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ON THE STABILITY ANALYSIS OF HYBRID COMPOSITE DYNAMICAL SYSTEMS

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On the stability analysis of hybrid

composite dynamical systems

by

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INTRODUCTION, BACKGROUND MATERIAL AND REVIEW OF LITERATURE

As stated in [12] control theory originally developed as a branch of engineering science, and has lately found applications elsewhere. The development of control theory passed through three stages.

(i) A classical stage originating with the study of speed control systems by Maxwell in 1868.

(ii) A more modern stage started during the 1950s when the attention of applied mathematicians was directed to aerospace and to complex industrial problems.

(iii) The most recent stage which emphasizes importance of uncertainty.

Since [12] was written it has become clear that control theory is entering a fourth stage. The use of computers (microcomputers) is emphasized in this most recent stage. Digital control systems are becoming increasingly common because they are often more flexible, compact, and reliable than the analog control systems [1].

Feedback control systems may be described by state space representations or by input-output representations. When a system is described by an input-output representation, then there arises an interesting kind of stability for that system which is referred to as "input-output stability". In order to describe input-output representations and to define the term "input-output stability" we require the following notations and definitions. Let R^n denote real n-space with any convenient norm. Let L_p^n denote the space of all measurable R^n-valued functions $f : (0,\infty) \to R^n$ such that $\int_0^\infty |f(t)|^p \, dt < \infty$ and define the norm $\|f\|_{L_p}$ by

$$\|f\|_{L_p} = \left(\int_0^\infty |f(t)|^p \, dt\right)^{1/p} .$$

The truncation of a function $f : (0,\infty) \to R^n$ is denoted by f_T and is defined by

$$f_T(t) = \begin{cases} f(t) , & \text{if} \quad t \leq T \\ 0 , & \text{if} \quad t > T \end{cases}$$

The extended space L_{pe}^n is defined by

$L_{pe}^n = \{f : (0,\infty) \to R^n \mid f_T \in L_p^n \quad \text{for all} \quad T > 0\}$, and the truncated norm $\|\cdot\|_T$ of $f \in L_{pe}^n$ is defined by $\|f\|_T = \left(\int_0^T |f(t)|^p\right)^{1/p}$ (see Michel and Miller [18, pp. 197]). Suppose that a system with n-inputs and m-outputs can be represented by a mapping $A : L_{pe}^n \to L_{pe}^m$. We say that the system is L_p-stable (input-output stable) if, whenever the input f belongs to the space L_p^n then the output Af belongs to the space L_p^m (see Vidyasagar [24] and C. A. Desoer and Vidyasagar [8]). For such systems the gain μ is defined by

$$\mu = \sup\left\{\frac{\|Af\|_T}{\|f\|_T},\ \ \|f\|_T \neq 0,\ \ \text{for all}\ \ T > 0\ \ \text{and for all}\ \ f \in L^n_{pe}\right\},$$

when this sup is finite. The subject of input-output stability is of recent origin. Important pioneering work was done during the 1960s by Sandberg and Zames (see Sandberg [22] and Zames [28]).

By a state space representation of a control system we mean a description of this control system by a system of differential equations, functional differential equations or difference equations. The state space description leads to concepts such as asymptotic stability in the sense of Lyapunov. Thus for state space representation one makes use of the classical theory of dynamical systems (see, e.g., J. Lasalle [15], R. K. Miller and A. N. Michel [21] and [18], T. Yoshizawa [27], W. Hahn [11], J. Lasalle and Solomon Lefschetz [14] and R. E. Kalman and J. E. Bertram [13]).

The subject of L_p-stability (input-output stability) has been extensively studied. Much has been published about this subject since it was originated in 1960s (see, e.g., M. T. Wu and C. A. Desoer [25], C. T. Chen [5], R. A. Baker and D. J. Vakharia [2] and C. A. Desoer and M. Vidyasagar [8]). It should be understood that an input-output representation and a state representation are two different ways of looking at the same system. Each of the two types of representation give a different perspective on how the system works. There exists a very close relationship between input-output stability and Lyapunov stability

[24, pp. 240-245]. Which of the two approaches is to be preferred depends on the particular application.

In this dissertation, we study the stability of hybrid composite dynamical systems. Such systems are composite systems consisting of a plant which is described by an input-output representation and a controller which is described by a state space representation (see Fig. 1). This is a new point of view. All previous work on control systems required that both the plant and the controller have the same type of representation. This restriction is mathematically convenient but is not always realistic in practice. In real life the plant is often described in terms of its input-output characteristics. On the other hand the controller is usually most conveniently described by a state space representation.

In Fig. 1 the signals u and y may be continuous time signals or discrete time signals. When the controller is described by a set of ordinary differential equations, this case is referred to as the continuous case and the signals u and y are continuous time signals. The case when the controller is described by a set of difference equations is referred to as the digital case. Then the signals u and y are discrete time signals.

Our definitions of stability used in Part I and Part II of this dissertation follow the standard definitions found in most texts on ordinary differential equations and control theory (see, e.g., R. K. Miller and A. N. Michel [21] and W. Hahn [11]). In particular, let the system of Fig. 1 be described by

$$\dot{x} = f(x,u)$$

$$y = c(x,u) \tag{1}$$

$$u = g(y,r_1,r_2)$$

where $x \in R^n$, $r_1 \in L_2^{\ell}$ and $r_2 \in L_2^m$, for fixed $p \triangleq (x_0,r_1,r_1) \in R^n \times L_2^{\ell} \times L_2^m$ let the solution of system (1) be $x(t,p)$. Then system (1) is called stable if given $\varepsilon > 0$, there is a $\delta > 0$ such that for any $p \in R^n \times L_2^{\ell} \times L_2^m$ with $\|p\| < \delta$ we get $|x(t,p)| < \varepsilon$ for all $t \geq 0$ where $\|\cdot\|$ denotes the norm of a point p in $R^n \times L_2^{\ell} \times L_2^m$. System (1) is called asymptotically stable if it is stable and if there is an $R > 0$ such that when $\|p\| < R$, then $x(t,p) \to 0$ as $t \to \infty$. Finally system (1) is called asymptotically stable in the large if it is stable and if for all $p \in R^n \times L_2^{\ell} \times L_2^m$, $x(t,p) \to 0$ as $t \to \infty$. This definition of stability for system (1) is close to the definition of stability under perturbations as defined by J. P. Lasalle [15, pp. 24]. In Part II essentially the same definition of stability with the obvious changes is applied to the digital case.

In the sequel we require the following definition of causality (also called nonanticipativity).

Definition 1.

The operator $L : L_{pe}^{\ell} \to L_{pe}^m$ is said to be causal (i.e., nonanticipative) if the future values of the input do not influence past values of the output.

Mathematically this is equivalent to requiring that $(Lf)_T = (Lf_T)_T$ for all $T < \infty$ and for all $f \in L^{\ell}_{pe}$. It is worth noting that causality is a basic property of physical systems.

Part I of this dissertation is concerned with the continuous case. Criteria for attractivity, uniform boundedness, asymptotic stability, asymptotic stability in the large, exponential stability and exponential stability in the large are established and proved. The results which we prove involve hypotheses which characterize the qualitative properties of the plant as well as the qualitative properties of the controller. For the plant our hypotheses requires some input-output properties to be satisfied such as causality, input-output stability and finite gain. For the controller, our hypotheses requires the existence of a Lyapunov function which satisfies certain properties. Hence our hypotheses require internal stability of the controller. A sufficient condition for the existence of such a Lyapunov function is stated and is proved in a converse Lemma (refer to T. Yoshizawa [27] for the statement and the proof of a similar theorem). In the digital case (Part II) a similar converse Lemma is proved (refer to S. P. Gordon [10] for converse theorems for difference equations).

Our stability results obtained in Part I are summarized in the form of stability criteria which can be easily checked. These stability criteria are given in Theorem 5 of Part I. The proof of the stability criteria given by Theorem 5 of Part I is similar to standard proof for "small-gain theorem" results (see C. A. DeSoer and M. Vidyasagar [8, pp. 40-45] and A. N. Michel and R. K. Miller [18, p. 207]). The stability criteria in Theorem 5 are rather simple and widely applicable. Part I ends with several examples which illustrate the wide applicability of the stability results obtained in that part.

In Part II, we study the digital case, i.e., the case where the controller is described by a set of difference equations. As mentioned before, the control system in this case consists of two parts, the plant (which is the analog part of the system) and the controller (which is the digital part of the system). Signal conversion is essential so that the digital and analog components can be interfaced. For instance, the output signals of the analog part of the system (the plant) must first undergo an analog-to-digital (A/D) conversion before they can be processed by the digital controller. Similarly, the output signals of the digital controller must undergo a digital to analog (D/A) conversion before they can be sent to the analog device for processing (see Fig. 3).

The simplest form of an A/D converter is illustrated in Fig. 2. It consists of a sample-and-hold device and a quantizer. The sample-and-hold device converts an analog signal into a train of a digital signals which are uniformly separated on the time scale by distance T. Each value of the digital signal is held or "frozen" for a prescribed time duration. Theoretically, the holding operation is not needed; however, the conversion time of an A/D is not zero. In order to reduce the effect of signal variation during conversion, the sampled signal is held until the conversion is completed. One of the major operations in the A/D conversion is quantization. Since the digital output can assume only a finite number of values, it is necessary to quantize or "round off" the analog number to the nearest digital value. Two important and commonly used quantizers are of the following types:

(1) Magnitude truncation quantizer

(2) Roundoff quantizer.

Fig. 4a illustrates the characteristic of a magnitude truncation quantizer with quantization level q while Fig. 4b illustrates the characteristic of a roundoff quantizer. An important fact, which is found to be useful in Part II, is that both of the quantizers listed above satisfy the relation $0 < xQ(x) < K_m x^2$, for all $x \in R$ where

$$K_m = \begin{cases} 1 & \text{for magnitude truncation} \\ 2 & \text{for roundoff truncation} \end{cases}$$

The basic elements of a D/A converter are illustrated in Fig. 5. It consists of a sample-and-hold device. In reality, the sampler is redundant in the functional representation of the D/A converter. However, since the sample-and-hold device is usually considered as one unit, the sampling operation is included even though it is not necessary. The type of hold that is used to model the sample-and-hold device is called the zero-order hold (ZOH). The unit impulse response of the zero-order hold is shown in Fig. 6. A data holding scheme whose phase delay is significantly less than that of the usual zero-order hold is presented by O. Yekutiel [26]. Thus lower sampling rates may be used in

A/D converters. As stated by A. H. Levis, R. A. Schlueter and M. Athans [16] the resulting sampling rate reduction can cause a significant degradation of control system performance. For the practical aspects and the details of the operations of A/D and D/A converters refer to M. Schmid [23].

Much has been published about the stability analysis of linear discrete systems (see, e.g., M. Mansour [17], C. A. Desoer and F. L. Lam [6] and C. A. Desoer and M. Y. Wu [7]). The second method of Liapunov is a powerful method of determining the stability of nonlinear discrete-data systems. The second method of Lyapunov is based on the determination of a function called the Lyapunov function. From the properties of that function one is able to show stability or instability of the system. However the main disadvantage is that there are no unique methods of determining the Lyapunov function. In Part II of this dissertation we study the stability analysis of the digital control system given in Fig. 3, which is not necessarily linear. In fact we consider the most general case where the controller is described by the nonlinear difference equation

$$x(k+1) = f(x(k),u(k)) \ . \tag{2}$$

In the sequel we require the following definition.

<u>**Definition 2.**</u>

Let $V(x)$ be a given continuous function $V : \mathbf{R}^n \to \mathbf{R}$, then the <u>first forward difference</u> of V along the solutions of (2) is defined by

$$DV_{(2)}[x(k)] = V[x(k+1)] - V[x(k)]$$
$$= V[f(x(k)] - V[x(k)] \ .$$

The definition of the first forward difference given above is a natural modification of the definition of the derivative for the continuous case. Our definitions of stability for the digital systems studied in Part II of this dissertation are the same as that given in Part I for the continuous case with the obvious necessary changes. Criteria for attractivity, uniform boundedness, asymptotic stability, asymptotic stability in the large and exponential stability are established and proved for the digital system of Fig. 3. The results which we prove in Part II involve hypotheses which characterize the qualitative properties of the plant as well as the qualitative properties of the controller. For the plant, our hypotheses require (as in Part I) some input-output properties to be satisfied. These properties include causality and finite gain. For the controller our hypotheses require the existence of a Lyapunov function with specified properties, i.e., our hypotheses require internal stability of the controller in the sense of Lyapunov. The main question which we address is: what conditions are required to insure that the digital system of Fig. 3 is attractive, asymptotically stable, or exponentially stable (in some appropriate sense), given that it is input-output stable? The attractivity conditions are given in Theorem 1 of Part II while Theorems 2, 3, and 4 give conditions which insure asymptotic stability in the large and exponential stability. In Theorems 5 and 6 we present easy-to-check stability criteria. The hypotheses of Theorem 5 are not only simple and natural but also imply that the hypotheses of Theorems 1, 2, and 4 are all true.

Part II ends with two examples to show the applicability of the stability results obtained in this part. In fact each example is really a wide class of examples. Example 1 presents the tools necessary for the study of stability of the digital systems in which the controller is a second order filter of the type studied in K. T. Erickson and A. N. Michel [9]. In Example 1 the digital controller is chosen to be a direct form digital filter with one quantizer. Other forms of filters which can be treated in a similar manner are, e.g.:

(a) several second order filters in cascade,

(b) several second order filters in parallel,

(c) combinations of (a) and (b).

In two recent papers, Brayton and Tong [3] and [4] established some significant results which make it possible to generate Lyapunov functions by computer. These Lyapunov functions can be used to analyze the stability of nonlinear systems. R. K. Miller and A. N. Michel [20] and [21] used the results of Brayton and Tong [3] and [4] to construct a Lyapunov function which is used in the stability analysis of interconnected systems. K. T. Erickson and A. N. Michel [9] used the results of Brayton and Tong [3] and [4] to construct a Lyapunov function for the fixed-point digital filters of the kind we study in Example 1 of Part II. In that example we use the results of K. T. Erickson and A. N.

Michel to show that the fixed-point digital filters satisfy the hypotheses required by our stability criteria given in Theorem 5.

In Example 2 the controller is described by a system of linear difference equations. For such systems the existence of a Lyapunov function with the required properties is guaranteed by means of Lemma 1 of Part II. This lemma is a converse theorem for stable systems which are described by difference equations (see S. P. Gordon [10] for examples of such converse theorems for difference equations).

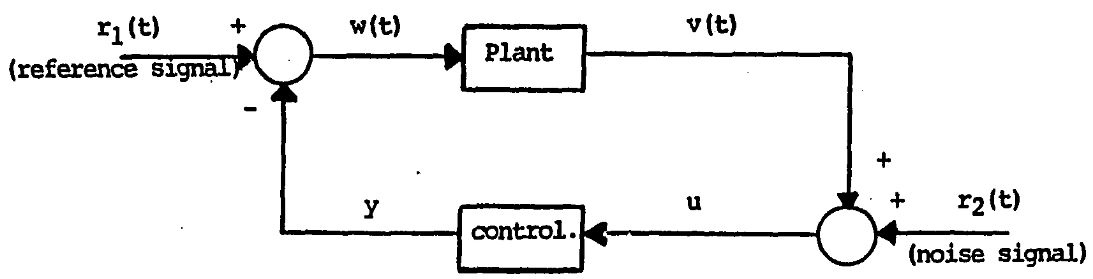

Figure 1. The Composite System

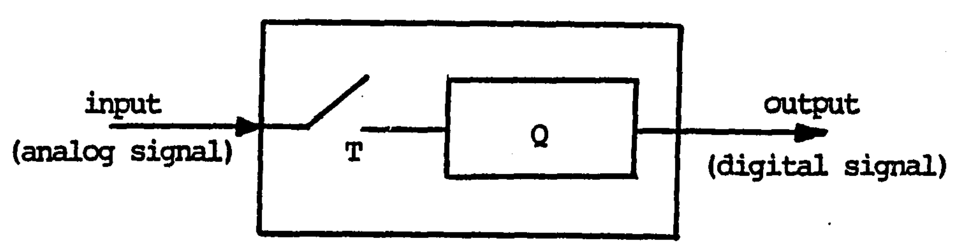

Figure 2. A/D Converter

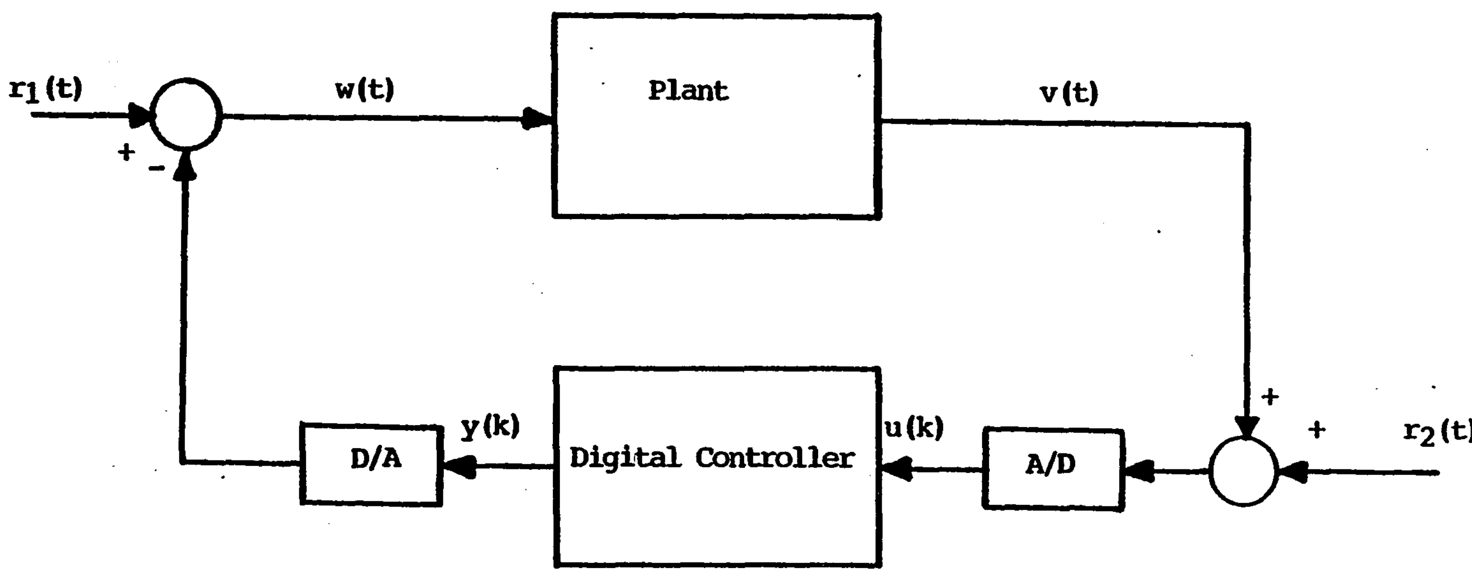

Figure 3. The Composite System (digital case)

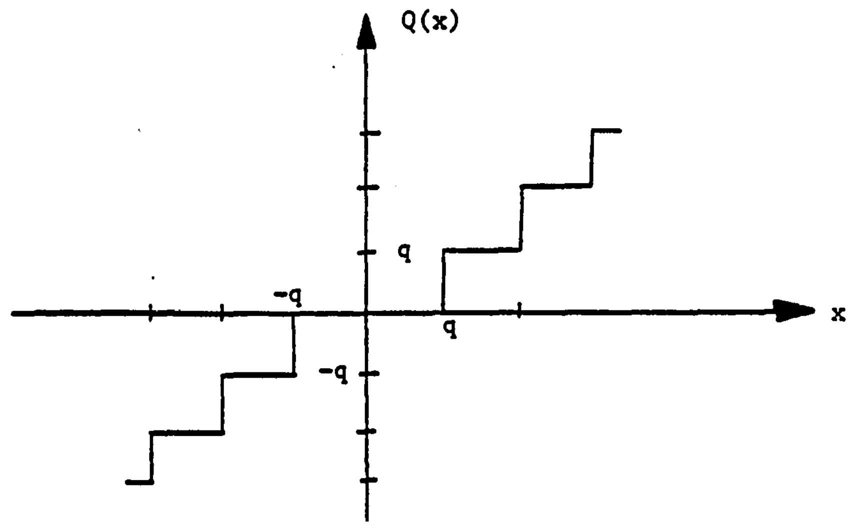

Figure 4a. Magnitude truncation quantization

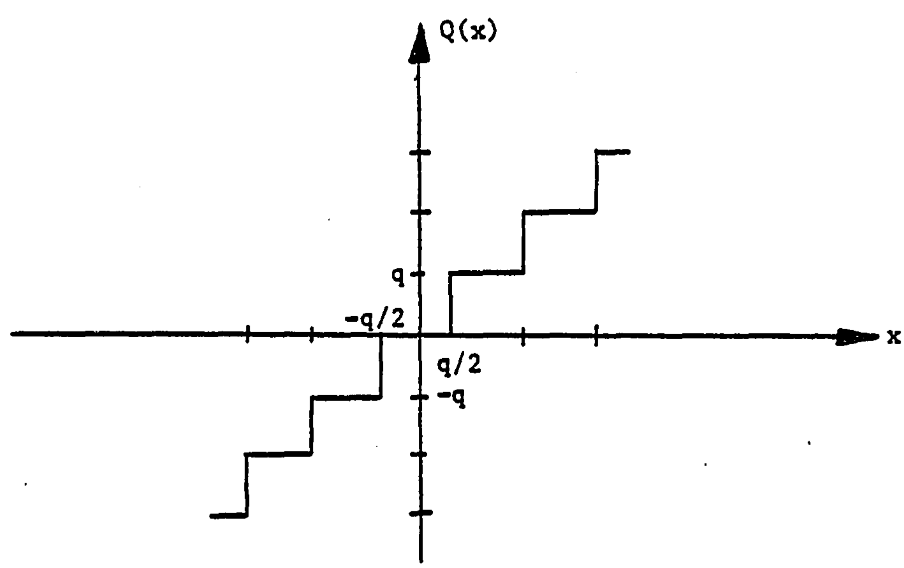

Figure 4b. Roundoff quantization

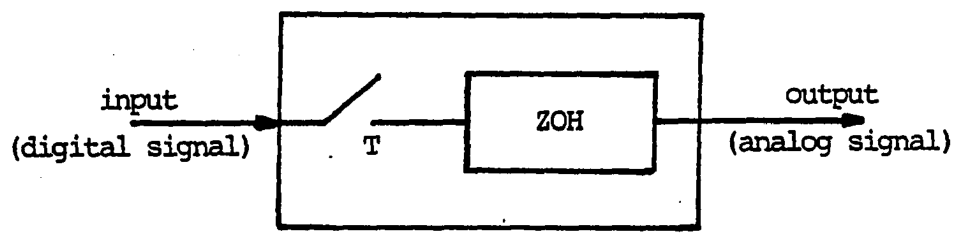

Figure 5. D/A Converter

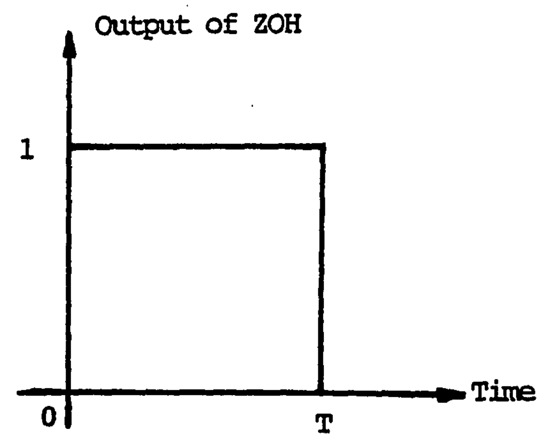

Figure 6. Unit Impulse Response of ZOH

EXPLANATION OF DISSERTATION FORMAT

This dissertation contains two papers, written by the author, which have been submitted for publication. These papers are labeled Part I and Part II. This dissertation consists of four distinct parts: the general introductory material preceding Part I; Part I; Part II; and the material following Part II. In each of these parts, equations and highlighted items such as theorems and figures are numbered consecutively, but separately from the other parts of the dissertation. In addition, the reference numbers in the introductory part and the conclusion following Part II refer to the list of references at the end of the dissertation. However, the reference numbers in Part I and Part II refer to the separate reference lists contained in those two parts.

PART I. STABILITY ANALYSIS OF HYBRID COMPOSITE DYNAMICAL SYSTEMS: DESCRIPTIONS INVOLVING OPERATORS AND DIFFERENTIAL EQUATIONS

ABSTRACT

We address the stability analysis of composite hybrid dynamical feedback systems of the type depicted in Figure 1, consisting of a block (usually the plant) which is described by an operator L and of a finite dimensional block described by a system of ordinary differential equations (usually the controller). We establish results for the well-posedness, attractivity, asymptotic stability, uniform boundedness, asymptotic stability in the large, and exponential stability in the large for such systems. The hypotheses of these results are phrased in terms of the I/O properties of L and in terms of the Lyapunov stability properties of the subsystem described by the indicated ordinary differential equations. The applicability of our results is demonstrated by means of general specific examples (involving C_0-semigroups, partial differential equations or integral equations which determine L).

I. INTRODUCTION

In the present paper we study the qualitative properties of hybrid interconnected systems of the type depicted in Fig. 1. Here, part of the system description is given in terms of an operator L (which is not necessarily linear) while the remaining part of the system is described by a system of ordinary differential equations (of appropriate dimension). The symbols r_1 and r_2 denote external inputs. Such systems arise naturally in applications. For example, the control of a flexible rocket booster may be described in this form, where L represents vehicle and engine actuator dynamics (described by partial differential equations and delay differential equations, respectively) while the indicated system of ordinary differential equations represents the finite dimensional model of the flight control system. Other examples include: the control of a plant where the operator L is determined by the heat equation while the control is represented by an appropriate set of ordinary differential equations; model of power systems, where it is sometimes desired to provide a detailed description of "local" synchronous machines (given by a set of ordinary differential equations) while the remaining power network is characterized by an appropriate operator; the control of large space structures (where L represents the dynamics of the large space structure) while the ordinary differential equations represent the model of the controller; and so forth. System models of the type depicted in Fig. 1 will also generally arise in cases where part of the system model can only be ascertained by means of input-output measurements (i.e., the operator L) while the remainder of the system model is rather precisely known (i.e., the controller described by the system of ordinary differential equations). We emphasize that in the system of Fig. 1, the operator L may represent a finite dimensional subsystem (described, e.g., by ordinary differential equations), or it may represent an infinite dimensional subsystem (described, e.g., by delay equations, functional differential equations, partial differential equations, Volterra integral equations, Volterra integrodifferential equations, etc.), or it may merely represent a memoryless nonlinearity, and the like.

In the sequel, we address the well-posedness and the stability properties of composite systems of the type shown in Fig. 1. For such systems, our results establish criteria for attractivity, uniform boundedness, asymptotic stability, asymptotic stability in the large, exponential stability and exponential stability in the large. (When the inputs $r_1 \equiv 0$ and/or $r_2 \equiv 0$, some of our results are in the usual (Lyapunov) stability sense while for $r_1 \neq 0$ and/or $r_2 \neq 0$, the stability definitions which we use involve obvious and reasonable modifications to the corresponding Lyapunov stability concepts.)

The stability results which we prove involve hypotheses which characterize the qualitative properties of the operator L as well as the qualitative properties of the subsystem described by the indicated ordinary differential equations. For L, these characterizations are given in terms of input-output properties (e.g., I/O stability, gain passivity, causality of L). On the other hand, the qualitative properties of the subsystem described by the ordinary differential equations are expressed via Lyapunov results (e.g., we may require that the system of differential equations be Lyapunov stable in some sense, or we may postulate the existence of some appropriate Lyapunov function which possesses certain properties along the solutions of the differential equations.) In addition, our results will also usually involve a hypothesis concerning the I/O stability of the entire system given in Figure 1. Indeed, a central question which we address is the following: what conditions ensure that the system of Fig. 1 is (in some appropriate sense) attractive, uniformly bounded, asymptotically stable, or exponentially stable, given that it is I/O stable (in some appropriate sense)?

We emphasize that whereas the results reported herein are tangentially related to existing work (see, e.g., Willems [15], [16], Hill and Moylan [3], [9], Vidyasagar [13], [14]), the problems which we address in this paper have not been addressed before in either form, scope or generality. (E.g., results which relate I/O stability and Lyapunov stability are usually confined to finite dimensional systems or to very specialized infinite dimensional systems, and they are usually of a global nature. Furthermore, existing results usually require some reachability (resp., controllability) conditions and/or some detectability (resp., observability) conditions, whereas our results do not.)

The remainder of this paper is organized as follows.

In Section II we provide the essential nomenclature for the paper.

In Section III we establish our basic results for attractivity (Theorem 1), for asymptotic stability (Theorem 2), for asymptotic stability in the large and for uniform boundedness (Corollary 1 and

Theorem 3) and for exponential stability in the large (Theorem 4). Some of these results make use of a procedure for constructing Lyapunov functions which is spelled out in the proof of Lemma 1.

There are two hypotheses ((A-5) and (A-6)) in Theorems 1, 2, 3, and 4 and Corollary 1 which can not always be established in an obvious manner. These difficulties are removed in Section IV (Theorems 5, 6, and 7).

The remainder of this paper is dedicated to applications involving specific classes of feedback systems.

In Section V we apply the results of Sections III and IV to a class of C_0-semigroups (Lemmas 2 and 3). These results in turn are applied to a feedback system for which the operator L is determined by the heat equation. In the rest of Section V we establish a stability result (Lemma 4) for a feedback system with a passive operator L (using the results of Sections III and IV) and we apply this result in the analysis of a feedback system for which the operator L is determined by the heat equation. Then we consider two specific examples of feedback systems for which L is determined by integral equations.

This paper is concluded in Section VI with some additional comments.

II. NOTATION

Let V and W be arbitrary sets. Then $V \cup W$, $V \cap W$ and $V \times W$ denote the union, intersection and Cartesian product of V and W, respectively. If V is a subset of W we write $V \subset W$ and if x is an element of V we write $x \in V$. If f is a function of V into W, we write $f : V \to W$.

We let R denote the real numbers and we let $R^+ = [0,\infty)$. We let R^n denote real n-space, we let $|\cdot|$ represent any one of the equivalent norms defined on R^n, and we let x^T denote the transpose of $x \in R^n$.

Unless otherwise specified, matrices are usually assumed to be real. If A is a real $m \times n$ matrix we write $A \in R^{m\times n}$ and we let A^T denote the transpose of a matrix A. Also, $|A|$ denotes the norm of a matrix A.

Given a Lipschitz continuous function $v : R^n \to R$ and a system of ordinary differential equations

$$\dot{x} = f(x) \tag{E}$$

where $f : R^n \to R^n$ and $\dot{x} = dx/dt$, we define the (Dini) derivative of v (with respect to t) for (E) by

$$\dot{v}_{(E)}(x) = \lim_{h \to 0^+} \sup \frac{v(x+hf(x)) - v(x)}{h} .$$

If in particular v is continuously differentiable, this reduces to

$$\dot{v}_{(E)}(x) = \nabla v(x)^T f(x)$$

where $\nabla v(x)$ denotes the gradient of $v(x)$. If $\phi(t,x_0)$ solves the initial-value problem

$$\dot{x} = f(x), \quad x(0) = x_0 ,$$

then it is known that

$$\dot{v}_{(E)}(x_0) = \lim_{t \to 0^+} \frac{v(\phi(t,x_0)) - v(x_0)}{t} .$$

We let

$$C[0,\infty) = \{f : R^+ \to R \mid f \text{ is continuous}\}.$$

for $p \in [1,\infty)$, we let

$$L_p^m(a,b) = \{f : (a,b) \to R^m \mid f \text{ is measurable and } \int_a^b |f(t)|^p \, dt < \infty\}$$

and we define the norm $\|\cdot\|_{L_p}$ of $f \in L_p^m(a,b)$ by

$$\|f\|_{L_p} = (\int_a^b |f(t)|^p \, dt)^{1/p} .$$

In particular, we shall find it convenient to write $L_2^m \triangleq L_2^m(0,\infty)$. Similarly, we let

$$L_\infty^m(a,b) = \{f : (a,b) \to R^m \mid f \text{ is essentially bounded over } (a,b)\}$$

and for $f \in L_\infty^m(a,b)$ we define the norm $\|\cdot\|_{L_\infty}$ by

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{t \in (a,b)} |f(t)| .$$

For the space $L_2^m = L_2^m(0,\infty)$ we define the extended space L_{2e}^m as

$$L_{2e}^m = \{f : (0,\infty) \to R^m \mid f \in L_2^m(0,T) \text{ for each } T > 0\}$$

and we define the (truncated) norm $\|\cdot\|_T$ of $f \in L_{2e}^m$ by

$$\|f\|_T = \left(\int_0^T |f(t)|^2\, dt\right)^{1/2} .$$

(the extended spaces L_{pe}^m with corresponding truncated norms $\|\cdot\|_T$ are defined in the obvious way). Also, spaces $L_p^{\ell\times\ell}(a,b)$ with matrix-valued elements are defined in the obvious way.) We also define the truncated inner product of $f,g \in L_{2e}^m$, $\langle\cdot,\cdot\rangle_T$, by

$$\langle f,g\rangle_T = \int_0^T f(t)^T g(t)dt.$$

Furthermore, for any $\sigma > 0$ and for $f \in L_{2e}^m$, we let

$$\|f\|_{\sigma,T} = \left(\int_0^T |f(t)|^2\, e^{2\sigma t}\, dt\right)^{1/2} .$$

Also, we let

$$L^m_{\sigma,2} = \{f \in L^m_{2e} : \int_0^\infty |f(t)|^2 e^{\sigma t} dt < \infty \}$$

and we define on this space

$$\|f\|_{\sigma,2} = \left(\int_0^\infty |f(t)|^2 e^{2\sigma t} dt\right)^{1/2}.$$

In this paper we will find it convenient to make use of the product space

$$X = R^n \times L^\ell_2 \times L^m_2 .$$

For $p = (x, r_1, r_2) \in X$, we define the norm

$$\|p\| = |x| + \|r_1\|_{L_2} + \|r_2\|_{L_2} .$$

Furthermore, we let

$$X_\sigma = R^n \times L^\ell_{\sigma,2} \times L^m_{\sigma,2}$$

and for $p = (x, r_1, r_2) \in X_\sigma$, we define

$$\|p\|_\sigma = |x| + \|r_1\|_{\sigma,2} + \|r_2\|_{\sigma,2} .$$

Let $f \in L_{2e}^{m}$. We define f_T, the truncation of f, by

$$f_T = \begin{cases} f(t), & t \in [0,T] \\ 0, & t > T \end{cases} \qquad \text{for all } f \in L_{2e}^{m} .$$

If L is an operator from L_{2e}^{m} into L_{2e}^{n}, then L is said to be causal if

$$(Lf)_T = (Lf_T)_T \quad \text{for all } T < \infty \text{ and for all } f \in L_{2e}^{m} .$$

Finally, if L is an operator from L_{2e}^{m} into L_{2e}^{n}, then we define the gain of L, gain (L), by

$$\text{gain } (L) = \sup\left\{ \frac{\|LF\|_T}{\|f\|_T} , \quad \|f\|_T \neq 0, \quad \text{for all } T > 0, \quad \text{for all } f \in L_{2e}^{m}\right\}.$$

III. BASIC STABILITY THEOREMS: ATTRACTIVITY, ASYMPTOTIC STABILITY AND EXPONENTIAL STABILITY

In the present section we establish some basic stability results for composite systems of the type depicted in Fig. 1. Such systems are governed by equations of the form

$$\begin{aligned}
\dot{x} &= f(x) + B(u), \quad x(0) = x_0 \\
y &= C(x,u) \\
w &= r_1 - y \\
v &= Lw \\
u &= v + r_2
\end{aligned} \tag{S}$$

or

$$\begin{aligned}
\dot{x} &= f(x) + B(u), \quad x(0) = x_0 \\
y &= C(x,u) \\
u &= r_2 + L(r_1 - y) \ .
\end{aligned} \tag{1}$$

We will find it useful to make the following reasonable assumptions for such systems.

Assumption (A-1)

$r_1 \in L_2^\ell = L_2^\ell(0,\infty)$, $r_2 \in L_2^m = L_2^m(0,\infty)$, $B : R^m \to R^n$, B is continuous with $|B(u)| \le K_B|u|$, $C : R^{n\times m} \to R^\ell$, C is continuous with $C(0,0) = 0$, $L : L_{2e}^\ell \to L_{2e}^m$, L maps the zero function into the zero function, $f : R^n \to R^n$, $f(0) = 0$, and f is Lipschitz continuous, i.e., there is a $K_1 > 0$ such that

$$|f(x) - f(\bar{x})| \le K_1|x - \bar{x}| \tag{2}$$

for all $x,\bar{x} \in R^n$.

Given $(x_0,r_1,r_2) = p \in X = R^n \times L_2^\ell \times L_2^m$, we assume that system (S) has a unique solution $(x(t,p), u(t,p)) \in C[0,\infty) \times L_{2e}^m$. ◇

Assumption (A-2)

There exist constants $k > 0$, $c > 0$ such that the solution $\phi(t,x_0)$ of the initial-value problem

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{E}$$

satisfies the bound $|\phi(t,x_0)| \le k\alpha e^{-ct}$ for all $t \ge 0$ and for all $x_0 \in R^n$ with $|x_0| \le \alpha$. ◇

We will require the following preliminary result.

Lemma 1

If f satisfies (1) (resp., (S)) and if (E) satisfies (A-2), then there exists a Lyapunov function $V : R^n \to R$ for system (E) which satisfies the following conditions:

(i) $V(0) = 0$ and $V(x) \geq |x|$ for all $x \in R^n$;

(ii) there exists $L_1 > 0$ such that for all $x, \bar{x} \in R^n$,

$$|V(x) - V(\bar{x})| \leq L_1 |x - \bar{x}| \ ;$$

and

(iii) $\dot{V}_{(E)}(x) \leq - qcV(x)$ for all $x \in R^n$ where q is a fixed constant in the range $0 < q < 1$ and c is given in (A-2).

Proof. The proof is similar to the one given in Yoshizawa [17, p. 94]. Since we shall later need some of the computations from the proof, we shall outline it here.

Fix $q \in (0,1)$ and define

$$V(x) = \sup_{\tau \geq 0} |\phi(\tau,x)| e^{qc\tau}$$

where $\phi(t,x)$ denotes the solution of (E) with $\phi(0,x) = x$. Taking $\tau = 0$ we see that $V(x) \geq |\phi(0,x)| = |x|$. Also, $\phi(\tau,0) = 0$ so that $V(0) = 0$. Given $x_0 \neq 0$, let $\alpha = |x_0|$.

Define T by the solution

$$K = (1/2)\ e^{(1-q)cT},$$

i.e.,

$$T = \frac{\ell n(2K)}{(1-q)c}\ .$$

Then,

$$\frac{\alpha}{2} < |x_0| \leq V(x_0) \leq \sup_{\tau \geq 0} |\phi(\tau,x_0)|e^{qc\tau}\ .$$

By (A-2) we see that

$$\frac{\alpha}{2} \leq \sup_{\tau \geq 0} (K\alpha e^{-c\tau})e^{qc\tau} = \sup_{T \geq 0} \frac{\alpha}{2} e^{(1-q)cT}\ e^{-c\tau}\ e^{qc\tau}$$

$$\leq \frac{\alpha}{2} \sup_{\tau \geq 0}\ e^{c(1-q)(T-\tau)}\ .$$

The sup in the definition of $V(x)$ is never taken on when $\tau > T$ since for $\tau > T$ we would have

$$\frac{\alpha}{2} \leq \frac{\alpha}{2} \sup_{\tau \geq T}\ e^{c(1-q)(T-\tau)} < \frac{\alpha}{2}\ .$$

Thus,

$$V(x) = \sup_{0 \leq \tau \leq T} |\phi(\tau,x)|e^{qc\tau}$$

where $T = \ell n(2K)/[(1-q)c]$ is a fixed constant independent of $x \in R^n$.

For $x,\bar{x} \in R^n$ we have

$$|V(x) - V(\bar{x})| = \left| \sup_{0 \leq \tau \leq T} |\phi(\tau,x)| e^{qc\tau} - \sup_{0 \leq \tau \leq T} |\phi(\tau,x)| e^{qc\tau} \right|$$
$$\leq \sup_{0 \leq \tau \leq T} |\phi(\tau,x) - \phi(\tau,\bar{x})| e^{qc\tau} . \tag{3}$$

Since ϕ satisfies the equation

$$\phi(\tau,x) = x + \int_0^\tau f(\phi(s,x))ds ,$$

and since f satifies (2), then

$$|\phi(\tau,x) - \phi(\tau,\bar{x})| \leq |x - \bar{x}| + \int_0^\tau |f(\phi(s,x)) - f(\phi(s,\bar{x}))| ds$$
$$\leq |x - \bar{x}| + K_1 \int_0^\tau |\phi(s,x) - \phi(s,\bar{x})| ds .$$

By the Gronwall inequality we obtain

$$|\phi(\tau,x) - \phi(\tau,\bar{x})| \leq |x - \bar{x}| e^{K_1 \tau} .$$

Substituting this relation into (3) we see that

$$|V(x) - V(\bar{x})| \leq \sup_{0 \leq \tau \leq T} e^{K_1 \tau} e^{qc\tau} |x - \bar{x}| \leq L_1 |x - \bar{x}|$$

where

$$\left.\begin{aligned} L_1 &= e^{(K_1+qc)T} \\ T &= \ln(2k)/[(1-q)c]. \end{aligned}\right\} \tag{4}$$

Given $x \in R^n$, let $\bar{x} = \phi(h,x)$ so that

$$V(\bar{x}) = \sup_{\tau \geq 0} |\phi(\tau,\bar{x})| e^{cq\tau} = \sup_{\tau \geq 0} |\phi(\tau+h,x)| e^{qc\tau}$$

$$= \sup_{\tau \geq h} \left(|\phi(\tau,x)| e^{qc\tau}\right) e^{-qch} \leq V(x)\, e^{-qch} .$$

Then

$$\frac{V\big(\phi(h,x)\big) - V(x)}{h} \leq V(x)\, \frac{e^{-qch} - 1}{h}$$

and

$$\lim_{h \to 0^+} \sup \frac{V\big(\phi(h,x)\big) - V(x)}{h} \leq V(x) \lim_{h \to 0} \frac{e^{-qch} - 1}{h}$$

$$= V(x)(-qc) . \qquad \diamond$$

In the sequel, we also require the following additional assumption.

Assumption (A-3)

System (S) is L_2-input-output stable. That is, for every $p \triangleq (x_0, r_1, r_2) \in X$, the solutions $u(t,p)$ are in L_2^m. $\diamond$

Theorem 1 (Attractivity)

If (A-1), (A-2) and (A-3) are true for system (S), then for any $p \in X$, the solution $x(t,p)$ tends to zero as $t \to \infty$.

Proof: By Lemma 1 there exists a Lyapunov function $V(x)$ for system (E) and constants $q \in (0,1)$ and $L_1 > 0$ such that (i), (ii) and (iii) of Lemma 1 are true. For the equation

$$\dot{x} = f(x) + B(u) \tag{5}$$

we have

$$\dot{V}_{(5)}(x) < \dot{V}_{(E)}(x) + L_1 |B(u)|$$

or

$$\dot{V}_{(5)}(x) < - cqV(x) + L_1 \; K_B |u| \; .$$

By the comparison principle (see, e.g., Miller and Michel [8], Hahn [2], Michel and Miller [6]), $V(x(t,p))$ will be less than or equal to the solution of

$$\dot{W} = - qcW + L_1 K_B |u(t,p)|, \quad W(0) = V(x_0) \; . \tag{6}$$

Since $|x(t,p)| < V(x(t,p)) < W(t)$, then

$$|x(t,p)| < V(x_0)e^{-qct} + \int_0^t e^{-qc(t-s)} L_1 K_B |u(s,p)| ds \; . \tag{7}$$

Since $u(t,p) \in L_2^m$, then (7) and the Schwarz inequality imply that

$$|x(t,p)| \leq V(x_0) + L_1 K_B \|u(\cdot,p)\|_{L_2} / \sqrt{2cq} \ . \tag{8}$$

Hence, $x(t,p)$ is bounded in t. Moreover, the first term on the right hand side in (7) tends to zero as $t \to \infty$. The second term on the right hand side in (7) is the convolution of two L_2-functions and thus, must tend to zero as $t \to \infty$ (see, e.g., Rudin [10, p. 4]). Hence, $x(t,p) \to 0$ as $t \to \infty$, as required. ◇

Remark 1

Theorem 1 shows that the origin $x = 0$ is attractive for system (S); however, it does not say that solutions must exhibit stable behavior in any reasonable sense. ◇

We now turn to the question of stability.

Definition 1

We call system (S) stable if given $\varepsilon > 0$, there is a $\delta > 0$ such that for any $p \in X$ with $\|p\| < \delta$ we have $|x(t,p)| < \varepsilon$ for all $t \geq 0$. (Here $\|\cdot\|$ denotes the norm for X defined in Section II.)

We call system (S) asymptotically stable if it is stable and if there is an $R > 0$ such that when $\|p\| \leq R$, then $x(t,p) \to 0$ as $t \to \infty$.

We call system (S) asymptotically stable in the large if it is stable and if for all $p \in X$, $x(t,p) \to 0$ as $t \to \infty$. ◇

In the next two results we make use of the following hypothesis.

Assumption (A-4)

For system (S) we have: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $p \in X$ and $\|p\| < \delta$, then $\|u(\cdot,p)\|_{L_2} < \varepsilon$. ◇

We are now in a position to consider asymptotic stability.

Theorem 2 (Asymptotic Stability)

Assume that (A-1), (A-2) and (A-4) are true. Then system (S) is asymptotically stable in the sense of Definition 1.

Proof: For $\varepsilon = 1$ use (A-4) to pick $\delta_1 > 0$. Let $R = \delta_1$. For $\|p\| < R$ we have $u(t,p) \in L_2$. Hence, the proof of Theorem 1 applies. By that proof, $x(t,p) \to 0$ as $t \to \infty$, and (8) is true.

Given $\varepsilon > 0$, by (A-4) we pick $\delta_2 > 0$ such that $\|p\| < \delta_2$ implies

$$\|u(\cdot,p)\|_{L_2} < \sqrt{2cq}\ \varepsilon/(2L_1K_B) \ . \tag{9}$$

Since $V(x) < L_1|x|$, then $V(x_0) < \varepsilon/2$ when $|x_0| < \varepsilon/(2L_1)$. Let $\delta = \min\{\delta_2, \varepsilon/2L_1, R\}$. If $\|p\| < \delta$, then by (8) and (9) we see that $|x(t,p)| < \varepsilon$ for all $t \geq 0$. ◇

Corollary 1 (Asymptotic Stability in the Large).

If (A-1)-(A-4) are true, then system (S) is asymptotically stable in the large in the sense of Definition 1.

Proof: The result follows by combining Theorems 1 and 2. ◇

Proceeding we note that (A-3) and (A-4) are true when the following hypothesis is true.

Assumption (A-5)

There exists $M > 0$ such that for any $p \in X$,

$$\|u(\cdot,p)\|_{L_2} < M\|p\| . \quad \diamond$$

When (A-5) is true, then Corollary 1 can be strengthened in the following manner.

Theorem 3 (Asymptotic Stability in the Large)

If (A-1), (A-2) and (A-5) are true, then system (S) is asymptotically stable in the large in the sense of Definition 1. Moreover, solutions of system (S) are uniformly bounded in the sense that there is a $K > 0$ with $|x(t,p)| < K\|p\|$ for all $t > 0$ and for all $p \in X$.

Proof. Only the uniform boundedness remains to be proved. By (8) and (A-5) we have

$$\begin{aligned} |x(t,p)| &< V(x_0) + L_1K_B(2cq)^{-1/2}M\|p\| \\ &< L_1|x_0| + L_1K_B(2cq)^{-1/2}M\|p\| \\ &< L_1(1 + K_BM/\sqrt{2cq})\|p\| . \quad \diamond \end{aligned}$$

In order to establish an exponential stability type result, we require the shifted version of (A-5) which assumes the following form.

Assumption (A-6)

There exists $\sigma > 0$ and $M > 0$ such that for any $p \in X$, we have

$$\|u(\cdot,p)\|_{\sigma,2} \leq M\|p\|_{\sigma} . \qquad \diamond$$

In the remaining sections of this paper we will establish conditions on system (S) (and on some special cases of (S)) which ensure that hypotheses (A-5) or (A-6) are true. First, however, we state and prove the last result of the present section.

Theorem 4 (Exponential Stability in the Large)

If (A-1), (A-2) and (A-6) are true, then for any $p \in X_\sigma$ and any $t > 0$, we have

$$|x(t,p)| \leq \{L_1 + 2L_1K_BM(2qc)^{-1/2}\}\|p\|_\sigma \; e^{-[\sigma qc/(\sigma+qc)]t} .$$

(In this case, system (S) is said to be <u>exponentially stable in the large in the sense of Theorem 4</u>.)

<u>Proof</u>. As in the proof of Theorem 1 (see inequality (7)), we see that

$$|x(t,p)| \leq V(x_0)e^{-qct} + L_1K_B \int_0^t e^{-qc(t-s)}|u(s,p)|ds$$

$$\leq L_1|x_0|e^{-qct} + L_1K_BI(t)$$

where $I(t)$ denotes the integral $\int_0^t e^{-qc(t-s)}|u(s,p)|ds$. Let

$Q = qc(qc+\sigma)^{-1}$. Then $I(t)$ can be written as

$$I(t) = \int_0^{tQ} e^{-qc(t-s)}|u(s,p)|ds + \int_{tQ}^{t} e^{-qc(t-s)}|u(s,p)|e^{\sigma s}e^{-\sigma s}\, ds$$

$$\leq e^{-qc(1-Q)t} \int_0^{Qt} e^{-qc(Qt-s)}|u(s,p)|ds$$

$$+ e^{-\sigma Qt} \int_{Qt}^{t} e^{-qc(t-s)}|u(s,p)|e^{\sigma s}\, ds$$

$$\leq e^{-qc(1-Q)t}(2qc)^{-1/2}\, M\|p\|_\sigma + e^{-\sigma Qt}(2qc)^{-1/2}\, M\|p\|_\sigma \,.$$

By the choice of Q, we have $qc(1-Q) = \sigma Q = \sigma qc(\sigma+qc)^{-1}$. Thus,

$$I(t) \leq 2(2qc)^{-1/2}\, M\|p\|_\sigma\, e^{-\sigma Qt}$$

and

$$|x(t,p)| \leq \{L_1 e^{-qct}\|p\|_\sigma + 2ML_1K_B(2qc)^{-1/2}\, e^{-\sigma Qt}\|p\|_\sigma\}$$

$$\leq \left(L_1 + \frac{2ML_1K_B}{\sqrt{2qc}}\right)\|p\|_\sigma\, e^{-\sigma qc(\sigma+qc)^{-1}t} \,. \qquad \diamond$$

Note that Theorem 4 provides a performance criterion for system (S); i.e., Theorem 4 yields estimates of the rate of decay of the solutions for system (S). This type of information, which is useful in applications, is not implied by Theorems 1-3.

Remark 2

Existing results which relate I/O stability and Lyapunov-type stabilities involve in general detectability (resp., observability) and/or reachability (resp., controllability) hypotheses (see, e.g., Vidyasagar [13], [14]). We emphasize that in general, assumptions of this type do not seem to be required in the present results. ◇

IV. STABILITY CONDITIONS: SMALL-GAIN THEOREM TYPE RESULTS

As indicated earlier, the purpose of this section is to present conditions on system (S) which will ensure that hypotheses (A-5) or (A-6) are true. Our results, which are of the "small-gain theorem" type (see, e.g., Desoer and Vidyasagar [1], Sandberg [11], [12], Zames [18], and Michel and Miller [6]) fall into one of two categories: (a) results which assume the existence of a Lyapunov function $V(x)$ for (E); and (b) results which assume that (E) is exponentially stable in the large and for which a Lyapunov function is constructed in some optimal manner.

Theorem 5

For system (S) assume that hypotheses (A-1) and (A-2) are true. In addition, assume that

(i) $|C(x,u)| \leq K_c|x| + K_c'|u|$ for all $(x,u) \in R^{n+m}$ where $K_c > 0$ and $K_c' > 0$ are constants; and

(ii) the operator L is nonanticipative (i.e., causal) and has gain μ.

If the "small gain" condition

$$\mu K_c' + \frac{\mu L_1 K_B K_c}{qc} < 1 \tag{10}$$

is true, then hypothesis (A-5) (and hence, hypothesis (A-3) and (A-4)) is true.

Proof: Let $T > 0$ be fixed. Given a solution of sytem (S), we have

$$\|y\|_T = \|C(x,u)\|_T \le K_c\|x\|_T + K_c'\|u\|_T ,$$

$$\|w\|_T \le \|r_1\|_T + \|y\|_T \le \|r_1\|_T + K_c\|x\|_T + K_c'\|u\|_T ,$$

$$\|v\|_T = \|Lw\|_T \le \mu\|w\|_T \le \mu\|r_1\|_T + \mu K_c\|x\|_T + \mu K_c'\|u\|_T$$

and

$$\|u\|_T = \|r_2\|_T + \|v\|_T$$

$$\le \|r_2\|_T + \mu\|r_1\|_T + \mu K_c\|x\|_T + \mu K_c'\|u\|_T , \qquad (11)$$

where $\|\cdot\|_T$ denotes the truncated norm defined in Section II.

We know that there is a Lyapunov function $V(x)$ which satisfies the conditions

$$|x| \le V(x) ,$$

$$|V(x) - V(\bar{x})| \le L_1|x - \bar{x}|, \text{ and}$$

$$\dot{V}_{(E)}(x) \le -qcV(x) .$$

For system (S) we now have

$$\dot{V}_{(S)}(x) \le -qcV(x) + L_1K_B|u(t)| .$$

Hence,

$$V(x(t)) < e^{-qct}V(x_0) + \int_0^t L_1K_Be^{-qc(t-s)}|u(s)|ds .$$

Since $|x| < V(x)$, then

$$\|x\|_T < \|V(x)\|_T < V(x_0)\frac{1}{\sqrt{2cq}} + L_1K_B\frac{1}{qc}\|u\|_T . \qquad (12)$$

Combining inequalities (11) and (12) yields

$$\|u\|_T < \|r_2\|_T + \mu\|r_1\|_T + \mu K_c'\|u\|_T + \mu K_cV(x_0)/\sqrt{2qc}$$
$$+ \left(\mu K_cL_1K_B/(qc)\right)\|u\|_T ,$$

or

$$\left(1 - \mu K_c' - \frac{\mu K_cL_1K_B}{qc}\right)\|u\|_T < \|r_2\|_T + \mu\|r_1\|_T + \frac{\mu K_c}{\sqrt{2qc}}V(x_0)$$
$$< K(p) \triangleq \|r_2\|_{L_2^m} + \mu\|r_1\|_{L_2^\ell}$$
$$+ \frac{\mu K_c}{\sqrt{2qc}}L_1|x_0| .$$

By (10), $1 - \mu K_c' - \mu K_cL_1K_B/(qc) > 0$. Hence,

$$\|u\|_T < \left(1 - \mu K_c' - \mu K_cL_1K_B/(qc)\right)^{-1}K(p)$$

for all $T > 0$. Letting $T \to \infty$, we see that

$$\|u\|_{L_2^m} \leq (1 - \mu K_c' - \mu K_c L_1 K_B/(qc))^{-1}[\max\{1,\mu, \frac{\mu K_c}{\sqrt{2qc}} L_1\}\|p\|] \ . \ \diamond$$

If the Lyapunov function $V(x)$ for (E) is given with $|V(x) - V(\bar{x})| \leq L_1|x - \bar{x}|$ and $\dot{V}_{(E)}(x) \leq -kV(x)$ for all x, $\bar{x} \in R^n$, then Theorem 5 remains true with (10) replaced by

$$\mu K_c' + \mu \frac{L_1 K_B K_c}{k} < 1 \ . \tag{13}$$

If the Lyapunov function $V(x)$ must be constructed by Lemma 1, then the constant L_1 depends on q where $0 < q < 1$. There arises then the question of how to choose q optimally, i.e., how to choose q so that the left side of (10) is minimized. The next result provides the answer.

<u>**Theorem 6**</u>

Assume that hypotheses (A-1) and (A-2) as well as assumptions (i) and (ii) of Theorem 5 are true. If the condition

$$\mu K_c' + \mu \frac{K_B K_c}{q^* c} \exp[\frac{K_1 + q^* c}{c(1-q^*)} \ln K] < 1$$

is true where $q^* = 1 + a - \sqrt{a^2 + 2a}$ and $a = \frac{c+K_1}{2c} \ln K$, then (A-5) is true.

<u>Proof</u>: If $V(x)$ is obtained by Lemma 1 and if $f(q)$ denotes the right-hand side of (10), then we require that $f(q) < 1$ where

$$f(q) = \mu K'_c + \mu \frac{L_1 K_B K_c}{qc} .$$

By (4) we have $T = \ln 2K/[c(1-q)]$, $L_1 = \exp(\frac{K_1+qc}{c-qc} \ln 2K)$ and

$$f(q) = \mu K'_c + \mu \frac{K_b K_c}{cq} \exp(\frac{K_1+qc}{c(1-q)} \ln\, 2K), \quad 0 < q < 1 .$$

If $\ln K > 0$, then it is an easy matter to check that $f(q)$ has a minimum at q^*. If $f(q^*) < 1$, then (10) will be true. By Theorem 5 it follows that (A-5) is true.

If $\ln K = 0$, then $q^* = 1$ and $f(q)$ has a minimum at $q^* = 1$. If $f(q^*) < 1$, then $f(q) < 1$ for all q sufficiently near $q^* = 1$. Hence, one can choose $q \in (0,1)$ so that (10) will be true and Theorem 5 implies that (A-5) is true. ◇

The last result of the present section yields conditions for (S) such that hypothesis (A-6) is true.

Theorem 7

Assume that hypotheses (A-1) and (A-2) are true and that assumption (i) of Theorem 5 is true. Also, assume that:

(ii) L is nonanticipative (i.e., causal) and there is a σ with $0 < \sigma < cq$ such that the operator L has shifted gain factor $\mu(\sigma)$, i.e.,

$$\|Lf\|_{\sigma,T} \leq \mu(\sigma)\|f\|_{\sigma,T}$$

for all $f \in L_{2e}^{\ell}$ and for all $T > 0$.

If the "small gain" condition

$$\mu(\sigma)K_c' + \mu(\sigma)\,\frac{L_1 K_B K_c}{qc-\sigma} < 1$$

is true, then hypothesis (A-6) is true.

Proof: The proof is similar to that of Theorem 5. As in Theorem 5 (see inequality (11)), we see that

$$\|u\|_{\sigma,T} \leq \|r_2\|_{\sigma,T} + \mu(\sigma)\|r_1\|_{\sigma,T} + \mu(\sigma)K_c\|x\|_{\sigma,T}$$

$$+ M(\sigma)K_c'\|u\|_{\sigma,T}\,. \tag{14}$$

Moreover, for system (S) we have

$$|x(t)| \leq V\big(x(t)\big) \leq V(x_0)e^{-qct} + L_1K_B \int_0^t e^{-qc(t-s)}|u(s)|ds.$$

Hence,

$$|x(t)|e^{\sigma t} \leq V(x_0)e^{(-qc+\sigma)t} + L_1K_B \int_0^t e^{(\sigma-qc)(t-s)}|u(s)|e^{\sigma s}ds$$

and

$$\|x\|_{\sigma,T} \leq V(x_0)(2cq - 2\sigma)^{-1/2} + \{L_1K_B/(qc-\sigma)\}\|u\|_{\sigma,T}\,. \tag{15}$$

Combining (14) and (15), we see that

$$\|u\|_{\sigma,T} \le \left[1 - \mu(\sigma)K_c' - \mu(\sigma)K_c L_1 K_B/(qc-\sigma)\right]^{-1}$$

$$\cdot \max\left\{1, \mu(\sigma), \mu(\sigma)\frac{K_c L_1}{\sqrt{2(qc-\sigma)}}\right\}\|p\|_{\sigma,T} .$$

Letting $T \to \infty$ completes the proof. ◇

In the remainder of this paper we consider specific applications of the results of Section III and of the present section.

V. APPLICATIONS AND EXAMPLES

A. Application: Results Involving C_0-Semigroups with Applications to the Heat Equation

We now turn our attention to applications of the results of Sections III and IV. We first consider a class of C_0-semigroups. Next, as a specific example, we consider a composite system (S) for which the operator L is determined by the heat equation.

1. An Example Involving General C_0-Semigroups

Let H be a Hilbert space with inner product $\langle\ ,\ \rangle$, let z^* be a continuous linear functional on H, and let B be a self adjoint, negative definite linear mapping defined on a dense subset of H into H with $\sup\langle Bz,z\rangle = -B_0 < 0$, $z \in H$, where $\langle z,z\rangle^{1/2} = \|z\|$. (For definitions of these concepts, refer, e.g., to Michel and Herget [5].)

Lemma 2

Consider the system

$$\left.\begin{aligned} \dot{z} &= Bz - n(y)g, \quad z(0) = z_0 \\ \dot{x} &= f(x) + bz^*(z), \quad x(0) = x_0 \\ y &= d^T x \end{aligned}\right\} \qquad (E_1)$$

where $b \in R^n$, $d \in R^n$, $x_0 \in R^n$, $z_0 \in H$, g is a fixed element of H and $f(x)$ is a continuous function which satisfies the conditions given in (A-1) and (A-2). Assume that $n(y)$ is a Lipschitz continuous function whose graph lies in the sector $[0,\gamma]$, i.e., we require that $0 \leq n(y)/y \leq \gamma$ for all $y \neq 0$. If

$$\frac{\gamma \|z^*\| \|g\| L_1 |b| |d|}{B_0 qc} < 1 \tag{16}$$

then system (E_1) is asymptotically stable in the large in the sense of Definition 1. (The constants L_1, q and c are defined in the previous sections.) Moreover, under these conditions, system (E_1) is also <u>asymptotically stable in the large</u> in the usual (Lyapunov) sense:

(i) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|z_0\| < \delta$, $|x_0| < \delta$, imply that $\|z(t,x_0,z_0)\| < \varepsilon$ and $|x(t,x_0,z_0)| < \varepsilon$ for all $t \geq 0$; and

(ii) For every x_0 and z_0,

$$\lim_{t \to \infty} \|z(t,x_0,z_0)\| = \lim_{t \to \infty} |x(t,x_0,z_0)| = 0 .$$

<u>Proof:</u> Let $U(t)$ be the C_0-semigroup generated by B (see, e.g., Krein [4, pp. 82-84] or [6, chapter 5]). Then

$$\|U(t)\| \leq e^{-B_0 t} \quad \text{for all} \quad t \geq 0 .$$

Moreover, mild solutions of (E_1) are given by

$$z(t) = U(t)z_0 - \int_0^t U(t-s)g\, n(y(s))\, ds$$

$$\dot{x} = f(x) + bz^*(z(t)), \quad x(0) = x_0 \tag{17}$$

$$y = d^T x .$$

Since f and n are Lipschitz continuous, the existence, uniqueness and continuation of solutions of (17) for all $t \geq 0$ can be established by standard contraction mapping arguments (see, e.g., [5]).

The above system can be put into the form of system (S) (resp., of System (1)), if we define

$$Lw(t) = z^*\left(\int_0^t U(t-s)g\, n(y(s))ds\right) = \int_0^t z^*(U(t-s)g)\, n(y(s))ds ,$$

$$r_1(t) \equiv 0 , \quad \text{and}$$

$$r_2(t) = z^*(U(t)z_0) .$$

Now since $\|Lw\|_T \leq \|z^*(U(t)g)\|_{L_1} \gamma\|y\|_T \leq \gamma\|z^*\|\| \frac{1}{B_0} \|y\|_T$, then $\mu = \gamma\|z^*\|\|g\|/B_0$ is a gain for L. Hence, by Theorem 5, if (16) is true, then system (E_1) is asymptotically stable in the large in the sense of Definition 1.

To see that system (E_1) is also asymptotically stable in the large in the usual (Lyapunov) sense, consider

$$z(t) = U(t)z_0 - \int_0^t U(t-s)g\, n(d^T x(s))ds .$$

Then

$$|z(t)| \leq e^{-B_0 t}|z_0| + \int_0^t e^{-B_0(t-s)} \|g\| \gamma |d||x(s)|ds \; . \tag{18}$$

By Theorem 1, we have $|x(t)| \to 0$ as $t \to \infty$ and thus, $|z(t)| \to 0$ as $t \to \infty$. Hence, the origin $(x,z) = (0,0)$ is attractive for system (E_1). To show that the origin $(0,0)$ is stable as well, fix $\varepsilon > 0$. Since by the first part of the proof, the system is asymptotically stable in the large in the sense of Definition 1, then there exists a $\delta > 0$ with $0 < \delta < \varepsilon/2$ such that if $|x_0| \leq \delta$ and $\|r_2\|_{L_2} \leq \delta$, then

$$|x(t)| \leq \frac{B_0 \varepsilon}{2\gamma\|g\||d|+2B_0} \triangleq \varepsilon_1 < \varepsilon \text{ for all } t \geq 0 \; . \tag{19}$$

But $|r_2(t)| = |z^*(U(t)z_0)| \leq \|z^*\| e^{-B_0 t}\|z_0\|$ which implies that $\|r_2\|_{L_2} \leq (\|z^*\|/B_0)\|z_0\|$. Hence, if

$$\|z_0\| \leq \min\{\delta, B_0\delta/\|z^*\|\} \triangleq \delta_1 \leq \delta$$

then

$$\|r_2\|_{L_2} \leq \delta \; .$$

Thus, if $|x_0| < \delta_1$ and $\|z_0\| < \delta_1$, then (19) implies that

$$|x(t)| < \varepsilon \text{ for all } t \geq 0$$

and (18) implies that

$$\|z(t)\| < \int_0^t e^{-B_0(t-s)} \|g\| \; \gamma \; |d| \; \varepsilon_1 \; ds + e^{-B_0 t} \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all} \quad t > 0 \; .$$

This proves stability in the usual sense. ◇

<u>Lemma 3</u>

Let all the assumptions of Lemma 2 be true for system (E_1). Then the zero solution $(x,z) = (0,0)$ of system (E_1) is also exponentially stable in the usual (Lyapunov) sense: there exists an $\alpha > 0$, and for $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|z(t,x_0,z_0)\| < \varepsilon \, e^{-\alpha t} \quad \text{for all} \quad t > 0$$

and

$$|x(t,x_0,z_0)| < \varepsilon \, e^{-\alpha t} \quad \text{for all} \quad t > 0$$

whenever $\|z_0\| < \delta$ and $|x_0| < \delta$.

<u>Proof:</u> Condition (16) implies that there exists $0 < \sigma < B_0$ such that

$$\frac{\gamma \|z^*\| \|g\| L_1 |b| |d|}{(B_0 - \sigma)qc} < 1 \; . \tag{20}$$

Since $|U(t)| < e^{-B_0 t}$ for all $t > 0$, we have for the above choice of σ,

$$|e^{\sigma t} L(y(t))| = |e^{\sigma t} z^*(\int_0^t U(t-s)g \; n(y(s))ds)|$$

which implies that

$$\|L(y(t))\|_{\sigma,T} \le \|z^*\|\|g\| \;\gamma\|\int_0^t e^{-(B_0-\sigma)(t-s)} |y(s)|e^{\sigma s}ds\|_T$$

$$\le \gamma \frac{\|z^*\|\|g\|}{B_0 - \sigma} \|y\|_{\sigma,T} .$$

Hence, a shifted gain for L is

$$\mu(\sigma) = \gamma \frac{\|z^*\|\|g\|}{B_0 - \sigma} .$$

If (20) is true, it follows from Theorem 7 that (A-6) is true for system (E_1) and by Theorem 4 we have

$$|x(t)| \le K(|x_0| + \|r_2\|_{\sigma,2})e^{\frac{-qc\sigma}{qc+\sigma} t} \quad \text{for all} \quad t \ge 0 . \tag{21}$$

But

$$\|r_2\|_{\sigma,2} = (\int_0^\infty |z^*(U(t)z_0)|^2 \; e^{2\sigma t})^{1/2}$$

$$\le (\int_0^\infty \|z^*\|^2\|z_0\|^2 \; e^{[-2(B_0-\sigma)]t} \; dt)^{1/2}$$

$$\le \|z^*\|\|z_0\|/\sqrt{2(B_0-\sigma)} .$$

This last inequality shows that $\|r_2\|_{\sigma,2}$ is small when $\|z_0\|$ is small. Thus, if $|x_0|$ and $\|z_0\|$ are small, then (21) shows that $|x(t)|$ is decaying exponentially. Also, inequality (18) shows that $z(t)$ is decaying exponentially with t. Therefore, the zero solution

$(x,z) = (0,0)$ of system (E_1) is exponentially stable in the usual (Lyapunov) sense. ◇

2. A Specific Example: Application to the Heat Equation

As a special case of (E_1), consider now the system

$$\left.\begin{aligned}
&\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} - n(y)g(x), \quad 0 < x < \ell \\
&z(0,x) = z_0(x), \quad z(t,0) = z(t,\ell) = 0 \\
&\dot{x} = f(x) + b\int_0^\ell \beta(x)\, z(t,x)dx, \quad x(0) = x_0 \\
&y = cx ,
\end{aligned}\right\} \tag{22}$$

where all the assumptions on $f(x)$ and $n(x)$ are as before. Here $H = L_2(0,\ell)$ while z^* denotes the functional defined by

$$z^*(\phi) = \int_0^\ell \beta(x)\phi(x)dx \,.$$

We assume that $\beta(x) \in H$ is fixed and thus,

$$\|z^*\| = \|\beta\|_{L_2(0,\ell)} \,. \tag{23}$$

In our particular case, the linear operator B is determined by the equation

$$B\phi = \frac{\partial^2\phi}{\partial x^2}, \quad \phi(0) = \phi(\ell) = 0 \ .$$

This operator is self adjoint and negative definite with

$$\sup_{\|z\| = 1} \langle Bz,z\rangle = -(\pi/\ell)^2 \triangleq -B_0 < 0 \ . \tag{24}$$

If we now apply Lemma 2, we see that if

$$\frac{\gamma\ell^2 L_1 |b| |d| \ \|g\|_{L_2} \|\beta\|_{L_2}}{qc\pi^2} < 1 \tag{25}$$

then the zero solution $(x,z) = (0,0)$ of system (22) is asymptotically stable in the large in the sense of Definition 1 as well as in the usual (Lyapunov) sense.

By Lemma 3 we see that if (25) is true, then there exists a σ with $0 < \sigma < B_0$ such that

$$\frac{\gamma\ell^2 L_1 |b| |d| \ \|g\|_{L_2} \|\beta\|_{L_2}}{qc(\pi^2 - \sigma\ell^2)} < 1 \tag{26}$$

and the system (22) is exponentially stable in the usual (i.e., Lyapunov) sense. ◇

We notice that for the above example the output from the heat equation is given by

$$z^*(z,(t,\cdot)) = \int_0^\ell \beta(x)\ z(t,x)dx$$

where β is a fixed element of H. This output is a weighted average of $z(t,x)$ at time t. For example, if we want the output to be the temperature at x_0, then we can choose $\beta(x) = 0$ except when x is near x_0 and $\beta(x)$ positive near x_0 with $\int_0^{\ell} \beta(x)dx = 1$. This will yield an average temperature which is nearly equal to $z(t,x_0)$. Notice also that the feedback control in (22) multiplies the function $g(x)$. Hence, this is a distributive type of control.

B. Application : A Passivity Result with Applications to the Heat Equation

In order to demonstrate the applicability of the results of Sections III and IV further, we now consider a result involving the concept of passivity. We first consider a general class of feedback systems (Figure 2) which are special cases of composite system (S). Next, as in Application (1), we treat a specific example where the operator L of system (S) is determined by the heat equation.

1. A Class of Feedback Systems

We begin by proving the following result.

Lemma 4

Consider the system of Figure 2 which is described by the equations

$$\left.\begin{aligned} x(t) &= r(t) + z(t) \\ y(t) &= n\big(x(t)\big) \\ w(t) &= -y(t) \\ z(t) &= L\big(w(t)\big) = \int_0^t a(t-\tau)w(\tau)d\tau \ . \end{aligned}\right\} \qquad (E_2)$$

Assume that for (E_2) the following hypotheses are true:

(i) $a(t)$ is a function in $L_1(0,\infty)$;

(ii) $n(x)$ is a continuous function which belongs to the sector $[0,\gamma]$, $\gamma > 0$, i.e.,

$$0 \leq \frac{n(x)}{x} \leq \gamma \quad \text{for all} \quad x \neq 0 \ ;$$

(iii) there exists constants $\lambda > 0$, $\Delta > 0$ such that

$$\inf_{\omega \geq 0} \mathrm{Re}\left[a^*(j\omega)(1 + j\omega/\lambda)\right] + \frac{1}{\gamma} \geq \Delta \tag{27}$$

where $a^*(s)$ denotes the Laplace transform of $a(t)$; and

(iv) $\mu \triangleq \sup_{\omega \geq 0} \left|a^*(j\omega)(1 + j\omega/\lambda)\right| < \infty$.

If $r_2 \triangleq r + \dot{r}/\lambda \in L_2$ then (A-5) is true and system (E_2) is asymptotically stable in the large in the sense of Definition 1.

Proof: The system of Figure 2 (i.e., system (E_2)) is equivalent to the system of Figure 3. We thus have

$$\dot{x} = -\lambda x + \lambda u \qquad x(0) = r(0) + z(0) = r(0)$$

$$y = n(x)$$

which implies that

$$\begin{aligned}
\langle u,y\rangle_T &= \langle x + \dot{x}/\lambda,\ n(x)\rangle_T \\
&= \int_0^T x\ n(x)\ dt + \frac{1}{\lambda}\int_0^T \dot{x}\ n(x)\ dt \\
&= \langle x,n(x)\rangle_T + \frac{1}{\lambda}\int_{x(0)}^{x(T)} n(\sigma)\ d\sigma \\
&= \langle x,n(x)\rangle_T + \frac{1}{\lambda}\int_0^{x(T)} n(\sigma)\ d\sigma - \frac{1}{\lambda}\int_0^{x(0)} n(\sigma)\ d\sigma \\
&\geq \langle x,n(x)\rangle_T - \frac{1}{\lambda}\int_0^{x(0)} n(\sigma)\ d\sigma\ .
\end{aligned}$$

But

$$\langle y,y\rangle_T = \langle n(x),n(x)\rangle_T = \int_0^T n(x)\ n(x)dt$$

$$\leq \gamma \int_0^T x\ n(x)dt = \gamma\langle x,n(x)\rangle_T \ ,$$

and thus,

$$\langle u,y\rangle_T \geq \frac{1}{\gamma} \langle y,y\rangle_T - \frac{1}{\lambda} \int_0^{x(0)} n(\sigma)\ d\sigma \ .$$

Now $u = r_2 - L_1 y$, and hence,

$$\langle r_2,y\rangle_T = \langle u + L_1 y,y\rangle_T$$

$$= \langle u,y\rangle_T + \langle L_1 y,y\rangle_T \ .$$

Since $\|r_2\|_T\|y\|_T \geq |\langle r_2,y\rangle_T|$, we have

$$\|r_2\|_T\|y\|_T \geq \langle u,y\rangle_T + \langle L_1 y,y\rangle_T$$

$$\geq \frac{1}{\gamma} \langle y,y\rangle_T - \frac{1}{\gamma} \int_0^{x(0)} n(\sigma)d\sigma + \langle L_1 y,y\rangle_T \ .$$

By assumption (iii), we have the (strict) passivity condition

$$\langle L_1 y,y\rangle_T \geq \left(\Delta - \frac{1}{\gamma}\right)\langle y,y\rangle_T \ .$$

Hence,

$$\|r_2\|_T\|y\|_T \geq \Delta\|y\|_T^2 - \frac{1}{\lambda}\int_0^{x(x)} n(\sigma)d\sigma$$

or

$$\|y\|_T^2 - \frac{1}{\Delta}\|r_2\|_T\|y\|_T \leq \frac{1}{\Delta\lambda}\int_0^{x(0)} n(\sigma)d\sigma \ .$$

Completing the square in the above inequality yields

$$\left(\|y\|_T - \frac{\|r_2\|_T}{2\Delta}\right)^2 \leq \frac{1}{\Delta\lambda}\int_0^{x(0)} n(\sigma)d\sigma + \left(\frac{\|r_2\|_T}{2\Delta}\right)^2 ,$$

and thus,

$$\|y\|_T \leq \frac{\|r_2\|_T}{\Delta} + \sqrt{\frac{1}{\lambda\Delta}\int_0^{x(0)} n(\sigma)d\sigma} \ .$$

But $n(\sigma) < \gamma\sigma$. Hence,

$$\|y\|_T \leq \frac{\|r_2\|_T}{\Delta} + \sqrt{\frac{\gamma}{\lambda\Delta}}\ |x(0)| \ .$$

By assumption (iv), the operator L_1 has a finite gain $\gamma > 0$. Therefore,

$$\|u\|_T = \|r_2+v\|_T \leq \|r_2\|_T + \|v\|_T \leq \|r_2\|_T + \mu\|y\|_T$$

$$\leq \left(1 + \frac{\mu}{\Delta}\right)\|r_2\|_T + \sqrt{\frac{\gamma}{\lambda\Delta}}\ |x(0)\mu \ .$$

If $r_2 = r_2 + \dot{r}/\lambda \in L_2$, then (A-5) is true and system (E_2) is asymptotically stable in the large in the sense of Definition 1. ◇

2. A Specific Example: Application to the Heat Equation

For the heat equation consider the feedback system given by

$$\left.\begin{aligned} &\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2} - n\big(w(t)\big)g(x) \\ &W(t,0) = W(t,\ell) = 0, \quad W(0,x) = W_0(x) \end{aligned}\right\} \tag{29}$$

where the output $w(t)$ is given by

$$w(t) = \int_0^{\ell} \beta(x)W(t,x)dx \tag{30}$$

for some fixed $\beta \in L_2(0,\ell)$. Assume that $\beta'' \in L_2(0,\ell)$, that $\beta(0) = \beta(\ell) = 0$, and that $n(\cdot)$ belongs to the sector $[0,\gamma]$.

Now let $\lambda_n = (\pi n/\ell)^2$ and $\phi_n(x) = \sqrt{2/\ell}\,\sin n\pi x/\ell$. Expanding W_0, β, and g in the Fourier series, we obtain

$$W_0(x) = \sum_{n=1}^{\infty} W_{0n}\,\phi_n(x)\ ,$$

$$\beta(x) = \sum_{n=1}^{\infty} \beta_n\,\phi_n(x), \quad \text{and}$$

$$g(x) = \sum_{n=1}^{\infty} g_n\,\phi_n(x)\ .$$

Then (29) has the mild solution

$$W(t,x) = \sum_{n=1}^{\infty} W_{0n}\, e^{-\lambda_n t}\, \phi_n(x)$$

$$- \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n (t-s)}\, n\big(w(s)\big)ds\; g_n\, \phi_n(x) \qquad (31)$$

and

$$w(t) = \sum_{n=1}^{\infty} W_{0n}\, \beta_n\, e^{-\lambda_n t} - \sum_{n=1}^{\infty} g_n \beta_n \int_0^t e^{-\lambda_n (t-s)}\, n\big(w(s)\big)ds\ .$$

We are now in a position to rewrite system (29) in the form of (E_2) as

$$\left.\begin{aligned}
r(t) &= \sum_{n=1}^{\infty} W_{0m}\, \beta_n\, e^{-\lambda_n t} \\
y(t) &= n\big(w(t)\big) \\
w(t) &= r(t) - L\big(y(t)\big) \\
L\big(y(t)\big) &= \sum_{n=1}^{\infty} g_n\, \beta_n\, e^{-\lambda_n (t-s)}\; y(s)\; ds \\
&= \int_0^t \Big(\sum_{n=1}^{\infty} g_n\, \beta_n\, e^{-\lambda_n (t-s)} \Big) y(s)\; ds\ .
\end{aligned}\right\} \qquad (32)$$

Now $\lambda_n > \lambda_1$ for all n implies that

$$|r(t)| \leq \sum_{n=1}^{\infty} |W_{0n}\ \beta_n| e^{-\lambda_n t}$$

$$\leq e^{-\lambda_1 t} \Big(\sum_{n=1}^{\infty} |W_{0n}|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |\beta_n|^2\Big)^{1/2}$$

$$= e^{-\lambda_1 t} \|W_0\|_{L_2} \|\beta\|_{L_2} ,$$

and thus, $r \in L_2$. Moreover,

$$\beta(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x), \quad \text{and}$$

$$\beta''(x) = -\sum_{n=1}^{\infty} \beta_n\ \lambda_n\ \phi_n(x) .$$

Therefore,

$$\Big(\sum_{n=1}^{\infty} |\beta_n \lambda_n|^2\Big)^{1/2} = \Big(\int_0^{\ell} |B''(x)^2 dx\Big)^{1/2} = \|B''\|_{L_2} < \infty ,$$

since we assumed that $\beta'' \in L_2(0,\ell)$ and $\beta(0) = \beta(\ell) = 0$. Hence,

$$|\dot{r}(t)| = \Big|\sum_{n=1}^{\infty} - W_{0n}\ \lambda_n\ \beta_n\ e^{-\lambda_n t}\Big|$$

$$\leq e^{-\lambda_1 t} \|W_0\|_{L_2} \|\beta''\|_{L_2} .$$

Thus, $\dot{r} \in L_2(0,\ell)$. Moreover,

$$|w(0)| = |r(0)| = \Big|\sum_{n=1}^{\infty} W_{0n}\ \beta_n\Big| \leq \|W_0\|_{L_2} \|\beta\|_{L_2} .$$

Therefore, if there exists a $\lambda > 0$ and $\Delta > 0$ such that

$$\inf_{\omega > 0} \mathrm{Re}\Big[\sum_{n=1}^{\infty} \frac{g_n \beta_n}{j\omega + \lambda_n} [1 + j\omega/\lambda)\Big] + \frac{1}{\gamma} > \Delta > 0 ,$$

or equivalently, such that

$$\inf_{\omega > 0} \sum_{n=1}^{\infty} g_n \beta_n \Big(\frac{\lambda_n + \omega^2/\lambda}{\lambda_n^2 + \omega^2}\Big) + \frac{1}{\gamma} > \Delta > 0 ,$$

then all the hypotheses of Lemma 4 are satisfied and system (32) (resp., system (29)) is asymptotically stable in the sense of Definition 1. ◇

C. Application: Applications to Integral Equations

Before concluding this paper, we demonstrate the applicability of the present results to composite systems (S) where the operator L is described by integral equations. We consider two specific examples. In the first of these, the operator need not be causal.

Example 1

If (A-1) is changed to require that $L : L_2^{\ell} \to L_2^m$ and that the solutions $(x,(t,p),\ u(t,p)) \in C[0,\infty) \times L_2^m$. then all the results in Section III remain true. This extension allows us to treat operators L which are noncausal. For example, consider the feedback system described by the equations

$$\left.\begin{aligned} Lw &= \int_0^\infty a(t-\tau)w(\tau)d\tau \\ \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx \\ u &= r_2 + Lw \\ w &= r_1 - y \end{aligned}\right\} \qquad (E_3)$$

where A, B and C are matrices of appropriate dimensions, A is stable, and $a \in L_1^{\ell\times\ell}(0,\infty)$. This system is a special case of composite system (S) for which the operator L is noncausal.

Since

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} Bu(s)\,ds$$

$$= e^{At} x_0 + e^{At} * (Bu) ,$$

and

$$u = r_2 + Lw = r_2 + L(r_1 - Cx) ,$$

then

$$u = r_2 + Lr_1 - L\left(Ce^{At} x_0\right) - L\left(Ce^{At} * Bu\right) . \qquad (33)$$

For any $u_1 \in L_2^n$ and $w_1 \in L_2^\ell$ it is true that

$$\|Lw_1\|_{L_2} \leq \|a\|_{L_1} \|w_1\|_{L_2} ,$$

and

$$\|e^{At} * u_1\|_{L_2} \leq \|e^{At}\|_{L_1} \|u_1\|_{L_2} .$$

Hence, the right hand side of (33) determines a contraction mapping on L_2^m where

$$\alpha \triangleq \|a\|_{L_1} |C| \|e^{At}\|_{L_1} |B| < 1 . \tag{34}$$

If (34) is true, then for any $p \in X$ there is a unique solution $u(t,p)$ of (33) in L_2^m which satisfies

$$\|u(\cdot,p)\|_{L_2} \leq (1-\alpha)^{-1}\left(\|r_2\|_{L_2} + \|a\|_{L_1} \|r_1\|_{L_2} + \|a\|_{L_1} |C| \|e^{At}\|_{L_1} |x_0|\right)$$

$$\leq M\|p\|$$

where

$$M = (1-\alpha)^{-1}\left(1 + \|a\|_{L_1} + |C| \|a\|_{L_1} \|e^{At}\|_{L_1}\right) .$$

Given u, we define x, y and w by the second, third and fifth lines in (E_3). This yields a solution of (E_3) with $x(t,p) \in C[0,\infty)$ and $u(t,p) \in L_2^\ell$. Moreover, (A-5) is true. Hence, system (E_3) is asymptotically stable in the large in the sense of Definition 1. ◇

Example 2

As a second example for composite system (S), where L is determined by integral equations, consider the system

$$\begin{aligned} \dot{x} &= f(x) + Bu, \quad x(0) = x_0 \\ y &= d^T x \\ \dot{z}(t) &= Dz(t) + \int_0^t E(t-s)z(s)ds - n(y(t))g, \quad z(0) = z_0 \\ u(t) &= Fz(t) \end{aligned} \tag{E_4}$$

where $x \in R^n$, $z(t) \in R^\ell$, $u(t) \in R^m$, $f : R^n \to R^n$, $B \in R^{n\times m}$, $d \in R^n$, $D \in R^{\ell\times\ell}$, $g \in R^\ell$, $E \in L_1^{\ell\times\ell}(0,\infty)$, $n : R \to R$, and $F \in R^{m\times\ell}$. We asssume that f is Lipschitz continuous, that $f(0) = 0$ and that hypothesis (A-2) is true. Also, we assume that $n(y)$ is Lipschitz continuous and that its graph belongs to the sector $[0,\gamma]$. Furthermore, we assume the determinant condition

$$\det(sI - D - E^*(s)) \neq 0 \tag{35}$$

in the right half complex plane $\mathrm{Re}\ x > 0$. In (35) $E^*(s)$ denotes the Laplace transform of $E(t)$ and I denotes the $\ell\times\ell$ identify matrix. Finally, we define

$$\mu \triangleq \sup F(j\omega I - D - E^*(j\omega))^{-1} g \ . \tag{36}$$

An easy Laplace transform argument shows that the third line of (E_4) can be written in the form

$$z(t) = R(t)z_0 - \int_0^t R(t-s)n(y(s))g\,ds \ . \tag{37}$$

Here $R(t)$ is the $\ell \times \ell$ matrix valued function whose Laplace transform is

$$R^*(s) = (sI - D - E^*(s))^{-1} \ .$$

By assumption (35) and results in Miller [7] it follows that $R \in L_p^{\ell\times\ell}(0,\infty)$ and $\dot{R} \in L_p^{\ell\times\ell}(0,\infty)$ for each p in $1 \leq p \leq \infty$. In particular $R \in L_1$ and $R \in L_2$. Also, $R(t)$ must tend to zero as $t \to \infty$. These facts will be needed below.

From (37) we see that (E_4) can be written in the form

$$\left.\begin{aligned} &\dot{x} = f(x) + Bu, \quad x(0) = x_0 \\ &y = d^T x \\ &z(t) = R(t)z_0 - \int_0^t R(t-s)\ n(y(s))g\,ds \\ &u(t) = Fz(t) \ . \end{aligned}\right\} \tag{38}$$

This system can be put into the form of composite system (S) (see Fig. 1) by letting

$$(Lw)(t) = \int_0^t FR(t-s)gw(s)ds$$

$$r_1(t) \equiv 0, \quad r_2(t) = FR(t)z_0, \quad \text{and} \quad y = n(d^T x) \ .$$

Notice that μ defined in (36) is an estimate for the gain of L. Thus, assumption (A-5) will be true if a and q^* are the quantities given in Theorem 6 and if

$$\mu \frac{\gamma|d||B|}{q^*c} \exp\left[\frac{K_1+q^*c}{c(1-q^*)} \ln K\right] < 1 \ . \tag{39}$$

We assume (39). By Theorems 6 and 3 system (38) is uniformly asymptotically stable in the large in the sense of Definition 1 and solutions are uniformly bounded.

Since for any x_0 and z_0 the solution $x(t) \to 0$ as $t \to \infty$, then $|y(t)| = |n(d^T x(t))| < \gamma|d||x(t)| \to 0$. Since $R(t)z_0$ also tends to zero and $R \in L_1$, then from (38) we see that $z(t) \to 0$ as $t \to \infty$. Since by Theorem 3 solutions are uniformly bounded, it is easy to see that (E_4) is uniformly asymptotically stable in the large in the usual sense.

VI. CONCLUDING REMARKS

In this paper we addressed the qualitative analysis of composite hybrid dynamical feedback systems of the type depicted in Figure 1 which consists of an operator L which may represent a finite or an infinite dimensional subsystem (usually the plant) and of a finite dimensional block described by a system of ordinary differential equations (usually the controller). Such system descriptions arise frequently naturally or they come about because of natural constraints (e.g., if only the input-output properties of the subsystem represented by L are known). We established conditions for the well-posedness and the stability of systems of this type. Specifically, we established new results for the attractivity, asymptotic stability, uniform boundedness, asymptotic stability in the large and exponential stability in the large for such systems. These results involve hypotheses which characterize the qualitative I/O properties of the operator L and of the entire system, and which express the stability properties of the finite dimensional block (described by the indicated system of ordinary differential equations) via the Lyapunov theory. Finally, we applied our results in the analysis of hybrid systems which are modeled by a variety of equations.

In a forthcoming paper, we address the stability analysis of discrete-time hybrid composite systems which are analogous to the continuous-time systems (Fig. 1) considered herein. Our ultimate goal is to develop stability results for digital control systems with structure similar to that shown in Fig. 1.

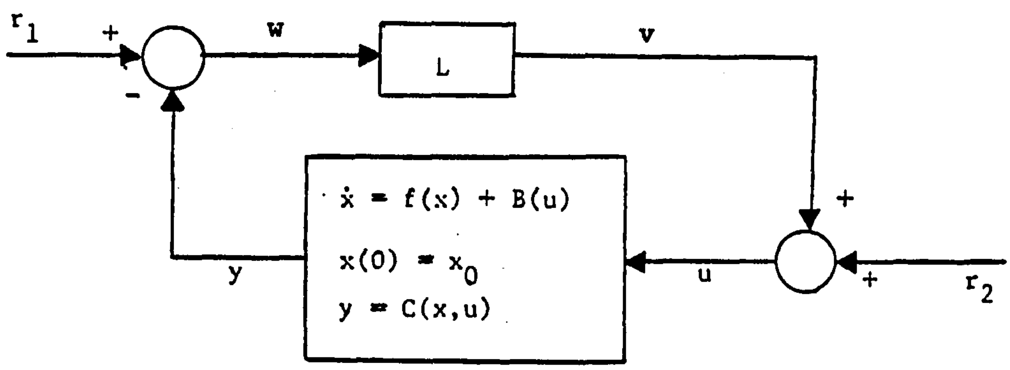

Figure 1. Composite Hybrid Dynamical System

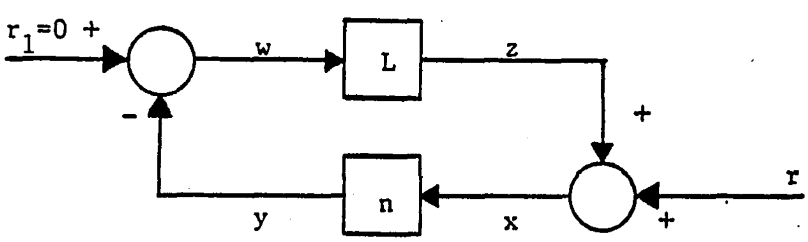

Figure 2. System (E2)

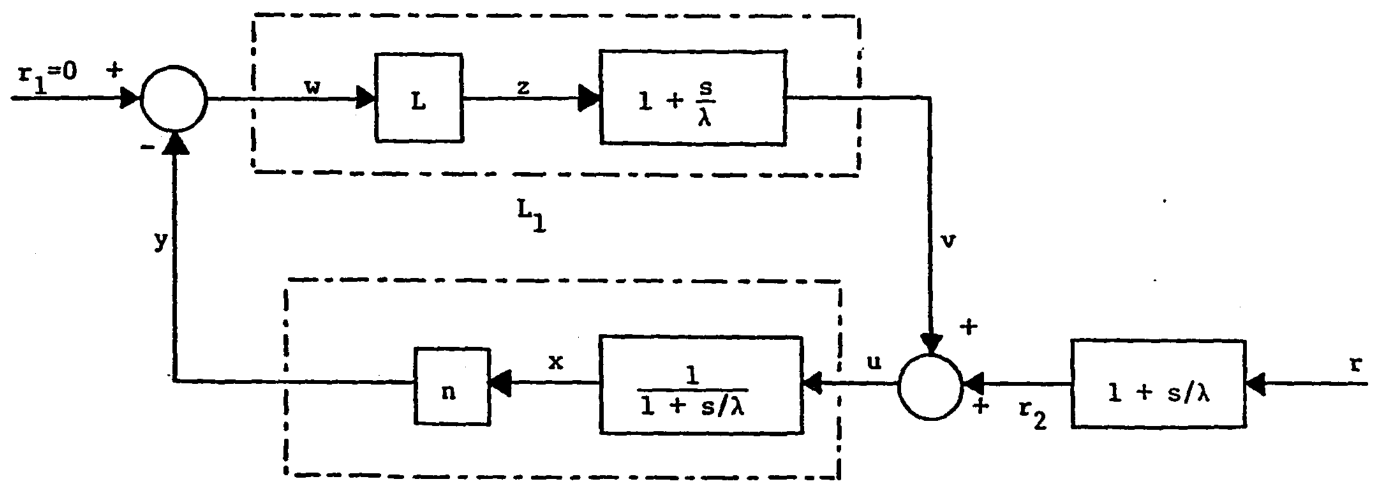

Figure 3. A System which is equivalent to System (E2)

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PART II. STABILITY ANALYSIS OF HYBRID COMPOSITE DYNAMICAL SYSTEMS: DESCRIPTIONS INVOLVING OPERATORS AND DIFFERENCE EQUATIONS

ABSTRACT

We address the stability analysis of composite hybrid dynamical feedback systems of the type depicted in Figure 1, consisting of a block (usually the plant) which is described by an operator L and of a finite dimensional block described by a system of difference equations (usually a digital controller). We establish results for the well-posedness, attractivity, asymptotic stability, uniform boundedness, asymptotic stability in the large, and exponential stability in the large for such systems. The hypotheses of our results are phrased in terms of the I/O properties of L and in terms of the Lyapunov stability properties of the subsystem described by the indicated difference equations. The applicability of our results is demonstrated by two specific examples.

I. INTRODUCTION

In a recent paper [8] we studied the qualitative properties of hybrid interconnected feedback systems consisting of an operator (which usually represents the plant) and a block described by a system of ordinary differential equations (which usually represents the controller). In the present paper we continue this work by studying hybrid interconnected systems of the type depicted in Fig. 1. Here, part of the system description is given in terms of an operator L (which is not necessarily linear, and which usually will represent the plant) while the remaining part of the system is described by a system of ordinary difference equations (which represent the controller). The symbols r_1 and r_2 denote external inputs, while the blocks labeled A/D and D/A denote analog-to-digital and digital-to-analog converters, respectively. Systems with this topology arise naturally in applications. Specifically, we have in mind feedback systems with digital controllers.

In the system of Fig. 1, the operator L may represent a finite dimensional subsystem (described, e.g., by ordinary differential equations), or it may represent an infinite dimensional subsystem (described, e.g., by delay equations, functional differential equations, partial differential equations, Volterra integral equations, Volterra integro-differential equations, etc.) or it may merely represent a memoryless nonlinearity, and the like.

In the sequel, we address the well-posedness and the stability properties of composite systems of the type shown in Fig. 1. For such systems, our results establish criteria for attractivity, uniform boundedness, asymptotic stability, asymptotic stability in the large, exponential stabilility and exponential stability in the large. (When the inputs $r_1 = 0$ and/or $r_2 = 0$, some of our results are in the usual (Lyapunov) stability sense while for $r_1 \neq 0$ and/or $r_2 \neq 0$, the stability definitions which we use involve obvious and reasonable modifications to the corresponding stability concepts.)

The stability results which we prove involve hypotheses which characterize the qualitative properties of the operator L as well as the qualitative properties of the subsystem described by the indicated difference equations. For L, these characterizations are given in terms of input-output properties (e.g., I/O stability, gain, passivity, causality of L). On the other hand, the qualitative properties of the subsystem described by the difference equations are expressed via Lyapunov results (e.g., we may require that the system of difference equations be Lyapunov stable in some sense, or we may postulate the existence of some appropriate Lyapunov function which possesses certain properties along the solutions of the difference equations.) In addition, our results will also usually involve a hypothesis concerning the I/O stability of the entire system given in Figure 1. Indeed, a central question which we address is the following: what conditions ensure that the system of Fig. 1 is (in some appropriate sense) attractive, uniformly bounded, asymptotically stable, or exponentially stable, given that it is I/O stable (in some appropriate sense)?

We emphasize that the results reported herein cannot be obtained from our earlier results [8], by making obvious "Continuous-time to discrete-time modifications." Furthermore, the present results provide a basis for analyzing digital feedback control systems in which the dynamic effects of quantization and overflow nonlinearities is taken into account. We also emphasize that whereas the results reported herein are tangentially related to existing work (see, e.g., Willems [13], [14], Moylan and Hill [9], Vidyasagar [12]), the problems which we address in this paper have not been addressed before in either form, scope or generality. (E.g., result which relate I/O stability and Lyapunov stability are usually confined to finite dimensional systems or to very specialized infinite dimensional systems, and they are usually of a global nature. Furthermore, existing results usually require some reachability (resp., controllability) conditions and/or some detectability (resp., observability) conditions, whereas our results do not.)

The remainder of this paper is organized as follows.

In Section II we provide the required notation and nomenclature.

In Section III we establish our principal results for attractivity (Theorem 1), for asymptotic stability (Theorem 2), for asymptotic stability in the large and for uniform boundedness (Corollary 1 and Theorem 3) and for exponential stability in the large (Theorem 4).

There are several hypotheses ((A-1) - (A-6)) in Theorems 1-4 and in Corollary 1 which can not always be established in an obvious manner. These difficulties are removed in Section IV.

We demonstrate the applicability of the present results by considering two examples in Section V.

In Example 1 we consider the system of Figure 5. In this case the operator L is characterized by a system of linear, time-invariant ordinary differential equations while the difference equations represent a specific second order fixed-point digital filter in direct form having a magnitude truncation quantizer Q and a saturation overflow nonlinearity P (see Figures 2a, 3 and 4a). In this system, the A/D converter is also endowed with a magnitude truncation quantizer Q.

In Example 2 we consider the system of Figure 7. In this case the operator L and the A/D and D/A converters are characterized in the same manner as in Example 1, while the remainder of the system is represented by a system of linear, time-invariant first-order difference equations.

In Example 1 we use some existing results of Erickson and Michel [4] to construct Lyapunov functions for the second order digital filter while in Example 2 we establish a result (Lemma 1) which enables us to construct Lyapunov functions for the subsystem described by the indicated equations in Figure 7.

This paper is concluded in Section VI with some additional comments.

II. NOTATION

Let V and W be arbitrary sets. Then $V \cup W$, $V \cap W$, $V - W$ and $V \times W$ denote the union, intersection, difference, and Cartesian product of V and W, respectively. If V is a subset of W we write $V \subset W$ and if x is an element of V we write $x \in V$. If f is a function of V into W we write $f : V \to W$.

We let R denote the real numbers, R^n real n-space, $|\cdot|$ any one of the equivalent norms defined on R^n, and x^T the transpose of $x \in R^n$. Also, we let Z^+ denote the set of nonnegative integers and we let Z_1^+ denote the set of positive integers.

Unless otherwise specified, matrices are usually assumed to be real. If A is a real $m \times n$ matrix we write $A \in R^{m\times n}$ and we let A^T denote the transpose of A. Furthermore, $|A|$ denotes the norm of the matrix A.

We will have occasion to consider systems described by difference equations of the form

$$x(k+1) = f(x(k)) \tag{A}$$

where $k \in Z^+$, $x \in R^n$, and $f : R^n \to R^n$. We let $x(k,x_0,k_0)$ denote the solution of (A) having the property that $x(k_0,x_0,k_0) = x_0$. When the initial data (x_0,k_0) are understood, we simply write $x(k,x_0,k_0) \triangleq x(k)$. We call any point x_e such that

$$x_e = f(x_e)$$

an equilibrium of (A) and we assume throughout this paper that such equilibrium points are isolated points in R^n.

Given a continuous function $V : R^n \to R$, we define the first forward difference of V along the solutions of (A) by

$$DV_{(A)}(x(k)) = V[x(k+1)] - V[x(k)]$$
$$= V[f(x(k))] - V[x(k)].$$

For $p \in R^+ = [0,\infty)$, we let

$$\ell_p^m = \{u : Z^+ \to R^m \mid \sum_{k=0}^{\infty} |u(k)|^p < \infty\}$$

and we define the norm $\|\cdot\|_{\ell_p}$ of $u \in \ell_p^m$ by

$$\|u\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |u(k)|^p\right)^{1/p}.$$

In particular, we write $\ell_2^m \triangleq \ell_2$ and for this space we define the extended space ℓ_{2e}^m as

$$\ell_{2e}^m = \{u : Z^+ \to R^m \mid \sum_{k=0}^{M-1} |u(k)|^2 < \infty \text{ for all } M \in Z^+\} .$$

Let $M \in Z^+$. We define the (truncated) norm $\|\cdot\|_M$ of $u \in \ell_{2e}^m$ by

$$\|u\|_M = \Big(\sum_{k=0}^{M-1} |u(k)|^2\Big)^{1/2} .$$

Also, we define the truncated inner product of $u,v \in \ell_{2e}^m$, $\langle\cdot,\cdot\rangle_M$, by

$$\langle u,v\rangle_M = \sum_{k=0}^{M-1} u(k)^T\, v(k) .$$

Furthermore, for any $\gamma > 0$ and for $u \in \ell_{2e}^m$, we let

$$\|u\|_{\gamma,M} = \Big(\sum_{k=0}^{M-1} |u(k)|^2\, e^{2\gamma k}\Big)^{1/2} .$$

We also let

$$\ell_{\gamma,2}^m = \{u \in \ell_{2e}^m : \sum_{k=0}^{\infty} |u(k)|^2\, e^{2\gamma k} < \infty\}$$

and we define on this space

$$\|u\|_{\gamma,2} = \Big(\sum_{k=0}^{\infty} |u(k)|^2\, e^{2\gamma k}\Big)^{1/2} .$$

The space $L_p^m(a,b)$ is defined by

$$L_p^m(a,b) = \{f : (a,b) \to R^m \mid f \text{ is measurable and } \int_a^b |f(t)|^p\, dt < \infty\}$$

and the norm of $f \in L_p^m(a,b)$ is defined by

$$\|f\|_{L_p} = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}.$$

For brevity we write $L_2^m = L_2^m(0,\infty)$ and we define the extended space L_{2e}^m with the corresponding truncated norm $\|\cdot\|_T$ $(T > 0)$ in the obvious way. For any $\sigma > 0$ and $f \in L_{2e}^m$ we let

$$\|f\|_{\sigma,T} = \left(\int_0^T |f(t)|^2 \, e^{2\sigma t}\right)^{1/2}.$$

Also, we let

$$L_{\sigma,2}^m = \left\{ f \in L_{2e}^m : \int_0^\infty |f(t)|^2 \, e^{2\sigma t} \, dt < \infty \right\}$$

and we define on this space

$$\|f\|_{\sigma,2} = \left(\int_0^\infty |f(t)|^2 \, e^{2\sigma t} dt\right)^{1/2}.$$

In the sequel we will find it convenient to make use of the product space

$$X = R^n \times L_2^\ell \times X_1$$

where X_1 is a subset of L_2^m. For $p = (x, r_1, r_2) \in X$, we define the norm

$$\|p\| = |x| + \|r_1\|_{L_2} + \|r_2\|_{L_2} .$$

Furthermore, we let

$$X_\gamma = R^n \times L^\ell_{\gamma,2} \times X_2$$

where X_2 is a subset of $L^m_{\gamma,2}$. For $p = (x,r_1,r_2) \in X_\gamma$ we define

$$\|p\|_\gamma = |x| + \|r_1\|_{\gamma,2} + \|r_2\|_{\gamma,2} .$$

If L is an operator from L^ℓ_{2e} to L^m_{2e}, then L is said to be <u>causal</u> if $(Lf)_T = (Lf_T)_T$ for all $T > 0$ and for all $f \in L^\ell_{2e}$. Finally, the <u>gain</u> of L is defined by

$$\text{gain}(L) = \sup\{\frac{\|Lf\|_T}{\|f\|_T} : T > 0, \ f \in L^\ell_{2e} \text{ and } \|f\|_T \neq 0\} .$$

Remark 1

If the sequence $\{u(k)\}_{k=1}^\infty$ is obtained by sampling the function $u_1(t)$ periodically every T seconds, then we write

$$u(k) = u_1(kT), \quad k = 0,1,2,\ldots$$

and it is understood that

$$\|u\|_{\gamma,2} = \left(\sum_{k=0}^\infty |u(k)|^2 \, e^{2\gamma Tk}\right)^{1/2} . \quad \diamond$$

For further details concerning the I/O theory and concerning some of the concepts discussed above, the reader should refer, e.g., to Sandberg [10], [11], Zames [16], Desoer and Vidyasagar [2] and Michel and Miller [5].

III. BASIC STABILITY THEOREMS: ATTRACTIVITY, ASYMPTOTIC STABILITY AND EXPONENTIAL STABILITY

In the present section we establish some basic stability results for a class of composite hybrid dynamical systems of the type depicted in Fig. 1. As indicated in Section I, system configurations of this kind arise naturally in digital control systems where the operator L represents the input/output description of the plant (which may be described, e.g., by integral equations, integro-differential equations, ordinary differential equations, functional differential equations, and the like) while the block containing the indicated difference equations characterizes the digital controller. We emphasize that our subsequent treatment will be general enough to accommodate digital signals (i.e., signals which are sampled and quantized). Thus, the variables $u(k)$ and $y(k)$ in Fig. 1 represent digital signals and the nonlinearities Q, Q_1, Q_2 in Figures 2a and 2b represent quantizers while the nonlinearities P in these figures represent overflow nonlinearities.

The system given in Fig. 1 is governed by equations of the form

$$\left.\begin{aligned}
&x(k+1) = f\big(x(k), u(k)\big), \quad x(0) = x_0 \\
&y(k) = C\big(x(k), u(k)\big) \\
&y_1(t) = y(k), \quad kT \leq t < (k+1)T \\
&w(t) = r_1(t) - y_1(t) \\
&v(t) = (Lw)(t) \\
&u_1(t) = r_2(t) + v(t) \\
&u(k) = Q[u_1(kT)], \quad k \in Z^+ \text{ and } u(0) = 0 \; .
\end{aligned}\right\} \tag{1}$$

We will find it useful to make the following reasonable assumptions for such systems.

Assumption (A-1)

The input signals $r_1(t)$ and $r_2(t)$ are L_2-functions. Specifically, we assume that $r_1 \in L_2^{\ell}$ and $r_2 \in X_1 \subset L_2^m$. The subset X_1 is not specified here; however, later we will see that X_1 cannot be the entire space $L_2(0,\infty)$. We also assume that $C : R^{n+m} \to R^{\ell}$, C is continuous with $C(0,0) = 0$, $L : L_{2e}^{\ell} \to L_{2e}^{m}$, L maps the zero function into the zero function, $f : R^{m+n} \to R^n$, $f(0,0) = 0$ f is continuous, and f is Lipschitz continuous in u, i.e., there exists $K_u > 0$ such that

$$|f(x,u) - f(x,\bar{u})| < K_u|u-\bar{u}| \tag{2}$$

for all $u,\bar{u} \in R^m$ and $x \in R^n$.

Given $(x_0,r_1,r_2) = p \in X = R^n \times L_2^{\ell} \times X_1$, we assume that the system (1) has a unique solution $\big(x(k,p),\ u(k,p)\big) \in R^n \times \ell_{2e}^m$.

For the quantizer nonlinearity Q we assume that there exists a constant $K_m > 0$ such that $|Q(u)| < K_m|u|$ for all $u \in R^m$. $\diamond$

Assumption (A-2)

There exists a Lyapunov function $V : R^n \to R$ which satisfies the following conditions:

(i) $V(0) = 0$ and there exists $K_v > 0$ such that $|x| < K_vV(x)$ for all $X \in R^n$;

(ii) there exists $L_1 > 0$ such that for all $x, \bar{x} \in R^n$,

$$|V(x) - V(\bar{x})| \leq L_1 |x - \bar{x}|; \quad \text{and}$$

(iii) there exists $c \in R$ such that $0 < c < 1$ and

$$DV_{(E)}(x(k)) \leq (c-1)V(x(k))$$

where (E) is given by

$$x(k+1) = f(x(k),0), \quad x(0) = x_0. \quad \diamond \tag{E}$$

__Remark 2__

a) The usual Lyapunov stability results for finite dimensional systems are treated, e.g., in Chapter 5 of Miller and Michel [7].

(b) System structures for digital controllers include, e.g., direct form digital filters, coupled form digital filters, and other kinds of topologies. Structures of this type may consists of several second order sections in cascade or in parallel, or they may consist of higher order filters. In Erickson and Michel [4] and in Michel and Miller [6] Lyapunov functions for a variety of such filter structures have been constructed (via a computer implemented algorithm due to Brayton and Tong [1]) which satisfy Assumption (A-2). ◇

Remark 3

A sufficient condition will be imposed on the solutions of the general system (E) to ensure that assumption (A-2) is satisfied. ◇

Assumption (A-3)

System (1) is L_2-input-output stable in the sense that for every $p \triangleq (x_0, r_1, r_2) \in X$, the solutions $u(k,p)$ are in ℓ_2^m. ◇

Remark 4

In Section IV we will present verifiable conditions on system (1) which guarantee that hypothesis (A-2) is true. ◇

We are now in a position to state and prove our first result.

Theorem 1 (Attractivity)

If (A-1), (A-2) and (A-3) are true, then for any $p \in X$ the solution $x(k,p)$ tends to zero as $k \to \infty$.

Proof: Along the solutions of the equation

$$x(k+1) = f\big(x(k), u(k)\big) \tag{3}$$

we have

$$\begin{aligned} DV_{(3)}\big(x(k)\big) &= V\big[f\big(x(k),u(k)\big)\big] - V\big[x(k)\big] \\ &= V\big[f\big(x(k),0\ \big)\big] - V\big[x(k)\big] + V\big[f\big(x(k),u(k)\big)\big] \\ &\qquad - V\big[f\big(x(k),0\big)\big] \\ &< DV_{(E)}\big[x(k)\big] + L_1\big|f\big(x(k),u(k)\big) - f\big(x(k),0\big)\big| \\ &< (c-1)V\big[x(k)\big] + L_1 K_u |u(k)| \ . \end{aligned}$$

Applying the definition $DV_{(3)}[x(k)] = V[x(k+1)] - V[x(k)]$ to the above inequality we obtain

$$V[x(K+1)] \leq cV[x(k)] + L_1K_u|u(k)| .$$

By the comparison principle (see, e.g., [5], [7]), $V[x(k)]$ will be less than or equal to the solution of the difference equation

$$w(k+1) = cw(k) + L_1K_u|u(k)|, \quad w(0) = V[x(0)] . \tag{4}$$

Since

$$|x(k,p)| \leq K_vV[x(k,p)] \leq K_vw(k) ,$$

then

$$|x(k,p)| \leq K_vc^kV[x(0)] + L_1K_uK_v \sum_{n=0}^{k-1} c^{k-1-n}|u(n,p)| . \tag{5}$$

Furthermore, since $u(k,p) \in \ell_2^m$, then (5) and the Schwarz inequality imply that

$$|x(k,p)| \leq K_vV(x_0) + L_1K_uK_v\|u(\cdot,p)\|_{\ell_2}/\sqrt{1-c^2} . \tag{6}$$

Hence, $x(k,p)$ is bounded in K. Moreover, the first term on the right hand side in (5) tends to zero as $k \to \infty$. The second term on the right hand side in (5) is the convolution of two ℓ_2-sequences and as such must tend to zero as $k \to \infty$. Hence, $x(k,p) \to 0$ as $k \to \infty$, as required. ◇

We now turn to the question of stability. We require the following definition.

Definition 1

We call system (1) <u>stable</u> if given $\varepsilon > 0$, there is a $\delta > 0$ such that for any $p \in X$ with $\|p\| < \delta$ we have $|x(k,p)| < \varepsilon$ for all $k \in Z^+$. We call system (1) <u>asymptotically stable</u> if it is stable and if there is an $R > 0$ such that when $\|p\| < R$ then $x(k,p) \to 0$ as $k \to \infty$. We call system (1) <u>asymptotically stable in the large</u> if it is stable and if for all $p \in X$, $x(k,p) \to 0$ as $k \to \infty$. ◇

In our next result we make use of the following hypothesis.

Assumption (A-4)

For system (1), the following is true: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $p \in X$ and $\|p\| < \delta$, then $\|u(\cdot,p)\|_{\ell_2} < \varepsilon$. ◇

Remark 5

In Section IV we will present <u>testable</u> conditions on system (1) which ensure that hypothesis (A-4) is true. ◇

We now state and prove a result for the asymptotic stability of system (1).

Theorem 2 (Asymptotic Stability)

Assume that (A-1), (A-2) and (A-4) are true. Then system (1) is <u>asymptotically stable</u> in the sense of Definition 1.

<u>Proof</u>: For $\varepsilon = 1$ we use (A-4) to pick $\delta_1 > 0$. Let $R = \delta_1$. For $\|p\| < R$ we have $u(k,p) \in \ell_2$. Hence, the proof of Theorem 1 applies. By that proof, $x(k,p) \to 0$ as $k \to \infty$, and (6) is true. Given $\varepsilon > 0$, by (A-4) we pick $\delta_2 > 0$ such that $\|p\| < \delta_2$ implies

$$\|u(\cdot,p)\|_{\ell_2} < \frac{\varepsilon\sqrt{1-c^2}}{2L_1K_uK_v} \tag{7}$$

Since $V(x) < L_1|x|$, then $V(x_0) < \frac{\varepsilon}{2K_v}$ when $|x_0| < \varepsilon/(2L_1K_v)$. Let $\delta = \min\{\delta_2, \varepsilon/(2L_1), R\}$. If $\|p\| < \delta$, then by (6) and (7) we see that $|x(k,p)| < \varepsilon$ for all $k \in Z^+$. ◇

<u>Corollary 1</u> (Asymptotic stability in the large)

If (A-1), (A-2), (A-3) and (A-4) are true, then system (1) is <u>asymptotically stable in the large</u> in the sense of Definition 1.

<u>Proof</u>: The proof of this result follows by combining Theorems 1 and 2. ◇

We can strengthen the above result by employing the following hypothesis.

<u>Assumption (A-5)</u>

There exists $M > 0$ such that for any $p \in X$, $\|u(\cdot,p)\|_{\ell_2} < M\|p\|$. ◇

Remark 6

When (A-5) is true, then (A-3) and (A-4) are clearly also true. We will present easily checked conditions on system (1) (in Section IV) which ensure that assumption (A-5) is true. ◇

Theorem 3 (Asymptotic stability in the large)

If (A-1), (A-2), and (A-5) are true, then system (1) is asymptotically stable in the large in the sense of Definition 1. Moreover, solutions of system (1) are uniformly bounded in the sense that there is a constant $K > 0$ such that

$$|x(k,p)| \leq K\|p\|$$

for all $k \in Z^+$ and for all $p \in X$.

Proof: Only the uniform boundedness remains to be proved. By (6) and (A-5) we have

$$\begin{aligned}|x(k,p)| &\leq K_v V(x_0) + L_1 K_u K_v (1-c^2)^{-1/2} M\|p\| \\ &\leq K_v L_1 |x_0| + L_1 K_u K_v (1-c^2)^{-1/2} M\|p\| \\ &\leq K_v L_1 [1 + K_u (1-c^2)^{-1/2} M]\|p\|. \quad \diamond\end{aligned}$$

In order to establish an exponential stability type of result, we require the shifted version of (A-5) which assumes the following form.

Assumption (A-6)

There exists a $\gamma > 0$ and $M > 0$ such that for any $p \in X_\gamma$ we have

$$\|u(\cdot,p)\|_{\gamma,2} \leq M\|p\|_\gamma \,. \qquad \diamond$$

Remark 7

In the next section of this paper we present conditions on system (1) which ensure that (A-6) is true. $\diamond$

Theorem 4 (Exponential stability in the large)

If (A-1), (A-2) and (A-6) are true, then for any $p \in X_\gamma$ and $k \in Z^+$ we have

$$|x(k,p)| \leq L_1 K_v \Big(1 + \frac{2K_u M}{\sqrt{1-c^2}}\Big)\|p\|_\gamma \, e^{-\frac{\gamma\sigma}{\gamma+\sigma}k}$$

where $\sigma = -\ell n\, c > 0$.

Proof: As in the proof of Theorem 1 (see inequality (5)), we have

$$K_v^{-1}|x(k,p)| \leq c^k V(x_0) + L_1 K_u \sum_{n=0}^{k-1} c^{k-1-n}|u(n,p)|$$

$$\leq L_1 |x_0|\, c^k + L_1 K_u S(k)$$

where $S(k)$ denotes the sum $\sum_{n=0}^{k-1} c^{k-1-n}|u(n,p)|$. Let $N = \frac{\sigma}{\gamma+\sigma} K$ and let $[N]$ denote the integer part of N. Then $S(k)$ can be written as

$$S(k) = \sum_{n=0}^{[N]} c^{(k-1-n)}|u(n,p)| + \sum_{n=[N]+1}^{k-1} c^{(k-1-n)}|u(n,p)|$$

$$< c^{k-N} \sum_{0}^{[N]} c^{N-1-n}|u(n,p)| + e^{-\gamma N} \sum_{[N]+1}^{k-1} c^{(k-1-n)}|u(n,p)|e^{\gamma n}$$

$$< c^{k-N} \frac{1}{\sqrt{1-c^2}} \|u(\cdot,p)\|_{\ell_2} + e^{-\gamma N} \frac{1}{\sqrt{1-c^2}} \|u(\cdot,p)\|_\gamma$$

$$< \frac{1}{\sqrt{1-c^2}} \|u(\cdot,p)\|_\gamma (c^{k-N} + e^{-\gamma N}) .$$

By the choice of N, we have

$$c^{k-N} = c^{(1 - \sigma/(\gamma+\sigma))k} = c^{[\gamma/(\gamma+\sigma)]k} = e^{-[\gamma\sigma/(\gamma+\sigma)]k} = e^{-\gamma N} .$$

Hence, we get

$$S(k) < \frac{2}{\sqrt{1-c^2}} \|u(\cdot,p)\|_\gamma \; e^{-\gamma N} = \frac{2}{\sqrt{1-c^2}} \|u(\cdot,p)\|_\gamma \; e^{-[\gamma\sigma/(\gamma+\sigma)]k}$$

$$< \frac{2M}{\sqrt{1-c^2}} \|p\|_\gamma \; e^{-[\gamma\sigma/(\gamma+\sigma)]k}$$

where we have made use of (A-6). Hence, for $x(k,p)$ we obtain

$$|x(k,p)| < L_1K_v|x_0|c^k + 2L_1MK_uK_v \frac{1}{\sqrt{1-c^2}} \|p\|_\gamma \; e^{-\gamma/(\gamma+\sigma)}$$

$$< L_1K_v\|p\|_\gamma \; e^{-\sigma k} + 2L_1MK_uK_v \frac{1}{\sqrt{1-c^2}} \|p\|_\gamma \; e^{-[\gamma\sigma/(\gamma+\sigma)]k} .$$

But $\frac{\gamma}{\gamma+\sigma} < 1$, and thus we have

$$|x(k,p)| \leq L_1 K_v \Big(1 + \frac{2M\ K_u}{\sqrt{1-c^2}}\Big) \|p\|_\gamma\ e^{-[\gamma\sigma/(\gamma+\sigma)]k} . \qquad \diamond$$

In contrast to Theorems 1, 2, 3, and Corollary 1, the above result yields estimates of the rate of decay of the solutions for system (1). This type of information is usually of great interest in design considerations.

When the solutions of system (1) satisfy the estimate given in the result above, we say that system (1) is <u>exponentially stable in the large in the sense of Theorem 4</u>.

IV. STABILITY CONDITIONS: SMALL-GAIN TYPE RESULTS

The purpose of this section is to present conditions on system (1) which ensure that hypotheses (A-5) (and hence, (A-3) and (A-4) or (A-6)) is true.

Theorem 5

For system (1) assume that hypotheses (A-1) and (A-2) are true. In addition, assume that:

(i) There exist constants $G_1 > 0$, $G_2 > 0$ such that

$$\|u\|_M \leq G_1 \|v\|_{MT} + G_2 \|r_2\|_{MT}$$

for all $M \in Z_1^+$, $r_2(t) \in X_1$ and for all solutions $v(t)$, where T is the sampling period.

(ii) $|C(x,u)| \leq K_c|x| + K_c'|u|$ for all $(x,u) \in R^{n+m}$.

(iii) The operator L is nonanticipative (i.e., causal) and has gain μ.

If the small gain condition given by

$$[G_1\sqrt{T}\,\mu K_c' + \frac{G_1\sqrt{T}\,\mu L_1 K_c K_v K_u}{1 - c}] < 1 \tag{8}$$

is true, then hypothesis (A-5) (and hence, hypotheses (A-3) and (A-4)) is true.

<u>Proof</u>: Let $M \in Z_1^+$ be fixed. Given a solution of system (1), we have

$$\|y\|_M = \|C(x,u)\|_M < K_c\|x\|_M + K_c'\|u\|_M ,$$

$$\|y_1\|_{MT} = \Big(\int_0^{MT} |y_1(t)|^2 dt\Big)^{1/2} = \Big(\sum_{k=0}^{M-1} \int_{kT}^{(k+1)T} |y_1(t)|^2 dt\Big)^{1/2}$$

$$= \sqrt{T}\,\|y\|_M < \sqrt{T}\,K_c\,\|x\|_M + \sqrt{T}\,K_c'\,\|u\|_M$$

$$\|w\|_{MT} < \|r_1\|_{MT} + \|y_1\|_{MT} < \|r_1\|_{MT} + \sqrt{T}\,K_c\|x\|_M + \sqrt{T}\,K_c'\,\|u\|_M$$

$$\|v\|_{MT} = \|Lw\|_{MT} < \mu\|w\|_{MT} < \mu\|r_1\|_{MT} + \sqrt{T}\,\mu\,K_c\|x\|_M + \sqrt{T}\,\mu\,K_c'\|u\|_M .$$

Hence,

$$\begin{aligned}\|u\|_M < G_2\|r_2\|_{MT} + G_1\mu\|r_1\|_{MT} + G_1\sqrt{T}\,\mu K_c\|x\|_M \\ + G_1\sqrt{T}\,\mu K_c'\|u\|_M .\end{aligned} \tag{9}$$

By assumption (A-2) there exists a Lyapunov function $V(x)$ which satisfies (i), (ii) and (iii) of assumption (A-2). For system (1) we have

$$DV_{(1)}[x(k)] < (c-1)V[x(k)] + L_1K_u|u(k)|$$

or

$$V[x(k+1)] < cV[x(k)] + L_1K_u|u(k)| \ .$$

Hence, we have

$$V[x(k)] < c^kV(x_0) + L_1K_u \sum_{n=0}^{k-1} c^{k-1-n}|u(n,p)| \ .$$

Since by assumption,

$$|x| < K_vV(x) \ ,$$

then

$$\|x\|_M < K_v\|V(x)\|_M \ ,$$

and thus

$$\|x\|_M < \frac{K_v}{\sqrt{1-c^2}} V(x_0) + L_1K_uK_v \frac{1}{1-c} \|u\|_M \ . \tag{10}$$

Combining inequalities (9) and (10) yields

$$\|u\|_M < G_2\|r_2\|_{MT} + G_1\mu\|r_1\|_{MT} + G_1\sqrt{T}K_c'\|u\|_M$$

$$+ G_1\sqrt{T}\mu K_cK_v[\frac{V(x_0)}{\sqrt{1-c^2}} + \frac{L_1K_u}{1-c} \|u\|_M] \ ,$$

or,

$$[1 - G_1\sqrt{T}\mu K_c' - \frac{G_1\sqrt{T}\mu L_1 K_c K_v K_u}{1-c}]\|u\|_M$$

$$< G_2\|r_2\|_{MT} + G_1\mu\|r_1\|_{MT} + \frac{G_1\sqrt{T}\mu K_c K_v}{\sqrt{1-c^2}} V(x_0) < K(p)$$

where

$$K(p) \triangleq G_2\|r_2\|_{MT} + G_1\mu\|r_1\|_{MT} + \frac{G_1\sqrt{T}\mu K_c K_v L_1}{\sqrt{1-c^2}} |x_0| .$$

From the small-gain condition (8) we now have

$$(1 - G_1\sqrt{T}\mu K_c' - \frac{G_1\sqrt{T}\mu L_1 K_c K_v K_u}{1-c}) > 0 .$$

Hence,

$$\|u\|_M < (1 - G_1\sqrt{T}\mu K_c - \frac{G_1\sqrt{T}\mu L_1 K_c K_v K_u}{1-c})^{-1} K(p)$$

for all $M \in Z_1^+$. Letting $M \to \infty$, we see that

$$\|u\|_{\ell_2^m} < (1 - G_1\sqrt{T}\mu K_c' - \frac{G_1\sqrt{T}\mu L_1 K_v K_u}{1-c})^{-1}$$

$$\cdot(\max\{G_2, G_1\mu, \frac{G_1\sqrt{T}\mu L_1 K_c K_v}{\sqrt{1-c^2}}\})\|p\| . \qquad \diamond$$

Remark 8

a) Assumption (i) of Theorem 5 imposes restrictions on the input signal $r_2(t)$. Specifically, $r_2(t)$ is not allowed to be an arbitrary L_2^m-function, but is restricted to L_2^m-functions having the property of resulting in ℓ_2^m-sequences when sampled periodically every T seconds. Furthermore, it is required that the ℓ_2-norm of the sampled sequence be smaller than or equal to the L_2-norm of the function $r_2(t)$ multiplied by a fixed constant $K > 0$. For this reason, we define $X_1(K)$ (for fixed $K > 0$) as

$$X_1(K) = \{f \in L_2^m : f \text{ is continuous and } \sum_{k=1}^{\infty} |f(kT)^-|^2 \leq K^2 \int_0^{\infty} |f(t)|^2 dt\}$$

where $f(kT)^- = \lim_{t \to (kT)^-} f(t)$ and $f(t) = 0$ for all $t < 0$.

b) Since the sampling period T is normally quite small, the small gain condition (8) is not very restrictive. $\diamond$

Remark 9

Not much seems to be known about the space $X_1(K)$. Given a sampling period $T > 0$, a sufficient condition for $r_2(t)$ to belong to $X_1(1/\sqrt{T})$ is that $r_2(t)$ be a continuous nonincreasing L_2^m-function which is zero for all $t < 0$. In this case we have

$$|r_2[(k+1)T]|^2 \leq \frac{1}{T} \int_{Tk}^{T(k+1)} |r_2(t)|^2 \, dt$$

which implies that

$$\sum_{k=0}^{\infty} |r_2(kT)|^2 < \frac{1}{T} \int_0^{\infty} |r_2(t)|^2 \, dt \; ,$$

since the sampled value of $r_2(t)$ at $k = 0$ is $r_2(0^-) = 0$. $\diamond$

Our next result, which is useful in applications and which will be employed in the next section of this paper, constitutes a generalization of Theorem 5.

Corollary 2

For system (1), assume that hypotheses (A-1) and (A-2) are true. In addition, assume that:

(i) There exist constants $G_2 > 0$, $G_3 > 0$, $G_4 > 0$ such that

$$\|u\|_M < G_4 \|y_1\|_{MT} + G_3 \|r_1\|_{MT} + G_2 \|r_2\|_{MT}$$

for all $M \in Z^+$, for all $r_2(t) \in X_1$, and for all solutions $w(t)$, where T is the sampling period.

(ii) $|C(x,u)| < K_c|x| + K_c'|u|$ for all $(x,u) \in R^{n+m}$.

If the small-gain condition

$$\left[G_4 \sqrt{T} K_c' + \frac{G_4 \sqrt{T} L_1 K_c K_v K_u}{1-c} \right] < 1 \tag{11}$$

is true, then hypothesis (A-5) is true.

Proof: The proof of this result is similar to the proof of Theorem 5. ◇

The last result of the present section yields conditions for system (1) under which hypothesis (A-6) is true.

Theorem 6

Assume that hypotheses (A-1) and (A-2) are true. In addition, assume that:

(i) There exist constants $G > 0$ and $\gamma > 0$, with $0 < \gamma T < \sigma \triangleq -\ell n\ c$, such that

$$\|u\|_{\gamma,M} \leq G\|u_1\|_{\gamma,MT} \quad \text{for} \quad M \in Z^+$$

where T denotes the sampling period.

(ii) $C(x,u) \leq K_c|x| + K_c'|u|$ for all $(x,u) \in R^{n+m}$ where $K_c > 0$ and $K_c' > 0$ are constants.

(iii) L is a nonanticipative (i.e., causal) operator and has shifted gain $\mu(\gamma)$, i.e.,

$$\|Lw\|_{\gamma,MT} \leq \mu(\gamma)\|w\|_{\gamma,MT}$$

for all $w \in L_{2e}^{\ell}$ and for all $M \in Z_1^+$.

If the small gain condition

$$G\sqrt{T}\, e^{\gamma T}\mu(\gamma)[K_c' + \frac{e^{\gamma T}L_1K_\mu K_v K_c}{1-e^{\gamma T-\sigma}}] < 1 \qquad (12)$$

is true, then hypothesis (A-6) is true.

Proof: The proof is similar to that of Theorem 5. Let $M \in Z^+$ be fixed. Given a solution of system (1), we have

$$\|y\|_{\gamma,M} \le K_c\|x\|_{\gamma,M} + K_c'\|u\|_{\gamma,M}\,, \quad \text{and}$$

$$\|y_1\|_{\gamma,MT} = \left(\int_0^{MT} |y_1(t)|^2 e^{2\gamma t}dt\right)^{1/2}$$

$$= \left(\sum_{k=0}^{M-1} \int_{kT}^{(k+1)T} |y_1(t)|^2 e^{2\gamma t}dt\right)^{1/2}$$

$$\le \left(\sum_{k=1}^{M-1} |y(k)|^2 e^{2\gamma(k+1)T}T\right)^{1/2}$$

where we have used the fact that in $[kT,(k+1)T]$, it is true that $e^{2\gamma t} \le e^{2\gamma(k+1)T}$. Thus,

$$\|y_1\|_{\gamma,MT} \le \sqrt{T}\, e^{\gamma T}\|y\|_{\gamma,M} \qquad (13)$$

$$\le \sqrt{T}\, e^{\gamma T}K_c\|x\|_{\gamma,M} + \sqrt{T}\, e^{\gamma T}K_c'\|u\|_{\gamma,M}\,,$$

$$\begin{aligned}\|w\|_{\gamma,MT} &= \|r_1\|_{\gamma,M} + \|y_1\|_{\gamma,MT} \\ &< \|r_1\|_{\gamma,MT} + \sqrt{T}\, e^{\gamma T} K_c \|x\|_{\gamma,M} + \sqrt{T}\, e^{\gamma T} K_c' \| \|u\|_{\gamma,M} ,\end{aligned}$$

$$\begin{aligned}\|v\|_{\gamma,MT} &< \mu(\gamma)\|w\|_{\gamma,MT} \\ &< \mu(\gamma)\|r_1\|_{\gamma,MT} + \mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c \|x\|_{\gamma,M} + \mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c' \|u\|_{\gamma,M} ,\end{aligned}$$

$$\begin{aligned}\|u_1\|_{\gamma,MT} &< \|r_2\|_{\gamma,MT} + \|v\|_{\gamma,MT} \\ &< \|r_2\|_{\gamma,MT} + \mu(\gamma)\|r_1\|_{\gamma,MT} + \mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c \|x\|_{\gamma,M} \\ &\quad + \mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c' \|u\|_{\gamma,M} .\end{aligned}$$

Now $\|u\|_{\gamma,M} < G\|u_1\|_{\gamma,MT}$, by assumption. Hence,

$$\begin{aligned}\|u\|_{\gamma,M} &< G\|r_2\|_{\gamma,MT} + G\mu(\gamma)\|r_1\|_{\gamma,MT} + G\mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c \|x\|_{\gamma,M} \\ &\quad + G\mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c' \|u\|_{\gamma,M} . \qquad (14)\end{aligned}$$

Furthermore, for system (1) we also have

$$|x(k)| < K_v V[x(k)] < K_v V(x_0) c^k + L_1 K_v K_u \sum_{n=1}^{k-1} c^{k-1-n} |u(n)| .$$

Hence,

$$x(k)e^{\gamma kT} \leq K_v V(x_0) e^{(\gamma T-\sigma)k}$$

$$e^{\gamma T} L_1 K_v K_u \sum_{n=0}^{k-1} e^{(k-1-n)(\gamma T-\sigma)} |u(n)| e^{n\gamma T}$$

and

$$\|x\|_{\gamma,M} \leq \frac{K_v V(x_0)}{\sqrt{1-e^{2(\gamma T-\sigma)}}} + \frac{e^{\gamma T} L_1 K_u K_v}{1-e^{\gamma T-\sigma}} \|u\|_{\gamma,M} . \tag{15}$$

Combining inequalities (14) and (15), we see that

$$\|u\|_{\gamma,M} \leq G\|r_2\|_{\gamma,MT} + G\mu(\gamma)\|r_1\|_{\gamma,MT} + G\mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c' \|u\|_{\gamma,M}$$

$$+ \frac{G\mu(\gamma)\sqrt{T}\, e^{\gamma T} K_c K_v L_1 |x_0|}{\sqrt{1-e^{2(\gamma T-\sigma)}}}$$

$$+ \frac{G\mu(\gamma)\sqrt{T}\, e^{2\gamma T} L_1 K_u K_v K_c}{1 - e^{\gamma T-\sigma}} \|u\|_{\gamma,M} .$$

Hence,

$$\|u\|_{\gamma,M} \leq \left[1 - G\sqrt{T}\, e^{\gamma T}\mu(\gamma)K_c' - \frac{G\sqrt{T}\, e^{2\gamma T}\mu(\gamma)L_1 K_u K_v K_c}{1 - e^{\gamma T-\sigma}}\right]^{-1}$$

$$\cdot \max\{G, G\mu(\gamma), \frac{G\sqrt{T}\, \mu(\gamma)e^{\gamma T} L_1 K_c K_v}{\sqrt{1-e^{2(\gamma T-\sigma)}}} \|p\|_{\gamma,M} .$$

Letting $M \to \infty$ completes the proof. ◇

Remark 10

Inequality (13) was derived using the estimate

$$\int_{kT}^{(k+1)T} e^{2\gamma T}\,dt \leq Te^{2\gamma(k+1)T} .$$

However, this integral can be calculated exactly. In doing so, inequality (13) is replaced by

$$\|y_1\|_{\gamma,MT} \leq \sqrt{\frac{e^{2\gamma T}-1}{2\gamma}}\,\|y\|_{\gamma,M} . \tag{13'}$$

Making use of (13'), the small gain condition (12) is then replaced by

$$\sqrt{\frac{e^{2\gamma T}-1}{2\gamma}}\,G\mu(\gamma)\Big[K_c' + \frac{e^{\gamma T}L_1K_uK_vK_c}{1-e^{\gamma T-\sigma}}\Big] < 1 . \quad ◇ \tag{16}$$

V. APPLICATIONS AND EXAMPLES

In the present section we consider two specific examples to demonstrate the applicability of the results of Sections III and IV.

Example 1

We consider the system depicted in Figure 5. The block labeled L denotes the plant and is described by the indicated set of differential equations while the block identified as D denotes a digital controller which is described by the indicated set of difference equations. It includes an analog-to-digital converter (A/D) and a digital-to-analog converter (D/A). We will consider the case when the D/A and A/D are operating synchronously with a periodic sample period T and also the case when the D/A and A/D are operating at different sampling periods.

In order to keep our discussion specific and manageable, we consider in particular a digital controller which is classified as a fixed-point second-order direct form digital filter with one or two quantizers Q, Q_1, Q_2 and with an overflow nonlinearity P, as shown in Figures 2a and 2b. Here z^{-1} represents a unit delay, x_1 and x_2 represent the filter states while a and b represent the filter parameters. The inputs $u(k)$ and the outputs $y(k)$ of these filters are digital signals (i.e., they exist only at discrete points in time and they are quantized).

Again, in order to be specific, we consider in the present case filters whose overflow nonlinearity P can be represented by the saturation characteristic depicted in Fig. 3. (I.e., the number that caused the overflow in the computer is replaced by a number having the same sign, but with magnitude corresponding to an overflow level p.)

The quantizers Q_1, Q_2 and Q are assumed here to represent either magnitude truncation or roundoff. The characteristic of the former is depicted in Fig. 4a while the graph of the latter nonlinearity is shown in Fig. 4b.

All of the above nonlinearities may be reviewed as belonging to a sector. Specifically, since Q, e.g., satisfies the conditions $0 \leq xQ(x) \leq K_m x^2$ for all $x \in R$, we say that Q belongs to the sector $[0,K_m]$, where

$$K_m = \begin{cases} 1 & \text{magnitude truncation .} \\ 2 & \text{roundoff .} \end{cases}$$

Similarly, the saturation overflow nonlinearity P of Figure 3 belongs to the sector $[0,1]$.

In Figure 5 we represent the A/D symbolically as consisting of a periodic sampler (with period T) and a quantizer Q. The input to the sampler is an analog signal $u_1(t)$, the output of the sampler is a sequence of numbers $\{\bar{u}_1(k) \triangleq u_1(kT)\}$ which may assume any value in R, while the output of the quantizer Q (i.e., the output of the A/D) is a digital signal (i.e., it is a sequence of numbers $\{u(k)\}$ which can assume only quantized values in R determined by the nonlinearities in Fig. 4a or Fig. 4b). Also, in Figure 5 we represent the D/A symbolically as consisting of a periodic sampler and a zero order hold. Thus, the input to the D/A is a digital signal while its output is an analog signal which is constant over the sampling periods.

For the remainder of this example we consider the digital controller to be a second order direct form filter with one quantizer (see Fig. 2a). The analysis involving other filter forms can be accomplished in a similar manner. Also, later, when we need to be specific, we will let the filter parameters assume the value $a = 1/2$ and $b = -1/2$.

We are now in a position to express the system of Fig. 5 by the following set of equations:

$$
\begin{aligned}
&x_1(k+1) = P\left[Q(ax_1(k) + bx_2(k)) + u(k)\right], \quad x(0) = x_0 \\
&x_2(k+1) = x_1(k) \\
&y(k) = x_1(k+1) \\
&y_1(t) = y(k), \quad kT \le t < (k+1)T \\
&w(t) = -y_1(t) + r_1(t) \\
&\dot{z}(t) = Fz(t) - DN(w(t)), \quad z(0) = z_0 \\
&u_1(t) = Hz(t) + n(t) \\
&u(k) = Q[u_1(kT)], \quad k \in Z^+
\end{aligned}
\tag{E-1}
$$

where $F \in R^{L\times L}$, $Z \in R^L$, $D \in R^L$, $H^T \in R^L$, and T is the sampling period. The function $N : R \to R$ is assumed to satisfy the conditions $N(0) = 0$ and $\alpha \le [N(w_1) - N(w_2)]/(w_1-w_2) \le \beta$ for all $w_1, w_2 \in R$ with $w_1 \neq w_2$.

The present system can be put into the form of composite system (1) (see Fig. 1) by letting $n = 2$, $m = \ell = 1$, $x(k) = [x_1(k), x_2(k)]^T$, and by letting $f : R^3 \to R^2$ be given by

$$f(x(k),u(k)) = \left(P[Q(ax_1(k) + bx_2(k)) + u(k)],\ x_1(k)\right)^T,$$

and

$$v(t) = (Lw)(t) = \int_0^t H\, e^{F(t-s)} DN(w(s))\,ds$$

and

$$r_2(t) = H\, e^{Ft} z_0 + n(t)\ .$$

Now

$$\begin{aligned} |f(x(k),u(k)) - f(x(k),\bar{u}(k))| &= \\ &\quad |P[Q(ax_1(k) + bx_2(k)) + u(k)] \\ &\qquad - P[Q(ax_1(k) + bx_2(k)) + \bar{u}(k)]| \\ &\leq |u(k) - \bar{u}(k)|\ . \end{aligned}$$

Hence, if we assume that the input $r_1(t) \in L_2$ and that the input signal $n(t)$ is a continuous nonincreasing L_2-function, then we see that

$n(t) \in X_1(1/\sqrt{T})$ (see Remark 9) and Assumption (A-1) is satisfied with

$$K_u = 1.$$

Next, in Erickson and Michel [4] it is shown that for

$$x(k+1) = f\big(x(k),0\big) \qquad \text{(E)}$$

there exists a norm Lyapunov functions $V : R^2 \to R$,

$$V(x) = \|x\|_1$$

such that

$$DV_{(E)}\big(x(k)\big) < \sigma\|x(k)\|_1 \ .$$

The unit ball for $\|\cdot\|_1$ (when $a = 1/2$ and $b = -1/2$) is shown in Fig. 6 and is found by using the constructive algorithm of Brayton and Tong [1]. (Note that when the locus determined by $V(x) = \|x\|_1$ is known, then we know $V(x)$ for all $x \in R^2$.) Furthermore, σ is also determined by the algorithm of Brayton and Tong and is found for our particular case to be [4],

$$\sigma = -0.29 \ .$$

Now let the norm notation given in Sections III and IV denote the following norm on R^2,

$$\|x\| = |x_1| + |x_2|$$

where $x = (x_1,x_2)^T \in R^2$. It is an easy matter to show that the norm $\|\cdot\|_1$ (see Fig. 6) is related to the norm $\|\cdot\|$ by

$$\|x\|_1 \le \|x\| \le (3/2)\|x\|_1 \ .$$

Hence,

$$\|x\| \le (3/2)\|x\|_1 = (3/2)V(x)$$

and

$$DV_{(E)}\big(x(k)\big) \le \sigma\|x(k)\|_1 \le \sigma\|x\| \ .$$

Furthermore,

$$|V(x) - V(\bar{x})| \le \big|\|x\|_1 - \|\bar{x}\|_1\big| \le \|x-\bar{x}\|_1 \le \|x-\bar{x}\| \ .$$

Thus, Assumption (A-2) is satisfied with

$$K_v = 1.5, \quad L_1 = 1, \quad \text{and} \quad c = 1 + \sigma = 0.71 \ .$$

Next we note that

$$
\begin{aligned}
|y(k)| = |x_1(k+1)| &= |P[Q(ax_1(k) + bx_2(k)) + u(k)]| \\
&= |P[Q(ax_1(k) + bx_2(k)) + u(k)] - P(0)| \\
&< |Q[ax_1(k) + bx_2(k)] + u(k)| \\
&< K_m \max\{|a|,|b|\}|x(k)| + |u(k)| .
\end{aligned}
$$

But by assumption, $|a| = |b| = 1/2$ and $K_m = 1$ since Q is by assumption a truncation quantizer. Hence,

$$
|y(k)| < (1/2)|x(k)| + |u(k)| .
$$

Thus, Assumption (ii) of Corollary 2 is satisfied with

$$
K_c = 1/2 \quad \text{and} \quad K_c' = 1 .
$$

Next, let us assume that the matrix F is a stable matrix. Then there exist constants $K_1 > 0$ and $F_1 > 0$ such that

$$
|e^{Ft}| < K_1 e^{-F_1 t} \quad \text{for all} \quad t > 0 .
$$

Hence, when sampling the L_2-function $g(t) \triangleq H\, e^{Ft} z_0$, we obtain the ℓ_2-sequence $S = \{H\, e^{FkT} z_0\}_{k=0}^{\infty}$. Consider the linear operator $\underset{\sim}{E} : Z_0 \subset L_2 \to \ell_2$ defined by

$$\underset{\sim}{E}(g(t)) = s, \quad g(t) \in Z_0 \subset L_2$$

where Z_0 is the finite dimensional subspace of L_2 given by

$$Z_0 = \{H\, e^{Ft} z_0 : z_0 \in R^{\ell}\} .$$

Since Z_0 is finite dimensional, then $\underset{\sim}{E}$ is continuous (see, e.g., Dunford and Schwarz [3]), i.e., there exists an $M > 0$ such that

$$\|s\|_{\ell_2} \leq M\|g\|_{L_2} \quad \text{for all} \quad g \in Z_0 .$$

Since $r_2(t) = g(t) + n(t)$ and $n(t) \in X_1(1/\sqrt{T})$, then $r_2(t) \in X_1(G_2)$ where

$$G_2 = \max\{1/\sqrt{T},\, M\} .$$

In order to apply Corollary 2, we need to calculate the factors G_3 and G_4 as well, From Fig. 5 we have

$$v(t) = \int_0^t H\, e^{F(t-s)} DN(w(s))ds .$$

After sampling we obtain

$$\bar{v}(k) \triangleq v(kT) = \int_0^{kT} H\, e^{F(kT-s)} DN(w(s))ds$$

$$= \int_0^{kT} H\, e^{F(kT-s)} DN(-y_1(s))ds + \int_0^{kT} H\, e^{F(kT-s)} D[N(w(s))$$

$$- N(-y_1(s))]ds \;,$$

$$|\bar{w}(k)| < |H||D|K_1K_N[s_1(k) + s_2(k)]$$

where

$$K_N = \max(|\alpha|,|\beta|) \;,$$

$$s_1(k) = \int_0^{kT} e^{-F_1(kT-s)} |y_1(s)|ds \;,$$

and

$$s_2(k) = \int_0^{kT} e^{-F_1(kT-s)} |r_1(s)|ds \;.$$

Hence

$$\|\bar{v}\|_M < |H||D|K_1K_N(\|s_1\|_M + \|s_2\|_M) \;. \tag{17}$$

Now

$$
\begin{aligned}
s_1(k) &= \int_0^{kT} e^{-F_1(kT-s)} |y_1(s)| ds \\
&= \sum_{J=0}^{k-1} \int_{JT}^{(J+1)T} e^{-F_1(kT-s)} |y_1(s)| ds \\
&= \sum_{J=0}^{k-1} \left(\int_{JT}^{(J+1)T} e^{-F_1(kT-s)} ds\right) |y_1(JT)| \\
&= \sum_{J=0}^{k-1} h(k-J)|y(J)| = h(k)*|y(K)|
\end{aligned}
$$

where

$$
h(k-J) = \int_{JT}^{(J+1)T} e^{-F_1(kT-s)} ds = \int_{(k-J-1)T}^{(k-J)T} e^{-F_1 s} ds .
$$

Hence,

$$
\|s_1\|_M < \|h\|_{\ell_1} \|y\|_M .
$$

But $\|y\|_M$ is $(1/\sqrt{T})\|y_1\|_{MT}$ because $y_1(t) = y(k)$ for $t \in [kT,(k+1)T)$, and

$$
\|h\|_{\ell_1} = \sum_{k=1}^{\infty} \int_{(k-1)T}^{kT} e^{-F_1 s} ds = \int_0^{\infty} e^{-F_1 s} ds = \frac{1}{F_1} .
$$

Then

$$
\|s_1\|_M < \frac{1}{\sqrt{T}} \frac{1}{F_1} \|y_1\|_{MT} .
$$

Now let $\alpha(t) = k$ on $KT < t \leq (k+1)T$. Then

$$\|s_2\|_M^2 = \sum_{k=0}^{M-1} \left|\int_0^{kT} e^{-F_1(kT-s)} |r_1(s)|ds\right|^2$$

$$= \int_0^{MT} \left|\int_0^t e^{-F_1(t-s)} |r_1(s)|ds\right|^2 d\alpha(t)$$

$$= \int_0^{MT} \left|\int_0^t e^{-(1/2)F_1(t-s)} (e^{-(1/2)F_1(t-s)} |r_1(s)|)ds\right|^2 d\alpha(t).$$

By the Schwarz inequality we get

$$\|s_2\|_M^2 \leq \int_0^{MT} \left(\int_0^t e^{-F_1(t-s)} ds\right)\left(\int_0^t e^{-F_1(t-s)} |r_1(s)|^2 ds\right) d\alpha(t)$$

$$\leq \frac{1}{F_1} \int_0^{MT} \int_0^t e^{-F_1(t-s)} |r_1(s)|^2 ds d\alpha(t) .$$

By Tonelli's theorem we get

$$\|s_2\|_M^2 \leq \frac{1}{F_1} \int_0^{MT} \int_s^{MT} e^{-F_1(t-s)} |r_1(s)|^2 d\alpha(t) ds$$

$$= \frac{1}{F_1} \int_0^{MT} \left(\int_0^{MT-s} e^{-F_1 t} d\alpha(t)\right) |r_1(s)|^2 ds$$

$$\leq \frac{1}{F_1} \int_0^{MT} \left(\int_0^{MT} e^{-F_1 t} d\alpha(t)\right) |r_1(s)|^2 ds$$

$$= \frac{1}{F_1} \int_0^{MT} \left(\sum_{k=0}^{M-1} e^{-F_1 kT}\right) |r_1(s)|^2 ds$$

$$\leq \frac{1}{F_1} \frac{1}{1-e^{-F_1 T}} \|r_1\|_{MT}^2 .$$

Then

$$\|s_2\|_M \le \frac{1}{\sqrt{F_1(1-e^{-F_1T})}} \|r_1\|_{MT} .$$

Using the estimates for $\|s_1\|_M$ and $\|s_2\|_M$ in (17) we obtain

$$\|\bar{v}\|_M \le |H|\,|D|K_1K_N\Big(\frac{1}{\sqrt{T}}\frac{1}{F_1} \|y_1\|_{MT} + \frac{1}{\sqrt{F_1(1-e^{F_1T})}} \|r_1\|_{MT}\Big) .$$

However,

$$u(k) = Q\big(\bar{v}(k) + \bar{r}_2(k)\big) .$$

Hence,

$$\begin{aligned}
\|u\|_M &\le K_m\|\bar{v}\|_M + K_m\|\bar{r}_2\|_M, \quad (K_m = 1) \\
&\le \|\bar{v}\|_M + \|\bar{r}_2\|_M \\
&\le \frac{|H|\,|D|K_1K_N}{F_1\,\sqrt{T}} \|y_1\|_{MT} + \frac{|H|\,|D|K_1K_N}{\sqrt{F_1(1-e^{-F_1T})}} \|r_1\|_{MT} + G_2\|r_2\|_{MT} .
\end{aligned}$$

Thus, Assumption (i) of Corollary 2 is true with

$$G_2 = \max\{\frac{1}{\sqrt{T}} , M\}$$

$$G_3 = \frac{|H|\,|D|K_1K_N}{\sqrt{F_1(1-e^{-F_1T})}}$$

$$G_4 = \frac{|H|\,|D|K_1K_N}{F_1\,\sqrt{T}} \,.$$

We have now established all conditions to apply Corollary 2 to the present example. We conclude that if

$$\frac{|H|\,|D|K_1K_N}{F_1} < 0.2788 \tag{18}$$

Then (A-5) is true and system (E-1) which is depicted in Figure 5 is asymptotically stable in the large in the sense of Definition 1. Furthermore, the solutions of this system are uniformly bounded in the sense of Theorem 3. ◇

__Remark 11__

If the D/A converter has a sampling period T_1 and the A/D converter (and the digital controller) has a sampling period $T_2 = T$, and if T_1 is an integer multiple of T_2, then the stability condition (18) for system (E-1) assumes the form

$$\frac{|H|\,|D|K_1K_N}{F_1}\sqrt{\frac{T_1}{T_2}} < 0.2788 \qquad \diamond$$

In our final example we require the following preliminary result in order to satisfy Assumption (A-2).

Lemma 1

Consider the difference equation

$$x(k+1) = f\big(x(k)\big), \quad x(0) = x_0 \tag{E}$$

and assume that:

1) $f : R^n \to R^n$, f is continuous on R^n, $f(0) = 0$, and f is Lipschitz continuous, i.e., there exists a constant $K_f > 0$ such that

$$|f(x) - f(\bar{x})| \leq K_f |x - \bar{x}| \tag{19}$$

for all $x, \bar{x} \in R^n$;

2) there exist constants $K_2 > 0$ and $0 < \sigma < 1$ such that the solution $\phi(k,x_0)$ of (E) satisfies the bound

$$|\phi(k,x_0)| \leq K_2 \alpha \sigma^k$$

for all $k \in Z^+$ and for all $x_0 \in R^n$ with $|x_0| \leq \alpha$.

Then there exists a Lyapunov function $V : R^n \to R$ which satisfies the following:

(i) $V(0) = 0$ and $V(x) \geq |x|$ for all $x \in R^n$;

(ii) there exists $L_1 > 0$ such that for all $x, \bar{x} \in R^n$,

$$|V(x) - V(\bar{x})| \leq L_1|x - \bar{x}|; \quad \text{and}$$

(iii) $\quad DV_{(E)}(x(k)) \leq (\sigma q - 1)V(x(k))$

for all $x(k) \in R^n$ where q is a fixed constant in the range $0 < q < 1$.

Proof: The proof is similar to a proof given in Yoshizawa [15, p. 94]. Fix $q \in (0,1)$ and define

$$V(x) = \sup_{k \in Z^+} |\phi(k,x)|\sigma^{-kq}$$

where $\phi(k,x)$ denotes the solution of (E) with $\phi(0,x) = x$. Taking $k = 0$ we see that

$$V(x) \geq |\phi(0,x)| = |x| .$$

Also, $\phi(k,0) = 0$ so that $V(0) = 0$.

Given $x_0 \neq 0$, let $\alpha = |x_0|$ and define M_1 by

$$M_1 \triangleq \inf\{N : N \in Z^+ \text{ and } K_2 \leq \tfrac{1}{2}\sigma^{-(1-q)N}\} .$$

Thus, M_1 denotes the positive integer in the interval

$$\left[\frac{\ln(2K_2)}{(1-q)\sigma_1}, \frac{\ln(2K_2)}{(1-q)\sigma_1} + 1\right)$$

where $\sigma_1 = -\ln \sigma > 0$. We have

$$\frac{\alpha}{2} < \alpha = |x_0| \leq V(x_0) \leq \sup_{k \in Z^+} |\phi(k,x_0)|\sigma^{-kq} .$$

Using assumption 2 of this lemma, we obtain

$$\frac{\alpha}{2} < \sup_{k \in Z^+} (K_2 \alpha \sigma^k)(\sigma^{-kq}) \leq \sup_{k \in Z^+} \frac{\alpha}{2} \sigma^{-(1-q)M_1} [\sigma^{(1-q)k}]$$

$$= \frac{\alpha}{2} \sup_{k \in Z^+} \sigma^{(1-q)(k-M_1)} .$$

The sup in the definition of $V(x)$ is never taken on when $k > M_1$ since for $k > M_1$ we would have

$$\frac{\alpha}{2} < \frac{\alpha}{2} \sup_{k > M_1} \sigma^{(1-q)(k-M_1)} < \frac{\alpha}{2} .$$

Thus,

$$V(x) = \sup_{\substack{0 \leq k \leq M_1 \\ k \in Z^+}} |\phi(k,0)|\sigma^{-kq} .$$

For $x,\bar{x} \in R^n$ we have

$$|V(x) - V(\bar{x})| = |\sup_{0 \leq k \leq M_1} |\phi(k,x)|\sigma^{-kq} - \sup_{0 \leq k \leq M} |\phi(k,\bar{x})|\sigma^{-kq}|$$

$$\leq \sup_{0 \leq k \leq M_1} |\phi(k,x) - \phi(k,\bar{x})|\sigma^{-kq} . \tag{20}$$

Since ϕ solves (E) and since f satisfies (19), then

$$|\phi(k+1,x) - \phi(k+1,\bar{x})| = |f(\phi(k,x)) - f(\phi(k,\bar{x}))|$$
$$\leq K_f|\phi(k,x) - \phi(k,\bar{x})| \ .$$

Hence

$$|\phi(k,x) - \phi(k,\bar{x})| \leq K_f^k|\phi(0,x) - \phi(0,\bar{x})| = K_f^k|x - \bar{x}| \ .$$

Substituting this relation into (20) we see that

$$|V(x) - V(\bar{x})| \leq \sup_{0 \leq k \leq M_1} \sigma^{-kq}K_f^k|x - \bar{x}| = L_1|x - \bar{x}| \ ,$$

where

$$L_1 = \sup_{0 \leq k \leq M_1} \sigma^{-kq} K_f^k \tag{21}$$

and M_1 is defined as before.

Given $x(k) \in R^n$, let

$$x(k+1) = \phi(1,x(k))$$

so that

$$
\begin{aligned}
V(x(k+1)) &= \sup_{N \in Z^+} |\phi(N,x(k+1))|\sigma^{-Nq} \\
&= \sup_{N \in Z^+} |\phi(N+1,x(k))|\sigma^{-Nq} \\
&= \sup_{N \in Z_1^+} (|\phi(N,x(k))|\sigma^{-Nq})\sigma^q \\
&\leq \sup_{N \in Z^+} (|\phi(N,x(k))|\sigma^{-Nq})\sigma^q = V(x(k))\sigma^q \ .
\end{aligned}
$$

Then

$$
\begin{aligned}
DV_{(E)}(x(k)) &= V[f(x(k))] - V[x(k)] \\
&= V[x(k+1)] - V[x(k)] \\
&\leq (\sigma^q - 1)\ V(x(k)) \ . \qquad \diamond
\end{aligned}
$$

If we define $V(k) \triangleq V[x(k)]$, then the difference inequality corresponding to (iii) of Lemma 1 is given by

$$
V(k+1) \leq \sigma^q V(k) \ .
$$

We are now in a position to consider the second example of the present section.

Example 2

We consider now the system depicted in Figure 7. This system, and the system of Example 1, which is given in Figure 5, differ only in the description of the controller. In the present example, the controller is described by a linear, n-th order difference equation, while in Example 1 the controller was represented by a second order nonlinear system which takes into account quantization effects and overflow effects. We assume that in the present example the quantizer associated with the A/D converter is a truncation quantizer and that the sampling for that A/D and D/A converters are synchronized with a sampling period equal to T seconds.

The system of Figure 7 is described by the set of equations

$$\left.\begin{aligned}
&x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \\
&y(k) = Cx(k) + Eu(k) \\
&y_1(t) = y(k), \quad KT \leq t < (k+1)T \\
&w(t) = r_1(t) - y_1(t) \\
&\dot{z}(t) = Fz(t) + Dw(t), \quad z(0) = z_0 \\
&u_1(t) = Hz(t) + n(t) \\
&u(k) = Q[u_1(kT)], \quad k \in Z^+
\end{aligned}\right\} \qquad \text{(E-2)}$$

where $F \in R^{L\times L}$, where F is assumed to be stable, where $Z \in R^L$, $D \in R^L$, $H^T \in R^L$, where $T > 0$ is the sampling period, where $A \in R^{n\times n}$, where it is assumed that $|A| < 1$, where $B \in R^n$ and $C^T \in R^n$, and where E is assumed to be a real constant.

This system can be put into the form of composite system (1) (see Fig. 1) by letting $m = \ell = 1$, and by defining $f : R^{n+1} \to R^n$, $v(t)$ and

$r_2(t)$ by

$$f(x(k),u(k)) = Ax(k) + Bu(k) ,$$

$$v(t) = (Lw)(t) = \int_0^t H\, e^{F(t-s)}\, Dw(s)ds ,$$

and

$$r_2(t) = H\, e^{Ft} z_0 + n(t) .$$

Now

$$|f(x(k),u(k)) - f(x(k),\bar{u}(k))| = |Bu(k) - B\bar{u}(k)| < |B|\,|u(k) - \bar{u}(k)| .$$

Hence, if we assume that $r_1(t) \in L_2$ and that the input signal $n(t)$ is a continuous nonincreasing L_2-function, then we see that $n(t) = X_1(1/\sqrt{T})$ (see Remark 9). Thus, Assumption (A-1) is satisfied with

$$K_u = |B| .$$

Next, consider the equation

$$x(k+1) = f(x(k),0) = Ax(k) . \tag{E}$$

We have in this case

$$|f(x(k),0) - f(\bar{x}(k),0)| = |Ax(k) - A\bar{x}(k)| \leq |A|\,|x(x) - \bar{x}(k)| \ .$$

Hence, Assumption 1 of Lemma 1 is satisfied with

$$K_f = |A| < 1 \ .$$

If we let $\phi(k,x_0)$ denote the solution of (E) with $\phi(0,x_0) = x_0$, then we obtain

$$\phi(k,x_0) = A^k x_0$$

and

$$|\phi(k,x_0)| \leq |A|^k \alpha_0$$

for all $x_0 \in R^n$ such that $|x_0| \leq \alpha_0$ and for all $k \in Z^+$. Hence, Assumption 2 of Lemma 1 is satisfied with

$$K_2 = 1 \quad \text{and} \quad \sigma = |A| < 1 \ .$$

Since all hypotheses of Lemma 1 are satisfied we can conclude that for $q \in (0,1)$ there exists a Lyapunov function $V : R^n \to R$ having the properties enumerated in (i), (ii) and (iii) of Lemma 1.

Therefore, Assumption (A-2) is satisfied with

$$K_v = 1, \quad C = \sigma^q = |A|^q$$

and

$$L_1 = \sup_{0 \le k \le M_1} \sigma^{-qk} K_f^k = \sup_{0 \le k \le M_1} |A|^{k(1-q)} = 1 .$$

The last relation follows because the sup is taken when $k = 0$, since $|A| < 1$.

Next, since

$$y(k) = Cx(k) + Eu(k) ,$$

we obtain

$$|y(k)| \le |C||x(k)| + |E||u(k)| .$$

Thus, assumption (ii) of Corollary 2 is satisfied with

$$K_c = |C| \quad \text{and} \quad K_c' = |E| .$$

Next, since F is a stable matrix, there exist constants $F_1 > 0$ and $K_1 > 0$ such that

$$|e^{Ft}| \le K_1 e^{-F_1 t} \quad \text{for all} \quad t \ge 0 .$$

Hence, the calculations of G_2, G_3 and G_4 required for Corollary 2 are identical to those for system (E-1) of Example 1. We thus obtain

$$G_2 = \max\{\frac{1}{\sqrt{T}} , M\} ,$$

$$G_3 = \frac{|H|\,|D|K_1}{\sqrt{F_1(1-e^{-F_1 T})}} ,$$

$$G_4 = \frac{1}{\sqrt{T}} \frac{|H|\,|D|\,|K_1|}{F_1} ,$$

where M is as defined for system (E-1).

We are now in a position to apply Corollary 2 to system (E-2) and conclude that if

$$\frac{|H|\,|D|K_1}{F_1} [\,|E| + \frac{|C|\,|B|}{1 - |A|^q}\,] < 1 \tag{22}$$

then (A-5) is true. Inequality (22) in turn can be expressed as $F(q) < 1$, where

$$F(q) \triangleq \frac{|H|\,|D|K_1}{F_1} [\,|E| + \frac{|C|\,|B|}{1-|A|^q}\,] .$$

We obtain the optimal choice of q by requiring that

$$\frac{dF(q)}{dq}\bigg|_{q=q^*} = 0$$

which yields $q^* = 1$. Of course we can not choose $q = 1$ but we can choose it as close to one as we wish. Therefore, if

$$\frac{|H|\,|D|K_1}{F_1}\left[\,|E| + \frac{|C|\,|B|}{1 - |A|}\,\right] < 1 \tag{23}$$

then (A-5) is true and system (E-2) is asymptotically stable in the large in the sense of Definition 1. Furthermore, the solutions of system (E-2) are uniformly bounded in the sense of Theorem 3.

For example, if in particular we let the controller be described by the equations

$$x(k+1) = \begin{bmatrix} 1/2 & -1/4 \\ 1/3 & 1/4 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0]\; x(k) + u(k)$$

and if we let $|A| = \|A\|_\infty = \max\limits_{1 \le i \le n} \sum\limits_{J=1}^{n} |a_{iJ}|$, then we obtain

$$|A| = 0.75, \quad |B| = 1, \quad |C| = 1, \quad \text{and} \quad |E| = 1 \,.$$

In this case the stability condition (23) assumes the form

$$\frac{\|H\|_\infty \|D\|_\infty K_1}{F_1} < 0.2 \,. \qquad \diamond \; (25)$$

Remark 12

If the D/A converter has a sampling period T_1 and the A/D converter (and the digital controller) has a sampling period $T_2 = T$, and if T_1 is an integer multiple of T_2, then the stability condition (22) for system (E-2) will assume the form

$$\frac{|H|\,|D|K_1}{F_1}\left(|E| + \frac{|C|\,|B|}{1-|A|}\right)\sqrt{\frac{T_1}{T_2}} < 1 . \quad \diamond \tag{24}$$

VI. CONCLUDING REMARKS

In this paper we addressed a qualitative analysis of composite hybrid dynamical feedback systems of the type depicted in Figure 1 which consist of an operator L which may represent an infinite dimensional subsystem (usually the plant) and of a finite dimensional block described by a system of difference equations (usually a digital controller).

We established conditions for the well-posedness and the stability of systems of this type (including attractivity, asymptotic stability, uniform boundedness, asymptotic stability in the large and exponential stability in the large) for such systems. These results involve hypotheses which characterize the qualitative I/O properties of the operator L and of the entire system, and which express the stability properties of the finite dimensional block (described by the indicated system of difference equations) via the Lyapunov theory. Finally, we applied our results to two specific examples.

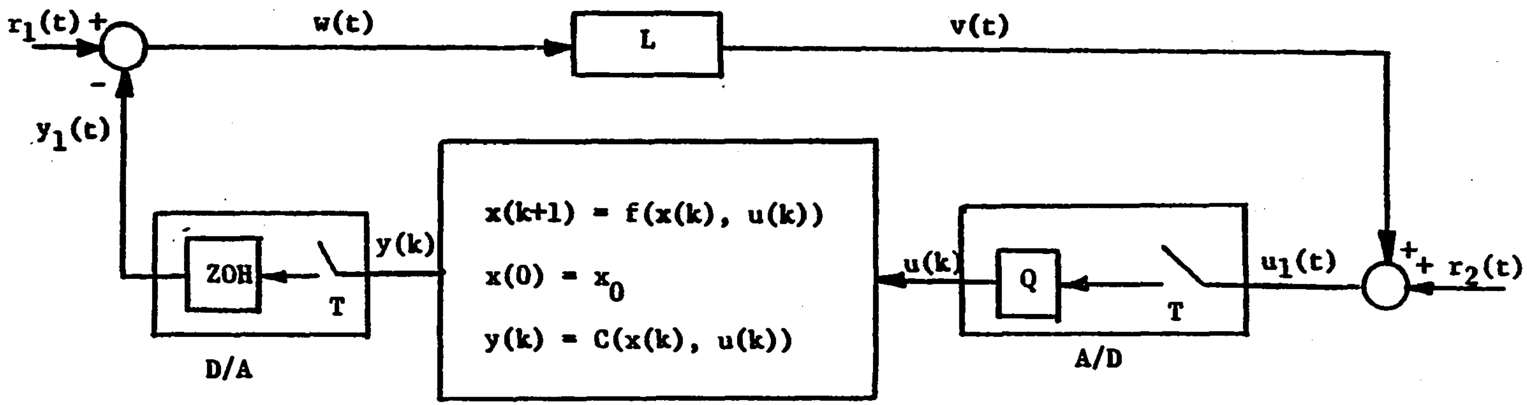

Figure 1. Composite System

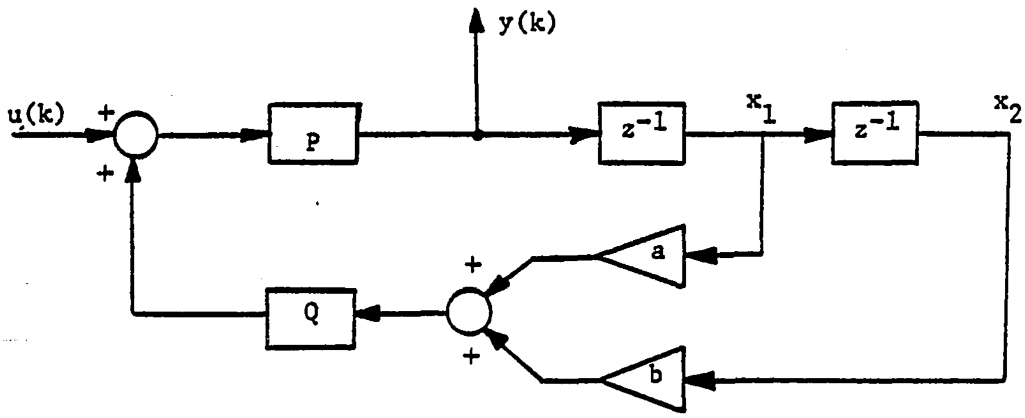

Figure 2a. Direct form fixed point digital filter with one quantizer

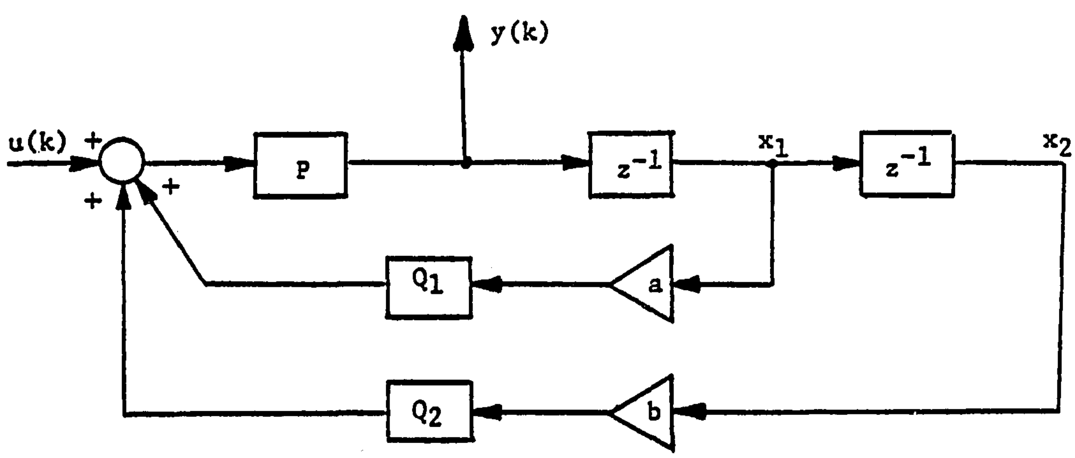

Figure 2b. Direct form fixed point digital filter with two quantizers

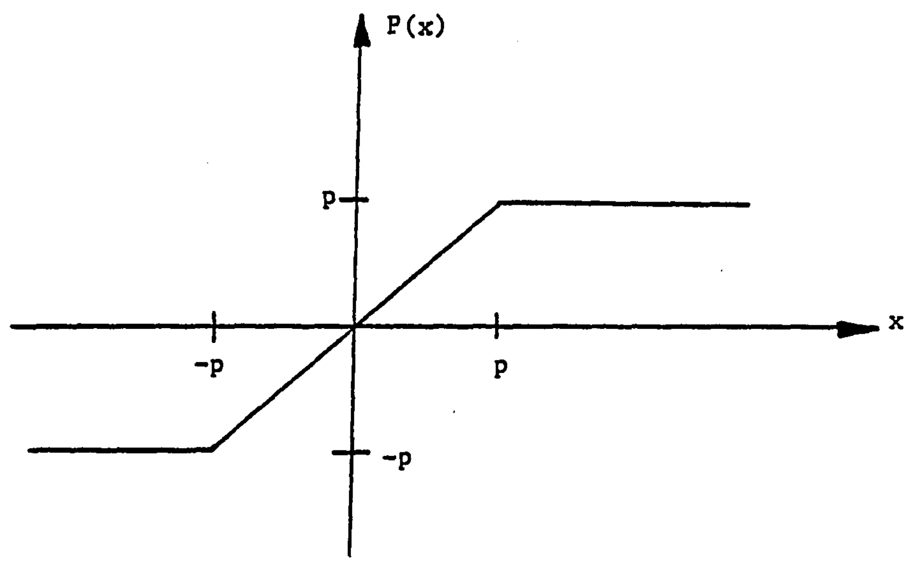

Figure 3. Saturation overflow characteristic

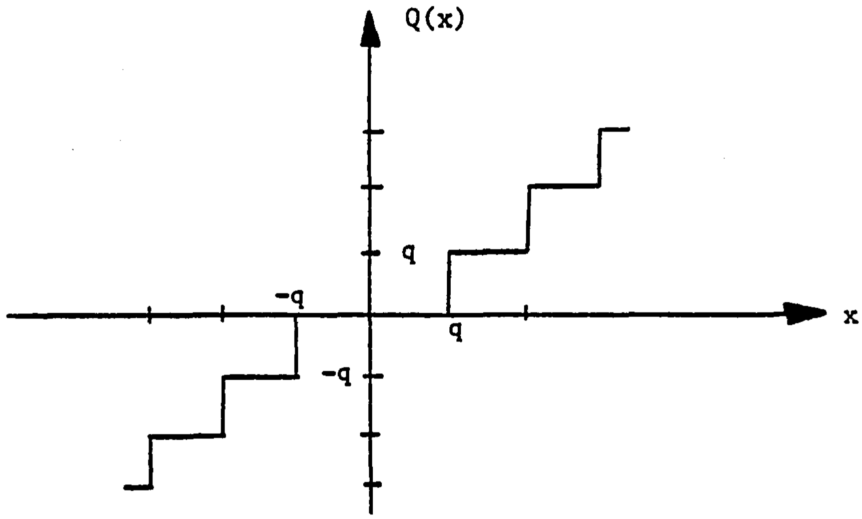

Figure 4a. Magnitude truncation quantization

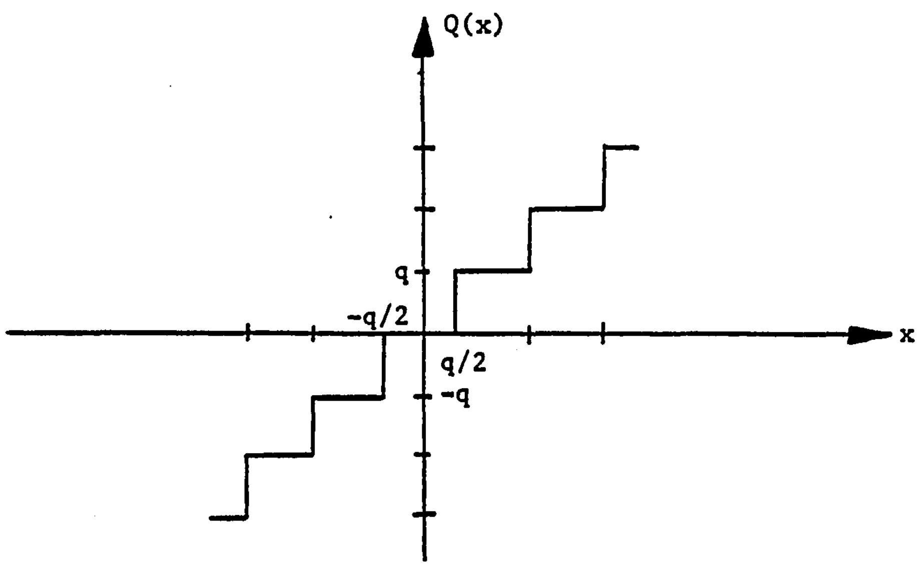

Figure 4b. Roundoff quantization

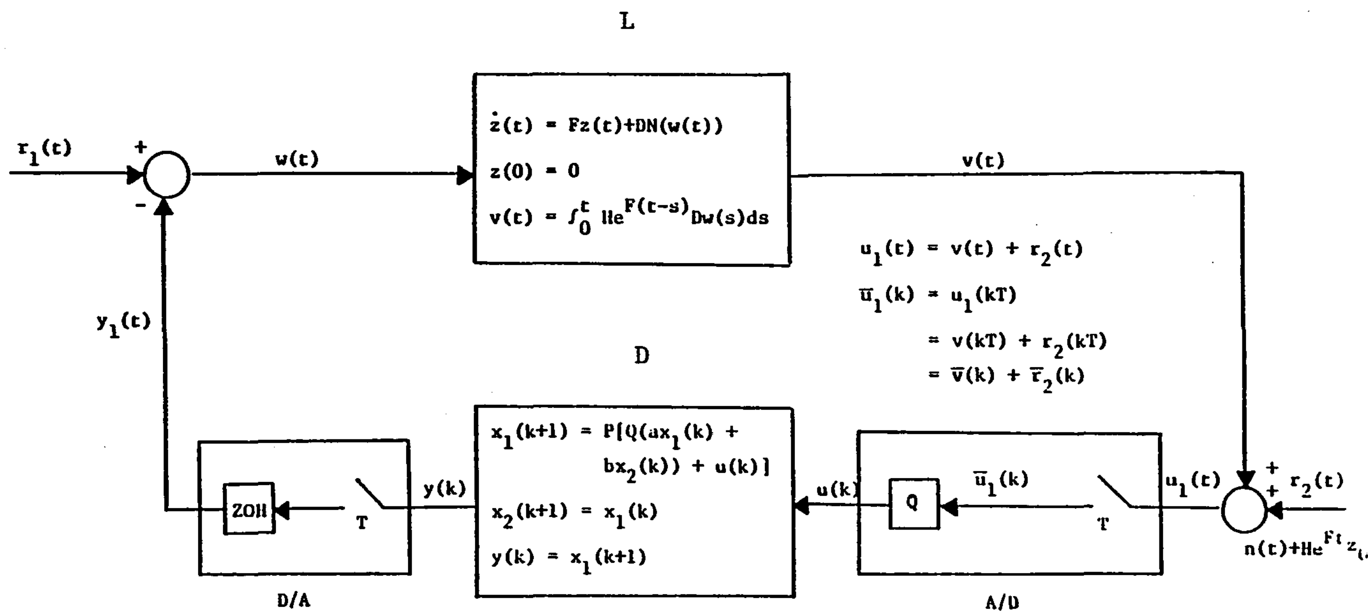

Figure 5. Composite System (E1)

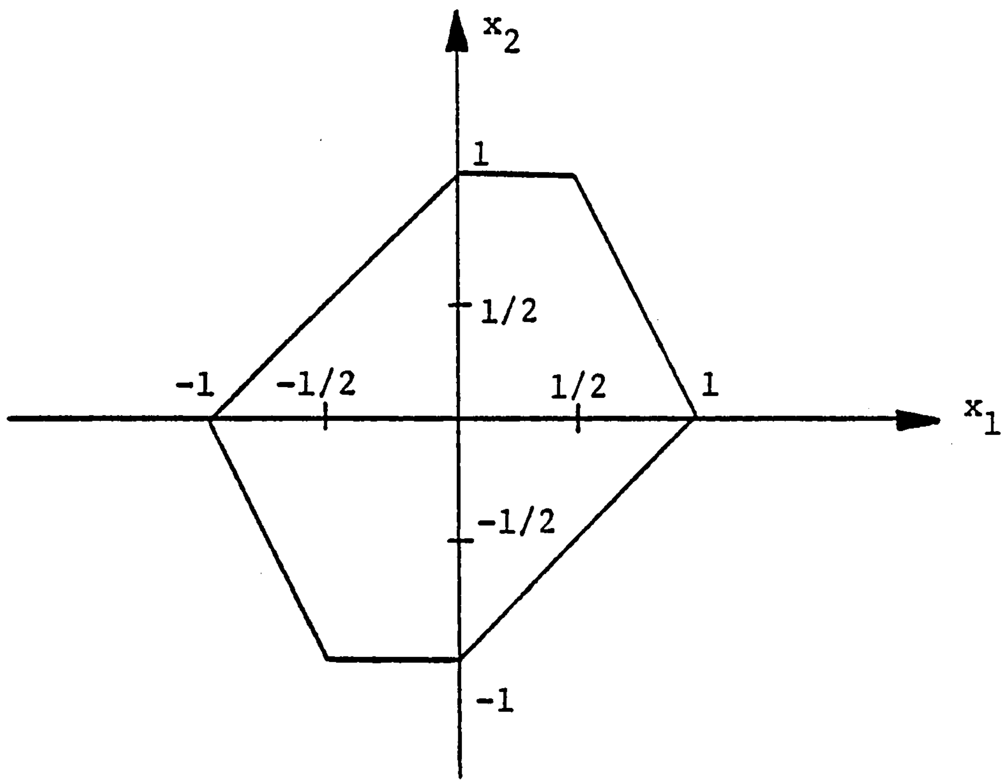

Figure 6. The unit ball determined by $\|\cdot\|_1$

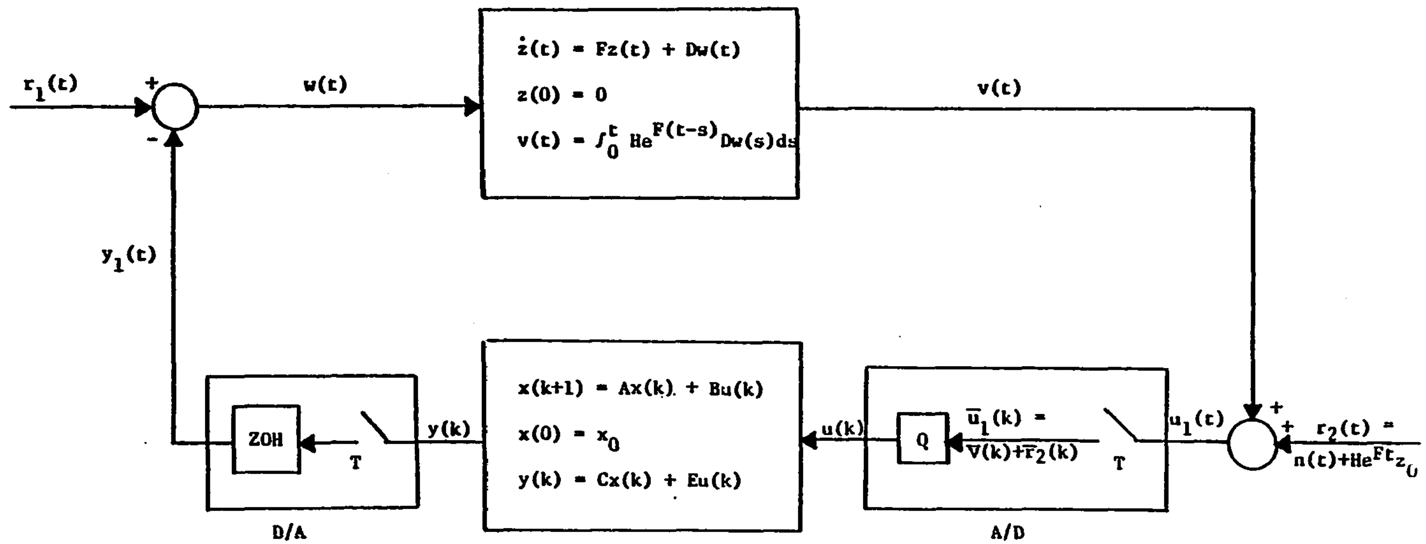

Figure 7. Composite System (E2)

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CONCLUSION

In Part I of this dissertation, we addressed the qualitative analysis of composite hybrid dynamical feedback systems of the type given in Figure 1 of Part I. We established conditions for the well-posedness and the stability of systems of this type. Conditions for the attractivity, asymptotic stability, asymptotic stability in the large, and exponential stability for such systems were found and proved. Some of the proofs given are long, however, the stability criteria are simple and can be verified easily. In fact theorem 5 of Part I gives <u>testable</u> conditions to ensure the stability of such systems.

In Part II of this dissertation, we studied the discrete-time hybrid composite systems which are analogous to the continuous-time systems (studied in Part I). Stability results for such digital control systems were developed. We emphasize that the results of Part II <u>cannot</u> be obtained from that of Part I, by making obvious "continuous-time to discrete-time modification."

In Parts I and II, our stability results involve hypotheses which characterize the qualitative I/O properties of the plant (the operator L) and of the entire system, and which express the stability properties of the finite dimensional block (the controller) via the Lyapunov theory. Also in both parts (I and II), the applicability of our results is demonstrated by means of specific examples.

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