

## INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.
2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.
3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of "sectioning" the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.
4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.
5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.

**University  
Microfilms  
International**

300 N. Zeeb Road  
Ann Arbor, MI 48106

8514428

**Park, Byung Sul**

**N-PERSON SIMPLE GAMES AND N-COMPONENT RELIABILITY STRUCTURES**

*Iowa State University*

PH.D. 1985

**University  
Microfilms  
International** 300 N. Zeeb Road, Ann Arbor, MI 48106

n-person simple games and n-component reliability structures

by

Byung Sul Park

A Dissertation Submitted to the  
Graduate Faculty in Partial Fulfillment of the  
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Department: Statistics

Major: Statistics (Applied Statistics)

Approved:

Signature was redacted for privacy.

In Charge of Major work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa state University

Ames, Iowa

1985

## TABLE OF CONTENTS

	PAGE
1. INTRODUCTION . . . . .	1
1.1. Motivation . . . . .	1
1.2. Basic Properties of Characteristic and Structure Functions . . . . .	3
1.3. Literature Review and Thesis Overview . . . . .	5
2. DIADIC FUNCTIONS . . . . .	10
2.1. Introduction . . . . .	10
2.2. Boolean Diadic Functions . . . . .	12
2.3. Monotone Diadic Functions . . . . .	14
2.4. Super-additive Diadic Functions . . . . .	23
2.5. Sub-additive Diadic Functions . . . . .	26
2.6. Inessential Diadic Functions . . . . .	28
2.7. Relationships among Sub-classes of Diadic Functions . . . . .	29
2.8. Cliques . . . . .	31
2.9. Characterization of Super-additivity for Special MDFs . . . . .	38
2.10. Inconsequential Coordinates . . . . .	44
3. CHARACTERISTIC FUNCTIONS AND STRUCTURE FUNCTIONS AS DIADIC FUNCTIONS . . . . .	51
3.1. Introduction . . . . .	51
3.2. The Principal Diadic Functions as Characteristic and Structure Functions . . . . .	52
3.3. Special Diadic Functions as Characteristic or Structure Functions . . . . .	56
4. DIADIC FUNCTION REPRESENTATIONS AND RELATIONS . . . . .	63
4.1. Introduction . . . . .	63
4.2. The Generating and Veto Vector Form of a MDF . . . . .	63

4.3.	The Shapley Form of a MDF . . . . .	66
4.4.	The Barlow-Proschan Form of a DF, and the Associated Multilinear Extension . . . . .	76
5.	IMPORTANCE INDICES . . . . .	79
5.1.	Introduction . . . . .	79
5.2.	The Shapley Value . . . . .	81
5.3.	The Banzhaf-Coleman Index and Birnbaum Index . . . . .	91
5.4.	The Barlow-Proschan Index . . . . .	95
5.4.1.	Basic concepts . . . . .	95
5.4.2.	Proportional hazard cases . . . . .	97
5.4.3.	Relationship between the Barlow-Proschan index and the Shapley value . . . . .	98
5.5.	The Deegan-Packel Index . . . . .	100
5.5.1.	Basic concepts . . . . .	100
5.5.2.	Extension . . . . .	101
6.	REFERENCES . . . . .	103
7.	ACKNOWLEDGEMENTS . . . . .	106

## 1. INTRODUCTION

## 1.1. Motivation

Let  $(N, v)$  be an  $n$ -person game structure, where  $N$  denotes the set  $\{1, \dots, i, \dots, n\}$  of all players and  $v$  denotes the characteristic function of this game structure, with domain equal to the power set  $2^N$  of  $N$ , whose elements are called "coalitions". Very often the phrase "game  $v$ " is used instead of "game structure  $(N, v)$ " (Owen (1982)).

A game  $v$  is said to be a "(0,1)  $n$ -person simple game", or simply a "(0,1) simple game" if, for each element  $S$  of  $2^N$ , we have either  $v(S) = 0$  or  $v(S) = 1$ . Essentially, a simple game is one in which every coalition is either "effective" (value 1) or "ineffective" (value 0), with nothing in between.

Let  $(N, f)$  be an  $n$ -component system structure, as indicated in the reliability literature, where  $N = \{1, \dots, i, \dots, n\}$  denotes the set of all components, and  $f$  is called the structure function of the system structure. This function  $f$  indicates the state of the system:

$$\begin{aligned} f &= 1 \text{ if the system is functioning,} \\ &= 0 \text{ if the system is failed.} \end{aligned} \tag{1.1}$$

Similarly, to indicate the state of the  $i$ -th component, a binary indicator variable  $x_i$  is assigned to component  $i$ ,  $i = 1, 2, \dots, n$ , as follows:

$$x_i = 1 \text{ if component } i \text{ is functioning,}$$

$$= 0 \text{ if component } i \text{ is failed.} \quad (1.2)$$

We assume that the state of the system is determined completely by the states of the components, so that we may write

$$f = f(X), \text{ where } X = (x_1, \dots, x_n) \text{ is called the state vector.}$$

Also, we will often use the phrase "structure  $f$ " in place of "system structure  $(N, f)$ ".

Let a vector  $X = (x_1, \dots, x_n)$ , with element  $x_i \in \{0, 1\}$  for  $i = 1, \dots, n$ , be called a diadic vector. The  $2^n$  possible diadic vectors are in obvious correspondence with the element  $S$  of the power set  $2^N$  of  $N$ :

$$S_X = \{j: x_j = 1\}, \quad (1.3)$$

component  $x_j$  of  $X_S = 1$  (resp., 0) if  $j$  is  
(resp., is not) a member of  $S$ .

$S_X$  in (1.3) is called the subset corresponding to the diadic vector  $X$ , and  $X_S$  is called the diadic vector corresponding to the subset  $S$ . With the one-to-one correspondence in mind, it becomes clear that the characteristic functions  $v$  of  $n$ -person simple games, and the structure functions  $f$  of reliability systems, are algebraically indistinguishable. This recognition is sufficient for us to seek to explore these sorts of functions in one vocabulary, as "diadic functions". To this end, let  $B$  be the doublet  $\{0, 1\}$ , and consider the set  $\xi$  composed of  $2^{2^n}$  functions  $\phi$  with domain  $B^N$  and range  $B$ .

We propose to simultaneously study properties of characteristic functions and structure functions through the study of such diadic functions. While such a study will reveal a few notions new to both areas, the primary emphasis will be on making sure that all findings on diadic functions, whether original to this thesis, or originating in one or the other of the two cognate disciplines, are viewed in the light both.

## 1.2. Basic Properties of Characteristic and Structure Functions

A physical system would be quite unusual (or perhaps poorly designed) if improving the performance of a component (that is, replacing a failed component by a functioning component) causes the system to deteriorate (that is, to change from the functioning state to the failed state). Thus, the structure function  $f$  of a system usually is assumed to satisfy

$$f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \geq f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \quad (1.4)$$

for all  $i = 1, \dots, n$ .

A structure function satisfying (1.4) is called a monotone structure function. Limiting the study of systems to structure functions satisfying (1.4) thus focuses attention on just those structures that do not deteriorate when a component improves. While

monotonicity is an attractive feature, it does not of course fit all interpretations of the phrase "system function" and "component function". There is, for example, a school of thought that thinks of war ( $f = 0$ ) more likely than peace ( $f = 1$ ) when one of the protagonists is weak and the other strong ( $(x_1, x_2) = (1, 0)$  or  $(0, 1)$ ).

The  $i$ -th component is called irrelevant to the structure function  $f$  if  $f$  is constant in  $x_i$ , that is,

$$f(1_i, X_{i\setminus i}) = f(0_i, X_{i\setminus i}) \text{ for all } (\bullet_i, X_{i\setminus i}), \quad (1.5)$$

where  $(1_i, X_{i\setminus i}) = (x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ ,  $(0_i, X_{i\setminus i}) = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ , and  $(\bullet_i, X_{i\setminus i}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ , otherwise the  $i$ -th component is called relevant to the structure.

In Barlow and Proschan (1975a), a system of components is called "coherent" if

- (a) its structure function  $f$  satisfies (1.4) and
- (b) each component is relevant.

While, in system theory, attention is focused on coherent systems, we will consider all possible cases, including systems whose state does not depend on the states of its components.

Usually, by an  $n$ -person game in characteristic functional form is meant a real-valued function  $v$ , defined on the subsets of  $N$ . Essentially,  $v(S)$  is the amount of utility that coalition  $S$  can obtain

from the game  $v$ , whatever the remaining players may do. From this definition, it is reasonable to assume that

$$v(\emptyset) = 0, \quad (1.6)$$

where  $\emptyset$  denotes the empty coalition. Now, if  $S$  and  $T$  are two disjoint coalitions, it is usual to assume that they can accomplish at least as much by joining forces as by remaining separate, leading to the "super-additivity" property:

$$v(S \cup T) \geq v(S) + v(T) \text{ if } S \cap T = \emptyset. \quad (1.7)$$

A less restrictive condition than that in (1.7) is "monotonicity":

$$v(S) \geq v(T) \text{ if } S \supseteq T, \quad (1.8)$$

which is the analogue of (1.4). If  $v$  is a simple game satisfying (1.6) as well as (1.8), then  $v$  is called "monotone simple game" (Deegan and Packel (1978)).

As indicated below in section 2.10, the game analogue of a component being irrelevant is that a player is a "dummy player".

### 1.3. Literature Review and Thesis Overview

It has been pointed out in section 1.1 that both characteristic functions and structure functions may be thought of as diadic functions; indeed, diadic functions of particular type; e.g., monotone, super-

additive, not-conditionally constant. A systematic examination of characteristic and structure functions therefore naturally begins with a careful examination of diadic functions as such, and sub-classes of diadic functions. Chapter 2 is devoted to such an examination, including a determination of inclusion relations linking the various sub-classes of interest. Other results of Chapter 2 explore facets of the individual sub-classes, as follows:

For the so-called "Boolean" sub-class of diadic functions, Rudeanu (1974) provides certain fundamental facts. For the examination of conditional constancy, the notion of carrier, as introduced in Shapley (1953), also is important, as well as the notion of "irrelevant component", as given in Barlow and Proschan (1975a), or "dummy player", as used in Owen (1982), or "inconsequential coordinate", as used in this thesis. We find it helpful, in addition, to introduce the notion of "minimal carrier", and its characterization as the complement of the set of inconsequential coordinates.

For monotone diadic functions, the notion of generating vector is introduced, which corresponds to the minimal path sets of system theory, and the minimal winning coalitions of game theory (Deegan and Packel (1978)). An algorithm is given for constructing the set of generating vectors of an arbitrary monotone diadic function. Linking the notions of conditional constancy and monotonicity is a theorem characterizing the carrier in several ways, including a characterization in terms of

generating vectors (and their dual "veto" vectors) that pertains to the monotone case. For the set of super-additive diadic functions, the main thrust is their characterization within the set of monotone functions. One corollary of that line of inquiry is a necessary and sufficient condition for the super-additivity of "weighted-majority games", and their specializations, the "k-out-of-n" systems of reliability theory.

As indicated in the introduction to that Chapter, Chapter 3 translates the concepts and results on diadic functions of Chapter 2 into statements about game characteristic functions and reliability structure functions. In so doing, various communalities between systems and games are pointed out.

Also, it should be noted that Chapter 3 may be of help in clarifying the domains of applicability of the various standard results of both game and reliability theory. Typically, authors have tended to be somewhat arbitrary in choosing the function domains applicable to their results, without much attention paid to the largest function domains to which these results might apply. Thus, some authors (Owen (1982)) in the literature of simple games deal with super-additive characteristic functions, while others (Deegan and Packel (1978)) assume monotonicity. On the other hand, while most authors in the literature on reliability structure functions deal with the coherent case, some (e.g., Block and Savits (1982)) treat the more general monotone case. We find that the monotone domain accommodates some of the results of

both disciplines, while, specifically in Chapter 5, the entire diadic domain accommodates others.

In Chapter 4, we survey several methods for expressing and relating characteristic functions and structure functions. Some of these originate in game theory, while others originate in reliability theory. Specifically, the valid domain for the "Shapley form" of an  $n$ -person game is extended to the full domain of all diadic functions. We exhibit a version of the Shapley form for monotone diadic functions, based solely on generating vectors, while the original form is based (in Shapley's terms) on all vectors dominated by a finite carrier. Finally, we exhibit the relationship between the "Barlow-Proschan form" (Barlow and Proschan (1975a)) and the "multilinear extension form" suggested by Owen (1972), involving not-necessarily diadic coordinates.

In Chapter 5, we survey several methods to measure the importance of a player (or component). Some of these methods originate in game theory, and deal largely with axiomatic rationale and static situations, while other methods originate in reliability theory, and deal with time dependence in dynamic settings. Further, the uniqueness of the "Shapley value" is considered over the entire domain of diadic functions, and its uniqueness over that domain is related to a set of "revised Shapley axioms". We also verify that the "Banzhaf-Coleman index" of game theory and the "Birnbaum structure index" of reliability theory represent the same value function in diadic function terms, and point out that these

indices can be defined over the entire domain of diadic functions. Also, we point out the equivalence of the "Birnbaum reliability function" for systems and the "multilinear extension" for games. All of the above importance indices are based on critical vectors. Some recently proposed game-theoretic indices (Deegan and Packel (1978)) are based on generating vectors. At the end of Chapter 5, these are interpreted in the system context.

## 2. DIADIC FUNCTIONS

### 2.1. Introduction

In the previous Chapter, it has been pointed out that the characteristic function of a (0,1) n-person simple game and the structure function of an n-component system belong to the class  $\xi$  of "diadic" functions.

In this Chapter we will consider the various restrictions on  $\xi$ , which produce the usual characteristic functions, structure functions and their sub-types. We thus consider five principal types of diadic functions:

- (1) Boolean diadic functions (BDF); i.e., Boolean functions over  $\{0,1\}^n$ ,
- (2) Monotone diadic functions (MDF),
- (3) Super-additive diadic functions (SPDF),
- (4) Sub-additive diadic functions (SBDF), and
- (5) Inessential diadic functions (IDF).

Also, relations among these classes will be explored, as well as certain pertinent details regarding their structures.

We begin with a glossary and some notations. First, let us consider the three Boolean operators  $\square$ ,  $\emptyset$ , ' on the Boolean Algebra B with the elements  $\{0,1\}$ , where the first two operators are binary and the third unary:

Table 1: Three Boolean operators on  $\{0,1\}$ 

$\square$	0	1	$\ominus$	0	1	$x$	0	1
0	0	1	0	0	0	$x'$	1	0
1	1	1	1	0	1			

The three operators are called disjunction, conjunction and complementation, respectively. One may define three corresponding operators  $U, \cap, c$  on the Boolean Algebra with elements in  $B^n$ : For any two vectors  $X = (x_1, \dots, x_n)$  and  $Y = (y_1, \dots, y_n)$  belonging to  $B^n$ ,

(a) the union of  $X$  and  $Y$ ,  $X \cup Y$ , is defined by

$$X \cup Y = (z_1, \dots, z_n), \text{ where } z_i = x_i \square y_i \text{ for any } i=1,2,\dots,n,$$

(b) the intersection of  $X$  and  $Y$ ,  $X \cap Y$ , is defined by

$$X \cap Y = (w_1, \dots, w_n), \text{ where } w_i = x_i \ominus y_i \\ \text{for } i = 1,2, \dots, n,$$

(c) the complement of  $X$ ,  $X^c$ , is defined by

$$X^c = (r_1, \dots, r_n) \text{ where } r_i = x_i' \\ \text{for } i = 1,2, \dots, n.$$

In addition, it is useful to also define the following:

- (d)  $X$  dominates  $Y$ , denoted by  $X \supseteq Y$ , if  $X \cup Y = X$ ,  
 $X$  dominates  $Y$  strictly, denoted by  $X \supset Y$ , if  $X \supseteq Y$  and  $X \neq Y$
- (e)  $X - Y = X \cap Y^c$ ,
- (f) the cardinality of a vector  $X$ , denoted by  $|X|$ , denotes the number of elements of  $X$  that equal 1,
- (g)  $E_i$  is the elementary vector corresponding to  $i$  if  $E_i = (x_1, \dots, x_n)$ , where  $x_k = 0$  for  $k \neq i$  and  $x_k = 1$  for  $k = i$ ,
- (h) given a particular diadic function  $f$ ,  $X_i$  is a critical vector corresponding to  $i$  if  $f(X_i) = 1$  and  $f(X_i - E_i) = 0$
- (i)  $E_0$  and  $E_N$  denote the zero vector and the unit vector, respectively.

## 2.2. Boolean Diadic Functions

From Rudeanu (1974), we have the following definitions.

### Definition 2.1

The class of Boolean diadic functions on  $B^n$  into  $B$  are determined by the following rules:

- (a) for either  $z \in B$ , the constant function  $f_z: B^n \rightarrow B$

is defined by

$$f_z(X) = z \text{ for any } X \in B^n,$$

(b) for any  $i = 1, 2, \dots, n$ , the projection function

$\tau_i: B^n \rightarrow B$  is defined by

$$\tau_i(X) = \tau_i(x_1, \dots, x_n) = x_i \text{ for all } X \in B^n,$$

(c) the class of Boolean diadic function is the class of constant functions defined under (a), plus functions constructed by a finite number of repeated application of the operators  $\square$ ,  $\emptyset$ , and  $'$  beginning with the projection functions defined under (b).

#### Example 2.1

The following functions all belong to the class of BDFs: for any  $X \in B^n$ ,

$$f_0(X) \equiv 0, f_1(X) \equiv 1,$$

$$\tau_1(X) = \tau_1(x_1, \dots, x_n) = x_1, \tau_2(X) = x_2,$$

$$g_1(X) = \tau_1(X) \square \tau_2(X), g_2(X) = \tau_1(X) \emptyset \tau_2(X), \text{ and}$$

$$g_3(X) = (\tau_1(X))'.$$

Rudeanu (1974, pp 23) has demonstrated the following lemma.

Lemma 2.1

The class of Boolean diadic functions coincides with the class of the diadic functions.

## 2.3. Monotone Diadic Functions

Monotonicity is to be interpreted in this section as denoting that a function is non-decreasing. Hence, a monotone diadic function (MDF) is defined by;

Definition 2.2

A diadic function  $\phi$  is a monotone diadic function (MDF) if  $\phi(X) \geq \phi(Y)$  for any  $X$  and  $Y$  such that  $X \supseteq Y$ .

Of special interest is the MDF  $\phi_{X_1}(X) \equiv \phi_1(X)$  satisfying

$$\begin{aligned} \phi_1(X) &= 1 \text{ if } X \supseteq X_1, \\ &= 0 \text{ if otherwise,} \end{aligned} \tag{2.1}$$

where we say that  $\phi_1$  corresponds to  $X_1$ , and  $X_1$  corresponds to  $\phi_1$ .

Consider now another vector  $X_2$  and its corresponding function  $\phi_{X_2}(X) \equiv \phi_2(X)$ , as well as the function  $\phi_{X_1 X_2} \equiv \phi_{12}(X)$  such that

$$\phi_{12}(X) = \phi_1(X) \square \phi_2(X), \tag{2.2}$$

$$= \max_{1 \leq i \leq 2} \phi_i(X), \tag{2.3}$$

$= I_{12}$ , where  $I_{12}$  is the indicator of

$$[x \supseteq x_1 \text{ or } x \supseteq x_2]. \quad (2.4)$$

Similarly, for points  $X_1, \dots, X_t$  with their corresponding functions  $\phi_1, \dots, \phi_t$ , respectively, we can consider the function

$\phi_{x_1 \dots x_t}(X) = \phi_B(X)$  satisfying

$$\phi_B(X) = \bigcap_{i=1}^t \phi_i(X), \quad (2.5)$$

$$= \max_{1 \leq i \leq t} \phi_i(X), \quad (2.6)$$

$$= I_B, \text{ where } I_B \text{ is the indicator of} \\ [X \supseteq x_1, \dots, \text{ or } X \supseteq x_t]. \quad (2.7)$$

### Definition 2.3

If none of  $X_1, \dots, X_t$  dominates any of the others in (2.7), we say that  $B = \{X_i\}_{i=1}^t$  is a basis of  $\phi_B$ , and  $X_1, \dots, X_t$  are called generating vectors (GV) of  $\phi_B$ . Trivially,  $X_1, \dots, X_t$  are generating vectors of, and constitute singleton basis of,  $\phi_1, \dots, \phi_t$ , respectively.

As is shown in the next lemma, bases are associated uniquely with MDFs, so that the article "the", rather than "a", should be used in the preceding definition.

### Lemma 2.2

Suppose both  $B_1 = \{X_i\}_{i=1}^t$  and  $B_2 = \{Y_j\}_{j=1}^r$  are bases for  $\phi$ ; then,  $r = t$  and  $B_1 = B_2$ .

Proof

By definition 2.3, for any  $X_i \in B_1$ ,  $\phi(X_i) = 1$ . Hence, since  $B_2$  is a basis,  $X_i$  either equals or dominates some  $Y_j$ . But, in the latter case, since  $Y_j$  is a basis element,  $X_i$  can not be, which is a contradiction, so that  $X_i = Y_j$ . Thus, for any  $X_i \in B_1$ , there is at least one  $Y_j \in B_2$  such that  $X_i = Y_j$ , so that  $B_1 \subseteq B_2$ . Similarly, for any  $Y_j \in B_2$ , there is at least one  $X_i \in B_1$ , such that  $X_i = Y_j$ , so that  $B_1 \supseteq B_2$ . Hence,  $B_1 = B_2$ . Q.E.D.

Now, given a MDF  $\phi$ , we need an algorithm to identify its basis:

Basis Algorithm

Exhibit each of the  $n$  possible projection functions  $(\tau_1, \dots, \tau_n)$  as a  $2^n$ -dimensional column vector, and construct a  $(2^n \times n)$  matrix  $M$  by listing  $\tau_1, \dots, \tau_n$  in order. This matrix has the following structure:

0. Its first row is  $E_0$ , which is denoted as the block  $\Omega_0$ .
1. Its next  $n$  rows, forming an  $n \times n$  block  $\Omega_1$ , each contain exactly  $(n - 1)$  zeros.
2. Its next  $\binom{n}{2}$  rows, forming a  $\binom{n}{2} \times n$  block  $\Omega_2$ , each contain exactly  $(n-2)$  zeros.
3. Its next  $\binom{n}{3}$  rows, forming a  $\binom{n}{3} \times n$  block  $\Omega_3$ , each contains exactly  $(n - 3)$  zeros.

•

- 
- 
- 

n. Its last row is  $E_N$ , which denoted as the block  $\Omega_n$ .

Given a MDF  $\phi$ , the algorithm generates a set of diadic vectors (to be called generating vectors) in the following way:

Step 0: If  $\phi(E_0) = 1$ ; then,  $\phi \equiv 1$ ,  $S_0 = E_0$  forms the basis of  $\phi$ , and the algorithm terminates.

If otherwise, proceed to step 1.

Step 1: Form the set  $S_1$  of all members of  $\Omega_1$ , for which  $\phi$  equals 1.

Step 2: Form the set  $S_2$  of all members of  $\Omega_2$ , for which  $\phi$  equals 1, that satisfy the additional requirement of not dominating any member of  $S_1$ .

Step 3: Form the set  $S_3$  of all members of  $\Omega_3$ , for which  $\phi$  equals 1, that satisfy the additional requirement of not dominating any member of  $(S_1 \cup S_2)$ .

- 
- 
- 
- 

Step n-1: Form the set  $S_{n-1}$  of all members of  $\Omega_{n-1}$ , for which  $\phi$

equals 1, that satisfy the additional requirement of not dominating any member of  $(S_1 \cup \dots \cup S_{n-2})$ .

Step n: If  $(S_1 \cup \dots \cup S_{n-1})$  is not empty; then, that is the basis of  $\phi$ , and the algorithm terminates.

If  $(S_1 \cup \dots \cup S_{n-1})$  is empty, and if  $\phi(E_N)$  equals 0, then,  $\phi \equiv 0$  and there is no basis of  $\phi$ .

If  $(S_1 \cup \dots \cup S_{n-1})$  is empty, and if  $\phi(E_N)$  equals 1, then,  $S_n = E_N$  forms the basis of  $\phi$ .

### Proof

In view of definition 2.3,  $D \equiv \{X_1, \dots, X_t\}$  is the basis of  $\phi$  if and only if

- ( $\alpha$ )  $\phi(X) = 1$  if  $X$  dominates at least one  $X_i$ ,  $i \in \{1, \dots, t\}$ ,  
 $= 0$  if otherwise,
- ( $\beta$ ) none of  $X_1, \dots, X_t$  dominates any other.

Let us check these two conditions for the following for cases:

(i) when  $\phi(X) = 0$  for all  $X \in B^n$ ,  $B = \{0,1\}$ :

It is trivially true that there is no vector satisfying ( $\alpha$ ) and ( $\beta$ ).

(ii) when  $\phi(X) = 1$  for all  $X \in B^n$ ,  $B = \{0,1\}$ :

It is clear that only  $\{E_0\}$  is the set satisfying ( $\alpha$ ) and ( $\beta$ ).

(iii) when  $\phi(E_N) = 1$  and  $\phi(X) = 0$  if  $X \subset E_N$ :

It is clear that only  $\{E_N\}$  is the set satisfying ( $\alpha$ ) and ( $\beta$ ).

(iv) when  $\phi(E_0) = 0$ ,  $\phi(E_N) = 1$ , and there is at least one  $X$  such that  $\phi(X) = 1$  and  $E_0 \subset X \subset E_N$ :

(a) let  $S_j$  be the first non-empty  $S$ -set; then, by construction of  $S_j$ ,

$$\begin{aligned}\phi(X) &= 1 \text{ if } X \text{ dominates at least one } X_i \in S_j, \\ &= 0 \text{ if otherwise,}\end{aligned}$$

and no member of  $S_j$  dominates any other. Also since it is impossible that a member of  $S_j$  dominates a member of  $(\Omega_{j+1} \cup \dots \cup \Omega_{n-1})$  (hence of  $(S_{j+1} \cup \dots \cup S_{n-1})$ ),  $(S_1 \cup \dots \cup S_j)$  is the set of all members  $X$  of  $D$  such that  $|X| \leq j$ .

(b) consider the induction hypothesis  $\Gamma_k$ ,  $k \geq j$ :  $(S_1 \cup \dots \cup S_k \cup S_{k+1})$  is non-empty, and is the set of all members  $X$  of  $D$  such that  $|X| \leq k$ . Now consider the set  $(S_1 \cup \dots \cup S_{k+1})$ . In view of  $\Gamma_k$ , and the construction of  $S_{k+1}$ , we can infer that

$$\begin{aligned}(\alpha') \phi(X) &= 1 \text{ if } X \text{ dominates at least one member of } (S_1 \cup \dots \cup S_{k+1}), \\ &= 0 \text{ if otherwise,}\end{aligned}$$

and  $(\beta')$  no member of  $(S_1 \cup \dots \cup S_{k+1})$  dominates any other. Moreover, since a member of  $(S_1 \cup \dots \cup S_{k+1})$  cannot dominate a member of  $(\Omega_{k+2} \cup \dots \cup \Omega_{n-1})$ ,  $(S_1 \cup \dots \cup S_{k+1})$  is the set of all members  $X$  of  $D$  such that  $|X| \leq k+1$ , which amounts to  $\Gamma_{k+1}$ . The argument in  $(\alpha')$  and  $(\beta')$  thus establishes that  $(S_1 \cup \dots \cup S_{n-1}) \equiv \zeta$  is non-empty, and is the set of all members  $X$  of  $D$  such that  $|X| \leq n-1$ . Moreover, since  $E_N$  dominates all members of  $\zeta$ , it cannot be in  $D$ , so that  $\zeta = D$ . Q.E.D.

We now turn to the following lemma.

Lemma 2.3

The class of BDFs includes, and does not coincides with, the class of MDFs.

Proof

While example 2.2 below, together with lemma 2.1, provides sufficient evidence for lemma 2.3, that {MDF} {BDF} is established here independently by an argument more accessible than that given by Rudeanu for lemma 2.1, (which of course cover {MDF} {BDF}, as well as {DF} {MDF}). To begin with, recalling equation (2.1) and definition 2.1. (b), the monotone diadic function  $\phi_{X_1}$  corresponding to the single generating vector  $X_1 = (x_{11}, \dots, x_{1n})$  may be expressed as:

$$\phi_{X_1}(X) = \bigcap_{\{j: x_{1j}=1\}} \tau_j(X). \quad (2.8)$$

Moreover, in view of equation (2.5), if the monotone diadic function  $\phi$  has basis  $\{X_1, \dots, X_t\}$ ; then,

$$\phi(X) = \bigcap_{i=1}^t \phi_{X_i}(X). \quad (2.9)$$

Hence, (2.9) and (2.10) imply that

$$\phi(X) = \bigcap_{i=1}^t \bigcap_{\{j: x_{ij}=1\}} \tau_j(X),$$

where  $x_{ij}$  is the  $j$ -th element in the  $i$ -th GV  $X_i$ .

Hence, by definition,  $\phi$  is a Boolean diadic function.

Further, if  $\phi$  is a constant function; then, it is also a BDF by definition. Finally, the following example shows that the class of BDFs does not coincide with the class of MDFs:

Example 2.2

Consider the following diadic function;

$$f(0,0) = 1,$$

$$f(0,1) = 1,$$

$$f(1,0) = 0,$$

$$f(1,1) = 0.$$

Let  $\tau'$  be a projection function such that  $\tau'(x_1, x_2) = x_1$ , then,  $f = (\tau')'$ , thus  $f$  is a BDF, but it is not a MDF. This completes the proof of lemma 2.3.

The next example illustrates the basis algorithm.

Example 2.3

Let us consider, for  $n = 3$ , the  $8 \times 3$  matrix  $M$ , together with the indicated MDF  $\phi$ :

	$\tau_1$	$\tau_2$	$\tau_3$		$\phi(X)$
$\Omega_0$	0	0	0		0
	1	0	0		0
$\Omega_1$	0	1	0		1
	0	0	1		1

-----					
	1	1	0		1
$\Omega_2$	1	0	1		1
	0	1	1		1
-----					
$\Omega_3$	1	1	1		1
-----					

The basis algorithm identifies the vectors  $(0,1,0)$  and  $(0,0,1)$  as constituting the basis of  $\phi$ . Moreover,  $\phi$  can be expressed as follows;

$$\phi(X) = (\tau_2 \square \tau_3)(X).$$

Structure functions in effect involve only the two Boolean operators  $\square$  and  $\emptyset$ . So it is of interest to consider BDFs whose construction does not involve the complementation operator. We call such a Boolean diadic function a "restricted Boolean diadic function"; it turns out (lemma 2.4) that the class of such diadic functions coincides with the MDFs.

Definition 2.4

A restricted Boolean diadic function (RBDF) is either a constant function (definition 2.1. (a)), or is constructed from the  $n$  projection functions  $\tau_i$  (definition 2.1. (b)) by a finite succession of operators  $\square$  or  $\emptyset$ .

Lemma 2.4

The class of RBDFs is the class of MDFs.

Proof (if part).

By the proof of lemma 2.3.

(only if part).

It is clear that, on the one hand, the projection function  $\tau_i$  are monotone, and, on the other hand, monotonicity is preserved under  $\square$  and  $\theta$ . Hence, the assertion is true. Q.E.D.

## 2.4. Super-additive Diadic Functions

Even more restrictive than monotonicity is the condition of super-additivity, which was introduced in section 1.2. Let us consider a diadic function with such a property.

Definition 2.5

A diadic function  $\phi$  is a super-additive diadic function (SPDF) if  $\phi(X \cup Y) \geq \phi(X) + \phi(Y)$  for any  $X$  and  $Y$  such that  $X \cap Y = E_0 = (0, \dots, 0)$ .

From definitions 2.3 and 2.5, the following lemma can be inferred.

Lemma 2.5

The class of MDFs includes, and does not coincide with, the class of SPDFs.

Proof

Suppose  $Y \supseteq X$ ; then,  $Y = X \cup (Y - X) \equiv X \cup Z$ , where  $X \cap Z = E_0$ .

Then, for any SPDF  $\phi$ ,

$$\begin{aligned}\phi(Y) &\geq \phi(X) + \phi(Z) \text{ by super-additivity} \\ &\geq \phi(X) \text{ since } \phi \text{ is diadic.}\end{aligned}$$

Thus,  $\phi$  belongs to the class of MDFs.

Finally, the following example shows that the class of MDFs does not coincide with the class of SPDFs.

Example 2.4

Consider a MDF such that

$$\begin{aligned}\phi(0,0) &= 0, \\ \phi(0,1) &= 1, \\ \phi(1,0) &= 1, \\ \phi(1,1) &= 1.\end{aligned}$$

Let  $X_1 = \{1,1\}$ ,  $X_2 = \{0,1\}$ , and  $X_3 = \{1,0\}$ ; then,

$$\phi(X_1) \leq \phi(X_2) + \phi(X_3), \text{ thus, } \phi \text{ is not super-additive.}$$

We next turn to a characterization of super-additivity within the class of MDFs.

Definition 2.6

The basis  $\{X_i\}_{i=1}^t$  of a MDF  $\phi$  is pairwise joint if  $X_i \cap X_j \neq E_0$  for all  $i, j = 1, \dots, t$ . When a basis is not pairwise joint; then, there is at least one pair of GVs such that  $X_i \cap X_j = E_0$ .

However, there is the possibility of equality between SPDF and MDF. The following lemma explores this.

Lemma 2.6

A MDF  $\phi$  possessing a basis (i.e., not identically zero) is super-additive if and only if the basis of a  $\phi$  is pairwise joint.

Proof (if part)

We consider two mutually exclusive and exhaustive cases (a) and (b) for  $\phi(X)$  and  $\phi(Y)$ , with  $X$  and  $Y$  such that  $X \cap Y = E_0$ ;

(a)  $\phi(X) = \phi(Y) = 1$ :

that this case can not happen under the condition of the lemma is shown as follows:

First we note that the evident fact that

$X \supseteq X_i, Y \supseteq X_j, X \cap Y = E_0$  implies that  $X_i \cap X_j = E_0$ .

Now  $\phi(X) = 1$  implies that there is a GV  $X_i$  such that  $X \supseteq X_i$ ;

also,  $\phi(Y) = 1$  implies that there is a GV  $X_j$  such that  $Y \supseteq X_j$ .

Hence,  $\phi(X) = \phi(Y) = 1$  and  $X \cap Y = E_0$  imply that  $X_i \cap X_j = E_0$ ,

which, in view of definition 2.6, contradicts the given condition.

(b)  $\phi(X) = 0$  or  $\phi(Y) = 0$ :

Supposing, without loss of generality, that  $\phi(X) = 0$ , we then have, by monotonicity,

$$\phi(X \cup Y) \geq \phi(Y) = \phi(X) + \phi(Y).$$

(only if part)

If  $\phi$  has the singleton basis; then, that basis clearly is pairwise joint. Now suppose that the basis is not a singleton basis, and that it is not pairwise joint; consider the GV pair  $(X_i, X_j)$  with  $X_i \cap X_j = E_0$ . Then,  $\phi(X_i) = \phi(X_j) = \phi(X_i \cup X_j) = 1$ , and super-additivity does not hold. Q.E.D.

## 2.5. Sub-additive Diadic Functions

Monotone diadic functions (that is, monotone non-decreasing diadic functions) typically are super-, as opposed to sub-, additive. Yet it is possible to identify, in terms of generating vectors, the few monotone diadic functions that are sub-additive. This section addresses that issue.

### Definition 2.7

A diadic function  $\phi$  is a sub-additive diadic function (SBDF) if

$$\phi(X_1 \cup X_2) \leq \phi(X_1) + \phi(X_2) \quad (2.10)$$

for any  $X_1$  and  $X_2$  such that  $X_1 \cap X_2 = E_0$ .

### Definition 2.8

A diadic vector  $X$  is said to be extremal if it is either an elementary vector or is the zero vector  $E_0$ .

Lemma 2.7

The following 3 assertions are equivalent for a diadic function  $\phi$ :

- (1) if  $X$  is a GV of  $\phi$ ; then,  $X$  is extremal.
- (2)  $\phi$  is sub-additive.
- (3)  $\phi$  is either (a) a constant function or (b) a diadic function all of whose GVs are elementary vectors.

Proof

(3) implies (2):

For any  $X_1$  and  $X_2$  such that  $X_1 \cap X_2 = E_0$ , let us check relation (2.10) for cases (a) and (b) in (3);

Case (a):

If  $\phi(X) = 0$  for any  $X$ ; then,  $\phi(X_1 \cup X_2) = \phi(X_1) = \phi(X_2) = 0$ , and if  $\phi(X) = 1$  for any  $X$ ; then,  $\phi(X_1 \cup X_2) = \phi(X_1) = \phi(X_2) = 1$ . Thus, in both cases  $\phi(X_1 \cup X_2) \leq \phi(X_1) + \phi(X_2)$ .

Case (b):

The only case for which sub-additivity does not hold is the case of  $\phi(X_1 \cup X_2) = 1$  and  $\phi(X_1) = \phi(X_2) = 0$ . However this case can not be happen.

For, if  $\phi$  has only elementary vectors as GVs; then,  $\phi(X_1) = \phi(X_2) = 0$  implies that  $X_1$  and  $X_2$  do not dominate any GV, and this fact implies that  $X_1 \cup X_2$  does not dominate any GV; i.e.,  $\phi(X_1 \cup X_2) = 0$ .

(2) implies (1):

Suppose  $\phi$  has a GV  $X$  that is not extremal; WLOG, let  $X = (1, 1, 0, \dots, 0)$ ; then,  $X = E_1 \cup E_2$ ,  $E_1 \cap E_2 = E_0$ ,  $\phi(X) = 1$  and  $\phi(E_1) = \phi(E_2) = 0$ . Hence,  $\phi(X) = \phi(E_1 \cup E_2) > \phi(E_1) + \phi(E_2)$ , which contradicts the fact that  $\phi$  is sub-additive.

(1) implies (3):

(1) means that  $\phi$  does not have any GV at all or that  $\phi$  has only GVs that are extremal. If  $\phi$  does not have any GV at all, or  $\phi$  has the the zero vector as its GV; then,  $\phi$  is a constant function. And, if  $\phi$  has elementary vectors as its GVs, then,  $\phi$  satisfies (b) in (3). Q.E.D.

## 2.6. Inessential Diadic Functions

The last special class of diadic functions to be considered is the intersection of the class of super-additive diadic functions and the class of sub-additive diadic functions.

### Definition 2.9

A diadic function  $\phi$  is an inessential diadic function if

$$\phi(X_1 \cup X_2) = \phi(X_1) + \phi(X_2)$$

for all  $X_1$  and  $X_2$  such that  $X_1 \cap X_2 = E_0$ .

Lemma 2.8

A diadic function  $\phi$  is an inessential diadic function if and only if the basis of  $\phi$  consists of singleton extremal vector.

Proof

Since  $\phi$  is inessential if and only if it is both super- and sub-additive, it follows, by lemma 2.4 and 2.5, that (i) the GVs of  $\phi$  are pairwise joint and (ii) if  $X$  is a GV of  $\phi$ ; then,  $X$  is extremal. But (i) and (ii) together are equivalent to the condition of the lemma.

## 2.7. Relationships among Sub-classes of Diadic Functions

We recall that, in sections 2.2 and 2.3, we have seen that the class of diadic functions coincides with the class of Boolean diadic functions, and that the class of monotone diadic functions coincides with the class of restricted Boolean diadic functions. The following example completes this sort of inclusion analysis, by showing that, for  $n \geq 3$ , the class of monotone diadic functions strictly includes the union of the class of super-additive and the class of sub-additive diadic functions.

Example 2.5

Let  $E_1, E_2, E_3$  denote three elementary vectors, and consider a MDF  $\phi$  which has  $E_1 \cup E_2$  and  $E_3$  as its GVs, i.e.,

$$\begin{aligned} \phi(X) &= 1 \text{ if } X \supseteq E_1 \cup E_2 \text{ or } X \supseteq E_3, \\ &= 0 \text{ if otherwise.} \end{aligned}$$

Then, letting  $n = 3$  WLOG,

$$(a) \phi(1,1,1) = \phi(1,1,0) = \phi(0,0,1) = 1,$$

so that  $\phi$  does not satisfy super-additivity,

$$(b) \phi(1,1,0) = 1 \text{ and } \phi(1,0,0) = \phi(0,1,0) = 0,$$

so that  $\phi$  does not satisfy sub-additivity, and

The following diagram summarizes the interrelationships among the diadic functions discussed in this Chapter. The theorem that goes along with the diagram is designated as the Diadic Function Inclusion Theorem: The class {DF} of diadic functions, and its six subclasses {BDF}, {RBDF}, {MDF}, {SPDF}, {ABDF}, and {IDF} are interrelated as indicated by figure 1.

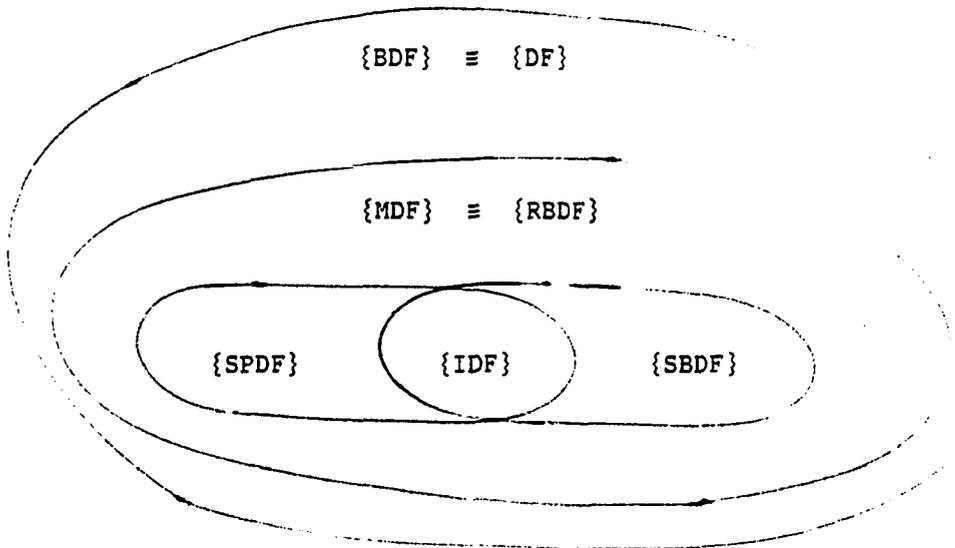


Figure 1: Relationships among classes of diadic functions

## 2.8. Cliques

Additionally, for certain special MDFs, there are more direct ways to check super-additivity than that in lemma 2.4. Before pursuing these ideas, we need certain additional definitions related to such special MDFs.

Definition 2.10

Given a diadic vector  $X_i$ , a MDF  $\psi_{X_i} \equiv \psi_i$  is defined by:

$$\begin{aligned}\psi_i(X) &= 1 \text{ if } X \cap X_i \neq E_0, \\ &= 0 \text{ if otherwise.}\end{aligned}\tag{2.11}$$

Then,  $X_i$  is said to be the "veto vector (VV)" of the MDF  $\psi_i$ . Similarly, given  $X_1, \dots, X_r$ , with none dominating any other, we can define a MDF  $\phi$  as follows;

$$\begin{aligned}\phi(X) &= 1 \text{ if } X \cap X_i \neq E_0 \text{ for all } i, \\ &= 0 \text{ if otherwise.}\end{aligned}\tag{2.12}$$

Then, the set of veto vectors  $\{X_i\}_{i=1}^r$  is said to be the "clique" of  $\phi$ .

The defining equation (2.12) has the following equivalent versions:

$$\phi(X) = \min_{1 \leq i \leq r} \psi_i(X).\tag{2.13}$$

$$\phi(X) = \prod_{i=1}^r \psi_i(X).\tag{2.14}$$

In analogy to the fact, explored in section 2.3, that bases are unique, it is equally true that cliques are unique. Furthermore, cliques can be identified by an algorithm, call it the "clique algorithm", analogous to the basis algorithm in section 2.3.

This algorithm is now presented.

### Clique Algorithm

Consider the matrix  $M$  in section 2.3 which contains  $2^n$  rows and  $n$  columns. Given a MDF  $\phi$ , the algorithm generates a set of vectors (to be called veto vectors) in the following way:

Step 0: If  $\phi(E_N)$  equals 0; then,  $T_0 = E_0$  is the clique of  $\phi$ , and the algorithm terminates.

If otherwise; then, proceed to step 1.

Step 1: Form the set  $T_1$  of all members of  $\Omega_{n-1}$ , for which  $\phi$  equals 0.

Step 2: Form the set of  $T_2$  of all members of  $\Omega_{n-2}$ , for which  $\phi$  equals 0, that satisfy the additional requirement of not being dominated by any member of  $T_1$ .

Step 3: Form the set of  $T_3$  of all members of  $\Omega_{n-3}$ , for which  $\phi$  equals 0, that satisfy the additional requirement of not being dominated by any member of  $(T_1 \cup T_2)$ .

•

•

- 
- 

Step n-1: Form the set  $T_{n-1}$  of all members of  $\Omega_1$ , for which  $\phi$  equals 0, that satisfy the additional requirement of not being dominated by any member of  $(T_1 \cup \dots \cup T_{n-2})$ .

Step n: If  $(T_1 \cup \dots \cup T_{n-1})$  is not empty; then, that is the clique of  $\phi$ .

If  $(T_1 \cup \dots \cup T_{n-1})$  is empty, and if  $\phi(E_0)$  equals 1, then, there is no clique for  $\phi$ .

If  $(T_1 \cup \dots \cup T_{n-1})$  is empty, and if  $\phi(E_0)$  equals 0, then,  $T_n = E_N$  constitutes the clique of  $\phi$ .

### Proof

In view of definition 2.10,  $C \equiv \{X_1, \dots, X_r\}$  is the clique of  $\phi$  if and only if

$$\begin{aligned} (\alpha) \quad \phi(X) &= 1 \text{ if } X \cap X_i \neq E_0, \quad i \in \{1, \dots, r\}, \\ &= 0 \text{ if otherwise,} \end{aligned}$$

$$(\beta) \quad \text{none of } X_1, \dots, X_r \text{ dominates any other.}$$

Let us check these two conditions for the following four cases:

(i) when  $\phi(X) = 0$  for all  $X \in B^n$ :

It is clear that only  $\{E_0\}$  is the set satisfying  $(\alpha)$  and  $(\beta)$ .

(ii) when  $\phi(X) = 1$  for all  $X \in B^n$ :

It is trivially true that there is no vector satisfying  $(\alpha)$  and  $(\beta)$ .

(iii) when  $\phi(E_0) = 0$  and  $\phi(X) = 1$  if  $X \subset E_0$ :

It is clear that only  $\{E_N\}$  is the set satisfying  $(\alpha)$  and  $(\beta)$ .

(iv) when  $\phi(E_0) = 0$ ,  $\phi(E_N) = 1$ , and there is at least one  $X$  such that  $\phi(X) = 0$  and  $E_0 \subset X \subset E_N$ :

(a) let  $T_j$  be the first non-empty T-set; then, by construction of  $T_j$ ,

$$\begin{aligned} \phi(X) &= 1 \text{ if } X \subset X_i \neq E_0 \text{ for all } X_i \in T_j, \\ &= 0 \text{ if otherwise,} \end{aligned}$$

and no member of  $T_j$  dominates any other. Also, since it is impossible that a member of  $T_j$  is dominated by a member of  $(\Omega_{n-j-1} \cup \dots \cup \Omega_1)$  (hence of  $(T_{j+1} \cup \dots \cup T_{n-1})$ ),  $(T_1 \cup \dots \cup T_j)$  is the set of all members  $X$  of  $C$  such that  $|X| \geq j$ .

(b) consider the induction hypothesis  $\Gamma_k$ ,  $k \geq j$ :  $(T_1 \cup \dots \cup T_k)$  is non-empty, and is the set of all members  $X$  of  $C$  such that  $|X| \geq k$ .

Consider the set  $(T_1 \cup \dots \cup T_k \cup T_{k+1})$ . In view of  $\Gamma_k$ , and the construction of  $T_{k+1}$ , we can infer that

$$\begin{aligned} (\alpha') \quad \phi(X) &= 1 \text{ if } X \subset X_i \neq E_0 \text{ for all } X_i \in (T_1 \cup \dots \cup T_{k+1}), \\ &= 0 \text{ if otherwise,} \end{aligned}$$

$(\beta')$  no member of  $(T_1 \cup \dots \cup T_{k+1})$  dominates any other.

Moreover, since a member of  $(T_1 \cup \dots \cup T_{k+1})$  cannot be dominated a member of  $(\Omega_{n-k-2} \cup \dots \cup \Omega_1)$ ,  $(T_1 \cup \dots \cup T_{k+1})$  is the set of all members  $X$  of  $C$  such that  $|X| \geq k+1$ , which amounts to  $\Gamma_{k+1}$ . The argument

in  $(\alpha')$  and  $(\beta')$  thus establishes that  $\xi \equiv (T_1 \cup \dots \cup T_{n-1})$  is non-empty, and is members  $X$  of  $C$  such that  $|X| \geq 1$ . Moreover, since  $E_0$  is dominated by all members of  $\xi$ , it cannot be in  $C$ , so that  $\xi = C$ .

Q.E.D.

The following table 2 shows bases and cliques for constant diadic functions, where  $\phi$  denotes the empty set.

Table 2: Bases and cliques for constant diadic functions

	Basis	clique
$\phi \equiv 0$	$\phi$	$\{E_0\}$
$\phi \equiv 1$	$\{E_0\}$	$\phi$

Not restricted to constant diadic functions is the more important relationship (duality) between basis and clique for MDFs, expressed by lemma 2.9: As a preliminaries, call  $\phi$  and  $\psi$  mutually dual if  $\phi(X) = \psi'(X^C)$ , or  $\psi(X) = \phi'(X^C)$ , then, we have:

Lemma 2.9

When  $\phi$  and  $\psi$  are mutually dual MDFs, then, the basis of one is the clique of the other.

Proof

To begin with, consider the following two mutually exclusive and exhaustive conditions, expressed in terms of a given collection  $D$  of diadic vectors  $\{X_i\}_{i=1}^S$ ,

condition 1:  $X$  satisfies condition 1 if

$$X \cap X_i \neq E_0 \text{ for all } x_i \in D,$$

condition 2:  $X$  satisfies condition 2 if

$$X \cap X_i = E_0 \text{ for at least one } x_i \in D.$$

Assume that  $D$  is the clique of the MDF  $\phi$ ; then,

$$\begin{aligned} \phi(X) &= 1 \text{ when } X \text{ satisfies condition 1,} \\ &= 0 \text{ when } X \text{ satisfies condition 2.} \end{aligned} \quad (2.15)$$

Expression (2.15) has the following equivalent versions:

$$\begin{aligned} \psi(X^C) &= 0 \text{ under condition 1,} \\ &= 1 \text{ under condition 2.} \end{aligned} \quad (2.16)$$

$$\begin{aligned} \psi(E_N - X) &= 0 \text{ under condition 1,} \\ &= 1 \text{ under condition 2.} \end{aligned} \quad (2.17)$$

$$\begin{aligned} \psi(X) &= 0 \text{ if } (E_N - X) \cap X_i \neq E_0 \text{ for all } i, \\ &= 1 \text{ if } (E_N - X) \cap X_i = E_0 \text{ at least one } i. \end{aligned} \quad (2.18)$$

$$\begin{aligned} \psi(X) &= 0 \text{ if } X \subset X_i \text{ for all } i, \\ &= 1 \text{ if } X \supseteq X_i \text{ for at least one } i. \end{aligned} \quad (2.19)$$

The step from (2.18) to (2.19) is valid since  $X \supseteq X_i$  if and only if  $(E_N - X) \cap X_i = E_0$ , and  $X \subset X_i$  if and only if  $(E_N - X) \cap X_i \neq E_0$ . The last expression (2.19) denotes, by (2.7), that  $D$  is the basis of  $\Psi$ .

In view of lemma 2.9, the definitions and results concerning cliques are in a sense implicit in the definitions and results concerning bases; however, the following lemma is naturally stated in terms of clique; it shows how to check the super-additivity of a MDF  $\phi$  through its clique.

Lemma 2.10

A MDF  $\phi$  is super-additive if its clique contains at least one elementary vector.

Proof

Let  $E_{i_0}$  be any one of the elementary vectors in the clique of  $\phi$ , and consider any  $X_1$  and  $X_2$  such that

$$X_1 \cap X_2 = E_0. \quad (2.20)$$

In view of (2.20) it must be true that one of  $X_1$  and  $X_2$ , say  $X_2$ , does not dominate  $E_{i_0}$ . Hence, since  $E_{i_0}$  belongs to the clique of  $\phi$ , by (2.12),

$$\phi(X_2) = 0. \quad (2.21)$$

Hence,  $\phi(X_1 \cup X_2) \geq \phi(x_1) = \phi(X_1) + \phi(X_2)$ , where the inequality follows from monotonicity, and the equality is due to (2.21). Q.E.D.

## 2.9. Characterization of Super-additivity for Special MDFs

We begin with a useful characterization of super-additivity for MDFs.

Lemma 2.11

A MDF  $\phi$  is super-additive if and only if there does not exist any  $X_1$  and  $X_2$  such that (a)  $X_1 \cap X_2 = E_0$ , and (b)  $\phi(X_1) = \phi(X_2) = 1$ .

Proof (if part).

If we assume the given conditions, there are two possibilities for a MDF  $\phi$ , and two diadic vectors  $X_1$  and  $X_2$  such that  $X_1 \cap X_2 = E_0$ :

$$(i) \phi(X_1) = \phi(X_2) = 0,$$

$$(ii) \phi(X_1) = 1 \text{ and } \phi(X_2) = 0.$$

Then, from (i) and (ii), and monotonicity of  $\phi$ , it follows that

$$\phi(X_1 \cup X_2) \geq \phi(X_1) + \phi(X_2).$$

(only if part).

Assume that there are  $X_1$  and  $X_2$  satisfying (a) and (b). Then, since  $\phi$  is diadic, it must be that

$$\phi(X_1 \cup X_2) < \phi(X_1) + \phi(X_2), \text{ violating super-additivity.}$$

Q.E.D.

Lemma 2.11 is now utilized to characterize super-additivity for two special classes of MDFs. The first such class of MDFs is that of MDFs whose clique  $C$  is composed of pairwise disjoint veto vectors, in which case a MDF  $\phi$  is called  $\phi_C$ . For MDFs  $\phi_C$ , the DFs  $\psi_i$  corresponding to the veto vector, and appearing in (2.11), (2.13) and (2.14), will be called component MDFs of  $\phi_C$ .

Example 2.6 illustrates the relationship between  $\phi_C$  and its component MDFs  $\psi_i$ , for the case  $n = 4$  and  $r = 2$ .

Example 2.6

Let  $X_1 = (1,1,0,0)$  and  $X_2 = (0,0,1,1)$ ; then,  $\psi_1$ ,  $\psi_2$ , and  $\phi_C$  are as follows;

$x_1$	$x_2$	$x_3$	$x_4$	$\psi_1$	$\psi_2$	$\phi_C$
0	0	0	0	0	0	0
1	0	0	0	1	0	0
0	1	0	0	1	0	0
0	0	1	0	0	1	0
0	0	0	1	0	1	0
1	1	0	0	1	0	0
1	0	1	0	1	1	1
1	0	0	1	1	1	1
0	1	1	0	1	1	1
0	1	0	1	1	1	1
0	0	1	1	0	1	0
1	1	1	0	1	1	1
1	1	0	1	1	1	1
1	0	1	1	1	1	1
0	1	1	1	1	1	1
1	1	1	1	1	1	1

One reason to focus on MDFs  $\phi_C$  is that the condition of lemma 2.10 is now necessary as well as sufficient.

Lemma 2.12

If  $\phi_C$  is not a constant function, it is super-additive if and only if its clique  $C = \{X_i\}_{i=1}^r$  contains at least one elementary vector.

Proof (if part).

Demonstrated by lemma 2.10.

(only if part).

Assume that the clique does not contain any elementary vectors at all; then, for every member  $X_i$  of the clique  $C$ , there are two disjoint partition vectors  $X_{i1}$  and  $X_{i2}$  of  $X_i$  such that  $X_{i1} \cup X_{i2} = X_i$ ,  $X_{i1} \cap X_{i2} = E_0$ ,  $X_{i1} \neq E_0$ , and  $X_{i2} \neq E_0$ . Also, if we put

$$Y_1 = \bigcup_{i=1}^r X_{i1} \quad \text{and} \quad Y_2 = \bigcup_{i=1}^r X_{i2}, \quad (2.22)$$

then,  $Y_1 \cap Y_2 = E_0$ ,  $Y_1 \cap X_i \neq E_0$  and  $Y_2 \cap X_i \neq E_0$  for all  $i = 1, \dots, r$ . Hence, from (2.12),

$$\phi_C(Y_1) = \phi_C(Y_2) = 1, \quad (2.23)$$

so that, by lemma 2.11,  $\phi_C$  can not be super-additive. It follows that, if  $\phi_C$  is super-additive; then, its clique must contain at least one elementary vector. Q.E.D.

The following example 2.7, and the previous example 2.6, serve to illustrate lemma 2.10. The MDF  $\phi_C$  of example 2.6 does not possess an elementary veto vector and is not super-additive, while the MDF  $\phi_C$  of

example 2.7 possesses the elementary veto vector  $(1,0,0,0)$  and is super-additive.

Example 2.7

Let  $C = \{x_1 = (1,0,0,0), x_2 = (0,1,1,1)\}$ ; then,  $\psi_1$ ,  $\psi_2$ , and  $\phi_C$  are as follows;

$x_1$	$x_2$	$x_3$	$x_4$	$\psi_1$	$\psi_2$	$\phi_C$
0	0	0	0	0	0	0
1	0	0	0	1	0	0
0	1	0	0	0	1	0
0	0	1	0	0	1	0
0	0	0	1	0	1	0
1	1	0	0	1	1	1
1	0	1	0	1	1	1
1	0	0	1	1	1	1
0	1	1	0	0	1	0
0	1	0	1	0	1	0
0	0	1	1	0	1	0
1	1	1	0	1	1	1
1	1	0	1	1	1	1
1	0	1	1	1	1	1
0	1	1	1	1	1	1
1	1	1	1	1	1	1

A second special class of MDFs is the class of "weighted-majority" MDFs  $\phi_M$ .

Definition 2.11

Let  $P = (p_1, \dots, p_n)$  be a non-negative vector, and let  $q$  satisfy  $0 < q \leq \sum_{i=1}^n p_i$ ; then,  $\phi_M$  is a weighted-majority MDF corresponding to  $P$  and  $q$  if, for any  $X = (x_1, \dots, x_n)$ ,

$$\begin{aligned} \phi_M(X) &= 1 \text{ if } \sum_{\{X: x_i=1\}} p_i \geq q, \\ &= 0 \text{ if otherwise.} \end{aligned} \quad (2.24)$$

The components of  $P$  are called the weights of  $\phi_M$ , while  $q$  is called the critical level of  $\phi_M$ . When all the weights of  $\phi_M$  equal 1 and the critical level equals  $n/2$ ; then,  $\phi_M$  is called a majority MDF.

The super-additivity of  $\phi_M$  is characterized by the following lemma.

Lemma 2.13

Let  $A$  denote a subset of  $N = \{1, \dots, n\}$ , and  $A^c$  denote  $N - A$ . Then, if  $\phi_M$  is not a constant function, it is super-additive if and only if  $q > \max \min \left( \sum_A p_i, \sum_{A^c} p_i \right)$ . (2.25)

Proof (only if part).

Assume that (2.25) does not hold, and let  $A_0$  be the maximizing  $A$  in the RHS of (2.25). Then,

$$\sum_{A_0} p_i = q + \varepsilon_1, \quad \sum_{A_0^c} p_i = q + \varepsilon_2; \quad \varepsilon_1, \varepsilon_2 \geq 0,$$

so that, using an obvious notation,  $X^{A_0} \cap X^{A_0^c} = E_0$  and  $\phi_M(X^{A_0}) = \phi_M(X^{A_0^c}) = 1$ , where  $X^{A_0}$  and  $X^{A_0^c}$  denote diadic vectors corresponding to  $A_0$  and  $A_0^c$ , respectively; hence,  $\phi_M$  is not super-additive in view of lemma 2.11.

(if part).

Assume that  $\phi_M$  is not super-additive; then, by lemma 2.11, there exist  $X_1 = (x_{11}, \dots, x_{1n})$  and  $X_2 = (x_{21}, \dots, x_{2n})$  satisfying (a) and (b) of lemma 2.11, where (b) translates, in the present instance, to

$$\sum_{\{i: x_{1i}=1\}} p_i \geq q, \quad \sum_{\{i: x_{2i}=1\}} p_i \geq q.$$

Hence, there exists  $A_1$ , corresponding to  $X_1$ , such that

$$\sum_{A_1} p_i \geq q, \quad \sum_{A_1^c} p_i \geq q, \text{ violating (2.25). Q.E.D.}$$

## 2.10. Inconsequential Coordinates

Given a diadic function  $\phi$ , a coordinate, say the first, is inconsequential if  $\phi$  is constant with respect to  $x_1$ ; i.e.,

$$\phi(0, x_2, \dots, x_n) = \phi(1, x_2, \dots, x_n)$$

for all  $2^{n-1}$  possible vectors  $(x_2, \dots, x_n)$ . When no coordinate is inconsequential, the diadic function involved will be called coherent, provided it is also monotone. Further, if  $D$  denotes the set of inconsequential coordinates of a diadic function  $\phi$ ; then,

$$\phi(X) = \phi^*(x_j: j \in (N - D)), \quad (2.26)$$

where  $\phi^*$  is defined over the non-inconsequential coordinates of  $\phi$ . We note that  $\phi$  and  $\phi^*$  are monotone, super-additive, sub-additive, or additive together.

The concept of "carrier" was introduced by Shapley (1953) in connection with n-person games. This concept is intimately related to the above notion of inconsequential coordinates. In keeping with Shapley's terminology, a diadic vector  $X$  is called a carrier of a MDF  $\phi$  if, for any diadic vector  $Y$ ,  $\phi(Y) = \phi(X \cap Y)$ . A consequence of this definition is that  $\phi(X) = \phi(E_N)$ . There may exist more than one carrier for a MDF  $\phi$ , since, if  $X$  is a carrier of  $\phi$ ; then, it is clear that all vectors dominating  $X$  also are carriers. This leads to the apparently new concept of minimality of a carrier.

Definition 2.12

Let  $\{X_i\}_{i=1}^p$  be the collection of carriers of a MDF  $\phi$ ; then,  $X$  is a minimal carrier of  $\phi$  if

- (a) for any  $Y$ ,  $\phi(Y) = \phi(X \cap Y)$  (carrier)
- (b)  $X \subseteq X_i$  for any  $i \in \{1, \dots, p\}$  (minimality).

Indeed, there is a unique minimal carrier  $Z$  for a MDF  $\phi$ , since, if  $Z_1$  and  $Z_2$  are minimal carriers of a MDF  $\phi$ ; then, both  $Z_1 \subseteq Z_2$  and  $Z_2 \subseteq Z_1$ , so that  $Z_1 = Z_2$ .

The following theorem exhibits the relationship between the minimal carrier and inconsequential coordinate set  $D$  of a DF, and the relationship among the minimal carrier, basis and clique of a MDF. Before stating the theorem, we need the following fact, which can be inferred from the definition of an inconsequential coordinate, that, for any non-inconsequential coordinate  $i_0$ , there is a critical vector  $X_{i_0}$  corresponding to  $i_0$ , with the  $i_0$ -th coordinate of  $X_{i_0}$  equal to one, such that  $\phi(X_{i_0}) = 1$  and  $\phi(X_{i_0} - E_{i_0}) = 0$ .

Theorem 2.1

Let  $\{Z_i\}_{i=1}^p$  be the set of carriers of a MDF  $\phi$ , and let  $\{X_i\}_{i=1}^t$  and  $\{Y_j\}_{j=1}^r$  be the basis and clique of the  $\phi$ , respectively; then, the following five assertions hold regarding  $Z$ , the minimal carrier of  $\phi$ ;

$$(a) Z = Z^1 \equiv (z_i = 1: i \in (N-D), z_j = 0: j \in D)$$

where  $D$  is the set of inconsequential coordinates, i.e.,

$Z$  is the vector corresponding to  $N - D$ .

$$(b) Z = Z^2 \equiv \bigcap_{i=1}^p Z_i.$$

(c)  $Z$  is a carrier such that  $\{[X \subset Z] \text{ or } [Z \not\subset X]\}$  implies that  $X$  is not a carrier.

$$(d) Z = Z^3 \equiv \bigcup_{i=1}^t X_i.$$

$$(e) Z = Z^4 \equiv \bigcup_{j=1}^r Y_j.$$

Proof for (a):

Let  $Z = (z_1, \dots, z_n)$  be the minimal carrier of  $\phi$ ;

(i) for all  $j \in D$ ,  $z_j = 0$ ; (2.27)

for, if we assume that, for any  $j_0 \in D$ ,  $z_{j_0} = 1$ ; then,

$Z - E_{j_0}$  is a carrier, which is in violation of the given condition.

(ii) for all  $i \in (N-D)$ ,  $z_i = 1$ ; (2.28)

for, if we assume that, for any  $i_0 \in (N-D)$ ,  $z_{i_0} = 0$ ; then,

for a critical vector  $X_{i_0}$  corresponding to  $i_0$ ,  $\phi(X_{i_0}) = 1$  and  $\phi(X_{i_0} \cap Z) = 0$ , so that  $\phi(X_{i_0}) \neq \phi(X_{i_0} \cap Z)$ ; i.e.,  $Z$  is not a carrier of  $\phi$ , which is in violation of the given condition.

Hence, from (2.27) and (2.28), (a) is true.

(b) part:

For two carriers  $Z_1$  and  $Z_2$  of a MDF  $\phi$ ,  $Z_1 \cap Z_2$  also is a carrier of the MDF  $\phi$ , since, for any  $Y$ ,

$$\begin{aligned} \phi(Y \cap Z_1 \cap Z_2) &= \phi(Y \cap Z_1) \text{ since } Z_2 \text{ is a carrier} \\ &= \phi(Y) \text{ since } Z_1 \text{ is a carrier.} \end{aligned}$$

Similarly, by induction,

$$Z^2 = \bigcap_{i=1}^p Z_i \text{ is a carrier of } \phi. \quad (2.29)$$

Also, (2.29) implies that

$$Z^2 \subseteq Z_i \text{ for all } i = 1, \dots, p. \quad (2.30)$$

hence, in view of (2.29), (2.30) and definition 2.12,  $Z^2$  is the minimal carrier of  $\phi$ .

(c) part:

Let  $X$  be a diadic vector such that (i)  $X = Z - Z^*$ ,  $Z^*$  is not the zero vector, or (ii)  $Z \leq X$ . If  $X$  is a carrier, then,  $Z$  violates (b) of definition 2.12, and hence can not be minimal. On the other hand, if  $Z$  satisfies (c), then, (b) of definition 2.12 must hold.

(d) is correct:

Firstly let us show that  $Z^3$  is a carrier. For any  $Y$ , there are two possibilities;

- (i)  $Y$  dominates at least one GV,
- (ii)  $Y$  does not dominate any GV.

For case (i);

assume, WLOG, that  $X_{i_0} \subseteq Y$ ; then,

$$1 \geq \phi(Y) \geq \phi(Y \cap Z^3) \geq \phi(Y \cap X_{i_0}) \geq \phi(X_{i_0}) = 1$$

where the equality follows from the fact that  $X_{i_0}$  is a generating vector of  $\phi$ . Thus,

$$\phi(Y) = \phi(Y \cap Z^3) = 1. \quad (2.31)$$

For case (ii);

$\phi(Y) = 0$  since  $Y$  does not dominate any GV, and

$$0 \leq \phi(Y \cap Z^3) \leq \phi(Y) = 0,$$

so that

$$\phi(Y) = \phi(Y \cap Z^3) = 0. \quad (2.32)$$

Relation (2.31) and (2.32) establish that  $Z^3$  is a carrier.

Secondly, let us show the minimality. For any elementary vector  $E_{i_0}$  and a GV  $X_{i_0}$  such that  $E_{i_0} \subseteq X_{i_0}$ , let  $Y = Z^3 - E_{i_0}$ ; then,

$$\begin{aligned} \phi(X_{i_0} \cap Y) &= \phi(X_{i_0} \cap (Z^3 - E_{i_0})) \\ &= \phi(X_{i_0} - E_{i_0}). \end{aligned}$$

However, it can be inferred from the definition of a GV that  $\phi(X_{i_0} - E_{i_0}) = 0$  and  $\phi(X_{i_0}) = 1$ ; i.e.,  $\phi(X_{i_0} \cap Y) \neq \phi(X_{i_0})$ . so that  $Y$  can not be a carrier.

Hence, in view of (c),  $Z^3$  is the minimal carrier of  $\phi$ .

(e) part:

It is enough to show that  $Z^4 = Z^1$ .

(i) by definition of a veto vector (VV), the set corresponding to a VV does not contain an inconsequential coordinate; hence, the set corresponding to  $Z^4$  does not contain an inconsequential coordinate.

(ii) for any non-inconsequential coordinate,  $i_0$  say, there exists at least one VV whose  $i_0$ -th coordinate is 1.

For, assume that this assertion is not true;

then,  $Y_j - E_{i_0} = Y_j$  for all  $j = 1, \dots, r$ .

so that,

$$\begin{aligned}
 \phi(X) &= \min_{1 \leq j \leq r} \phi(X \cap Y_j) \quad \text{by (2.12)} \\
 &= \min_{1 \leq j \leq r} \phi(X \cap (Y_j - E_{i_0})) \\
 &= \min_{1 \leq j \leq r} \phi((X - E_{i_0}) \cap Y_j) \\
 &= \phi(X - E_{i_0}).
 \end{aligned}$$

Thus,  $i_0$  is an inconsequential coordinate, which violates the given condition.

Hence, from (i) and (ii),  $Z^4 = Z^1$ . Q.E.D.

#### Remark

We note that the notion of carrier and minimal carrier is meaningful, and may be thought of, in the context of the full domain of diadic functions (or, equivalently, in view of lemma 2.11, the domain of BDF's). Moreover, parts (a), (b) and (c) of theorem 2.1 apply in that full domain.

### 3. CHARACTERISTIC FUNCTIONS AND STRUCTURE FUNCTIONS AS DIADIC FUNCTIONS

#### 3.1. Introduction

In this Chapter, we translate the concepts and results on diadic functions  $\phi$ , as discussed in the previous Chapter, into statements about characteristic functions  $\nu$  of  $(0,1)$  simple games, and structure functions  $f$  of reliability system structures. In doing so, we implicitly point to communalities between game and reliability structures.

Before proceeding with this agenda, it ought to be pointed out that the literature on games and reliability systems is not uniform in the restrictions placed on  $\nu$  and  $f$ . Some authors (Owen (1982)) in the literature of simple games deal with super-additive characteristic functions  $\nu$ , while others (Deegan and Packel (1978)) assume monotonicity. On the other hand, while most authors in the literature on reliability structure functions  $f$  deal with the coherent cases, some (e.g., Block and Savits (1982)) treat the more general monotone cases. Given this lack of uniformity in both areas, Chapter 2 may be thought of as clarifying the interrelationship between the various types of diadic functions to which given assertions about structure and characteristic functions in fact apply. Most assertions about structure and characteristic functions, in the literatures and in this thesis, pertain to the domain of monotone diadic functions, while some pertain to the entire diadic domain.

### 3.2. The Principal Diadic Functions as Characteristic and Structure Functions

In this section, we will interpret the principal types of diadic functions as characteristic functions  $v$  of  $(0,1)$  simple games and as structure functions  $f$  of  $n$ -component systems.

The diadic vector  $X$  in the domain of the diadic function  $\phi$  is to be interpreted as a coalition when  $\phi$  is a characteristic function, and as a state vector when  $\phi$  is a structure function.

The operators  $U, \cap, c, -, \supseteq, \supset, \subseteq, \subset$ , and  $|X|$ , which have been applied to diadic vectors can be considered as set operators with the usual meaning when diadic vectors are interpreted as coalitions.

An elementary diadic vector  $E_{i_0}$  denotes the coalition containing only player  $i_0$ , when  $\phi$  is a characteristic function, and the system state which only the  $i_0$ -th component functions, call it an elementary state vector, for which  $\phi$  is a structure function.

A generating vector of a MDF  $\phi$  can be interpreted as a minimal winning coalition of a  $(0,1)$  simple game whose characteristic function is  $\phi$ , or as a minimal path vector whose structure function is  $\phi$ . Moreover, vectors dominating a generating vector can be considered as winning coalitions when  $\phi$  is a characteristic function, and as path vectors when  $\phi$  is a structure function. The set corresponding to a (minimal) path vector is called a (minimal) path set. Thus, the basis

of a MDF  $\phi$  is to be considered as the collection of minimal winning coalitions of a characteristic function  $\phi$ , or as the collection of minimal path vectors of a structure function  $\phi$ . Given a characteristic function or structure function, we can collect its minimal winning coalitions or minimal path vectors by applying the basis algorithm; also, this collection is unique in both cases, by lemma 2.2.

We can infer, from lemma 2.5, that the class of characteristic or structure functions satisfying super-additivity is included in the class of characteristic or structure functions satisfying monotonicity. In order to check whether or not a characteristic or structure function is super-additive, it is enough to check, in view of lemma 2.6, whether or not the minimal winning coalitions or minimal path vectors of the characteristic or structure function are pairwise joint.

Given that diadic functions  $\phi$  can be thought of as either characteristic functions or structure functions, examples and counter-examples will simultaneously apply to both areas. This will be especially useful when the diagrams common to structure functions provide unexpected insights for characteristic functions. As an example, while super-additivity is generally thought of in connection with characteristic functions, the structure function corresponding to example 3.1 (a) below immediately translates to the not-super-additive characteristic function of example 3.1 (b).

Example 3.1

(a) Consider the following 4-component system;

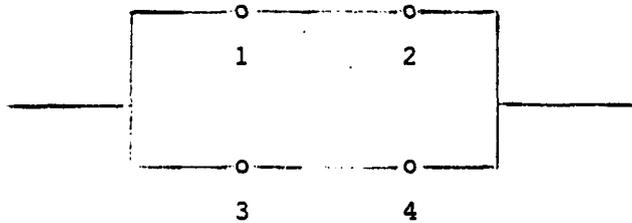


Figure 2: A 4-component system

The not-super-additive structure function of this system is given by

$$f(X) = 1 \text{ if } X \supseteq (1,1,0,0) \text{ or } X \supseteq (0,0,1,1), \\ = 0 \text{ if otherwise.}$$

(b) Consider the corresponding 4-person (0,1) simple game  $v$  such that

$$v(S) = 1 \text{ if } S \supseteq \{1,2\} \text{ or } S \supseteq \{3,4\}, \\ = 0 \text{ if otherwise.}$$

That  $v$  is not super-additive follows from the equivalence of (a) and (b), or from the fact that  $v(\{1,2,3,4\}) = v(\{1,2\}) = v(\{3,4\})$ , so that, by lemma 2.11,  $v$  is not super-additive.

In order to characterize characteristic or structure functions satisfying sub-additivity, it is helpful to note that an extremal diadic vector can be considered as either the "empty coalition" or a singleton

coalition in the game context, and as either the "total breakdown state" vector or elementary state vector in the reliability context. Also, a constant diadic function can be interpreted as either the characteristic function of a  $(0,1)$  simple game in which all coalitions win (or lose), and as the structure function of a system which operates (or does not) regardless of the state of any of its components.

With these interpretations in mind, lemma 2.7 says that  $v$  is sub-additive if  $v$  is the characteristic function of either

(i) a game in which all coalitions win (or lose),

or

(ii) a game that has no minimal winning coalitions other than singleton coalitions.

Similarly, we can say that  $f$  is sub-additive if  $f$  is the structure function of either

(i) a system which operates (or does not) regardless of the state of any of its components,

or

(ii) a system which has not minimal path vectors other than elementary vectors.

Inessential diadic functions can be thought of as characteristic or structure functions substituting equality for the inequalities of super- and sub-additivity.

Example 3.2

Inessential (0,1) simple games are of only two types:

$$(i) v(\{i_j\}) \equiv 0,$$

and

$$(ii) v(\{i_1\}) = 1 \text{ and } v(\{i_j\}) = 0 \text{ if } j \neq 1. \quad (3.1)$$

The system corresponding to (3.1) is one with  $(n - 1)$  inconsequential coordinates (i.e.,  $(n - 1)$  irrelevant components, in usual system terminology), whose diagram is as follows;

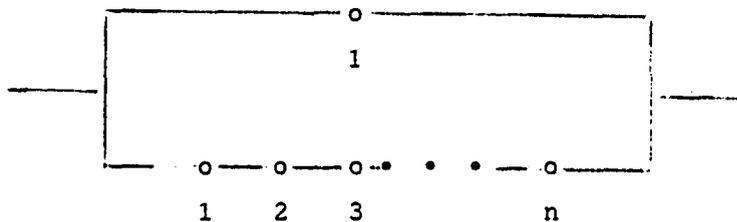


Figure 3: A system with  $(n - 1)$  irrelevant components

### 3.3. Special Diadic Functions as Characteristic or Structure Functions

Let us interpret the veto vectors and the clique of a MDF  $\phi$  in game and reliability terms. A veto vector of a MDF  $\phi$  is to be interpreted as a minimal cut vector when  $\phi$  is a structure function. The corresponding concept for games does not seem to occur in the literature, which leads us to the following definition.

Definition 3.1

A subset  $S$  of  $N$  is called a minimal losing "rotten egg" coalition of a  $(0,1)$  simple game  $v$  if

$$\begin{aligned} v(N - T) &= 1 \text{ if } T \subset S, \\ &= 0 \text{ if otherwise.} \end{aligned} \quad (3.2)$$

We note that, if  $S$  is the only coalition satisfying (3.2), the form has the following equivalent expression; for any coalition  $M$ ,

$$\begin{aligned} v(M) &= 1 \text{ if } M \cap S \neq \emptyset, \\ &= 0 \text{ if otherwise,} \end{aligned}$$

where  $\emptyset$  denotes the empty set. In view of definition 2.10, it is clear that a veto vector of a MDF  $\phi$  can be interpreted as a minimal losing coalition of a characteristic function  $\phi$ .

Similarly, let  $\{X_i\}_{i=1}^r$  be the clique of a MDF  $\phi$ ; then, this clique can be interpreted as the set  $\{C_i\}_{i=1}^r$  of minimal losing coalitions of a characteristic function, and, with  $C_1, \dots, C_r$  the coalitions corresponding to  $X_1, \dots, X_r$ , the characteristic function  $\phi$  can be expressed as follows;

$$\begin{aligned} \phi(T) &= 1 \text{ if } T \cap C_i \neq \emptyset \text{ for all } i = 1, \dots, r, \\ &= 0 \text{ if otherwise.} \end{aligned} \quad (3.3)$$

From the fact that every MDF  $\phi$  possesses a unique clique, it can be inferred that every characteristic (or structure) function possesses a

unique set of minimal losing coalitions (or minimal cut vectors). Moreover, given a characteristic (or structure) function, the set of minimal losing coalitions (or minimal cut vectors) can be obtained by applying the clique algorithm of section 2.8. Also, the super-additivity of a characteristic (or structure) function can be checked through the set of its minimal losing coalitions (or minimal cut vectors), in view of lemma 2.10.

The duality between basis and clique, as expressed in lemma 2.9, implies the corresponding duality between the set of minimal winning coalitions and the set of minimal losing coalitions, as well as the duality between the set of minimal path vectors and the set of minimal cut vectors. The latter duality is mentioned in Barlow and Proschan (1975a, pp 12). Since definition 3.1 appears to be new, so, it would seem, is mention of the duality between minimal winning and losing coalitions.

Section 2.9 introduced the special monotone diadic functions  $\phi_C$  whose cliques  $C$  are composed of pairwise disjoint veto vectors, with the diadic functions corresponding to these veto vectors called component MDFs of  $\phi_C$ . These ideas of section 2.9 translate to the notion of modular decomposition of system structures into structures possessing only one minimal cut vector, and  $\nu$ -composition of (0,1) simple games possessing only one minimal losing coalition. This is now verified by citing the relevant definitions in the two areas, and showing their equivalence to the constructs of section 2.9.

Definition 3.2 (Definition IX.2.7, Owen (1982))

Let  $C_1, \dots, C_r$  be  $r$  pairwise disjoint non-empty sets of players, and let  $\omega_1, \dots, \omega_r$  be  $(0,1)$  simple games, with player sets  $C_1, \dots, C_r$ , respectively. Let  $v$  be a  $(0,1)$  simple game with player set  $R = \{1, \dots, r\}$ . Then, the  $v$ -composition of  $\omega_1, \dots, \omega_r$ , denoted by  $\mu = v(\omega_1, \dots, \omega_r)$ , is a game with player set  $C = \bigcup_{i=1}^r C_i$  and characteristic function

$$\mu(S) = v_j(\{j: \omega_j(S \cap C_j) = 1\}) \text{ for } S \subseteq C. \quad (3.4)$$

If we put a restriction on  $v$  such that

$$v(R) = 1 \text{ and } v(\text{other coalitions}) = 0, \quad (3.5)$$

then,  $\mu(S) = \min_{1 \leq j \leq r} \omega_j(S \cap C_j)$  for  $S \subseteq C$ .  $v$ -composition under restriction (3.5) is now related to the functions  $\phi_C$  of section 2.9, with the following correspondence of terms:

$$S \sim X,$$

$$C_i \sim \text{veto vector, and}$$

$$\mu(S) \sim \phi_C(X).$$

The following example shows a  $v$ -composition under assumption (3.5):

Example 3.3

Consider  $(0,1)$  simple games  $\omega_1, \omega_2$ , and  $\omega_3$  with  $C_1 = \{1,2\}$ ,  $C_2 = \{3\}$ , and  $C_3 = \{4,5\}$  such that

$$\omega_1(\{1\}) = \omega_1(\{2\}) = \omega_1(\{1,2\}) = 1,$$

$$\omega_2(\{3\}) = 1,$$

$$\omega_3(\{4\}) = \omega_3(\{5\}) = \omega_3(\{4,5\}) = 1, \text{ and}$$

$\omega_i(\text{other sets}) = 0$  for  $i = 1, 2, 3$ .

Then,  $\mu = \nu(\omega_1, \omega_2, \omega_3)$  satisfies

$\mu(S) = 1$  if  $\omega_i(S \cap C_j) = 1$  for all  $i = 1, 2, 3$ ,  
 $= 0$  if otherwise.

Definition 3.3 (Definition 4.1 and 4.3, Barlow and Proschan (1975a))

The system structures  $(A_1, \mathfrak{F}_1), \dots, (A_r, \mathfrak{F}_r)$  are called modules of a system structure  $(N, f)$  if

$$f(X) = \Psi(\mathfrak{F}_1(X^{A_1}), \dots, \mathfrak{F}_r(X^{A_r})), \quad (3.6)$$

where (a)  $\bigcup_{i=1}^r A_i = N$  and  $A_i \cap A_j = \emptyset$  for  $i \neq j$ ,

(b)  $f$  is said to be decomposed into  $\mathfrak{F}_1, \dots, \mathfrak{F}_r$ , and the sets  $A_1, \dots, A_r$  are "modular sets" of  $(N, f)$ .

(c)  $\Psi$  is a structure function with arguments  $(x_1, \dots, x_r)$ .

(d)  $X^{A_i}$  denotes the vector with elements  $x_j; j \in A_i$ .

Analogously to the case of  $\nu$ -composition, the notion of modular decomposition is captured by the function  $\phi_C$ , with the following correspondence;

$$\begin{aligned} A_i &\sim \text{veto vector,} \\ \{A_i\}_{i=1}^r &\sim \text{clique, and} \\ f(X) &\sim \phi_C(X). \end{aligned}$$

The next example shows modular sets and modules of a system structure.

Example 3.4

Consider the following coherent system:

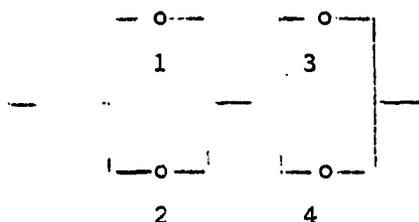


Figure 4: A 4-component coherent system

Then, the structure function  $f$  satisfies

$$\begin{aligned} f(x) &= 1 \text{ if } x \supseteq (1,1,0,0) \text{ or } (0,0,1,1), \\ &= 0 \text{ if otherwise.} \end{aligned}$$

Also, one module of  $(N, f)$  is  $(A_1, \vartheta_1)$  with  $A_1 = \{1,2\}$  and

$$\begin{aligned} \vartheta_1(x^{A_1}) &= 1 \text{ if } x^{A_1} \supseteq (1,0) \text{ or } (0,1), \\ &= 0 \text{ if otherwise,} \end{aligned}$$

and the other module is  $(A_2, \vartheta_2)$  with  $A_2 = \{3,4\}$  and

$$\begin{aligned} \vartheta_2(x^{A_2}) &= 1 \text{ if } x^{A_2} \supseteq (1,0) \text{ or } (0,1), \\ &= 0 \text{ if otherwise.} \end{aligned}$$

A second class of special MDFs, the weighted-majority MDFs  $\phi_M$ , can be considered as characteristic functions of weighted-majority games, and as structure functions of q-out-of-n systems, whose super-additivity can be checked by lemma 2.13.

Inconsequential coordinates of a MDF  $\phi$  can be thought of as dummy players when  $\phi$  represents a characteristic function, and, as already mentioned in connection with example 3.2, as irrelevant components when  $\phi$  represents a structure function. Also, when  $\phi$  is monotone, its related  $\phi^*$  can be thought of as a characteristic function whose domain does not contain dummy players (i.e., as a characteristic function whose domain is the minimal carrier), and as the structure function of a coherent system.

The concept of carrier is a new one in the system context, and the concept of minimal carrier appears to be new for both systems and games. Moreover, in view of theorem 2.1, the minimal carrier of a MDF  $\phi$  can be interpreted as the set of non-dummy players or the union of minimal winning coalitions when  $\phi$  is a characteristic function, and as the set of relevant components or the union of minimal path sets when  $\phi$  is a structure function.

#### 4. DIADIC FUNCTION REPRESENTATIONS AND RELATIONS

##### 4.1. Introduction

In the previous chapters, several types of diadic functions have been explored, and, by interpreting these functions in the game and reliability contexts, various properties of characteristic and structure functions have been found. In this chapter, we survey several representations and relations for diadic functions. Many of these functional relationships have been introduced in game theory, and some in reliability theory. However, in view of the communalities between these two areas, by introducing these relationships in diadic function terms, we can see that some of them are new for games, and others are new for reliability theory.

##### 4.2. The Generating and Veto Vector Form of a MDF

One sort of representation of a MDF  $\phi$  can be found in Barlow and Proschan (1975a), in which a structure function is expressed through minimal path or cut sets. This method, of course, can be applied to a (0,1) simple n-person games in view of the communalities between minimal path (or cut) sets and minimal winning (or losing) coalitions. To cover both areas, we introduce the ideas involved in terms of diadic functions.

Let  $\{Z_i\}_{i=1}^t$  be the set of GVs of a MDF  $\phi$ ; then, as in section 2.3, with the  $i$ -th GV  $Z_i = (z_{i1}, \dots, z_{in})$  we may associate a diadic function  $\phi_{Z_i}$  such that

$$\begin{aligned}\phi_{Z_i}(X) &= 1 \text{ if } X \supseteq Z_i, \\ &= 0 \text{ if otherwise.}\end{aligned}\tag{4.1}$$

This can be expressed as:

$$\phi_{Z_i} = \prod_{\{j: z_{ij}=1\}} x_j,\tag{4.2}$$

where  $x_j$  and  $z_{ij}$  denote  $j$ -th elements of the diadic vector  $X$  and the generating vector  $Z_i$ , respectively.

We have seen, from (2.1) and definition 2.3, that  $\phi_{Z_i}$  is the MDF which has  $Z_i$  as its only GV. Now, since the MDF  $\phi$  has GVs  $Z_1, \dots, Z_t$ , from (2.3) we have that

$$\begin{aligned}\phi(X) &= \max_{1 \leq i \leq t} \phi_{Z_i}(X) \\ &= 1 - \prod_{i=1}^t (1 - \phi_{Z_i}(X)) \\ &= 1 - \prod_{i=1}^t (1 - \prod_{\{j: z_{ij}=1\}} x_j).\end{aligned}\tag{4.3}$$

Thus, we can say that, in equation (4.3) the above  $\phi(X)$  is expressed through its GVs.

The following example illustrates equations (4.2) and (4.3) for a MDF  $\phi$ .

Example 4.1

Let us consider a MDF  $\phi$  which has  $Z_1 = (1,1,0,0)$  and  $Z_2 = (0,1,1,1)$  as its GVs; then, from (4.2),

$$\phi_{Z_1}(X) = x_1 x_2,$$

$$\phi_{Z_2}(X) = x_2 x_3 x_4,$$

and, from (4.3),

$$\begin{aligned} \phi(X) &= 1 - (1 - x_1 x_2)(1 - x_2 x_3 x_4) \\ &= x_1 x_2 + x_2 x_3 x_4 - x_1 x_2 x_3 x_4. \end{aligned} \quad (4.4)$$

Similarly, let  $\{Y_j\}_{j=1}^r$  be the set of veto vectors (VV) of  $\phi$ ; then, we may associate a diadic function with the  $i$ -th veto vector  $Y_i = (y_{i1}, \dots, y_{in})$  such that, for any  $X = (x_1, \dots, x_n)$ ,

$$\begin{aligned} \psi_{Y_i}(X) &= \max_{\{j: y_{ij}=1\}} x_j \\ &= 1 - \prod_{\{j: y_{ij}=1\}} (1 - x_j), \end{aligned} \quad (4.5)$$

where  $y_{ij}$  denotes the  $j$ -th element of  $i$ -th VV  $Y_i$ . Also, it can be checked from (2.11) that  $\psi_{Y_i}$  is the MDF which has  $Y_i$  as its unique veto vector. However, since the MDF  $\phi$  has veto vectors  $Y_1, \dots, Y_r$ , we have from (2.13) that

$$\phi(X) = \min_{1 \leq i \leq r} \psi_{Y_i}(X)$$

$$\begin{aligned}
&= \prod_{i=1}^r \psi_{Y_i}(X) \\
&= \prod_{i=1}^r \left(1 - \prod_{\{j: Y_{ij}=1\}} (1 - x_j)\right). \quad (4.6)
\end{aligned}$$

Thus, we can say that, in equation (4.6)  $\phi(X)$  is expressed through its VVs.

#### Example 4.2

Consider a MDF  $\phi$  whose veto vectors are  $Y_1 = (0,1,0,0)$ ,  $Y_2 = (1,0,1,0)$ , and  $Y_3 = (1,0,0,1)$ ; then, from (4.5) and (4.6),

$$\phi(X) = x_2(x_1 + x_3 - x_1x_3)(x_1 + x_4 - x_1x_4). \quad (4.7)$$

It is to be noted that (4.4) and (4.6) coincide on  $\{0,1\}^4$ , illustrating the fact that bases and cliques determine each other uniquely.

### 4.3. The Shapley Form of a MDF

Shapley (1953) provided a formula to express a super-additive characteristic function as a linear combination of characteristic functions  $\phi_Y$  of "symmetric games":

$$\begin{aligned}
\phi_Y(X) &= 1 \text{ if } X \supseteq Y, \\
&= 0 \text{ if otherwise.} \quad (4.8)
\end{aligned}$$

It is clear that  $Y$  is a carrier of  $\phi_Y$ ; indeed, it is the minimal carrier. The following lemma for game characteristic functions is due to Shapley (1953). It may be noted from Shapley's argument, reproduced below, that, as indicated in the statement of the lemma, its validity extends to the domain of DFs, with the diadic functions  $\phi_Y$  thought of in diadic function terms. Before stating the lemma we introduce the following terminology:  $x, y, z, \dots$  stand for the number of non-zero coordinates of  $X, Y, Z, \dots$ , respectively.

Lemma 4.1

Any (not necessarily monotone) DF  $\phi$ , with carrier  $T$ , has the following unique linear representation in terms of the  $2^t$  DFs  $\phi_Y$  corresponding to the  $2^t$  diadic vector  $Y$  such that  $Y \subseteq T$ :

$$\phi = \sum_{Y \subseteq T} C_Y(\phi) \phi_Y, \quad (4.9)$$

where the coefficients  $C_Y(\phi)$  are given by

$$C_Y(\phi) = \sum_{Z \subseteq Y} (-1)^{y-z} \phi(Z). \quad (4.10)$$

Proof

We must verify that

$$\phi(X) = \sum_{Y \subseteq T} C_Y(\phi) \phi_Y(X) \text{ holds for all } X.$$

In view of (4.9) and (4.10), we may proceed WLOG under the assumption that  $\phi$  is not constant function with value 0.

(i) to begin with, for  $X \subseteq T$ ,

$$\begin{aligned}
\sum_Y C_Y(\phi) \phi_Y(X) &= \sum_{Y \subseteq X} \left( \sum_{Z \subseteq Y} (-1)^{Y-Z} \phi(Z) \right) \phi_Y(X) \\
&= \sum_{Y \subseteq X} \left( \sum_{Z \subseteq Y} (-1)^{Y-Z} \phi(Z) \right) \\
&= \sum_{Z \subseteq X} \left( \sum_{Z \subseteq Y \subseteq X} (-1)^{Y-Z} \right) \phi(Z), \quad (4.11)
\end{aligned}$$

where the first and the second equalities follow from (4.8), and the third equality follows by switching the summation orders. Now consider the inner parenthesis in (4.11); for any value  $y$  between  $x$  and  $z$  there will be  $\binom{x-z}{y-z}$  vectors  $Y$  such that  $Z \subseteq Y \subseteq X$ . Hence, the inner parenthesis may be replaced by  $\sum_{y=z}^x \binom{x-z}{y-z} (-1)^{Y-Z}$ , so that expression (4.11) reduces to

$$\begin{aligned}
&\sum_{Z \subseteq X} \left( \sum_{y=z}^x \binom{x-z}{y-z} (-1)^{Y-Z} \right) \phi(Z) \\
&= \sum_{Z \subseteq X} ((1 - 1)^{x-z}) \phi(Z) \\
&= \phi(X),
\end{aligned}$$

where the first equality follows from the fact that  $\sum_{y=z}^x \binom{x-z}{y-z} (-1)^{Y-Z}$  is precisely the binomial expression of  $(1 - 1)^{x-z}$ , and the second equality follows from the fact that  $(1 - 1)^{x-z}$  will be zero for all  $z < x$  and will be unity for  $z = x$ , which, in view of  $Z \subseteq X$ , amounts to  $Z = X$ .

(ii) also, for  $X \supseteq T$ ,

$$\begin{aligned}
\phi(X) &= \phi(X \cap T) \\
&= \phi(T) \\
&= \sum_{Y \subseteq T} C_Y(\phi) \phi_Y(T)
\end{aligned}$$

$$= \sum_{Y \subseteq T} C_Y(\phi) \phi_Y(X),$$

where the first equality follows from the fact that  $T$  is a carrier, the second one from the fact  $X \cap T = T$ , the third one from the argument of (i), and the fourth one from the fact, in view of (4.8), that  $\phi_Y(T) = \phi_Y(X) = 1$  for all  $X$  and  $Y$  such that  $Y \subseteq T \subseteq X$ .

To show the uniqueness of this expression, we note that the expression (4.9) is a system of  $2^t$  linear equations in the  $2^t$  unknowns  $C_Y$ . Indeed, there will be a permutation such that the corresponding matrix of coefficients for the system will be a lower triangular matrix with all diagonal elements non-zero. Such a matrix is of course non-singular, so that the expression (4.9) is unique.

#### Corollary 4.1

Any (not necessarily monotone) DF  $\phi$ , with carrier  $T$ , has the following unique representation in terms of  $2^t$  coefficients  $C_Y$ :

$$\phi = \sum_{Y \subseteq T} C_Y.$$

The above lemma and corollary are now illustrated with a specific general (i.e., not monotone) DF:

#### Example 4.3

Let us consider a diadic function  $\phi$  such that

$$\begin{aligned} \phi(0,0,0) &= 0, \quad \phi(1,0,0) = 1, \quad \phi(0,1,0) = 0, \quad \phi(0,0,1) = 0, \\ \phi(1,1,0) &= 1, \quad \phi(1,0,1) = 0, \quad \phi(0,1,1) = 1, \quad \text{and } \phi(1,1,1) = 1. \end{aligned}$$

It is clear that  $\phi$  is not a MDF since  $\phi(1,0,0) > \phi(1,0,1)$ .

Let  $Y_1 = (0,0,0)$ ,  $Y_2 = (1,0,0)$ ,  $Y_3 = (0,1,0)$ ,  $Y_4 = (0,0,1)$ ,

$Y_5 = (1,1,0)$ ,  $Y_6 = (1,0,1)$ ,  $Y_7 = (0,1,1)$ , and  $Y_8 = (1,1,1)$ .

Then, from  $\phi(X) = \sum_{Y \subseteq X} C_Y(\phi) \phi_Y(X)$ , by putting  $C_Y(\phi) = C_Y$  for convenience,

$$\begin{aligned}
 0 &= \phi(0,0,0) = C_{Y_1} \\
 1 &= \phi(1,0,0) = C_{Y_1} + C_{Y_2} \\
 0 &= \phi(0,1,0) = C_{Y_1} + C_{Y_3} \\
 0 &= \phi(0,0,1) = C_{Y_1} + C_{Y_4} \\
 1 &= \phi(1,1,0) = C_{Y_1} + C_{Y_2} + C_{Y_5} \\
 0 &= \phi(1,0,1) = C_{Y_1} + C_{Y_2} + C_{Y_4} + C_{Y_6} \\
 1 &= \phi(0,1,1) = C_{Y_1} + C_{Y_3} + C_{Y_4} + C_{Y_7} \\
 1 &= \phi(1,1,1) = C_{Y_1} + C_{Y_2} + C_{Y_3} + C_{Y_4} + C_{Y_5} + C_{Y_6} + C_{Y_7} + C_{Y_8}.
 \end{aligned}$$

Hence, we can obtain an unique solution from the above equations such that

$$\begin{aligned}
 C_{Y_1} &= 0, C_{Y_2} = 1, C_{Y_3} = 0, C_{Y_4} = 0, C_{Y_5} = 0, C_{Y_6} = -1, C_{Y_7} = 1, \text{ and} \\
 C_{Y_8} &= 0.
 \end{aligned}$$

We note that a difficulty with the Shapley form is that there are a great many summation terms. However, if  $\phi$  is monotone; then, we can obtain a form with fewer summation terms, as in the following theorem. And, as is indicated below, this form turns out to be the generating

vector form corresponding to the familiar path set representation of system theory.

Theorem 4.1

The Shapley form of an MDF  $\phi$  which has  $\{Z_i\}_{i=1}^t$  as its GVs can be reduced to

$$\phi = \sum_{i=1}^t \phi_{Z_i} - \sum_{1 \leq i < j \leq t} \phi_{Z_i \cup Z_j} + \dots + (-1)^{t+1} \phi_{\bigcup_{i=1}^t Z_i}. \quad (4.12)$$

Proof

If we put  $T = \bigcup_{i=1}^t Z_i$ , we know, from lemma 4.1, that,

for all  $Y$  such that  $Y \subseteq T$ ,

$$\phi = \sum_Y C_Y(\phi) \phi_Y,$$

and we can consider the following 3 possible cases for any  $Y$ ;

(i)  $Y$  is strictly dominated by all  $Z_i$ ,  $i \in \{1, \dots, t\}$ ; then, from (4.10),  $C_Y(\phi) = 0$  since  $\phi(Z) = 0$  for all  $Z \subseteq Y$ .

(ii)  $Y$  dominates at least one  $Z_i$ , and is not equal to the union of the GVs that it dominates. In particular, assume WLOG that

$Z_1, \dots, Z_k$ ,  $k \leq t$ , are dominated by  $Y$  and

$Y \neq \bigcup_{i=1}^k Z_i$ ; then,

$$C_Y(\phi) = \sum_{Z \subseteq Y} (-1)^{|Y-Z|} \phi(Z)$$

$$\begin{aligned}
 &= \sum_{z_1 \subseteq Z \subseteq Y} (-1)^{Y-z} + \dots + \sum_{z_k \subseteq Z \subseteq Y} (-1)^{Y-z} \\
 &- \sum_{(z_1 \cup z_2) \subseteq Z \subseteq Y} (-1)^{Y-z} - \dots - \sum_{(z_{k-1} \cup z_k) \subseteq Z \subseteq Y} (-1)^{Y-z} \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &(-1)^{k+1} \sum_{\substack{z \\ \cup_{i=1}^k z_i \subseteq Z \subseteq Y}} (-1)^{Y-z}
 \end{aligned}$$

since  $\phi(Z) = 1$  for all  $Z \supseteq Z_i$ , and there will be  $\binom{Y-X}{Z-X}$  vectors such that  $X \subseteq Z \subseteq Y$ . Let  $z_{i_1 i_2 \dots i_r}$  denote the cardinality of  $Z_{i_1} \cup Z_{i_2} \cup \dots \cup Z_{i_r}$ ; then, the above equation reduces to

$$\begin{aligned}
 &= \sum_{z=z_1}^Y \binom{Y-z_1}{z-z_1} (-1)^{Y-z} + \dots + \sum_{z=z_k}^Y \binom{Y-z_k}{z-z_k} (-1)^{Y-z} \\
 &- \sum_{z=z_{12}}^Y \binom{Y-z_{12}}{z-z_{12}} (-1)^{Y-z} - \dots - \sum_{z=z_{(k-1)k}}^Y \binom{Y-z_{(k-1)k}}{z-z_{(k-1)k}} (-1)^{Y-z} \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &(-1)^{k+1} \sum_{z=z_{12\dots k}}^Y \binom{Y-z_{12\dots k}}{z-z_{12\dots k}} (-1)^{Y-z}
 \end{aligned}$$

$$\begin{aligned}
 &= (-1+1)^{Y-z_1} + \dots + (-1+1)^{Y-z_k} \\
 &\quad - (-1+1)^{Y-z_{12}} - \dots - (-1+1)^{Y-z_{(k-1)k}} \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot \\
 &(-1)^{k-1} (-1+1)^{Y-z_{12\dots k}}. \tag{4.13}
 \end{aligned}$$

But all  $z_{i_1 \dots i_r} < Y$ , in view of the current assumption regarding  $Y$ . Hence, all terms in (4.13) are zero; i.e.,  $C_Y(\phi) = 0$ .

Thus, from (i) and (ii),  $\phi$  can be reduced to

$$\phi = \sum_{i=1}^t C_{Z_1}(\phi) + \sum_{1 < i < j \leq t} C_{Z_i \cup Z_j} \dots + C_t(\phi) \phi_t, \tag{4.14}$$

$\begin{matrix} \cup Z_i & \cup Z_i \\ i=1 & i=1 \end{matrix}$

where the  $C_Y(\phi)$  are as in (4.9).

(iii)  $Y = \bigcup_{i=1}^k Z_i$  for some  $k \in \{1, \dots, t\}$ ;

$$\begin{aligned}
 C_Y(\phi) &= \sum_{Z_1 \subseteq Z \subseteq Y} (-1)^{Y-Z} + \dots + \sum_{Z_K \subseteq Z \subseteq Y} (-1)^{Y-Z} \\
 &\quad - \sum_{(Z_1 \cup Z_2) \subseteq Z \subseteq Y} (-1)^{Y-Z} - \dots - \sum_{(Z_{K-1} \cup Z_K) \subseteq Z \subseteq Y} (-1)^{Y-Z} \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot
 \end{aligned}$$

$$\begin{aligned}
& (-1)^{k+1} \sum_{\substack{U \subseteq Z \subseteq Y \\ i=1}}^k (-1)^{Y-Z} \text{ by similar arguments as in (ii)} \\
& = (-1)^{k+1} (-1+1)^{Y-Z}_{12\dots k} \\
& = (-1)^{k+1} (-1+1)^0 \text{ since } Y=Z_{12\dots k} \\
& = (-1)^{k+1}. \tag{4.15}
\end{aligned}$$

Hence, (4.14) can be reduced to (4.12).

We note that (4.12) in theorem 4.1 and (4.3) both express the MDF  $\phi$  through its GVs, even though the expressions are formally different.

However, we note that

(a) if the GVs are pairwise disjoint, then, the expressions are formally the same,

(b) if the GVs are not pairwise disjoint, then,  $\phi_{Z_1 U \dots U Z_k} (X)$  in (4.12) does not coincide formally with  $\prod_{i=1}^k \phi_{Z_i} (X)$  in (4.3).

However, for any diadic vector  $X$ , both forms, as they must, have the same value, since, for diadic  $x_i$ ,  $x_i^k = x_i$  for all real numbers  $k$ . The following example shows this.

Example 4.4

Consider the following coherent system;

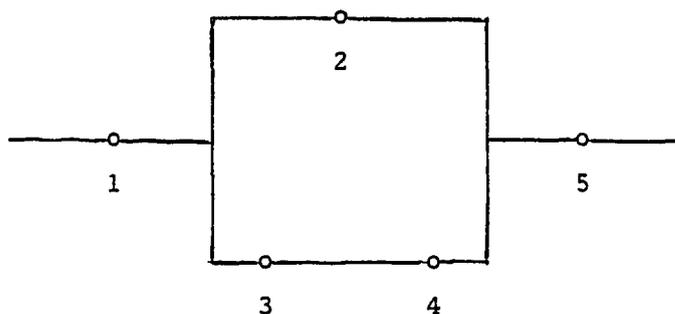


Figure 5: A 5-component coherent system

It is clear that  $Z_{11} = (1,1,0,0,1)$  and  $Z_2 = (1,0,1,1,1)$  can be thought of as the two GVs of the above system, which are not disjoint; then we obtain,

from (4.12),

$$\begin{aligned}\phi(X) &= \phi_{Z_1} + \phi_{Z_2} - \phi_{Z_1 \cup Z_2} \\ &= x_1 x_2 x_5 + x_1 x_3 x_4 x_5 - x_1 x_2 x_3 x_4 x_5.\end{aligned}$$

and, from (4.3),

$$\begin{aligned}\phi(X) &= 1 - (1 - \phi_{Z_1})(1 - \phi_{Z_2}) \\ &= x_1 x_2 x_5 + x_1 x_3 x_4 x_5 - x_1^2 x_2 x_3 x_4 x_5^2.\end{aligned}$$

#### 4.4. The Barlow-Proschan Form of a DF, and the Associated Multilinear Extension

Barlow and Proschan (1975a, pp5) showed the following identity;

$$\phi(X) = x_i \phi(x_i, X_{i'}) + (1-x_i) \phi(0_i, X_{i'}) \text{ for all } X, \quad (4.16)$$

and introduced a method for expressing a coherent  $\phi$  through repeated application of (4.16). Analogously to what we have seen in lemma 4.1, this method in fact applies to any DF  $\phi$ , as follows:

$$\begin{aligned} \phi(X) &= x_1 \phi(1, x_2, x_3, \dots, x_n) + (1-x_1) \phi(0, x_2, x_3, \dots, x_n) \\ &= x_1 (x_2 \phi(1, 1, x_3, \dots, x_n) + (1-x_2) \phi(1, 0, x_3, \dots, x_n)) \\ &\quad + (1-x_1) (x_2 \phi(0, 1, x_3, \dots, x_n) + (1-x_2) \phi(0, 0, x_3, \dots, x_n)) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &= \sum_Y \prod_{i=1}^n x_i^{y_i} (1-x_i)^{1-y_i} \phi(Y) \end{aligned} \quad (4.17)$$

This equation (4.17) will be called the "Barlow-Proschan form".

We further consider an extension, called "multilinear" in Owen (1972), which is simply achieved by interpreting  $X = (x_1, \dots, x_n)$  to be an arbitrary point of the unit cube in n-dimensional space in the equation (4.17). In order to distinguish the extension from its related DF, we denote it

$$g_{\phi}(X) = \sum_Y \prod_{i=1}^n x_i^{y_i} (1-x_i)^{1-y_i} \phi(Y), \quad X \in [0,1]^n. \quad (4.18)$$

To see the uniqueness of (4.17) and (4.18),  $g_{\phi}(X)$  in (4.18) can be rewritten as follows;

$$g_{\phi}(X) = \sum_Y C_Y \prod_{\{i: y_i=1\}} x_i. \quad (4.19)$$

Also, since it is clear that, for any diadic vector  $X$ ,  $g_{\phi}(X) = \phi(X)$ , (4.17) and (4.19) reduce to the expression of corollary 4.1:

For all  $X$ ,

$$\phi(X) = \sum_{Y \subseteq X} C_Y = g_{\phi}(X). \quad (4.20)$$

This is a system of  $2^n$  linear equations in the  $2^n$  unknowns  $C_Y$ , and, from lemma 4.1 and corollary 4.1, the corresponding matrix of coefficients  $C_Y$  is non-singular.

Finally, following Owen (1972), the extension can be interpreted probabilistically, i.e., if  $X_i$  denotes a random variable representing the state of coordinate  $i$  such that  $P(X_i=1) = x_i$ ,  $P(X_i=0) = 1-x_i$  for all  $i = 1, \dots, n$ , then,  $g_{\phi}(X)$  in (4.18) can be thought of as the mathematical expectation of  $\phi(X)$  under the assumption that  $X_1, \dots, X_n$  are pairwise independent.

The following example shows the Barlow-Proschan form and multilinear extension of a DF  $\phi$ .

Example 4.5

Let  $\phi$  be a DF such that

$$\begin{aligned}\phi(X) &= 1 \text{ if } X = (1,0,0), (1,0,1), (0,1,1), \text{ or } (1,1,1), \\ &= 0 \text{ if otherwise,}\end{aligned}$$

then, the Barlow-Proschan form is, from (4.17),

$$\phi(X) = x_1(1-x_2)(1-x_3) + x_1(1-x_2)x_3 + (1-x_1)x_2x_3 + x_1x_2x_3,$$

where  $x_i \in \{0,1\}$ ,  $i = 1,2,3$ . And, from (4.18),

$$g_\phi(X) = x_1(1-x_2)(1-x_3) + x_1(1-x_2)x_3 + (1-x_1)x_2x_3 + x_1x_2x_3,$$

where  $x_i \in [0,1]$ ,  $i = 1,2,3$ .

## 5. IMPORTANCE INDICES

## 5.1. Introduction

The previous chapter has been concerned with evaluations of, and functional relations among, diadic functions, these evaluations and interrelations arising for the most part in game and reliability theory.

Now we are interested in measuring the importance of each coordinate in a DF structure  $(N, \phi)$ . It is clearly of value to the game analyst, system designer or reliability analyst to have a quantitative measure of the importance of an individual coordinate (called "importance index"). Some coordinates typically will be more important than others in determining whether a DF  $\phi$  has the value 1 or 0. For example, most would agree that, for the system in figure 6 below, or a simple 4-person game  $v$  whose only winning coalitions contain player 1, coordinate 1 is more important than the others.

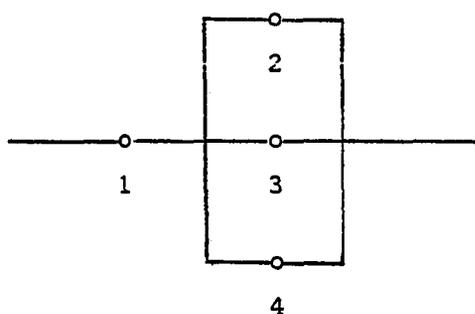


Figure 6: A 4-component coherent system

Thus, in this chapter we survey several methods of obtaining importance indices of coordinates. Some of these indices originate in game theory and others in reliability theory. The former largely deal with axiomatic rationale, while some of the latter deal with time dependent properties. Of course, it is in keeping with the general theme of this dissertation that all of these methods can be applied in both areas.

We recall the notion, in section 2.1, of a critical vector corresponding to a coordinate. Both in game theory and in reliability theory, most authors base their suggested importance indices on these critical vectors. Since, for the two constant diadic functions  $\phi(X) \equiv 0$  and  $\phi(X) \equiv 1$ , there are no critical vectors corresponding to any coordinate, such authors exempt the constant functions from their importance analysis.

In addition, Deegan and Packel (1978) introduced an alternative method for obtaining importance indices, in which the importance index of a coordinate is defined in terms of its role as a member of a generating vector rather than of a critical one. Moreover we will suggest an extension of this method to the time-dependent and other cases.

## 5.2. The Shapley Value

Shapley (1953) proved the following: there is a unique "value function" (called "Shapley value"), defined over the set of characteristic functions of  $n$ -person games, and taking values in  $\mathbb{R}^N$ , which satisfies a certain set of axioms called "Shapley axioms", where  $\mathbb{R}$  denotes the set of real numbers. When he proposed the Shapley value as an importance index for players in an  $n$ -person game, Shapley considered a game to be a super-additive set-function from the subsets of  $N$  to the real numbers. However, his proof does not introduce super-additivity considerations, so that it in fact pertains to the general class of functions from the subsets of  $N$  to  $\mathbb{R}^N$ .

Dubey (1975) showed that the Shapley value function is the unique one on the domain of monotone (not necessarily simple, i.e., not necessarily diadic)  $n$ -person game characteristic functions, satisfying the Shapley axioms. For the diadic case, Dubey (1975) showed that, under a variant  $S3'$  of Shapley's third axiom  $S3$ , together with Shapley's remaining axioms  $S1$  and  $S2$ , the Shapley value function is unique on the domain of monotone simple  $n$ -person game characteristic functions, i.e., on the domain of monotone diadic functions. He further demonstrated the analogous result for super-additive diadic functions. With the monotone and super-additive diadic case, as well as the monotone not-necessarily-diacic case, treated by Dubey (1975), it thus seems to the writer that the question of value function uniqueness on the domain of super-additive  $n$ -person game characteristic functions is still unsolved.

What Dubey and Shapley also seem to have left undone is the treatment of the domain of arbitrary, and arbitrary diadic, functions, and it is the purpose of this section to show, respectively in theorem 5.1 and 5.2, by arguments essentially contained in Shapley (1953), that the Shapley value function on these two types of domains is unique under the "revised Shapley axioms S1', S2', and S3'" given below.

Let  $\alpha(\cdot)$  denote the Shapley value function on the set of DFs. In order to state the revised Shapley axioms for  $\alpha(\cdot)$ , it is necessary to first define two concepts, the first of which occurs in Shapley (1953), while the second essentially appears in Shapley (1953).

(a) If  $\pi: N \rightarrow N$  is a permutation of  $N$ ; then, the  $\pi\phi$  is defined by

$$\pi\phi(\pi X) = \phi(X) \text{ for all } X. \quad (5.1)$$

(b) Given any two DFs  $\phi_1$  and  $\phi_2$  with constants  $c_1$  and  $c_2$ ,

then,  $c_1\phi_1 + c_2\phi_2$  is defined by

$$(c_1\phi_1 + c_2\phi_2)(X) = c_1\phi_1(X) + c_2\phi_2(X) \text{ for all } X.$$

The revised Shapley axioms are;

S1': if  $\phi$  is monotone; then, for any carrier  $T = (t_1, \dots, t_n)$  of  $\phi$ ,

$$\sum_{\{i: t_i=1\}} \alpha_i(\phi) = \phi(T) = \phi(E_N) = 1.$$

S2': If  $\phi$  is monotone; then, for any permutation  $\pi$  and  $i \in N$ ,

$$\alpha_{\pi i}(\pi\phi) = \alpha_i(\phi).$$

S3': For any MDFs  $\phi_1, \dots, \phi_r$  and constants  $c_1, \dots, c_r$  such that

$$\sum_{j=1}^r c_j \phi_j \text{ is diadic, } \alpha\left(\sum_{j=1}^r c_j \phi_j\right) = \sum_{j=1}^r c_j \alpha(\phi_j).$$

These axioms are sufficient to determine a value function  $\alpha(\bullet)$  uniquely for any diadic function. Before proving this we need the following lemma.

Lemma 5.1

For any  $Y = (y_1, \dots, y_n)$ , let us consider a symmetric MDF  $\phi_Y$  as in (4.1), then, for  $\alpha(\bullet)$  satisfying S1' and S2', it is true that

$$\begin{aligned} \alpha_i(\phi_Y) &= 1/y \text{ for } i \text{ such that } y_i = 1, \\ &= 0 \text{ for } i \text{ such that } y_i = 0. \end{aligned}$$

Proof

Consider the minimal carrier  $Y$  of  $\phi_Y$ , and let  $Z = Y \cup E_j$ , with  $E_j$  such that  $y_j = 0$  (i.e., such that the  $j$ -th coordinate is inconsequential).

Then, by S1',

$$\sum_{\{i: z_i=1\}} \alpha_i(\phi_Z) = \sum_{\{i: y_i=1\}} \alpha_i(\phi_Y),$$

so that

$$\alpha_j(\phi_Y) = 0. \tag{5.2}$$

Next, let  $i$  and  $k$  be such that  $y_i = y_k = 1$  (i.e., such that the  $i$ -th and  $k$ -th coordinates are not inconsequential). Also, let  $\pi$  be any permutation such that  $\pi Y = Y$  and  $\pi i = k$ ; then, we have  $\alpha_i(\phi_Y) = \alpha_{\pi i}(\pi \phi_Y)$

$= \alpha_k(\phi_Y)$ , where the first equality follows from S2', and the second from the assumed. It follows, in view of S1', that

$$\alpha_i(\phi_Y) = 1/Y. \quad (5.3)$$

Hence, from (5.2) and (5.3), the assertion is true. Q.E.D.

### Theorem 5.1

For any diadic function  $\phi$  defined on  $\{0,1\}^n$ , there is a unique value function  $\alpha(\phi)$  satisfying S1' - S3'. It is given by the Shapley value function

$$\alpha_i(\phi) = \sum_{\{Z: z_i=1\}} d_n(Z)(\phi(Z) - \phi(Z - E_i)),$$

where  $d_n(Z) = \frac{(z-1)!(n-z)!}{n!}$ .

### Proof

Since  $E_N = (1,1, \dots, 1)$  is a carrier of any DF  $\phi$  defined on  $\{0,1\}^n$ , we know, from lemma 4.1, that, for all  $n$  dimensional diadic vectors  $Y$ ,

$$\phi = \sum_Y C_Y \phi_Y \text{ with } C_Y(\phi) = \sum_{Z \subseteq Y} (-1)^{Y-Z} \phi(Z),$$

where  $C_Y(\phi)$  are as defined in lemma 4.1.

Hence, if there is a value function  $\alpha(\cdot)$  satisfying S1' - S3', then,

$$\begin{aligned} \alpha_i(\phi) &= \sum_Y C_Y(\phi) \alpha_i(\phi_Y) \\ &= \sum_{\{Y: Y_i=1\}} C_Y(\phi) (1/Y) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\{Y: Y_i=1\}} (1/Y) \left( \sum (-1)^{Y-Z} \phi(Z) \right) \\
&= \sum_Z \left( \sum_{(Z \cup E_i) \subseteq Y} (-1)^{Y-Z} (1/Y) \right) \phi(Z). \tag{5.4}
\end{aligned}$$

where the first equality follows from S3', the second one from lemma 5.1, and the third one from definition of  $C_Y$ . Let us write

$$d_i(Z) = \sum_{(Z \cup E_i) \subseteq Y} (-1)^{Y-Z} (1/Y). \tag{5.5}$$

Then, it is clear that, if  $Z' \cap E_i = E_0$  and  $Z = Z' \cup E_i$ , and  $d_i(Z') = -d_i(Z)$ , where the last assertion follows from the fact that all the terms on the right side of (5.5) will be the same for  $Z$  and  $Z'$ , except  $z = z' + 1$ , so that there is "a change of sign throughout" (Owen (1982)). This means we will have, from (5.4) and (5.5),

$$\alpha_i(\phi) = \sum_{\{Z: z_i=1\}} d_i(Z) (\phi(Z) - \phi(Z - E_i)). \tag{5.6}$$

Now, following Owen (1982), it is clear, with  $z_i = 1$ , that there are exactly  $\binom{n-z}{y-z}$  vectors  $Y$  with cardinality  $y$  such that  $Z \subseteq Y$ . Thus, we have, for  $i$  such that  $z_i = 1$ ,

$$\begin{aligned}
d_i(Z) &= \sum_{y=z}^n (-1)^{Y-Z} \binom{n-z}{y-z} (1/Y) \\
&= \sum_{y=z}^n (-1)^{Y-Z} \binom{n-z}{y-z} \int_0^1 x^{Y-1} dx \\
&= \int_0^1 \sum_{y=z}^n (-1)^{Y-Z} \binom{n-z}{y-z} x^{Y-1} dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 x^{z-1} \sum_{y=z}^n (-1)^{y-z} \binom{n-z}{y-z} x^{y-z} dx \\
&= \int_0^1 x^{z-1} (1-x)^{n-z} dx \\
&= \frac{(z-1)!(n-z)!}{n!}, \tag{5.7}
\end{aligned}$$

where the second last equality follows from the fact that

$\sum_{y=z}^n (-1)^{y-z} \binom{n-z}{n-y} x^{y-z} = (1-x)^{n-z}$ , and the last equality follows by the properties of the gamma function. Also, we note that  $d_i(Z)$  do not depend on  $i$ , it can be written as  $d_n(Z)$ . So that, from (5.6),

$$\alpha_i(\phi) = \sum_{\{Z: z_i=1\}} \frac{(z-1)!(n-z)!}{n!} (\phi(Z) - \phi(Z - E_i)). \tag{5.8}$$

It remains to show that  $\alpha(\phi)$  in (5.6) or (5.8) satisfies  $S1' - S3'$ .

To this end, one notes that  $S3'$  is evident, so that only  $S1'$  and  $S2'$  need to be addressed in detail. We know, from lemma 4.1, that, for any DF, in particular an MDF  $\phi$ ,

$$\phi = \sum_{Y \subseteq T} C_Y(\phi) \phi_Y, \text{ where } T \text{ is a carrier of } \phi.$$

Hence,

$$\begin{aligned}
\phi(T) &= \sum_{Y \subseteq T} C_Y(\phi) \phi_Y(T) \\
&= \sum_{Y \subseteq T} C_Y(\phi) \text{ by (4.8)}. \tag{5.9}
\end{aligned}$$

Also, since  $\alpha(\phi)$  in (5.6) satisfies  $S3'$ ,

$$\alpha_i(\phi) = \sum_{Y \subseteq T} C_Y(\phi) \sum_{\{i: t_i=1\}} \alpha_i(\phi_Y),$$

so that

$$\begin{aligned} \sum_{\{i: t_i=1\}} \alpha_i(\phi) &= \sum_{Y \subseteq T} C_Y(\phi) \sum_{\{i: t_i=1\}} \alpha_i(\phi_Y) \\ &= \sum_{Y \subseteq T} C_Y(\phi), \end{aligned} \quad (5.10)$$

by lemma 5.1, and since all  $Y$ 's are dominated by  $T$ . However, for  $\alpha(\bullet)$  in (5.6), it is clear that, for any carrier  $(t_1^*, \dots, t_n^*)$ ,

$$\sum_{\{i: t_i^*=1\}} \alpha_i(\phi) = \sum_{\{i: t_i=1\}} \alpha_i(\phi)$$

since, if there is a  $k$  such that  $t_k^* = 0$  and  $t_k = 1$ , then, the  $k$ -th coordinate is inconsequential in view of theorem 2.1, so that  $(\phi(Z) - \phi(Z - E_k))$  will be zero. Hence, it is clear from (5.9) and (5.10) that, for any carrier  $T$ ,

$$\sum_{\{i: t_i=1\}} \alpha_i(\phi) = \phi(T) = \phi(E_N) = 1$$

since  $\phi$  is a MDF, where the middle equality is from section 2.10.

Finally, let us show that  $\alpha(\bullet)$  in (5.6) satisfies  $S2'$ :

From (5.6), we can obtain that

$$\alpha_i(\pi\phi) = \sum_{\{Z: z_i=1\}} d_i(Z)(\pi\phi(Z) - \pi\phi(Z - E_i)),$$

so that,

$$\begin{aligned} \alpha_{\pi i}(\pi\phi) &= \sum_{\{\pi Z: z_{\pi i}=1\}} d_{\pi i}(\pi Z)(\pi\phi(\pi Z) - \pi\phi(\pi Z - \pi E_i)) \\ &= \sum_{\{Z: z_i=1\}} d_i(Z)(\phi(Z) - \phi(Z - E_i)) \\ &= \alpha_i(\phi), \end{aligned}$$

where the next-to last equality follows from the following three facts;

- (i) since  $|\pi Z| = |Z|$ , from (5.7), we have  $d_{\pi i}(\pi Z) = d_i(Z)$ ,
- (ii)  $\pi\phi(\pi Z) = \phi(Z)$  from (a) in (5.1),
- (iii)  $\pi i$ -th coordinate of  $\pi Z$  is the same coordinate as  $i$ -th coordinate of  $Z$ .

Thus,  $S2'$  is satisfied.

Since  $\alpha(\cdot)$  satisfies  $S1'$ ,  $S2'$ , and  $S3'$ , and has been derived from  $S1'$ ,  $S2'$ , and  $S3'$ , it is indeed the unique value function satisfying  $S1'$ ,  $S2'$ , and  $S3'$ . Q.E.D.

We note that, if  $\phi$  is monotone, then, the terms  $(\phi(Z) - \phi(Z - E_i))$  will always equal 0 or 1, taking the value 1 when  $Z$  is a critical vector corresponding to  $i$ ; hence, in the monotone case, one has

$$\alpha_i(\phi) = \sum_{\{Z_i\}} \frac{(z-1)!(n-z)!}{n!}, \quad (5.11)$$

where the summation is taken over all critical vectors  $Z_i$  corresponding to  $i = 1, \dots, n$ .

### Theorem 5.2

For any real valued function  $\phi^*$  defined on  $\{0,1\}^N$ , there is a unique value function  $\alpha^*(\phi)$  satisfying  $S1' - S3'$ , namely the Shapley value function:

$$\alpha_i^*(\phi^*) = \sum_{\{Z: z_i=1\}} d_n(Z) (\phi^*(Z) - \phi^*(Z - E_i)). \quad (5.12)$$

Proof

Since  $E_N$  is a carrier of any real valued function  $\phi^*$  defined on  $\{0,1\}^N$ , we can infer from lemma 4.1 that  $\phi^*$  has the following unique representation in terms of the  $2^n$   $\phi_Y$  corresponding to the  $2^n$  possible  $Y$ :

$$\phi^* = \sum_Y C_Y(\phi^*) \phi_Y, \text{ where } C_Y(\phi^*) = \sum_{Z \subseteq Y} (-1)^{Y-Z} \phi^*(Z).$$

Hence, the assertion can be verified using arguments similar to those in theorem 5.1.

We note that, for the value function  $\alpha(\cdot)$  and  $\alpha^*(\cdot)$ ,

- (i) if  $\phi$  (resp.,  $\phi^*$ ) is monotone; then,  $\sum_{i=1}^n \alpha_i(\phi) = \phi(E_N) = 1$   
 (resp.,  $\sum_{i=1}^n \alpha_i^*(\phi^*) = \phi^*(E_N)$ ).

- (ii) if  $\phi$  is not monotone, then, the above assertion (i) is not necessarily true, and, indeed,  $\alpha_i^*(\phi)$  or  $\alpha_i(\phi^*)$  may be negative for some  $i \in \{1, \dots, n\}$ .

As pointed out in Owen (1975), there is a relationship between the multilinear extension  $g_\phi$  and the Shapley value  $\alpha(\phi)$  of a MDF  $\phi$ , as follows; if  $g_i(X)$  is the  $i$ -th partial derivative of  $g_\phi(X)$ , then,

$$\int_0^1 g_i(t, \dots, t) dt = \alpha_i(\phi). \quad (5.13)$$

Thus, we can obtain the Shapley value from the multilinear extension, by integrating the partial derivative of  $g_\phi(X)$  on the main diagonal  $x_1 = x_2 = \dots = x_n$  of the unit cube  $[0,1]^N$ . Also, the proof of (5.13) does

not rely on the monotonicity of  $\phi$ , so that (5.13) is in fact valid for any DF  $\phi$ .

The following example illustrate the Shapley value of a DF  $\phi$ , and the relationship between multilinear extension and the Shapley value.

Example 5.1

Consider a DF  $\phi$  such that

$$\begin{aligned}\phi(X) &= 1 \text{ if } X = (0,0) \text{ or } (0,1), \\ &= 0 \text{ if otherwise,}\end{aligned}$$

then, from (4.6),

$$g_{\phi}(X) = (1-x_1)(1-x_2) + (1-x_1)x_2,$$

and

$$\begin{aligned}g_1(X) &= - (1-x_2) - x_2 = -1, \\ g_2(X) &= - (1-x_1) + (1-x_1) = 0.\end{aligned}$$

Also, from (5.8),

$$\begin{aligned}\alpha_1(\phi) &= -1, \\ \alpha_2(\phi) &= 0,\end{aligned}$$

and

$$\begin{aligned}\int_0^1 g_1(t,t)dt &= \alpha_1(\phi), \\ \int_0^1 g_2(t,t)dt &= \alpha_2(\phi),\end{aligned}$$

so that (5.13) is verified.

## 5.3. The Banzhaf-Coleman Index and Birnbaum Index

Banzhaf (1965) introduced an importance index for a simple game, which is essentially the same as that given by Coleman (1971). Also, Birnbaum (1969) introduced an importance index for n-component systems, which turns out to coincide with that introduced by Banzhaf and Coleman, if we consider the corresponding characteristic and structure functions as MDFs. Indeed, we can extend these indices to the class of all DFs  $\phi$ , since, like the Shapley value, they are based on the critical vectors of  $\phi$ .

We now introduce these indices in our DF notation. Letting  $\theta_i(\phi)$  be the number of critical vectors corresponding to the coordinate  $i$  for a non-constant DF, the Banzhaf-Coleman (or Birnbaum "structure") index is defined by

$$\beta_i(\phi) = \frac{\theta_i(\phi)}{2^{n-1}}, \quad (5.14)$$

where the coefficient  $1/(2^{n-1})$  may be thought of as the (uniform) probability of the critical vectors  $(1_i, X_{i\setminus i})$ ; moreover, if  $\phi$  is a MDF; then,  $\beta_i(\phi)$  is the mathematical expectation of the quantity  $\phi(1_i, X_{i\setminus i}) - \phi(0_i, X_{i\setminus i})$  under the uniform distribution over  $X_{i\setminus i}$ .

In general,  $\sum_{i=1}^n \beta_i(\phi) \neq 1$  since the number of critical vectors is not necessarily  $2^{n-1}$ . In view of that, Owen (1978) introduced the normalized Banzhaf-Coleman index, which, in DF notation, is given by

$$\beta_i^*(\phi) = \frac{\theta_i(\phi)}{\sum_{i=1}^n \theta_i(\phi)} \quad (5.15)$$

Further, Birnbaum (1969) introduced other importance indices for n-component systems, under the assumption that random variables  $X_1, \dots, X_n$ , as the states of components 1,  $\dots$ , n, respectively, have binary distributions such that

$$P(X_i = 1) = p_i, \quad P(X_i = 0) = 1 - p_i, \quad \text{for } i = 1, \dots, n. \quad (5.16)$$

If  $X_1, \dots, X_n$  are independent; then, the value of  $P = (p_1, \dots, p_n)$  determine the probability that  $\phi(X)$  has the value 1, i.e.,

$$P(\phi(X)=1|P) = E(\phi(X)|P) \equiv h_\phi(P).$$

$h_\phi(P)$  is called the reliability function of  $\phi$ .

In DF terms, the Birnbaum reliability index  $\gamma_i(\phi)$  of coordinate i is defined by

$$\gamma_i(\phi) = \frac{\partial h_\phi(P)}{\partial p_i}. \quad (5.17)$$

On the other hand, in section 4.4 we have seen that the multilinear extension of  $\phi$ ,  $g_\phi(X)$ , can be thought of as the mathematical expectation of the DF  $\phi$  if coordinate  $i = 1, \dots, n$  has probability  $x_i$  of equaling 1, with coordinates independent. Thus,

$$g_\phi(X) |_{x_i=p_i; i=1, \dots, n} = h_\phi(P). \quad (5.18)$$

Also, from the fact that  $\phi(X) = x_i \phi(1_i, X_{i\setminus i}) + (1-x_i) \phi(0_i, X_{i\setminus i})$ ,

$$\begin{aligned}
h_\phi(P) &= E(\phi(X)/P) \\
&= E(x_i(1_i, X_{i\setminus i})/P) + E((1-x_i)(0_i, X_{i\setminus i})/P) \\
&= p_i h_\phi(1_i, P_{i\setminus i}) + (1-p_i) h_\phi(0_i, P_{i\setminus i}),
\end{aligned}$$

so that

$$\frac{\partial h_\phi(P)}{\partial P_i} = h_\phi(1_i, P_{i\setminus i}) - h_\phi(0_i, P_{i\setminus i});$$

hence,

$$\begin{aligned}
\gamma_i(\phi) &= h_\phi(1_i, P_{i\setminus i}) - h_\phi(0_i, P_{i\setminus i}) \\
&= E(\phi(1_i, X_{i\setminus i}) - \phi(0_i, X_{i\setminus i})). \tag{5.19}
\end{aligned}$$

If  $\phi$  is monotone, (5.19) reduces to

$$P(\phi(1_i, X_{i\setminus i}) - \phi(0_i, X_{i\setminus i}) = 1). \tag{5.20}$$

It is further true that

- (i) in view of (5.20),  $\gamma_i(\phi)$  is the probability that the MDF  $\phi$  equals 1 when  $x_i = 1$ , and equals 0 when  $x_i = 0$ .
- (ii) in view of (5.17), if  $p_j = 1/2$  for all  $j = 1, \dots, i-1, i+1, \dots, n$ , then,  $\gamma_i(\phi) = \beta_i(\phi)$ .
- (iii) in view of (ii), if  $p_i = 1/2$  for all  $i = 1, \dots, n$ ; then, the Birnbaum reliability index and Banzhaf-Coleman (or Birnbaum structure) index coincide; i.e.:

$$\beta_i(\phi) = \frac{\partial h_\phi(P)}{\partial p_i} \Big|_{p_1=p_2=\dots=p_n=1/2} \quad (5.21)$$

Example 5.2

Consider a MDF  $\phi$  such that

$$\begin{aligned} \phi(X) &= 1 \text{ if } X = (1,0,1), (1,1,0), (0,1,1) \text{ or } (1,1,1), \\ &= 0 \text{ if otherwise,} \end{aligned}$$

then,

$$g_\phi(X) = x_1(1-x_2)x_3 + x_1x_2(1-x_3) + (1-x_1)x_2x_3 + x_1x_2x_3,$$

and

$$h_\phi(P) = p_1(1-p_2)p_3 + p_1p_2(1-p_3) + (1-p_1)p_2p_3 + p_1p_2p_3.$$

Also, we can see that  $\{(1,0,1), (1,1,0)\}$ ,  $\{(1,1,0), (0,1,1)\}$ , and  $\{(1,0,1), (0,1,1)\}$  are critical vectors corresponding to 1, 2, and 3, respectively. Hence,

(i) from (5.14), the Banzhaf-Coleman index of  $\phi$  is given by

$$\beta_1(\phi) = \beta_2(\phi) = \beta_3(\phi) = 1/2,$$

(ii) from (5.15) the normalized Banzhaf-Coleman index is given by

$$\beta_1^* = \beta_2^* = \beta_3^* = 1/3,$$

(iii) from (5.17) the birnbaum reliability index is given by

$$\begin{aligned} r_1(\phi) &= p_2 + p_3 - 2p_2p_3, \\ r_2(\phi) &= p_1 + p_3 - 2p_1p_3, \text{ and} \\ r_3(\phi) &= p_1 + p_2 - 2p_1p_2. \end{aligned}$$

#### 5.4. The Barlow-Proschan Index

##### 5.4.1. Basic concepts

We have seen several methods to measure the importance of coordinates. All of these methods were in a sense "static", in that they did not take time into account. However, Barlow and Proschan (1975b) took a time-dependent approach in describing importance indices of n-component systems. They assumed that components fail sequentially in time, so that, given a specific assignment of failure times to components, plus the structure function of the system, system failure time is determined, and equals failure time of a specific component.

Expanding on this ideas, and assuming monotonicity, Barlow and Proschan (1975b) defined the reliability importance of a component as the probability that its failure time coincides with that of the system, i.e., that the component is "critical". This notion is related to, but does not coincides with, property (i) of the Birnbaum reliability index.

We now present a simple derivation of the value of the Barlow-Proschan index, assuming independence (but not identical distribution) of system components in DF terms.

Let coordinate  $i$  have life distribution  $F_i(t)$ , and let  $X_i(t) = 1$  ( 0 ) as long (soon) as coordinate  $i$  alive (dead).

Then,

$$\begin{aligned}
E(X_i(t)) &= P(X_i(t) = 1) \\
&= 1 - F_i(t) \\
&= \bar{F}_i(t), \text{ say.}
\end{aligned}$$

Now, with  $\bar{F}(t)$  denoting the vector  $(\bar{F}_1(t), \dots, \bar{F}_n(t))$ , define the function

$$\begin{aligned}
h(\bar{F}(t)) &= P(\phi(X(t)) = 1) \\
&= E(\phi(X(t))),
\end{aligned}$$

which is like the function  $h_\phi(P)$  of section 5.3, except for the explicit incorporation of time. Then, the probability that coordinate  $i$  causes the system to fail, given that  $i$  fails at time  $t$ , is

$$\begin{aligned}
P(\phi(1_i, X_{i^c}(t)) - \phi(0_i, X_{i^c}(t)) = 1) \\
= h(1_i, \bar{F}_{i^c}(t)) - h(0_i, \bar{F}_{i^c}(t)),
\end{aligned}$$

so that the probability that  $i$  causes the system to fail, i.e., the Barlow-Proschan index of coordinate  $i$  is

$$P_i(h) = \int_0^\infty (h(1_i, \bar{F}_{i^c}(t)) - h(0_i, \bar{F}_{i^c}(t))) dF_i(t). \quad (5.22)$$

We note that, in the game context, the Barlow-Proschan index has the interpretation of a Shapley-type value that measures the importance of a player by the probability that, in departing, he will render some coalition ineffective.

### 5.4.2. Proportional hazard cases

It is difficult to compute  $P_i(h)$  for arbitrary failure distributions. However, if, as in Barlow and Proschan (1975b), we assume proportional hazards, i.e.,

$$\bar{F}_i(t) = \exp^{-\lambda_i R(t)} = \exp^{-\lambda_i \int_0^t r(u) du} \quad \text{for } i = 1, \dots, n,$$

where  $R(t)$  is common to all coordinates; then, with a change of variable,  $\bar{F}_i(t)$  may be set equal to  $\exp^{-\lambda_i t}$ , so that

$$P_i(h) = \int_0^1 (h(p^{\lambda_i}, \dots, p^{\lambda_{i-1}}, 1, p^{\lambda_{i+1}}, \dots, p^{\lambda_n}) - h(p^{\lambda_1}, \dots, p^{\lambda_{i-1}}, 0, p^{\lambda_{i+1}}, \dots, p^{\lambda_n})) \lambda_i p^{\lambda_i - 1} dp. \quad (5.23)$$

#### Example 5.3

Consider an MDF  $\phi$  such that

$$\begin{aligned} \phi(X) &= 1 \text{ if } X \supseteq (1, 1, 1, 0) \text{ or } X \supseteq (0, 1, 0, 1), \\ &= 0 \text{ if otherwise,} \end{aligned}$$

then, from (4.17),  $\phi(X)$  can be expressed equivalently as follows:

$$\begin{aligned} \phi(X) &= x_1 x_2 x_3 (1 - x_4) + (1 - x_1) x_2 (1 - x_3) x_4 \\ &\quad + (1 - x_1) x_2 x_3 x_4 + x_1 x_2 x_3 x_4. \end{aligned}$$

Let us assume that  $F_i(t) = \exp^{-it}$ ; then, by putting  $p = \exp^{-t}$ , we obtain, for  $i = 2$ ,

$$\begin{aligned} h(1_2, \bar{F}_{2\vee}(t)) &= E(\phi(1_2, X_{2\vee}(t))) \\ &= 2p^4 - p^8, \end{aligned}$$

$$h(0_2, \bar{F}_{2\vee}(t)) = 0.$$

Hence, with  $dF_i(t) = d(1 - \bar{F}_i(t))$ ,

$$P_2(h) = 7/15 \text{ from (5.23).}$$

Similarly, we can get indices for the other coordinates such that

$$P_1(h) = 1/15, \quad P_3(h) = 3/15, \quad \text{and} \quad P_4(h) = 4/15.$$

#### 5.4.3. Relationship between the Barlow-Proschan index and the Shapley value

We have seen that the Barlow-Proschan index and the Shapley value are based on the critical vectors; hence, it may be possible to find a relationship between these indices. The following lemma, suggested by theorem 4.1 of Barlow and Proschan (1975b) in the system context, address this possibility.

#### Lemma 5.3

When  $F_i(t) = F(t)$  for all  $i = 1, \dots, n$ , the Barlow-Proschan index is the Shapley value.

#### Proof

By the transformation  $\bar{F}_i(t) = p_i$  for all  $i = 1, \dots, n$ ,

$$P_i(h) = \int_0^1 (h(1_i, p_{i\vee}) - h(0_i, p_{i\vee})) d(1 - p_i). \quad (5.24)$$

Also, it can be seen from (4.17) that

$$\begin{aligned}
 h(p_1, \dots, p_n) &= E(\phi(X)/P) \\
 &= \sum_Y \prod_{i=1}^n p_i^{y_i} (1-p_i)^{1-y_i} \phi(Y),
 \end{aligned}$$

so that

$$h(p, \dots, p) = \sum_Y p^{Y-1} (1-p)^{n-Y} \phi(Y),$$

and

$$h(1_i, P_{i'}) = \sum_Y p^{Y-i} (1-p)^{n-Y-1} \phi(1_i, Y_{i'}),$$

$$h(0_i, P_{i'}) = \sum_Y p^{Y-1} (1-p)^{n-Y-1} \phi(0_i, Y_{i'}).$$

Hence, (5.24) reduces to

$$P_i(h) = \int_0^1 \sum_Y p^{Y-1} (1-p)^{n-Y} (\phi(1_i, Y_{i'}) - \phi(0_i, Y_{i'})) d(1-p)$$

$$= \sum_Y (\phi(1_i, Y_{i'}) - \phi(0_i, Y_{i'})) \int_0^1 p^{Y-1} (1-p)^{n-Y} d(1-p)$$

$$= \sum_Y (\phi(1_i, Y_{i'}) - \phi(0_i, Y_{i'})) \frac{(Y-1)!(n-Y)!}{n!}$$

$$= \sum_{\{Z: z_i=1\}} \frac{(Y-1)!(n-Y)!}{n!} (\phi(Y) - \phi(Y - E_i)),$$

the Shapley value function. Q.E.D.

## 5.5. The Deegan-Packel Index

### 5.5.1. Basic concepts

Deegan and Packel (1978) proposed an index which, in DF terms, is based on generating vectors (GVs), as opposed to the critical vectors underlying the previous indices. Since GV's can be considered only for MDFs, this index is valid only for MDFs.

Let  $M(\phi) = \{Z_i\}_{i=1}^t$  be the set of GV's of a MDF  $\phi$  which is not a constant function. For any  $i \in \{1, \dots, n\}$ , define

$$M_i(\phi) = \{Z_j : z_{ji}=1\},$$

where  $z_{ji}$  denotes the value of the  $i$ -th coordinate of the  $j$ -th GV  $Z_j$ , so that  $M_i(\phi)$  is the set of GV's whose  $i$ -th coordinate is 1.

Then, the Deegan-Packel index  $\delta(\cdot)$  is defined by

$$\delta_i(\phi) = \frac{1}{t} \sum_{Z_j \in M_i} \frac{1}{z_j}, \quad (5.25)$$

where  $t$  is the number of GV's of  $\phi$ .

The Deegan-Packel index has the following probabilistic interpretation: It is (for non-inconsequential coordinates) the probability of being selected, when

- (i) GV is selected at random from the set of all GV's whose  $i$ -th coordinate is 1, followed by

(ii) a random selection from among the non-zero coordinate of the selected GV.

For the uniqueness of this index, Deegan and Packel suggested 4 axioms, and proved that  $\delta(\bullet)$  is the unique value function defined over the set of MDFs satisfying these 4 axioms. The following example illustrate this index.

Example 5.4

Consider again MDF  $\phi$  in example 5.3 such that

$$\begin{aligned} \phi(X) &= 1 \text{ if } X \supseteq (1,1,1,0) \text{ or } X \subseteq (0,1,0,1), \\ &= 0 \text{ if otherwise.} \end{aligned}$$

It is readily checked that there are two GVs  $Z_1 = (1,1,1,0)$  and  $Z_2 = (0,1,0,1)$ , and  $M_1 = \{(1,1,1,0)\}$ ,  $M_2 = \{(1,1,1,0), (0,1,0,1)\}$ ,  $M_3 = \{(1,1,1,0)\}$ , and  $M_4 = \{(0,1,0,1)\}$ , hence, from (5.25), the following are obtained;

$$\delta_1(\phi) = 1/6, \delta_2(\phi) = 5/12, \delta_3(\phi) = 1/6, \text{ and } \delta_4(\phi) = 1/4.$$

5.5.2. Extension

As we have seen in the previous sub-section, there is a probabilistic interpretation for the Deegan-Packel index, involving a uniformity assumption. However, if we consider the  $X_i$  as independent binomial random variables such that

$$P(X_i = 1) = p_i, P(X_i = 0) = 1 - p_i \text{ for } i = 1, \dots, n,$$

then, that uniformity is no longer in effect, and the probability of forming a particular GV  $Z_j$  is

$$P_{Z_j} = \prod_{i=1}^n p_i^{z_{ji}} (1-p_i)^{1-z_{ji}}, \quad (5.26)$$

and the Deegan-Packel index becomes

$$\delta_i^*(\phi) = \sum_{Z_j} \frac{P_{Z_j}}{\sum_{j=1}^n P_{Z_j}} \frac{1}{z_j}, \quad (5.27)$$

The probabilities  $p_i$  can be made to reflect a dynamic aspect if we think of the coordinates arriving independently according to  $F_i(t)$  and staying independently according to residence distributions  $G_i(t)$ , in which case a time-dependent version  $\delta_i^*(\phi|t)$  of  $\delta_i^*(\phi)$  simply replaces  $p_i$  by  $P(U_i \leq t, U_i + V_i > t)$ , where  $U_i$  and  $V_i$  are random variables distributed independently, respectively according to  $F_i$  and  $G_i$ .

## 6. REFERENCES

- Banzhaf, J. F. 1965. Weighted voting doesn't work. Rutgers Law Review 19: 317-343.
- Barlow, R. E. and Proschan, F. 1965. Mathematical Theory of Reliability. Wiley, New York.
- Barlow, R. E. and Proschan, F. 1975a. Statistical Theory of Reliability and life testing. Holt, Rinehart and Winston, Inc. New York.
- Barlow, R. E. and Proschan, F. 1975b. Importance of system components and fault tree events. Stochastic Process and Their Application 3: 153-173.
- Birnbaum, Z. W. 1969. "On the importance of different components in a multi-component system" in multivariate analysis - II. Krishnaiah, P. R. Academic Press, New York.
- Block, H. W. and Savits, T. H. 1982. A decomposition for multi-state system. Journal of Applied Probability 19: 391-402.
- Bolger, E. M. 1980. A class of power indices for voting games. International Journal of Game Theory 9: 217-232.
- Bolger, E. M. 1982. Characterization of the Banzhaf and Shapley values assuming limited linearity. Int. Journal of Game Theory 11: 1-12.
- Charnes, A. and Sorensen, S. 1974. Constrained n-person games. Int. Journal of Game Theory 3: 141-158.
- Coleman, J. S. 1971. Control of collectivities and power of a collectivity to act. Pp 269-300 in B.Lieberman, ed. Social Choice. Gordon and Breach, New York.
- Deegan, J. and Packel, E. W. 1978. A new index of power for simple n-person games. Int. Journal of Game Theory 7: 113-123.
- Dubey, P. 1975. On the uniqueness of the Shapley value. Int. Journal of Game Theory 4: 131-139.
- Dubey, P. and Shapley, L. S. 1979. Mathematical properties of the Banzhaf power index. Mathematics of Operation Research 4: 99-131.
- Gambarelli, G. 1983. Common behavior of power indices. Int. Journal of Game Theory 12: 237-244.

- Kikuta, K. 1976. On the contribution of a player to a game. *Int. Journal of Game Theory* 5: 199-208.
- Lambert, H. E. 1975. Measures of importance of events and cut sets in fault trees. Pp. 77-100 in R. E. Barlow, J. B. Fussell, and N. D. Singpurwalla, eds. *Reliability and Fault Tree Analysis*. SIAM, Philadelphia. Pa.
- Leminaire, J. 1973. A new value for games without transferable utilities. *Int. Journal of Game Theory* 2: 205-213.
- Merterns, J. F. 1980. A note on the characteristic function of super-games. *Int. Journal of Game Theory* 9: 189-190.
- Myerson, R. B. 1977. Values of games in partition function form. *Int. Journal of Game Theory* 6: 23-31.
- Neumann, J. V. and Morgenstern, O. 1953. *Theory of games and economic behavior*. 3rd ed. Princeton University Press, Princeton, N. J.
- Owen, G. 1968. A note on the Shapley value. *Management Science* 11: 731-732.
- Owen, G. 1972. Multilinear extension of games. *Management Science* 18: 64-79.
- Owen, G. 1975. Multilinear extensions and the Banzhaf value. *Naval Research Logistics Quarterly* 22: 741-758.
- Owen, G. 1978. Characterization of the Banzhaf-Coleman index. *SIAM Journal of Applied Mathematics* 35, No 2: 315-327.
- Owen, G. 1982. *Game theory*. 2nd ed. Academic Press, New York.
- Packel, E. W. 1981. Semi-simple games, enactment power and generalized indices. *Int. Journal of Game Theory* 10: 137-146.
- Peleg, B. 1981. Coalition formation in simple games with dominant player. *Int. Journal of Game Theory* 10: 11-33.
- Roth, A. E. 1977. A note on values and multilinear extensions. *Naval Research Logistics Quarterly* 24: 517-520.
- Rudeanu, S. 1974. *Boolean functions and equations*. American Elsevier Publishing Co., New York.
- Shapley, L. S. 1953. A value for n-person games. *Annals of Mathematics Study* 28: 307-317.
- Shapley, L. S. 1977. A comparison of power indices and a non-symmetric

generalization. P-5872. The Rand Corporation, Santa Monica, California.

Sorin, S. 1979. A note on the value of zero-sum sequential repeated games with incomplete information. *Int. Journal of Game Theory* 8: 217-223.

Thomson, W. and Myerson, R. B. 1980. Monotonicity and independence axioms. *Int. Journal of Game Theory* 9: 37-49.

Tijs, S. H. and Jansen, M. J. M. 1982. On the existence of values for arbitration games. *Int. Journal of Game Theory* 11: 87-104.

## 7. ACKNOWLEDGEMENTS

I would like to express my thanks and deepest appreciation to my major professor, Dr. Herbert T. David, for his constant guidance, advice, encouragement, and, above all, for his time during the course of this research. It was a privilege working with such a scholar of knowledge and insight. I can say presumably that only a few professors in the world, including my major professor, would willingly come to his office to meet a student on a below-zero-cold Iowa Winter weekend.

I wish to thank my other committee members, Dr. Sposito, Dr. Vardeman, Dr. Meeks, and Dr. Even, for their willingness to serve on my committee and various ways they contributed to my education.

I am deeply thankful to Dr. William J. Kennedy for helping me in the production of the thesis document.

I am specially thankful to Korean Army officials for permitting and supporting me to study for a long time.

I would also like to thank Mr. Criag Beam for his helpful participation in conversations, in the initial stages of this research.

I am thankful to my parents for encouraging me during this hard time. Finally, my thanks go to my wife, Jung Sim, for persevering with me during the many difficult times we passed through. As for my daughter and son, Shin Young and Jung Shin, I can only say that I will, henceforth, be a better daddy.