ANDERSON, Ronald Myles, 1935–
SOLUTIONS OF MIXED TYPE PARTIAL DIFFERENTIAL EQUATIONS.

Iowa State University of Science and Technology
Ph.D., 1962
Mathematics

SOLUTIONS OF MIXED TYPE PARTIAL DIFFERENTIAL EQUATIONS

by

Ronald Myles Anderson

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1962

TABLE OF CONTENTS

I. INTRODUCTION

In this thesis the existence of weak solutions is established for a certain class of boundary value problems involving partial differential equations of mixed elliptic-hyperbolic type. We consider an equation of the form

$$(1.1) \quad Lu = K(y)u_{xx} + u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = g(x,y) \quad ,$$

where $yK(y) > 0$ if $y \neq 0$. This equation is elliptic for $y > 0$ and hyperbolic for $y < 0$. The coefficients K, a and b are assumed to be of class C'' and the coefficient c of class C' in a prescribed region D of the xy-plane. Some additional restrictions on the coefficients are required in the existence proof. They will be given in Chapter III.

For $y > 0$, the region is bounded by a piecewise smooth arc C_o (see Figure 1) which intersects the x-axis at the two points x_l and x_r, with $x_l < x_r$. For $y < 0$, the region is bounded by two piecewise smooth arcs C_l and C_r which issue from x_l and x_r, respectively, and by two characteristic arcs Γ_l and Γ_r. We require that the arcs C_l and C_r be non-characteristic and satisfy the conditions:

$$(1.2) \quad 0 \leq dy/dx < 1/\sqrt{-K} \text{ along } C_l,$$

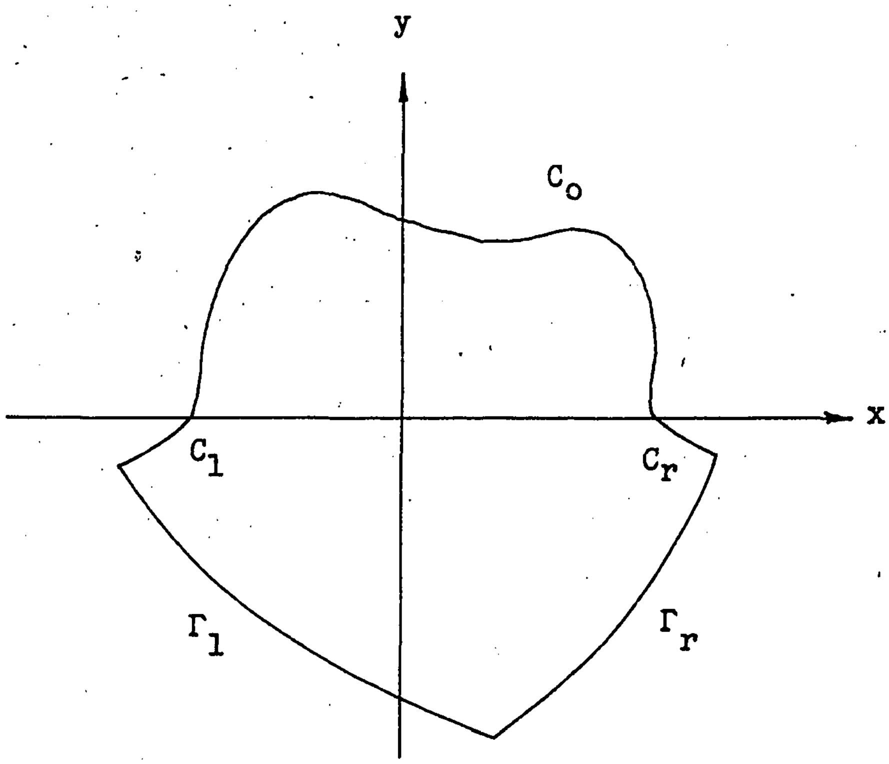

Figure 1. Domain

(1.3) $\quad 0 > dy/dx > -1/\sqrt{-K}$ along C_r .

The characteristics Γ_l and Γ_r satisfy the conditions:

(1.4) $\quad dy/dx = -1/\sqrt{-K}$ along Γ_l ,

(1.5) $\quad dy/dx = 1/\sqrt{-K}$ along Γ_r .

We impose the boundary condition

(1.6) $\quad u = 0$ along $C_o + C_l + C_r + \Gamma_l$.

No condition is imposed on u along Γ_r. We also include the cases where either C_l or C_r or both C_l and C_r are degenerate arcs.

We shall employ Hilbert space methods in proving the existence of weak solutions to Equation 1.1 under the boundary condition 1.6. This technique has previously been used for mixed type equations by Morawetz (6), Friedrichs (5), and Berezanskiĭ (2,3). Morawetz and Friedrichs considered systems of simultaneous first order equations. Berezanskiĭ considered an equation equivalent to Equation 1.1, but imposed the restriction that the coefficients a, b and c vanish for $y < 0$.

In Chapter II, we introduce an adjoint problem and define the weak solution of 1.1 under the boundary condition 1.6 in terms of the adjoint problem. We then show that if u is a weak solution and is sufficiently differentiable, then u is a solution in the usual sense.

In Chapter III, we establish an inequality for the adjoint problem and on the basis of this inequality prove a uniqueness theorem for the adjoint problem. We then employ a well-known theorem from Hilbert space theory to establish our existence theorem.

In Chapter IV, we compare our results with other known results and conclude by giving two examples.

II. WEAK SOLUTIONS

Let H be the Hilbert space of all (equivalence classes of) real valued measurable functions for which the norm

$$(2.1) \qquad ||u|| = \left[\iint_D u^2 \, dx \, dy\right]^{\frac{1}{2}}$$

is finite; the inner product is

$$(2.2) \qquad (u,v) = \iint_D u \, v \, dx \, dy \quad .$$

Let W be the set of all real valued functions w with continuous second derivatives in D and continuous first derivatives on the boundary B of D which vanish on the exterior of D and in addition satisfy the conditions:

$$(2.3) \qquad w = 0 \text{ on } C_o + C_1 + C_r + \Gamma_r$$

$$(2.4) \qquad ||Mw|| < \infty \quad ,$$

where

$$(2.5) \qquad Mw = K(y)w_{xx} + w_{yy} - (aw)_x - (bw)_y + cw \quad .$$

The derivatives w_x and w_y for points on B are one-sided derivatives, taken from the interior of D. The differential operator M is called the formal adjoint of the operator L and the boundary value problem Mw = g under the boundary condition 2.3 is called the adjoint problem associated with Equation 1.1

under the boundary condition 1.6.

Definition: We call $u \in H$ a weak solution of $Lu = g$ under the boundary condition 1.6 if, for every $w \in W$,

$$(2.6) \quad (w,g) = (Mw,u) \quad .$$

To justify this definition we consider the following theorem.

Theorem 1: If u is a weak solution and in addition u has continuous second derivatives in D and continuous first derivatives on B and g is continuous in D, then u is a solution in the usual sense of Equation 1.1 under the boundary condition 1.6.

Proof: We investigate the inner product (Mw,u). We note the identities:

$$Kw_{xx}u = K[u_{xx}w + (w_xu)_x - (wu_x)_x] \quad ,$$

$$w_{yy}u = u_{yy}w + (w_yu)_y - (wu_y)_y \quad ,$$

$$-(aw)_xu = au_xw - (awu)_x \quad ,$$

$$-(bw)_yu = bu_yw - (bwu)_y \quad .$$

Thus, if w is any element of W, we have the identity

$$(2.7) \quad (Mw,u) = (w,Lu) + I \quad ,$$

where

$$I = \iint_D [(Kw_x u - Kwu_x - awu)_x + (w_y u - wu_y - bwu)_y] \, dx \, dy \quad .$$

However, since u is a weak solution, we may replace the left hand side of Equation 2.7 by its value from Equation 2.6. We then have

$$(2.8) \qquad (w,g) = (w,Lu) + I \quad .$$

We now apply Green's theorem to I. We obtain

$$I = \int_B (wu_y - w_y u + buw) \, dx + (Kw_x u - Kwu_x - awu) \, dy$$

$$\equiv I(C) + I(\Gamma_r) + I(\Gamma_l) \quad ,$$

where $C = C_o + C_l + C_r$.

We assert that Identity 2.8 implies u is a solution in the usual sense to Equation 1.1 under the boundary condition 1.6. We write Identity 2.8 in the form

$$(2.9) \qquad \iint_D (Lu - g)w \, dx \, dy = I(C) + I(\Gamma_r) + I(\Gamma_l) \quad .$$

We first show that $Lu = g$ in D. The proof is by contradiction. Suppose there exists a point P in D for which $Lu(P) \neq g(P)$. Then either $Lu(P) - g(P) > 0$ or $Lu(P) - g(P) < 0$. For definiteness, we consider the case $Lu(P) - g(P) > 0$. Since $Lu - g$ is continuous in D, there exists a neighborhood N(P) of P in D throughout which $Lu - g > 0$. We define a function w with the properties that $w > 0$ in a neighborhood $N'(P) \subset N(P)$; $w \equiv 0$ in $D - N'(P)$; with the further restriction that w have continuous second derivatives in D. Clearly $w \in W$ and is such that the boundary integrals in Identity 2.9 vanish. Thus, for this choice of w, we obtain from Identity 2.9 the identity

$$\iint_{N'(P)} (Lu - g)w \, dx \, dy = 0 .$$

But this implies that $Lu - g \equiv 0$ in $N'(P)$ and hence contradicts the assumption that $Lu(P) - g(P) > 0$.

We next show that u satisfies the boundary condition 1.6. Since $Lu - g \equiv 0$, we may rewrite Identity 2.9 as

$$(2.10) \quad I(C) + I(\Gamma_r) + I(\Gamma_1) = 0 ,$$

for all $w \in W$. We note that by Equation 1.6 $w = 0$ along $C + \Gamma_r$. We thus may write

$$I(C) = \int_C u(Kw_x \, dy - w_y \, dx) ,$$

$$I(\Gamma_r) = \int_{\Gamma_r} u(Kw_x \, dy - w_y \, dx) ,$$

where the integrations are in a counterclockwise direction along the appropriate arcs.

Along Γ_r, $dx = \sqrt{-K} \, dy$ and thus we find that

$$Kw_x\,dy - w_y\,dx = -\sqrt{-K}\,(w_x\,dx + w_y\,dy) = -\sqrt{-K}\,dw = 0 ,$$

since $w = 0$. Therefore $I(\Gamma_r) = 0$, independent of u.

We now show that $u = 0$ on C. The proof is by contradiction. Suppose there exists a point P on C for which $u \neq 0$, say $u(P) > 0$. Further, suppose P is a point at which C is smooth and $dx \neq 0$. Since $w = 0$ on C we have $dw = w_x\,dx + w_y\,dy = 0$. We then may write $w_x = -w_y\,(dy/dx)$ at P. Moreover, since C is smooth at P, there exists an arc $C(P)$ containing P for which the last equation holds. Thus, along $C(P)$, we find

$$Kw_x\,dy - w_y\,dx = -[K(dy/dx)^2 + 1]\,w_y\,dx.$$

From Equations 1.2 and 1.3 we have that $(dy/dx)^2 < -1/K$ along C_l and C_r while K is positive along C_o. Therefore $K(dy/dx)^2 + 1 > 0$ on $C(P)$. Moreover, since u is continuous on C, there exists an arc $C'(P) \subset C(P)$ containing P for which u is positive. We construct a circle $R(P)$ with center at P such that $R(P) \cap C \subset C'(P)$. We define a function w with continuous second derivatives with the properties that $w = 0$ on C and on the boundary of $R(P)$ and all exterior points of $R(P) \cap D$; $w > 0$ in $R(P) \cap D$. Then w_y is non-zero on $C'(P) \cap R(P)$ and is of one sign. Moreover, $w \in W$. For this choice of w Identity 2.10 reduces to

$$\int_{C'(P)} u[K(dy/dx)^2 + 1]\, w_y\, dx = 0 \ .$$

But this implies that $u \equiv 0$ on $C'(P)$ and consequently $u(P) = 0$. If P is a point of which $dx = 0$ and C is smooth we use the relation $w_y = -w_x(dx/dy)$ and proceed as in the above argument. We have thus shown that $u = 0$ on C except possibly at points where C is not smooth. But C is piecewise smooth and hence these points are isolated points. Consequently, by the continuity of u, we may conclude that $u = 0$ at these points also.

We complete the proof by showing that $u = 0$ on Γ_1. Since $u = 0$ on C and $I(\Gamma_r) = 0$, Identity 2.10 yields $I(\Gamma_1) = 0$ for all $w \in W$. Along Γ_1, $dx = -\sqrt{-K}\, dy$. We then may write

$$I(\Gamma_1) = \int_{\Gamma_1} [-w\,(\sqrt{-K}\, u_y + Ku_x) + u(\sqrt{-K}\, w_y + Kw_x)$$

$$- u\, w\,(\sqrt{-K}\, b + a)]\, dy \ .$$

Moreover, $dw/dy = -\sqrt{-K}\, w_x + w_y$ along Γ_1. We thus may write the above integral in the form

$$I(\Gamma_1) = \int_{\Gamma_1} \{-\sqrt{-K}\, w \frac{du}{dy} + \sqrt{-K}\, u \frac{dw}{dy} - u\, w\,(\sqrt{-K}\, b + a)\}\, dy \ .$$

However, $\sqrt{-K}\, u \dfrac{dw}{dy} = \dfrac{d}{dy}(\sqrt{-K}\, u\, w) - w\sqrt{-K}\dfrac{du}{dy} - u\, w\dfrac{d(\sqrt{-K})}{dy}$.

We insert this identity into the above integral to obtain

$$(2.11)\quad I(\Gamma_1) = -\int_{\Gamma_1} w\left[2\sqrt{-K}\,\frac{du}{dy} + u\{\frac{d(\sqrt{-K})}{dy} + \sqrt{-K}\,b + a\}\right] dy + \sqrt{-K}\,u\,w\,\Big|_{y_o}^{y_1} ,$$

where y_o and y_1 are the values of y at the end points of Γ_1. But w = 0 at y_o and y_1 and thus $\sqrt{-K}\,u\,w\,\Big|_{y_o}^{y_1} = 0$. Hence, since $I(\Gamma_1) = 0$, we conclude that the integral in Equation 2.11 vanishes for all $w \in W$. But each $w \in W$ vanishes at the endpoints of Γ_1 and u is of class C' on Γ_1. We therefore may apply the fundamental lemma of calculus of variations (Courant and Hilbert (4, pp. 185, 200)) to the above integral. We find

$$2\sqrt{-K}\,\frac{du}{dy} + u\{\frac{d}{dy}(\sqrt{-K}) + \sqrt{-K}\,b + a\} \equiv 0 \text{ on } \Gamma_1 .$$

We then have the differential equation

$$2\,\frac{du}{u} + \frac{d(\sqrt{-K})}{\sqrt{-K}} + [a/\sqrt{-K} + b]\,dy = 0 ,$$

which has the solution

$$u^2\sqrt{-K} = A \exp\left[\int (a/\sqrt{-K} + b)\,dy\right] ,$$

where A is a constant. Since u vanishes at one end of Γ_1 we conclude that A = 0. Therefore u = 0 on Γ_1.

Hence we have shown that Lu = g in D and u = 0 on $C_o + C_1 + C_r + \Gamma_1$ and the proof is complete.

III. AN EXISTENCE THEOREM

We begin this chapter by deriving an inequality involving the adjoint operator M, which is given by Equation 2.5. We investigate the expression

$$(3.1) \qquad 2(Mw,\ pw_x + qw_y + rw)$$

where $w \in W$ and $p(x,y)$, $q(x,y)$ and $r(x,y)$ are functions with continuous first derivatives and are yet to be determined. We note the following identities:

$$2pKw_x w_{xx} = (pKw_x^2)_x - (pK)_x w_x^2$$

$$2pW_{yy}w_x = 2(pw_y w_x)_y - 2p_y w_x w_y - (pw_y^2)_x + p_x w_y^2$$

$$-2(aw)_x pw_x = -2apw_x^2 - (a_x pw^2)_x + (a_x p)_x w^2$$

$$-2(bw)_y pw_x = -2bpw_x w_y - (b_y pw^2)_x + (b_y p)_x w^2$$

$$2cpww_x = (cpw^2)_x - (cp)_x w^2$$

$$2Kqw_y w_{xx} = 2(Kqw_y w_x)_x - 2(Kq)_x w_x w_y - (Kqw_x^2)_y + (Kq)_y w_x^2$$

$$2qw_y w_{yy} = (qw_y^2)_y - q_y w_y^2$$

$$-2(aw)_x qw_y = -2aqw_x w_y - (a_x qw^2)_y + (a_x q)_y w^2$$

$$- 2(bw)_y q w_y = - 2bqw_y^2 - (b_y q w^2)_y + (b_y q)_y w^2$$

$$2cqww_y = (cqw^2)_y - (cq)_y w^2$$

$$2rKww_{xx} = 2(rKww_x)_x - 2rKw_x^2 - (r_x K w^2)_x + (r_x K)_x w^2$$

$$2rww_{yy} = 2(rww_y)_y - 2rw_y^2 - (r_y w^2)_y + r_{yy} w^2$$

$$- 2(aw)_x rw = - 2ra_x w^2 - (raw^2)_x + (ra)_x w^2$$

$$- 2(bw)_y rw = - 2rb_y w^2 - (rbw^2)_y + (rb)_y w^2 \quad .$$

We substitute these identities into Expression 3.1 and apply Green's theorem to obtain the identity

$$(3.2) \quad 2(Mw, pw_x + qw_y + rw)$$

$$= \iint_D \{[K'q - K(p_x - q_y + 2r) - 2ap]w_x^2 - 2[Kq_x + p_y + aq + bp]w_x w_y + [p_x - q_y - 2r - 2bq]w_y^2\} \, dx \, dy$$

$$+ \iint_D \{-[(c - a_x - b_y)p]_x - [(c - a_x - b_y)q]_y + Kr_{xx} + r_{yy} + ar_x + br_y + (2c - a_x - b_y)r\} w^2 \, dx \, dy$$

$$+ \int_B \{q(Kw_x^2 - w_y^2) - q(c - a_x - b_y)w^2 - 2pw_x w_y - 2rww_y + r_y w^2 + brw^2\} \, dx + \{p(Kw_x^2 - w_y^2)$$

$$+ p(c - a_x - b_y)w^2 + 2Kqw_xw_y + 2rKww_x - r_xKw^2$$

$$- raw^2\} \, dy$$

$$\equiv I_1 + I_2 + I_3 \quad .$$

We shall show that by properly choosing the arbitrary functions p, q and r and by placing certain restrictions on the coefficients of Equation 1.1, each of the above integrals is non-negative for all $w \in W$.

We begin by simplifying the expression for the boundary integral I_3. Along $C = C_o + C_1 + C_r$ we have $w = 0$. We thus may write

$$I_3(C) \;=\; \int_C q\{Kw_x^2 - w_y^2) \, dx + 2Kw_x(w_y \, dy)\}$$

$$+ p\{(Kw_x^2 - w_y^2) \, dy - w_y(w_x \, dx)\}$$

Moreover, $w = 0$ on C implies that $- w_x \, dx = w_y \, dy$ along C. We substitute this relation into the above integral to obtain

$$I_3(C) \;=\; \int_C -q(Kw_x^2 + w_y^2) \, dx + p(Kw_x^2 + w_y^2) \, dy$$

or

$$(3.3) \qquad I_3(C) \;=\; \int_C (Kw_x^2 + w_y^2)(p \, dy - q \, dx) \quad .$$

Along Γ_r, $dx = \sqrt{-K} \, dy$. Moreover, $w = 0$ and consequently $dw = w_x \, dx + w_y \, dy = (\sqrt{-K} \, w_x + w_y) \, dy = 0$. We substitute these relations into the expression for I_3 to obtain

$$(3.4)\qquad I_3(\Gamma_r) = \int_{\Gamma_r} (Kw_x^2 + w_y^2)(p\,dy - q\,dx)$$

$$= -\int_{\Gamma_r} (\sqrt{-K}\,w_x + w_y)(\sqrt{-K}\,w_x - w_y)(p\,dy - q\,dx)$$

$$= 0 \quad .$$

Along Γ_1, $dx = -\sqrt{-K}\,dy$. We employ this relation to write the line integral along Γ_1 as

$$I_3(\Gamma_1) = \int_{\Gamma_1} [\sqrt{-K}\,q - p)(\sqrt{-K}\,w_x + w_y)^2 + (c - a_x - b_y)$$

$$(\sqrt{-K}\,q + p)w^2 - r(a + \sqrt{-K}\,b)w^2 + \sqrt{-K}\,(\sqrt{-K}\,r_x - r_y)w^2$$

$$+ 2rw\sqrt{-K}\,(-\sqrt{-K}\,w_x + w_y)]\,dy \quad .$$

If we take y as a parameter in the above integral we find

$dy < 0$ and $\frac{dw}{dy} = -\sqrt{-K}\,w_x + w_y$. Consequently we may rewrite

the above integral in the form

$$I_3(\Gamma_1) = \int_{\Gamma_1} [(\sqrt{-K}\,q - p)(\sqrt{-K}\,w_x + w_y) + (c - a_x - b_y)$$

$$(\sqrt{-K}\,q + p)w^2 - r(a + \sqrt{-K}\,b)w^2 - \sqrt{-K}\,w^2\frac{dr}{dy}$$

$$+ r\sqrt{-K}\,\frac{d}{dy}(w^2)]\,dy \quad .$$

We now perform an integration by parts on the last term of the above integral, observing that w vanishes at the end points of Γ_1. We find

$$(3.5)\qquad I_3(\Gamma_1) = \int_{\Gamma_1} [(\sqrt{-K}\, q - p)(\sqrt{-K}\, w_x + w_y)^2 + \{(c - a_x - b_y)(\sqrt{-K}\, q + p) - r(a + \sqrt{-K}\, b) - 2\sqrt{-K}\,\frac{dr}{dy} - r\frac{d}{dy}(\sqrt{-K})\}w^2]\, dy \quad .$$

We now eliminate the first term in the above integral by choosing $p = \sqrt{-K}\, q$ for $y < 0$. In addition we choose $q = p = 0$ for $y \geq 0$. We then may combine Equations 3.3, 3.4 and 3.5 to obtain

$$(3.6)\qquad I_3 = \int_{C_1 + C_r} q(Kw_x^2 + w_y^2)(dy - \sqrt{-K}\, dx) + \int_{\Gamma_1} \Big[2(c - a_x - b_y)\sqrt{-K}\, q - r\{a + \sqrt{-K}\, b + \frac{d}{dy}(\sqrt{-K})\} - 2\sqrt{-K}\,\frac{dr}{dy}\Big] w^2\, dy \quad .$$

We now turn our attention to I_1. We denote by D^+ the part of D above the x-axis and by D^- the part of D below the x-axis. We use the fact that $p = \sqrt{-K}\, q$ in D^- and $p = q = 0$ in D^+ to write

$$(3.7) \quad I_1 = \iint_{D^+} - 2r(Kw_x^2 + w_y^2)\,dx\,dy$$

$$+ \iint_{D^-} [\{K'q - K(\sqrt{-K}\,q_x - q_y + 2r)$$

$$- 2a\sqrt{-K}\,q\}w_x^2 - 2\{Kq_x + \sqrt{-K}\,q_y - K'q/(2\sqrt{-K})$$

$$+ aq + b\sqrt{-K}\,q\}w_x w_y + \{\sqrt{-K}\,q_x - q_y - 2r$$

$$- 2bq\}w_y^2]\,dx\,dy$$

$$\equiv I_1^+ + I_1^- \quad .$$

We regroup terms in I_1^- to obtain

$$(3.8) \quad I_1^- = \iint_{D^-} [(\sqrt{-K}\,q_x - q_y)(\sqrt{-K}\,w_x + w_y)^2$$

$$+ (\sqrt{-K}\,w_x + w_y)\,\{w_x(K'q/\sqrt{-K} - 2aq + 2r\sqrt{-K})$$

$$+ w_y(-2bq - 2r)\}]\,dx\,dy \quad .$$

We now require that q and r satisfy the pair of equations

$$(3.9) \quad \begin{aligned} (K'/\sqrt{-K} - 2a)q + 2r\sqrt{-K} &= \sqrt{-K}\,A(x,y) \\ -2bq - 2r &= A(x,y) \quad , \end{aligned}$$

where $A(x,y)$ is a function to be determined. We solve for q and A in terms of r to obtain

$$(3.10)\quad q = 4Kr/(K' + 2Kg)$$

$$A = -2r(K' + 2Kh)/(K' + 2Kg)$$

where

$$(3.11)\quad g = a/\sqrt{-K} - b \quad \text{and} \quad h = a/\sqrt{-K} + b \quad .$$

For convenience, we introduce two differential operators D_λ and D_μ, which are defined by

$$(3.12)\quad D_\lambda(\ ) = \sqrt{-K}(\ )_x - (\ )_y$$

$$D_\mu(\ ) = \sqrt{-K}(\ )_x + (\ )_y \quad .$$

We substitute the values of q and A from Equation 3.10 into Equation 3.8 to obtain

$$(3.13)\quad \bar{I}_1 = \iint_D -(\sqrt{-K}\,w_x + w_y)\ \{4KD_\lambda(r)/(K' + 2Kg)$$

$$+ 2r[2D_\lambda(K/(K' + 2Kg)) - (K' + 2Kh)/$$

$$(K' + 2Kg)]\}\ dx\,dy \quad .$$

If we now replace p and q by their appropriate values in the integral I_2 and combine the results of Equations 3.6, 3.7 and 3.13, the equation 3.2 becomes

$$(3.14)\quad 2(Mw,\ pw_x + qw_y + rw) = J_1 + J_2 + J_3 + J_4 + J_5 + J_6\ ,$$

where

$$J_1 = \iint_{D^-} (\sqrt{-K}\, w_x + w_y)^2 \{4KD_\lambda (r)/(K' + 2Kg)$$

$$+ 2r[2D_\lambda (K/(K' + 2Kg)) - (K' + 2Kh)/(K' + 2Kg)\} \, dx\, dy$$

$$J_2 = \iint_{D^+} - 2r(Kw_x^2 + w_y^2)\, dx\, dy$$

$$J_3 = \iint_{D^-} D_\mu [-4rK(c - a_x - b_y)/(K' + 2Kg)]w \, dx\, dy$$

$$J_4 = \iint_D [Kr_{xx} + r_{yy} + ar_x + br_y + (2c - a_x - b_y)r]w^2 \, dx\, dy$$

$$J_5 = \int_{C_1 + C_r} 4Kr[(Kw_x^2 + w_y^2)/(K' + 2Kg)](dy - \sqrt{-K}\, dx)$$

$$J_6 = \int_{\Gamma_1} 2\sqrt{-K}\, \{r[4K(c - a_x - b_y)/(K' + 2Kg) -$$

$$(K' + 2Kh)/4K] - dr/dy\}w^2 \, dy \quad .$$

Lemma 1: Suppose the coefficients K, a, b and c of Equation 1.1 satisfy the following set of conditions:

(3.15) $2c - a_x - b_y \leq 0$ in D

(3.16) $K' + 2Kg > 0$ in $\overline{D^-} - (x_1, x_r)$, where $g = a/\sqrt{-K} - b$

(3.17) $2D_\lambda [K/(K' + 2Kg)] - (K' + 2Kh)/(K' + 2Kg) \leq - \delta$

< 0 in D^-, where $D_\lambda (\ ) = \sqrt{-K}(\ )_x - (\ )_y$ and

$h = a/\sqrt{-K} + b$

(3.18) $D_\mu\,[K(c - a_x - b_y)/(K' + 2Kg)] \geq 0$ in D^-, where

$$D_\mu(\ \ ) = \sqrt{-K}\,(\ \ )_x + (\ \ )_y$$

(3.19) $16K^2(c - a_x - b_y) - (K' + 2Kh)(K' + 2Kg) \leq 0$ on Γ_L

Then there exist functions $p(x,y)$, $q(x,y)$ and $r(x,y)$ and a positive constant A, independent of w, such that

(3.20) $$(Mw,\ pw_x + qw_y + rw) \geq A||w||_1^2 \quad ,$$

where

(3.21) $$||w||_1^2 = \iint_{D^-} (\sqrt{-K}\,w_x + w_y)^2\,dx\,dy + \iint_{D^+} (Kw_x^2 + w_y^2)\,dx\,dy \quad .$$

<u>Proof</u>: Equation 3.14 has been shown to hold for the selections $p = \sqrt{-K}\,q$ in D^-, $q = 4Kr/(K' + 2Kg)$ in D^- and $p = q = 0$ in D^+. We now select $r = -1$ and consider each of the integrals appearing on the right hand side of Equation 3.14.

In view of Condition 3.17 we have

$$J_1 \geq \delta \iint_{D^-} (\sqrt{-K}\,w_x + w_y)^2\,dx\,dy \quad .$$

For J_2, we immediately have

$$J_2 = 2\iint_D (Kw_x^2 + w_y^2)\,dx\,dy \quad .$$

We now observe that the remaining integrals are non-negative. Condition 3.18 implies $J_3 \geq 0$. Condition 3.15 implies $J_4 \geq 0$. Conditions 3.16 and 3.19 together imply $J_6 \geq 0$. To show that $J_5 \geq 0$, we recall that $w = 0$ and consequently $dw = w_x\, dx + w_y\, dy = 0$ along $C_l + C_r$. Therefore, $Kw_x^2 + w_y^2 = w_y^2[K(dy/dx)^2 + 1]$. However, by Equations 1.2 and 1.3, $(dy/dx)^2 < -1/K$ along C_l and C_r and so $K(dy/dx)^2 + 1 > 0$. Moreover, Equations 1.2 and 1.3 imply $dy - \sqrt{-K}\, dx > 0$ along $C_l + C_r$. Hence the integrand in J_5 is non-negative and so $J_5 \geq 0$.

In view of the above results, we may replace Equation 3.14 with the inequality

$$2(Mw,\ pw_x + qw_y + rw) \geq \delta \iint_{D^-} (\sqrt{-K}\, w_x + w_y)^2\, dx\, dy + 2\iint_{D^+} (Kw_x^2 + w_y^2)\, dx\, dy \quad .$$

Hence, Inequality 3.21 is satisfied by any A such that $0 < A \leq \min(\delta/2, 1)$.

Lemma 1 immediately yields a uniqueness theorem for the adjoint problem.

<u>Theorem 2</u>: Let the coefficients K, a, b and c satisfy the conditions of Lemma 1. Then, for each continuous function g, there is at most one function $w \in W$ satisfying $Mw = g$.

<u>Proof</u>: Suppose there exist two functions w_1 and w_2

in W such that $Mw_1 = g$ and $Mw_2 = g$. Let $v = w_1 - w_2$. Clearly $v \epsilon W$ and satisfies $Mv = 0$. We observe by Lemma 1 that

$$\iint_{D^-} (\sqrt{-K}\, v_x + v_y)^2\, dx\, dy + \iint_{D^+} (Kv_x^2 + v_y^2)\, dx\, dy = 0 \quad .$$

Since each of the above integrands is clearly non-negative, we conclude that they must be zero. Thus $\sqrt{-K}\, v_x + v_y \equiv 0$ in D^- and $Kv_x^2 + v_y^2 \equiv 0$ in D^+. We thus find that $v \equiv$ constant in D^+. But since $v = 0$ on C_o and is continuous we have that $v \equiv 0$ in D^+. The condition $\sqrt{-K}\, v_x + v_y \equiv 0$ implies that $v =$ constant along any characteristic satisfying $dy/dx = \sqrt{-K}$. Furthermore, as $y \to 0^-$, $\sqrt{-K}\, v_x + v_y \to v_y$ and since v_y is continuous we find $v_y(x,0) = 0$. Consequently $v = 0$ on the x-axis. Because of the fact that the characteristics with slope $dy/dx = \sqrt{-K}$ terminate either on C_1, C_r or the x-axis we may conclude that $v \equiv 0$ in D^-. Finally, since v is continuous in the closure of D, we conclude that $v \equiv 0$ on Γ_1 and consequently in D closure. Thus $w_1 \equiv w_2$ and the uniqueness is proved.

We now prove two additional lemmas which are needed in the proof of the existence theorem.

<u>Lemma 2</u>: There exists a positive constant A_1, independent of w, such that

$$||w||_1^2 \geq A_1 ||w||^2$$

holds for all $w \in W$, where the norms $||w||$ and $||w||_1$ are given by Equations 2.1 and 3.21, respectively.

<u>Proof</u>: Let $\psi(y) = \int_0^y \sqrt{-K}\, dy$. Let x_l' and x_r' be the points of intersection of the characteristics Γ_l and Γ_r, respectively, with the x-axis. The arcs Γ_l and Γ_r are then given by

(3.23) Γ_l: $x = -\psi(y) + x_l'$

(3.24) Γ_r: $x = \psi(y) + x_r'$.

We consider the identity

(3.25) $$w(x,y) = \int_{(x_0,0)}^{(x,y)} w_x\, dx + w_y\, dy + w(x_0,0) \quad ,$$

where the path of integration is any smooth arc in D from $(x_0,0)$ to (x,y). If $(x,y) \in D^-$, we select as the path of integration the characteristic arc

(3.26) $x = \psi(y) + x_0$.

Along any such arc, $dx = \sqrt{-K}\, dy$ and Equation 3.25 may be rewritten as

$$w(x,y) = \int_0^y (\sqrt{-K}\, w_x + w_y)\, dy + w(x_0,0) \quad .$$

We now square both sides of the above equation and apply the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ to obtain

$$w^2(x,y) \leq 2\left[\int_0^y (\sqrt{-K}\, w_x + w_y)\, dy\right]^2 + 2\, w^2\,(x_0,0) \quad .$$

We next apply Schwarz's inequality to the above integral to get

$$(3.27) \quad w^2(x,y) \leq 2 \int_0^y 1^2\, dy \int_0^y (\sqrt{-K}\, w_x + w_y)^2\, dy$$

$$+ 2\, w^2(x_0,0) \quad .$$

Let $m = \sup|y|$ in D^-. Let $y_B(x,y)$ be the ordinate of the point of intersection of the curve of integration 3.26, and the characteristic Γ_1, 3.23. A simple calculation yields

$$y_B(x,y) = \psi^{-1}\left[\frac{x_1' + \psi(y) - x}{2}\right]$$

where we have used the fact that $x_0 = x - \psi(y)$. We then replace Inequality 3.27 with

$$w^2(x,y) \leq 2m \int_{y_B(x,y)}^{0} (\sqrt{-K}\, w_x + w_y)^2\, dy$$

$$+ 2\, w^2(x - \psi(y),0) \quad .$$

We next integrate this inequality with respect to x from $x = -\psi(y) + x_1'$ to $x = \psi(y) + x_r$. We find

$$(3.28)\quad \int_{-\psi(y)+x_1'}^{\psi(y)+x_r'} w^2(x,y)\,dx$$

$$\leq 2m \int_{-\psi(y)+x_1'}^{\psi(y)+x_r'} dx \int_{y_B(x,y)}^{0} (\sqrt{-K}\, w_x + w_y)^2\, dy$$

$$+2 \int_{-\psi(y)+x_1'}^{\psi(y)+x_r'} w^2(x-\psi(y),0)\,dx \quad .$$

We now observe that the first integral on the right hand side of Inequality 3.28 may be replaced by the double integral

$$\iint_{D(y)^-} (\sqrt{-K}\, w_x + w_y)^2\, dx\, dy \quad ,$$

where $D(y)^- \subset D^-$. In the second integral, we make the change of variables $z = x - \psi(y)$, $dz = dx$. This integral then becomes

$$\int_{x_1'-2\psi(y)}^{x_r'} w^2(z,0)\,dz \quad .$$

We then may replace Inequality 3.28 with

$$\int_{-\psi(y)+x_l'}^{\psi(y)+x_r'} w^2(x,y)\,dx \;\leq\; 2m \iint_{D^-} (\sqrt{-K}\, w_x + w_y)^2\,dx\,dy$$

$$+2\int_{x_l}^{x_r} w^2(x,0)\,dx \quad .$$

We now integrate this inequality with respect to y from y = -m to y = 0, noting that the right side is a constant. We find

$$(3.29) \quad \iint_{D^-} w^2(x,y)\,dx\,dy \;\leq\; 2m^2 \iint_{D^-} (\sqrt{-K}\, w_x + w_y)^2\,dx\,dy$$

$$+2m\int_{x_l}^{x_r} w^2(x,0)\,dx \quad .$$

We next consider the identity

$$w(x,y) = -\int_y^{\infty} w_y\,dy \quad .$$

For $y \geq 0$, we square both sides of this identity and apply Schwarz's inequality to the result to obtain

$$w^2(x,y) \leq n\int_0^\infty w_y^2\, dy \quad ,$$

where $n = \sup y$ in D^+. We then integrate both sides of this inequality with respect to x from $-\infty$ to $+\infty$ to obtain

$$(3.30) \quad \int_{-\infty}^{\infty} w^2(x,y)\, dx \leq n\iint_{D^+} w_y^2\, dx\, dy$$

$$\leq n\iint_{D^+} (Kw_x^2 + w_y^2)\, dx\, dy \quad .$$

In particular, if $y = 0$, Inequality 3.30 becomes

$$(3.31) \quad \int_{x_1}^{x_r} w^2(x,0)\, dx \leq n\iint_{D^+} (Kw_x^2 + w_y^2)\, dx\, dy \quad .$$

We next integrate 3.30 with respect to y from $y = 0$ to $y = n$, observing that the right hand side is a constant. We find

$$(3.32) \quad \iint_{D^+} w^2(x,y)\, dx\, dy \leq n^2\iint_{D^+} (Kw_x^2 + w_y^2)\, dx\, dy \quad .$$

Finally, we substitute from Inequality 3.31 into 3.29 and add the resulting inequality to 3.32 to obtain

$$\iint_D w^2\,dx\,dy \le 2m^2 \iint_{D^-} (\sqrt{-K}\,w_x + w_y)^2\,dx\,dy$$

$$+ (2mn + n^2) \iint_{D^+} (Kw_x^2 + w_y^2)\,dx\,dy$$

$$\le ||w||_1^2/A_1 \quad ,$$

where A_1 is any constant satisfying

$$1/A_1 \ge \max(2m^2,\ 2mn + n^2) \quad .$$

Lemma 3: Let the coefficients a, b, c and K satisfy the conditions of Lemma 1. Then, there exists a positive constant A_2, independent of w, such that $||Mw|| \ge A_2||w||$ for all $w \in W$.

Proof: From Lemma 2, we may write

$$||w||_1^2 \ge \tfrac{1}{2}\left[||w||_1^2 + A_1\ ||w||^2\right] \ge A_3\left[||w||_1^2 + ||w||^2\right] \quad ,$$

where $2A_3 = \min(1,A_1)$. In Lemma 1 we found that

$$(Mw,\ pw_x + qw_y + rw) \ge A||w||_1^2 \quad .$$

If we combine these last two inequalities we obtain

$$(3.33)\quad (Mw,\ pw_x + qw_y + rw) \ge AA_3\left[||w||_1^2 + ||w||^2\right] \quad .$$

However, by Schwarz's inequality,

$$(3.34)\quad (Mw,\ pw_x + qw_y + rw) \ \leq\ ||Mw||\cdot||pw_x + qw_y + rw|| \ .$$

Moreover, the triangle inequality implies

$$(3.35)\quad ||pw_x + qw_y + rw|| \ \leq\ ||pw_x + qw_y|| + ||rw||$$

$$= \left[\iint_{D^-} \{-4K(\sqrt{-K}\ w_x + w_y)/(K' + 2Kg)\}^2 dx\ dy\right]^{\frac{1}{2}} + ||w||$$

$$\leq\ A_4\left[||w||_1 + ||w||\right] \quad ,$$

where $A_4 \ = \ \sup\left[-4K/(K' + 2Kg), 1\right]$ in D^-.

We substitute from Inequality 3.35 into Inequality 3.34 to obtain

$$(3.36)\quad (Mw,\ pw_x + qw_y + rw) \ \leq\ A_4||Mw||\cdot\left[||w||_1 + ||w||\right] \ .$$

We now combine Inequalities 3.33 and 3.36. We find

$$(3.37)\quad A_4||Mw||\left[||w||_1 + ||w||\right] \ \geq\ AA_3\left[||w||_1^2 + ||w||^2\right] \ .$$

Moreover, $||w||_1^2 + ||w||^2 \ \geq\ \frac{1}{2}\left[||w||_1 + ||w||\right]^2$.

Consequently, Inequality 3.37 may be replaced by

$$A_4||Mw||\left[||w||_1 + ||w||\right] \ \geq\ \frac{1}{2}\ AA_3\left[||w||_1 + ||w||\right]^2 \ .$$

For $w \neq 0$, we divide out the common factor $||w||_1 + ||w||$ to obtain $||Mw|| \geq A_2 \left[||w||_1 + ||w|| \right] \geq A_2||w||$,

where $A_2 = AA_3/2A_4$. If $w \equiv 0$, $||Mw|| = A_2|| \; ||$

We next state as a lemma a well known theorem from Hilbert space theory (Taylor (8, p. 245)).

Lemma 4: If G is a continuous linear functional defined over a Hilbert space H, then there exists a unique $y \in H$ such that $G(x) = (x,y)$ for each x in H.

We now state and prove our existence theorem.

Theorem 3: Let the coefficients K, a, b and c satisfy the conditions of Lemma 1. Then, for every $g \in H$, there exists a $u \in H$ which is a weak solution of $Lu = g$ under the boundary condition 1.6.

Proof: We consider the class of functions MW which consists of all functions of the form Mw where $w \in W$. Since W is a linear manifold and M is a linear operator, MW is a linear manifold (Taylor (8, p. 18)). Moreover, by Condition 2.4, $MW \subset H$.

For each $g \in H$ we define a linear functional G on MW by $G(Mw) = (w,g)$. From Theorem 2 we have that G is single valued and from Lemma 3 we have that $w \in H$. By Schwarz's inequality and Lemma 3 we have that

$$|G(Mw)| = |(w,g)| \leq ||w|| \cdot ||g|| \leq A_2||g|| \cdot ||Mw||$$

Hence, for fixed $g \in H$, G is a continuous linear functional over the linear manifold MW. Consequently, by the Hahn-Banach theorem (Taylor (8, p. 186)), there exists a continuous linear functional G_1 defined on H such that $G_1(x) = G(x)$ if $x \in MW$.

We now apply Lemma 4 to G_1. In particular, if $Mw \in MW$, there exists an element $u \in H$ such that $G(Mw) = (u, Mw)$ for all w in W. That is, there exists an element $u \in H$ such that $(w,g) = (u, Mw)$ for all $w \in W$. But this is precisely the requirement that u be a weak solution to our problem.

IV. CONCLUSION

In this final chapter we make certain observations concerning our results.

We first observe that if the coefficients of the lower order terms in Equation 1.1 vanish for $y \leq 0$, the conditions 3.15 through 3.19 of Lemma 1 reduce to

(4.1) $2c - a_x - b_y \leq 0$ in D^+

(4.2) $K' > 0$ in D^-

(4.3) $2 \frac{d}{dy}\left[\frac{K}{K'}\right] + 1 \geq \delta > 0$ in D^- .

The above set of conditions was imposed by Berezanskiĭ (2,3) in his investigation of Equation 1.1. In addition, he states that Condition 4.3 can be violated on a finite number of y intervals provided $K''' \leq 0$ on such intervals.

In Theorem 2 we established a uniqueness theorem for the adjoint problem. A uniqueness theorem has previously been given for Equation 1.1 in a paper by Agmon, Nirenberg and Protter (1). Their result is based on a maximum principle for hyperbolic equations. They consider a region bounded by the arcs C_o, Γ_1 and Γ_r. Their conditions for uniqueness, when applied to the adjoint problem, are

(4.4) $K' + 2Kg \geq 0$ in D^-

(4.5) $D_\lambda [(K' + 2Kg)/K] + (K' + 2Kg)(K' + 2Kh)/(4K^2)$

$+ 4(c - a_x - b_y) \geq 0$ in D^-

(4.6) $c - a_x - b_y \leq 0$ in D .

In the special case $a = b = c = 0$, these conditions become

(4.7) $K' \geq 0$ in D^-

(4.8) $4 \frac{d}{dy} \left[\frac{K}{K'}\right] + 1. \geq 0$.

Hence their conditions are more restrictive than those required for our uniqueness theorem.

Protter (7) has established a uniqueness theorem for Equation 1.1 using methods similar to those used in this thesis in a region bounded by C_0, Γ_1 and Γ_r for the special case where $a = b = c = 0$ in D. His theorem requires that $K''' \leq 0$ whenever Condition 4.3 is not satisfied. Thus his results are stronger than those established in this thesis.

We conclude by giving two examples of mixed type equations.

<u>Example 1</u>: We consider the equation

$$yu_{xx} + u_{yy} - \frac{1}{18} y^2 u_x + \frac{1}{12} xu_y =$$

$$= 42y(A - B) + 72y(9x^2 + 9x + 4y^3) - y^2A \quad ,$$

where $A(x,y) = 9 - 9x^2 - 4y^3$ and $B(x,y) = 9(x + 1)^2 + 4y^3$.

We consider a region D bounded by the arcs C_o, Γ_1 and Γ_r given by

$$C_o: \quad 9x^2 + 4y^3 = 9$$

$$\Gamma_1: \quad x = \frac{2}{3}(-y)^{\frac{3}{2}} - 1$$

$$\Gamma_r: \quad x = -\frac{2}{3}(-y)^{\frac{3}{2}} + 1 \quad .$$

We specify $u = 0$ on $C_o + \Gamma_1$.

We observe that the coefficients of this differential equation satisfy the conditions of Lemma 1. Moreover, this boundary value problem has the solution $u = AB$. Thus the class of equations considered in this thesis is not empty.

Example 2: We consider the differential equation

$$yu_{xx} + u_{yy} - 2y^2u_x + 3xu_y$$

$$= 42y(A - B) + 72y(9x^2 + 9x + 4y^3) - 36y^2A \quad ,$$

where A, B and the region D are as given in Example 1.

The coefficients of this equation do not satisfy Condition 3.17 of Lemma 1, but the boundary value problem has the solution $u = AB$. Thus the conditions given in this thesis are only sufficient conditions for the existence of weak solutions.

V. BIBLIOGRAPHY

1. Agmon, S., Nirenberg, L. and Protter, M. H. A maximum principle for a class of hyperbolic equations and applications to equations of mixed elliptic-hyperbolic type. Comm. Pure and Applied Math. 6: 455-470. 1953.

2. Berezanskiĭ, Yu. M. Energy inequalities for some classes of equations of mixed type. Soviet Mathematics--Doklady 1: 447-451. 1960.

3. ________. Some examples of nonclassical boundary-value problems for partial differential equations. Soviet Mathematics--Doklady 1: 259-262. 1960.

4. Courant, R. and Hilbert, D. Methods of Mathematical Physics. 1st English ed. New York, N.Y., Interscience Publishers, Inc. 1953.

5. Friedrichs, K. O. Symmetric positive linear differential equations. Comm. Pure and Applied Math. 11: 333-418. 1958.

6. Morawetz, C. S. A weak solution for a system of equations of elliptic-hyperbolic type. Comm. Pure and Applied Math. 11: 315-331. 1958.

7. Protter, M. H. Uniqueness theorems for the Tricomi problem. II. J. Rational Mech. Anal. 4: 721-732. 1955.

8. Taylor, Angus E. Introduction to Functional Analysis. New York, N.Y., John Wiley and Sons, Inc. c1958.

VI. ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. C. G. Maple for his helpful suggestions and encouragement during the preparation of this thesis.