NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION WITH SMALL PARAMETER

by

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INTRODUCTION

In the last few decades, mathematicians have become increasingly interested in the behavior of solutions of partial differential equations in which the coefficient of the highest ordered term depends on a parameter. The study of such parameter dependent solutions seems to be important for the following reasons:

1) it may justify the construction of solutions in terms of solutions of lower ordered equations;

2) it may justify the construction of "generalized solutions" (that is, solutions with lower class properties) in terms of limits of regular solutions of higher ordered equations; and

3) it may shed some light on the general question of the nature of the dependence of the solutions on the coefficients of the equation.

Linear partial differential equations in two independent variables have been rather completely analysed. Levinson [8] has considered the boundary problem for the linear elliptic partial differential equation

$$\delta u_{xx} + \delta u_{yy} + Au_x + Bu_y + Cu = D, \tag{1.1}$$

where A, B, C, and D are functions of x and y. He has shown, under fairly general conditions, that in the region interior to the prescribed boundary the solution $u(x,y,\delta)$ approaches the solution of the first order part of the equation, as δ

goes to zero through positive values. Aronson [1] studied the first boundary value problem for the linear parabolic partial differential equation

$$\delta u_{xx} + Au_x + Bu_y + Cu = D, \tag{1.2}$$

and has obtained results analogous to those of Levinson. Blondel [3] has announced complete results for the linear hyperbolic equation

$$\delta[Au_{xx} + Bu_{xy} + Cu_{yy}] + au_x + bu_y + cu = 0, \tag{1.3}$$

where A, B, C, a, b, and c are functions of x and y.

The results for nonlinear partial differential equations are not nearly so complete. Due to the absence of a general theory concerning the behavior of solutions of nonlinear problems, we agree with Hopf [5] in the opinion that continued study of special problems is a commendable way to approach the subject. Hopf himself has studied the behavior of the solution of the initial value problem for the nonlinear parabolic equation

$$\mu u_{xx} = uu_x + u_t \tag{1.4}$$

as μ goes to zero, and has shown that in general the solution approaches a discontinuous function; but except for these discontinuities, the limit function satisfies the reduced equation

$$uu_x + u_t = 0. \tag{1.5}$$

In two later papers, Lax [6,7] considered generalizations of this equation of the form

$$(1.6) \qquad \mu u_{xx} = a(u)\, u_x + u_t$$

in his studies of hyperbolic systems of "conservation laws". Both Lax and Hopf were interested in the parameterized equation from the standpoint of finding generalized solutions to the first order equation as limits of regular solutions to the second order equations. Hopf's analysis was greatly facilitated by his observation that the equation 1.4 can be transformed into the linear heat equation

$$(1.7) \qquad \phi_t = \mu \phi_{xx},$$

about which a great deal is known.

It is the purpose of this paper to study the behavior of the solution of the Goursat problem for the nonlinear hyperbolic equation

$$(1.8) \qquad \delta u_{xy} - Auu_x - Bu_x u_y - Cu_x = 0$$

where δ is a small parameter and A, B, and C are functions of y only.

We first prove, in the next section, a special case for one of Blondel's theorems [3] in order to give the reader some feeling for the hyperbolic problem, as well as to illustrate certain differences which arise between the linear and nonlinear problems. Since Blondel's results were published without proofs, the arguments given are necessarily original, and so may be quite divergent from the methods employed by Blondel. In the third section we establish sufficient conditions that the solution of equation 1.8 which takes on prescribed values on the coordinate axes converge to the solution of the reduced equation

(1.9) $$Auu_x + Bu_x u_y + Cu_x = 0,$$

as δ goes to zero.

A THEOREM OF BLONDEL

In this section we prove the following special case of Blondel's results [3] . We consider the linear hyperbolic problem:

$$\text{I}\qquad \begin{aligned} \delta u_{xy} - Au_x - Bu_y &= 0, \\ u(x,0) &= \alpha(x), \\ u(0,y) &= \beta(y), \\ \alpha(0) &= \beta(0), \end{aligned}$$

where A and B are constants and δ is a parameter, and the associated linear first order problem:

$$\text{II}\qquad \begin{aligned} Av_x + Bv_y &= 0, \\ v(x,0) &= \alpha(x). \end{aligned}$$

Then we establish the following theorem.

Theorem.

Let

(1) R be the closed triangle

$$0 < x_1 \leq x \leq x_2,$$

$$0 \leq y \leq \frac{B}{A} x - \frac{B}{A} x_1,$$

for some positive constants x_1 and x_2, $x_1 \neq x_2$,

(2) $A > 0$,

(3) $B > 0$,

(4) $\delta < 0$,

(5) $\alpha(x)$ be of class C^2,

(6) $\beta(y)$ be of class C^1,

(7) $u(x,y,\delta)$ be the solution to problem I in R,

and

(8) $v(x,y)$ be the solution to problem II in R.

Then $u(x,y,\delta)$ converges uniformly to $v(x,y)$ for all (x,y) in R as δ goes to zero through negative values.

Proof.

As in the works of Hopf [5] and Lax [6], this proof relies heavily on the explicit solution of I. Let (x,y) be in R, and let conditions (1) through (8) be satisfied. Then $u(x,y,\delta)$ is given by

(2.1) $u(x,y,\delta) =$

$$\alpha(x)+\beta(y)-\alpha(0)+\frac{1}{\delta}\exp\left\{\frac{Bx+Ay}{\delta}\right\}\int_0^y\int_0^x [A\alpha'(s) + B\beta'(t)]\exp\left\{-\frac{Bs+At}{\delta}\right\} I_0\left\{\sqrt{\frac{4AB}{\delta^2}(x-s)(y-t)}\right\} dsdt,$$

where $I_0(x)$ is the modified Bessel function. Through the change of variables of integration from s and t to

$$\sigma = \frac{B}{\delta}(s-x)$$

and

$$\tau = \frac{A}{\delta}(t-y)$$

we may write equation 2.1 as

(2.2) $u(x,y,\delta) = \alpha(x)+\beta(y)-\alpha(0)$

$$+\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_0^{-\frac{Bx}{\delta}} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\} d\sigma d\tau$$

$$+\frac{\delta}{A}\int_0^{-\frac{Ay}{\delta}}\int_0^{-\frac{Bx}{\delta}} I_0(2\sqrt{\sigma\tau})\,\beta'(y+\frac{\delta}{A}\tau)\exp\left\{-(\sigma+\tau)\right\} d\sigma d\tau.$$

If we recall that the Laplace transform of $I_0(2\sqrt{kt})$ is e^k at $s = 1$ [4], we may write

$$(2.3)\quad \frac{\delta}{A}\int_0^{-\frac{Ay}{\delta}}\int_0^{-\frac{Bx}{\delta}} I_0(2\sqrt{\sigma\tau})\,\beta'(y+\frac{\delta}{A}\tau)\exp\left\{-(\sigma+\tau)\right\}\,d\sigma d\tau$$

$$= \frac{\delta}{A}\int_0^{-\frac{Ay}{\delta}} \beta'(y+\frac{\delta}{A}\tau)\,d\tau$$

$$- \frac{\delta}{A}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\beta'(y+\frac{\delta}{A}\tau)\exp\left\{-(\sigma+\tau)\right\}\,d\sigma d\tau$$

$$= \beta(0)-\beta(y) - \frac{\delta}{A}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\beta'(y+\frac{\delta}{A}\tau)\exp\left\{-(\sigma+\tau)\right\}\,d\sigma d\tau .$$

Hence the solution may be written as

$$(2.4)\quad u(x,y,\delta) =$$

$$\alpha(x) + \frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_0^{-\frac{Bx}{\delta}} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\sigma d\tau$$

$$- \frac{\delta}{A}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\beta'(y+\frac{\delta}{A}\tau)\exp\left\{-(\sigma+\tau)\right\}\,d\sigma d\tau .$$

If we change the order of integration in the first of the double integrals in equation 2.4, we may write

$$(2.5)\quad \frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_0^{-\frac{Bx}{\delta}} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\sigma d\tau$$

$$= \frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_0^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$- \frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Ay}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$+ \frac{\delta}{B}\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}}\int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma.$$

We again make use of our knowledge of the Laplace transform of $I_0(2\sqrt{\sigma\tau})$ to obtain

$$(2.6)\quad \frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_0^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$= \frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}} \alpha'(x+\frac{\delta}{B}\sigma)\,d\sigma = \alpha(x - \frac{A}{B}y) - \alpha(x).$$

The second double integral on the right in equation 2.5 we resolve as follows:

$$(2.7)\quad -\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Ay}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$= -\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$-\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma.$$

If we make the change of variable $\sigma=\tau$ in the first double integral on the right of equation 2.7 and then change the order of integration on that integral, we get

$$(2.8)\quad -\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Ay}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$= -\frac{\delta}{B}\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}}\int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\tau)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$-\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma.$$

We may combine the results of equations 2.6 and 2.8 with equation 2.5 to obtain

$$(2.9)\quad \frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_0^{-\frac{Bx}{\delta}} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\} d\sigma d\tau$$

$$= \alpha(x-\frac{A}{B}y) - \alpha(x)$$

$$-\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B}\sigma)\exp\left\{-(\sigma+\tau)\right\} d\tau d\sigma$$

$$+\frac{\delta}{B}\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}}\int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\left[\alpha'(x+\frac{\delta}{B}\sigma)-\alpha'(x+\frac{\delta}{B}\tau)\right]\exp\left\{-(\sigma+\tau)\right\} d\tau d\sigma,$$

and then use equation 2.9 with 2.4 to get

$$(2.10)\quad u(x,y,\delta) = (x-\frac{A}{B}y)$$

$$-\frac{\delta}{B}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\alpha'(x+\frac{\delta}{B})\exp\left\{-(\sigma+\tau)\right\} d\tau d\sigma$$

$$-\frac{\delta}{A}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,\beta'(y+\frac{\delta}{A}\tau)\exp\left\{-(\sigma+\tau)\right\} d\sigma d\tau$$

$$+\frac{\delta}{B}\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}}\int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\left[\alpha'(x+\frac{\delta}{B}\sigma)-\alpha'(x+\frac{\delta}{B}\tau)\right]\exp\left\{-(\sigma+\tau)\right\} d\tau d\sigma.$$

The first two integrals on the right in equation 2.10 may be combined to yield

$$(2.11)\quad u(x,y,\delta) = \alpha(x-\frac{A}{B}y)$$

$$-\frac{\delta}{AB}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\left[A\alpha'(x+\frac{\delta}{B}\sigma)\right.$$
$$\left.+B\beta'(y+\frac{\delta}{A}\sigma)\right]\exp\left\{-(\sigma+\tau)\right\}\ d\tau d\sigma$$

$$+\frac{\delta}{B}\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}}\int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\left[\alpha'(x+\frac{\delta}{B}\sigma)\right.$$
$$\left.-\alpha'(x+\frac{\delta}{B}\tau)\right]\exp\left\{-(\sigma+\tau)\right\}\ d\tau d\sigma.$$

Now the function $\alpha(x - \frac{A}{B}y)$ is the solution to problem II. Thus if we can show that the integrals in equation 2.11 go to zero uniformly as δ goes to zero through negative values, for (x,y) in R, we will have proved the theorem. We shall establish that each of the integrals of equation 2.11 has the limit zero as δ goes to zero, the convergence being uniform in R. Since we have assumed that $\alpha(x)$ is of class C^2, we may use the identity

$$\alpha'(x + \frac{\delta}{B}\sigma) - \alpha'(x + \frac{\delta}{B}\tau) = \frac{\delta}{B}(\sigma - \tau)\ \alpha''(z)$$

where

$$0 \leq z \leq x.$$

Let

$$M = \max\frac{|\alpha''(z)|}{B}$$

for $0 \leq z \leq x_2$. Then it follows that

$$(2.12)\quad \left| \frac{\delta}{B} \int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}} \int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\,[\alpha'(x+\frac{\delta}{B}\sigma) - \alpha'(x+\frac{\delta}{B}\tau)]\exp\left\{-(\sigma+\tau)\right\} d\tau d\sigma \right|$$

$$\leq \delta^2 M \int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}} \int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\exp\left\{-(\sigma+\tau)\right\} d\tau d\sigma .$$

Next we recall that

$$(2.13)\qquad I_0(2\sqrt{\sigma\tau}) = \sum_{k=0}^{\infty} \frac{\sigma^k\,\tau^k}{(k!)^2} .$$

If we use equation 2.13 and perform the integration term by term, we may write

$$(2.14)\quad \int_0^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\,e^{-\tau}(\sigma-\tau)\,d\tau$$

$$= \sum_{k=0}^{\infty} \frac{\sigma^{k+1}}{(k!)^2} \int_0^{-\frac{Ay}{\delta}} \tau^k e^{-\tau} d\tau - \sum_{k=0}^{\infty} \frac{\sigma^k}{(k!)^2} \int_0^{-\frac{Ay}{\delta}} \tau^{k+1} e^{-\tau} d\tau$$

$$= \sum_{k=0}^{\infty} \frac{\sigma^{k+1}}{k!}\left[1 - e^{\frac{Ay}{\delta}} \sum_{n=0}^{k} \frac{(-\frac{Ay}{\delta})^n}{n!}\right]$$

$$- \sum_{k=0}^{\infty} \frac{\sigma^k (k+1)}{k!}\left[1 - e^{\frac{Ay}{\delta}} \sum_{n=0}^{k+1} \frac{(-\frac{Ay}{\delta})^n}{n!}\right].$$

We treat the double integral on the right side of inequality 2.12 as an iterated integral and perform repeated term by term integration, utilizing the results of equation 2.14, to get

$$(2.15)\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}}\int_{0}^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\,e^{-(\sigma+\tau)}(\sigma-\tau)\;d\tau\,d\sigma$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left[1 - e^{\frac{Ay}{\delta}} \sum_{n=0}^{k} \frac{(-\frac{Ay}{\delta})^n}{n!}\right]\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}} \sigma^{k+1}e^{-\sigma}d\sigma$$

$$- \sum_{k=0}^{\infty} \frac{k+1}{k!}\left[1 - e^{\frac{Ay}{\delta}} \sum_{n=0}^{k+1} \frac{(-\frac{Ay}{\delta})^n}{n!}\right]\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}} \sigma^{k}e^{-\sigma}d\sigma$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left[1 - e^{\frac{Ay}{\delta}} \sum_{n=0}^{k} \frac{(-\frac{Ay}{\delta})^n}{n!}\right]\cdot\left[e^{\frac{Ay}{\delta}} (k+1)! \sum_{m=0}^{k+1} \frac{(-\frac{Ay}{\delta})^m}{m!} - e^{\frac{Bx}{\delta}} (k+1)! \sum_{m=0}^{k+1} \frac{(-\frac{Bx}{\delta})^m}{m!}\right]$$

$$- \sum_{k=0}^{\infty} \frac{k+1}{k!}\left[1- e^{\frac{Ay}{\delta}} \sum_{n=0}^{k+1} \frac{(-\frac{Ay}{\delta})^n}{n!}\right]\left[e^{\frac{Ay}{\delta}} k! \sum_{m=0}^{k} \frac{(-\frac{Ay}{\delta})^m}{m!} - e^{\frac{Bx}{\delta}} k! \sum_{m=0}^{k} \frac{(-\frac{Bx}{\delta})^m}{m!}\right].$$

We may simplify equation 2.15 to obtain

$$(2.16)\quad 0 \leq \int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}} \int_{0}^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\, e^{-(\sigma+\tau)} (\sigma-\tau)\, d\tau\, d\sigma$$

$$= e^{\frac{Ay}{\delta}} \left(-\frac{Ay}{\delta}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{Ay}{\delta}\right)^k}{k!} - e^{\frac{Bx}{\delta}} \left(-\frac{Bx}{\delta}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{Bx}{\delta}\right)^k}{k!}$$

$$+ \; e^{\frac{Ay+Bx}{\delta}} \sum_{k=0}^{\infty} \frac{\left(-\frac{Bx}{\delta}\right)^{k+1}}{(k+1)!} \cdot \sum_{n=0}^{k} \frac{\left(-\frac{Ay}{\delta}\right)^n}{n!}$$

$$- \; e^{\frac{Ay+Bx}{\delta}} \sum_{k=0}^{\infty} \frac{\left(-\frac{Ay}{\delta}\right)^{k+1}}{(k+1)!} \cdot \sum_{n=0}^{k} \frac{\left(-\frac{Bx}{\delta}\right)^n}{n!}$$

$$\leq \; -\frac{Ay}{\delta} - \frac{Bx}{\delta} + 2$$

$$\leq \; -\frac{2Bx_2}{\delta} + 2.$$

It follows that

$$0 \leq \lim_{\delta\to 0^-} \delta^2 \cdot M \cdot \int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}} \int_{0}^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\, e^{-(\sigma+\tau)} (\sigma-\tau)\, d\tau\, d\sigma$$

$$\leq \lim_{\delta\to 0^-} \delta^2 M \cdot \left[-\frac{2Bx_2}{\delta} + 2 \right]$$

$$= 0,$$

uniformly for every (x,y) in R. By inequality 2.12, this implies that the expression

$$\frac{\delta}{B}\int_{-\frac{Ay}{\delta}}^{-\frac{Bx}{\delta}}\int_{0}^{-\frac{Ay}{\delta}} I_0(2\sqrt{\sigma\tau})\,[\alpha'(x+\frac{\delta}{B}\,\sigma)-\alpha'(x+\frac{\delta}{B}\,\tau)]\,e^{-(\sigma+\tau)}\,d\tau d\sigma$$

goes to zero uniformly in R as δ goes to zero.

Next, let K be given by

$$K = 2\max\left[\frac{|\alpha'(x)|}{B},\ \frac{|\beta'(y)|}{A}\right]$$

for (x,y) in R. Then it is clear that

$$(2.17)\quad 0 \leq -\frac{\delta}{B}\int_{0}^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\,[A\alpha'(x+\frac{\delta}{B}\,\sigma)$$

$$+ B\beta'(x+\frac{\delta}{A}\,\sigma)]\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma$$

$$\leq -\delta K\int_{0}^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\exp\left\{-(\sigma+\tau)\right\}\,d\tau d\sigma.$$

We recall that

$$(2.18)\quad 0 < I_0(2\sqrt{\sigma\tau})$$

$$= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}[\exp(2\sqrt{\sigma\tau}\,\sin\Theta)+\exp(-2\sqrt{\sigma\tau}\,\sin\Theta)]\,d\Theta$$

$$< \frac{1}{2}\,[\exp(2\sqrt{\sigma\tau})\ +1].$$

Therefore

$$(2.19)\quad 0<\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\exp\left\{-(\sigma+\tau)\right\}\,d\sigma d\tau$$

$$<\frac{1}{2}\int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty}\exp\left\{-(\tau-\sqrt{\sigma})^2\right\}d\tau d\sigma$$

$$+\frac{1}{2}\int_0^{-\frac{Ay}{\delta}}\exp\left\{\frac{Bx}{\delta}-\sigma\right\}d\sigma$$

$$=\frac{1}{2}\int_0^{-\frac{Ay}{\delta}}\int_p^{\infty}(\theta+\sqrt{\sigma})\exp(-\theta^2)\,d\theta d\sigma$$

$$+\frac{1}{2}\exp\left(\frac{Bx}{\delta}\right)\left[1-\exp\left(\frac{Ay}{\delta}\right)\right],$$

where $p=\sqrt{-\frac{Bx}{\delta}}-\sqrt{\sigma}$.

Now,

$$(2.20)\quad 0<\int_0^{-\frac{Ay}{\delta}}\int_p^{\infty}(\theta+\sqrt{\sigma})\exp(-\theta^2)\,d\theta d\sigma$$

$$=\int_0^{-\frac{Ay}{\delta}}\exp(-p^2)\,d\sigma+\int_0^{-\frac{Ay}{\delta}}\sqrt{\sigma}\int_p^{\infty}\exp(-\theta^2)\,d\theta d\sigma$$

$$\leqslant\frac{1}{2}\int_0^{-\frac{Ay}{\delta}}\exp(-p^2)\,d\sigma+\frac{1}{2}\int_0^{-\frac{Ay}{\delta}}\frac{\sqrt{\sigma}}{p}\exp(-p^2)\,d\sigma$$

$$= \frac{1}{2}\int_0^{-\frac{Ay}{\delta}} \frac{\sqrt{-\frac{Bx}{\delta}}}{\sqrt{-\frac{Bx}{\delta}}-\sqrt{\sigma}} \exp\left\{-[\sqrt{-\frac{Bx}{\delta}}-\sqrt{\sigma}\,]^2\right\} \quad d\sigma$$

$$= \frac{1}{2}\sqrt{-\frac{Bx}{\delta}}\int_0^{-\frac{Ay}{\delta}} \frac{\sqrt{\sigma}}{[\sqrt{-\frac{Bx}{\delta}}-\sqrt{\sigma}\,]^2} \cdot \frac{\sqrt{-\frac{Bx}{\delta}}-\sqrt{\sigma}}{\sqrt{\sigma}} \exp\left\{-[\sqrt{-\frac{Bx}{\delta}}-\sqrt{\sigma}\,]^2\right\} d\sigma$$

$$\leq \frac{1}{2}\frac{\sqrt{-\frac{Bx}{\delta}}\ \sqrt{-\frac{Ay}{\delta}}}{[\sqrt{-\frac{Bx}{\delta}}-\sqrt{-\frac{Ay}{\delta}}\,]^2}\left[\exp\left\{-[\sqrt{-\frac{Bx}{\delta}}-\sqrt{-\frac{Ay}{\delta}}\,]^2\right\} -\exp\left(\frac{Bx}{\delta}\right)\right]$$

$$= \frac{1}{2}\ \frac{\sqrt{ABxy}}{[\sqrt{Bx}-\sqrt{Ay}\,]^2}\left[\exp\left\{\frac{[\sqrt{Bx}-\sqrt{Ay}\,]^2}{\delta}\right\} - \exp\left(\frac{Bx}{\delta}\right)\right].$$

Since for (x,y) in R, we have

$$Ay \leq Bx - Bx_1 < Bx,$$

we may conclude from inequality 2.20 that

$$0 \leq \lim_{\delta\to 0^-} - \delta K \int_0^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})\exp\left\{-(\sigma+\tau)\right\} d\sigma d\tau$$

$$= 0,$$

uniformly for (x,y) in R. Therefore it follows from inequality 2.17 that the expression

$$-\frac{\delta}{AB}\int_{0}^{-\frac{Ay}{\delta}}\int_{-\frac{Bx}{\delta}}^{\infty} I_0(2\sqrt{\sigma\tau})[\,A\alpha'(x+\frac{\delta}{B}\sigma) + B\beta'(y+\frac{\delta}{A}\sigma)\,]\exp\left\{-(\sigma+\tau)\right\}d\tau d\sigma$$

converges to zero uniformly in R as δ goes to zero through negative values. The proof of the theorem is complete.

A similar theorem holds in a closed triangle in the first quadrant above the line

$$Ay = Bx,$$

provided the class properties ascribed to $\alpha(x)$ and $\beta(y)$ are interchanged. In this case, the solution of I approaches the solution of the problem

$$\text{III}\qquad \begin{aligned} Av_x + Bv_y &= 0, \\ v(0,y) &= \beta(y), \end{aligned}$$

as δ goes to zero through negative values. Analogous results may clearly be obtained in the third quadrant.

It should be noted that the line

$$Ay = Bx$$

is the characteristic of the first order equation

$$Au_x + Bu_y = 0$$

which passes through the origin. Since by hypothesis

$$A > 0,$$

and

$$B > 0,$$

this line divides the first and third quadrants each into two sections. The results of Blondel [3] then assert that in any closed region contained in one of these four sections, the solution of problem I for negative δ tends uniformly to the solution of the first order equation which takes on the values prescribed by problem I on the axis adjacent to the section. Blondel further reports that under the same conditions on A, B, and δ, these results will in general not hold in quadrants two and four.

A NONLINEAR PROBLEM

We consider the Goursat problem

$$\text{IV} \qquad \begin{aligned} \delta u_{xy} - Auu_x - Bu_x u_y - Cu_x &= 0, \\ u(x,0) &= \alpha(x), \\ u(0,y) &= \beta(y), \\ \alpha(0) &= \beta(0), \end{aligned}$$

where A, B, and C are functions of y only and δ is a real-valued parameter. We restrict A, B, and C to be continuous functions and both $\alpha(x)$ and $\beta(y)$ to be of class C^1. The existence and uniqueness of the solution of IV, in a neighborhood of the origin, is assured [2]. The object of this section is to study the behavior of the solution, $u(x,y,\delta)$ of IV as δ goes to zero. Let us note that the reduced differential equation in IV, that is,

$$Auu_x + Bu_x u_y + Cu_x = 0, \tag{3.1}$$

may be factored and written as the two differential equations

$$Au + Bu_y + C = 0, \tag{3.2}$$

and

$$u_x = 0. \tag{3.3}$$

Hence there would appear to be more possibilities for the limit $u(x,y,\delta)$ as δ goes to zero than in the linear case.

In the proof of the theorem given in the second section for the linear hyperbolic problem, we used the explicit solution for fixed δ as the basis for the analysis. In Hopf's work [5] with the nonlinear parabolic problem, he relied on a linearizing transformation to facilitate his studies. Lacking both of these tools, we resort to the construction of a problem which is equivalent to IV, but which appears in the form of a nonlinear integral equation. The equivalence is established in the following lemma.

Lemma.

When $B(y) \neq 0$ and $\delta \neq 0$, problem IV is equivalent to the integral equation

V

$$u(x,y,\delta) = \exp\left\{-\int_0^y \frac{A(s)}{B(s)}\,ds\right\}\left(\alpha(x)+\int_0^y\left[-\frac{\delta A(t)}{B^2(t)}-\frac{C(t)}{B(t)}\right.\right.$$
$$\left.\left.+P(t,\delta)\exp\left\{\frac{B(t)}{\delta}\,[u(x,t,\delta)-\beta(t)]\right\}\right]\exp\left\{\int_0^t \frac{A(s)}{B(s)}\,ds\right\}dt\right)$$

where

$$P(t,\delta) = \frac{A(t)}{B(t)}\,\beta(t)+\beta'(t)+\frac{\delta A(t)}{B^2(t)}+\frac{C(t)}{B(t)}. \tag{3.4}$$

Proof of lemma.

It is clear from the statement of problem V that every function $u(x,y,\delta)$ which satisfies V is such that $u(x,0,\delta) = \alpha(x)$. Let us suppose that $u_1(x,y,\delta)$ and $u_2(x,y,\delta)$ are two functions which satisfy V, and further that

$$u_2(0,y,\delta) = \beta(y). \tag{3.5}$$

Then we define the function $w(x,y,\delta)$ by the relation

$$w(x,y,\delta) = u_1(x,y,\delta) - u_2(x,y,\delta). \tag{3.6}$$

From V, it follows that

(3.7) $w(x,y,\delta) =$

$$\exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y P(t,\delta)\exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\}\left[\exp\left\{\frac{B(t)}{\delta}[u_1(x,t,\delta) -\beta(t)]\right\} -\exp\left\{\frac{B(t)}{\delta}[u_2(x,t,\delta)-\beta(t)]\right\}\right] dt.$$

Combining equation 3.5 and 3.6, we see that

(3.8) $$w(0,y,\delta) = u_1(0,y,\delta) - \beta(y),$$

and thus we may write

(3.9) $w(0,y,\delta) =$

$$\exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y P(t,\delta)\exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\}\left[\exp\left\{\frac{B(t)}{\delta}w(0,y,\delta)\right\} -1\right] dt.$$

We define the function $v(y,\delta)$ by

(3.10) $$v(y,\delta) = w(0,y,\delta).$$

Then from equations 3.9 and 3.10, we get

(3.11) $v(y,\delta) =$

$$\exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y P(t,\delta)\exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\}\left[\exp\left\{\frac{B(t)}{\delta}v(y,\delta)\right\} -1\right] dt,$$

and

(3.12) $$v_y(y,\delta) = -\frac{A(y)}{B(y)}v(y,\delta) + P(y,\delta)\left[\exp\left\{\frac{B(y)}{\delta}v(y,\delta)\right\} -1\right].$$

Since both $u_1(x,y,\delta)$ and $u_2(x,y,\delta)$ are solutions of V, we have that

$$u_1(x,0,\delta) = \alpha(x),$$

and

$$u_2(x,0,\delta) = \alpha(x).$$

Then we obtain

$$\begin{aligned} v(0,\delta) &= w(0,0,\delta) \\ &= u_1(0,0,\delta) - u_2(0,0,\delta) \\ &= \alpha(0) - \alpha(0) \\ &= 0. \end{aligned} \tag{3.13}$$

Now the identically zero function satisfies equation 3.12 and 3.13 and since the solution of equation 3.12 which satisfies equation 3.13 is unique [2] , we must have

$$v(y,\delta) \equiv 0. \tag{3.14}$$

From equations 3.14, 3.10, and 3.8, we see that

$$u_1(0,y,\delta) - \beta(y) = 0,$$

from which we conclude that if problem V has a solution $u(x,y,\delta)$ such that

$$u(0,y,\delta) = \beta(y),$$

then every solution of V assumes the value $\beta(y)$ when $x = 0$. The fact that there is a solution $u(x,y,\delta)$ such that $u(0,y,\delta) = \beta(y)$ will be clear when we show that every solution of IV satisfies V, since the existence and uniqueness of the solution of IV is known [2] . To see that the solution of V satisfies the differential equation of IV, we solve V for

$$u(x,y,\delta)\exp\left\{\int_0^y \frac{A(s)}{B(s)}ds\right\}$$

and differentiate with respect to y to obtain

$$(3.15)\ [u_y(x,y,\delta)+\frac{A(y)}{B(y)}\ u(x,y,\delta)]\ \exp\left\{\int_0^y \frac{A(s)}{B(s)}ds\right\}$$

$$= \exp\left\{\int_0^y \frac{A(s)}{B(s)}ds\right\}\ [-\frac{\delta A(y)}{B^2(y)} - \frac{C(y)}{B(y)}$$

$$+ P(y,\delta)\exp\left\{\frac{B(y)}{\delta}[\ u(x,y,\delta)-\beta(y)\ ]\right\}\ ],$$

and hence

$$(3.16)\quad u_y(x,y,\delta)+\frac{A(y)}{B(y)}\ u(x,y,\delta)$$

$$= -\frac{\delta A(y)}{B^2(y)} - \frac{C(y)}{B(y)} + P(y,\delta)\exp\left\{\frac{B(y)}{\delta}\ [u(x,y,\delta)-\beta(y)]\right\}.$$

If we differentiate equation 3.16 with respect to x, we get

$$(3.17)\ u_{xy}(x,y,\delta) + \frac{A(y)}{B(y)}\ u_x(x,y,\delta)$$

$$= \frac{B(y)}{\delta}\ u_x(x,y,\delta)\left\{u_y(x,y,\delta)+\frac{A(y)}{B(y)}\ u(x,y,\delta)\right.$$

$$\left. + \frac{\delta A(y)}{B^2(y)} + \frac{C(y)}{B(y)}\right\}.$$

Equation 3.17 may be written in the form

$$(3.18)\quad \delta u_{xy}(x,y,\delta) = A(y)u_x(x,y,\delta)u(x,y,\delta) + B(y)u_x(x,y,\delta)u_y(x,y,\delta) + C(y)u_x(x,y,\delta),$$

and this is precisely the differential equation of IV. Therefore every solution of problem V satisfies problem IV, provided there exists one solution of problem V which assumes the value $\beta(y)$ when x is zero.

To see that the converse is true, we suppose that $u = u(x,y,\delta)$ is a solution of problem IV. We define

$$(3.19)\qquad w = Au + Bu_y + C.$$

Then w_x is given by

$$(3.20)\qquad w_x = Au_x + Bu_{xy},$$

and the differential equation in IV may be written as

$$(3.21)\qquad \frac{\delta}{B} w_x - \frac{\delta A}{B} u_x - u_x w = 0,$$

whose solution is

$$(3.22)\qquad \log\left|\frac{w(x,y,\delta) + \frac{\delta A(y)}{B(y)}}{w(0,y,\delta) + \frac{\delta A(y)}{B(y)}}\right| = \frac{B(y)}{\delta}\,[u(x,y,\delta) - u(0,y,\delta)]\ .$$

From equation 3.19 and the fact that $u(x,y,\delta)$ satisfies IV, we obtain

$$(3.23)\quad w(0,y,\delta) = A(y)\beta(y) + B(y)\beta'(y) + C(y).$$

Using equation 3.23 we may write equation 3.22 as

$$(3.24)\quad w(x,y,\delta) = -\frac{\delta A(y)}{B(y)} + [A(y)\beta(y) + B(y)\beta'(y)$$

$$+ \frac{\delta A(y)}{B(y)} + C(y)]\exp\left\{\frac{B(y)}{\delta}\,[u(x,y,\delta) - \beta(y)]\right\}.$$

Next, we solve equation 3.19 for u as a function of w to get

$$(3.25)\ u(x,y,\delta) = \exp\left\{\int_0^y \frac{A(s)}{B(s)}\,ds\right\}\left[\alpha(x)\right.$$

$$\left. + \int_0^y \exp\left\{\int_0^t \frac{A(s)}{B(s)}\,ds\right\}\left[\frac{w(x,t,\delta) - C(t)}{B(t)}\right]dt\right].$$

Elimination of the function w between equations 3.25 and 3.24 yields V, and the proof of the lemma is complete.

We define the problem:

$$\text{VI}\qquad \begin{aligned} &Av + Bv_y + C = 0,\\ &v(x,0) = \alpha(x). \end{aligned}$$

In the next theorem we establish sufficient conditions that the limit of the solution of problem IV as δ goes to zero be the solution of VI.

Theorem 1.

Let (1) R be the closed rectangle:

$0 < x_1 \leq x \leq x_2,$

$0 \leq y \leq y_2,$

for some positive constants x_1, x_2, and y_2, $x_1 \neq x_2$;

(2) $B(y) > 0$;

(3) $\delta > 0$;

(4) $\beta'(y) \geq 0$;

(5) $\alpha'(x) < 0$;

(6) $P(y,\delta) > 0$, where $p(y,\delta)$ is defined by equation 3.4;

(7) T be the transformation defined by

$$Tf(x,y,\delta) = \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\left[\alpha(x) + \int_0^y\left(-\frac{\delta A(t)}{B^2(t)} - \frac{C(t)}{B(t)}\right.\right.$$

$$\left.\left. + P(t,\delta)\exp\left\{\frac{B(t)}{\delta}[f(x,t,\delta) - \beta(t)]\right\}\right)\exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\}dt\right]$$

for $\delta \neq 0$, and

$$= \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\left[\alpha(x) - \int_0^y \exp\left\{\int_0^t \frac{A(s)}{B(s)}\right\}\frac{C(t)}{B(t)}dt\right],$$

for $\delta = 0$;

and

(8) the sequence $f_n(x,y,\delta)$ be defined by

$f_1(x,y,\delta) = \beta(y)$,

$f_n(x,y,\delta) = Tf_{n-1}(x,y,\delta)$, for $n=2,3,4,\ldots$

Then there exists a positive number δ_0 such that

(a) the sequence $f_n(x,y,\delta)$ converges uniformly for all (x,y) in R and all δ such that $0 \leq \delta \leq \delta_0$,

(b) the function $f(x,y,\delta) = \lim_{n\to\infty} f_n(x,y,\delta)$ is the solution of IV for (x,y) in R and $0 < \delta \leq \delta_0$,

and

(c) the limit $\lim_{\delta \to 0+} f(x,y,\delta)$ converges uniformly in R to the function $v(x,y)$ which satisfies problem VI.

Proof.

Let conditions (1) through (8) be satisfied and let (x,y) be in R. We first note that, for $\delta \neq 0$,

$$
\begin{aligned}
(3.26) \qquad f_2(x,y,\delta) &= T\beta(y) \\
&= \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\left[\alpha(x) + \int_0^y \left[\frac{A(t)}{B(t)}\beta(t)+\beta'(t)\right]\exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\}dt\right] \\
&= \beta(y) - [\beta(0)-\alpha(x)]\exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\} \\
&< \beta(y).
\end{aligned}
$$

Further, if $\delta \neq 0$, and

$$f_n(x,y,\delta) < \beta(y),$$

then it follows that

(3.27)

$$
\begin{aligned}
f_{n+1}(x,y,\delta) &= Tf_n(x,y,\delta) \\
&= \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\left[\alpha(x) + \int_0^y \left(-\frac{\delta A(t)}{B^2(t)} - \frac{C(t)}{B(t)}\right.\right. \\
&\quad \left.\left. + P(t,\delta)\exp\left\{\frac{B(t)}{\delta}[f_n(x,t,\delta)-\beta(t)]\right\}\right)\exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\}dt\right]
\end{aligned}
$$

$$< \exp\left\{ -\int_0^y \frac{A(s)}{B(s)} ds \right\} \left[\alpha(x) + \int_0^y \left[\frac{A(t)}{B(t)} \beta(t) + \beta'(t) \right] \exp\left\{ \int_0^t \frac{A(s)}{B(s)} ds \right\} dt \right]$$

$$= f_2(x,y,\delta)$$

$$< \beta(y).$$

Thus by mathematical induction, we have that when $\delta \neq 0$,

$$f_n(x,y,\delta) \leq f_2(x,y,\delta) < \beta(y), \text{ for } n=2,3,4,\ldots \tag{3.28}$$

Then there exists a positive constant K such that

$$f_n(x,y,\delta) - \beta(y) \leq -K < 0, \tag{3.29}$$

for all (x,y) in R and all $\delta \neq 0$.

Next, let us consider the series $\sum_{n=1}^{\infty} S_n(x,y,\delta)$, where

$$S_1(x,y,\delta) = f_1(x,y,\delta)$$

and

$$S_n(x,y,\delta) = f_n(x,y,\delta) - f_{n-1}(x,y,\delta),$$

for $n \geq 2$. Then we have

$$\sum_{n=1}^{k} S_n(x,y,\delta) = f_k(x,y,\delta). \tag{3.30}$$

We propose to show that this series converges uniformly and absolutely in R. Let a positive number δ_1 be chosen and let M be defined by

$$(3.31)\quad M = \max\left[\exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y P(t,\delta)\exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\}dt\right]$$

for all (x,y) in R and all δ such that

$$0 \leq \delta \leq \delta_1.$$

For $\delta = 0$, condition (2) and the continuity of $B(y)$ imply that $B(y)$ is bounded away from zero. Therefore, we have that

$$(3.32)\qquad \lim_{\delta\to 0^+} M \cdot \frac{B(y)}{\delta}\exp\left\{-K\frac{B(y)}{\delta}\right\} = 0.$$

This implies that there exists a positive number δ_0, with $\delta_0 \leq \delta_1$, such that for all (x,y) in R and all δ satisfying the condition

$$0 < \delta \leq \delta_0 \leq \delta_1$$

it is true that

$$(3.33)\qquad 0 < M \cdot \frac{B(y)}{\delta}\exp\left\{-K\frac{B(y)}{\delta}\right\} < \theta < 1,$$

for some positive constant θ. Then for

$$0 < \delta \leq \delta_0,$$

and for some t' satisfying

$$0 < t' < y,$$

we use a mean value theorem to get

$$\begin{aligned}(3.34)\qquad &\left|S_n(x,y,\delta)\right|\\ &= \left|f_n(x,y,\delta) - f_{n-1}(x,y,\delta)\right|\\ &= \left|Tf_{n-1}(x,y,\delta) - Tf_{n-2}(x,y,\delta)\right|\end{aligned}$$

$$= \left| \exp\left\{-\int_0^y \frac{A(s)}{B(s)} ds\right\} \int_0^y \exp\left\{\int_0^t \frac{A(s)}{B(s)} ds\right\} P(t,\delta) \left[\exp\left\{\frac{B(t)}{\delta}[f_{n-1}(x,t,\delta) - \beta(t)]\right\} - \exp\left\{\frac{B(t)}{\delta}[f_{n-2}(x,t,\delta) - \beta(t)]\right\}\right] \right|$$

$$\leq M \cdot \left| \exp\left\{\frac{B(t')}{\delta}[f_{n-1}(x,t',\delta) - \beta(t')]\right\} - \exp\left\{\frac{B(t')}{\delta}[f_{n-2}(x,t',\delta) - \beta(t')]\right\} \right| .$$

Again using a mean value theorem, we see that

$$(3.35) \quad \exp\left\{\frac{B(t')}{\delta}[f_{n-1}(x,t',\delta) - \beta(t')]\right\} - \exp\left\{\frac{B(t')}{\delta}[f_{n-2}(x,t',\delta) - \beta(t')]\right\}$$

$$= \frac{B(t')}{\delta}[f_{n-1}(x,t',\delta) - f_{n-2}(x,t',\delta)]\exp\left\{\frac{B(t')}{\delta} z(x,t',\delta)\right\} ,$$

where $z(x,t',\delta)$ is some value intermediate to $[f_{n-1}(x,t',\delta)-\beta(t')]$ and $[f_{n-2}(x,t',\delta)-\beta(t')]$, both of which are negative for $n \geq 4$. In fact, for $n \geq 4$, we have by equation 3.29

$$(3.36) \qquad z(x,t',\delta) \leq -K < 0.$$

Combining equations 3.34, 3.35, and 3.36, we get

$$(3.37) \quad \left|S_n(x,y,\delta)\right| \leq \left|f_{n-1}(x,t',\delta) - f_{n-2}(x,t',\delta)\right| M \frac{B(t')}{\delta} \exp\left\{-K \frac{B(t')}{\delta}\right\}$$

$$< \Theta \cdot \left| f_{n-1}(x,t',\delta) - f_{n-2}(x,t',\delta) \right|$$

$$= \Theta \left| S_{n-1}(x,t',\delta) \right| ,$$

for $n \geqslant 4$ and $0 < \delta \leq \delta_0$. From the definition of T for $\delta = 0$, we note that

$$f_n(x,y,0) = f_{n-1}(x,y,0)$$

for $n > 2$, and thus we see that

$$S_n(x,y,0) = 0$$

for $n > 2$. Therefore, for $0 \leq \delta \leq \delta_0$,

$$\sum_{n=1}^{\infty} \left| S_n(x,y,\delta) \right| < \left| S_1(x,y,\delta) \right| + \left| S_2(x,y,\delta) \right| + \max_{(x,y) \text{ in } R} \left| S_3(x,y,\delta) \right| \sum_{n=0}^{\infty} \Theta^n \tag{3.38}$$

$$= \left| S_1(x,y,\delta) \right| + \left| S_2(x,y,\delta) \right| + \max_{(x,y) \text{ in } R} \left| S_3(x,y,\delta) \right| \frac{1}{1-\Theta} .$$

We conclude that the series $\sum_{n=0}^{\infty} S_n(x,y,\delta)$ is absolutely convergent, and since the contraction constant Θ suffices for all (x,y) in R and all δ such that

$$0 \leq \delta \leq \delta_0$$

it follows that the series converges uniformly for these values of x, y, and δ. Since the series converges uniformly, the sequence of partial sums must also converge uniformly; that is, by equation 3.30, the sequence $f_n(x,y,\delta)$ converges uniformly for all (x,y) in R and all δ such that

$$0 \leq \delta \leq \delta_0.$$

Now because of the class properties ascribed to A, B, C, α, and β, it is clear that for each n, $f_n(x,y,\delta)$ is a continuous function in R for each $\delta \neq 0$. For $n > 2$,

$$(3.39) \qquad \lim_{\delta \to 0^+} f_n(x,y,\delta) = \lim_{\delta \to 0^+} Tf_{n-1}(x,y,\delta)$$

$$= \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\left[\alpha(x) - \int_0^y \frac{C(t)}{B(t)} \exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\} dt\right]$$

$$- \lim_{\delta \to 0^+} \delta \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y \frac{A(t)}{B(t)} \exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\} dt$$

$$+ \lim_{\delta \to 0^+} \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y P(t,\delta)$$

$$\text{(times)} \exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\} \exp\left\{\frac{B(t)}{\delta}[f_{n-1}(x,t,\delta) - \beta(t)]\right\} dt.$$

Using equation 3.29, we see that for $n > 2$,

$$(3.40) \quad 0 \leq \lim_{\delta \to 0^+} \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y P(t,\delta)$$

$$\text{(times)} \exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\} \exp\left\{\frac{B(t)}{\delta}[f_{n-1}(x,t,\delta) - \beta(t)]\right\} dt$$

$$\leq \lim_{\delta \to 0^+} \exp\left\{-\int_0^y \frac{A(s)}{B(s)}ds\right\}\int_0^y P(t,\delta)$$

$$\text{(times)} \exp\left\{\int_0^t \frac{A(s)}{B(s)}ds\right\} \exp\left\{-K \frac{B(t)}{\delta}\right\} dt$$

$$= \lim_{\delta \to 0^+} y \exp\left\{-\int_0^y \frac{A(s)}{B(s)} ds\right\} P(t',\delta)$$

$$\text{(times)} \exp\left\{\int_0^{t'} \frac{A(s)}{B(s)} ds\right\} \exp\left\{-K \frac{B(t')}{\delta}\right\}$$

for $0 \leq t' \leq y$. Since $P(y,\delta)$ is bounded and $B(y)$ is bounded away from zero for all (x,y) in R and all δ such that $0 \leq \delta \leq \delta_0$, we deduce that

$$(3.41) \quad \lim_{\delta \to 0^+} y \exp\left\{-\int_0^y \frac{A(s)}{B(s)} ds\right\} P(t'\delta)$$

$$\text{(times)} \exp\left\{\int_0^{t'} \frac{A(s)}{B(s)} ds\right\} \exp\left\{-K \frac{B(t')}{\delta}\right\} = 0.$$

Using equation 3.41 in inequality 3.40, and this result in equation 3.39, we get

$$(3.42) \quad \lim_{\delta \to 0^+} f_n(x,y,\delta) =$$

$$\exp\left\{-\int_0^y \frac{A(s)}{B(s)}\right\} [\alpha(x) - \int_0^y \frac{C(t)}{B(t)} \exp\left\{\int_0^t \frac{A(s)}{B(s)} ds\right\} dt]$$

$$= f_n(x,y,0),$$

for $n > 2$. Thus the functions $f_n(x,y,\delta)$ are continuous from the right at $\delta = 0$ for $n > 2$. We conclude that for $n > 2$, the functions $f_n(x,y,\delta)$ form a sequence of continuous, uniformly convergent functions for (x,y) in R and all δ such that

$0 \leq \delta \leq \delta_0$. The proof of part (a) is complete.

Let us now define

$$(3.43) \qquad f(x,y,\delta) = \lim_{n\to\infty} f_n(x,y,\delta).$$

Because the convergence is uniform in R, then for $\delta \neq 0$, we may write

$$(3.44) \qquad \begin{aligned} f(x,y,\delta) &= \lim_{n\to\infty} f_n(x,y,\delta) \\ &= \lim_{n\to\infty} Tf_{n-1}(x,y,\delta) \\ &= T \lim_{n\to\infty} f_{n-1}(x,y,\delta) \\ &= Tf(x,y,\delta). \end{aligned}$$

From the definition of T, we see that equation 3.44 implies that $f(x,y,\delta)$ satisfies problem V. By the preceding lemma, $f(x,y,\delta)$ satisfies problem IV for (x,y) in R and all δ such that

$$0 < \delta \leq \delta_0,$$

which is assertion (b).

We have established that for $n > 2$, the functions $\{ f_n(x,y,\delta) \}$ form a sequence of continuous, uniformly convergent functions for (x,y) in R and $0 \leq \delta \leq \delta_0$. Thus the limit function $f(x,y,\delta)$ is continuous for these values of x, y, and δ. In particular for $\delta = 0$, $f(x,y,\delta)$ has the value given by

$$(3.45) \qquad f(x,y,0) = \lim_{\delta\to 0^+} f(x,y,\delta)$$

$$= \lim_{\delta \to 0^+} \lim_{n \to \infty} f_n(x,y,\delta)$$

$$= \lim_{n \to \infty} f_n(x,y,0)$$

$$= \exp\left\{\int_0^y \frac{A(s)}{B(s)}\,ds\right\} \left[\alpha(x) - \int_0^y \frac{C(t)}{B(t)} \exp\left\{\int_0^t \frac{A(s)}{B(s)}\,ds\right\} dt\right].$$

By direct computation,

$$v(x,y) = f(x,y,0)$$

is shown to satisfy problem VI. This completes the proof of theorem 1.

The natural question that one asks after reading theorem 1 is: Are all of the conditions imposed by the hypothesis necessary? We attempt to answer this question, at least in part, by the remaining theorem and two examples. We note that (7) and (8) of theorem 1 are definitions rather than restrictions on the generality of problem IV. We concentrate then on the conditions (1) through (6), and first look at condition (1), which restricts the region R.

It is quite clear from the proof of theorem 1 that the line $x = 0$ represents a natural boundary, for the function $f(x,y,\delta)$ satisfies the condition

$$f(0,y,\delta) = \beta(y),$$

when $\delta \neq 0$; yet the function $f(x,y,0)$ was shown to satisfy problem VI in R. The function $f(x,y,0)$ which satisfies VI will not in general take on the value $\beta(y)$ when $x = 0$, as is well known.

One might suspect that theorem 1 would be true for (x,y) in the second quadrant, so long as conditions (2) through (6) of theorem 1 were true. The following example illustrates the difficulty there.

Example 1.

We consider the following problem:

$$\text{VII} \qquad \begin{aligned} &\delta u_{xy} - u_x u_y = 0, \\ &u(x,0) = \alpha(x), \\ &u(0,y) = \beta(y), \\ &\alpha(x) = -x + \alpha(0), \\ &\beta(y) = y + \alpha(0), \\ &\alpha(0) > 0, \end{aligned}$$

and

$$\delta > 0.$$

Comparing this problem with IV, we see that it corresponds to the case

$$A(y) = C(y) = 0,$$

and

$$B(y) = 1.$$

Further, we note that conditions (2) through (6) of theorem 1 are satisfied. The solution of problem VII, when it exists, is given by

$$(3.46) \quad u(x,y,\delta) = -\delta \ln \left| \exp\left\{-\frac{\alpha(x)}{\delta}\right\} + \exp\left\{-\frac{\beta(y)}{\delta}\right\} - \exp\left\{-\frac{\alpha(0)}{\delta}\right\} \right| .$$

We notice that the function $u(x,y,\delta)$ defined by equation 3.46 does not have derivatives with respect to x and y when the expression

$$(3.47)\qquad \exp\left\{-\frac{\alpha(x)}{\delta}\right\} + \exp\left\{-\frac{\beta(y)}{\delta}\right\} - \exp\left\{-\frac{\alpha(0)}{\delta}\right\}$$

is zero, and thus at points (x,y) such that expression 3.47 vanishes, the solution to problem VII does not exist. We propose to show that for each negative x, and for each positive δ less than some positive δ_0, there exists a positive y such that expression 3.47 is zero, and that for fixed x these values of y tend to zero as δ goes to zero.

Let $x = -K < 0$. Then

$$\alpha(-K) = K + \alpha(0)$$
$$> 0.$$

Inserting the explicit values for $\alpha(x)$ and $\beta(y)$ at $x = -K$, we may write expression 3.47 as

$$(3.48)\qquad \left[\exp\left\{-\frac{\alpha(0)}{\delta}\right\}\right]\left[\exp\left\{-\frac{y}{\delta}\right\} - \left(1-\exp\left\{-\frac{K}{\delta}\right\}\right)\right].$$

We see that expression 3.48 will be zero provided that

$$(3.49)\qquad y = -\delta \ln\left[1-\exp\left\{-\frac{K}{\delta}\right\}\right].$$

Since $-\frac{K}{\delta} < 0$, there exists a δ_0 such that for all δ satisfying the condition

$$0 < \delta < \delta_0,$$

it is true that

$$0 < 1\text{-exp}\left\{-\frac{K}{\delta}\right\} < 1.$$

Therefore, for each δ_1 such that $0 < \delta_1 < \delta_0$, we have a positive number y_1 defined by equation 3.49, such that the partial derivatives of $u(x,y,\delta)$ as given by equation 3.46 do not exist at the point $(-K,y_1,\delta_1)$. It is clear from equation 3.49 that as δ goes to zero, the y defined by equation 3.49 also goes to zero. This implies that for each negative x, the point (x,0) is a limit point of points where the solution of VII fails to exist. We see then that the region adjoining the x-axis in the second quadrant in which the solution to problem VII exists shrinks to include only points on the x-axis as δ goes to zero. This example illustrates that the conclusions of theorem 1 may not be valid when conditions (2) through (6) are satisfied and the region R is replaced by a similar rectangle in the second quadrant.

We note that theorem 1 placed conditions on the signs of the derivatives of the prescribed functions, $\alpha(x)$ and $\beta(y)$. To see that some such condition is necessary we cite the following example.

Example 2.

Suppose that all of the conditions of theorem 1 are satisfied except the condition that $\alpha'(x)$ be negative. In particular, suppose $\alpha'(x)$ is zero. Then $\alpha(x)$ is the constant

$\beta(0)$. In this case, the solution to problem IV is

$$u(x,y,\delta) = \beta(y),$$

which in general will not satisfy the differential equation of VI.

The conditions

$$B(y) > 0$$

and

$$P(y,\delta) > 0$$

both depend on the nonvanishing of $B(y)$. Let us then consider the case $B(y) = 0$.

<u>Theorem 2</u>.

Let

(1) R be the closed rectangle

$$0 \leq x \leq x_2$$

$$0 < y_1 \leq y \leq y_2;$$

(2) $B(y) \equiv 0$;

(3) $A(y)\beta(y)+C(y) \leq -K < 0$;

(4) $\alpha'(x) \leq 0$;

(5) $A(y) > 0$;

and

(6) $\delta > 0$.

Then if $u(x,y,\delta)$ is the solution of problem IV in R, it is true that

$$\lim_{\delta \to 0^+} u(x,y,\delta) = \beta(y).$$

Proof.

In the case $B(y) \equiv 0$, the differential equation of problem IV reduces to

$$\delta u_{xy} - Auu_x - Cu_x = 0. \tag{3.50}$$

We find the explicit form of the solution to problem IV for this case as follows. Integration of equation 3.50 with respect to x and application of the condition $u(0,y) = \beta(y)$ to the resulting relation yields

$$\delta u_y - \delta\beta' = \frac{A}{2}(u^2-\beta^2) + C(u-\beta). \tag{3.51}$$

Let $w = u-\beta$. Then we get

$$\delta w_y = \frac{A}{2} w(w+2\beta) + cw, \tag{3.52}$$

or

$$\frac{\delta w_y}{w^2} - \frac{1}{w}(A\beta + C) = \frac{A}{2}. \tag{3.53}$$

Let us define $v = \frac{1}{w}$. Then we obtain

$$v_y + \frac{A\beta + C}{\delta} v = -\frac{A}{2\delta}, \tag{3.54}$$

which is linear in v. The solution of equation 3.54 is given by

$$v(x,y,\delta) \exp\left\{\int_0^y \frac{A(s)\beta(s)+C(s)}{\delta} ds\right\} \tag{3.55}$$

$$= v(x,0,\delta) - \int_0^y \frac{A(t)}{2\delta} \exp\left\{\int_0^t \frac{A(s)\beta(s)+C(s)}{\delta} ds\right\} dt.$$

Writing equation 3.55 in terms of $u(x,y,\delta)$, we get

$$(3.56) \qquad u(x,y,\delta) = \beta(y)$$

$$+ \frac{[\alpha(x)-\beta(0)] \exp\left\{\int_0^y \frac{A(s)\beta(s)+C(s)}{\delta} ds\right\}}{1-\frac{[\alpha(x)-\beta(0)]}{2\delta}\int_0^y A(t) \exp\left\{\int_0^t \frac{A(s)\beta(s)+C(s)}{\delta} ds\right\} dt} .$$

We consider (x,y) in R as defined by condition (1) and assume that conditions (2) through (6) are satisfied. Then we may write

$$(3.57) \qquad \alpha(x)-\beta(0) \leq 0,$$

using condition (4). From inequality 3.57 and conditions (5) and (6), it follows that

$$(3.58) \quad 1 - \frac{[\alpha(x)-\beta(0)]}{2\delta} \int_0^y A(t) \exp\left\{\int_0^t \frac{A(s)\beta(s)+C(s)}{\delta} ds\right\} dt$$
$$\leq 1.$$

Using condition (3), we see that

$$(3.59) \qquad 0 \leq \operatorname*{limit}_{\delta \to 0^+} \exp\left\{\int_0^y \frac{A(s)\beta(s)+C(s)}{\delta} ds\right\}$$
$$\leq \operatorname*{limit}_{\delta \to 0^+} \exp\left\{-K \frac{y}{\delta}\right\}$$
$$= 0.$$

From equation 3.56 and inequalities 3.58 and 3.59 we conclude that

$$(3.60) \qquad \operatorname*{limit}_{\delta \to 0^+} u(x,y,\delta) = \beta(y).$$

We note that all of the conditions set forth in the hypothesis of theorem 2 are not necessary. For a case in point, see example 2. Any set of conditions which insure that the second term on the right in equation 3.56 goes to zero as δ goes to zero suffices. The conditions (1) through (6) were chosen because they most closely parallel the conditions in theorem 1. In fact, by properly choosing A, B, C, α, β, and δ, we may satisfy conditions (2), (3), (4), (5), and (6) of theorem 1 and for the same choice of A, C, α, β, and δ, but with $B = 0$, we may satisfy (2), (3), (4), (5), and (6) of theorem 2 in a closed rectangle in the first quadrant. For example,

$$A = 1,$$

$$B = 1,$$

$$C = -2y,$$

$$\alpha = -x,$$

$$\beta = 2y,$$

and

$$0 < \delta < 1,$$

satisfies (2) through (6) of theorem 1 if $0 \leq y \leq 1$; and

$$A = 1,$$

$$B = 0,$$

$$C = -2y,$$

$$\alpha = -x,$$

$$\beta = 2y,$$

and

$$0 < \delta < 1$$

satisfies (2) through (6) of theorem 2 when $0 \lesssim y \lesssim 1$. Thus we see something of the general nature of the dependence of the solution on $B(y)$.

We note that the same type of arguments used in this theorem would suffice for the case $y < 0$, provided that the inequality restricting A, β, C, and δ were reversed. Note also that among the functions A, β, and C for which $A(y)\beta(y) + C(y)$ is not negative are included the functions $A(y) = C(y) = 0$. For this case problem IV reduces to simply

$$\text{VIII} \qquad \begin{aligned} \delta u_{xy} &= 0 \\ u(x,0) &= \alpha(x) \\ u(0,y) &= \beta(y) \end{aligned}$$

whose solution

$$u(x,y) = \alpha(x) + \beta(y) - \alpha(0), \tag{3.61}$$

is independent of δ and for which the conclusions of theorem 2 do not hold.

We note that the reduced equation, that is, when $\delta = 0$, in problem IV may be written as two equations

$$Au + Bu_y + C = 0 \tag{3.62}$$

and

$$u_x = 0. \tag{3.63}$$

We have given, in theorem 1, sufficient conditions that the solution to problem IV approach the solution of equation 3.62 which takes on the values prescribed by problem IV along the x-axis. In theorem 2, we have given sufficient conditions that the solution of problem IV approach the solution of equation 3.63 which takes on the prescribed conditions along the y-axis. It would have been quite interesting, but rather improbable, had we found sufficient conditions of a general nature which insure that the solution of problem IV approaches the solution of equation 3.62 which takes on prescribed values along the y-axis, because the conditions

$$\text{IX} \qquad \begin{aligned} Au + Bu_y + C &= 0, \\ u(0,y) &= \beta(y), \end{aligned}$$

do not uniquely determine the function $u(x,y)$. This is so because the line $x = 0$ is a characteristic line for equation 3.62. A similar remark may be made about the problem

$$\text{X} \qquad \begin{aligned} u_x &= 0, \\ u(x,0) &= \alpha(x), \end{aligned}$$

for which $y = 0$ is a characteristic line.

Finally, we note that the two characteristic lines

$$x = 0$$

and

$$y = 0$$

enter our theorems as boundaries, in a manner similar to that in which the characteristic line

$$Ay = Bx$$

enters the linear hyperbolic problem considered in the second section of this paper.

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