MODULI OF CONTINUITY OF QUASI-SMOOTH FUNCTIONS

by

Sidney Dean Nolte

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1959

TABLE OF CONTENTS

I. INTRODUCTION

A function f(x) is uniformly continuous on an interval [a,b] if and only if the first difference f(x + h) - f(x) goes to zero uniformly on the interval. Also, if f(x) is uniformly continuous on [a,b] then the second difference f(x + h) - 2f(x) + f(x - h) will go to zero uniformly on that interval. However, Hamel (4) has shown that if the second difference goes to zero uniformly the function need not be continuous. Kodres (5) has shown that if the second difference goes to zero uniformly, then the function is either uniformly continuous or non-measurable.

It follows then that if the condition

$$(1.1) \qquad |f(x + h) - 2f(x) + f(x - h)| \leq C|h| , \qquad C > 0$$

be imposed upon the second difference, the function f is either uniformly continuous or non-measurable. The class of continuous functions which satisfy (1.1) are called quasi-smooth functions. The modulus of continuity of quasi-smooth functions was investigated by Timan (6) and Zygmund (7).

Condition 1.1 can be written

$$(1.2) \qquad \left| \frac{f(x + h) + f(x - h)}{2} - f(x) \right| \leq \frac{C}{2}|h| , \qquad C > 0 .$$

Written in this form, it is seen that the arithmetic mean of f(x + h), f(x - h) is compared with f(x). The problem considered in this thesis is that of studying the behavior of functions satisfying

$$(1.3) \qquad \left|M[f(x + h), f(x - h)] - f(x)\right| \le C|h| , \qquad C > 0$$

where M is any mean function.

In Chapter II the mean functions and modulus of continuity are defined and the properties of these functions required in this study are given. A generalization of convexity is considered and it is shown that if

$$(1.4) \qquad pf(x + h) + qf(x - h) - f(x) \ge 0 , \quad x + h, x - h \in [a,b],$$
$$p,q,h > 0, \qquad p + q = 1 ,$$

then f must be monotone on $[a,b]$ if $p \ne q$.

Timan (6) and Brudnyĭ (3) have considered the maximum modulus of the sub-class of quasi-smooth functions on the interval $[-1,1]$ which take on the value zero at + 1 and - 1. They have shown that this maximum modulus is less than or equal to $\frac{4}{3}$. In Chapter III this bound has been slightly improved. When the generalized arithmetic mean is applied to this class the bound of $\frac{4}{3}\left\{\max[p,q] + \frac{1}{2}\right\}$ is also established in Chapter III.

In Chapter IV the main results of this thesis are given. It is established that when the generalized arithmetic mean is applied to the second difference to form a class of generalized quasi-smooth functions, then every element of this class satisfies a Lipschitz condition. This result is then used to establish bounds on the modulus of continuity of generalized quasi-smooth functions, that is to say, functions which satisfy 1.3.

II. DEFINITIONS AND PRELIMINARIES

A. The Modulus of Continuity

Definition 2.1. Let $f(x)$ be a function defined on a finite or infinite interval I. Then the modulus of continuity of f is defined by the equation

$$\omega(f,h) = \sup_{|x_1 - x_2| \le h} |f(x_1) - f(x_2)| , \qquad x_1, x_2 \in I .$$

It is obvious from the definition that $\omega(f,h)$ is a monotone function of h. That is to say, if $h_1 < h_2$, then $\omega(f,h_1) \le \omega(f,h_2)$. The domain of definition of $\omega(f,h)$ is the interval $0 < h \le L$ where L is the length of I and $\lim_{h \to 0} \omega(f,h) = 0$ if and only if f is uniformly continuous on I. In addition to these elementary properties of the modulus of continuity function we will also use the properties described by the following three lemmas (1).

Lemma 2.1. $\omega(f,mh) \le m\,\omega(f,h)$, where m is a positive integer.

Proof: The validity of this inequality follows from the fact that

$$f(x + mh) - f(x) = \sum_{k=0}^{m-1} f[x + (k + 1)h] - f(x + kh) .$$

By use of the triangle inequality we obtain

$$|f(x + mh) - f(x)| \le \sum_{k=0}^{m-1} |f[x + (k-1)h] - f(x + kh)| \le m\,\omega(f,h) .$$

Hence,

$$\omega(f,mh) = \sup_{|x_1-x_2| \leq mh} |f(x_1) - f(x_2)| \leq m\,\omega(f,h) .$$

Lemma 2.2. If $\lim_{h\to 0} \frac{\omega(f,h)}{h} = 0$, then f is a constant function on I.

Proof: Let m be any positive integer. Then by Lemma 2.1 it follows that

$$\omega(f,h) = \omega(f,m\,\tfrac{h}{m}) \leq m\,\omega(f,\tfrac{h}{m}) = \frac{h\,\omega(f,\,h/m)}{h/m} .$$

Therefore,

$$\omega(f,h) = \lim_{m\to\infty} \omega(f,h) \leq h \lim_{m\to\infty} \frac{\omega(f,\,h/m)}{h/m} = 0 .$$

Since $\omega(f,h) \geq 0$, it follows that $\omega(f,h) = 0$ for all positive h. Therefore, f is a constant function.

Lemma 2.3. Let f be continuous and strictly monotone on [a,b], that is to say, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. Then $\omega(f,h)$ is strictly monotone.

Proof: Since f is strictly monotone, it follows that

$$\omega(f,h) = \sup_{|x_1-x_2| = h} |f(x_1) - f(x_2)| = \sup_{x} [f(x+h) - f(x)]$$

Now suppose $h_2 > h_1$. Since $\omega(f,h_2) \geq \omega(f,h_1)$ we must show that equality cannot hold. Suppose $\omega(f,h_2) = \omega(f,h_1)$. Then since $f(x+h_1) - f(x)$ is uniformly continuous, there exists an x_1 such that

$$\omega(f,h_1) = f(x_1+h_1) - f(x_1) < f(x_1+h_2) - f(x_1) \leq \omega(f,h_2) .$$

This is a contradiction. Hence

$$\omega(f,h_1) < \omega(f,h_2) .$$

Definition 2.2. If f is defined on I and if there exists an $M > 0$ such that

$$|f(x_1) - f(x_2)| \le M|x_1 - x_2|^\alpha , \qquad x_1, x_2 \in I ,$$

then f(x) satisfies a Lipschitz condition of order α on I ($f \in \text{Lip}\,\alpha$). M is said to be the Lipschitz constant.

If $\omega(t)$ is a monotone, non-decreasing function of t for $t > 0$ and if $\lim_{t \to 0} \omega(t) = 0$, then H_ω will denote the class of all functions f defined on I satisfying the relation

$$|f(x + h) - f(x)| \le M \ \omega(h)$$

for some $M > 0$. M is a constant depending only on f. If $\omega_1(h) \le \omega_2(h)$ for all h, then it is clear that $H\,\omega_1 \subset H\,\omega_2$. In particular, if $\omega(h) < Mh^\alpha$ and if $f \in H_\omega$, then $f \in \text{Lip}\,\alpha$. This leads us to consider the following lemma which will be used later.

Lemma 2.4. If $\omega(f,h) \le Mh \ln \frac{1}{h}$ and if $0 < \epsilon < 1$, then $f \in \text{Lip}(1 - \epsilon)$.

Proof: Let ϵ be given such that $0 < \epsilon < 1$. Since

$$Mh \ln \frac{1}{h} < \frac{2M}{\epsilon} h^{1-\epsilon}$$

for all h, it follows that

$$\sup_{|x_1-x_2|=h} |f(x_1) - f(x_2)| \le \sup_{|x_1-x_2|\le h} |f(x_1) - f(x_2)|$$

$$\le Mh \ln \frac{1}{h} < \frac{2M}{\epsilon} h^{1-\epsilon} \quad .$$

Therefore,

$$|f(x_1) - f(x_2)| < \frac{2M}{\epsilon} |x_1 - x_2|^{1-\epsilon}$$

and $f \in \text{Lip}\ (1-\epsilon)$ with Lipschitz constant $\frac{2M}{\epsilon}$. This completes the proof.

B. Generalized Means

A generalized mean is defined to be a single valued function $M(x,y)$ of two variables x and y ($\alpha \le x,\ y \le \beta$) if $M(x,y)$ satisfies the following postulates:

(i) Strictly monotonic: This means that if $x < x'$, then $M(x,y) < M(x',y)$ and likewise for y.

(ii) Continuous:

(iii) Bisymmetric: This means that

$$M[M(x_1,x_2),\ M(y_1,y_2)] = M[M(x_1,y_1),\ M(x_2,y_2)] \quad .$$

(iv) Reflexive: $M(x,x) = x$.

(v) Symmetric: $M(x,y) = M(y,x)$.

It follows immediately from postulates (i) and (iv) that any $M(x,y)$ will have the property that if $x < y$, then $x < M(x,y) < y$.

Aczel (2) has proved that postulates (i) through (v) are necessary and sufficient conditions for the existence of a strictly monotone, continuous function $\psi(x)$ $(\alpha \leq x \leq \beta)$ by which M(x,y) has the form

$$M(x,y) = \psi^{-1}\left[\frac{\psi(x) + \psi(y)}{2}\right] .$$

Further, a necessary and sufficient condition for the function M(x,y) to satisfy postulates (i) through (iv) is that there exists a strictly monotone, continuous function $\psi(x)$ $(\alpha \leq x \leq \beta)$ and a pair of positive numbers p,q such that $p + q = 1$ and by which M(x,y) has the form

$$M(x,y) = \psi^{-1}\left[p\,\psi(x) + q\,\psi(y)\right] .$$

The well known arithmetic, geometric and harmonic means are means satisfying postulates (i) through (v) and can be generated by the functions x, log x, $\frac{1}{x}$ respectively. Non-symmetric means are referred to as weighted means. The weighted arithmetic mean is written $M_x(x,y) = px + qy$, the weighted geometric mean is written $M_{\log x}(x,y) = x^p y^q$ and the weighted harmonic mean is written $M_{1/x}(x,y) = \frac{xy}{py + qx}$, where $p + q = 1$ and $p,q > 0$ in all these cases.

The above mentioned examples of mean functions are but special cases of a family of means generated by the family of functions

$$\psi_r(x) = x^r \quad , \qquad r \neq 0$$

$$\psi_r(x) = \log x \ , \qquad r = 0 .$$

Thus, it can be seen that there exists a one to one correspondence between the means generated by members of this family and the real numbers.

<u>Definition 2.3</u>. Means that satisfy postulates (i) through (v) will be called symmetric means. Those which satisfy postulates (i) through (iv) but not (v) will be called non-symmetric means.

<u>Definition 2.4</u>. The first difference of a function f at x_0 is defined to be $\Delta(f;\, x_0,\, h) = f(x_0 + h) - f(x_0)$.

<u>Definition 2.5</u>. The second difference of a function f at x_0 is defined to be $\Delta^2(f;\, x_0,\, h) = f(x_0 + h) - 2f(x_0) + f(x_0 - h)$.

<u>Definition 2.6</u>. If f satisfies the conditions

(a) f(x) is continuous for $a \leq x \leq b$,

(b) $|\Delta^2(f;\, x_0,\, h)| \leq 2M|h|$, M a fixed constant,

$$x_0 + h, \quad x_0 - h \in [a,b] ,$$

then f is said to be quasi-smooth. We will use the symbol L(a,b)M to denote the class of quasi-smooth functions on the interval [a,b] .

It is the purpose of this thesis to examine a generalized form of quasi-smooth functions. This generalized form will be one in which the condition (b) of the definition is replaced by

$$|M_\psi [f(x + h),\, f(x - h)] - f(x)| \leq Mh , \qquad h > 0 .$$

We will denote by $L_\psi(a,b)M$ this class of generalized quasi-smooth functions.

C. The Generalized Mean Applied to Convex Functions

Definition 2.6. A real valued function f, defined on the interval $[a,b]$ and satisfying

$$f(x_1) - 2f\left(\frac{x_1 + x_2}{2}\right) + f(x_2) \geq 0 , \qquad x_1, x_2 \in [a,b]$$

is said to be convex.

Definition 2.7. The generalized second difference of a function f, defined on the interval $[a,b]$, is defined to be

$$\Delta^2_{\psi} (f;x,h) = M_{\psi} [f(x+h), f(x-h)] - f(x), \quad x+h, x-h \in [a,b] .$$

Since a convex function f satisfies $\Delta^2(f;x,h) \geq 0$, we can use the generalized second difference to define a new type of convexity.

Definition 2.8. A real valued function f defined on an interval $[a,b]$ is said to be convex with respect to M_{ψ} if and only if

$$M_{\psi} [f(x_1), f(x_2)] - f\left(\frac{x_1 + x_2}{2}\right) \geq 0, \qquad x_1, x_2 \in [a,b] .$$

It is understood that the domain of ψ includes the range of f.

In this section we will obtain a result concerning convex functions with respect to the weighted arithmetic mean. This result will be expanded to include any weighted mean.

It is easy to see that for continuous f,

$$f(x_1) + f(x_2) - 2f\left(\frac{x_1 + x_2}{2}\right) = 0$$

if and only if $f(x) = Ax + B$ and

$$pf(x_1) + qf(x_2) - f\left(\frac{x_1 + x_2}{2}\right) = 0$$

if $f(x) = C$. This leads to the question of whether there exist non-constant solutions to the functional equation

$$(2.1) \quad pf(x) + qf(y) = f\left(\frac{x+y}{2}\right), \quad p+q = 1, \quad p,q > 0, \quad x > y.$$

Let f be a non-constant solution of 2.1 and let f be defined on $[a,b]$. Let $x_0 \varepsilon (a,b)$. Then for a positive h sufficiently small, $x_0 - 2h$, $x_0 + 2h \;\varepsilon\; (a,b)$. Also, $g(x) = f(x) - f(x_0)$ is a solution of 2.1. Hence,

$$(2.2) \quad \begin{aligned} pg(x_0 + 2h) + qg(x_0 - 2h) &= 0 \\ pg(x_0 + 2h) - g(x_0 + h) &= 0 \\ qg(x_0 - 2h) - g(x_0 - h) &= 0 \\ pg(x_0 + h) + qg(x_0 - h) &= 0 . \end{aligned}$$

Since there exists a non-trivial solution of this system of equations, the determinant of the coefficients $\Delta = pq(q - p) = 0$. Therefore, if 2.1 has a non-constant solution, then $p = q = 1/2$. If $p \neq q$, then 2.1 has only a constant solution.

Now since there are no non-constant solutions of 2.1 in the case $p \neq q$, what types of functions are such that the inequality holds? This question is answered by the following theorem.

<u>Theorem 2.1</u>. If f is continuous on $[\alpha, \beta]$ and if

$$(2.3) \quad pf(x_1) + qf(x_2) \geq f\left(\frac{x_1+x_2}{2}\right), \quad x_1 > x_2, \quad x_1, x_2 \in [\alpha, \beta],$$

then f is monotone.

<u>Proof</u>: Let $a,b \in (\alpha, \beta)$, $a < b$. Then subdivide the interval (a,b) by the points $a+h$, $a+2h$, $a+3h$, ..., $a+nh = b$ and let h be such that $a-h$, $b+h \in [\alpha, \beta]$. Then,

$$(2.4) \quad \begin{aligned} pf(a+h) + qf(a-h) &\geq f(a) \\ pf(a+2h) + qf(a) &\geq f(a+h) \\ pf(a+3h) + qf(a+h) &\geq f(a+2h) \\ &\vdots \\ pf(b) + qf(b-2h) &\geq f(b-h) \\ pf(b+h) + qf(b-h) &\geq f(b) . \end{aligned}$$

Addition of these inequalities yields

$$qf(a-h) + qf(a) + pf(b) + pf(b+h) \geq f(a) + f(b) .$$

This inequality is equivalent to

$$q[f(a-h) - f(a)] + p[f(b+h) - f(b)] \geq (p-q)[f(a) - f(b)] .$$

By continuity of f, for each $\epsilon > 0$, h can be made sufficiently small so that $\epsilon \geq (p-q)[f(a) - f(b)]$. Therefore,

$(p - q)\,[f(a) - f(b)] \le 0$ and f is monotone.

Corollary 2.1. If M_ψ is any weighted mean generated by ψ, and if f is such that $M_\psi\,[f(x + h), f(x - h)] \ge f(x)$, then if $\psi[f(x)]$ is continuous it will be monotone.

Proof: Since $\psi^{-1}\{p\,\psi[f(x + h)] + q\,\psi[f(x - h)]\} \ge f(x)$ and since $\psi(t)$ is strictly monotone, we have $p\,\psi[f(x + h)] + q\,\psi[f(x - h)] \ge \psi[f(x)]$. Application of the previous theorem yields the desired result.

If we denote by C_p the class of functions f defined and continuous on $[a,b]$ and satisfying

$$pf(x + h) + qf(x - h) \ge f(x), \quad p + q = 1, \quad p, q, h > 0,$$

then for $p > q$, $f \varepsilon C_p$ implies f is monotone increasing whereas if $p < q$, $f \varepsilon C_p$ implies f is monotone decreasing. If $p = q = 1/2$, then $f \varepsilon C_{1/2}$ need not imply that f is monotone. Therefore, we see that the values of the parameter p are critical at 1/2. This unusual property occurs again in the consideration of the modulus of continuity of a continuous function satisfying the relation

$$|pf(x + h) + qf(x - h) - f(x)| \le Mh, \quad p + q = 1, \quad p, q, M, h > 0.$$

III. THE MAXIMUM MODULUS OF QUASI-SMOOTH FUNCTIONS

A. The Class $L^*(-1,1)1$

If $f(x) \varepsilon L(a,b)M$, then clearly $g(x) = f(x) + Ax + B$, where A and B are arbitrary constants, is also an element of this class. In particular, consider

$$g(x) = f(x) - f(a) - (x - a)\left[\frac{f(b) - f(a)}{b - a}\right].$$

Then $g(x) \varepsilon L(a,b)M$ and further, $g(a) = g(b) = 0$. Now let

$$h(x) = \frac{2}{b - a}\, g\left(\frac{b - a}{2}x + \frac{b + a}{2}\right) .$$

It can easily be verified that $h(x) \varepsilon L(-1,1)M$. Finally, let $\varphi(x) = \frac{1}{M} h(x)$. Then $\varphi(x) \varepsilon L(-1,1)1$. This construction of $\varphi(x)$ from $f(x)$ has been, in effect, a normalization of the class $L(a,b)M$ into the class $L(-1,1)1$ with the additional property that $\varphi(1) = \varphi(-1) = 0$. Let us denote this class by the symbol $L^*(-1,1)1$. This class was examined by Timan (6) and Brudnyĭ (3) and it was shown that

(3.1) $$\sup_{f \varepsilon L^*(-1,1)1} \left\{ \max_{-1 \le x \le 1} |f(x)| \right\} \le \frac{4}{3} .$$

It is the purpose of this section to obtain a slight improvement of this bound.

Consider the function

(3.2) $$\varphi_1(x) = \begin{cases} \frac{5}{3} + x , & -1 \le x \le -\frac{1}{3} \\ \frac{4}{3} , & \frac{1}{3} \le x \le \frac{1}{3} \\ \frac{5}{3} - x , & \frac{1}{3} \le x \le 1 . \end{cases}$$

Lemma 3.1. Let $f \varepsilon L^*(-1,1)1$. Then $|f(x)| \le \varphi_1(x)$ for all $x \varepsilon [-1,1]$.

Proof: Let $x_0 \varepsilon [\frac{1}{3},1]$. Then

$$|f(1) - 2f(x_0) + f(2x_0 - 1)| \le 2 - 2x_0 .$$

Therefore,

$$|f(x_0)| \le \frac{2-2x_0+|f(1)+f(2x_0-1)|}{2} \quad \frac{2-2x_0+\frac{4}{3}}{2} = \frac{5}{3} - x_0 .$$

Since $f(x) \varepsilon L^*(-1,1)1$ implies $f(-x) \varepsilon L^*(-1,1)1$, we obtain $|f(-x_0)| \le \frac{5}{3} + x_0$ for $x_0 \varepsilon [-1,-\frac{1}{3}]$. This completes the proof.

Lemma 3.2. Let $f \varepsilon L^*(-1,1)1$ and let $-1 \le x_1 < x_2 \le 1$ and $|f(x_1)| \le z_1$, $|f(x_2)| \le z_2$. Then

$$(3.3) \quad |f(x)| \le \frac{x_2-x_1}{2} \varphi_1\left(\frac{2x}{x_2-x_1} - \frac{x_2+x_1}{x_2-x_1}\right) + z_2 + (x-x_2)\left(\frac{z_2-z_1}{x_2-x_1}\right),$$

$$x_1 \le x \le x_2 .$$

Proof: Consider the function

$$g(x) = \frac{2}{x_2-x_1}\left\{ f\left(\frac{x_2-x_1}{2} x + \frac{x_2+x_1}{2}\right) - f(x_2) - \left(\frac{x_2-x_1}{2}x + \frac{x_2-x_1}{2} - x_2\right) \cdot \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)\right\}, \quad -1 \le x \le 1 .$$

We note that $g(1) = g(-1) = 0$ and that g is continuous and $|g(x + h) - 2g(x) + g(x - h)| \le 2h$. Therefore, $g(x) \varepsilon L^*(-1,1)1$. By lemma 3.1, $|g(x)| \le \varphi_1(x)$ for $-1 \le x \le 1$.

Hence

$$\left|\frac{2}{x_2-x_1}\left\{f(x)-f(x_2)-(x-x_2)\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)\right\}\right| \le \varphi_1\left(\frac{2x}{x_2-x_1} - \frac{x_2+x_1}{x_2-x_1}\right),$$

$$x_1 \le x \le x_2 .$$

This inequality can be written in the form

$$|f(x)| \leq \frac{x_2-x_1}{2}\,\varphi_1\left(\frac{2x}{x_2-x_1} - \frac{x_2+x_1}{x_2-x_1}\right) + \left|f(x_2) + (x-x_2)\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)\right| .$$

Since

$$\left|f(x_2) + (x-x_2)\left(\frac{f(x_2)- f(x_1)}{x_2-x_1}\right)\right| \leq \left|z_2 + (x-x_2)\left(\frac{z_2-z_1}{x_2-x_1}\right)\right| \quad ,$$

we obtain the desired result. This completes the proof.

We now establish bounds on the class $L^*(-1,1)1$ at specific points in $[-1,1]$. As has been pointed out earlier, if $f(x) \varepsilon L^*(-1,1)1$, then $f(-x) \varepsilon L^*(-1,1)1$. Hence, if it is established that

$$\sup_{f \varepsilon L^*(-1,1)1} |f(x_0)| \leq z_0 ,$$

then it is clear that

$$\sup_{f \varepsilon L^*(-1,1)1} |f(-x_0)| \leq z_0 .$$

It is for this reason that we need only to consider a bound on $f(x)$ for $0 \leq x \leq 1$.

By letting $x = 0$, $h = 1$ in

$$(2.4) \qquad |f(x + h) - 2f(x) + f(x - h)| \leq 2h$$

we obtain $|f(0)| \leq 1$. Then by letting $x = \frac{1}{2}$, $h = \frac{1}{2}$ in 2.4 we obtain $|f(\frac{1}{2})| \leq 1$. Similarly, we obtain $|f(1/4)| \leq 5/4$, $|f(1/8)| \leq 5/4$, $|f(3/8)| \leq 5/4$.

Now let f be any element of $L^*(-1,1)1$.

It follows from 2.4 that

$$\left|f\left(\frac{1}{6}\right) - 2f(0) + f\left(-\frac{1}{6}\right)\right| \leq \frac{1}{3}$$

$$\left|f(1) - 2f\left(\frac{5}{12}\right) + f\left(-\frac{1}{6}\right)\right| \leq \frac{7}{6}$$

$$\left|f\left(\frac{5}{12}\right) - 2f\left(\frac{5}{24}\right) + f(0)\right| \leq \frac{5}{12}$$

$$\left|f\left(\frac{5}{24}\right) - 2f\left(\frac{1}{6}\right) + f\left(\frac{1}{8}\right)\right| \leq \frac{1}{12} .$$

Hence, by use of the triangle inequality,

$$\left|9f\left(\frac{1}{6}\right)\right| = \left|f\left(\frac{1}{6}\right) - 2f(0) + f\left(-\frac{1}{6}\right) - f(1) + 2f\left(\frac{5}{12}\right) - f\left(-\frac{1}{6}\right)\right.$$

$$- 2f\left(\frac{5}{12}\right) + 4f\left(\frac{5}{24}\right) - 2f(0) - 4f\left(\frac{5}{24}\right) + 8f\left(\frac{1}{6}\right) - 4f\left(\frac{1}{8}\right)$$

$$\left. + 4f(0) + 4f\left(\frac{1}{8}\right)\right| \leq \frac{1}{3} + \frac{7}{6} + \frac{5}{6} + \frac{1}{3} + 4\left|f(0)\right| + 4\left|f\left(\frac{1}{8}\right)\right|$$

$$\leq \frac{8}{3} + 4 + 5 = \frac{35}{3} .$$

Therefore $\left|f\left(\frac{1}{6}\right)\right| \leq \frac{35}{27}$.

Similarly,

$$\left|5f\left(\frac{1}{5}\right)\right| = \left|f\left(\frac{1}{5}\right) - 2f(0) + f\left(-\frac{1}{5}\right) - f(1) + 2f\left(\frac{2}{5}\right)\right.$$

$$\left. - f\left(-\frac{1}{5}\right) - 2f\left(\frac{2}{5}\right) + 4f\left(\frac{1}{5}\right) - 2f(0) + 4f(0)\right|$$

$$\leq \left|f\left(\frac{1}{5}\right) - 2f(0) + f\left(-\frac{1}{5}\right)\right| + \left|f(1) - 2f\left(\frac{2}{5}\right) + f\left(-\frac{1}{5}\right)\right|$$

$$+ 2\left|f\left(\frac{2}{5}\right) - 2f\left(\frac{1}{5}\right) + f(0)\right| + 4\left|f(0)\right| \leq \frac{2}{5} + \frac{6}{5} + \frac{4}{5} + 4 = \frac{32}{5}$$

Therefore $\left|f\left(\frac{1}{5}\right)\right| \leq \frac{32}{25}$.

By a proper combination of the inequalities

$$\left|f\left(\frac{1}{2}\right) - 2f\left(\frac{1}{3}\right) + f\left(\frac{1}{6}\right)\right| \leq \frac{1}{3}$$

$$\left|f\left(\frac{1}{6}\right) - 2f(0) + f\left(-\frac{1}{6}\right)\right| \leq \frac{1}{3}$$

$$\left|f\left(-\frac{1}{6}\right) - 2f\left(-\frac{1}{3}\right) + f\left(-\frac{1}{2}\right)\right| \leq \frac{1}{3}$$

$$\left|f(1) - 2f\left(\frac{1}{3}\right) + f\left(-\frac{1}{3}\right)\right| \leq \frac{2}{3} \quad ,$$

the inequality $\left|f\left(\frac{1}{3}\right)\right| \leq \frac{23}{18}$ can be obtained. Finally, the inequalities

$$\left|f\left(\frac{1}{2}\right) - 2f\left(\frac{3}{10}\right) + f\left(\frac{1}{10}\right)\right| \leq \frac{2}{5}$$

$$\left|f\left(\frac{1}{10}\right) - 2f(0) + f\left(-\frac{1}{10}\right)\right| \leq \frac{1}{5}$$

$$\left|f\left(-\frac{1}{10}\right) - 2f\left(-\frac{3}{10}\right) + f\left(-\frac{1}{2}\right)\right| \leq \frac{2}{5}$$

$$\left|f(1) - 2f\left(\frac{7}{20}\right) + f\left(-\frac{3}{10}\right)\right| \leq \frac{13}{10}$$

$$\left|f\left(\frac{7}{20}\right) - 2f\left(\frac{3}{10}\right) + f\left(\frac{1}{4}\right)\right| \leq \frac{1}{10}$$

yield the inequality $\left|f\left(\frac{3}{10}\right)\right| \leq \frac{13}{10}$.

By use of Lemma 3.2 and the fact that $\left|f\left(\frac{1}{6}\right)\right| \leq \frac{35}{27}$, $\left|f\left(\frac{1}{5}\right)\right| \leq \frac{32}{25}$, we obtain

$$|f(x)| \leq \frac{1}{60}\, \varphi_1(60x - 11) + \frac{32}{25} + \left(x - \frac{1}{5}\right)\left(-\frac{22}{45}\right) , \quad \frac{1}{6} \leq x \leq \frac{1}{5} .$$

The right hand side of this inequality has a maximum when $60x - 11 = \frac{1}{3}$. Therefore,

$$|f(x)| \leq \frac{1}{60} \cdot \frac{4}{3} + \frac{32}{25} + \left(\frac{8}{45} - \frac{1}{5}\right)\left(-\frac{22}{45}\right) = \frac{2659}{2025}, \quad \frac{1}{6} \leq x \leq \frac{1}{5} .$$

In a similar manner the bounds

$$|f(x)| \le \frac{283}{216}, \qquad \frac{1}{8} \le x \le \frac{1}{6}$$

$$|f(x)| \le \frac{391}{300}, \qquad \frac{1}{5} \le x \le \frac{1}{4}$$

$$|f(x)| \le \frac{79}{60}, \qquad \frac{1}{4} \le x \le \frac{3}{10}$$

$$|f(x)| \le \frac{71}{54}, \qquad \frac{3}{10} \le x \le \frac{1}{3}$$

$$|f(x)| \le \frac{35}{27}, \qquad \frac{1}{3} \le x \le \frac{3}{8}$$

can be obtained.

Since $|f(x)| \le \frac{1}{4}\ \varphi_1(4x - 1)$ for $0 \le x \le \frac{1}{2}$ and since $|f(x)| \le \frac{79}{60}$ for $\frac{1}{8} \le x \le \frac{3}{8}$, it follows that $|f(x)| \le \frac{79}{60}$ for $-1 \le x \le 1$.

B. The Class $L_p^*(-1,1)1$

In this section the class of continuous f on $[-1,1]$ which satisfy the conditions

(3.5) $$f(1) = f(-1) = 0$$

(3.6) $$|pf(x + h) + qf(x - h) - f(x)| \le h, \qquad p,\ q,\ h > 0,\ p+q=1$$

is considered. This class will be denoted by $L_p^*(-1,1)1$. The class considered in the previous section was $L_{\frac{1}{2}}^*(-1,1)1$.

Theorem 3.1. Let $K = \sup\limits_{f \in L_p(-1,1)1} \left\{ \max |f(x)| \right\}$.

Then $K \le \frac{4}{3}\left\{ \max [p,q] + \frac{1}{2} \right\} = \frac{4}{3}\left[1 + |p - \frac{1}{2}|\right]$.

<u>Proof</u>: Take $f(x) \varepsilon L_p^*(-1,1)1$ and let $\max\limits_{-1 \leq x \leq 1} f(x) = f(x_0) = K-\epsilon$, $\epsilon > 0$. Let $x_1 < x_2 < x_3 \ldots\ldots < x_n$ be the points in $[-1,1]$ at which $f(x) = L$, where $0 < L < K - \epsilon$. Then there exists two points x_i, x_{i+1} such that $x_i \leq x_0 \leq x_{i+1}$. Now consider the function

$$(3.7) \quad \Psi(x) = \frac{2}{x_{i+1} - x_i}\left\{ f\left[\frac{x_{i+1} - x_i}{2} x + \frac{x_{i+1} + x_i}{2}\right] - L \right\} .$$

Then $\Psi(1) = \Psi(-1) = 0$ and for $h > 0$,

$$|p \Psi(x + h) + q \Psi(x - h) - \Psi(x)| \leq h, \quad -1 \leq x \leq 1 .$$

Therefore $\Psi(x) \varepsilon L_p^*(-1,1)1$. Hence,

$$\max_{-1 \leq x \leq 1} \Psi(x) = \frac{2}{x_{i+1} - x_i}\left\{ \max_{-1 \leq x \leq 1} f(x) - L \right\}$$

$$= \frac{2}{x_{i+1} - x_i}\left\{ K - \epsilon - L \right\} \leq K .$$

Therefore,

$$(3.8) \qquad K \leq \frac{2(L + \epsilon)}{2 - (x_{i+1} - x_i)} .$$

Now let $x = 0$, $h = 1$ in 3.6 to obtain $|f(0)| \leq 1$. Set $x = \frac{1}{2}$, $h = \frac{1}{2}$ in 3.6 to obtain $|f(\frac{1}{2})| \leq q + \frac{1}{2}$, and set $x = -\frac{1}{2}$, $h = \frac{1}{2}$ in 3.6 to obtain $|f(-\frac{1}{2})| \leq p + \frac{1}{2}$. Hence, if in 3.8, $L = \max\,[p,q] + \frac{1}{2}$, then $x_{i+1} - x_i \leq \frac{1}{2}$ and

$$K \leq \frac{2\left\{\max\,[p,q] + \frac{1}{2} + \epsilon\right\}}{2 - \frac{1}{2}} = \frac{4}{3}\left\{\max\,[p,q] + \frac{1}{2} + \epsilon\right\} .$$

Since ϵ was arbitrary, it follows that

$$K \leq \frac{4}{3}\left\{\max\ [p,q] + \frac{1}{2}\right\}.$$

This completes the proof.

IV. THE MODULUS OF CONTINUITY OF A GENERALIZED QUASI-SMOOTH FUNCTION

A. The Weighted Arithmetic Mean

Let f be defined on $[a,b]$ and satisfy the condition that

(4.1) $$|f(x + h) - f(x)| \leq M|h|, \quad x+h, \quad x \in [a,b], \quad M > 0.$$

Then by use of the triangle inequality, it follows that

(4.2) $$|f(x + h) - 2f(x) + f(x - h)| \leq 2M|h|, \quad x+h, \quad x-h \in [a,b] .$$

However, if 4.2 is satisfied for all x in the interval this need not imply that 4.1 holds. Hamel (4) has constructed a function which satisfies

$$|f(x + h) - 2f(x) + f(x - h)| \equiv 0$$

but which is non-measurable. Kodres (5) has shown that if 4.2 holds, then either f is uniformly continuous or non-measurable. The question arises as to what can be said about the class of continuous functions which satisfies 4.2, that is, the class of quasi-smooth functions. This question was answered by Timan (6) and it was shown that

$$\omega(h) = \sup_{f \in L(a,b)M} \omega(f,h) = \frac{M}{\ln 2} h \log \frac{1}{h} + O(h) .$$

Now the question arises as to the possibility of obtaining a bound on the modulus of continuity of a continuous function f which satisfies the condition

(4.3) $$\left|M_\psi \left[f(x+h),\ f(x-h)\right] - f(x)\right| \le Mh \ .$$

In other words, for what types of means can we obtain a bound on the modulus of continuity of a generalized quasi-smooth function? This question is partially answered in this chapter. We begin by letting M_ψ be the weighted arithmetic mean.

Let $L_p(a,b)M$ be the class of functions f which satisfy

(4.4) f is continuous on $[a,b]$

(4.5) $|pf(x+h) + qf(x-h) - f(x)| \le Mh, \quad p + q = 1,\ M,h,p,q>0.$

We will assume in this section that $p \ne \frac{1}{2}$. In other words, we are using the non-symmetric mean.

Theorem 4.1. If $f \varepsilon L_p(a,b)M$, $p \ne \frac{1}{2}$, then

$$\omega(h) = \sup_{f \varepsilon L_p(a,b)M} \omega(f,h) = \frac{Mh}{|p-q|} \ .$$

Proof: Let $x_1 < x_2$, x_1, $x_2 \varepsilon [a,b]$. Since f(x) is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$. Hence, let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ whenever $|x - x'| < \delta$. Choose n, a positive integer, such that $\frac{x_2 - x_1}{n} = h' < \delta$. Then

$$- Mh' \le pf(x_2) - f(x_2 - h') + qf(x_2 - 2h') \le Mh'$$

$$- Mh' \leq pf(x_2 - h') - f(x_2 - 2h') + qf(x_2 - 3h') \leq Mh'$$

$$- Mh' \leq pf(x_2 - 2h') - f(x_2 - 3h') + qf(x_2 - 4h') \leq Mh'$$

$$\vdots$$

$$- Mh' \leq pf(x_1 + 3h') - f(x_1 + 2h') + qf(x_1 + h') \leq Mh'$$

$$- Mh' \leq pf(x_1 + 2h') - f(x_1 + h') + qf(x_1) \leq Mh' .$$

Addition of these inequalities yields

$$-(n-2)Mh' \leq pf(x_2) + (p-1)f(x_2-h') + qf(x_1) + (q-1)f(x_1+h')$$

$$\leq (n-2)Mh' .$$

Since $p + q = 1$, this inequality can be written

$$|pf(x_2) - qf(x_2- h') + qf(x_1) - pf(x_1 + h')| \leq (n - 2)Mh' .$$

By use of this inequality and the triangle inequality we note that

$$|q-p| \cdot |f(x_1) - f(x_2)| = |qf(x_1) - pf(x_1) - qf(x_2) + pf(x_2)|$$

$$= |pf(x_2) - qf(x_2-h') + qf(x_1) - pf(x_1+ h') + qf(x_2-h')$$

$$- qf(x_2) + pf(x_1 + h') - pf(x_1)|$$

$$\leq |pf(x_2) - qf(x_2- h') + qf(x_1) - pf(x_1+ h')| + q|f(x_2-h')$$

$$- f(x_2)| + p|f(x_1 + h') - f(x_1)|$$

$$\leq (n-2)Mh' + q\epsilon + p\epsilon = (n-2)Mh' + \epsilon \leq M|x_2 - x_1| + \epsilon \ .$$

Therefore,

$$|f(x_2) - f(x_1)| \leq \frac{M\,|x_2 - x_1|}{|p-q|}$$

and

$$\omega(f,h) = \sup_{|x_2-x_1| \leq h} |f(x_2) - f(x_1)| \leq \frac{Mh}{p-q} \ .$$

Now consider the function

$$g(x) = \frac{Mx}{|q-p|}\ , \qquad a \leq x \leq b\ .$$

Then

$$|pg(x+h) - g(x) + qg(x-h)| = Mh\ ,$$

$g(x)$ is continuous and hence $g(x) \varepsilon L_p(a,b)M$. Also,

$$|g(x_1) - g(x_2)| = \frac{M|x_1 - x_2|}{|q-p|}$$

and hence,

$$\sup_{|x_1-x_2| \leq h} |g(x_1) - g(x_2)| = \frac{Mh}{|q-p|}\ .$$

Therefore,

$$\omega(h) = \sup_{f\,\varepsilon\, L_p(a,b)M} \left\{ \sup_{|x_1-x_2| \leq h} |f(x_2) - f(x_2)| \right\} = \frac{Mh}{|q-p|}\ .$$

This completes the proof.

We see from Theorem 4.1 that if $f\,\varepsilon\, L_p(a,b)M$, $p \neq \frac{1}{2}$, then $f \varepsilon \text{Lip } 1$ with Lipschitz constant, $\frac{M}{|p - q|}$. However, if $f \varepsilon L_{\frac{1}{2}}(a,b)M$ then it follows from Lemma 1.4 that $f \varepsilon \text{Lip}(1-\epsilon)$ for any ϵ such that $0 < \epsilon < 1$. For each real number $p \varepsilon (0,1)$ there corresponds a class $L_p(a,b)M$ and the behavior of the modulus of continuity is radically different for the class $L_{\frac{1}{2}}(a,b)M$ than that of the class $L_{\frac{1}{2}\pm\epsilon}(a,b)M$. In other words, the parameter p is critical at 1/2. We have observed this unusual property previously with regard to convex functions.

B. Non-Symmetric Means

Let M_ψ be a non-symmetric mean generated by $\psi(t)$, $\alpha \leq t \leq \beta$. That is to say,

$$M_\psi(x,y) = \psi^{-1}[p\,\psi(x) + q\,\psi(y)], \quad p \neq q, \quad p + q = 1, \quad p,q > 0,$$

where $\psi(t)$ is strictly monotone and continuous for $\alpha \leq t \leq \beta$.

Let $L_\psi(a,b)M$ be the class of functions f which satisfy

(4.6) if $x \varepsilon [a,b]$, then $f(x) \varepsilon [\alpha, \beta]$

(4.7) f is continuous on $[a,b]$

(4.8) $|M_\psi[f(x+h), f(x-h)] - f(x)| \leq Mh, \quad h > 0, \; x+h, x-h \varepsilon [a,b]$.

It is the purpose of this section to obtain a bound on the modulus of continuity of any $f \varepsilon L_\psi(a,b)M$ under suitable restrictions upon ψ.

First we note from Lemma 1.2 that if $\lim_{h\to 0} \frac{\omega(\Psi,h)}{h} = 0$, then Ψ is a constant function on $[\alpha,\beta]$. Therefore, since Ψ is strictly monotone, it is clear that $\lim_{h\to 0} \frac{\omega(\Psi,h)}{h} > 0$. It will be necessary to further restrict this limit to be finite.

<u>Lemma 4.1</u>. If $f \in L_{\Psi}(a,b)M$ and if $\lim_{h\to 0} \frac{\omega(\Psi,h)}{h} = C < \infty$, then $\omega(\Psi f,h) \leq \frac{MCh}{|q - p|}$ where Ψf means $\Psi[f(x)]$.

<u>Proof</u>: Let $f \in L_{\Psi}(a,b)M$. Then by 4.8,

$$f(x) - Mh \leq M_{\Psi}[f(x + h), f(x - h)] \leq f(x) + Mh, \quad h > 0 .$$

Since Ψ is strictly monotone,

$$(4.9) \quad \Psi[f(x) - Mh] \leq p\,\Psi[f(x+h)] + q\,\Psi[f(x-h)] \leq \Psi[f(x) + Mh].$$

We now observe that

$$(4.10) \qquad |\Psi[f(x) + Mh] - \Psi[f(x)]| \leq \omega(\Psi, Mh)$$

and

$$(4.11) \qquad |\Psi[f(x) - Mh] - \Psi[f(x)]| \leq \omega(\Psi, Mh).$$

From 4.10 we obtain

$$(4.12) \qquad \Psi[f(x) + Mh] \leq \omega(\Psi, Mh) + \Psi[f(x)]$$

and from 4.11 we obtain

$$(4.13) \qquad \Psi[f(x) - Mh] \geq -\omega(\Psi, Mh) + \Psi[f(x)] .$$

Inequality 4.9 together with 4.12 and 4.13 yield

(4.14) $|p\,\Psi[f(x+h)] + q\,\Psi[f(x-h)] - \Psi[f(x)]| \leq \omega(\Psi, Mh)$.

Since Ψf is continuous on $[a,b]$, it is uniformly continuous there. Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that $|\Psi[f(x)] - \Psi[f(x')]| < \epsilon$ whenever $|x - x'| < \delta$. Let $x_1 < x_2$, $x_1, x_2 \in [a,b]$ and let $h' = \frac{x_1 - x_2}{n}$, n a positive integer such that $h' < \delta$. Then by use of 4.14 we obtain

$$-\omega(\Psi, Mh') \leq p\,\Psi[f(x_1+2h')] + q\,\Psi[f(x_1)] - \Psi[f(x_1+h')] \leq \omega(\Psi, Mh')$$

$$-\omega(\Psi, Mh') \leq p\,\Psi[f(x_1+3h')] + q\,\Psi[f(x_1+h')] - \Psi[f(x_1+2h')] \leq \omega(\Psi, Mh')$$

$$-\omega(\Psi, Mh') \leq p\,\Psi[f(x_1+4h')] + q\,\Psi[f(x_1+2h')] - \Psi[f(x_1+3h')] \leq \omega(\Psi, Mh')$$

$$\vdots$$

$$-\omega(\Psi, Mh') \leq p\,\Psi[f(x_2-h')] + q\,\Psi[f(x_2-3h')] - \Psi[f(x_2-2h')] \leq \omega(\Psi, Mh')$$

$$-\omega(\Psi, Mh') \leq p\,\Psi[f(x_2)] + q\,\Psi[f(x_2-2h')] - \Psi[f(x_2-h')] \leq \omega(\Psi, Mh').$$

Addition of these inequalities yields

$$-(n-2)\,\omega(\Psi, Mh') \leq q\,\Psi[f(x_1)] - p\,\Psi[f(x_1+h')] + p\,\Psi[f(x_2)]$$
$$- q\,\Psi[f(x_2-h')] \leq (n-2)\,\omega(\Psi, Mh').$$

We can write this inequality in the form

(4.15) $|q\,\Psi[f(x_1)] - p\,\Psi[f(x_1+h')] + p\,\Psi[f(x_2)] - q\,\Psi[f(x_2-h')]|$

$$\leq (n-2)\,\omega(\Psi, Mh') = (n-2)h'M\,\frac{\omega(\Psi, Mh')}{Mh'} \leq M\,|x_2 - x_1|\,\frac{\omega(\Psi, Mh')}{Mh'}.$$

Now by use of the triangle inequality and 4.15 we obtain,

$$|q-p| \cdot \big|\Psi[f(x_1)] - \Psi[f(x_2)]\big| = \big|q\Psi[f(x_1)] - p\Psi[f(x_1)]$$

$$- q\Psi[f(x_2)] + p\Psi[f(x_2)]\big|$$

$$= \big|q\Psi[f(x_1)] - p\Psi[f(x_1+h')] + p\Psi[f(x_2)] - q\Psi[f(x_2-h')]$$

$$+ p\Psi[f(x_1+h')] - p\Psi[f(x_1)] - q\Psi[f(x_2)] + q\Psi[f(x_2-h')]\big|$$

$$\leq M|x_1-x_2|\,\frac{\omega(\Psi, Mh')}{Mh'} + p\big|\Psi[f(x_1+h')] - \Psi[f(x_1)]\big|$$

$$+ q\big|\Psi[f(x_2-h')] - \Psi[f(x_2)]\big|$$

$$\leq M|x_1-x_2|\,\frac{\omega(\Psi, Mh')}{Mh'} + p\epsilon + q\epsilon = M|x_1-x_2|\,\frac{\omega(\Psi, Mh')}{Mh'} + \epsilon .$$

Therefore,

$$\big|\Psi[f(x_1)] - \Psi[f(x_2)]\big| \leq \frac{M|x_1-x_2|}{|p-q|} \cdot \frac{\omega(\Psi, Mh')}{Mh'}$$

and for h' sufficiently small, $\frac{\omega(\Psi, Mh')}{Mh'}$ is arbitrarily near C. Hence,

$$\big|\Psi[f(x_1)] - \Psi[f(x_2)]\big| \leq \frac{MC|x_1-x_2|}{|q-p|}$$

and it follows that

$$\omega(\Psi f, h) \leq \frac{MCh}{|q-p|} .$$

This completes the proof.

Thus we see that Ψf satisfies a Lipschitz condition with Lipschitz constant $\frac{MC}{|p - q|}$. By the use of this lemma, it is possible to make a statement concerning the modulus of continuity of f.

Theorem 4.2. If $f \varepsilon L_{\Psi}(a,b)M$ and if $\lim_{h \to 0} \frac{\omega(\Psi,h)}{h} = C < \infty$, then $\omega(f,h) \leq \omega^{-1}(\Psi, \frac{MC}{|p - q|} h)$, where the symbol $\omega^{-1}(\Psi,h)$ means the inverse of $\omega(\Psi,h)$ considered as a function of h alone.

Proof: Consider the function

$$\omega(\Psi, \omega(f,h)) = \sup_{|z_1 - z_2| \leq \omega(f,h)} |\Psi(z_1) - \Psi(z_2)| .$$

Since Ψ is strictly increasing, this equation can be written

$$\omega(\Psi, \omega(f,h)) = \sup_{|z_1 - z_2| = \omega(f,h)} |\Psi(z_1) - \Psi(z_2)|$$

$$= \sup_{|z_1 - z_2| = \sup_{|x_1 - x_2| \leq h} |f(x_1) - f(x_2)|} \left\{ |\Psi(z_1) - \Psi(z_2)| \right\}$$

Since $g(x,y) = |f(x) - f(y)|$ is continuous in x and y, there exists an x_1^* and an x_2^* such that $\sup_{|x_1 - x_2| \leq h} |f(x_1) - f(x_2)| = |f(x_1^*) - f(x_2^*)|$. Hence

$$\omega(\Psi, \omega(f,h)) = \sup_{|z_1 - z_2| = |f(x_1^*) - f(x_2^*)|} \left\{ |\Psi(z_1) - \Psi(z_2)| \right\}$$

$$= \sup_{|x_1^* - x_2^*| \leq h} |\Psi[f(x_1^*)] - \Psi[f(x_2^*)]| \leq \frac{MC}{|p - q|} h .$$

Hence,

$$\omega(f,h) \leq \omega^{-1}(\psi, \frac{MC}{|p - q|} h) .$$

This completes the proof.

V. BIBLIOGRAPHY

1. Achieser, N. I. Theory of Approximation. New York, N. Y., Frederick Ungar Publishing Company, 1956.

2. Aczel, J. On Mean Values. Bull. Am. Math. Soc. 54: 392-410. 1948.

3. Brudnyĭ, Y. A. On the Maximum Modulus of a Quasi-Smooth Function (in Russian, translated title). Uspehi Mat. Nauk 12, No. 4: 273-275.

4. Hamel, G. Eine Basis aller Zahlen und die unstetige Lösungen der Funktionalgleichung: f(x+y) = f(x)+f(y). Math. Annalen 60: 459-462. 1905.

5. Kodres, U. R. Characterization of Functions Whose Second Differences Approach Zero. Unpublished Ph.D. Thesis. Ames, Iowa, Library, Iowa State University of Science and Technology. 1958.

6. Timan, A. F. On Quasi-Smooth Functions (in Russian, translated title). Izvestiya Akad. Nauk SSSR. Ser. Mat. 15: 243-254. 1951.

7. Zygmund, A. Smooth Functions. Duke Math. Journal 12: 47-76. 1945.

VI. ACKNOWLEDGMENT

The author wishes to express his sincere appreciation for the inspiration and the helpful suggestions given him by Dr. H. P. Thielman during the preparation of this thesis and throughout the author's graduate studies.