

RECONSTRUCTION OF THE ELECTROMAGNETIC WAVEFIELD FROM SCATTERING DATA

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INTRODUCTION

Inverse scattering theory concerns itself with determining the properties of a scatterer (e.g., the spatial variation of the scatterer's dielectric constant and conductivity) from measured scattering data. Two general approaches to this problem exist. The first approach is a direct variational method. One starts by computing the scattering amplitude for some assumed properties of the scatterer. The resulting theoretical scattering amplitude is compared with the measured scattering data. If these results differ, then one varies the properties of the assumed scatterer and recomputes the scattered field for the new properties. This process is truncated when the measured and computed fields agree to a specified accuracy. The properties of the assumed scatterer are then supposed to coincide to within some accuracy with the unknown scatterer which generated the measured data.

The second approach is more direct. One develops linear integral (or differential) equations whose input is the scattered field [1,2]. The solution of these equations then either yields the properties of the scatterer directly or after simple post-processing.

Both approaches have been successful in solving the inverse problem for one-dimensional scatterers; i.e., scatterers whose properties depend only on a single spatial variable. Examples are layered or spherically symmetric scatterers. Both approaches suffer great difficulties for scatterers whose properties vary in a general three-dimensional way. In the first method the computational effort of solving the direct problem often becomes prohibitive. The second approach is not generally fully developed for the three-dimensional case. Nevertheless, approximations to it such as the inverse Born approximation and optical imaging indicate that good progress can be made. In particular, the exact general equations which govern the reconstruction of three-dimensional scatterers are currently unknown for acoustics, electromagnetics and elastodynamics.

The purpose of this paper is to propose a candidate for such a general reconstruction equation in the case of electromagnetics. The structure of the paper is as follows. First, we briefly review the current situation. Then we introduce some of the elements of scattering theory. Finally, we derive the new equation.

Review

Recently, Newton [3] has developed the exact equations which govern three-dimensional inverse scattering for Schrödinger's equation. Newton's approach proceeds in two steps. First, he showed that the wavefield everywhere in space (including the interior of the scatterer) could be deduced from a linear integral equation whose input is the far-field scattering amplitude. Second, the potential is recovered straightforwardly from the wavefield using wavefront conditions.

Recently, Rose, Cheney, and DeFazio [4] have shown that the integral equation used by Newton is also valid for a wide range of scalar wave equations. For example, this linear integral equation relates the wavefield to the scattering amplitude for the acoustic wave equation. (Note, however, that it is currently unknown if the equation has unique solutions in this case). The derivation of the integral equation in Ref. [4] is very general and relies only on such features of the scattering process as linearity, causality and the far-field decay of the wavefield.

The form of the derivation makes it clear that the same integral equation (modified slightly) will also hold for linear, hyperbolic vector wave equations. In this report, the derivation is generalized for electromagnetic scattering. A similar derivation is also possible for elastodynamics.

Elements of Electromagnetic Scattering Theory

The propagation of electromagnetic waves in a linear medium is governed by [2]

$$\vec{\nabla}_x(\mu^{-1}(\omega, \vec{x})\vec{\nabla}_x\vec{E}(\omega, \vec{x})) - k^2 n^2(\omega, \vec{x})\vec{E}(\omega, \vec{x}) = 0. \quad (1)$$

Here ω is the angular frequency, k is the wavevector of light in free space, and $\omega = k$, since we choose the speed of light $c=1$. The electric field is denoted by $\vec{E}(\omega, \vec{x})$ and the magnetic permeability by μ . Finally,

$$n^2(\omega, \vec{x}) = \epsilon(\omega, \vec{x}) + \frac{4\pi i}{\omega} \sigma(\omega, \vec{x}). \quad (2)$$

Here $\epsilon(\omega, \vec{x})$ denotes the dielectric permittivity and $\sigma(\omega, \vec{x})$ the conductivity. In general, μ , ϵ and σ are tensors.

The geometry for the scattering experiment is described. The scatterer has finite spatial extent. It is enclosed by some finite region of space called R which also contains the origin of coordinates. Within the region R the material properties (μ , ϵ and σ) vary as functions of \vec{x} and ω and may, in fact, be discontinuous. However, they are assumed to be bounded functions. Exterior to the scattering region the conductivity is zero and ϵ and μ take on constant frequency independent values, e.g., ϵ and μ could be chosen to take on their free space values exterior to R . Thus, the geometry presupposes a finite (possibly metallic) object embedded in an infinite otherwise homogeneous space. Finally, the entire scattering region is surrounded by a large ball S which is centered about the origin of coordinates.

The derivation to be given is clearest in the time-domain. The time-domain electromagnetic wave equation is obtained by taking the Fourier

transform of the quantities in Eq. (1) with respect to ω . For example,

$$\vec{E}(t, \vec{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \vec{E}(\omega, \vec{x}). \quad (3)$$

Exterior to the region R the wave equation reduces to

$$\nabla_x \nabla_x \vec{E}(t, \vec{x}) - \frac{\partial^2}{\partial t^2} \vec{E}(t, \vec{x}) = 0. \quad (4)$$

Here we have set the speed of light in the surrounding material equal to one and used the fact that $c^{-2} = \epsilon\mu$.

The scattering experiment is assumed to proceed as follows. At early times, a transversely polarized plane wave delta pulse is incident propagating in the \hat{e} direction with polarization $\hat{\alpha}$. This incident field is a solution of Eq. (4) and is given explicitly by

$$E_i^0(t, \hat{e}, \hat{\alpha}, \vec{x}) = \delta(t - \hat{e} \cdot \vec{x}) \hat{\alpha}_i. \quad (5)$$

Here \hat{e} and $\hat{\alpha}$ are unit vectors which are indicated by carets. Note $\hat{e} \cdot \hat{\alpha} = 0$ since E_i^0 is transversely polarized. Tensor notation will be used from here on in denoting the component of field quantities.

The incident pulse propagates until it strikes the scattering region where it interacts with the scatterer in a complicated way. Finally, at late times the field propagates outward from the region R and is measured on the surface of the large sphere S which is centered about the origin. These measurements are the scattering data.

The scattering data can be defined in terms of the field $E_i^+(t, \hat{e}, \hat{\alpha}, \vec{x})$ which result from the incident field E_i^0 . We take $x = |\vec{x}|$ large and define the impulse response functions R^{\perp} and R^{\parallel} by

$$E_i^+(t, \hat{e}, \hat{\alpha}, \vec{x}) = \delta(t - \hat{e} \cdot \vec{x}) \hat{\alpha}_i + \frac{R^{\perp}(t-x, \hat{e}, \hat{\alpha}, \hat{x})}{x} \hat{y}_i^{\perp} + \frac{R^{\parallel}(t-x, \hat{e}, \hat{\alpha}, \hat{x})}{x} \hat{y}_i^{\parallel} + o(x^{-1}) \quad (6)$$

Here $\hat{x} \equiv \vec{x}/x$ denotes the direction of scattering, while \hat{y}^{\perp} and \hat{y}^{\parallel} denote transverse polarization directions on the surface of S. The unit vector \hat{y}^{\parallel} lies in the plane of scattering and has $\hat{x} \cdot \hat{y}^{\parallel} = 0$, while $\hat{y}^{\perp} \equiv \hat{x} \times \hat{y}^{\parallel}$.

Integral Equation Derivation

The idea behind the derivation is quite simple. The field everywhere in space can be computed given the initial data (i.e., the wavefield at very early times) and the properties of the scatterer ϵ , μ , and σ . The result is formally denoted by $E_i^+(t, \hat{e}, \hat{\alpha}, \vec{x})$. However, the field can also be computed given the final data (i.e., the wavefield at very late times which is known from the scattering data) and ϵ , μ and σ . When these two ways of computing the field are equated, the integral equation

is obtained. Perhaps surprisingly ϵ , μ and σ do not appear, and a relation strictly between the wavefields and impulse response functions results.

The derivation rests on the following assumptions concerning the time-domain electromagnetic wave equation. (1) The solution of the wave equation which yields the scattering data in Eq. (6) is unique. This rules out the existence of solutions which decay more rapidly than x^{-1} at large x . (2) The signal (wave train) received at an observation point on the surface of the sphere S is assumed to decay sufficiently rapidly to zero outside a time interval which contains $t=x$. A closely related further assumption is that the wavefield approaches zero in any bounded region of space as $t \rightarrow \infty$. (3) It is assumed that the amplitude of the scattered field decays as x^{-1} and varies continuously as a function of angle for sufficiently large x .

We start with the second method of computing $E_i^+(t, \hat{e}, \hat{\alpha}, \vec{x})$. Namely, we compute it for all \vec{x} and t from the scattering data. First we define $E_i^{+sc} = E_i^+ - \delta \hat{\alpha}_i$. Then we re-express the scattering data in Eq. (6) as $(\vec{x} \rightarrow \infty)$

$$E_i^{+sc}(t, \hat{e}, \hat{\alpha}, \vec{x}) = \sum_P \int_{-\infty}^{\infty} d\tau x^{-1} \delta(t-\tau-x) R^P(\tau, \hat{e}, \hat{\alpha}, \hat{x}) \hat{y}_i^P + o(x^{-1}). \quad (7)$$

Here the sum is over the polarization $P = (\perp, \parallel)$. We note that the Dirac delta contributes only when $t-\tau=x$, so that without loss of generality we may restrict the τ integration to $t-\tau$ positive.

Next we will express the outgoing spherical wave, $x^{-1}\delta(t-\tau-x)$, as a sum of plane waves $\delta(t-\tau-\hat{e} \cdot \vec{x})$ via the formula

$$x^{-1}[\delta(s+x)-\delta(s-x)] = (2\pi)^{-1} \frac{d}{ds} \int_{S^2} d^2e' \delta(s-\hat{e}' \cdot \vec{x}). \quad (8)$$

We use (8) in (7) with $s>0$:

$$E_i^{sc}(t, \hat{e}, \hat{\alpha}, \vec{x}) = - (2\pi)^{-1} (d/dt) \int_{-\infty}^{\infty} d\tau \int_{S^2} d^2e' \sum_P \delta(t-\tau-\hat{e}' \cdot \vec{x}) R^P(\tau, \hat{e}, \hat{\alpha}, \hat{x}) + o(x^{-1}). \quad (9)$$

Here S^2 denotes the two-sphere and the integration is over all angles. We see from this expression that the Dirac δ contributes only when $\tau = t-\hat{e}' \cdot \vec{x}$. Since we know from (3) that this must be the same as $\tau = t-x$, it follows that the Dirac δ contributes only when $\hat{e}' = \hat{x}$. This implies that $R^P(\tau, \hat{e}, \hat{\alpha}, \hat{x})$ which appears on the right-hand side can be replaced with $R^P(\tau, \hat{e}, \hat{\alpha}, \hat{e}')$ in Eq. (9). The result is

$$E_i^+(t, \hat{e}, \hat{\alpha}, \vec{x}) = \delta(t-\hat{e} \cdot \vec{x}) \hat{\alpha}_i - (2\pi)^{-1} (d/dt) \int_{-\infty}^{\infty} d\tau \int_{S^2} d^2e' \sum_P \delta(t-\tau-\hat{e}' \cdot \vec{x}) \hat{y}_i^P R^P(\tau, \hat{e}, \hat{\alpha}, \hat{e}') + o(x^{-1}). \quad (10)$$

Thus, the entire \vec{x} dependence on the right-hand side resides in the Dirac delta, a result needed below.

Equation (10) is a rewriting of the final (i.e., scattering) data as a weighted sum of plane waves. Now we need to answer the following question. What solution of the wave equation corresponds to the final data defined by Eq. (10)? By construction E_i^+ is such a solution.

However, a different representation of the same solution can be obtained from the right-hand side of Eq. (10). In order to state this clearly, a second type of solution is needed for the electromagnetic wave equation. Namely, we define $E_i^-(t, \hat{e}, \hat{\alpha}, \vec{x})$ to be that solution which for sufficiently late times is given by $\delta(t - \hat{e} \cdot \vec{x}) \hat{\alpha}_i$. Thus, the solution E_i^- is defined by its behavior at late times (i.e., by its final data).

Now consider the final (late time) data defined by the right-hand side of Eq. (10). Each Dirac delta plane wave, $\delta(t - \tau - \hat{e}' \cdot \vec{x}) \hat{y}_i^P$, can be considered to have evolved from a solution $E_i^-(t - \tau, \hat{e}', \hat{y}_i^P, \vec{x})$. Consequently, the whole right-hand side of (6) is seen to have evolved from the solution

$$E_i^-(t, \hat{e}, \hat{\alpha}, \vec{x}) - (2\pi)^{-1} (d/dt) \sum_P \int_{-\infty}^{\infty} d\tau \int_{S^2} d^2 e' E_i^-(t - \tau, \hat{e}', \hat{y}_i^P, \vec{x}) R^P(\tau, \hat{e}, \hat{\alpha}, \hat{e}'). \quad (11)$$

Here we have used the fact that the solutions of a linear wave equation can be superimposed to form a new solution.

Both E_i^+ and the expression given in (11) are solutions of the electromagnetic wave equation which evolve into the same scattering data for sufficiently late times. Since the solutions of the wave equation are assumed to be unique, these solutions can be equated to give our basic result

$$E_i^+(t, \hat{e}, \hat{\alpha}, \vec{x}) = E_i^-(t, \hat{e}, \hat{\alpha}, \vec{x}) - (2\pi)^{-1} (d/dt) \int_{-\infty}^{\infty} d\tau \int_{S^2} d^2 e' \sum_P R^P(\tau, \hat{e}, \hat{\alpha}, \hat{e}') E_i^-(t - \tau, \hat{e}', \hat{y}_i^P, \vec{x}). \quad (12)$$

We have shown that subject to a few mild assumptions, Eq. (12) holds for electromagnetic scattering. We note that the scalar wave analogue of Eq. (12) is the basis of Newton's exact solution of the inverse problem for Schrödinger's equation. Substantial work is underway to see if Eq. (12) can be made to yield unique solutions for E_i^+ given the scattering data for electromagnetic scattering.

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