Slot-specific Priorities with Capacity Transfers*

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Abstract

In many real-world matching applications, there are restrictions for institutions either on priorities of their slots or on the transferability of unfilled slots over others (or both). Motivated by the need in such real-life matching problems, this paper formulates a family of practical choice rules, *slot-specific priorities with capacity transfers* (SSPwCT). These practical rules invoke both slot-specific priorities structure and transferability of vacant slots. We show that the *cumulative offer mechanism* (COM) is *stable*, *strategy-proof* and *respects improvements* with regards to SSPwCT choice rules. Transferring the capacity of one more unfilled slot, while all else is constant, leads to *strategy-proof Pareto improvement* of the COM. Following *Kominer*'s (2020) formulation, we also provide comparative static results for expansion of branch capacity and addition of new contracts in the SSPwCT framework. Our results have implications for resource allocation problems with diversity considerations.

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1 Introduction

The slot-specific priorities framework of *Kominers and Sönmez* (2016) is an influential many-to-one matching model with contracts. Each contract is between an agent and an institution, and specifies some terms and conditions. Slots have their own (potentially independent) rankings for contracts. Within each institution, a linear order – referred to as the *precedence order* – determines the sequence in which slots are filled. This framework provides a powerful tool for market designers and policymakers to handle diversity and affirmative action. Institutions choose contracts by filling their slots sequentially. An agent might have different priority at different slots of the same institution . In the context of cadet-branch matching (*Switzer and Sönmez*, 2013 and *Sönmez*, 2013), for example, some slots for each service branch grant higher priority for cadets who are willing to serve additional years of service. *Kominers and Sönmez* (2016) develop a general framework to handle these types of slot-specific priority structures. The slot-specific priorities framework embeds the settings of the following works among others: *Balinski and Sönmez* (1999), *Abdulkadiroğlu and Sönmez* (2003), *Kojima* (2012), *Hafalir et al.* (2013), and *Aygün and Bó* (2020), *Switzer and Sönmez* (2013), and *Sönmez* (2013).

The slot-specific priorities offer flexible solutions to important real-world matching problems. *Aygün and Bó* (2020), for example, design slot-specific priorities choice rules for the Brazilian college admission problem, where students have multidimensional privileges. In 2012 Brazilian public universities were mandated to use affirmative action policies for candidates from racial and income minorities. The law established that certain fractions of the accepted students in each program should have studied in public high schools, come from a low-income family, and/or belong to racial minorities. This objective was implemented by partitioning the positions in each program, earmarking them for different combinations of these affirmative action characteristics. *Aygün and Bó* (2020) analyze several of these choice rules, present their shortcomings and correct their shortcomings by designing slot-specific priorities choice rules for programs.

More recently, *Pathak et al.* (2020a) invoke the slot-specific priorities framework to design triage protocol for ventilator rationing.[1] The authors analyze the consequences of using a reserve system—in which ventilators are partitioned into multiple categories—for rationing medical resources. The authors propose sequential reserve matching rules, which are first introduced in the slot-specific framework of *Kominers and Sönmez* (2016).

When slot priorities are restricted due to institutional constraints, the full potential of the slot-specific priorities framework might not be achieved. Another useful tool that allows interactions across slots is *capacity transfers* —introduced by *Westkamp* (2013). Capacity transfers allow more general forms of interactions across slots than slot specific priorities do. However, *Westkamp* (2013) does not allow the variation in contractual terms. For applications such as cadet-branch matching (*Switzer and Sönmez*, 2013 and *Sönmez*, 2013), airline upgrade allocation (*Kominers and Sönmez*, 2016), and admissions for publicly funded educational institutions and government-sponsored jobs

[1] The most recent version of *Pathak et al.* (2020a) considers more general framwork. See *Pathak et al.* (2020b).

in India (*Aygün and Turhan*, 2017, 2020a, 2020b), allowing the variation in contractual terms is necessary.

In many real-world applications, either slot priorities (for some slots if not all) or the ability to transfer vacant slots may be restricted due to institutional constraints especially in diversity and affirmative action in school choice, college admissions, government-sponsored job matching, and also faculty hiring. When both are restricted, some slots might remain unfilled. This, in turn, might lead to a Pareto inferior outcome. A real-world example of this case can be seen in India. For admissions to publicly funded educational institutions and government-sponsored jobs in India, each institution reserves 15 percent of its slots for people from Scheduled Castes (SC), 7.5 percent for people from Scheduled Tribes (ST), and 27 percent for people from Other Backward Classes (OBC). People who do not belong to any of these categories are referred to as members of the *General Category* (GC). The remaining slots are called *open-category* slots and are available to everyone, including those from SC, ST, and OBC. In each institution, for slots that are reserved for SC, ST and OBC, only applicants who declare they belong to these respective categories are considered. If there is low demand from either SC or ST applicants, some slots remain vacant. Vacant SC/ST slots can potentially be utilized by two ways: (1) other candidates can be made eligible for SC/ST slots, but at a lower priority than all SC/ST applicants, or (2) vacant SC/ST slots can be reverted into, say, open-category slots. Currently, none of these possibilities are allowed. Each year many SC/ST slots remain vacant.

There are instances where slot priorities are restricted but transferability is allowed. *Baswana et al.* (2018) designed and implemented a new joint seat allocation process for technical universities in India. Since 2008, following a Supreme Court decision, unfilled OBC slots are required to be made available to GC candidates of publicly funded educational institutions. However, institutions are *prohibited* from modifying the priorities of OBC slots. This possibility was offered to Indian authorities but ultimately rejected. On the other hand, reverting vacant OBC slots into GC slots is allowed. *Baswana et al.* (2018) report their interaction with the Indian policy makers as follows:

> "Business rule 5 required unfilled OBC seats to be made available to Open category candidates. *The approach we initially suggested involved construction of augmented Merit Lists making Open category candidates eligible for OBC seats but at a lower priority than all OBC candidates*, and modification of virtual preference lists so that general candidates now apply for both the OPEN and the OBC virtual programs. We showed that running our algorithm on these modified inputs would produce the candidate optimal allocation satisfying the business rules. However, *the authorities feared that this approach may cause issues* with computing the closing rank correctly (see Design Insight 6), or have some other hidden problem. An authority running centralized college or school admissions is typically loathe to modify, add complexity to, or replace software that is tried and tested."

In this paper, we invoke both of these powerful tools –i.e., capacity transfers across slots and independent slot priorities– to formulate a larger set of practical choice rules. Our aim is to expand the

toolkit of market designers to be able to implement comprehensive selection criteria, especially when there are institutional restrictions. We construct institutional choices as follows: Each institution has two types of slots: *original* slots and *shadow* slots. Each shadow slot has an initial capacity of 0 and is associated with an original slot. Each slot (original and shadow) has a linear order (potentially independent) over contracts. Institutions have precedence orders over original and shadow slots. Each institution is also endowed with a *location vector* for shadow slots. The exact order at which slots are processed is determined by the precedence orders over original and shadow slots together with the location vector. The interaction between associated original and shadow slots is as follows: If an original slot is assigned a contract, then the capacity of the associated shadow slot remains 0, i.e., the shadow slot will be inactive. If an original slot cannot be filled, the institution has the option to transfer its capacity to the associated shadow slot. In this case, the capacity of the shadow slot becomes 1. A capacity transfer vector of the institution determines for which original slots such reversion is allowed and for which ones it is not.

When the transferability of all original slots is prohibited, our model reduces to that of *Kominers and Sönmez* (2016). Given the exact precedence order (i.e., precedence orders over original and shadow slots, location vector of shadow slots), capacity transfer vector, and slot priorities over contracts, slots are filled *sequentially* in a straightforward manner. We call this family of choice rules that are constructed this way *Slot-specific Priorities with Capacity Transfers Choice Rules* (SSPwCT). We show how markets with SSPwCT choice rules can be cleared by the COM. To do so, we borrowed the hidden substitutes theory of *Hatfield and Kominers (2019).* The COM is the unique stable and strategy-proof mechanism in the SSPwCT environment (Theorem 1) that also respects improvements (Theorem 2).

Finally, we provide several comparative static regarding the outcome of the COM with respect to SSPwCT choice rules. We first show that when an institution reverts one more original slot into a shadow slot in the case of a vacancy, if all else is fixed, we obtain a strategy-proof Pareto improvement under the COM (Theorem 3). Then, building on *Kominers'* (2020) analysis, we show that the outcome of the COM is (weakly) improved for all agents when (1) an original slot is added to an institution, while all else remains fixed (Theorem 4), (2) new contracts are added at the bottom of slots' priority orders right before the null contract (Theorem 5), and (3) the new contracts of a single agent are added anywhere in the slots' priority orders (Theorem 6).

SSPwCT choice rules may be used in cadet-branch matching in USMA and ROTC (*Sönmez and Switzer,* 2013 and *Sönmez,* 2013), resource allocation in India with comprehensive affirmative action (*Boswana et al.,* 2019, *Aygün and Turhan,* 2020a and 2020b), Chilean school choice and college admissions with affirmative action (*Rios et al.,* 2018 and *Correa et al., 2019*) , and Brazilian college admissions with multidimensional reserves (*Aygün and Bó*, 2020).

Related Literature

Our model is built on that of *Kominers and Sönmez* (2016). The SSPwCT choice rules expands the slot-specific priorities choice rules of *Kominers and Sönmez* (2016) by allowing transferability

of vacant slots. When transferability is not allowed, the SSPwCT choice rules are equivalent to the slot-specific priorities choice rules. As opposed to *Kominers and Sönmez* (2016), in the presence of capacity transfers, it is not possible to define associated one-to-one model of agent-slot market in our setting. We instead invoke *Hatfield and Kominers'* (2019) *hidden substitutes* theory to show that the COM with respect to SSPwCT choice rules is stable and strategy-proof.

Recently, *Aygün and Turhan* (2020) show that each slot-specific priorities choice rule can be written as a *dynamic reserves choice rule*. The family of SSPwCT choice rules embeds *Aygün and Turhan*'s (2020) dynamic reserves choice rules family. There are two important advantages of the SSPwCT rules over the dynamic reserves choice rules. The first one is that the SSPwCT functions are simpler to describe. The capacity functions in the setting of dynamic reserves are complicated, whereas transfer functions in the SSPwCT environment is just a vector of binary variables. The second one is that in the dynamic reserves setting there is a master priority ordering for each institution, and the priority orderings of different groups of slots within the institution are obtained from the master priority ordering in a straightforward manner by removing some agents. In the current paper, following the model of *Kominers and Sönmez* (2016), slots have more general priorities (possibly independent) over contracts.

Kominers (2020) gives a novel proof of the entry comparative static via the respecting improvement property, and also shows that his proof extends to yield comparative static results in matching with slot-specific priorities framework. By adapting his notation and formulation, we further extend his results to the SSPwCT environment. Papers that study entry comparative static include *Kelso and Crawford* (1982), *Gale and Sotomayor* (1985), *Crawford* (1991), *Hatfield and Milgrom* (2005), *Biró et al.* (2008), *Ostrovsky* (2008), and *Chambers and Yenmez* (2017), among others.

Our paper uses the matching with contracts framework that is introduced by *Fleiner* (2003) and *Hatfield and Milgrom* (2005). Important work on matching with contracts, among many others, include *Hatfield and Kojima* (2010), *Aygün and Sönmez* (2013), *Afacan* (2017), *Hatfield and Kominers (2019)*, and *Hatfield et al.* (2017, 2019).

We study a family of lexicographic choice rules with transfers. Lexicographic choice rules (without transfers) are recently studied in *Chambers and Yenmez* (2018) and *Doğan et al.* (2018), among others.

Finally, our paper also contributes the matching literature on distributional constraints. There is extensive literature on this line of literature. A partial list includes *Abdulkadiroğlu and Sönmez* (2003), *Abdulkadiroğlu* (2005), *Biró et al.* (2010), *Kojima* (2012), *Hafalır et al.* (2013), *Westkamp* (2013), *Echenique and Yenmez* (2015), *Kamada and Kojima* (2015, 2017, and 2018), *Fragiadakis and Troyan* (2017), *Aygün and Turhan (2017, 2020a, 2020b), Jagadeesan* (2019), *Nguyen and Vohra* (2019), *Afacan* (2020), and *Dur et al.* (2020).

2 Matching with Contracts Framework

There is a finite set of *agents* $\mathcal{I} = \{i_1, ..., i_n\}$ and a finite set of *branches* $\mathcal{B} = \{b_1, ..., b_m\}$. There is a finite set of *contracts* $\mathcal{X}$. Each contract $x \in \mathcal{X}$ is associated with an agent $\mathbf{i}(x)$ and a branch $\mathbf{b}(x)$. There may be many contracts for each agent-branch pair. We call a set of contracts $X \subseteq \mathcal{X}$ an *outcome*, with $\mathbf{i}(X) \equiv \underset{x \in X}{\cup}\{\mathbf{i}(x)\}$ and $\mathbf{b}(X) \equiv \underset{x \in X}{\cup}\{\mathbf{b}(x)\}$. For any $i \in \mathcal{I}$, we let $X_i \equiv \{x \in X \mid \mathbf{i}(x) = i\}$. Similarly, for any $b \in \mathcal{B}$, we let $X_b \equiv \{x \in X \mid \mathbf{b}(x) = b\}$.

Each agent $i \in \mathcal{I}$ has unit demand over contracts in $\mathcal{X}_i \equiv \{x \in \mathcal{X} \mid \mathbf{i}(x) = i\}$ and an outside option $\emptyset_i$. The strict preference of agent i over $\mathcal{X}_i \cup \{\emptyset_i\}$ is denoted by P_i. A contract $x \in \mathcal{X}_i$ is *acceptable for* i (with respect to P_i) if $xP_i\emptyset_i$. Agent preferences over contracts are naturally extended to preferences over outcomes. For each individual $i \in \mathcal{I}$ and set of contracts $X \subseteq \mathcal{X}$, we denote by $\underset{P_i}{max}X$ the maximal element of X according to preference order P_i, and we assume that $\underset{P_i}{max}X = \emptyset$ if $\emptyset_i P_i x$ for all $x \in X$. Each individual always chooses the best available contract according to his preferences, so that choice rule $C^i(X)$ of an individual $i \in \mathcal{I}$ from contract set $X \subseteq \mathcal{X}$ is defined by $C^i(X) = \underset{P_i}{max}X$.

Each branch $b \in \mathcal{B}$, on the other hand, has multi-unit demand and is endowed with a choice rule C^b. We let q_b denote the capacity of branch b. The choice rule C^b describes how branch b would choose from any offered set of contracts. We assume throughout that for all $X \subseteq \mathcal{X}$ and for all $b \in \mathcal{B}$, the choice rule C^b satisfies (1) $C^b(X) \subseteq X_b$, and (2) $C^b(X)$ is feasible in the sense that it contains at most one contract of any given agent. For any $X \subseteq \mathcal{X}$ and $b \in \mathcal{B}$, we denote by $R^b(X) \equiv X \setminus C^b(X)$ the set of contracts that b *rejects* from X.

A set of contracts $X \subseteq \mathcal{X}$ is a *feasible outcome* if $\mid X_i \mid \leq 1$, for every $i \in \mathcal{I}$, and $\mid X_b \mid \leq q_b$, for each $b \in \mathcal{B}$.

A feasible outcome $X \subseteq \mathcal{X}$ is **stable** if

1. $C^i(X) = X_i$ for all $i \in \mathcal{I}$, and $C^b(X) = X_b$, for all $b \in \mathcal{B}$, and
2. there is no $b \in \mathcal{B}$ and set of contracts $Y \neq C^b(X)$ such that $Y_b = C^b(X \cup Y)$ and $Y_i = C^i(X \cup Y)$ for all $i \in \mathbf{i}(Y)$.

The first condition is known as *individual rationality* and requires that no agent or branch would rather drop one of the signed contracts. The second condition is the *no blocking* requirement. If fails, then there is an alternative set of contracts that a branch and agents associated with a contract in that set strictly prefers.

2.1 Slot-specific Priorities with Capacity Transfers (SSPwCT) Choice Rules

Each branch $b \in \mathcal{B}$ has two types of seats: **original** seats and **shadow** seats. Let $O_b = \{o_b^1, o_b^2, ..., o_b^{n_b}\}$ and $E_b = \{e_b^1, e_b^2, ..., e_b^{n_b}\}$ be branch b's set of original seats and shadow seats, respectively, where n_b denotes the physical capacity of branch b. Each seat in both O_b and E_b has priority orders over contracts in $\mathcal{X}_b \cup \{\emptyset\}$ denoted by Π_b^o for $o \in O_s$ and Π_b^e for $e \in E_s$ (the weak orders are denoted by Γ^o and Γ^e) and can be assigned at most one contract in $\mathcal{X}_b \equiv \{x \in \mathcal{X} \mid \mathbf{b}(x) = b\}$. Let $\Pi_b = (\Pi_b^o, \Pi_b^e)$

denote the priority profile of branch b and $\Pi = (\Pi_b)_{b\in\mathcal{B}}$ denote the priority profiles of all branches. We denote by $\underset{\pi^o}{max}X$ the maximal element of X according to priority ordering Π^o and by $\underset{\pi^e}{max}X$ the maximal element of X according to priority ordering Π^e. We assume $\underset{\pi^o}{max}X = \emptyset$ if $\emptyset\Pi^o x$ for all $x \in X$ and $\underset{\pi^e}{max}X = \emptyset$ if $\emptyset\Pi^e x$ for all $x \in X$.

Each branch $b \in \mathcal{B}$ has two linear precedence orders, one over original seats, $\rhd_b^O$, and one over shadow seats, $\rhd_b^E$. We denote $O_b = \{o_b^1, o_b^2..., o_b^{n_b}\}$ with $o_b^l \rhd_b^O o_b^{l+1}$ and $E_b = \{e_b^1, e_b^2..., e_b^{n_b}\}$ with $e_b^l \rhd_b^E e_b^{l+1}$ unless otherwise stated. The interpretation of $\rhd_b^O$ is that if $o \rhd_b^O o'$, then, whenever possible, branch b fills seat o before $o^{'}$. Each shadow seat is associated with an original seat. If the original seat remains empty, then branch b can decide whether to transfer its capacity to its associated shadow seat, which initially has no capacity, through a capacity transfer scheme q_b defined below.

A *capacity transfer scheme* is an integer-valued vector $q_b = (q_b^1, q_b^2, ..., q_b^{n_b})$ such that for every $k = 1, ..., n_b$:

$$q_b^k = \begin{cases} 0 & \text{if branch } b \text{ does not transfer capacity from } o_b^k \text{ to } e_b^k \text{ when } o_b^k \text{ is not filled.} \\ 1 & \text{if branch } b \text{ transfers capacity from } o_b^k \text{to } e_b^k \text{ when } o_b^k \text{ is not filled.} \end{cases}$$

Since a capacity transfer from o_b^k to e_b^k is possible *only when* o_b^k is not filled, the physical capacity of branch b is never violated. We define an *indicator function* for the original seats as follows:

$$\mathbf{1}_{o_b^l} = \begin{cases} 0 & \text{if seat } o_b^l \text{ remains empty.} \\ 1 & \text{if seat } o_b^l \text{ is filled.} \end{cases}$$

Given precedence orders $\rhd_b^O$ and $\rhd_b^E$, a *location vector* for the shadow seats of branch b is an integer-valued vector $L_b = (l_1, ..., l_{n_b})$ where l_k is the number of original seats that precede shadow seat e_b^k that satisfy the following condition:

$$L_b = \{(l_1, ..., l_{n_b}) \mid k \leq l_k \leq n_b \; \forall k = 1, ..., n_b \text{ and } l_k \geq l_{k-1} \; \forall k = 2, ..., n_b\}.$$

The condition in the definition of L_b ensures that for every shadow seat, the number of preceding original seats is greater than the number of preceding shadow seats. Hence, a shadow seat will never come before its associated original seat in this order. The location vector L_b together with precedence orders $\rhd_b^O$ and $\rhd_b^E$ gives us the exact order in which the original and shadow seats are filled. Let $\rhd_b \equiv (\rhd_b^O, \rhd_b^E, L_b)$ denote the exact order of branch b's slots. We illustrate this with an example below.

Example 1. Consider a branch with three original seats with $o_b^1 \rhd_b^O o_b^2 \rhd_b^O o_b^3$ and three shadow seats with $e_b^1 \rhd_b^E e_b^2 \rhd_b^E e_b^3$ together with the location vector $L_b = (1, 3, 3)$. The order $\rhd_b$ in which the original and shadow seats are filled is as follows:

Description of SSPwCT Choice Rules

For branch $b \in \mathcal{B}$, $C^b(\cdot, \triangleright_b, q_b, \Pi_b) : 2^X \to 2^X$ denotes the choice rule of branch b given the precedence order of slots $\triangleright_b$, the capacity transfer function q_b, and the priority profile of slots Π_b. Given a set of contracts $X \subseteq \mathcal{X}$, $C^b(X, \triangleright_b, q_b, \Pi_b)$ denotes the set of chosen contracts for branch b from the set of contracts X.

To formulate the choice rule, we first rename the slots as $S = (s^1, s^2, ..., s^{2n_b})$ where s^k is either an original or a shadow seat, depending on $\triangleright_b^O$, $\triangleright_{\mathrm{b}}^E$, and L_b. In Example 1 above with $L_b = (1, 3, 3)$, we can rename slots as follows: $s_b^1 = o_b^1$, $s_b^2 = e_b^1$,$s_b^3 = o_b^2$, $s_b^4 = o_b^3$, $s_b^5 = e_b^2$, and $s_b^6 = e_b^3$. It is important to note that $\Pi_b^{s_b^1} = \Pi_b^{o_b^1}$, $\Pi_b^{s_b^2} = \Pi_b^{e_b^1}$, etc...

We are now ready to describe the choice procedure. Given $X \subseteq \mathcal{X}$:

- Start with the original seat s_b^1. Assign the contract x^1 that is $\Pi_b^{s_b^1} - maximal$ among the contracts in X.
- If s_b^2 is either an original or a shadow seat such that $\mathbf{1}_{o_b^1} = 0$ and $q_b^1 = 1$, assign the contract x^2 that is $\Pi_b^{s_b^2} - maximal$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1\})}$. Otherwise, assign the empty set.
- This process continues in sequence. If s_b^k is an original seat or a shadow seat such that $\mathbf{1}_{o_b^r} = 0$, where o_b^r is the original seat that is associated with the shadow seat s_b^k, and $q_b^r = 1$, then assign contract x^k that is $\Pi_b^{s_b^k} - maximal$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1,...,x^{k-1}\})}$. Otherwise, assign the empty set.

Given n_b, $(\triangleright_b, q_b, \Pi_b)$ parametrizes the family of SSPwCT choice rules for branch b.

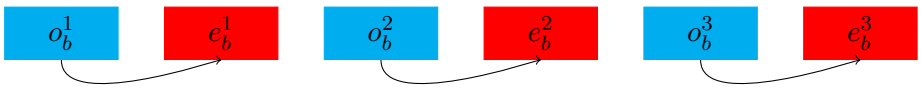
Examples of SSPwCT Choice Rules

Example 2. Consider $b \in \mathcal{B}$ with $n_b = 3$, $L_b = (1, 2, 3)$, and $q_b = (1, 1, 1)$. The capacity transfer scheme allows branch b to transfer capacities from original seats to shadow seats whenever they remain unfilled. Given the location vector and capacity transfer scheme, the choice procedure for branch b is as follows. Given an offer set $X \subseteq \mathcal{X}$:

- Assign o_b^1 the contract x^1 that is $\Pi_b^{o_b^1} - maximal$ among the contracts in X.
- If $\mathbf{1}_{o_b^1} = 0$, assign e_b^1 the contract x^2 that is $\Pi_b^{e_b^1} - maximal$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1\})}$. Otherwise, assign e_b^1 the empty set.
- Assign o_b^2 the contract x^3 that is $\Pi_b^{o_b^2} - maximal$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1,x^2\})}$.
- If $\mathbf{1}_{o_b^2} = 0$, assign e_b^2 the contract x^4 that is $\Pi_b^{e_b^2} - maximal$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1,x^2,x^3\})}$. Otherwise, assign e_b^2 the empty set.
- Assign o_b^3 the contract x^5 that is $\Pi_b^{o_b^3} - maximal$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1,x^2,x^3,x^4\})}$.

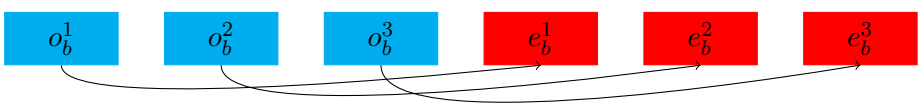
- If $\mathbf{1}_{o_b^3} = 0$, assign e_b^3 the contract x^6 that is $\Pi_b^{e_b^3} - maximal$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1,x^2,x^3,x^4,x^5\})}$. Otherwise, assign e_b^3 the empty set.

The following picture depicts the order of slots and the capacity transfer scheme where arrows indicate that the capacity is transferred if the original seat remains empty:



In the previous example each shadow seat appears right after its corresponding original seat. It is common in practice for the institution to try to fill its original seats first before the shadow seats. We provide such an example below.

Example 3. Consider branch $b \in \mathcal{B}$ with $n_b = 3$, $L_b = (3,3,3)$, and $q_b = (1,1,1)$. The following picture depicts the order of slots and the capacity transfer scheme where arrows indicate that the capacity is transferred if the original seat remains empty:



Note that for any location vector, if the capacity transfer scheme is a vector of zeros, the SSPwCT choice rules are equivalent to the slot-specific priorities choice rules in *Kominers and Sönmez* (2016).

Substitutable Completion of SSPwCT Choice Rules

A choice function C^b satisfies the **irrelevance of rejected contracts** (IRC) condition if for all $Y \subset X$, for all $z \in X \setminus Y$, and $z \notin C^b(Y \cup \{z\})$ implies $C^b(Y) = C^b(Y \cup \{z\})$. A choice function C^b satisfies **substitutability** if for all $z, z' \in X$, and $Y \subseteq X$, $z \notin C^b(Y \cup \{z\}) \implies z \notin C^b(Y \cup \{z, z'\})$. A choice function C^b satisfies the **law of aggregate demand** (LAD) if $Y \subseteq Y' \implies | C^b(Y) | \leq | C^b(Y') |$.

The following definitions are from *Hatfield and Kominers* (2019):

> A **completion** of a choice function C^b of branch $b \in \mathcal{B}$ is a choice function $\overline{C}^b$, such that for all $X \subseteq \mathcal{X}$, either $\overline{C}^b(X) = C^b(X)$ or there exists a distinct $x, x' \in \overline{C}^b(X)$ such that $\mathbf{i}(x) = \mathbf{i}(x')$. If a choice function C^b has a completion that satisfies the substitutability and *IRC* condition, then C^b is said to be **substitutably completable**. If every choice function in a profile $C = (C^b)_{b \in \mathcal{B}}$ is substitutably completable, then we say that C is **substitutably completable**.

Given the precedence order of slots $\triangleright_b$, the capacity transfer function q_b, and the priority profile of slots Π_b, and a set of contracts $X \subseteq \mathcal{X}$, we define a related choice procedure $\overline{C}^b$. To formulate this related choice rule, we first rename the slots as $S = (s^1, s^2, ..., s^{2n_b})$ where s^k is either an original or a shadow seat, depending on $\triangleright_b^O$, $\triangleright_b^E$, and L_b.

- Start with the original seat s_b^1. Assign the contract x^1 that is $\Pi_b^{s_b^1} - maximal$ among the contracts in X.
- If s_b^2 is either an original or a shadow seat such that $\mathbf{1}_{o_b^1} = 0$ and $q_b^1 = 1$, assign the contract x^2 that is $\Pi_b^{s_b^2} - maximal$ among the contracts in $X \setminus \{x^1\}$. Otherwise, assign the empty set.
- This process continues in sequence. If s_b^k is an original seat or a shadow seat such that $\mathbf{1}_{o_b^r} = 0$, where o_b^r is the original seat that is associated with the shadow seat s_b^k, and $q_b^r = 1$, then assign contract x^k that is $\Pi_b^{s_b^k} - maximal$ among the contracts in $X \setminus \{x^1, ..., x^{k-1}\}$. Otherwise, assign the empty set.

The difference between the SSPwCT choice rule C^b we defined in the main text and $\overline{C}^b$ defined above is as follows: In the computation of C^b, if a contract of an agent is chosen by some slot, then her other remaining contracts are removed for the rest of the choice procedure. On the other hand, in the computation of $\overline{C}^b$, if a contract of an agent is chosen by a slot, then her other contracts will still be available for the following slots.

The following proposition shows that $\overline{C}^b$ defined above is the completion of the SSPwCT choice rule C^b.

Proposition 1. *$\overline{C}^b$ is a completion of C^b.*

Our next results shows that $\overline{C}^b$ satisfies the IRC condition, the substitutability and the LAD.

Proposition 2. *$\overline{C}^b$ is substitutable, satisfies the IRC condition, and the LAD.*

2.2 Cumulative Offer Mechanism

A mechanism $\Psi(\cdot, C)$, where $C = (C^b)_{b \in \mathcal{B}}$ is a given profile of choice rules for branches, is a mapping from preference profiles of agents $P = (P_i)_{i \in I}$ to outcomes. A mechanism $\Psi(\cdot, C)$ is *stable* if $\Psi(P, C)$ is a stable outcome for every preference profile P. A mechanism $\Psi(\cdot, C)$ is *strategy-proof* if for every preference profile P, and for each individual $i \in \mathcal{I}$, there is no reported preference $\widetilde{P}_i$, such that

$$\Psi((\widetilde{P}_i, P_{-i}), C) P_i \Psi(P, C).$$

We now introduce the cumulative offer mechanism (COM), whose outcome is found with the following cumulative offer algorithm:

Step 1. An arbitrarily chosen agent propose her first choice contract x_1. The branch $\mathbf{b}(x_1)$ holds the contract x_1 if $C^{\mathbf{b}(x_1)}(\{x_1\}) = \{x_1\}$, and rejects it otherwise. Let $A^1_{\mathbf{b}(x_1)} = \{x_1\}$, and $A^1_b = \emptyset$ for all $b \in \mathcal{B} \setminus \{\mathbf{b}(x_1)\}$.

In general,

Step t. An arbitrarily chosen agent, for whom no branch currently holds a contract, proposes her favorite contract, call it x_t, among the ones that have not been rejected in the previous steps. The branch $\mathbf{b}(x_t)$ holds x_t if $x_t \in C^{\mathbf{b}(x_t)}\left(A^{t-1}_{\mathbf{b}(x_t)} \cup \{x_t\}\right)$ and rejects it otherwise. Let $A^t_{\mathbf{b}(x_t)} = A^{t-1}_{\mathbf{b}(x_t)} \cup \{x_t\}$, and $A^t_b = A^{t-1}_b$ for all $b \in \mathcal{B} \setminus \{\mathbf{b}(x_t)\}$.

This algorithm terminates when every agent is matched to a branch or every unmatched agent has all acceptable contracts rejected. Since there are finitely many contracts, the algorithm terminates in some finite step T. The final outcome is $\bigcup_{b\in\mathcal{B}} C^b\left(A^T_b\right)$.

Theorem 1. *The COM with respect to SSPwCT choice rules is stable and strategy-proof.*

3 Respecting Improvements

Respect for improvements is an intuitive and desirable property in practice. It suggest that agents should have no incentive to try to lower their standings in branches' priority orders. This natural property becomes crucial, especially in merit-based systems where branches' priority orderings are determined through exam scores. To formally define it in our framework, fix the precedence order $\rhd_b \equiv (\rhd^O_b, \rhd^E_b, L_b)$ and the capacity transfer function q_b of branch b.

Definition 1. We say that a choice rule $C^b(\cdot, \rhd_b, q_b, \overline{\Pi}_b)$ of branch b is an *improvement over* $C^b(\cdot, \rhd_b, q_b, \Pi_b)$ for agent $i \in \mathcal{I}$ if for all slots $s \in O_b \cup E_b$ the following conditions hold:

1. for all $x \in \mathcal{X}_i$ and $y \in \left(\mathcal{X}_{\mathcal{I}\setminus\{i\}} \cup \{\emptyset\}\right)$, if $x\Pi^s_b y$ then $x\overline{\Pi}^s_b y$; and
2. for all $y, z \in \mathcal{X}_{\mathcal{I}\setminus\{i\}}$, $y\Pi^s_b z$ if and only if $y\overline{\Pi}^s_b z$.

That is, $C^b(\cdot, \rhd_b, q_b, \overline{\Pi}_b)$ is an improvement over $C^b(\cdot, \rhd_b, q_b, \Pi_b)$ for agent i if $\overline{\Pi}_b$ is obtained from Π_b by increasing the priority of some of i's contracts while leaving the relative priority of other agents' contracts unchanged. We say that a profile of branch choices $\overline{C} \equiv (C^b(\cdot, \rhd_b, q_b, \overline{\Pi}_b))_{b\in\mathcal{B}}$ is an **improvement** over $C \equiv (C^b(\cdot, \rhd_b, q_b, \Pi_b))_{b\in\mathcal{B}}$ for agent $i \in \mathcal{I}$ if, for each branch $b \in \mathcal{B}$, $C^b(\cdot, \rhd_b, q_b, \overline{\Pi}_b)$ is an improvement over the choice rule $C^b(\cdot, \rhd_b, q_b, \Pi_b)$.

We say that a mechanism φ **respects improvements** for $i \in I$ if for any preference profile $P \in \times_{i\in I}\mathcal{P}^i$

$$\varphi_i(P; \overline{C}) R^i \varphi_i(P; C)$$

whenever $\overline{C} \equiv (C^b(\cdot, \rhd_b, q_b, \overline{\Pi}_b))_{b\in\mathcal{B}}$ is an improvement over $C \equiv (C^b(\cdot, \rhd_b, q_b, \Pi_b))_{b\in\mathcal{B}}$.

Theorem 2. *The COM with respect to SSPwCT choice rules respects improvements.*

4 Comparative Statics

In this section, we first look at the effect of increasing the transferability of original seats on agents' welfare under the COM with respect to SSPwCT choice rules. We, then, extend the comparative static results of *Kominers* (2020) to the SSPwCT family. *Kominers* (2020) provides a new proof

of the entry comparative static, by way of the respect for improvements property. The author sheds light on a strong relationship between several different entry comparative statics and the respecting improvement property in many-to-one matching markets with contracts. Building on his formulation, we analyze the effect of expanding the capacity of a single branch on agents' welfare agent-proposing under the COM in SSPwCT environment. We also examine the effect of adding contracts on agents' welfare under the COM in our setting.

4.1 Increasing Transferability

SSPwCT is a large family of choice rules. If transferability of all original slots is prohibited, we obtain the slot-specific priorities choice rules of *Kominers and Sönmez* (2016). Allowing transferability of an original slot, while everything else remains fixed, is welfare-improving for agents. We analyze the monotonicity of improvements on agents' welfare by only changing the transferability of original slots, while all else remains fixed.

Let $\widetilde{q}_b$ and q_b be two capacity transfer schemes for branch b. We say that $\widetilde{q}_b$ is **more flexible** than q_b if $\widetilde{q}_b > q_b$, i.e., $\widetilde{q}_b^k \geq q_b^k$ for all $k = 1, ..., n_b$ and $\widetilde{q}_b^l > q_b^l$ for some $l = 1, ..., n_b$. Suppose that $\rhd_b$ and Π_b are fixed. Then, the SSPwCT choice rule $C^b(\cdot, \rhd_b, \widetilde{q}_b, \Pi_b)$ can be interpreted as an expansion of the SSPwCT choice rule $C^b(\cdot, \rhd_b, q_b, \Pi_b)$ if $\widetilde{q}_b$ is more flexible than q_b. We are now ready to present our result.

Theorem 3. *Suppose that Z is the outcome of the COM at (P, C), where $P = (P_{i_1}, ..., P_{i_n})$ is the profile of agent preferences and $C = (C^{b_1}, ..., C^{b_m})$ is the profile of branches' SSPwCT choice rules. Fix a branch $b \in \mathcal{B}$. Suppose that $\widetilde{C}^b$ takes as an input capacity transfer scheme that is more flexible than that of C^b, holding all else constant. Then, the outcome of the COM at $(P, (\widetilde{C}^b, C_{-b}))$, $\widetilde{Z}$, Pareto dominates Z.*

Theorem 3 is intuitive and indicates that making an untransferable original slot of a branch transferable leads to strategy-proof Pareto improvement of the COM. One should note that expanding a branch's choice rule changes the stability definition. However, Theorem 3 provides a normative foundation for such a change, as it increases agents' welfare.[2]

4.2 Expanding Capacity

We follow the formulation of *Kominers* (2020) in this section. We look at how the COM outcome changes when an original slot is added to branch b in the SSPwCT environment. Suppose the choice rules of all branches other than b are fixed. We extend the set of original slots in branch b from O_b to $\widetilde{O}_b = O_b \cup \{\widetilde{o}\}$, where $\widetilde{o}$ is the newly added original slot. We assume there is no change in the priorities of slots in O_b. We write $\widetilde{\Pi}_b = (\Pi_b, \pi^{\widetilde{o}})$, where $\Pi_b = (\Pi_b^o, \Pi_b^e)$ is the priority profile of slots in $O_b \cup E_b$ and $\pi^{\widetilde{o}}$ is the priority ordering of new original slot $\widetilde{o}$.

[2]This result does not contradict the findings of *Alva and Manjunath* (2019), because transferring the capacity of an original slot to an associated shadow slot changes the feasible set in their context.

As pointed out by *Kominers* (2020), adding a new original slot $\widetilde{o}$ is similar to boosting the ranking of contracts of all agents that were deemed unacceptable in $\pi^{\widetilde{o}}$, keeping constant all other slots' rankings. Hence, by our Theorem 2, adding $\widetilde{o}$ leads to an improvement for all agents. We state this result formally as follows:

Theorem 4. *Let Z and $\widetilde{Z}$ be the outcomes of the COM in the markets with the set of slots $\{O_b \cup E_b\}_{b\in\mathcal{B}}$ and $\{O_b \cup E_b\}_{b\in\mathcal{B}} \cup \{\widetilde{o}_b\}$, respectively, where $\widetilde{o}_b$ is an original slot added to branch b. Then, each agent $i \in \mathcal{I}$ (weakly) prefers her assignment under $\widetilde{Z}$ to her assignment under Z.*

This result expands Theorem 2 of *Kominers* (2020) to the SSPwCT family. Our Theorems 3 and 4 together imply the following: if a new original slot and its associated shadow slot are added and transferability is allowed, then the COM outcome is improved for agents.

4.3 Adding Contracts

Kominers (2020) also shows that adding new contracts at the bottom of some slots' priority orders (that is, right before the null contract $\emptyset$) improves outcomes for all agents. We follow his terminology to extend his result to the SSPwCT family. The following formulation is adapted to the SSPwCT environment from *Kominers* (2020): Let X be an initial set of contracts and $\widetilde{X}$ is a newly added set of contracts, expanding the set to $X \cup \widetilde{X}$. Let $\widetilde{P} = (\widetilde{P}_i)_{i\in\mathcal{I}}$ and $\widetilde{\Pi} = (\widetilde{\Pi}_b)_{b\in\mathcal{B}}$ denote the vector of agent preferences and slot priorities over $X \cup \widetilde{X}$, respectively. Suppose that (1) $x\widetilde{P}_i x'$ if and only if $xP_i x'$ for all $i \in \mathcal{I}$ and (2) $x, x' \in X$, and $x\widetilde{\Pi}^s_b x'$ if and only if $x\Pi^s_b x'$ for all slots $s \in \{O_b \cup E_b\}_{b\in\mathcal{B}}$ and $x, x' \in X$. If $x\widetilde{\Pi}^s_b \widetilde{x}$ for all slots $s \in \{O_b \cup E_b\}_{b\in\mathcal{B}}$, $x \in X$, and $\widetilde{x} \in \widetilde{X}$, then $\widetilde{\Pi}$ is an *improvement* over Π under $X \cup \widetilde{X}$. Each slot $s \in \{O_b \cup E_b\}_{b\in\mathcal{B}}$ deems all the contracts in $\widetilde{X}$ as unacceptable in Π^s_b. Hence, $\widetilde{\Pi}$ can be obtained from Π by improving the ranking of contracts in $\widetilde{X}$ above the outside option. Then, our Theorem 2 implies the following result.

Theorem 5. *Let Z and $\widetilde{Z}$ be the outcomes of the COM in the markets with the set of contracts X and $X \cup \widetilde{X}$, respectively. Then, each agent $i \in \mathcal{I}$ (weakly) prefers her assignment under $\widetilde{Z}$ to her assignment under Z.*

Our Theorem 5 expands Theorem 3 of *Kominers* (2020) to the SSPwCT family.

In our final comparative static, we show that an agent receives a (weakly) better outcome in the COM when new contracts are added for this agent. The following formulation is adapted to the SSPwCT environment from *Kominers* (2020): Let $\widetilde{x}$ be a new contract that is added to the contract set X, expanding it to $X \cup \{\widetilde{x}\}$. Let $\widetilde{P} = (\widetilde{P}_i)_{i\in\mathcal{I}}$ and $\widetilde{\Pi} = (\widetilde{\Pi}_b)_{b\in\mathcal{B}}$ denote the vector of agent preferences and slot priorities over $X \cup \{\widetilde{x}\}$, respectively. Suppose (1) $x\widetilde{P}_i x'$ if and only if $xP_i x'$ for all $i \in \mathcal{I}$ and $x, x' \in X$, and (2) $x\widetilde{\Pi}^s_b x'$ if and only if $x\Pi^s_b x'$ for all slots $s \in \{O_b \cup E_b\}_{b\in\mathcal{B}}$ and $x, x' \in X$. Then, our Theorem 2 implies the following result.

Theorem 6. *Let Z and $\widetilde{Z}$ be the outcomes of the COM in the market with the set of contracts X and $X \cup \{\widetilde{x}\}$, respectively. Then, each agent $\mathbf{i}(\widetilde{x})$ (weakly) prefers her assignment under $\widetilde{Z}$ to her assignment under Z.*

This result expands Theorem 4 of *Kominers* (2020) to the SSPwCT family.

5 Conclusions

This paper introduces a practical family of SSPwCT choice rules. We show that when these choice rules are used by institutions the COM is stable, strategy-proof, and respects improvements. Moreover, we show that transferring the capacity of one more unfilled slot, if all else remains constant, leads to strategy-proof Pareto improvement of the COM. We also show that both expansion of branch capacities and adding new contracts (weakly) increase agents' welfare under the COM.

The SSPwCT choice rules expands the toolkit available to market designers and may be used in real-world matching markets to accommodate diversity concerns. We believe SSPwCT choice rules may be invoked in cadet-branch matching in USMA and ROTC, resource allocation problems in India with comprehensive affirmative action, Chilean school choice with affirmative action constraints, and Brazilian college admissions with multidimensional reserves, among many others.

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Appendices

Proof of Proposition 1. Let $\rhd_b$ the precedence order of slots, q_b be the capacity transfer function, and Π_b be the priority profile of slots. Consider an offer set $X \subseteq \mathcal{X}$. If there are distinct $x, x^{'} \in \overline{C}^b(X)$ such that $\mathbf{i}(x) = \mathbf{i}(x^{'})$, then we are done. Now, suppose that there is no pair of contracts $x, x^{'} \in \overline{C}^b(X)$ such that $\mathbf{i}(x) = \mathbf{i}(x^{'})$. We need to show that $C^b(X) = \overline{C}^b(X)$ in this case.

We prove our claim by induction on slots' indexes $k = 1, ..., 2n_b$. We show that for each slot the contracts chosen by C^b and $\overline{C}^b$ is the same. For the first slot, s_b^1, both C^b and $\overline{C}^b$ chooses the $\Pi_b^{s_b^1} - maximal$ among the contracts in X. By way of induction, assume that each of the first $k-1$ slots, $s_b^1, ..., s_b^{k-1}$, selects the same contract under choice procedures C^b and $\overline{C}^b$, respectively. Call

these contracts $x^1, ..., x^{k-1}$, with the possibility that some of x^js, $j = 1, ..., k-1$, might be the null contract $\emptyset$. We need to show that the kth slot s_b^k, selects the same contract under C^b and $\overline{C}^b$, respectively. Let x^k be the contract C^b selects at slot k among the contracts in $X \setminus X_{\mathbf{i}(\{x^1,...,x^{k-1}\})}$. That is, x^k is the $\Pi_b^{s_b^k} - maximal$ contract among contracts in $X \setminus X_{\mathbf{i}(\{x^1,...,x^{k-1}\})}$. Under $\overline{C}^b$, slot s_b^k selects $\Pi_b^{s_b^k} - maximal$ contract among $X \setminus \{x^1, ..., x^{k-1}\}$. Since $\Pi_b^{s_b^k}$ is a linear order and no contract of agents $\mathbf{i}\left(\{x^1, ..., x^{k-1}\}\right)$ is chosen at any other slot by our initial supposition, $\Pi_b^{s_b^k} - maximal$ contract among the contracts in $X \setminus \{x^1, ..., x^{k-1}\}$ is x^k. Hence, the same contract x^k is chosen at slot s_b^k under both C^b and $\overline{C}^b$. This completes the proof of Proposition 1.

Proof of Proposition 2. We first show that $\overline{C}^b$ satisfies the IRC condition. For any $X \subseteq \mathcal{X}$ such that $X \neq \overline{C}^b(X)$, let x be one of the rejected contracts, i.e., $x \in X \setminus \overline{C}^b(X)$. We need to show that

$$\overline{C}^b(X) = \overline{C}^b(X \setminus \{x\}).$$

Let x^k and $\widetilde{x}^k$ be the contracts chosen by the slot s_b^k under $\overline{C}^b$ from the set X and $X \setminus \{x\}$, respectively. We will show that $x^k = \widetilde{x}^k$ for each $k = 1, ..., 2n_b$ by induction. For the first slot s_b^1, since we know that $x$is not chosen by $\overline{C}^b$—and, hence by s_b^1— and $\Pi^{s_b^1}$ is a linear order we have $x^1 = \widetilde{x}^1$. Suppose that $x^j = \widetilde{x}^j$ for $j = 1, ..., k-1$. We need to show that $x^k = \widetilde{x}^k$. Notice that the set of remaining contracts for slot s_b^k from the choice processes starting with X and $X \setminus \{x\}$ are $X \setminus \{x^1, ..., x^{k-1}\}$ and $X \setminus \{x, x^1, ..., x^{k-1}\}$, respectively. Since we know that x is not chosen, and hence, is not $\Pi_b^{s_b^k} - maximal$ among the contracts $X \setminus \{x^1, ..., x^{k-1}\}$, then we have $x^k = \widetilde{x}^k$. Therefore, at each slot, the same contract is chosen from the choice processes starting with X and $X \setminus \{x\}$, respectively. This completes our proof.

Substitutability. Consider an offer set $X \subseteq \mathcal{X}$ such that $X \neq \overline{C}^b(X)$. Let x be one of the rejected contracts, i.e., $x \in X \setminus \overline{C}^b(X)$, and let z be an arbitrary contract in $\mathcal{X} \setminus X$. To show substitutability, we need to show that

$$x \notin \overline{C}^b(X \cup \{z\}).$$

There are two cases to consider:

Case 1: $z \notin \overline{C}^b(X \cup \{z\})$.

Since $\overline{C}$ satisfies the IRC condition, we have

$$\overline{C}^b(X) = \overline{C}^b(X \cup \{z\}).$$

Since $x \notin \overline{C}^b(X)$ is given, we have $x \notin \overline{C}^b(X \cup \{z\})$, which is the desired conclusion.

Case 2: $z \in \overline{C}^b(X \cup \{z\})$.

We call the choice processes starting with the contract sets X and $X \cup \{z\}$, respectively, as the **initial** and **new** processes. Note that the precedence order $\triangleright_b$, the capacity transfer scheme q_b, and the priority profile of slots Π_b are the same in the initial and new processes. Let Y^i and $\overline{Y}^i$ be the sets of remaining contracts after slot s_b^i, $i = 1, ..., 2n_b - 1$, selects its contract in the initial and new processes, respectively.

Let s_b^j be the slot that selects the contract z in the new process. For each slot $i = 1, ..., j-1$, we have $\overline{Y}^i = Y^i \cup \{z\}$ as each slot s_b^i, $i = 1, ..., j-1$, selects the same contract under the initial and new processes. Since slot s_b^j selects z, x is not chosen in any of $s_b^1, ..., s_b^j$ in the new process. We have the following possibilities:

1. s_b^j is an *original* seat with $q_b^j = 0$ and selects $\emptyset$ in the initial process.
2. s_b^j is an *original* seat with $q_b^j = 1$ and selects $\emptyset$ in the initial process.
3. s_b^j is an *original* seat and selects a different contract—call it y—in the initial process.
4. s_b^j is a *shadow* seat and selects $\emptyset$ in the initial process.
5. s_b^j is a *shadow* seat and selects a different contract—call it y—in the initial process.

In cases 1 and 4, $Y^j = \overline{Y}^j$ and the rest of the initial and the new process will be the same. Therefore, for these cases we can conclude that $x \notin \overline{C}(X \cup \{z\})$. In case 2, $Y^j = \overline{Y}^j$ and the associated shadow seat of s_b^j is passive in the new process while it is active in the initial process. In cases 3 and 5, $Y^j \cup \{y\} = \overline{Y}^j$. Note that in case 5, the associated shadow seat of s_b^j is passive as $\mathbf{1}_{s_b^j} = 1$.

For every seat s_b^κ, $\kappa = j, ..., 2n_b$, we have either (i) $Y^\kappa = \overline{Y}^\kappa$ and the capacity of s_b^{k+1} is either the same in the initial and new processes or 1 in the initial process and 0 in the new process, or (ii) $Y^\kappa \cup \{y\} = \overline{Y}^\kappa$, and the capacity of s_b^{k+1} is the same under both processes. We have already showed it for $\kappa = j$ above. By the way of induction, suppose that the assertion holds for slots $\kappa = j, ..., k-1$. We need to show that it holds for $\kappa = k$.

If $Y^{k-1} = \overline{Y}^{k-1}$ and the capacity of the kth slot is 0 in both initial and new processes, then we will have $Y^k = \overline{Y}^k$, and the capacity of s_b^{k+1} will be the same under the initial and the new processes. Similarly, if $Y^{k-1} = \overline{Y}^{k-1}$ and the capacity of s_b^k is 1 in both initial and new processes, then the same contract will be chosen at s_b^k . Hence, we will have $Y^k = \overline{Y}^k$, and the capacity of s_b^{k+1} will be the same under the both processes.

Now suppose $Y^{k-1} = \overline{Y}^{k-1}$ and the capacity of s_b^k is 1 in the initial process and 0 in the new processes. Note that s_b^k must be a shadow seat in this case. If seat s_b^k selects $\emptyset$ in the initial process, then we have $Y^k = \overline{Y}^k$ and the capacity of s_b^{k+1} is the same under both process. If seat s_b^k selects contract y in the initial process, then we have $Y^k \cup \{y\} = \overline{Y}^k$. Moreover, the capacity of s_b^{k+1} is the same under both processes.

Finally, suppose $Y^{k-1} \cup \{y\} = \overline{Y}^{k-1}$, and the capacity of s_b^k is the same under the both processes. If the capacity of s_b^k is 0, then $Y^{k-1} \cup \{y\} = \overline{Y}^{k-1}$, and the capacity of s_b^{k+1} is the same under the both processes. If the capacity of s_b^k is 1 and $Y^{k-1} \cup \{y\} = \overline{Y}^{k-1}$, either the same contract is chosen

under both processes, or $\emptyset$ is chosen in the initial process and y is chosen in the new process. In the former case, we have $Y^k \cup \{z\} = \overline{Y}^k$ for some contract z and the capacity of s_b^{k+1} is the same under both processes. In the latter case, $Y^k = \overline{Y}^k$ and the capacity of s_b^{k+1} is either the same under both processes—in the cases, where s_b^k is a shadow seat or is an original seat with $q_b^k = 0$—or the capacity of s_b^{k+1} is 1 in the initial process and 0 in the new process—in the case where s_b^k is an original slot with $q_b^k = 1$. This ends the proof of our claim.

Since x is not chosen by any seat in the initial process, it will not be chosen under the new process either as a result of the above claim. Therefore, $\overline{C}^b$ is substitutable.

Law of Aggregate Demand. Consider two sets of contracts X and Y such that $X \subseteq Y \subseteq \mathcal{X}$. We want to show that

$$\mid \overline{C}^b(X) \mid \leq \mid \overline{C}^b(Y) \mid .$$

We call the choice process starting with the contract set X as "process X", and the choice process starting with the contract set Y as "process Y". Let X^j and Y^j be the set of remaining contracts after slot s_b^j selects its contract in processes X and Y, respectively, for $j = 1, ..., 2n_b$. We will prove that for each $j = 1, ..., 2n_b$, $X^j \subseteq Y^j$.

The first slot s_b^1 is an original slot. Let x^1 be the contract chosen by s_b^1 in process X. Since $X \subset Y$, $x^1 \in Y$. In process Y, s_b^1 either selects x^1 or another contract $y \in Y \setminus X$. In both cases, we have $X^1 \subseteq Y^1$. By the way of induction, suppose that $X^j \subseteq Y^j$, for all $j = 1, ..., k-1$. We need to show $X^k \subseteq Y^k$.

First suppose that s_b^k is an original seat. Since $X^{k-1} \subseteq Y^{k-1}$, if s_b^k chooses a contract $x^k \in X^{k-1} \subseteq Y^{k-1}$ in process X, then s_b^k chooses either x^k or another contract $z \in Y^{k-1} \setminus X^{k-1}$. In both cases, $X^k \subseteq Y^k$.

Now suppose that, s_b^k is a shadow seat. Note that the capacity of the shadow seat in process X is either the same as the capacity of this seat in process Y (both 0 or 1), or the capacity of s_b^k is 1 in process X and 0 in process Y—that is, the associated original seat is filled in process Y, but remained vacant in process X. If they are both 0, then $X^{k-1} \subseteq Y^{k-1}$, and $X^{k-1} = X^k$ and $Y^{k-1} = Y^k$ imply $X^k \subseteq Y^k$. If they are both 1 and the contract x^k is chosen by seat s_b^k in process X, then either contract x^k or another contract $w \in Y^{k-1} \setminus X^{k-1}$ is chosen in process Y. Both imply $X^k \subseteq Y^k$. Finally, suppose that s_b^k is active in process X and passive in process Y. This is the case only when the associated original seat is filled in process Y with a contract in $Y \setminus X$ and remains vacant in process X. Given that $X^{k-1} \subseteq Y^{k-1}$, if a contract x^k is chosen in process X, we have $X^k \subseteq Y^k$. It is important to note that it cannot be the case where the shadow seat s_b^k is passive in process X and active in process Y. It contradicts with the inductive assumption.

Since for all seats s_b^j, $j = 1, ..., 2n_b$, we have $X^{j-1} \subseteq Y^{j-1}$, then we can conclude that $\overline{C}^b(X) \mid \leq \mid \overline{C}^b(Y) \mid$.

Proof of Theorem 2. Assume, toward a contradiction, that the COM with regard to SSPwCT does not respect improvements. Then, there exists an agent $i \in I$, a preference profile of agents

$P \in \times_{i \in I} \mathcal{P}^i$, and choice profiles $\overline{C}$ and C such that $\overline{C}$ is an improvement over C for agent i and

$$\mathcal{C}_i(P; C) P^i \mathcal{C}_i(P; \overline{C}).$$

Let $\mathcal{C}_i(P; C) = x$ and $\mathcal{C}_i(P; \overline{C}) = \overline{x}$. Consider a preference $\widetilde{P}^i$ of agent i according to which the only acceptable contract is x, i.e., $\widetilde{P}^i : x - \emptyset_i$. Let $\widetilde{P} = (\widetilde{P}^i, P_{-i})$. We will first prove the following claim:

Claim: $\mathcal{C}_i(\widetilde{P}; C) = x \implies \mathcal{C}_i(\widetilde{P}; \overline{C}) = x$**.**

Proof of the Claim: Consider the outcome of the COM under choice profile C given the preference profile of agents $\widetilde{P}$. By Hirata and Kasuya (2014), the cumulative offer is order-independent. We can first completely ignore agent i and run the COM until it stops. Let Y be the outcome. At this point, agent i makes an offer for his only contract x. This might create a chain of rejections, but it does not reach agent i since we assumed $\mathcal{C}_i(\widetilde{P}; C) = x$. Let the kth slot with respect to precedence order $\rhd_{\mathbf{b}(x)}$ be the slot that chose contract x.

Now consider the COM under choice profile $\overline{C}$. Again, we completely ignore agent i and run the COM until it stops. The same outcome Y is obtained, because the only difference between the two COMs is agent i's position in the priority rankings. At this point, agent i makes an offer for his only contract x. If x is chosen by the same slot, i.e., k^{th} slot with respect to $\rhd_{\mathbf{b}(x)}$, then the same rejection chain (if there was one in the COM under the choice profile C) will occur and it does not reach agent i; otherwise, we would have a contradiction with the case under choice profile C. The only other possibility is the following: since agent i's ranking is now (weakly) better under $\overline{\Pi}_{\mathbf{b}(x)}$ compared to $\Pi_{\mathbf{b}(x)}$, his contract x might be chosen by slot l which precedes slot k with respect to $\rhd_{\mathbf{b}(x)}$. Then, it must be the case that by selecting x slot l must reject some other contract it was holding. Let us call this contract y. If no contract of agent $\mathbf{i}(y) = j$ is chosen between slots l and k, then the slots between l and k choose the same contracts under both priority profiles. In this case, y is chosen by slot k. Thus, if a rejection chain starts, it will not reach agent i; otherwise, we could have a contradiction, due to the fact that x was chosen at the end of the COM under choice profile C. A different contract of agent j cannot be chosen between groups l and k; otherwise, the observable substitutability[3] of branch $\mathbf{b}(x)$'s SSPwCT choice rule would be violated. Therefore, if any contract of agent j is chosen by slots between l and k, it must be y. If y is chosen by a slot that precedes k, then it must replace a contract—we call this contract z. By the same reasoning, no other contract of agent $\mathbf{i}(z)$ can be chosen before slot k; otherwise, we would violate the observable substitutability of branch $\mathbf{b}(x)$'s SSPwCT choice rule. Proceeding in this fashion causes the same contract in slot k to be rejected and initiates the same rejection chain that occurs under choice profile C. Since the same rejection chain does not reach agent i under choice profile C, it will not reach agent i under choice profile $\overline{C}$, which ends our proof for the claim.

[3]By Hatfield, Kominers and Westkamp (2019), we know that any choice function that has a substitutable and size monotonic completion must be observably substitutable. Observable substitutability simply says that branch choices along the cumulative offer process satisfy substitutes property, i.e., for the offer sets that can arise during the cumulative offer process.

Since $\mathcal{C}_i(P;C) = x$ and $\mathcal{C}_i(P;\overline{C}) = \overline{x}$ such that $xP^i\overline{x}$, if agent i misreports and submits $\widetilde{P}^i$ under choice profile $\overline{C}$, then she can successfully manipulate the COM. This is a contradiction as we have already established that the COM is strategy-proof.

Proof of Theorem 3. Suppose that Z is the outcome of the COM at (P, C), where $P = (P_{i_1}, ..., P_{i_n})$ is the profile of agent preferences and $C = (C^{b_1}, ..., C^{b_m})$ is the profile of branches' SSPwCT choice rules. Consider a branch $b \in \mathcal{B}$. Suppose that $\widetilde{C}^b$ and C^b take as an input capacity transfer schemes $\widetilde{q}_b$ and q_b, respectively, where $\widetilde{q}_b^k = q_b^k$ for all $k = 1, ..., s-1, s+1, ..., n_b$. For slot s, let $\widetilde{q}_b^s = 1$ and $q_b^s = 0$. That is, the capacity of the original seat s is transferred to the associated shadow seat under capacity function $\widetilde{q}_b$, but not under the capacity function q_b. We need to prove that the outcome of the COM at $(P, (\widetilde{C}^b, C_{-b}))$, $\widetilde{Z}$, Pareto dominates Z.

In the computation of COM, if the original slot s is filled, then we have $\widetilde{Z} = Z$ because, under both $\widetilde{q}_b$ and q_b the shadow slot associated with the original slot s will become inactive.

We now consider the case where the original slot s remains vacant in the computation of COM under (P, C). Then, under $(P, (\widetilde{C}^b, C_{-b}))$, the shadow slot associated with the original slot s – we call it $\widetilde{s}$ – will be active, i.e., it will have a capacity of 1. There are two cases to consider. If the shadow slot $\widetilde{s}$ remains vacant in the computation of COM under $(P, (\widetilde{C}^b, C_{-b}))$, then we again have $\widetilde{Z} = Z$, as the only difference between the two COMs, under (P, C) and $(P, (\widetilde{C}^b, C_{-b}))$, is the capacity of the shadow slot $\widetilde{s}$.

The non-trivial case is the one where the shadow slot $\widetilde{s}$ is assigned a contract in COM under $(P, (\widetilde{C}^b, C_{-b}))$. We now define an *improvements chain* algorithm that starts with the outcome Z.

Step 1. Let x_1 be the contract that is assigned to slot $\widetilde{s}$ in the SSPwCT choice procedure of branch $\mathbf{b}(x_1)$. If agent $\mathbf{i}(x_1)$ is assigned $\emptyset$ under Z, then the improvement process ends and we have $\widetilde{Z} = Z \cup \{x_1\}$. Otherwise, set as z_1 the contract that agent $\mathbf{i}(x_1)$ is assigned under Z. Note that $x_1 P_{\mathbf{i}(x_1)} z_1$.

Step 2. Let x_2 be the contract that is chosen by the slot vacated by z_1 (or the shadow seat that is associated with it). If agent $\mathbf{i}(x_2)$ is assigned $\emptyset$ under Z, then the improvement process ends and we have $\widetilde{Z} = Z \cup \{x_1, x_2\} \setminus \{z_1\}$. Otherwise, set as z_2 the contract that agent $\mathbf{i}(x_2)$ is assigned under Z. Note that $x_2 P_{\mathbf{i}(x_2)} z_2$.

Step n. Let x_n be the contract that is chosen by the slot vacated by z_{n-1} (or the shadow seat that is associated with it). If agent $\mathbf{i}(x_n)$ is assigned $\emptyset$ under Z, then the improvement process ends and we have $\widetilde{Z} = Z \cup \{x_1, ..., x_n\} \setminus \{z_1, ..., z_{n-1}\}$. Otherwise, set as z_n the contract that agent $\mathbf{i}(x_n)$ is assigned under Z. Note that $x_n P_{\mathbf{i}(x_n)} z_n$.

In every step of the improvement chain algorithm a contract is replaced by a more preferred contract. Since there are finitely many contracts the improvement chain algorithm must end. Therefore, we reach $\widetilde{Z}$, which Pareto dominates Z, in finitely many step. This ends our proof.