
&= \sqrt{2}A\big[\sqrt{2}A\boldsymbol{\varphi}^L(x_0) + \sqrt{2}B_0\boldsymbol{\varphi}(x_0)\big] + \sqrt{2}B_0\boldsymbol{\varphi}(\frac{x_0}{2}) \\
&= (\sqrt{2}A)^2\boldsymbol{\varphi}^L(x_0) + \sqrt{2}A\sqrt{2}B_0\boldsymbol{\varphi}(x_0) + \sqrt{2}B_0\sqrt{2}H_0\boldsymbol{\varphi}(x_0) \\
&= (\sqrt{2}A)^2\boldsymbol{\varphi}^L(x_0) + (\sqrt{2}A\sqrt{2}B_0 + \sqrt{2}B_0\sqrt{2}H_0)\boldsymbol{\varphi}(x_0), \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\varphi}^L(2^{-n}x_0) = (\sqrt{2}A)^n\boldsymbol{\varphi}^L(x_0) + \big[(\sqrt{2}A)^{n-1}\sqrt{2}B_0 + (\sqrt{2}A)^{n-2}(\sqrt{2}B_0)(\sqrt{2}H_0) + \dots \\
+ \dots + (\sqrt{2}A)(\sqrt{2}B_0)(\sqrt{2}H_0)^{n-2} + (\sqrt{2}B_0)(\sqrt{2}H_0)^{n-1}\big]\boldsymbol{\varphi}(x_0).
\end{aligned} \tag{3.34}$$

More precisely,

$$\boldsymbol{\varphi}^L(2^{-n}x_0) = [\sqrt{2}A]^n\boldsymbol{\varphi}^L(x_0) + M_{n-1}\boldsymbol{\varphi}(x_0), \tag{3.35}$$

where

$$\begin{aligned}
M_{n-1} &= (\sqrt{2}A)^{n-1}\sqrt{2}B_0 + (\sqrt{2}A)^{n-2}\sqrt{2}B_0\sqrt{2}H_0 + \dots \\
&+ \sqrt{2}B_0(\sqrt{2}H_0)^{n-1}
\end{aligned}$$

We now investigate what happens to this sum as $n \to \infty$.

Case 1:$\rho(\sqrt{2}A) > 1$

In this case, the cascade algorithm will blow up near 0 for most initial guesses. Thus, $\boldsymbol{\varphi}^L$ will most likely become unbounded in this case. We will not consider this further since we do not expect that there are useful solutions.

Case 2:$\rho(\sqrt{2}A) = 1$

This case will produce some interesting results as shown below.

Case 3:$\rho(\sqrt{2}A) < 1$

In this case, the cascade algorithm will converge to a continuous function and $\boldsymbol{\varphi}^L(0) = \mathbf{0}$. This is not useful for applications.

Recall that a matrix A satisfies *Condition E(p)* if it has a non-degenerate $p-$fold eigenvalue 1, and all other eigenvalues are smaller than 1 in magnitude.

That is,

$$\lambda_1 = \lambda_2 = \ldots = \lambda_p = 1 \quad \text{and} \quad |\lambda_j| < 1, \text{ for } \ j > p.$$

Notation:

$\mathbf{r}_j$ and $\mathbf{l}_j^*$ are right and left eigenvectors of $\sqrt{2}A$, respectively.

Without loss of generality we assume that $\mathbf{r}_1$, $\mathbf{r}_2$, $\ldots$, $\mathbf{r}_p$ are the eigenvectors of $\sqrt{2}A$ corresponding to eigenvalue 1.

$$V = (\mathbf{r}_1, \ldots, \mathbf{r}_L), \qquad \Lambda = \left(\begin{array}{c|ccc} I_p & 0 & 0 & 0 \\ \hline 0 & \lambda_{p+1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_L \end{array}\right), \qquad V^{-1} = \begin{pmatrix} \mathbf{l}_1^* \\ \vdots \\ \mathbf{l}_L^* \end{pmatrix},$$

$$\Gamma = (\mathbf{r}_1, \ldots, \mathbf{r}_p) \begin{pmatrix} \mathbf{l}_1^* \\ \vdots \\ \mathbf{l}_p^* \end{pmatrix}$$

Theorem 3.2. *Assume that $\sqrt{2}A$ satisfies condition $E(p)$ and the cascade algorithm for $\boldsymbol{\varphi}(x)$ converges for every initial guesses. Then*

$$M_n \to \Gamma(\sqrt{2}B_0)(I - \sqrt{2}H_0)^{-1}.$$

Proof. By assumption,

$$\begin{aligned}(\sqrt{2}A)V &= V\Lambda,\\ \sqrt{2}A &= V\Lambda V^{-1}.\end{aligned}$$

Since $(\sqrt{2}A)^n = V\Lambda^n V^{-1}$, it is clear that

$$(\sqrt{2}A)^n \longrightarrow V \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right) V^{-1} = (\mathbf{r}_1, \ldots, \mathbf{r}_p) \begin{pmatrix} \mathbf{l}_1^* \\ \vdots \\ \mathbf{l}_p^* \end{pmatrix} = \Gamma.$$

For $j = 1, 2, \ldots, p$, we have

$$\mathbf{l}_j^*(\sqrt{2}A) = \mathbf{l}_j^*.$$

Thus,

$$\begin{aligned}\mathbf{l}_j^* M_n &= \mathbf{l}_j^*[(\sqrt{2}A)^n\sqrt{2}B_0 + (\sqrt{2}A)^{n-1}\sqrt{2}B_0\sqrt{2}H_0 + \ldots + \sqrt{2}B_0(\sqrt{2}H_0)^n]\\ &= \mathbf{l}_j^*(\sqrt{2}B_0)[I + \sqrt{2}H_0 + (\sqrt{2}H_0)^2 + \ldots + (\sqrt{2}H_0)^n]\\ &\longrightarrow \mathbf{l}_j^*(\sqrt{2}B_0)(I - \sqrt{2}H_0)^{-1}.\end{aligned}$$

For $j > p$, we obtain

$$\mathbf{l}_j^* M_n = \mathbf{l}_j^*(\sqrt{2}B_0)[\lambda_j^n + \lambda_j^{n-1}(\sqrt{2}H_0) + \ldots + (\sqrt{2}H_0)^n]. \tag{3.36}$$

We now use the fact that given $\epsilon > 0$ and a matrix A, there is a norm such that $\|A\| \leq \rho(A) + \epsilon$ [17].

The convergence of cascade algorithm implies that there is a norm such that $\|\sqrt{2}H_0\| < 1$. Let

$$m = \max(\|\sqrt{2}H_0\|,\ |\lambda_{p+1}|,\ \ldots,\ |\lambda_L|).$$

Taking the norm of the identity (3.36), we get

$$\begin{aligned}\|\mathbf{l}_j^* M_n\| &\leq \|\mathbf{l}_j^*\|\|\sqrt{2}B_0\| \sum_{l=0}^{n} |\lambda_j|^l (\|\sqrt{2}H_0\|)^{n-l}\\ &\leq \|\mathbf{l}_j^*\|\|\sqrt{2}B_0\|(n+1)m^n\\ &\longrightarrow 0.\end{aligned}$$

We have thus shown that

$$V^{-1}M_n = \begin{pmatrix} \mathbf{l}_1^* \\ \vdots \\ \mathbf{l}_L^* \end{pmatrix} M_n \longrightarrow \begin{pmatrix} \mathbf{l}_1^* \\ \vdots \\ \mathbf{l}_p^* \\ 0 \\ \vdots \\ 0 \end{pmatrix} (\sqrt{2}B_0)(I-\sqrt{2}H_0)^{-1}.$$

This implies that

$$M_n \longrightarrow VV^{-1}M_n = \Gamma(\sqrt{2}B_0)(I-\sqrt{2}H_0)^{-1}.$$

$\square$

With this theorem, we conclude that as $n \to \infty$,

$$\boldsymbol{\varphi}^L(2^{-k}x_0) \to \Gamma[\boldsymbol{\varphi}^L(x_0) + (\sqrt{2}B_0)(I-\sqrt{2}H_0)^{-1}\boldsymbol{\varphi}(x_0)]. \tag{3.37}$$

Note that the continuity of $\boldsymbol{\varphi}^L$ at $x = 0$ will require the right hand side of (3.37) to be independent of x_0.

Indeed, $\boldsymbol{\varphi}^L$ is continuous at $x = 0$ if and only if the right hand side of (3.37) is independent of $x_0 \in (0, 1]$.

If there is only a single boundary function, we have $r = 1$, and

$$\varphi^L(x) = \begin{cases} \varphi^L(0) - \sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}\boldsymbol{\varphi}(x) & \text{if } x \in [0,1]; \\ \sqrt{2}[\mathbf{b}_0^*\boldsymbol{\varphi}(2x) + \mathbf{b}_1^*\boldsymbol{\varphi}(2x-1)] & \text{if } x \in [1,2]. \end{cases} \tag{3.38}$$

In particular, the continuity of φ^L at $x = 1$ gives

$$\begin{aligned} \varphi^L(1) &= \boldsymbol{\varphi}^L(0) - \sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}\boldsymbol{\varphi}(1), \\ \varphi^L(1) &= \sqrt{2}[\mathbf{b}_0^*\boldsymbol{\varphi}(2) + \mathbf{b}_1^*\boldsymbol{\varphi}(1)]. \end{aligned} \tag{3.39}$$

Thus,

$$\varphi^L(0) = \sqrt{2}\mathbf{b}_0^*\boldsymbol{\varphi}(2) + [\sqrt{2}\mathbf{b}_1^* + \sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}]\boldsymbol{\varphi}(1). \tag{3.40}$$

An interesting question is whether every regular boundary function is a linear combination of boundary-crossing functions.

In general the answer is no, as illustrated by the example in section 5.5. In the case where there is only a single boundary function, the following theorem shows that the answer is yes.

Theorem 3.3. *If there is only a single boundary function, the only regular solution (up to a normalization factor) is the one corresponding to*

$$a = \frac{\sqrt{2}}{2}, \quad \mathbf{b}^* = (\boldsymbol{\mu}_0^* H_2 \,, \boldsymbol{\mu}_0^* H_3), \quad \text{and} \quad \mathbf{c}^* = (\boldsymbol{\mu}_0^* \,, \boldsymbol{\mu}_0^*).$$

Proof. First notice that if there is only one boundary function and $\varphi^L \neq 0$ then $A = a = \frac{1}{\sqrt{2}}$. We assume that we have (3.38) and the recursion relation

$$\varphi^L(x) = \varphi^L(2x) + \sqrt{2}\mathbf{b}_0^* \boldsymbol{\varphi}(2x) + \sqrt{2}\mathbf{b}_1^* \boldsymbol{\varphi}(2x-1). \tag{3.41}$$

For $x \in [1\,,2]$, the identities (3.38) and (3.41) are equal.

For $x \in [0\,, \frac{1}{2}]$, (3.41) implies that

$$\begin{aligned} \varphi^L(x) &= \varphi^L(2x) + \sqrt{2}\mathbf{b}_0^* \boldsymbol{\varphi}(2x) \\ &= \varphi^L(0) - \sqrt{2}\mathbf{b}_0^*(I - \sqrt{2}H_0)^{-1}\boldsymbol{\varphi}(2x) + \sqrt{2}\mathbf{b}_0^* \boldsymbol{\varphi}(2x). \end{aligned} \tag{3.42}$$

and (3.38) reads

$$\begin{aligned} \varphi^L(x) &= \varphi^L(0) - \sqrt{2}\mathbf{b}_0^*(I - \sqrt{2}H_0)^{-1}\boldsymbol{\varphi}(x) \\ &= \varphi^L(0) - \sqrt{2}\mathbf{b}_0^*(I - \sqrt{2}H_0)^{-1}(\sqrt{2}H_0)\boldsymbol{\varphi}(2x) \end{aligned} \tag{3.43}$$

Comparing (3.42) and (3.43), we need to show that

$$(I - \sqrt{2}H_0)^{-1} - I = (I - \sqrt{2}H_0)^{-1}(\sqrt{2}H_0). \tag{3.44}$$

This follows by multiplying (3.44) from the left by $(I - \sqrt{2}H_0)$.

Notice that

$$H_0(I - \sqrt{2}H_0) = (I - \sqrt{2}H_0)H_0, \tag{3.45}$$

$$H_0(I - \sqrt{2}H_0)^{-1} = (I - \sqrt{2}H_0)^{-1}H_0. \tag{3.46}$$

For $x \in [1/2\,,1]$, (3.38) becomes

$$\begin{aligned}
\varphi^L(x) &= \varphi^L(0) - \sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}[\sqrt{2}H_0\boldsymbol{\varphi}(2x) + \sqrt{2}H_1\boldsymbol{\varphi}(2x-1)] \\
&= \varphi^L(0) - \sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}(\sqrt{2}H_0)^2\boldsymbol{\varphi}(4x) \\
&\quad -\sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}\sqrt{2}H_0\sqrt{2}H_1\boldsymbol{\varphi}(4x-1) \qquad (3.47)\\
&\quad -\sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}[\sqrt{2}H_0\sqrt{2}H_2 + \sqrt{2}H_1\sqrt{2}H_0]\boldsymbol{\varphi}(4x-2) \\
&\quad -\sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}[\sqrt{2}H_0\sqrt{2}H_3 + (\sqrt{2}H_1)^2]\boldsymbol{\varphi}(4x-3).
\end{aligned}$$

On the other hand, (3.41) says

$$\begin{aligned}
\varphi^L(x) &= \varphi^L(2x) + \sqrt{2}\mathbf{b}_0^*\boldsymbol{\varphi}(2x) + \sqrt{2}\mathbf{b}_1^*\boldsymbol{\varphi}(2x-1) \\
&= \sqrt{2}\mathbf{b}_0^*\boldsymbol{\varphi}(4x) + \sqrt{2}\mathbf{b}_1^*\boldsymbol{\varphi}(4x-1) \\
&\quad +\sqrt{2}\mathbf{b}_0^*[\sqrt{2}H_0\boldsymbol{\varphi}(4x) + \sqrt{2}H_1\boldsymbol{\varphi}(4x-1) + \sqrt{2}H_2\boldsymbol{\varphi}(4x-2) + \sqrt{2}H_3\boldsymbol{\varphi}(4x-3)] \\
&\quad +\sqrt{2}\mathbf{b}_1^*[\sqrt{2}H_0\boldsymbol{\varphi}(4x-2) + \sqrt{2}H_1\boldsymbol{\varphi}(4x-3)].
\end{aligned}$$

Combining like terms, we get

$$\begin{aligned}
\varphi^L(x) &= [\sqrt{2}\mathbf{b}_0^* + \sqrt{2}\mathbf{b}_0^*\sqrt{2}H_0]\boldsymbol{\varphi}(4x) \\
&+ [\sqrt{2}\mathbf{b}_1^* + \sqrt{2}\mathbf{b}_0^*\sqrt{2}H_1]\boldsymbol{\varphi}(4x-1) \qquad (3.48)\\
&+ [\sqrt{2}\mathbf{b}_0^*\sqrt{2}H_2 + \sqrt{2}\mathbf{b}_1^*\sqrt{2}H_0]\boldsymbol{\varphi}(4x-2) \\
&+ [\sqrt{2}\mathbf{b}_0^*\sqrt{2}H_3 + \sqrt{2}\mathbf{b}_1^*\sqrt{2}H_1]\boldsymbol{\varphi}(4x-3).
\end{aligned}$$

Equating (3.47) and (3.48), we obtain

$$\begin{aligned}
\varphi^L(0) &= [\sqrt{2}\mathbf{b}_0^* + \sqrt{2}\mathbf{b}_0^*\sqrt{2}H_0 + \sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}(\sqrt{2}H_0)^2]\boldsymbol{\varphi}(4x) \\
&\quad +[\sqrt{2}\mathbf{b}_1^* + \sqrt{2}\mathbf{b}_0^*\sqrt{2}H_1]\boldsymbol{\varphi}(4x-1) \qquad (3.49)\\
&\quad +[\sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}(\sqrt{2}H_0)(\sqrt{2}H_1)]\boldsymbol{\varphi}(4x-1) \qquad (3.50)\\
&\quad +[\sqrt{2}\mathbf{b}_0^*\sqrt{2}H_2 + \sqrt{2}\mathbf{b}_1^*\sqrt{2}H_0]\boldsymbol{\varphi}(4x-2) \qquad (3.51)\\
&\quad +\sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}(\sqrt{2}H_0\sqrt{2}H_2 + \sqrt{2}H_1\sqrt{2}H_0)]\boldsymbol{\varphi}(4x-2) \\
&\quad +[\sqrt{2}\mathbf{b}_0^*\sqrt{2}H_3 + \sqrt{2}\mathbf{b}_1^*\sqrt{2}H_1]\boldsymbol{\varphi}(4x-3) \qquad (3.52)\\
&\quad +\sqrt{2}\mathbf{b}_0^*(I-\sqrt{2}H_0)^{-1}(\sqrt{2}H_0\sqrt{2}H_3 + (\sqrt{2}H_1)^2)]\boldsymbol{\varphi}(4x-3).
\end{aligned}$$

For the left hand side of (3.49) to be constant, the coefficients $\boldsymbol{\varphi}(x)$ through $\boldsymbol{\varphi}(x-3)$ must be equal to $d\boldsymbol{\mu}_0^*$ for some scalar d. We now show that the coefficient of $\boldsymbol{\varphi}(4x)$ leads to the condition

$$\mathbf{b}_0^* = d\boldsymbol{\mu}_0^* H_2.$$

We first recall the fact that

$$\boldsymbol{\mu}_0^*(H_0 + H_2) = \frac{1}{\sqrt{2}}\boldsymbol{\mu}_0^*. \tag{3.53}$$

We then need to show that

$$\sqrt{2}\mathbf{b}_0^*[I + \sqrt{2}\sqrt{2}H_0 + (I - \sqrt{2}H_0)^{-1}(\sqrt{2}H_0)^2](I - \sqrt{2}H_0) = d\boldsymbol{\mu}_0^*(I - \sqrt{2}H_0). \tag{3.54}$$

Since H_0 commutes with $(I - \sqrt{2}H_0)^{-1}$, (3.54) means

$$\sqrt{2}\mathbf{b}_0^* = d\boldsymbol{\mu}_0^*(I - \sqrt{2}H_0). \tag{3.55}$$

(3.53) and (3.55) together lead to

$$\mathbf{b}_0^* = d\boldsymbol{\mu}_0^* H_2.$$

We now use the coefficient of $\boldsymbol{\varphi}(4x-1)$ to show

$$\mathbf{b}_1^* = d\boldsymbol{\mu}_0^* H_3.$$

If we first substitute $d\boldsymbol{\mu}_0^* H_2$ for $\mathbf{b}_0^*$ into

$$\sqrt{2}\mathbf{b}_1^* + \sqrt{2}\mathbf{b}_0^*\sqrt{2}H_1 + \sqrt{2}\mathbf{b}_0^*(I - \sqrt{2}H_0)^{-1}(\sqrt{2}H_0)(\sqrt{2}H_1) = d\boldsymbol{\mu}_0^*,$$

we get

$$\begin{aligned}
d\boldsymbol{\mu}_0^* &= \sqrt{2}\mathbf{b}_1^* + \sqrt{2}d\boldsymbol{\mu}_0^* H_2(\sqrt{2}H_1) + d\boldsymbol{\mu}_0^*(\sqrt{2}H_0)(\sqrt{2}H_1) \\
d\boldsymbol{\mu}_0^* &= \sqrt{2}\mathbf{b}_1^* + d\underbrace{(\sqrt{2}\boldsymbol{\mu}_0^* H_2 + \sqrt{2}\boldsymbol{\mu}_0^* H_0)}_{\boldsymbol{\mu}_0^*}\sqrt{2}H_1 \\
d\boldsymbol{\mu}_0^*(I - \sqrt{2}H_1) &= \sqrt{2}\mathbf{b}_1^*.
\end{aligned}$$

The fact that

$$\sqrt{2}\boldsymbol{\mu}_0^* H_3 = \mathbf{b}_0^*(I - \sqrt{2}H_1)$$

leads

$$\mathbf{b}_1^* = d\boldsymbol{\mu}_0^* H_3.$$

It follows from (3.17) that $\mathbf{c}^* = (\boldsymbol{\mu}_0^*, \boldsymbol{\mu}_0^*)$ is a solution and leads to $\mathbf{b}^*$. □

3.5 Approximation Order

We have so far described and examined boundary functions based on two approaches. The first approach was to assume boundary functions are linear combinations of boundary crossing functions and other approach was to assume boundary functions satisfy a recursion relation. We have also shown how these two approaches are related. In this section we investigate approximation order 1 for each approach independently and then seek the interrelation between them.

Approach 1:

Assume that the boundary functions satisfy the following recursion relation

$$\boldsymbol{\varphi}^L(x) = \sqrt{2}A\boldsymbol{\varphi}^L(2x) + \sqrt{2}\sum_{l=0}^{N-2} B_l \boldsymbol{\varphi}(2x-l). \tag{3.56}$$

If we have approximation order 1, then $\exists\ \mathbf{d}^*$ with

$$\begin{aligned} \mathbf{d}^*\boldsymbol{\varphi}^L(x) &= \boldsymbol{\mu}_0^* \sum_{j=-N+1}^{-1} \boldsymbol{\varphi}(x-j) \quad x \in [0,\ N-1], & (3.57) \\ \mathbf{d}^*\boldsymbol{\varphi}^L(2x) &= \boldsymbol{\mu}_0^* \sum_{j=-N+1}^{-1} \boldsymbol{\varphi}(2x-j) \quad x \in [0,\ \frac{N-1}{2}]. & (3.58) \end{aligned}$$

Using the interior relation and the fact that

$$\boldsymbol{\mu}_0^* \sum_k H_{2k} = \boldsymbol{\mu}_0^* \sum_k H_{2k+1} = \frac{1}{\sqrt{2}}\boldsymbol{\mu}_0^*$$

in (3.57) and comparing the result with (3.56), we obtain

$$
\begin{aligned}
\mathbf{d}^*(\sqrt{2}A) &= \mathbf{d}^*, \\
\boldsymbol{\mu}_0^* \sum_{k=1}^{\frac{N-1}{2}} H_{2k+j} &= \mathbf{d}^* B_j, \quad j = 0, 1, \\
\boldsymbol{\mu}_0^* \sum_{k=1}^{\frac{N-1-j}{2}} H_{j+2k} &= \mathbf{d}^* B_j, \quad 2 \le j \le N-4, \text{ and } j \text{ is even }, \\
\boldsymbol{\mu}_0^* \sum_{k=1}^{\frac{N-1}{2}} H_{j+2k} &= \mathbf{d}^* B_j, \quad 2 \le j \le N-4, \text{ and } j \text{ is odd }, \\
\boldsymbol{\mu}_0^* H_{j+2} &= \mathbf{d}^* B_j \quad j = N-3, N-2.
\end{aligned}
$$

Our particular interest is the case where $N = 3$. Then the above conditions reduces to

$$
\begin{aligned}
\mathbf{d}^*(\sqrt{2}A) &= \mathbf{d}^*, \\
\boldsymbol{\mu}_0^* H_2 &= \mathbf{d}^* B_0, \\
\boldsymbol{\mu}_0^* H_3 &= \mathbf{d}^* B_1.
\end{aligned}
\tag{3.59}
$$

Approach 2:

Assume that the boundary functions are linear combinations of boundary crossing functions. Since $\boldsymbol{\varphi}$ has approximation order 1, there exists a vector $\boldsymbol{\mu}_0^*$ so that

$$1 = \sum_j \boldsymbol{\mu}_0^* \boldsymbol{\varphi}_j$$

We work with orthogonal multiwavelets supported on $[0, N]$.

For $x \ge 0$, we only need $\boldsymbol{\varphi}_j, \quad j = -N+1, \ldots, \infty$. Then

$$1 = \sum_{j=-N+1}^{\infty} \boldsymbol{\mu}_0^* \boldsymbol{\varphi}_j, \quad x \ge 0, \tag{3.60}$$

where $\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1, \ldots,$ are interior functions and $\varphi_{-N+1}, \varphi_{-N+2}, \ldots, \varphi_{-1}$ are boundary crossing functions. Equation (3.60) can be written

$$1 = \sum_{j=-N+1}^{-1} \boldsymbol{\mu}_0^* \boldsymbol{\varphi}_j + \underbrace{\sum_{j=0}^{\infty} \boldsymbol{\mu}_0^* \boldsymbol{\varphi}_j,}_{\text{interior functions}} \qquad x \ge 0.$$

Thus, the first part of above sum must be taken care of by left end point functions.

For some $\mathbf{e}^*$,

$$\begin{aligned} \mathbf{e}^*\boldsymbol{\varphi}^L(x) &= \sum_{j=-N+1}^{-1} \boldsymbol{\mu}_0^*\boldsymbol{\varphi}_j, \quad x \geq 0 \\ \mathbf{e}^* \sum_{j=-N+1}^{-1} C_j\boldsymbol{\varphi}_j &= \sum_{j=-N+1}^{-1} \boldsymbol{\mu}_0^*\boldsymbol{\varphi}_j. \end{aligned}$$

This implies,

$$\mathbf{e}^*C_j = \boldsymbol{\mu}_0^*, \quad j = -N+1, \ldots, -1,$$

or

$$\mathbf{e}^*C = \begin{bmatrix} \boldsymbol{\mu}_0^* & \boldsymbol{\mu}_0^* & \ldots & \boldsymbol{\mu}_0^*. \end{bmatrix} \tag{3.61}$$

This is only possible when we can represent the right hand side of (3.61) by rows of C.

CHAPTER 4. Wavelets on Finite Intervals

4.1 Introduction

We have described, in the preceding sections, the techniques and methods for finding basis functions defined on the whole space $\mathbb{R}$. We have also shown that DWT and IDWT act on infinite sequences of coefficients and hence the matrices in these transforms are infinite matrices. However, in many practical applications one needs to know how to handle problems defined on some finite intervals. To be more precise, our concern is how to handle the boundary.

As we have shown in (2.10), the infinite length DWT can be interpreted as

$$(\mathbf{sd})_{n-1} = T\mathbf{s}_n,$$

where T is an infinite matrix. It is clear that when a sequence of coefficients has finite length, the infinite matrix T must be replaced by a finite matrix T_n which has a similar structure as T. We want T_n to satisfy

$$(\mathbf{sd})_{n-1} = T_n\mathbf{s}_n,$$

where $(sd)_{n-1}$ and s_n are finite.

A reasonable choice for T_n is

$$T_n = \begin{pmatrix} T_l & & \\ & T_i & \\ & & T_r \end{pmatrix},$$

where T_i is formed by complete rows of infinite matrix T, T_l, and T_r are to be determined. We want T_l and T_r to be small in size and independent of levels. To reconstruct the signal $\mathbf{s}_n$, we require T_n to be invertible. If the wavelet is orthogonal, we also want T_n to be orthogonal. There are available approaches in the literature for finding T_l and T_r [20], [25]. We mention some of these techniques for both scalar and multiwavelet cases, and then concentrate on approaches to endpoint construction.

4.2 Data Extension Approach

The given signal is extended across the boundaries in such a way that extended coefficients are linear combinations of known coefficients. This has the effect of folding parts of the original matrix T that "stick over the edge" back inside. Although this method usually works, non-singularity of the obtained matrix is not guaranteed. This usually also destroys orthogonality. Since the extension will also be based upon the data or signal, we include some specific data extension methods.

Periodic Extension

In this method, one expands a finite data into a periodic bi-infinite sequence. It is well suited for periodic data, otherwise it introduces discontinuity at the boundary points. Meanwhile, it preserves orthogonality and approximation order 1. The matrix representation enjoys being circulant.

Zero Extension

This is the "laziest" technique among all. The extension is achieved by zero-padding or truncating the matrix without any modifications at the boundary. It preserves neither orthogonality nor any approximation orders.

Symmetric Extension

In this case, the aim is to extend the data without having boundary artifacts. The data is extended by reflecting it about its initial and end point. This method is best for symmetric data. The detailed explanation for this method can be found in [6].

Polynomial Extrapolation

The idea in this method is to extrapolate the data at the boundaries by estimating the coefficients of best fit polynomial passing through data points near the boundary. It doesn't preserve orthogonality. This method is explained in [32].

4.3 Boundary Function Approach

In this approach, one adds appropriate boundary functions at each end in the sense that they, together with the internal scaling functions, maintain both orthogonality and approximation order. The details and worked-out examples can be found in [9], [20].

4.4 Matrix Completion Approach

In this approach, we use linear algebra techniques to search for right and left end blocks such that T_n is orthogonal. That is,

$$T_n(T_n)^* = I.$$

In general, the approximation order is not preserved. The details and examples are elaborated in [21], [25].

4.5 Madych Approach for Scalar Wavelets

This is a particular implementation of the Matrix Completion approach. Madych [21] started from the periodic extension matrix for a scalar wavelet, and modified it into the desired form by orthogonal matrices.

Under certain conditions, we will show that it also works for orthogonal multiwavelets. The following is a generalization of Madych' approach.

At first, we assume that there are only two block matrices T_0 and T_1 such that

$$T_0 = \begin{pmatrix} H_0 & H_1 \\ G_0 & G_1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} H_2 & H_3 \\ G_2 & G_3 \end{pmatrix}.$$

If we let

$$S_0 = \begin{pmatrix} H_0 & H_1 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} H_2 & H_3 \end{pmatrix}, \qquad W_0 = \begin{pmatrix} G_0 & G_1 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} G_2 & G3 \end{pmatrix}.$$

The above matrices can be more conveniently represented by

$$T_0 = \begin{pmatrix} S_0 \\ W_0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} S_1 \\ W_1 \end{pmatrix}.$$

We know that the matrix

$$\tilde{T}_n = \begin{pmatrix} T_0 & T_1 & 0 & \dots & \dots \\ 0 & T_0 & T_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & T_0 & T_1 \\ T_1 & 0 & \dots & \dots & T_0 \end{pmatrix}$$

is orthogonal. Our objective is to find orthogonal matrices U, V so that

$$U\tilde{T}_n V = \begin{pmatrix} L_0 & L_1 & 0 & \dots & \dots & \\ 0 & T_0 & T_1 & 0 & \dots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & T_0 & T_1 & \\ 0 & 0 & \dots & 0 & R_0 & R_1 \end{pmatrix}$$

has the desired structure and is orthogonal.

Definition 4.1. ***Condition B***

$$\begin{pmatrix} S_0 \\ S_1 \end{pmatrix} = \begin{pmatrix} H_0 & H_1 \\ H_2 & H_3 \end{pmatrix} \textit{ has full rank.}$$

Note: This condition is automatic in the scalar case, because S_0 and S_1 are row vectors satisfying $S_0 S_1^* = 0$. This implies that S_0 and S_1 are orthogonal and hence linearly independent. The following example shows that this condition may not hold for multiwavelets.

Example 4.1. Consider the standard Chui-Lian CL2 multiwavelet. The recursion coefficients are

$$H_0 = \begin{pmatrix} \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}t/2 & \sqrt{2}t/2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & \sqrt{4-8t^2}/2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \sqrt{2}/4 & -\sqrt{2}/4 \\ -\sqrt{2}t/2 & \sqrt{2}t/2 \end{pmatrix},$$

where $t = -\sqrt{7}/4$.

The matrix

$$\begin{pmatrix} S_0 \\ S_1 \end{pmatrix} = \begin{pmatrix} H_0 & H_1 \\ H_2 & 0_2 \end{pmatrix}$$

doesn't have full rank. Thus, Condition B does not hold in this case.

Proposition 4.2. *If Condition B is satisfied, then we can find $r \times r$ nonsingular matrices R_0, R_1 so that*

$$\begin{pmatrix} R_0 S_0 \\ R_1 S_1 \end{pmatrix}$$

is othogonal.

Proof. If $r = 1$ (Madych case), then

$$\begin{aligned} S_0 &= (h_0 \quad h_1), \\ S_1 &= (h_2 \quad h_3), \end{aligned}$$

and $S_0 S_1^* = 0$. Simply take $R_0 = \frac{1}{\sqrt{h_0^2+h_1^2}}$ and $R_1 = \frac{1}{\sqrt{h_2^2+h_3^2}}$.
If $r > 1$, S_0, S_1 are $r \times 2r$ matrices.
Simply take

$$R_0 = L_0^{-1} \text{ and } R_1 = L_1^{-1},$$

where L_0 and L_1 are obtained from the LQ factorization of S_0 and S_1, respectively. □

Let us get back now to our main objective. We start with orthogonal

$$\tilde{T}_n = \begin{pmatrix} T_0 & T_1 & 0 & \dots & \dots \\ 0 & T_0 & T_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & T_0 & T_1 \\ T_1 & 0 & \dots & \dots & T_0 \end{pmatrix}.$$

Let

$$V = \left(\begin{array}{c|c|c} (R_0 S_0)^* & 0 & (R_1 S_1)^* \\ \hline 0 & I & 0 \end{array}\right),$$

then

$$\tilde{T}_n V = \left(\begin{array}{c|cccc|c} T_0(R_0S_0)^* & T_1 & 0 & \dots & 0 & \overbrace{T_0(R_1S_1)^*}^{=0} \\ \hline 0 & T_0 & T_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & T_0 & T_1 & 0 \\ \hline \underbrace{T_1(R_0S_0)^*}_{=0} & 0 & \dots & \dots & T_0 & T_1(R_1S_1)^* \end{array}\right).$$

This already has the desired form. Multiply TV from left by

$$U = \left(\begin{array}{c|c|c} U_L & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & U_R \end{array}\right),$$

where U_L, U_R are arbitrary orthogonal matrices, to obtain

$$T_n = U\tilde{T}_n V = \left(\begin{array}{c|cccc|c} L_0 & L_1 & 0 & \dots & \dots & 0 \\ \hline 0 & T_0 & T_1 & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & T_0 & T_1 & 0 \\ \hline 0 & 0 & \dots & \dots & R_0 & R_1 \end{array}\right)$$

with

$$\begin{aligned} L_0 &= U_L T_0 S_0^* R_0^* \\ L_1 &= U_L T_1 \\ R_0 &= U_R T_0 \\ R_1 &= U_R T_1 S_1^* R_1^* \end{aligned}$$

For simplicity, we made an assumption that there are only two block matrices when we began this section. The general case follows similar steps, except we will have larger block matrices as we show in the following.

Assume there are $2(n+1)$ coefficients. We then have $H_0, H_1, \dots, H_{2n+1}$, each of size $r \times r$. The matrix representation becomes

$$\check{T} = \begin{pmatrix} \check{T}_0 & \check{T}_1 & 0 & \dots & \dots \\ 0 & \check{T}_0 & \check{T}_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \check{T}_0 & \check{T}_1 \\ \check{T}_1 & 0 & \dots & \dots & \check{T}_0 \end{pmatrix},$$

where

$$\check{T}_0 = \begin{pmatrix} T_0 & T_1 & T_2 & \dots & T_{n-1} \\ 0 & T_0 & T_1 & \dots & T_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & T_1 \\ 0 & 0 & 0 & 0 & T_0 \end{pmatrix}, \qquad \check{T}_1 = \begin{pmatrix} T_n & 0 & 0 & \dots & 0 \\ T_{n-1} & T_n & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ T_1 & T_2 & \dots & \dots & T_n \end{pmatrix},$$

and

$$T_i = \begin{pmatrix} H_{2i} & H_{2i+1} \\ G_{2i} & G_{2i+1} \end{pmatrix}, i = 0, 1, 2, \dots, n.$$

4.6 A New Approach to Multiwavelet Endpoint Modification

We again first assume that there are only T_0, T_1 of size $2r \times 2r$ and

$$\tilde{T}_n = \begin{pmatrix} T_0 & T_1 & 0 & \dots & \dots \\ 0 & T_0 & T_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & T_0 & T_1 \\ T_1 & 0 & \dots & \dots & T_0 \end{pmatrix}.$$

To explain the idea, it suffices to look at

$$\tilde{T}_3 = \begin{pmatrix} T_0 & T_1 & 0 \\ 0 & T_0 & T_1 \\ T_1 & 0 & T_0 \end{pmatrix}.$$

Everything bigger just has more copies of T_0, T_1 in the middle.

In our approach, we make use of theorem A.8. That is, there exist orthogonal U and V such that $\rho_0 = \text{rank}(T_0), \quad \rho_1 = \text{rank}(T_1), \quad \rho_0 + \rho_1 = 2r \quad$ and

$$T_0 = U \left(\begin{array}{c|c} I & 0 \\ \hline \underbrace{0}_{\rho_0} & \underbrace{0}_{\rho_1} \end{array} \right) V^*, \qquad T_1 = U \left(\begin{array}{c|c} 0 & 0 \\ \hline \underbrace{0}_{\rho_0} & \underbrace{I}_{\rho_1} \end{array} \right) V^*. \tag{4.1}$$

Let us briefly discuss the idea by considering the DGHM multiwavelet before we introduce it for a more general setting.

Heuristic Argument:

For the DGHM example, $\rho_0 = 3$ and $\rho_1 = 1$. Since the actual support is $[0,2]$ instead of $[0,3]$, we expect that there are 3 right endpoint functions and 1 left endpoint function. In general, we suspect ρ_1, and ρ_0 left and right endpoint functions, respectively. Thus, we want to end up with

$$\left(\begin{array}{c|c|c|c} L_0 & L_1 & 0 & 0 \\ \hline 0 & T_0 & T_1 & 0 \\ \hline 0 & 0 & R_0 & R_1 \end{array} \right)$$

where the matrices L_0, L_1, R_0, and R_1 are of size $2\rho_1 \times \rho_1, \quad 2\rho_1 \times 2r, \quad 2\rho_0 \times 2r$, and $2\rho_0 \times \rho_0$, respectively.

It will be shown later (theorem 4.3) that the choice $L = \rho_1$ and $R = \rho_0$ is unique.

Our completion approach here is based on the assumption that the number of left boundary functions L is equal to ρ_1.

Consider

$$U_3^* = \left(\begin{array}{c|c|c} U^* & 0 & 0 \\ \hline 0 & U^* & 0 \\ \hline 0 & 0 & U^* \end{array} \right), \qquad V_3 = \left(\begin{array}{c|c|c} V & 0 & 0 \\ \hline 0 & V & 0 \\ \hline 0 & 0 & V \end{array} \right),$$

where U and V are those defined in (4.1).

We multiply $\tilde{T}_3$ from the left by U_3^* and from right by V_3. We obtain

$$U_3^* \tilde{T}_3 V_3 = \left(\begin{array}{c|c||c|c||c|c}
I & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & I & 0 & 0 \\ \hline\hline
0 & 0 & I & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & I \\ \hline\hline
0 & 0 & 0 & 0 & I & 0 \\ \hline
\underbrace{0}_{\rho_0} & \underbrace{I}_{\rho_1} & \underbrace{0}_{\rho_0} & \underbrace{0}_{\rho_1} & \underbrace{0}_{\rho_0} & \underbrace{0}_{\rho_1}
\end{array}\right).$$

By inspection, one technique that works is that we move the first ρ_0 columns to the end, and then interchange the first ρ_0 rows with the last ρ_1 rows. That amounts to multiplying from the right with

$$P_R = \left(\begin{array}{c|c}
0 & I \\ \hline
\underbrace{I}_{rest} & \underbrace{0}_{\rho_0}
\end{array}\right)$$

and from the left with

$$P_R = \left(\begin{array}{c|c|c}
0 & 0 & I \\ \hline
0 & I & 0 \\ \hline
I & 0 & 0
\end{array}\right),$$

then

$$P_L U_3^* T_3 V_3 P_R = \left(\begin{array}{c||cc||cc||c}
I & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & I & 0 & 0 & 0 \\ \hline\hline
0 & I & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & I & 0 \\ \hline\hline
0 & 0 & 0 & I & 0 & 0 \\ \hline
\underbrace{0}_{\rho_1} & 0 & 0 & 0 & 0 & \underbrace{I}_{\rho_0}
\end{array}\right) \tag{4.2}$$

Now let

$$Q_L = (\underbrace{Q_{L,0}}_{\rho_1} \,|\, \underbrace{Q_{L,1}}_{\rho_1})$$
$$Q_R = (\underbrace{Q_{R,0}}_{\rho_0} \,|\, \underbrace{Q_{R,1}}_{\rho_0})$$

be arbitrary orthogonal matrices of size $2\rho_1 \times 2\rho_1, 2\rho_0 \times 2\rho_0$.

Likewise,

$$U = (\underbrace{U_0}_{\rho_0} \mid \underbrace{U_1}_{\rho_1})$$
$$V = (\underbrace{V_0}_{\rho_0} \mid \underbrace{V_1}_{\rho_1})$$

We now multiply (4.2) from the left with

$$\tilde{P}_L = \left(\begin{array}{c|c|c} Q_L & 0 & 0 \\ \hline 0 & U & 0 \\ \hline \underbrace{0}_{2\rho_1} & 0 & \underbrace{Q_R}_{2\rho_0} \end{array}\right),$$

and from the right with

$$\tilde{P}_R = \left(\begin{array}{c|cc|c} I & 0 & 0 & 0 \\ \hline 0 & V^* & 0 & 0 \\ 0 & 0 & V^* & 0 \\ \hline \underbrace{0}_{\rho_1} & 0 & 0 & \underbrace{I}_{\rho_0} \end{array}\right),$$

then

$$\tilde{P}_L P_L U_3^* T_3 V_3 P_R \tilde{P}_R = \left(\begin{array}{c|cc|c} L_0 & L_1 & 0 & 0 \\ \hline 0 & T_0 & T_1 & 0 \\ \hline 0 & 0 & R_0 & R_1 \end{array}\right), \text{ where}$$

$$\begin{aligned} L_0 &= (Q_{L,0})_{2\rho_1 \times \rho_1} \\ L_1 &= (Q_{L,1} V_1^*)_{2\rho_1 \times 2r} \\ R_0 &= (Q_{R,0} V_0^*)_{2\rho_0 \times 2r} \\ R_1 &= (Q_{R,1})_{2\rho_0 \times \rho_0}. \end{aligned}$$

In the following theorem, we show that this construction is unique up to the choices of Q_L and Q_R. In addition, we show that the choice of ρ_0 and ρ_1 is unique.

Theorem 4.3. *If*

$$T_3 = \left(\begin{array}{c|cc|c} L_0 & L_1 & 0 & 0 \\ \hline 0 & T_0 & T_1 & 0 \\ \hline 0 & 0 & R_0 & R_1 \end{array}\right)$$

is an endpoint modification for

$$\tilde{T}_3 = \begin{pmatrix} T_0 & T_1 & 0 \\ 0 & T_0 & T_1 \\ T_1 & 0 & T_0 \end{pmatrix},$$

then L_0, L_1, R_0, *and* R_1 *must have sizes* $2\rho_1 \times \rho_1$, $2\rho_1 \times 2r$, $2\rho_0 \times 2r$, *and* $2\rho_0 \times \rho_0$, *respectively. Moreover, if*

$$\hat{T}_3 = \left(\begin{array}{c|cc|c} \hat{L}_0 & \hat{L}_1 & 0 & 0 \\ \hline 0 & T_0 & T_1 & 0 \\ \hline 0 & 0 & \hat{R}_0 & \hat{R}_1 \end{array}\right)$$

is another endpoint modification for $\tilde{T}_3$, *then*

$$\hat{T}_3 = \left(\begin{array}{c|c|c} U_L & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & U_R \end{array}\right) T_3$$

for some orthogonal U_L, U_R.

Proof. Assume that

$$T_3 = \left(\begin{array}{c|cc|c} L_0 & L_1 & 0 & 0 \\ \hline 0 & T_0 & T_1 & 0 \\ \hline 0 & 0 & R_0 & R_1 \end{array}\right) = \begin{pmatrix} T_l \\ T_i \\ T_r \end{pmatrix}$$

is an endpoint modification for $\tilde{T}_3$, where

$$T_l = \left(\begin{array}{c|cc|c} L_0 & L_1 & 0 & 0 \end{array}\right),$$

$$T_i = \left(\begin{array}{c|cc|c} 0 & T_0 & T_1 & 0 \end{array}\right),$$

$$T_r = \left(\begin{array}{c|cc|c} 0 & 0 & R_0 & R_1 \end{array}\right).$$

Assume also that L_0 and R_1 have sizes $2r_1 \times r_1$ and $2r_0 \times r_0$ respectively.

Since $T_i T_i^* = I_{2r}$, $\mathrm{rank}(T_i) = 2r$. The aim now is to find $4r$ linearly independent rows that are also orthogonal to the T_i. To do this, we follow the argument used in [25] for filter bank completions. Consider first

$$T_i^* T_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & T_0^* T_0 & T_0^* T_1 & 0 \\ 0 & T_1^* T_0 & T_1^* T_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & T_0^* T_0 & 0 & 0 \\ 0 & 0 & T_1^* T_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, let

$$P = I - T_i^* T_i = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I - T_0^* T_0 & 0 & 0 \\ 0 & 0 & I - T_1^* T_1 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

The fact that $T_0^* T_0 + T_1^* T_1 = I$, implies that

$$P = I - T_i^* T_i = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & T_1^* T_1 & 0 & 0 \\ 0 & 0 & T_0^* T_0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{4.3}$$

Multiplying P from left by T_i yields

$$T_i P = T_i - T_i T_i^* T_i = 0.$$

Hence, the columns of P are orthogonal to the rows of T_i. It is obvious from (4.3) that $\mathrm{rank}(P) = 4r$. That is exactly the number that we need to complete the matrix T_i into the T_3. For short, the rows of P will be used to go from T_i to T_3. The rows of T_l must be chosen from the first two block-rows of P. Similarly, the rows of T_r must be selected from the last two block-rows of P. Otherwise, it would be impossible to obtain T_3 with the desired structure.

Notice that $\mathrm{rank}(T_1) = \rho_1$ and $\mathrm{rank}(T_0) = \rho_0$ imply that there are $r_1 + \rho_1$ linearly independent rows in the first two block-rows of P and $r_0 + \rho_0$ linearly independent rows in the last two block-rows of P.

Consider the following cases:

If $r_1 > \rho_1$, there are not enough rows so that we can have $(L_0\,, L_1)$ with size $2r_1 \times (r_1 + 2r)$ and rank $2r_1$

If $r_1 < \rho_1$, then $r_0 > \rho_0$. This will in turn imply that there cannot be $(R_0\,, R_1)$ with size $2r_0 \times (r_0 + 2r)$ and rank $2r_0$.

It is clear that everything works out nicely if $r_1 = \rho_1$.

We now assume that $\hat{T}_3$ is another endpoint modification for $\tilde{T}_3$. It suffices to show that

$$\hat{T}_3 T_3^* = \left(\begin{array}{c|c|c} U_L & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & U_R \end{array} \right),$$

for some orthogonal U_L, U_R. Since $\hat{T}_3$ and T_3 are orthogonal, we have

$$T_3 T_3^* = T_3^* T_3 \quad = \quad I, \tag{4.4}$$

$$\hat{T}_3 \hat{T}_3^* = \hat{T}_3^* \hat{T}_3 \quad = \quad I, \tag{4.5}$$

where I is the identity matrix. From (4.4), we obtain the following set of identities:

$$\begin{array}{rl} L_0 L_0^* + L_1 L_1^* = I, & R_0^* R_1 = 0, \\ R_0 R_0^* + R_1 R_1^* = I, & T_0 L_1^* = 0, \\ T_0 T_0^* + T_1 T_1^* = I, & T_1 R_0^* = 0, \\ L_1^* L_1 + T_0^* T_0 = I, & L_0^* L_0 = I, \\ T_1^* T_1 + R_0^* R_0 = I, & L_0^* L_1 = 0, \\ R_1^* R_1 = I, & T_0^* T_1 = 0. \end{array}$$

Likewise from (4.5), we have

$$
\begin{aligned}
\hat{L}_0\hat{L}_0^* + \hat{L}_1\hat{L}_1^* &= I, & \hat{R}_0^*\hat{R}_1 &= 0,\\
\hat{R}_0\hat{R}_0^* + \hat{R}_1\hat{R}_1^* &= I, & T_0\hat{L}_1^* &= 0,\\
T_0T_0^* + T_1T_1^* &= I, & T_1\hat{R}_0^* &= 0,\\
\hat{L}_1^*\hat{L}_1 + T_0^*T_0 &= I, & \hat{L}_0^*\hat{L}_0 &= I,\\
T_1^*T_1 + \hat{R}_0^*\hat{R}_0 &= I, & \hat{L}_0^*\hat{L}_1 &= 0,\\
\hat{R}_1^*\hat{R}_1 &= I, & T_0^*T_1 &= 0.
\end{aligned}
$$

Consider

$$
\begin{aligned}
\hat{T}_3T_3^* &= \left(\begin{array}{c|cc|c} \hat{L}_0 & \hat{L}_1 & 0 & 0 \\ \hline 0 & T_0 & T_1 & 0 \\ \hline 0 & 0 & \hat{R}_0 & \hat{R}_1 \end{array}\right) \left(\begin{array}{c|c|c} L_0^* & 0 & 0 \\ \hline L_1^* & T_0^* & 0 \\ 0 & T_1^* & R_0^* \\ \hline 0 & 0 & R_1^* \end{array}\right) \\
&= \left(\begin{array}{c|c|c} \hat{L}_0L_0^* + \hat{L}_1L_1^* & \hat{L}_1T_0^* & 0 \\ \hline T_0L_1^* & T_0T_0^* + T_1T_1^* & T_1R_0^* \\ \hline 0 & \hat{R}_0T_1^* & \hat{R}_0R_0^* + \hat{R}_1R_1^* \end{array}\right).
\end{aligned} \tag{4.6}
$$

If we use some of the identities obtained above, (4.6) turns out to be of the following form:

$$
\left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & C \end{array}\right),
$$

where

$$
A = \hat{L}_0L_0^* + \hat{L}_1L_1^*, \quad \text{and} \quad C = \hat{R}_0R_0^* + \hat{R}_1R_1^*.
$$

It remains for us to show that the matrices A and C are orthogonal.

$$\begin{aligned}
AA^* &= (\hat{L}_0 L_0^* + \hat{L}_1 L_1^*)(\hat{L}_0 L_0^* + \hat{L}_1 L_1^*)^* \\
&= (\hat{L}_0 L_0^* + \hat{L}_1 L_1^*)(L_0 \hat{L}_0^* + L_1 \hat{L}_1^*) \\
&= \hat{L}_0 \underbrace{L_0^* L_0}_{I} \hat{L}_0^* + \hat{L}_0 \underbrace{L_0^* L_1}_{0} \hat{L}_1^* + \hat{L}_1 \underbrace{L_1^* L_0}_{0} \hat{L}_0^* + \hat{L}_1 L_1^* L_1 \hat{L}_1^* \\
&= \hat{L}_0 \hat{L}_0^* + \hat{L}_1 L_1^* L_1 \hat{L}_1^*.
\end{aligned}$$

We know that $L_1^* L_1 + T_0^* T_0 = I$. Multiplying this identity from the right by $\hat{L}_1$ and from the left by $\hat{L}_1^*$ and using the fact that $\hat{L}_1 T_0^* = 0$ yields

$$\hat{L}_1 L_1^* L_1 \hat{L}_1^* = \hat{L}_1 \hat{L}_1^*.$$

This implies that

$$AA^* = \hat{L}_0 \hat{L}_0^* + \hat{L}_1 \hat{L}_1^* = I.$$

Thus, A is an orthogonal matrix. That C is also an orthogonal matrix is proved in a similar manner. $\square$

Note:

The theorem indicates that the Madych construction is essentially unique, except for the choice of orthogonal matrices U_L, U_R. If we reconsider Madych's approach in this light when $L = r = \rho_0$, it turns out that the second set of columns of

$$U_3^* T_3 V_3 = \begin{array}{c}
\left(\begin{array}{cc||cc||cc}
I & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & I & 0 & 0 \\
\hline\hline
0 & 0 & I & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & I \\
\hline\hline
0 & 0 & 0 & 0 & I & 0 \\
\hline
0 & I & 0 & 0 & 0 & 0
\end{array}\right) \\
\begin{array}{cccccc}
\rho_0 & \rho_1 & \rho_0 & \rho_1 & \rho_0 & \rho_1
\end{array}
\end{array}$$

is moved to the end so that we have

$$\left(\begin{array}{c||cc||cc||c}
I & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & I & 0 & 0 & 0 \\ \hline\hline
0 & I & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & I & 0 \\ \hline\hline
0 & 0 & 0 & I & 0 & 0 \\ \hline
\underbrace{0}_{\rho_0} & \underbrace{0}_{\rho_0} & \underbrace{0}_{\rho_1} & \underbrace{0}_{\rho_0} & \underbrace{0}_{\rho_1} & \underbrace{I}_{\rho_1}
\end{array}\right).$$

This only yields matrices of the correct size if $\rho_0 = \rho_1$ which is equivalent to Condition B from definition 4.1.

In order to have useful boundary coefficients in our completion approach, we need to reduce the arbitrariness in the choice of orthogonal matrices Q_L and Q_R. To this purpose, we devote the next chapter to help elucidate the understanding of the two approaches with some particular wavelets.

CHAPTER 5. Elaborated Examples

5.1 Introduction

This chapter involves application of our results to some specific classic wavelets. The boundary functions for the scalar wavelet D_4 and the multiwavelets CL2, DGHM, and CL3 are constructed. Their properties are analyzed as well.

5.2 D_4 Scalar Wavelet

Example 5.1. Consider the orthogonal Daubechies D_4 scalar wavelet with its recursion coefficients

$$h_0 = \tfrac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 = \tfrac{3+\sqrt{3}}{4\sqrt{2}}, \quad h_2 = \tfrac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 = \tfrac{1-\sqrt{3}}{4\sqrt{2}},$$

$$g_0 = \tfrac{1-\sqrt{3}}{4\sqrt{2}}, \quad g_1 = \tfrac{\sqrt{3}-3}{4\sqrt{2}}, \quad g_2 = \tfrac{3+\sqrt{3}}{4\sqrt{2}}, \quad g_3 = \tfrac{\sqrt{3}-1}{4\sqrt{2}}.$$

Boundary Set-up

The scaling function $\varphi(x)$ has support length 3, so $N = 3$. We know from (3.1) that

$$L + R = (N-1)r = 2.$$

For orthogonal scalar wavelets with an even number of recursion coefficients, we can always take the same number of boundary functions at each end. We therefore take a single boundary function at each end and analyze the left boundary function, whose support is assumed to be the interval $[0\,,2]$. We also note that

$$\rho_0 = \mathrm{rank}(T_0) = 1, \text{ and } \ \rho_1 = \mathrm{rank}(T_1) = 1,$$

where

$$T_0 = \begin{pmatrix} h_0 & h_1 \\ g_0 & g_1 \end{pmatrix}, \text{ and } T_1 = \begin{pmatrix} h_2 & h_3 \\ g_2 & g_3 \end{pmatrix}.$$

Recursion Relation:

$$\phi^L(x) = \sqrt{2}a\phi^L(2x) + \sqrt{2}b_0\phi(2x) + \sqrt{2}b_1\phi(2x-1).$$

Assuming that the left boundary function is the linear combination of boundary-crossing functions will produce the following relation.

Boundary Crossing:

$$\phi^L(x) = c_{-2}\phi(x+2) + c_{-1}\phi(x+1).$$

As we pointed out earlier, the relation between these two representations can be obtained by solving the following identities:

$$\begin{aligned} CY &= AC, \\ CZ &= B, \end{aligned}$$

where

$$C = \mathbf{c}^* = (c_{-2}\,, c_{-1}),\ B = \mathbf{b}^* = (b_0\,, b_1),\ A = a,\ Y = \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix}, \text{and } Z = \begin{pmatrix} 0 & 0 \\ h_2 & h_3 \end{pmatrix}.$$

There are two solutions.

The first solution, normalized to $\| \varphi^L \|_2 = 1$, is

$$a = \frac{\sqrt{2}}{2},\ \mathbf{c}^* = (1+\sqrt{3}\,, 1+\sqrt{3}),\ \mathbf{b}^* = (\frac{\sqrt{6}}{4}\,, -\frac{\sqrt{2}}{4}).$$

Since

$$\varphi(1) = \frac{1+\sqrt{3}}{2},\quad \varphi(2) = \frac{1-\sqrt{3}}{2},$$

it follows from (3.40) and (3.39) that

$$\varphi^L(0) = \sqrt{3}+1,\quad \varphi^L(1) = -1.$$

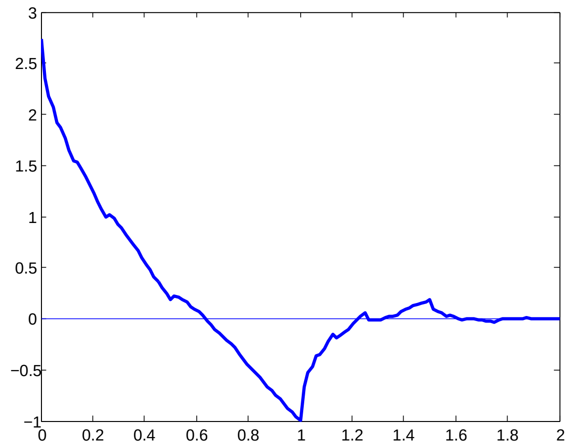

Figure 5.1 Daubechies D_4 first solution

From theorem 3.3, we know that this is the only regular solution.

Approximation Order

Based on Boundary Crossing:

Since there is only one boundary function,

$$C = (\boldsymbol{\mu}_0^*, \boldsymbol{\mu}_0^*)$$

is always a solution with approximation order 1.

Based on Recursion Relation:

We need to show that there is a constant d so that

$$\begin{aligned} \mu_0 h_2 &= db_0, \\ \mu_0 h_3 &= db_1. \end{aligned}$$

It is easy to see that

$$d = \frac{\sqrt{3}-1}{2}$$

satisfies these equations. We now investigate the continuity of ϕ^L and obtain an explicit formula for it.

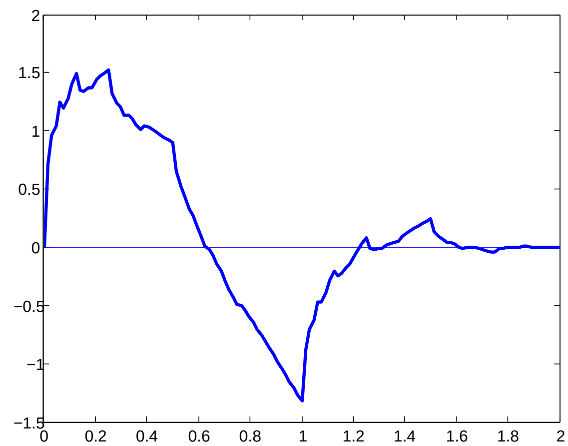

Figure 5.2 Daubechies D_4 second solution

Continuity

By iterating ϕ^L and using the continuity of interior functions, we know that ϕ^L is continuous over the interval $(0, 2]$. In order for ϕ^L to be continuous at $x = 0$, the following identity, which is proved in Chapter 3, must hold.

$$\frac{b_0}{b_1} = \frac{h_2}{h_3}.$$

This is true for $\mathbf{b}^* = (\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4})$. Thus, ϕ^L is continuous at $x = 0$. Using (3.32), we now have an explicit formula for $\phi^L(x)$, which is

$$\phi^L(x) = \begin{cases} (1+\sqrt{3})(1-\phi(x)), & x \in [0,1]; \\ \frac{\sqrt{3}}{2}\phi(2x) - \frac{1}{2}\phi(2x-1), & x \in [1,2]. \end{cases}$$

The second solution, normalized to $\| \varphi^L \|_2 = 1$, is

$$a = \frac{\sqrt{2}}{4},\ \mathbf{c}^* = (\frac{\sqrt{7}(5+\sqrt{3})}{2}, \frac{\sqrt{7}(1+\sqrt{3})}{2}),\ \mathbf{b}^* = (\frac{\sqrt{42}}{8}, -\frac{\sqrt{14}}{8}).$$

Since $a < \frac{\sqrt{2}}{2}$, $\varphi^L(0) = 0$. The solution is therefore not useful.

In chapter 4, we introduced an approach for finding boundary functions. We have also shown that the construction is unique up to the choices of orthogonal matrices Q_L and Q_R. For not every Q_L and Q_R produce a useful set of coefficients, it is therefore of interest to find a way to decrease the degree of freedom in Q_L and Q_R so that the construction results in an effective and useful set of coefficients. Following the construction introduced in Chapter 4, we have the following matrix.

$$\left(\begin{array}{c|cc|cc|c} q_1 & -\frac{\sqrt{3}}{2}q_2 & \frac{1}{2}q_2 & 0 & 0 & 0 \\ q_3 & -\frac{\sqrt{3}}{2}q_4 & \frac{1}{2}q_4 & 0 & 0 & 0 \\ \hline 0 & h_0 & h_1 & h_2 & h_3 & 0 \\ 0 & g_0 & g_1 & g_2 & g_3 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2}q_5 & -\frac{\sqrt{3}}{2}q_5 & q_6 \\ 0 & 0 & 0 & -\frac{1}{2}q_7 & -\frac{\sqrt{3}}{2}q_7 & q_8 \end{array}\right), \tag{5.1}$$

where $q_i's$ are the entries of

$$Q_L = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}, \text{ and } Q_R = \begin{pmatrix} q_5 & q_6 \\ q_7 & q_8 \end{pmatrix}.$$

Because the first row of (5.1) generates ϕ^L and we already have a useful set of coefficients from the first approach, we are now going to restrict Q_L in such a way that this approach too produces a useful set of coefficients, indeed the same set of coefficients. To be more precise, we want that

$$q1 = a, \quad (-\frac{\sqrt{3}}{2}q_2\,, \frac{1}{2}q_2) = \mathbf{b}^*,$$

where $a = \frac{\sqrt{2}}{2}, \mathbf{b}^* = (\frac{\sqrt{6}}{4}\,, -\frac{\sqrt{2}}{4})$. This determines the first row of Q_L. Since Q_L is an orthogonal matrix, then it is either

$$Q_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \text{ or } \; Q_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

5.3 CL(2) Multiwavelet

Example 5.2. Consider the standard orthogonal Chui-Lian $CL(2)$ multiwavelet with its recursion coefficients

$$H_0 = \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & 1 \\ -\frac{\sqrt{7}}{2} & -\frac{\sqrt{7}}{2} \end{pmatrix}, \; H_1 = \frac{\sqrt{2}}{4}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; H_2 = \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & -1 \\ \frac{\sqrt{7}}{2} & -\frac{\sqrt{7}}{2} \end{pmatrix},$$

$$G_0 = \frac{\sqrt{2}}{4}\begin{pmatrix} -1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; G_1 = \frac{\sqrt{2}}{4}\begin{pmatrix} 2 & 0 \\ 0 & \sqrt{7} \end{pmatrix}, \; G_2 = \frac{\sqrt{2}}{4}\begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Boundary Set-up

In this case

$$\rho_0 = \mathrm{rank}(T_0) = 3, \text{ and } \; \rho_1 = \mathrm{rank}(T_1) = 1.$$

Thus, we have only a single boundary function at the left end.

Recursion Relation:

$$\varphi^L(x) = \sqrt{2}a\varphi^L(2x) + \sqrt{2}B_0\boldsymbol{\varphi}(2x).$$

If the boundary function is a combination of boundary crossing functions, we have

Boundary Crossing:

$$\varphi^L(x) = C\boldsymbol{\varphi}(x+1).$$

The following identities

$$\begin{aligned} CY &= AC, \\ CZ &= B, \end{aligned}$$

where

$$C = \mathbf{c}^* = (c_{-2}\ c_{-1}),\ B_0 = \mathbf{b}^* = (b_0\ b_1),\ A = a,\ Y = H_1,\ Z = H_2,$$

provide solutions.

The first solution, normalized to $\| \varphi^L \|_2 = 1$, is

$$a = \frac{\sqrt{2}}{2},\ \mathbf{c}^* = [\sqrt{2}\,, 0],\ \mathbf{b}^* = [\frac{1}{2}\,, -\frac{1}{2}].$$

Since

$$\boldsymbol{\varphi}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\varphi}(2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

it follows from (3.40) and (3.39) that

$$\varphi^L(0) = \sqrt{2}, \quad \varphi^L(1) = 0.$$

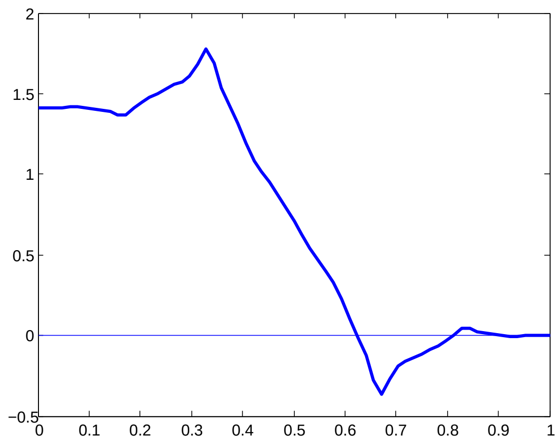

Figure 5.3 CL2 first solution

From theorem 3.3, we know that this is the only regular solution.

Approximation Order

Based on Boundary Crossing:

Since there is only one boundary function,

$$C = (\boldsymbol{\mu}_0^*) = (1\,,0)$$

is always a solution with approximation order 1.

Based on Recursion Relation:

We need to show that there is a constant d so that

$$\mu_0 H_2 \quad = \quad d\mathbf{b}^*.$$

It is easy to verify that

$$d = \frac{\sqrt{2}}{2}$$

satisfies this equation.

We now investigate the continuity of ϕ^L and obtain an explicit formula for it.

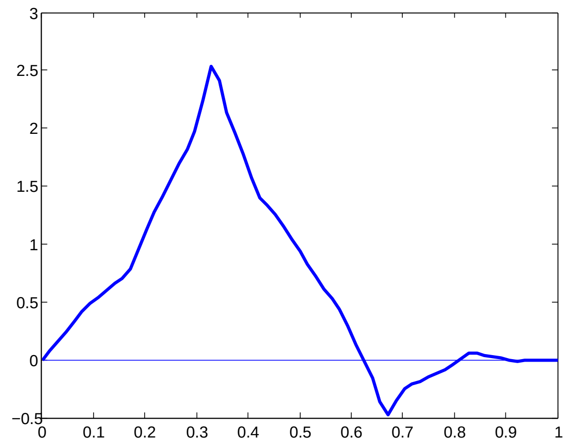

Figure 5.4 CL2 second solution

Continuity

By iterating $\boldsymbol{\varphi}^L$ and using the continuity of interior functions, $\boldsymbol{\varphi}^L$ is continuous over the interval $(0,1]$. $\boldsymbol{\varphi}^L$ is continuous at $x=0$ if and only if the right hand side of the following identity is constant for every $x_0 \in (0,1]$. We need to show that

$$\varphi^L(x_0) + (\sqrt{2}\mathbf{b}^*)(I - \sqrt{2}H_0)^{-1}\boldsymbol{\varphi}(x_0)$$

is independent of $x_0 \in (0,1]$. This amounts to

$$(\sqrt{2}\,,0)\boldsymbol{\varphi}(x+1) + \sqrt{2}(\frac{1}{2}\,,-\frac{1}{2})(\frac{1}{2+\sqrt{7}})\begin{pmatrix} 4+\sqrt{7} & 2 \\ -\sqrt{7} & 2 \end{pmatrix}\boldsymbol{\varphi}(x),$$

which is equivalent to

$$(\sqrt{2}\,,0)[\boldsymbol{\varphi}(x+1) + \boldsymbol{\varphi}(x)] = \sqrt{2}.$$

The second solution, normalized to $\| \varphi^L \|_2 = 1$, is

$$a = \frac{\sqrt{2}}{4},\ \mathbf{c}^* = (0\,,\sqrt{2}),\ \mathbf{b}^* = (\frac{\sqrt{7}}{4}\,,-\frac{\sqrt{7}}{4}).$$

Since $a < \frac{\sqrt{2}}{2}$, $\varphi^L(0) = 0$. This solution is not of interest.

We now use the second approach to find the coefficients for boundary functions. Following the steps explained in chapter 4 for completing the matrix, the left boundary functions must have the following coefficients.

$$\left(\begin{array}{c|cc} q_1 & -\frac{\sqrt{2}}{2}q_2 & \frac{\sqrt{2}}{2}q_2 \\ q_3 & -\frac{\sqrt{2}}{2}q_4 & \frac{\sqrt{2}}{2}q_4 \end{array}\right), \tag{5.2}$$

where $q_i's$ are the entries of

$$Q_L = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}.$$

The first row of (5.2) forms ϕ^L and there is already a set of useful coefficients obtained in the first approach, so we want to obtain the same set with this construction. That means in (5.2) we want that $q_1 = \frac{\sqrt{2}}{2}$ and that $q_2 = -\frac{\sqrt{2}}{2}$. Therefore, Q_L must be either

$$Q_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \text{ or } \; Q_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

5.4 DGHM Multiwavelet

Example 5.3. Consider the orthogonal DGHM multiwavelet with its recursion coefficients

$$H_0 = \begin{pmatrix} \frac{3\sqrt{2}}{10} & 4/5 \\ \frac{-1}{20} & \frac{-3\sqrt{2}}{20} \end{pmatrix}, \; H_1 = \begin{pmatrix} \frac{3\sqrt{2}}{10} & 0 \\ \frac{9}{20} & \frac{\sqrt{2}}{2} \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 \\ \frac{9}{20} & \frac{-3\sqrt{2}}{20} \end{pmatrix}, \; H_3 = \begin{pmatrix} 0 & 0 \\ \frac{-1}{20} & 0 \end{pmatrix},$$

$$G_0 = \begin{pmatrix} \frac{-1}{20} & \frac{-3\sqrt{2}}{20} \\ \frac{\sqrt{2}}{20} & \frac{3}{10} \end{pmatrix}, \; G_1 = \begin{pmatrix} \frac{9}{20} & \frac{-\sqrt{2}}{2} \\ \frac{-9\sqrt{2}}{20} & 0 \end{pmatrix}, G_2 = \begin{pmatrix} \frac{9}{20} & \frac{-3\sqrt{2}}{20} \\ \frac{9\sqrt{2}}{20} & \frac{-3}{10} \end{pmatrix}, \; G_3 = \begin{pmatrix} \frac{-1}{20} & 0 \\ \frac{-\sqrt{2}}{20} & 0 \end{pmatrix}.$$

Boundary Set-up

In this case, we have

$$\rho_0 = \text{rank}(T_0) = 3, \text{ and } \; \rho_1 = \text{rank}(T_1) = 1.$$

Therefore, there is only a single boundary function at the left end.

Recursion Relation:

$$\varphi^L(x) = \sqrt{2}a\varphi^L(2x) + \sqrt{2}\mathbf{b}_0^*\boldsymbol{\varphi}(2x) + \sqrt{2}\mathbf{b}_1^*\boldsymbol{\varphi}(2x-1)$$

If the boundary function is a combination of boundary crossing functions, we have

Boundary Crossing:

$$\varphi^L(x) = \mathbf{c}_{-2}^*\boldsymbol{\varphi}(x+2) + \mathbf{c}_{-1}^*\boldsymbol{\varphi}(x+1)$$

The following identities

$$\begin{aligned} CY &= AC, \\ CZ &= B, \end{aligned}$$

where

$$C = \mathbf{c}^* = (\mathbf{c}_{-2}^*\ \mathbf{c}_{-1}^*),\ B = \mathbf{b}^* = (\mathbf{b}_0^*\,,\mathbf{b}_1^*),\ A = a,$$

provide four solutions.

The first solution, normalized to $\parallel \varphi^L \parallel_2 = 1$, is

$$a = \frac{\sqrt{2}}{2},\ \mathbf{c}^* = [2\,,\sqrt{2}\,,2\,,\sqrt{2}],\ \mathbf{b}^* = [\frac{9\sqrt{2}}{20}\,,-\frac{6}{20}\,,-\frac{\sqrt{2}}{20}\,,0].$$

Since

$$\boldsymbol{\varphi}(1) = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix},\quad \boldsymbol{\varphi}(2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

it follows from (3.40) and (3.39) that

$$\varphi^L(0) = \sqrt{6},\quad \varphi^L(1) = 0.$$

From theorem 3.3, we know that this is the only regular solution.

Approximation Order

Based on Boundary Crossing:

There is only one boundary function, so

$$C = (\boldsymbol{\mu}_0^*, \boldsymbol{\mu}_0^*) = (\frac{\sqrt{6}}{3}\,,\frac{\sqrt{3}}{3}\,,\frac{\sqrt{6}}{3}\,,\frac{\sqrt{3}}{3})$$

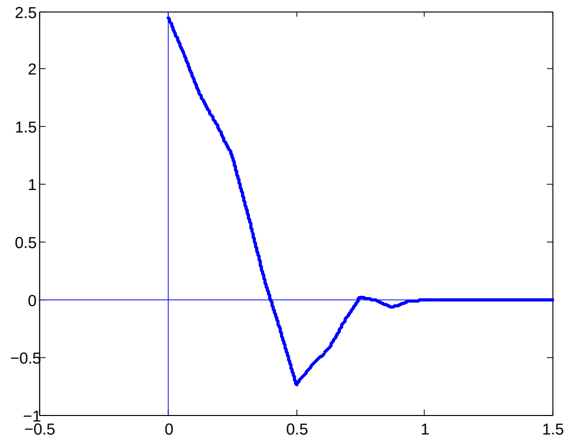

Figure 5.5 DGHM first solution

is always a solution with approximation order 1.

Based on Recursion Relation:

We need to show that there is a constant d so that

$$\begin{aligned} \mu_0 H_2 &= d\mathbf{b}_0^* \\ \mu_0 H_3 &= d\mathbf{b}_1^* \end{aligned}$$

It is easy to show that

$$d = \frac{\sqrt{6}}{6}$$

satisfies these equations.

We now investigate the continuity of ϕ^L and obtain an explicit formula for it.

Continuity

By iterating $\boldsymbol{\varphi}^L$ and using the continuity of interior functions, $\boldsymbol{\varphi}^L$ is continuous over the interval $(0, 2]$. $\boldsymbol{\varphi}^L$ is continuous at $x = 0$ if and only if the right hand side of following identity is constant for every $x_0 \in (0, 1]$. That is, we need to verify that

$$\varphi^L(x_0) + (\sqrt{2}\mathbf{b}_0^*)(I - \sqrt{2}H_0)^{-1}\boldsymbol{\varphi}(x_0)$$

is independent of $x_0 \in (0\,,1]$. This is equivalent to

$$[2\,,\sqrt{2}]\boldsymbol{\varphi}(x+2)+[2\,,\sqrt{2}]\boldsymbol{\varphi}(x+1)+\sqrt{2}[\frac{9\sqrt{2}}{20}\,,-\frac{6}{20}]\begin{pmatrix} \frac{13}{6} & \frac{4\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{12} & \frac{2}{3} \end{pmatrix}\boldsymbol{\varphi}(x+1).$$

After some algebra, this simplifies to

$$\sqrt{6}[\frac{\sqrt{6}}{3}\,,\frac{\sqrt{3}}{3}][\boldsymbol{\varphi}(x+2)+\boldsymbol{\varphi}(x+1)+\boldsymbol{\varphi}(x)]=\sqrt{6}.$$

The other solutions are

- $a=-\frac{\sqrt{2}}{10}\,,\mathbf{b}^*=(0\,,0\,,0\,,0),\mathbf{c}^*=(-39\,,8\sqrt{2}\,,1\,,0)$,
- $a=0\,,\mathbf{b}^*=(0\,,0\,,0\,,0),\mathbf{c}^*=(1\,,0\,,0\,,0)$,
- $a=\frac{\sqrt{2}}{4}\,,\mathbf{b}^*=(0\,,0\,,0\,,0),\mathbf{c}^*=(1\,,\frac{\sqrt{2}}{3}\,,\frac{1}{3}\,,0)$.

These three solutions are not useful since they all lead to functions which are identically zero.

From the completion approach, the left boundary function is obtained from the first row of the following matrix.

$$\left(\begin{array}{c|cccc} q_1 & -\frac{9}{10}q_2 & \frac{3\sqrt{2}}{10}q_2 & \frac{1}{10}q_2 & 0 \\ q_3 & -\frac{9}{10}q_4 & \frac{3\sqrt{2}}{10}q_4 & \frac{1}{10}q_4 & 0 \end{array}\right), \tag{5.3}$$

where $q_i's$ are the entries of

$$Q_L=\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}.$$

We want

$$q_1=a=\frac{1}{\sqrt{2}}\,,\quad (-\frac{9}{10}q_2\,,\frac{3\sqrt{2}}{10}q_2\,,\frac{1}{10}q_2\,,0)=\mathbf{b}^*.$$

This is only possible when Q_L is either

$$Q_L=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\text{ or }\;Q_L=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

5.5 CL(3) Multiwavelet

Example 5.4. Consider the standard orthogonal Chui-Lian $CL(3)$ multiwavelet with its recursion coefficients

$$H_0 = \frac{\sqrt{2}}{80}\begin{pmatrix} 10-3\sqrt{10} & 5\sqrt{6}-2\sqrt{15} \\ 5\sqrt{6}-3\sqrt{15} & 5-3\sqrt{10} \end{pmatrix}, \; H_1 = \frac{\sqrt{2}}{80}\begin{pmatrix} 30+3\sqrt{10} & 5\sqrt{6}-2\sqrt{15} \\ -5\sqrt{6}-7\sqrt{15} & 15-3\sqrt{10} \end{pmatrix},$$

$$H_2 = \frac{\sqrt{2}}{80}\begin{pmatrix} 30+3\sqrt{10} & -5\sqrt{6}+2\sqrt{15} \\ 5\sqrt{6}+7\sqrt{15} & 15-3\sqrt{10} \end{pmatrix}, \; H_3 = \frac{\sqrt{2}}{80}\begin{pmatrix} 10-3\sqrt{10} & -5\sqrt{6}+2\sqrt{15} \\ -5\sqrt{6}+3\sqrt{15} & 5-3\sqrt{10} \end{pmatrix},$$

$$G_0 = \frac{\sqrt{2}}{80}\begin{pmatrix} 5\sqrt{6}-2\sqrt{15} & -10+3\sqrt{10} \\ 5+3\sqrt{10} & 5\sqrt{6}-3\sqrt{15} \end{pmatrix}, \; G_1 = \frac{\sqrt{2}}{80}\begin{pmatrix} 5\sqrt{6}+2\sqrt{15} & 30+3\sqrt{10} \\ 15-3\sqrt{10} & 5\sqrt{6}+7\sqrt{15} \end{pmatrix},$$

$$G_2 = \frac{\sqrt{2}}{80}\begin{pmatrix} -5\sqrt{6}+2\sqrt{15} & -30-3\sqrt{10} \\ -15+3\sqrt{10} & 5\sqrt{6}+7\sqrt{15} \end{pmatrix}, \; G_3 = \frac{\sqrt{2}}{80}\begin{pmatrix} 5\sqrt{6}-2\sqrt{15} & 10-3\sqrt{10} \\ 5-3\sqrt{10} & 5\sqrt{6}-3\sqrt{15} \end{pmatrix}.$$

Boundary Set-up

Since

$$\rho_0 = \mathrm{rank}(T_0) = 2, \text{ and } \; \rho_1 = \mathrm{rank}(T_1) = 2,$$

there is a vector of two boundary functions at the left end.

Recursion Relation:

$$\boldsymbol{\varphi}^L(x) = \sqrt{2}A\boldsymbol{\varphi}^L(2x) + \sqrt{2}B_0\boldsymbol{\varphi}(2x) + \sqrt{2}B_1\boldsymbol{\varphi}(2x-1)$$

If the boundary function is a combination of boundary crossing functions, we have

Boundary Crossing:

$$\boldsymbol{\varphi}^L(x) = C_{-2}\boldsymbol{\varphi}(x+2) + C_{-1}\boldsymbol{\varphi}(x+1)$$

The task is again to find a set of useful coefficients to form the $\boldsymbol{\varphi}^L$ by solving the following equations.

$$\begin{aligned} CY &= AC \\ CZ &= B, \end{aligned}$$

where

$$C = (C_{-2}\ C_{-1}),\ B = (B_0\ B_1),\ T = \begin{pmatrix} H_2 & H_3 \\ H_0 & H_1 \end{pmatrix},\ W = \begin{pmatrix} 0_2 & 0_2 \\ H_2 & H_3 \end{pmatrix}$$

Notice that $CY = AC$ implies that the rows of C must be linear combinations of left-eigenvectors of Y. The left eigenvectors with corresponding eigenvalues of Y are as follows:

$$\begin{aligned} \lambda_1 &= \frac{\sqrt{2}}{2}, \quad v_1^* = (1\,,0\,,1\,,0), \\ \lambda_2 &= \frac{\sqrt{2}}{8}, \quad v_2^* = (\sqrt{5}\,,-\sqrt{3}\,,\sqrt{5}\,,\sqrt{3}), \\ \lambda_3 &= \frac{\sqrt{2}}{4}, \quad v_3^* = (\sqrt{10}-10\,,4\sqrt{15}-5\sqrt{6}\,,10-\sqrt{10}\,,4\sqrt{15}-5\sqrt{6}) \\ & \qquad\quad\ v_4^* = (35\sqrt{6}-16\sqrt{15}\,,90-36\sqrt{10}\,,-35\sqrt{6}+16\sqrt{15}\,,90-36\sqrt{10}) \end{aligned}$$

v_4^* is a generalized eigenvector to the eigenvalue λ_3. Therefore, there are four solutions instead of six.

To find the first solution, we begin with

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{8} \end{pmatrix}, \quad C = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix}.$$

This is a valid solution but it is not orthogonal. We need to find a suitable matrix M so that $M\boldsymbol{\varphi}^L(x)$ is orthogonal. This leads to

$$\begin{aligned} A &\rightarrow MAM^{-1}, \\ B &\rightarrow MB, \\ C &\rightarrow MC. \end{aligned}$$

If we use the following transformation matrix

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{\sqrt{15}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix},$$

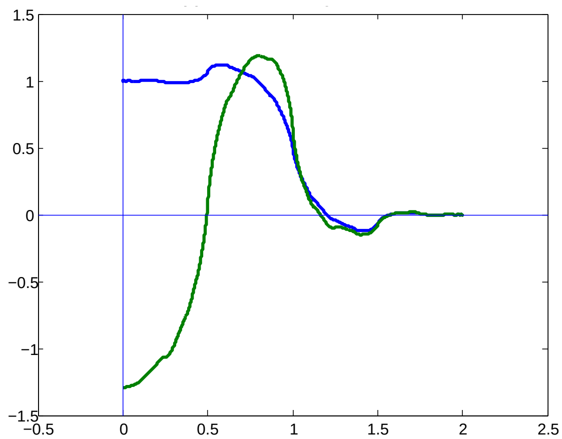

Figure 5.6 CL3 first solution

then the matrix (MAM^{-1}, MB) has orthonormal rows. Therefore, the first solution is

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{30}}{8} & \frac{\sqrt{2}}{8} \end{pmatrix},$$

$$B = \frac{\sqrt{2}}{80} \begin{pmatrix} 30+3\sqrt{10} & 2\sqrt{15}-5\sqrt{6} & 10-3\sqrt{10} & 2\sqrt{15}-5\sqrt{6} \\ 5\sqrt{6}+7\sqrt{15} & 15-3\sqrt{10} & 3\sqrt{15}-5\sqrt{6} & 5-3\sqrt{10} \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Since

$$\boldsymbol{\varphi}(1) = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{15}}{6} \end{pmatrix}, \quad \boldsymbol{\varphi}(2) = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{15}}{6} \end{pmatrix},$$

we have

$$\boldsymbol{\varphi}^L(0) = \begin{pmatrix} 1 \\ -\frac{\sqrt{15}}{3} \end{pmatrix}, \quad \boldsymbol{\varphi}^L(1) = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{15}}{6} \end{pmatrix}.$$

This solution is regular.

Approximation Order

Based on Boundary Crossing:

$$\boldsymbol{\mu}_0^* = (1\,,0),\ \mathbf{d}^* C = (\boldsymbol{\mu}_0^*\,,\boldsymbol{\mu}_0^*)$$

Based on Recursion Relation:

$$\begin{aligned} \mathbf{d}^*(\sqrt{2}A) &= \mathbf{d}^* \\ \boldsymbol{\mu}_0^* H_2 &= \mathbf{d}^* B_0 \\ \boldsymbol{\mu}_0^* H_3 &= \mathbf{d}^* B_1 \end{aligned}$$

It is easy to verify that

$$\mathbf{d} = \boldsymbol{\mu}_0$$

satisfies these equations.

To find the second solution, we start out with

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad C = \begin{pmatrix} v_1^* \\ v_3^* \end{pmatrix}.$$

If we use the following transformation matrix

$$M = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{4390-1240\sqrt{10}}(100-19\sqrt{10})}{-4390+1240\sqrt{10}} & \frac{3\sqrt{4390-1240\sqrt{10}}}{2195-620\sqrt{10}} \end{pmatrix},$$

then the matrix $(MAM^{-1}\,, MB)$ has orthonormal rows. Thus, the second solution is

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{4390-1240\sqrt{10}}(100\sqrt{2}-38\sqrt{5})}{-17560+4960\sqrt{10}} & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & B_1 \end{pmatrix}, \text{and}\, C = \begin{pmatrix} C_0 & C_1 \end{pmatrix},$$

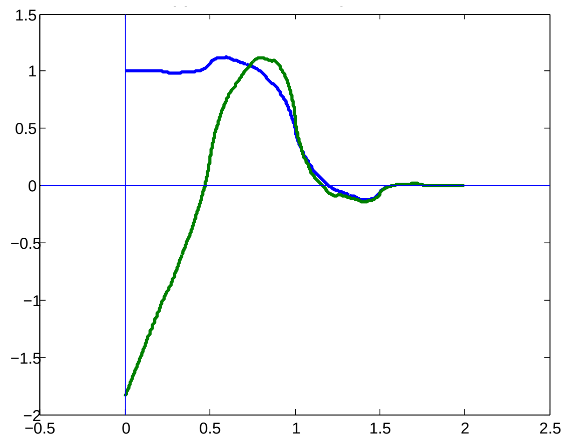

Figure 5.7 CL3 second solution

where

$$B_0 = \frac{\sqrt{2}}{80}\begin{pmatrix} 30+3\sqrt{10} & 2\sqrt{15}-5\sqrt{6} \\ \frac{81\sqrt{4390-1240\sqrt{10}}}{439-124\sqrt{10}} & \frac{(48\sqrt{6}-33\sqrt{15})\sqrt{4390-1240\sqrt{10}}}{-439+124\sqrt{10}} \end{pmatrix},$$

$$B_1 = \frac{\sqrt{2}}{80}\begin{pmatrix} 10-3\sqrt{10} & 2\sqrt{15}-5\sqrt{6} \\ \frac{(-119+38\sqrt{10})\sqrt{4390-1240\sqrt{10}}}{\sqrt{-439+124\sqrt{10}}} & \frac{(18\sqrt{6}-9\sqrt{15})\sqrt{4390-1240\sqrt{10}}}{-439+124\sqrt{10}} \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 1 & 0 \\ \frac{(32-5\sqrt{10})\sqrt{4390-1240\sqrt{10}}}{-878+248\sqrt{10}} & \frac{(15\sqrt{6}-12\sqrt{15})\sqrt{4390-1240\sqrt{10}}}{-2195+620\sqrt{10}} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 \\ \frac{(40-13\sqrt{10})\sqrt{4390-1240\sqrt{10}}}{-4390+1240\sqrt{10}} & \frac{(15\sqrt{6}-12\sqrt{15})\sqrt{4390-1240\sqrt{10}}}{-2195+620\sqrt{10}} \end{pmatrix}.$$

Since

$$\boldsymbol{\varphi}(1) = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{15}}{6} \end{pmatrix}, \quad \boldsymbol{\varphi}(2) = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{15}}{6} \end{pmatrix},$$

we have

$$\boldsymbol{\varphi}^L(0) = \begin{pmatrix} 1 \\ -\frac{(2260+451\sqrt{10})\sqrt{4390-1240\sqrt{10}}}{43290} \end{pmatrix}, \quad \boldsymbol{\varphi}^L(1) = \begin{pmatrix} \frac{1}{2} \\ \frac{(-80+17\sqrt{10})\sqrt{4390-1240\sqrt{10}}}{-8780+2480\sqrt{10}} \end{pmatrix}.$$

This solution is also regular.

Approximation Order

Based on Boundary Crossing:

$$\boldsymbol{\mu}_0^* = (1\,,0),\ \mathbf{d}^* C = (\boldsymbol{\mu}_0^*\,,\boldsymbol{\mu}_0^*)$$

Based on Recursion Relation:

$$\begin{aligned} \mathbf{d}^*(\sqrt{2}A) &= \mathbf{d}^* \\ \boldsymbol{\mu}_0^* H_2 &= \mathbf{d}^* B_0 \\ \boldsymbol{\mu}_0^* H_3 &= \mathbf{d}^* B_1 \end{aligned}$$

It is also easy to show that

$$\mathbf{d} = \boldsymbol{\mu}_0$$

satisfies these equations.

To find the third solution, we begin with

$$A = \begin{pmatrix} \frac{\sqrt{2}}{8} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad C = \begin{pmatrix} v_2^* \\ v_3^* \end{pmatrix}.$$

If we use the following transformation matrix

$$M = \begin{pmatrix} \frac{\sqrt{2}}{4} & 0 \\ \frac{(89\sqrt{5}-85\sqrt{2})\sqrt{4390-1240\sqrt{10}}}{-8780+2480\sqrt{10}} & \frac{3\sqrt{4390-1240\sqrt{10}}}{878-248\sqrt{10}} \end{pmatrix},$$

then the matrix $(MAM^{-1}\,, MB)$ has orthonormal rows. The third solution is

$$A = \begin{pmatrix} \frac{\sqrt{2}}{8} & 0 \\ \frac{(89\sqrt{5}-85\sqrt{2})\sqrt{4390-1240\sqrt{10}}}{17560-4960\sqrt{10}} & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & B_1 \end{pmatrix}, \text{and} \quad C = \begin{pmatrix} C_0 & C_1 \end{pmatrix},$$

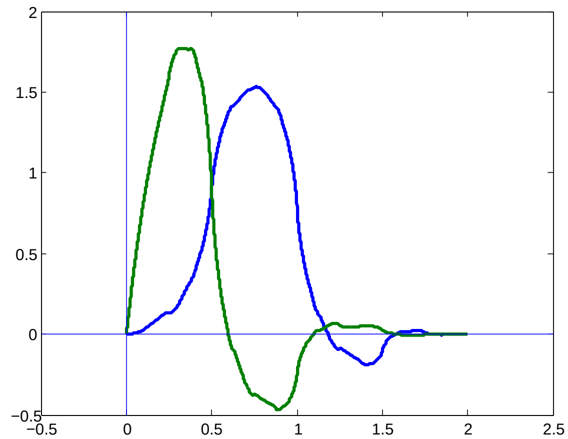

Figure 5.8 CL3 third solution

where

$$B_0 = \frac{1}{160}\begin{pmatrix} 51\sqrt{5}+30\sqrt{2} & -8\sqrt{30}+25\sqrt{3} \\ \frac{(-63\sqrt{20}+279\sqrt{2})\sqrt{4390-1240\sqrt{10}}}{-878+248\sqrt{10}} & \frac{(-3\sqrt{30}+66\sqrt{3})\sqrt{4390-1240\sqrt{10}}}{-878+248\sqrt{10}} \end{pmatrix},$$

$$B_1 = \frac{1}{160}\begin{pmatrix} 19\sqrt{5}-30\sqrt{2} & -8\sqrt{30}+15\sqrt{3} \\ \frac{(13\sqrt{20}-49\sqrt{2})\sqrt{4390-1240\sqrt{10}}}{-878+248\sqrt{10}} & \frac{(59\sqrt{30}-194\sqrt{3})\sqrt{4390-1240\sqrt{10}}}{-878+248\sqrt{10}} \end{pmatrix},$$

$$C_0 = \frac{1}{4}\begin{pmatrix} \sqrt{10} & -\sqrt{6} \\ \frac{(149-23\sqrt{10})\sqrt{4390-1240\sqrt{10}}}{-439+124\sqrt{10}} & \frac{(235\sqrt{6}-209\sqrt{15})\sqrt{4390-1240\sqrt{10}}}{-2195+620\sqrt{10}} \end{pmatrix},$$

$$C_1 = \frac{1}{4}\begin{pmatrix} \sqrt{10} & \sqrt{6} \\ \frac{(29-11\sqrt{10})\sqrt{4390-1240\sqrt{10}}}{-439+124\sqrt{10}} & \frac{(65\sqrt{6}-31\sqrt{15})\sqrt{4390-1240\sqrt{10}}}{-2195+620\sqrt{10}} \end{pmatrix}.$$

In this case, we get

$$\boldsymbol{\varphi}^L(0) = \mathbf{0}, \quad \boldsymbol{\varphi}^L(1) = \begin{pmatrix} \frac{\sqrt{10}}{4} \\ \frac{(\sqrt{10}-1)\sqrt{4390-1240\sqrt{10}}}{-1756+496\sqrt{10}} \end{pmatrix}.$$

To find the last solution, we begin with

$$A = \begin{pmatrix} \frac{\sqrt{2}}{4} & 0 \\ \frac{3\sqrt{3}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad C = \begin{pmatrix} v_3^* \\ v_4^* \end{pmatrix}.$$

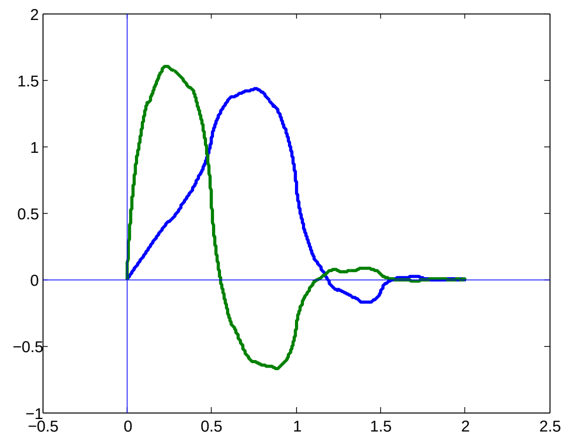

Figure 5.9 CL3 fourth solution

If we use the following transformation matrix

$$M = \begin{pmatrix} \frac{\sqrt{75+21\sqrt{10}}}{90} & 0 \\ \frac{(5\sqrt{6}-\sqrt{15})\sqrt{75+21\sqrt{10}}}{270} & \frac{\sqrt{75+21\sqrt{10}}}{135} \end{pmatrix},$$

then the matrix (MAM^{-1}, MB) has orthonormal rows. Therefore, we have

$$A = \begin{pmatrix} \frac{\sqrt{2}}{4} & 0 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & B_1 \end{pmatrix}, \text{and } C = \begin{pmatrix} C_0 & C_1 \end{pmatrix},$$

where

$$B_0 = \begin{pmatrix} \frac{(12\sqrt{2}-2\sqrt{5})\sqrt{75+21\sqrt{10}}}{160} & \frac{(8\sqrt{30}-26\sqrt{3})\sqrt{75+21\sqrt{10}}}{480} \\ \frac{(\sqrt{30}-7\sqrt{3})\sqrt{75+21\sqrt{10}}}{240} & \frac{(\sqrt{2}-\sqrt{5})\sqrt{75+21\sqrt{10}}}{80} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \frac{(92\sqrt{2}-29\sqrt{20})\sqrt{75+21\sqrt{10}}}{1440} & \frac{(16\sqrt{30}-58\sqrt{3})\sqrt{75+21\sqrt{10}}}{1440} \\ \frac{(\sqrt{30}-3\sqrt{3})\sqrt{75+21\sqrt{10}}}{240} & \frac{(19\sqrt{2}-11\sqrt{5})\sqrt{75+21\sqrt{10}}}{720} \end{pmatrix},$$

$$C_0 = \begin{pmatrix} \frac{(-10+\sqrt{10})\sqrt{75+21\sqrt{10}}}{90} & \frac{(4\sqrt{15}-5\sqrt{6})\sqrt{75+21\sqrt{10}}}{90} \\ \frac{(5\sqrt{6}-4\sqrt{15})\sqrt{75+21\sqrt{10}}}{90} & \frac{(-10+\sqrt{10})\sqrt{75+21\sqrt{10}}}{90} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} \frac{(10-\sqrt{10})\sqrt{75+21\sqrt{10}}}{90} & \frac{(4\sqrt{15}-5\sqrt{6})\sqrt{75+21\sqrt{10}}}{90} \\ \frac{(4\sqrt{15}-5\sqrt{6})\sqrt{75+21\sqrt{10}}}{90} & \frac{(-10+\sqrt{10})\sqrt{75+21\sqrt{10}}}{90} \end{pmatrix}.$$

In this case, we get

$$\varphi^L(1) = \begin{pmatrix} \frac{(5-\sqrt{10})\sqrt{75+21\sqrt{10}}}{30} \\ \frac{(\sqrt{15}-5\sqrt{6})\sqrt{75+21\sqrt{10}}}{270} \end{pmatrix}.$$

5.5.1 Comparison of two approaches

We now construct a regular boundary function vector that cannot be written as a linear combination boundary crossing-functions. This shows that interesting solutions can be obtained from the recursion relationship that cannot be found by the usual approach based on linear combinations of boundary-crossing functions.

Consider the first solution above. Setting the second row of B to zero yields

$$B = \frac{1}{80}\begin{pmatrix} 30\sqrt{2}+3\sqrt{20} & -10\sqrt{3}+2\sqrt{30} & 10\sqrt{2}-6\sqrt{5} & -10\sqrt{3}+2\sqrt{30} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the original solution solution is regular, it satisfies the continuity condition (3.37). We note that

$$\Gamma = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{15}}{3} & 0 \end{pmatrix}.$$

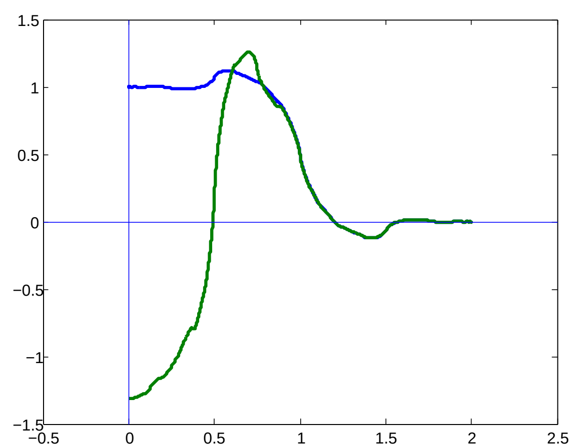

Figure 5.10 CL3 regular, but not a linear combination of boundary-crossing functions

We only need to check what happens to ΓB. For given Γ as above, it is easy to see that ΓB has not changed. Thus, continuity is still valid. It remains to orthonormalize this solution. We obtain

$$M = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{62}}{8} & \frac{7\sqrt{930}}{120} \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{31}}{8} & \frac{\sqrt{2}}{8} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ \frac{\sqrt{62}}{8} & \frac{-7\sqrt{790}}{120} & \frac{\sqrt{62}}{8} & \frac{7\sqrt{790}}{120} \end{pmatrix}$$

$$B_0 = \begin{pmatrix} \frac{30\sqrt{2}+6\sqrt{5}}{80} & \frac{-10\sqrt{3}+2\sqrt{30}}{80} \\ \frac{30\sqrt{31}+3\sqrt{310}}{320} & \frac{-5\sqrt{186}+2\sqrt{465}}{320} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{10\sqrt{2}-6\sqrt{5}}{80} & \frac{2\sqrt{30}-10\sqrt{3}}{80} \\ \frac{10\sqrt{31}-3\sqrt{310}}{320} & \frac{-5\sqrt{186}+2\sqrt{465}}{320} \end{pmatrix}.$$

Since

$$\boldsymbol{\varphi}(1) = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{15}}{6} \end{pmatrix}, \quad \boldsymbol{\varphi}(2) = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{15}}{6} \end{pmatrix},$$

we have

$$\boldsymbol{\varphi}^L(0) = \begin{pmatrix} 1 \\ -\frac{\sqrt{62}}{6} \end{pmatrix}, \quad \boldsymbol{\varphi}^L(1) = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{62}}{6} \end{pmatrix}.$$

CHAPTER 6. Concluding Remarks

6.1 Applications

For signal analysis, existing ad hoc methods work fairly well. This includes padding the signal with zeros, windowing the signal, or extending it by symmetry or extrapolation.

We expect our new approach to be useful in situations where one needs to reconstruct the processed signal again, especially if it is desirable to keep all signals at constant length. This should be especially useful in applications in numerical analysis [5].

6.2 Future Research

- *Larger Orthogonal Wavelets:* We have investigated and obtained results for orthogonal wavelets with 4 recursion coefficients. For complete generality, we need to analyze the case for arbitrary N. For wavelets with 6 recursion coefficients, the following idea can be used to reduce this number to 4.

  We start out with
  $$\boldsymbol{\varphi}(x) = \sqrt{2}\sum_{k=0}^{5} H_k \boldsymbol{\varphi}(2x-k),$$
  then let
  $$\mathbf{\Phi}(x) = \begin{pmatrix} \boldsymbol{\varphi}(2x) \\ \boldsymbol{\varphi}(2x-1) \end{pmatrix}.$$
  The function $\mathbf{\Phi}(x)$ with 4 recursion coefficients solves the following two-scale refinement equation.

$$\mathbf{\Phi}(x) = \sqrt{2}\sum_{k=0}^{3} \tilde{H}_k \mathbf{\Phi}(2x-k),$$

where

$$\tilde{H}_0 = \begin{pmatrix} H_0 & H_1 \\ 0_2 & 0_2 \end{pmatrix}, \ \tilde{H}_1 = \begin{pmatrix} H_2 & H_3 \\ H_0 & H_1 \end{pmatrix}, \ \tilde{H}_2 = \begin{pmatrix} H_4 & H_5 \\ H_2 & H_3 \end{pmatrix}, \ \tilde{H}_3 = \begin{pmatrix} 0_2 & 0_2 \\ H_4 & H_5 \end{pmatrix}.$$

We expect that a similar idea will work for an arbitrary N.

- *Biorthogonal Wavelets:* For orthogonal wavelets, the inverse of the associated wavelet transform is the conjugate transpose of the original one. Because we want to be able reconstruct what we decompose, we always want to keep the invertibility condition. However, the orthogonality condition can be relaxed. It has been shown that such a relaxation, replacing orthogonality condition by a milder one, produces biorthogonal wavelets. They form an important class of wavelets. We, therefore, think that analyzing endpoint functions for biorthogonal wavelets will produce useful results.

- *Numerical Experiments:* In this dissertation our focus has been mainly theoretical. Numerical experiments and searching for new applications in areas where wavelets are applicable can be a direction for future work.

- *Preconditioning:* The preconditioning step needed for multiwavelets also corresponds to multiplying by an infinite banded block matrix. Similar techniques to our endpoint modification techniques should apply.

APPENDIX A. Linear Algebra Results

Most of these results are known, and collected here for easier reference. Throughout this section we will assume that A is an $r \times s$ matrix. We also use $A : \mathbb{R}^s \mapsto \mathbb{R}^r$ to denote the corresponding linear transform.

Notation:

$R(A)$: range of A,

rank(A): rank of $A =$ dimension of $R(A)$,

$N(A)$: nullspace of A.

Theorem A.1.

$$\begin{aligned} \text{rank}(A) + \dim(N(A)) &= s, \\ \text{rank}(A^*) + \dim(N(A^*)) &= r. \end{aligned}$$

Lemma A.2.

$$N(A^*A) = N(A)$$

Theorem A.3. [17] *Assume A is an $r \times s$ matrix. Then*

$$\text{rank}(A) = \text{rank}(A^*) = \text{rank}(AA^*) = \text{rank}(A^*A) \leq \min(r, s).$$

If B is an $s \times k$ matrix, then

$$\begin{aligned} \text{rank}(AB) &\leq \min(\text{rank}(A), \text{rank}(B)), \\ \text{rank}(A + B) &\leq \text{rank}(A) + \text{rank}(B). \end{aligned}$$

Condition O: Let T_0 and T_1 be of size $r \times s$. We say that T_0 and T_1 satisfy Condition O if

$$\begin{aligned} T_0 T_0^* + T_1 T_1^* &= I_r, \\ T_0 T_1^* &= 0_r, \end{aligned}$$

where I_r is the $r \times r$ identity matrix and 0_r is the $r \times r$ zero matrix.

Lemma A.4. *Condition O is only possible if* $r \leq s$.

Theorem A.5. *Assume* T_0 *and* T_1 *are of size* $r \times s$ *with* $r \leq s$, *and satisfy Condition O. Then*

$$r \leq \operatorname{rank}(T_0) + \operatorname{rank}(T_1) \leq s.$$

Corollary A.6. *If* T_0 *and* T_1 *are of size* $r \times r$ *and satisfy Condition O, then*

$$\operatorname{rank}(T_0) + \operatorname{rank}(T_1) = r.$$

Lemma A.7. *Assume* T_0 *and* T_1 *are of size* $r \times r$ *and satisfy Condition O. Then,*

$$R(T_0) \text{ and } R(T_1) \text{ are orthogonal and } R(T_0) \bigoplus R(T_1) = \mathbb{R}^r,$$
$$N(T_0) \text{ and } N(T_1) \text{ are orthogonal and } N(T_0) \bigoplus N(T_1) = \mathbb{R}^r.$$

Theorem A.8. [30] *If* T_0 *and* T_1 *are of size* $r \times r$ *and satisfy Condition O, then there exist orthogonal* U *and* V *with*

$$T_0 = U \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array} \right) V^*, \quad I_{\rho_0 \times \rho_0}, \quad \rho_0 = \operatorname{rank}(T_0),$$

$$T_1 = U \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array} \right) V^*, \quad I_{\rho_1 \times \rho_1}, \quad \rho_1 = \operatorname{rank}(T_1), \text{ where } \quad \rho_0 + \rho_1 = r.$$

Corollary A.9. *There exist orthogonal matrices*

$$V = \begin{pmatrix} V_0 & V_1 \end{pmatrix} \text{ and } U = \begin{pmatrix} U_0 & U_1 \end{pmatrix},$$

with

$$T_0 V_1 = T_1 V_0 = U_0^* T_1 = U_1^* T_0 = 0,$$

where U_0, V_0 *and* U_1, V_1 *are obtained from the reduced singular value decomposition of* T_0 *and* T_1, *respectively.*

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