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BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SYSTEMS.

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Boundary value problems for second order systems

by

Walter Edward Stennes Will

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I. INTRODUCTION

Let $X$ be a Banach Space, $[X]$ the algebra of bounded linear operators from $X$ into itself, and $\alpha, \beta \in X$. Let $T$ be a positive real number, and $f$ a function from $[0,T] \times X \times X$ into $X$ which is continuous. We consider the problem,

\begin{align}
(1.1) \quad y'' &= f(t,y,y'), \quad y(0) = \alpha, \quad y(T) = \beta.
\end{align}

Let $U$ denote the set of twice continuously differentiable functions with domain the closed interval $[0,T]$ and range in $X$, which satisfy the boundary data in (1.1)\(^1\); i.e.

\begin{align}
(1.2) \quad U &= \{ u \in C^2([0,T],X) : u(0) = \alpha, u(T) = \beta \}.
\end{align}

We suppose throughout this paper that there is a function $\varphi \in U$ which "almost" solves (1.1) in the sense that, for $t \in [0,T]$, $\| \varphi''(t) - f(t,\varphi(t),\varphi'(t)) \|$ is small, where the norm is that of $X$. We then consider conditions on $f$ near the graph of $\varphi$ which guarantee the existence of a solution of

---

\(^1\)Herein, the equations will be represented by ( ) and the references will be represented by [ ].
(1.1). The results are constructive in that they give, at least in principle, algorithms for obtaining successive approximations to the solution whose existence is established. Since the hypotheses are local in nature, we cannot claim uniqueness of solutions except, at most, within a certain set.

Suppose $y$ is a solution of (1.1), and $\varphi$ is in $U$. Let $z = y - \varphi$; then $z$ is a twice continuously differentiable function taking the compact interval $[0, T]$ into $X$ with $z(0) = z(T) = 0$, a set of properties we summarize by stating that $z \in U_0$, where we define $U_0$ by

$$U_0 = \{ u \in C^2([0,T], X) : u(0) = u(T) = 0 \}.$$ 

Furthermore, it is clear that $z$ solves the boundary value problem,

$$z''(t) = f(t, z(t) + \varphi(t), z'(t) + \varphi'(t)) - \varphi''(t), z(0) = z(T) = 0.$$  

Conversely, if $z$ solves (1.4) and $y = z + \varphi$, where $\varphi$ is as above, then $y$ solves (1.1).
We have observed that the principal hypothesis on \( \varphi \) is that the residual function, \( \varphi''(t) - f(t, \varphi(t), \varphi'(t)) \), be small in norm for \( 0 \leq t \leq T \). Then, in the event that a solution, say \( y \), exists, it is reasonable to hope that \( y \) is close to \( \varphi \) with respect to some natural metric. If an a priori bound on the distance between \( y \) and \( \varphi \) can be made, then the other hypotheses on \( f \) need apply only locally, near \( \varphi \). The solution, \( z \), of (1.4) is then expected to be close to zero. We might, therefore, consider as a possible condition on \( f \) that the right hand side of (1.4) nearly agree with the residual function,

\[
\varphi''(t) - f(t, \varphi(t), \varphi'(t)); \text{ i.e. that }\]

\[
\|f(t, z(t) + \varphi(t), z'(t) + \varphi'(t)) - f(t, \varphi(t), \varphi'(t))\| \text{ be small.}\]

This suggests a Lipschitz condition on \( f \). We shall use a stronger condition on \( f \).

**Definition 1.1:** Let \( X \) and \( Y \) be Banach spaces, and let \( g : X \to Y \). \( g \) is said to have a Frechet derivative at \( x \in X \) in case there is a continuous linear operator, \( D(x) \), taking \( X \) into \( Y \) such that for every positive number \( \varepsilon \) there exists a positive number \( \delta \) such that

\[
\|g(x + h) - g(x) - D(x)h\| \leq \varepsilon \|h\|, \text{ for } h \in X \text{ with } 0 \leq \|h\| \leq \delta. \quad D(x) \text{ is called the Frechet derivative of } g
\]
at \( x \). The necessary calculus may be found in [7], pp.58-123. We will use the notation \( f_2(t,y,y') \) to denote the Frechet derivative of \( f \) with respect to the second variable, \( y \), at \( (t,y,y') \). More precisely, \( f_2(t,y,y') \) is the bounded linear operator having the property that for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that
\[
\|f(t,y+h,y') - f(t,y,y') - f_2(t,y,y')h\| \leq \varepsilon \|h\| \quad \text{whenever} \quad h \in X \text{ is such that } 0 \leq \|h\| \leq \delta.
\]
Similarly, \( f_3(t,y,y') \) will denote the Frechet derivative with respect to the third variable, \( y' \). We shall see that these Frechet derivatives are related to the difference,
\[
f(t,z(t)+\varphi(t),z'(t)+\varphi'(t)) - f(t,\varphi(t),\varphi'(t)),
\]
which quantity we want to be small in norm. Thus we require the concept of operator norms.

**Definition 1.2**: Let \( X \) and \( Y \) be normed linear spaces, and \( M : X \rightarrow Y \) a continuous linear mapping. The non-negative real number, \( \sup\{\|Mx\| : \|x\| \leq 1\} \), is called the norm of \( M \) and is denoted \( \|M\| \).

An inequality commonly used in the paper is
\[
\|Mx\| \leq \|M\| \|x\| \quad \text{for all } x.
\]

In many of the applications, and in all the examples
presented here, the Banach space $X$ is finite dimensional.

Suppose $X$ is real $n$-space, $\mathbb{R}^n$, and $Y$ is $\mathbb{R}^m$. If

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

the Frechet derivative $D(x)$, if it exists, is the Jacobian matrix,

$$D(x) = \begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n}
\end{pmatrix},$$

where all the partial derivatives above are evaluated at $x$.

In many of our examples we choose to use the maximum norm of $\mathbb{R}^n$; i.e.

$$\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Recall that the corresponding operator norm is the maximum absolute row sum,
\[ ||M|| = \max_{1 \leq i \leq n} \sum_{j} |m_{ij}|, \]

where the \( n \) by \( n \) matrix \( M = (m_{ij}) \) takes \( \mathbb{R}^n \) into itself.

The technique used for establishing existence of solutions of (1.1) involves the notions of quasilinearization and successive approximations. To illustrate, suppose we have an initial estimate, \( \phi \). We made the change of variables \( z = y - \phi \), and consider the boundary value problem (1.4). A sequence of functions, \( \{z_n\} \), in \( U_\phi \) is defined as follows: let \( z_0 \) be defined by \( z_0(t) = 0 \) for \( 0 \leq t \leq T \). For \( n = 0,1,2,\ldots \), once \( z_n \) has been defined, consider a linear differential equation of the form,

\[ (1.5) \quad z'' = A(t;z_n)z' + B(t;z_n)z + h(t;z_n), \]

where \( A(t;z_n) \) and \( B(t;z_n) \) are in \( [X] \) and \( h(t;z_n) \in X \) for \( 0 \leq t \leq T \). The notation points out the dependence of the equation on \( z_n \). \( z_{n+1} \) is then defined to be the function in \( U_\phi \) that solves (1.5).
A corresponding sequence \( \{y_n\} \) in \( U \) is defined by
\[
y_n = z_n + \varphi \quad \text{for} \quad n = 0, 1, \ldots.
\]
We are interested in conditions which force this latter sequence to converge to a solution of (1.1).

There is some latitude available in the choices of \( A, B, \) and \( h \) in (1.5). We refer to each choice as a quasilinearization, and consider in Chapters 2, 3, and 4 three such. In Chapter 2 we examine the easiest choice, \( A = B = 0 \). This approach has a long history, and has led to some of the standard theorems in the theory of boundary value problems. (See, for example, Bailey, Shampine, Waltman [1], Chapter 3). Our work here involves the imposition of some strong local hypotheses on a special case of the problem. Taking \( X \) to be \( \mathbb{R}^d \), real \( d \)-dimensional space, with vectors partially ordered in a coordinatewise manner, we restrict \( f \) to be independent of \( y' \). We are able to establish two theorems, Theorem 2.3 and Theorem 2.6, which guarantee existence of solutions of some problems which do not yield to other theory.

In Chapters 3 and 4 we choose quasilinearizations which involve the Frechet derivatives \( f_2 \) and \( f_3 \). In Chapter 3 we apply a mean value type of quasilinearization, introduced by...
Lees [11] for scalar problems which are independent of \( y' \).

Another quasilinearization, which is sometimes used in practice to obtain numerical solutions of boundary value problems, is applied in Chapter 4. The idea in both quasilinearizations is to incorporate into \( A \) and \( B \) as much of \( f \) as is practical. If the remainder, \( h \), is small we might expect the quasilinearizations to be relatively successful. Theorems (3.5) and (4.1) show that this is true if the operators \( A(t;z) \) and \( B(t;z) \) are only slightly sensitive to small changes in \( y \). The numerical technique which uses the quasilinearization of Chapter 4 is known as Newton's Method. Theorem 4 extends the theory which is available at present.

In Chapter 5 we discuss other quasilinearizations which appear to be potentially helpful in establishing existence of solutions of (1.1). These lead to a discussion of Green's functions, and an explicit representation is given in Theorem 5.5 for a very special case.

Throughout this paper we adopt the notation that if \( x \) is any continuous function from the compact interval \([0,T]\) to the Banach space \( X \), the number 

\[
\max\{\|x(t)\| : 0 \leq t \leq T\}
\]

is denoted by \( \|x\|_\infty \).
We require a mean value theorem for Frechet derivatives, and the contraction mapping theorem in order to prove our theorems. They are stated here. The proof of the former is given, as the idea of the proof is used in the quasilinearization of Lees. The latter is stated without proof.

**Lemma 1.3:** Suppose $f$ takes the set $[0,T] \times S$ into $X$ continuously, where $S$ is a convex subset of $X$, and that $f$ has a continuous Frechet derivative $f_2$ at each point of its domain. Consider any $\varphi \in U$, $z \in U_0$, $u \in U_0$. Then for any $t \in [0,T]$, $f(t,z(t) + \varphi(t)) - f(t,u(t) + \varphi(t)) = \int_0^1 f_2(t, \xi z(t) + (1 - \xi)u(t) + \varphi(t)) d\xi (z(t) - u(t))$.

**Proof:** Define $\psi : [0,1] \to X$ by $\psi(\xi) = f(t, \xi z(t) + (1 - \xi)u(t) + \varphi(t))$. Then $\psi(1) = f(t, z(t) + \varphi(t))$, $\psi(0) = f(t, u(t) + \varphi(t))$, and $\psi'(\xi) = f_2(t, \xi z(t) + (1 - \xi)u(t) + \varphi(t))(z(t) - u(t))$. Integration from $\xi = 0$ to $\xi = 1$ gives the result. Q.E.D.

**Theorem 1.4:** (Contraction Mapping Theorem): Let $d$ be a metric defined on a set $X$ so that the pair $(X,d)$ forms a complete metric space. A mapping $A$ which takes $X$ into
itself is called a contraction if there is a positive number $\rho < 1$ having the property that if $y$ and $z$ are any points in $X$ then $d(Ay,Az) \leq \rho d(y,z)$. If $A$ is a contraction mapping, then there is exactly one point $x \in X$ such that $Ax = x$. If $x_0$ is any point in $X$, and if $x_{n+1} = Ax_n$ for $n = 0,1,2,\ldots$, then $\lim_{n \to \infty} x_n = x$, and if $n \geq 1$, then $d(x_n,x) \leq d(x_0,x_1)\rho^{n+1}/(1-\rho)$. 
II. THE QUASILINEARIZATION OF PICARD

More is known regarding existence, uniqueness, and properties of solutions of two-point boundary value problem (1.1) in the case that \( f \) is linear in \( y \) and \( y' \); more precisely, in the case that the problem can be expressed as

\[
y''(t) = A(t)y'(t) + B(t)y(t) + h(t), y(0) = \alpha, y(T) = \beta,
\]

where \( A \) and \( B \) map the interval \([0,T]\) into the algebra \([X]\) of bounded linear operators from \( X \) into \( X \). Therefore it is often advantageous, given a problem, to suppress the nonlinearity of \( f \), thus considering a related linear equation, which is equivalent to the original. This technique results in a problem that is said to be quasilinearized. One can choose \( A, B \) and \( h \) in a variety of ways. The most elementary, however, is the classical choice \( A = B = 0, h(t) = f(t,y(t),y'(t)) \). The method of establishing existence, first used by Picard [15], is that of successive approximations. A sequence of functions, \( \{y_n\} \) is constructed by choosing \( y_0 = \varphi \), then letting \( y_{n+1} \) solve

\[
y''(t) = f(t,y_n(t),y'_n(t)), y(0) = \alpha, y(T) = \beta.
\]

Sufficient conditions are given so that the sequence converges to a function which can be shown to solve the problem. In order to obtain bounds on the iterates, one can
let \( z_n = y_n - \phi \). Then \( z_0 = 0 \), and \( z_{n+1} \) solves

\[
(2.1) \quad z''(t) = f(t, z_n(t) + \phi(t), z'_n(t) + \phi'(t)) - \phi''(t), z(0) = z(T) = 0.
\]

We define the scalar function \( G_0 \) by

\[
G_0(t,s) = \begin{cases} 
  s(t-T)/T & 0 \leq s \leq t \leq T \\
  t(s-T)/T & 0 \leq t \leq s \leq T.
\end{cases}
\]

\( G_0 \) is the familiar Green's function for the scalar equation \( z'' = 0 \). The solution of (2.1) is precisely that given by

\[
z(t) = \int_0^T G_0(t,s) \left[ f(s, z_n(s) + \phi(s), z'_n(s) + \phi'(s)) - \phi''(s) \right] ds.
\]

In fact, the boundary value problem, (1.4) is equivalent to the integral equation,

\[
z(t) = \int_0^T G_0(t,s) \left[ f(s, z(s) + \phi(s), z'(s) + \phi'(s)) - \phi''(s) \right] ds.
\]

Furthermore,
Some easily derived and well known properties of \( G_0 \) are now given without proof.

Lemma 2.1: Let \( G_0 \) be the function defined in (2.2). Then, for all \( t, s \in [0, T] \),

\[
\begin{align*}
(a) & \quad -\frac{T}{4} \leq G_0(t, s) \leq 0 \\
(b) & \quad |G_0(t, s)| \leq s(T - s)/T \\
(c) & \quad \left| \frac{\partial G_0}{\partial t}(t, s) \right| \leq \max\{s/T, (T - s)/T\} \\
(d) & \quad \int_0^T |G_0(t, s)| ds \leq T^2/8 \\
(e) & \quad \int_0^T \left| \frac{\partial G_0}{\partial t}(t, s) \right| ds \leq T/2.
\end{align*}
\]

The new results presented in this chapter are restricted to the case in which \( f \) does not depend upon \( y' \); hence we can write the problem as

\[(2.3) \quad y'' = f(t, y), \quad y(0) = \alpha, \quad y(T) = \beta, \quad \text{or}\]
(2.4) \[ z(t) = \int_0^T G_0(t,s)[f(s,z(s) + \phi(s)) - \phi''(s)]ds, \]

where \( \phi \in U, \ U \) as defined in (1.2).

We further restrict ourselves to the finite-dimensional case, letting \( X \) be \( \mathbb{R}^d \), real d-space, and we induce a coordinatewise partial ordering on the space.

**Definition 2.2:** Let \( x,y \in \mathbb{R}^d \), and \( u,v \) be functions from \([0,T]\) into \( \mathbb{R}^d \). Then we define

\[ x \leq y \text{ if and only if } x_j \leq y_j \text{ for } j = 1,2,\ldots,d \]

\[ u \leq v \text{ if and only if } u(t) \leq v(t) \text{ for } 0 \leq t \leq T. \]

We write \( x \geq y, u \geq v \) if and only if \( y \leq x, v \leq u \), respectively.

Suppose that \( \phi \in U \) has the property that on \([0,T]\), \( \phi''(t) - f(t,u(t)) \) is either always nonnegative or always nonpositive whenever \( u \in U \) is near \( \phi \) with respect to the uniform metric. Since \( G_0 \) is a nonpositive function, the integral

\[ \int_0^T G_0(t,s)[f(s,u(s)) - \phi''(s)]ds \]
always has the algebraic sign of $\varphi''(t) - f(t,u(t))$. As a result, hypotheses need only be given for some positive or negative cone in the space $U_0$ defined in (1.3). We give the following.

**Theorem 2.3:** Let $X$ be real $d$-space, $\mathbb{R}^d$, and impose on it the partial ordering of Definition 2.2. Assume that there is a function $\varphi \in U$ and a positive number $L < T_2/8$ such that the function defined by $f(t,\varphi(t))$ is continuous for $0 \leq t \leq T$, and, if we define a positive number $K$ by

$$K = \max \left\| \int_0^T G_0(t,s) \{ f(s,\varphi(s)) - \varphi''(s) \} ds \right\|,$$

and a set $S$ by $S = \{ u \in U : u \leq \varphi, \| u - \varphi \| \leq 8K/(8 - T^2L) \}$, then $f$ and its Frechet derivative $f_2(t)$ are continuous functions from $[0,T] \times S$ into $\mathbb{R}^d$ and $[\mathbb{R}^d]$, respectively. Assume, further, that $\varphi$ and $L$ can be chosen so that, for any $y \in S$,

(i) $\varphi''(t) \leq f(t,y(t))$ for $0 \leq t \leq T$

(ii) $\| f_2(t,y(t)) \| \leq L$ for $0 \leq t \leq T$.

Then the problem (2.3) has a solution in $S$. 
Proof: Suppose first that condition (ii) were given to hold on all of $U$. Then $\|f_2(t, z(t) + \varphi(t))\| \leq L$ on $V_0 = \{ u \in C^0([0,T], X) : u(0) = u(T) = 0 \}$. Define a mapping $M$ on $V_0$ by

$$(Mz)(t) = \int_0^T G_0(t,s)\{f(s, z(s) + \varphi(s)) - \varphi''(s)\}ds.$$ 

Clearly $Mz \in V_0$ if $z \in V_0$, and if $u$ and $z$ are any functions in $U_0$ and $t$ is any point in $[0,T]$, then

$$(Mz - Mu)(t) = \int_0^T G_0(t,s)\{f(s, z(s) + \varphi(s)) - f(s, u(s) + \varphi(s))\}ds.$$ 

Applying Lemma 1.3, we find

$$(Mz - Mu)(t) = \int_0^T G_0(t,s)\left[\int_0^1 f_2(s, \xi z(s) + (1-\xi)u(s) + \varphi(s))d\xi\right] (z(s) - u(s))ds.$$ 

Since $V_0$ is a convex set, $\xi z + (1-\xi)u \in V_0$, so, by (ii),
Thus, \( \| (Mz-Mu)(t) \| \leq \frac{T^2}{8} \| z - u \|_\infty \). Since the above inequality holds for all \( t \in [0,T] \), we have

\[ \| Mz-Mu \|_\infty \leq \rho \| z - u \|_\infty, \]

where \( \rho = \frac{T^2}{8} < 1 \). Hence \( M \) is a contraction map from the Banach space \( V_o \) into itself, so has a unique fixed point, \( z \). Surely \( y = z + \varphi \) solves (2.3). The contraction mapping theorem further guarantees that \( \| z \|_\infty \leq \frac{K}{1-\rho} \). To see this, define a sequence \( \{ z_n \} \) by \( z_0 = 0, \quad z_{n+1} = Mz_n \) for \( n = 0,1,2,... \). Then

\[ z_n(t) = \int_0^T G_0(t,s)\{ f(s,\varphi(s)) - \varphi''(s) \}ds, \]

so \( \| z_n \|_\infty = K \). The contraction mapping theorem establishes the claim. The result is that the condition (ii) need not hold on all of \( V_o \); that the restriction

\[ \| u \|_\infty \leq \frac{K}{1-\rho} = \frac{8K}{8 - T^2} \]

is permissible.
We now show that (i) allows us to restrict the hypotheses to $S$. Recalling that $G_0$ is a nonpositive function, we apply condition (i) to (2.5) to see that $z_1(t) \leq 0$ for $0 \leq t \leq T$, so if we define $y_n = z_n + \phi$ for $n = 0, 1, 2, \ldots$, we have $y_1 \in S$. Consider any $n$ for which $y_n \in S$. Then

$$z_{n+1}(t) = \int_0^T G_0(t,s)\{f(s, z_n(s) + \phi(s)) - \phi''(s)\} ds < 0,$$

for $0 \leq t \leq T$, and since $\|z_{n+1}\|_{\infty} \leq K/(1 - \rho)$ we see that $y_{n+1} \in S$. Hypotheses may therefore be eliminated outside of $S$ and, since $S$ is closed, the theorem is established.

Q.E.D.

It is clear that a companion result holds. We state it without proof.

**Theorem 2.4:** Let $X$ be real $d$-space, $R^d$, and impose on it the partial ordering of Definition 2.2. Assume that there is a function $\phi \in \Phi$ and a positive real number $L$ such that the function defined by $f(t, \phi(t))$ is continuous for $0 \leq t \leq T$, and if we define a positive number $K$ by
\[ K = \max_{0 \leq t \leq T} \left\| \int_0^T G_0(t,s) \{ f(s,\varphi(s)) - \varphi''(s) \} ds \right\|, \quad \text{and} \]

a set \( S' \) by \( S' = \{ u \in U : u \geq \varphi, \| u - \varphi \| \leq 8K/(8 - T^2L) \} \),

then \( f \) and its Frechet derivative \( f_2 \) are continuous functions from \([0,T] \times S' \) into \( \mathbb{R}^d \) and \( [\mathbb{R}^d] \), respectively. Assume, further, that \( \varphi \) and \( L \) can be chosen so that, for any \( y \in S' \) and any \( t \in [0,T] \),

(i') \( \varphi''(t) \geq f(t,y(t)) \)

(ii) \( \| f_2(t,y(t)) \| \leq L \).

Then the problem (2.3) has a solution in \( S' \).

We give an example of Theorem 2.4.

Example 2.5: We let \( X \) be the set of all ordered pairs of real numbers, with norm, \( \| (x,y) \| = \max \{ |x|, |y| \} \). Consider the boundary value problem,

\[
\begin{cases}
  x''(t) = 1/(1-x-y), & x(0) = x(T) = 0 \\
  y''(t) = (x-y)^2/2, & y(0) = y(T) = 0,
\end{cases}
\]

(2.6)

where \( T = 3/2 \). Take \( \varphi(t) = (0,0) \) for all \( t \). With
\( f(t, (x,y)) \) defined by \( f(t, (x,y)) = \left( \frac{1}{1-x-y}, \frac{1}{2}(x-y)^2 \right) \), we observe that for \( x \leq 0, y \leq 0 \), we have \( f(t, (x,y)) \geq (0,0) \). Also,

\[
K = \max \left\| \int_0^T G_o(t,s)f(s,(0,0))ds \right\| = \max \int_0^T |G_o(t,s)|ds\| (1,0) \| = \frac{T^2}{8}.
\]

If we express the Frechet derivative as a Jacobian matrix, we have

\[
f_2(t, (x,y)) = \begin{pmatrix} 1/(1-x-y)^2 & 1/(1-x-y)^2 \\ x-y & y-x \end{pmatrix}.
\]

To estimate the norm of the above matrix, we recall that it is given by

\[
\|f_2(t, (x,y))\| = \max\{2/(1-x-y)^2, 2|x-y|\}.
\]

If we restrict \( (x,y) \) to the third quadrant of the ball of radius \( r \) about \((0,0)\); i.e. to the set \( \{(x,y); -r \leq x \leq 0, -r \leq y \leq 0\} \), we find \( \|f_2(t,x,y)\| = \max\{2,2r\} \). If \( r \leq 1 \), we are assured
\|f_2(t,(x,y))\| \leq 2. In that case we can take \( L \) to be 2. We have \( K = T^2/8 = 9/32 \) and \( L = 2 \). Surely \( L \leq 8/T^2 \). Further, \( 8K/(8-T^2L) = 9/14 < 1 \), and the hypotheses of Theorem 2.4 are satisfied. We conclude that the system (2.6), with \( T = 3/2 \) has a solution \( x(t),y(t) \), and that 

\[-9/14 \leq x(t),y(t) \leq 0 \text{ for } 0 \leq t \leq 3/2.\]

We point out that the classical theorems involving Picard's iteration approach fail to guarantee existence of a solution of this problem. (See, for example, Hartman [4], Chapter XII.) The difficulty is that the function \( f \) is badly behaved near \( x + y = 1 \), a problem that turns out to be unimportant, as the approximations stay in the third quadrant of the \( x-y \) plane.

By imposing on \( f_2 \) a condition that it is either "positive" (or "negative") for all functions \( u \in U \) near \( \varphi \) and all \( t \in [0,T] \), we are able to obtain a sequence of approximations which is monotonic. In that case the bounding of the sequence is sufficient to guarantee existence of a solution of our problem. The following theorem makes this idea precise.

**Theorem 2.6:** Let \( X \) be \( \mathbb{R}^d \) and suppose that there exists a \( \varphi \in U \) such that for any \( x \in X \) with \( x \leq 0 \), any
\( z \in U_0 \) with \( z(t) \leq 0 \) on \([0,T]\), and for any \( t \in [0,T] \), \( f \) and its Fréchet derivative \( f_2 \) are continuous, and

(i) \( \psi''(t) \leq f(t, z(t) + \varphi(t)) \)

(ii) \( 0 \leq f_2(t, z(t) + \varphi(t))x, \)

where coordinatewise partial ordering of \( \mathbb{R}^d \) is assumed.

Define a sequence in \( U_0 \) by \( z_0 = 0 \), and

\[
(2.7) \quad z_{n+1}(t) = \int_{0}^{T} G_o(t,s)[f(s, z_n(s) + \varphi(s)) - \psi''(s)]ds, \quad n \geq 0, \quad 0 \leq t \leq T.
\]

If the sequence is bounded, it converges. If the limit of the sequence is \( z \), then \( y = z + \varphi \) solves (2.3).

Proof: By (i), since \( z_0(t) = 0 \) and \( G_o(t,s) \leq 0 \), we have \( z_1(t) \leq 0 \) for \( 0 \leq t \leq T \). Furthermore, if \( z_n(t) \leq 0 \) for \( 0 \leq t \leq T \), the same is true of \( z_{n+1}(t) \). Hence the sequence consists of "negative" functions.

Suppose that \( z_n(t) \leq z_{n-1}(t) \) on \([0,T]\). Then

\[
z_{n+1}(t) - z_n(t) = \int_{0}^{T} G_o(t,s)[f(s, z_n(s) + \varphi(s)) - f(s, z_{n-1}(s) + \varphi(s))]ds.
\]
Appealing to Lemma 1.3, we have

\[ z_{n+1}(t) - z_n(t) = \int_0^T G_\varphi(t, s) \left[ \int_0^1 f(s, \xi z_n(s) + (1-\xi) z_{n-1}(s) + \psi(s)) \, d\xi \right] \right] (z_n(s) - z_{n-1}(s)) \, ds. \]

Now, \( \xi z_n(s) + (1-\xi) z_{n-1}(s) \leq 0 \) for all \( \xi \in [0,1], s \in [0,T] \). Hence by (ii), \( z_{n+1}(t) \leq z_n(t) \) for \( 0 \leq t \leq T \). Now we have seen that \( z_1(t) \leq 0 = z_0(t) \) on \([0,T] \), so by induction we have

\[ 0 \geq z_1(t) \geq z_2(t) \geq z_3(t) \geq \ldots \] for \( t \in [0,T] \).

By hypothesis the sequence is bounded. Since \( f \) is a continuous function, and since \( \varphi'' \) and \( G_\varphi \) do not depend on \( n \), we see that the sequence of derivatives,

\[ z_{n+1}'(t) = \int_0^T \frac{\partial G_\varphi}{\partial t}(t, s)[f(s, z_n(s) + \psi(s)) - \psi''(s)] \, ds, \]

is also uniformly bounded; hence the sequence \( \{z_n\} \) is uniformly bounded and equicontinuous, so by Ascoli's Lemma,
has a uniformly convergent subsequence. But the monotone
property of the sequence itself insures that the sequence
must converge uniformly, say to \( z \). We claim that \( z \)
solves the integral equation, (2.4). For, consider any
\( t \in [0,T], \) and any positive integer \( n \). Then

\[
z(t) - \int_0^T G_0(t,s) \left[ f(s,z(s) + \psi(s)) - \phi'(s) \right] ds = 0
\]

\[
= (z(t) - z_{n+1}(t)) + \int_0^T G_0(t,s) \left[ f(s,z_n(s) + \psi(s)) - f(s,z(s) + \psi(s)) \right] ds,
\]

so

\[
\|z(t) - \int_0^T G_0(t,s) \left[ f(s,z(s) + \psi(s)) - \phi'(s) \right] ds\| \leq (T^2/8) \max_{0 \leq s \leq T} \|f(s,z_n(s) + \psi(s)) - f(s,z(s) + \psi(s))\|.
\]

The first term on the right is arbitrarily small as a result
of the uniform convergence of the sequence \( \{z_n\} \); the
second is arbitrarily small because of the same uniform
convergence together with the continuity of \( f \). Hence the
claim, and the result follows. Q.E.D.
Again we state, without proof a companion result.

**Theorem 2.7:** Let $X$ be $\mathbb{R}^d$ and suppose that there exists a $\phi \in \mathcal{U}$ such that for any $x \in X$ with $x \leq 0$, any $z \in \mathcal{U}_0$ with $z(t) \leq 0$ on $[0,T]$, and for any $t \in [0,T]$, $f$ and its Frechet derivative $f_2$ are continuous, and

(i') $f(t,z(t) + \phi(t)) \leq \phi''(t)$

(ii') $f_2(t,z(t) + \phi(t))x \leq 0$,

where coordinatewise partial ordering is assumed on $\mathbb{R}^d$.

Define a sequence in $\mathcal{U}_0$ by $z_0 = 0$, and (2.7). If the sequence is bounded, it converges. If the limit of the sequence is $z$, then $y = z + \phi$ solves (2.3).

We give an example of Theorem 2.6.

**Example 2.8:** We let $X$ be the set of all ordered pairs of real numbers, with norm, $\|(x,y)\| = \max\{|x|,|y|\}$. Consider the boundary value problem,

\[
x''(t) = 10 - e^{x+y}, \quad x(0) = 0, \quad x(1) = 1.
\]

\[
y''(t) = 1.2 - \arctan(x+y), \quad y(0) = 0, \quad y(1) = 1.
\]
We take for our initial approximation, the segment, 
\[ \varphi(t) = (t, t). \] Then \[ \varphi'(t) = (0, 0). \] Letting

\[ f(t, (x, y)) = (10 - e^{x+y}, 1.2 - \arctan(x+y)), \]
we see that for \((x(t), y(t)) \leq (t, t) = \varphi(t),\)

\((0,0) \leq (10 - e^2, 1.2 - \arctan 2) \leq f(t, (x(t), y(t))) \leq (10, 1.2 + \pi/2).\)

Writing the Fréchet derivative \( f_2 \) as a matrix, we have

\[ f_2(t, (x, y)) = \begin{pmatrix}
-e^{x+y} & -e^{x+y} \\
-1/(1 + (x+y)^2) & -1/(1 + (x+y)^2)
\end{pmatrix}. \]

We observe that each component is negative; thus condition (ii) is satisfied. The sequence defined by \( z_0 = 0 \) and (2.7) consequently is a negative sequence which is decreasing. Since \( f(t(x,y)) \) is bounded above by \((10, 1.2 + \pi/2)\) for \((x,y) \leq (t,t),\) we see that

\[ z_{n+1}(t) \geq (-1^2/8)(10, 1.2 + \pi/2) \] for all \( n. \) In particular the sequence is bounded. The result is that the problem in
question has a solution \((x,y)\) such that

\[(t,t) - (5/4,(12+5\pi)/80) \leq (x(t),y(t)) \leq (t,t) \text{ for } 0 \leq t \leq 1.\]

Lasota and Yorke [9] give a set of conditions which can be used to bound the sequence \(\{z_n\}\). We state their lemma without proof.

**Lemma 2.9:** Let \(K\) and \(a\) be nonnegative constants. Let \(u(t)\) be a nonnegative real function satisfying the inequalities, \(u'(0) \leq 0\), \(u'(T) \leq 0\), \(u(0) \leq a u'(0)\),

\[u''(t) \geq -K[1 + (2 u(t))^{1/2} + |u'(t)|].\]

Then there exists a constant \(B_o\) (depending on \(K\) and \(a\) only) such that \(u(t) \leq B_o\), \(|u'(t)| \leq B_o\) for all \(t \in [0,T]\). We combine Lemma 2.9 and Theorem 2.6 to obtain

**Theorem 2.10:** Let \(R^d\) have the Euclidean norm, and suppose \(f\) satisfies the hypotheses of Theorem 2.6. Suppose further that there is a positive number \(K\) such that for any \(t \in [0,T]\), any \(x \leq 0\), and \(y \in R^d\), and any \(z \in U_o\) with
$z(t) \leq 0$ on $[0, T]$, 

(iii) $\|y\|^2 + x \cdot [f(t, z(t) + \varphi(t)) - \varphi''(t)] \geq -K[1 + \|x\| + |x \cdot y|],

where "•" indicates the usual inner product in real Euclidean space. Then (2.3) has a solution.

Proof: With the monotone sequence defined by $z_0 = 0$ and (2.7) we define scalar functions $u_n$ by

$u_n(t) = \frac{1}{2} z_n(t) \cdot z_n(t)$. Then $u_n'(t) = z_n(t) \cdot z'_n(t)$, and $u_n''(t) = \|z'_n(t)\|^2 + z_n(t) \cdot [f(t, z_n(t) + \varphi(t)) - \varphi''(t)],$

$n = 1, 2, \ldots$. By (iii) we have

$u_n''(t) \geq -K[1 + \|z_n(t)\| + |z_n(t) \cdot z'_n(t)|], \text{ or}

u_n''(t) \geq -K(1 + (2u_n(t))^{1/2} + |u_n'(t)|).$ Furthermore,

$u(0) = u'(0) = u(T) = 0$, and $u$ is surely a nonnegative function. Hence $\|z_n\|_{\infty} \leq (2 B_0)^{1/2}$. The remaining hypotheses guarantee the convergence of $\{z_n\}$ to a function $z$ such that $y = z + \varphi$ solves (2.8). Q.E.D.
In Chapter 2 results were obtained by regarding the function $f$ as a linear function, so that the iteration equation for $y'' = f(t,y,y')$ was the nonhomogeneous linear equation

$$y'' = A(t)y' + B(t)y + h(t),$$

in which $A$ and $B$ were the zero operators. Of course the function $h$ could not be dependent on $t$ alone if (3.1) were to be equivalent to the original equation. For that reason we refer to the technique as a quasilinearization. We might better adopt a notation which indicates the dependence of $A$, $B$, and $h$ upon the function $y$ and its derivative. We write

$$y'' = A(t;y)y' + B(t;y)y + h(t;y).$$

In this chapter and the next we make two other choices for $A$, $B$, and $h$ of the equation (3.2). The iteration technique of Chapter 2 will then be applied to develope a
sequence of approximations to the solution of (1.1) using as a recursion relation that $y_{n+1}$ solves the linear problem,

$$(3.3) \ y''(t) = A(t; y_n)y'(t) + B(t; y_n)y(t) + h(t; y_n), y(0) = \alpha, y(T) = \beta.$$ 

In order for the sequence defined here to converge to a solution of the original problem, (1.1), several conditions seem to be required. First, equation (3.2) must be equivalent to the original equation. This affects the choices of $A$, $B$, and $h$. Next, the linear problem (3.3) must have a unique solution, in order that the sequence be well defined. The conditions required here are those which satisfy the hypotheses of theorems involving existence and uniqueness theorems for linear problems, and most often require bounds on the norms of $A$ and $B$. Finally, we will look for $A$, $B$, and $h$ to change slowly as $y$ changes near the solution. This will involve some kind of Lipschitz requirements on $A$ and $B$.

We give two theorems of Heimes [5] which guarantee the existence and uniqueness of solutions of the homogeneous case of (3.1).
Theorem 3.1: For $A$ continuously differentiable and $B$ continuous on $[0,T]$ into the algebra of continuous linear operators on $X$, let

$$C(t) = R^{-1}(t)\left[\frac{1}{4} A^2(t) + b(t) - \frac{1}{2} A'(t)\right]R(t),$$

Where $R(t)$ is the solution in the operator space of $R' = \frac{1}{2} A R$, $R(0) = I$, where $I$ is the multiplication identity operator of $X$. If $\int_0^T s(T-s)\|C(s)\|ds < T$, the boundary value problem,

$$(3.4) \quad y'' = A(t)y' + B(t)y, y(0) = a, y(T) = b,$$

has a unique solution for every choice of $a, b \in X$.

Theorem 3.2: Let $h(s) = \|A(s)\| + s\|B(s)\|$, and assume

$$\max\left\{\int_0^T s h(s)ds, \int_0^T (T-s)h(s)ds\right\} < T. \quad \text{Then the boundary value problem (3.4) is uniquely solvable for all } a, b \in X.$$

In order to accommodate nonlinear equations, we require

Lemma 3.3: Let $h$ be a continuous function from the compact interval $[0,T]$ into $X$, and let $A$ and $B$ be
continuous functions from \([0,T]\) into the algebra of continuous linear operators on \(X\). If solutions of homogeneous boundary value problems (3.4) exist uniquely for all \(a, b \in X\), then the nonhomogeneous problem, (3.1), \(y(0) = \alpha, y(T) = \beta\) has a unique solution.

Proof: Suppose \(y_1, y_2\) solve the problems,

\[
y''_1(t) = A(t)y'_1(t) + B(t)y_1(t) + h(t), y_1(0) = y'_1(0) = 0.
\]

\[
y''_2(t) = A(t)y'_2(t) + B(t)y_2(t), y_2(0) = \alpha, y_2(T) = \beta - y_1(T),
\]

respectively. Then \(y_3 = y_1 + y_2\) solves (3.1), \(y(0) = \alpha, y(T) = \beta\). If \(y_4\) solves the same problem, then \(y_4 - y_3\) is a solution of the homogeneous equation.

\[
y''(t) = A(t)y'(t) + B(t)y(t), y(0) = y(T) = 0.
\]

Since by hypothesis, the trivial function is the only solution of that boundary value problem, we have \(y_4 = y_3\), which establishes uniqueness. Q.E.D.
We now consider a quasilinearization first applied by Lees [11] for the case in which there is no \( y' \) -dependence, and \( X \) is the real line. As before, let \( \varphi \) be in \( U \); i.e. \( \varphi \) takes the compact interval \([0,T]\) continuously into the Banach space \( X \), and satisfies \( \varphi(0) = \alpha, \varphi(T) = \beta \).

Suppose that \( f \) has Frechet derivatives \( f_2 \) and \( f_3 \), such that if \( y \) is any twice continuously differentiable function from \([0,T]\) to \( X \), then \( f_j(t,y(t),y'(t)) \) is continuously differentiable in \( t \) for \( j = 1,2 \). We summarize the above conditions on \( f \) by

**Definition 3.4:** \( f \) is said to be **smooth** at \( y \) in case it is as described above.

Suppose \( y \) solves the problem \((1.1)\). If \( z = y - \varphi \) we have observed that \( z \) solves

\[
(3.5) \quad z''(t) = f(t,z(t)+\varphi(t),z'(t)+\varphi'(t))-\varphi''(t), z(0)=z(T)=0.
\]

For any \( t \in [0,T] \), any \( \varphi \in U \), and any \( z \in U_0 \), define \( \psi : [0,1] \to X \) by

\[
\psi(\xi;t,z,\varphi) = f(t,\xi z(t) + \varphi(t),\xi z'(t) + \varphi'(t)) - \varphi''(t).
\]

Then \( \psi \) is differentiable with respect to \( \xi \) on \([0,1]\), and
\[ \psi'(\xi; t, z, \phi) = f_2(t, \xi z(t) + \phi(t), \xi z'(t) + \phi'(t)) z(t) + f_3(t, \xi z(t) + \phi(t), \xi z'(t) + \phi'(t)) z'(t). \]

If we integrate the above equation from \( \xi = 0 \) to \( \xi = 1 \), observing that \( \psi(1) = f(t, z(t) + \phi(t), z'(t) + \phi'(t)) - \phi''(t) \) and \( \psi(0) = f(t, \phi(t), \phi'(t)) - \phi''(t) \), we get

\[
f(t, z(t) + \phi(t), z'(t) + \phi'(t)) - \phi''(t) = A(t; z, \phi) z'(t) + B(t; z, \phi) z(t) + h(t; \phi),
\]

where

\[
\begin{align*}
A(t; z, \phi) &= f_3(t, \xi z(t) + \phi(t), \xi z'(t) + \phi'(t)) \\
B(t; z, \phi) &= f_2(t, \xi z(t) + \phi(t), \xi z'(t) + \phi'(t)) \\
h(t; \phi) &= f(t, \phi(t), \phi'(t)) - \phi''(t).
\end{align*}
\]

Note: Because our applications of the above operators will not involve the changing of \( \phi \) at any time during the proof of a theorem or the demonstration of an example, we shall suppress the dependence upon \( \phi \) by using the notation \( A(t; z), B(t; z), \) and \( h(t) \). If \( z \) is a solution of (3.5) we have
(3.7) \( z''(t) = A(t;z)z'(t) + B(t;z)z(t) + h(t), z(0) = z(T) = 0. \)

Conversely, a solution of (3.7) also solves (1.4).

We can now give

**Theorem 3.5:** Suppose there is a \( \varphi \in U \) and positive numbers \( \eta < 1 \) and \( L \) such that \( f(s,\varphi(s),\varphi'(s)) \) is continuous for \( s \in [0,T] \), and such that

\[
(i) \int_0^T \left| \frac{\partial G}{\partial t}(t,s) \right| \left( \|A(s;z)\| + s\|B(s;z)\| \right) ds \leq \eta
\]

for \( 0 \leq t \leq T \),

\[
(ii) \int_0^T \left| \frac{\partial G}{\partial t}(t,s) \right| \left( \|A(s;z) - A(s;u)\| + s\|B(s;z) - B(s;u)\| \right) ds \leq L\|z' - u'\|_\infty
\]

(iii) \( f \) is smooth at \( z + \varphi \),

for all \( z,u \in S = \{ u \in C'([0,T],X) : u(0) = u(T) = 0, \|u'\|_\infty \leq K/(1-\eta) \} \), where

\[
K = \max \| \int_0^T \frac{\partial G}{\partial t}(t,s) \{ f(s,\varphi(s),\varphi'(s)) - \varphi''(s) \} ds \|,
\]

and the operators \( A(s;z), B(s;z) \) are those defined in (3.6). If \( \varphi, \eta, L \) can be chosen so that \( KL < (1-\eta)^2 \), then problem
Proof: We first claim that for any $z \in S$ the boundary value problem,

$$
\dddot{z}(t) = A(t; z) \dot{z}'(t) + B(t; z) \dot{z}(t), \quad z(0) = a, \quad z(T) = b,
$$

has a unique solution for every $a, b \in X$. For we have seen in Lemma 2.1 that

$$
\frac{\partial G}{\partial t}(t, s) \leq \max\left\{ \frac{s}{T}, \frac{T-s}{T} \right\} \quad \text{for} \quad 0 \leq t \leq T.
$$

Since $\eta < 1$, it is clear that (i) implies that the hypotheses of Theorem 3.2 are satisfied, and hence the claim. By Lemma 3.3 we are assured that the problem,

$$
\dddot{z}(t) = A(t; z) \dot{z}'(t) + B(t; z) \dot{z}(t) + f(t, \varphi(t), \varphi'(t)) - \varphi''(t),
$$

$$
z(0) = z(T) = 0,
$$

is uniquely solvable, and hence, so is the equivalent integral equation,
We define a map $M$ on $S$ by $\hat{z} = Mz$, where $\hat{z}$ solves (3.8).

Then for each $t \in [0,T]$,

\[
\|\dot{\hat{z}}'(t)\| = \left\| \int_0^T g(t,s) \left[ A(s;z)\dot{\hat{z}}'(s) + B(s;z)\dot{\hat{z}}(s) + f(s,\varphi(s),\varphi'(s)) - \varphi''(s) \right] ds \right\|
\]

\[
\leq \left\| \int_0^T g(t,s) \left[ A(s;z)\dot{\hat{z}}'(s) + B(s;z)\dot{\hat{z}}(s) \right] ds \right\|
\]

\[
+ \left\| \int_0^T g(t,s) \left[ f(s,\varphi(s),\varphi'(s)) - \varphi''(s) \right] ds \right\|
\]

\[
\leq \int_0^T \left| g(t,s) \right| \left( \left\| A(s;z) \dot{\hat{z}}'(s) \right\| + \left\| B(s;z) \dot{\hat{z}}(s) \right\| \right) ds + K
\]

\[
\leq \int_0^T \left| g(t,s) \right| \left( \left\| A(s;z) \right\| \left\| \dot{\hat{z}}' \right\|_{\infty} + \left\| B(s;z) \right\| \left\| \dot{\hat{z}} \right\|_{\infty} \right) ds + K.
\]

\[
\|\dot{\hat{z}}'(t)\| \leq \eta\|\dot{\hat{z}}\|_{\infty} + K.
\]

Since the inequality holds for all $t \in [0,T]$, we have
\[(1 - \eta)\|z'\|_\infty \leq K, \text{ and dividing by the positive}\]

\[1 - \eta, \|z'\|_\infty \leq \frac{K}{1 - \eta}.\] We see that the mapping \(M\) takes \(S\) into itself.

Now consider any \(z, u \in S\), and let \(\hat{z} = Mz, \hat{u} = Mu\).

Then for any \(t \in [0, T]\), we have

\[
\|\hat{z}'(t) - \hat{u}'(t)\| = \left\| \int_0^T \frac{\partial G}{\partial t}(t, s) \left[ A(s; z)\hat{z}'(s) - A(s; u)\hat{u}'(s) + B(s; z)\hat{z}(s) - B(s; u)\hat{u}(s) \right] ds \right\|
\]

\[
= \left\| \int_0^T \frac{\partial G}{\partial t}(t, s) \left[ (A(s; z) - A(s; u))\hat{z}'(s) + (B(s; z) - B(s; u))\hat{z}(s) + A(s; u)(\hat{z}'(s) - \hat{u}'(s)) + B(s; u)(\hat{z}(s) - \hat{u}(s)) \right] ds \right\|
\]

\[
\leq \int_0^T \left| \frac{\partial G}{\partial t}(t, s) \right| \left\{ \|A(s; z) - A(s; u)\| \|\hat{z}'\|_\infty + s\|B(s; z) - B(s; u)\| \|\hat{z}'\|_\infty \right\} ds +
\]

\[
+ \int_0^T \left| \frac{\partial G}{\partial t}(t, s) \right| \left\{ \|A(s; u)\| \|\hat{z}' - \hat{u}'\|_\infty + s\|B(s; u)\| \|(\hat{z} - \hat{u})'\|_\infty \right\} ds.
\]
where we have used $\hat{z}(s) = \int_0^s \hat{z}'(r)dr$ in obtaining the last inequality. If we factor $\|\hat{z}'\|_\infty$ from the first integral, and $\|(\hat{z} - \hat{u})'\|_\infty$ from the second, then apply hypotheses (i), (ii), we have

$$\|\hat{z}'(t) - \hat{u}'(t)\|_\infty \leq L\|\hat{z} - \hat{u}\|_\infty + \eta\|(\hat{z} - \hat{u})'\|_\infty \text{ for } t \in (0,T].$$

Hence, recalling that $\hat{z} \in S$, $(1-\eta)\|(\hat{z} - \hat{u})'\|_\infty \leq \frac{LK}{1-\eta}\|(z-u)'\|_\infty$, or $\|(\hat{z} - \hat{u})'\|_\infty \leq \frac{LK}{(1-\eta)^2}\|(z-u)'\|_\infty$.

Now $S$, together with the metric $\|(z-u)'\|_\infty$, forms a complete metric space. Consequently the map $M$ has a unique fixed point $z$ in $S$, which is a solution of (3.5). It follows that $y = z + \phi$ solves (1.1).

**Example 3.6:** Let $X = \mathbb{R}^2$ and consider the scalar system,

$$x'' = \frac{1}{4}(xx' + y), \quad x(0) = 0, \quad x(1) = \frac{1}{8}$$

$$\dot{y}'' = \frac{1}{5}(x' - y^2), \quad y(0) = 1, \quad y(1) = .9.$$
We use the notation \( \vec{y} = (x,y) \), let \( ||\vec{y}|| = \max{|x|, |y|} \)

\[
\vec{f}(t,\vec{y},\vec{y}') = \left( \frac{1}{4}(xx' + y), \frac{1}{5}(x' - y^2) \right).
\]

For our initial approximation we choose \( \vec{\varphi}(t) = \left( \frac{t^2}{8}, 1 - \frac{t^2}{10} \right) \). Then

\[
\begin{align*}
\vec{\varphi}'(t) &= \left( \frac{t}{4}, -\frac{t}{5} \right), \\
\vec{\varphi}''(t) &= \left( \frac{1}{4}, -\frac{1}{5} \right),
\end{align*}
\]

and

\[
\begin{align*}
\vec{f}(t,\vec{\varphi}(t),\vec{\varphi}'(t)) &= \left( \frac{1}{4} \left( \frac{t^2}{8} + 1 - \frac{t^2}{10} \right), \frac{1}{5} \left( 4 - \left( 1 - \frac{t^2}{10} \right)^2 \right) \right) \\
&= \left( \frac{1}{4} \left( \frac{3}{32} - \frac{t^2}{10} + 1 \right), \frac{1}{5} \left( -\frac{t^4}{100} + \frac{t^2}{5} + \frac{t}{4} - 1 \right) \right).
\end{align*}
\]

We have

\[
\begin{align*}
\vec{f}(s,\vec{\varphi}(s),\vec{\varphi}'(s)) - \vec{\varphi}''(s) &= \left( \frac{1}{4} \left( \frac{s^2}{32} - \frac{s^2}{10} \right), \frac{1}{5} \left( -\frac{s^4}{10} + \frac{s^2}{5} + \frac{s}{4} \right) \right),
\end{align*}
\]

so from (2.2),

\[
\begin{align*}
&\int_0^1 \frac{\partial G^0}{\partial t}(t,s) \left[ \vec{f}(s,\vec{\varphi}(s),\vec{\varphi}'(s)) - \vec{\varphi}''(s) \right] ds = \\
&\int_0^t \left( \frac{1}{4} \left( \frac{s^2}{32} - \frac{s^2}{10} \right), \frac{1}{5} \left( -\frac{s^4}{10} + \frac{s^2}{5} + \frac{s}{4} \right) \right) ds + \\
&\int_0^1 (s - 1) \left( \frac{1}{4} \left( \frac{s^3}{32} - \frac{s^2}{10} \right), \frac{1}{5} \left( -\frac{s^4}{10} + \frac{s^2}{5} + \frac{s}{4} \right) \right) ds = \\
&\left( \frac{1}{4} \left( \frac{t^4}{128} - \frac{t^3}{30} + \frac{13}{1920} \right), \frac{1}{5} \left( -\frac{t^5}{50} + \frac{t^3}{30} + \frac{t^2}{8} - \frac{11}{200} \right) \right). \quad \text{The derivative of}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{4} \left( \frac{t^4}{128} - \frac{t^3}{30} + \frac{13}{1920} \right)
\end{align*}
\]

is seen to be negative on \([0,1]\). The extreme values are \( \frac{13}{7680} \) and \( -\frac{9}{1920} \).

The derivative of \( \frac{1}{5} \left( -\frac{t^5}{50} + \frac{t^3}{30} + \frac{t^2}{8} - \frac{11}{200} \right) \) is positive on...
[0,1], so extreme values are at endpoints also. They are $-\frac{11}{1000}$ and $\frac{1}{60}$. Hence the $K$ of Theorem 3.5 can be taken to be $\frac{1}{60}$. Let $\overrightarrow{z}(t) = (z_1(t), z_2(t))$; then since

$$\overrightarrow{f}_2(t, y) = \begin{pmatrix} \frac{1}{4} x'(t) & 1 \\ 0 & -\frac{2}{5} y(t) \end{pmatrix},$$

we have

$$B(s; \overrightarrow{z}) = \int_0^1 \begin{pmatrix} \frac{1}{4} \left(\frac{s}{4} z_1'(s) + \frac{s}{4}\right) & 1 \\ 0 & -\frac{2}{5} \left(\frac{s}{2} z_2(s) + 1 - \frac{s^2}{10}\right) \end{pmatrix} \, d\xi =$$

$$\begin{pmatrix} \frac{z_1'(s)}{8} + \frac{s}{16} & \frac{1}{4} \\ 0 & -\frac{2}{5} \left(\frac{z_2(s)}{2} + 1 - \frac{s^2}{10}\right) \end{pmatrix}.$$ 

Recalling that the norm of a matrix is the largest of the sums of the absolute values of the elements in each row, we have

$$\|B(s; \overrightarrow{z})\| = \max\left\{ \left| \frac{2z_1'(s) + s}{16} \right| + \frac{1}{4}, \left| -\frac{2}{5} \left( \frac{z_2(s)}{2} + 1 - \frac{s^2}{10} \right) \right| \right\}$$

for $0 \leq s \leq 1$. Similarly,
\[ f_3(t, y) = \begin{pmatrix} \frac{x}{4} & 0 \\ \frac{1}{5} & 0 \end{pmatrix}, \text{ so} \]

\[
A(s; z) = \int_0^1 \begin{pmatrix} \frac{\xi z_1(s) + \frac{s^2}{8}}{4} & 0 \\ \frac{1}{5} & 0 \end{pmatrix} d\xi = \begin{pmatrix} \frac{z_1(s)}{8} + \frac{s^2}{32} & 0 \\ \frac{1}{5} & 0 \end{pmatrix}.
\]

Hence \( \|A(s; z)\| = \max\left\{ \frac{1}{32} |s^2 + 4z(s)|, \frac{1}{5} \right\} \) for each \( s \).

At this point we make the restriction \( \frac{k}{1 - \eta} \leq 1 \), so \( \eta \leq \frac{59}{60} \), where \( \eta \) is to be as in the hypotheses of the theorem. This means that for \( z \in S, \|z'\|_\infty \leq 1 \). Consequently \( \|z\|_\infty \leq \frac{1}{2} \), so \( \|A(s; z)\| = \max\left\{ \frac{s^2 + 1}{32}, \frac{1}{5} \right\} = \frac{1}{5} \) for all \( s \), and

\[
\|B(s; z)\| = \max\left\{ \frac{2 + s}{16}, \frac{4}{16}, \frac{1}{2} - \frac{s^2}{25} \right\} = \frac{1}{2} - \frac{s^2}{25}. \]

Let

\[
h(s) = \|A(s; z)\| + s\|B(s; z)\| = \frac{7}{10} - \frac{s^2}{25}. \]

Then

\[
\int_0^1 sh(s) = \frac{34}{100} < \frac{7}{20}, \text{ and } \int_0^1 (1-s)h(s)ds = \frac{104}{300} < \frac{7}{20}.
\]

We recall that in Lemma 2.1 we observed
\[
\left| \frac{\partial G}{\partial t}(t,s) \right| \leq \max\{s,1-s\} \text{ for each } t \in [0,1]. \text{ It follows that we may take } \eta = \frac{7}{20} \text{ in condition (i) of the theorem.} \\
\text{We note that } \eta \leq \frac{59}{60}. \text{ Now if } \vec{u} \in S, \text{ then}
\]
\[
\|A(s;\vec{z}) - A(s;\vec{u})\| = \begin{bmatrix}
    \frac{z_1(s) - u_1(s)}{8} & 0 \\
    0 & 0 \\
    0 & 0
\end{bmatrix}.
\]
\[
\frac{|z_1(s) - u_1(s)|}{8} \leq \frac{|z'_1(s) - u'_1(s)|}{16} \leq \frac{1}{16} \|\vec{z}' - \vec{u}'\|_\infty, \text{ and}
\]
\[
\|B(s;\vec{z}) - B(s;\vec{u})\| = \begin{bmatrix}
    \frac{|z'_1(s) - u'_1(s)|}{8} & 0 \\
    0 & \frac{u_2(s) - z_2(s)}{5}
\end{bmatrix}.
\]
\[
\leq \max\left\{\frac{|z'_1(s) - u'_1(s)|}{8}, \frac{|u'_2(s) - z'_2(s)|}{16}\right\} \leq \frac{1}{8} \|\vec{z}' - \vec{u}'\|_\infty, \text{ so}
\]
\[
\int_0^1 \left| \frac{\partial G}{\partial t}(t,s) \right| \|A(s;\vec{z}) - A(s;\vec{u})\| + s\|B(s;\vec{z}) - B(s;\vec{u})\| ds \leq \int_0^1 \left| \frac{\partial G}{\partial t}(t,s) \right| (1 + \frac{s}{8}) ds \|\vec{z}' - \vec{u}'\|_\infty = \frac{1}{16} \int_0^1 \left| \frac{\partial G}{\partial t}(t,s) \right| ds \|\vec{z}' - \vec{u}'\|_\infty.
\]
Once again we recall that $|\frac{\partial G^0(t,s)}{\partial t}| \leq \max\{s, 1-s\}$. Since

$$\frac{1}{16} \int_0^1 (2s^2 + s) ds = \frac{9}{96} < \frac{1}{10}, \quad \text{and} \quad \frac{1}{16} \int_0^1 (1-s)(2s+1) ds < \frac{1}{10},$$

we have

$$\int_0^1 \left| \frac{\partial G^0(t,s)}{\partial t} \right| \left( \|A(s; \overrightarrow{z}) - A(s; \overrightarrow{u})\| + s \|B(s; \overrightarrow{z}) - B(s; \overrightarrow{z})\| \right) ds \leq \frac{1}{10} \|\overrightarrow{z'} - \overrightarrow{u'}\|_\infty.$$ Let $L = \frac{1}{10}$. We have three positive numbers, $K = \frac{1}{60}$, $\eta = \frac{7}{20}, L = \frac{1}{10}$, for which the hypotheses of the theorem are satisfied. The result follows.
IV. NEWTON'S METHOD

A numerical technique for approximating solutions of boundary value problems is based on another quasi-linearization. The technique is known as Newton's method, for reasons that will be made clear presently, and was first used by Bellman [2]. The original applications by Bellman, and by Bellman and Kalaba [3] were to dynamic programming. The technique has since been applied to boundary value problems (see, for example, Lee [10]). Other numerical schemes are shooting methods and finite difference methods. The former uses classical methods for solving initial value problems in which the correct data is used for the function value at the left end point. Adjustments are made to the initial "slope" until the boundary value at the right end point is well approximated. The finite difference schemes involve the approximation of the boundary value problem by a large linear system of algebraic equations. Quasi-linearization is used when solutions of initial value problems are too sensitive to initial conditions to allow the shooting methods to be effective, yet when the preparation necessary for the finite difference methods is
excessive (see Roberts and Shipman [14]).

In order to motivate the particular quasilinearization and its name, we recall Newton's Method for approximating the solution of a system of two nonlinear algebraic equations in two unknowns,

\[ f(x,y) = 0 \]
\[ g(x,y) = 0, \]

where \( x, y, f(x,y), g(x,y) \) are real numbers. To proceed, one chooses an initial estimate \((x_0, y_0)\) of the solution \((x^*, y^*)\) of the system, then writes the Taylor expansions of \( f \) and \( g \) about \((x_0, y_0)\), neglecting all but the linear terms. Let us write

\[
F(x,y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)
\]
\[
G(x,y) = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0).
\]

An assumption is made that \( F \) and \( G \) are reasonably good approximations of \( f \) and \( g \), respectively, when \((x,y)\) is close to \((x^*,y^*)\). We define the point \((x_1, y_1)\) to be such that \( F(x_1, y_1) = G(x_1, y_1) = 0 \). This gives a system of
two linear, algebraic equations in two unknowns,

\[ 0 = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x_1 - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y_1 - y_0) \]

\[ (4.1) \]

\[ 0 = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x_1 - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y_1 - y_0). \]

This system has a solution if the Jacobian matrix,

\[
\begin{pmatrix}
    \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
    \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\]

is nonsingular.

If \((x_0, y_0)\) is close enough to the solution \((x^*, y^*)\), there is reason to hope that \(F, G\) are good enough approximations to \(f, g\) respectively, so that \((x_1, y_1)\) is an even better approximation of the solution.

To obtain the next approximation, \((x_0, y_0)\) is replaced in the preceding discussion by \((x_1, y_1)\), and \((x_1, y_1)\) by \((x_2, y_2)\). The principal concerns that occur in using this method are that (4.1) be solvable, and that \((x_0, y_0)\) be near enough to the solution \((x^*, y^*)\).
Now consider the vector differential equation
\[ q(t,y,y',y'') = 0, \] where \( q : [0,T] \times X \times X \times X \to X \) has Frechet derivatives wherever required for the discussion. Regarding \( t \) as a parameter, we write the first order approximation of \( q \) about \( (t,y_o,y'_o,y''_o) \) as
\[ Q(t,y,y',y'') = q(t,y'_o,y'_o,y''_o) + q_2(y-y'_o) + q_3(y'-y'_o) + q_4(y''-y''_o), \]
where \( q_2, q_3, \) and \( q_4 \) denote the Frechet derivatives of \( q \) with respect to \( y, y', \) and \( y'' \), respectively, at \( (t,y_o,y'_o,y''_o) \). We intend to find \( y_1 \) so that the left hand side of the above equation vanishes. The task can be made somewhat less difficult by observing that we are interested in dealing with the differential equation, \( y'' = f(t,y,y') \). We thus want \( q(t,y,y',y'') = y'' - f(t,y,y') \), so
\[ q_2 = -f_2, \quad q_3 = -f_3, \quad \text{and} \quad q_4 \] is the identity operator, \( I \). This gives the following equation to solve for \( y_1 \):
\[ 0 = y'' - f - f_2 \cdot (y_1 - y'_o) - f_3 \cdot (y'_1 - y'_o) + I(y''_1 - y''_o), \]
where \( f, f_2, \) and \( f_3 \) are all evaluated at \( (t,y'_o,y'_o) \). Rearranging, we have
\[ y''_1 = f_2 \cdot (y_1 - y_0) + f_3 \cdot (y'_1 - y'_0) + f, \quad \text{or} \]
\[ y''_1 = f_3 \cdot y'_1 + f_2 \cdot y_1 + (f - f_3 \cdot y'_0 - f_2 \cdot y_0). \]

We therefore define a sequence \{y_n\} of approximations by \( y_0(t) = \varphi(t) \) for \( 0 \leq t \leq T \), and for \( n = 0, 1, \ldots \), \( y_n \) is the solution of the boundary value problem,

\[(4.2) \quad y'' = f_3(t, y_n(t), y'_n(t))y' + f_2(t, y_n(t), y'_n(t))y + h_n(t), \]
\[
y(0) = \alpha, \quad y(T) = \beta,
\]
where \( h_n(t) = f(t, y_n(t), y'_n(t)) - f_3(t, y_n(t), y'_n(t))y'_n(t) - f_2(t, y_n(t), y'_n(t))y_n(t). \)

McGill and Kenneth [12] have considered the case in which \( f \) does not depend upon \( y' \) (\( f_3 \) is the zero operator), and \( X = \mathbb{R}^d \). They conclude that if \( f \) is continuous and if \( f_2 \) is Lipschitz and continuous, then, for small enough \( T \) a solution exists, and the sequence defined converges to it. Since existence can be established for small enough \( T \) at relatively low cost by other means, and since \( T \) is not a part of the problem that can be varied
at will, the existence part of the theorem is not very help-
ful. In the theorem that follows we give a result more in
keeping with Newton's method in that the principal concerns
are that certain linear equations be solvable (uniquely),
and that our initial estimate, \( \varphi \), come close enough to
solving the problem with respect to some natural metric.

In the presentation we adopt the notation,

\[
f(t; z) = f(t, z(t) + \varphi(t), z'(t) + \varphi'(t))
\]

\[
f_j(t; z) = f_j(t, z(t) + \varphi(t), z'(t) + \varphi'(t)), \quad j = 1, 2.
\]

**Theorem 4.1:** Suppose that there is a \( \varphi \in \mathcal{U} \) and positive
real numbers \( K, L, \eta \), such that \( \eta < 1 \) and such that if
\( z, u, \) are any continuously differentiable functions from
the compact interval \([0, T]\) to the Banach space \( X \) with
\( z(0) = u(0) = z(T) = u(T) = 0 \) and \( \|z'\|_\infty \leq B, \|u'\|_\infty \leq B, \)
where \( B = K/(1 - \eta) \), then the following hold:

\[
(i) \left\| \int_0^T \frac{\partial G}{\partial t}(t, s) \{\varphi''(s) - f(s; z) - f_2(s; z) z(s) - f_3(s; z) z'(s)\} \right\| ds \leq K
\]

\[
(ii) \int_0^T \left| \frac{\partial G}{\partial t}(t, s) \right| \left[ s\|f_2(s; z)\| + \|f_3(s; z)\| \right] ds \leq \eta,
\]
If $2BL < 1 - 3\eta$, then a solution exists, and the sequence (4.2) converges to it.

Proof: Let $S = \{v \in \mathbb{C}'(\left[0,T\right],X) : v(0) = v(T) = 0, \|v'\|_\infty \leq B\}$. Define a map $M$ on $S$ by $Mz = \hat{z}$, where $\hat{z}$ is the unique solution of

$$
(4.3) \quad \hat{z}(t) = \int_0^T G_0(t,s) \left\{ f_2(s;\hat{z}) (\hat{z}-z)(s) + f_3(s;z)(\hat{z}-z')(s) \right\} ds + f(s;z)-\varphi''(s) \}
ds.
$$

(the existence and uniqueness of a solution of (4.3) is established by applying the first two inequalities in the hypotheses in a manner identical to that of equation (3.8) in the proof of theorem 3.5.) We now show that $M$ takes $S$ back into itself. If $z$ is any function in $S$ and $t$ is any number in the interval $[0,T]$, we have
\[ \| \dot{z}'(t) \| = \left\| \int_0^T \frac{\partial G}{\partial t}(t,s) \left[ f_2(s;z) \dot{z}(s) + f_3(s;z) \dot{z}'(s) + [f(s;z) - f_2(s;z) z(s) - f_3(s;z) z'(s) - \varphi''(s)] \right] ds \right\| \]

\[ \leq \int_0^T \left\| \frac{\partial G}{\partial t}(t,s) \left[ f_2(s;z) \dot{z}(s) + f_3(s;z) \dot{z}'(s) \right] \right\| ds + \int_0^T \left\| \frac{\partial G}{\partial t}(t,s) \left[ \varphi''(s) - (f(z;s) - f_2(z;s) z(s) - f_3(z;s) z'(s)) \right] ds \right\| . \]

Applying condition (i) and usual properties of norms, we have

\[ \| z'(t) \| \leq \int_0^T \left\| \frac{\partial G}{\partial t}(t,s) \left[ \| f_2(s;z) \| \| \dot{z}(s) \| + \| f_3(s;z) \| \| \dot{z}'(s) \| \right] ds + K \]

\[ \leq \int_0^T \left\| \frac{\partial G}{\partial t}(t,s) \left[ \| f_2(s;z) \| s \| \dot{z}'(s) \| _\infty + \| f_3(s;z) \| \| \dot{z}'(s) \| _\infty \right] ds + K, \]

and by condition (ii), we have, \( \| z'(t) \| \leq \eta \| z' \| _\infty + K \), for each \( t \in [0,T] \). Hence, \( \| z' \| _\infty \leq K/(1 - \eta) \), so \( z \in S \), and the mapping \( M \) is indeed into. Consider now \( z, u \in S \), and \( t \in [0,T] \).
\[ \dot{z}'(t) - \dot{u}'(t) = \int_0^T \frac{\partial G}{\partial t}(t,s) \{ f_2(s;z)(\dot{z}(s)-z(s)) - f_2(s;u)(\dot{u}(s)-u(s)) + f_3(s;z)(\dot{z}'(s)-z'(s)) - f_3(s;u)(\dot{u}'(s)-u'(s)) + f(s;z)-f(s;u) \} ds. \]

\[ \dot{z}'(t) - \dot{u}'(t) = \int_0^T \frac{\partial G}{\partial t}(t,s) \{ f_2(s;z)(\dot{z}(s)-\dot{u}(s)) + f_2(s;u)(u(s)-z(s)) + [f_2(s;z)-f_2(s;u)](\dot{u}(s)-z(s)) + f_3(s;z)(\dot{z}'(s)-\dot{u}'(s)) + f_3(s;u)(u'(s)-z'(s)) + [f_3(s;z)-f_3(s;u)](\dot{u}'(s)-\dot{z}'(s)) + [f(s;z)-f(s;u)] \} ds. \]

We apply the triangle inequality to get
(4.4) \[ \| \hat{z}'(t) - \hat{u}'(t) \| \leq W_1 + W_2 + W_3 + W_4, \]

where

\[ W_1 = \left\| \int_0^T \frac{\partial G}{\partial t}(t,s) \left[ f_2(s;z)(\hat{z}(s) - \hat{u}(s)) + f_3(s;z)(\hat{z}'(s) - \hat{u}'(s)) \right] ds \right\|, \]

\[ W_2 = \left\| \int_0^T \frac{\partial G}{\partial t}(t,s) \left[ f_2(s;u)(u(s) - z(s)) + f_3(s;u)(u'(s) - z'(s)) \right] ds \right\|, \]

\[ W_3 = \left\| \int_0^T \frac{\partial G}{\partial t}(t,s) \left[ [f_2(s;z) - f_2(s;u)](\hat{u}(s) - z(s)) + [f_3(s;z) - f_3(s;u)](\hat{u}'(s) - z'(s)) \right] ds \right\|, \]

and

\[ W_4 = \left\| \int_0^T \frac{\partial G}{\partial t}(t,s) \left[ f(s;z) - f(s;u) \right] ds \right\|. \]

To get bounds on \( W_1 \) and \( W_2 \), we notice that for \( v \in S \), \( \| v(s) \| \leq s \| v' \|_{\infty} \), then apply familiar inequalities with (ii) to get

(4.5) \[ W_1 \leq \eta \| \hat{z}' - \hat{u}' \|_{\infty}, \quad W_2 \leq \eta \| z' - u' \|_{\infty}. \]
is bounded by applying the same observations to condition (iii). We obtain

\[ W_3 \leq L\|z' - u'\|_{\infty}[\|\hat{u}'\|_{\infty} + \|z'\|_{\infty}]. \]

Since \( \hat{u} \) and \( z \) are in \( S \), we can further bound \( W_3 \) to \( W_3 \leq 2BL\|z' - u'\|_{\infty} \).

In order to bound \( W_4 \) we appeal to a mean value theorem of the type that was stated and proved as Lemma 1.3. The proof is similar. Thus

\[
\begin{align*}
  f(s;z) - f(s;u) &= \int_{0}^{1} \left[ f_2(s, \xi z(s) + (1-\xi)u(s) + \varphi(s), \xi z'(s) + (1-\xi)u'(s)\right] (z(s) - u(s)) + \\
  &\quad \left[ \int_{0}^{1} f_3(s, \xi z(s) + (1-\xi)u(s) + \varphi(s), \xi z'(s) + (1-\xi)u'(s)\right] (z'(s) - u'(s)),
\end{align*}
\]

for each \( s \in [0,T] \). Since \( \xi z + (1-\xi)u \in S \), we have \( W_4 \leq \eta\|z' - u'\|_{\infty} \). Applying this bound and those in inequalities (4.5) and (4.6) to (4.4) we find that for each \( t \in [0,T] \),
\[ ||\dot{z}'(t) - \dot{u}'(t)|| \leq \eta ||\dot{z}' - \dot{u}'||_\infty + [2\eta + 2BL]||z' - u'||_\infty. \]

Consequently, we have \[ ||\dot{z}' - \dot{u}'||_\infty \leq \rho ||z' - u'||_\infty, \] where \[ \rho = 2(\eta + BL)/(1 - \eta), \] which is strictly less than 1 by hypothesis. \( M \) is contraction map on the complete metric space \( S \), where the distance between functions \( z \) and \( u \) is given by \( ||z' - u'||_\infty \). The result follows. Q.E.D.

If \( f \) is independent of \( y' \), so that the problem can be expressed as

\[
(4.7) \quad y'' = f(t,y), \quad y(0) = a, \quad y(T) = \beta,
\]

we can apply the same kind of argument to obtain

**Theorem 4.2:** Suppose that there is a \( \varphi \in U \) and positive real numbers \( K, L, \eta \), such that \( \eta < 1 \) and such that if \( z, u \) are any continuously differentiable functions from the compact interval \([0,T]\) to the Banach space \( X \), with \( z(0) = z(t) = u(0) = u(T) = 0 \), and \( ||z'||_\infty \leq B, ||u'||_\infty \leq B \), where \( B = K/(1 - \eta) \), then the following holds:
If $2BL < 1 - 3\eta$, then a solution of (4.7) exists, and the sequence defined by $z_0 = 0$,

$$z_{n+1}(t) = \int_0^T G_o(t,s) \left[ f_2(s,z_n(s)+\varphi(s)) \left( z_{n+1}(s) - z_n(s) \right) + f(s,z_n(s)+\varphi(s)) - \varphi''(s) \right] ds \quad n = 0, 1, \ldots,$$

converges to a function $z$ having the property that $\varphi + z$ is that solution.

We give an example of Theorem 4.2.
Example 4.3: We let $X$ be the set of ordered pairs of real numbers $(x,y)$, and consider the problem,

$$(4.8) \quad (x,y)^{\prime\prime} = (tx+y, t^2 + xy), \quad (x,y)(0) = (x,y)(1) = 0.$$

We take $(0,0)$ to be the initial estimate, $\varphi(t)$, and write the Frechet derivative as $f_2(t,(x,y)) = \begin{pmatrix} t & 1 \\ y & x \end{pmatrix}$.

where, as usual, $f$ represents the right hand side of the differential equation. Then, using the norm $\| (x,y) \| = \max\{|x|,|y|\}$, we have

$$\| \int_0^1 G_0(t,s) \{ -f(s,z(s)+\varphi(s)) + f_2(s,z(s)+\varphi(s)) z(s) \} ds \| = \| \int_0^t G_0(t,s) (0, s^2 - x(s)y(s)) ds \| = \| (0, \int_0^t s(t-1)[s^2 - x(s)y(s)] ds) + \int_0^t t(s-1)[s^2 - x(s)y(s)] ds \| \leq (t-1) \int_0^t [s^3 - sx(s)y(s)] ds + t \int_0^t [s^3 - s^2 - (s-1)x(s)y(s)] ds \|.$$

Suppose, for a moment, that the $B$ of the theorem is no
larger than $1/2$. Then we will have $|x(s)y(s)| \leq 1/4$ for $0 \leq s \leq 1$. Write the last expression above as

$$
\sum_{t}^{t} (s^3)ds + t \sum_{s,t}^{s^3-s^2} ds - t \sum_{s}^{s} xs(s)y(s)ds \sum_{0}^{t} xs(s)y(s)ds + t \sum_{0}^{t} x(s)y(s)ds.
$$

Evaluating, and using the simplifying assumption that $x(s)y(s) < 1$, we bound the quantity by

$$
\left| \frac{t^4 - t}{12} + \frac{(1/2)^2}{2} (3t - t^2) \right|,
$$

which is bounded on the interval, $0 \leq t \leq 1$ by $3/10$. Accordingly, we tentatively choose $K = 3/10$. Viewing $f_2$, we see that $\|f_2(t, (x,y))\| \leq 2$, if $0 < |x|, |y| < 1/2$, so $\eta$ may be chosen as $1/4$. Again $\|f_2(s, z(s)+\phi(s)) - f_2(s, u(s)+\phi(s))\| < 2\|z - u\|_\infty$ so we take $L = 1/4$, also. Then, $B = K/(1 - \eta) = 4/10 < 1/2$. We now have positive numbers, $K = 3/10$, $L = \eta = 1/4$, such that if $z, u \in S$, then (i), (ii), (iii) follow. Since $2BL = 2/10 < 1/4 = 1 - 3\eta$, we see that a solution of the problem (4.8) exists.
Consider the homogeneous linear differential equation,

\[ y'' = A(t)y' + B(t)y, \tag{5.1} \]

where \( y \) takes the compact interval \([0,T]\) into the Banach space \( X \), and \( A \) and \( B \) take \([0,T]\) continuously into the algebra \([X]\) of bounded linear operators on \( X \).

**Definition 5.1:** A function \( G : [0,T] \times [0,T] \to [X] \) is called a Green's function for the equation (5.1) if, for every continuous function \( g \) from \([0,T]\) into \( X \), the solution of the boundary value problem,

\[ y'' = A(t)y' + B(t)y + g(t), \quad y(0) = y(T) = 0, \]

is given by \( y(t) = \int_0^T G(t,s)g(s)\, ds \).

As an example, let \( G(t,s) = G_0(t,s)I \), where \( G_0(t,s) \) is defined in (2.1), and \( I \) is the multiplicative identity in \([X]\). Then \( G \) is the Green's function for (5.1) when \( A(t) \) and \( B(t) \) are the zero operator for \( 0 \leq t \leq T \).
Let us review the approach that has been used in establishing the results of Chapters 2-4. Quasi-linearizations of the original equation,

\[ y'' = f(t,y,y') , \]

are effected as

\[(5.2) \quad y'' = A(t;y)y' + B(t;y)y + h(t;y),\]

where the notation suggests the dependence of \( A, B, \) and \( h \) upon \( y \). If, for simplification of the immediate discussion, we assume the zero boundary conditions \( y(0) = y(T) = 0 \), the next step can be thought of as defining a sequence \( \{y_n\} \) of approximate solutions by

\[ y_{n+1} = 0, \quad y_{n+1} = \text{solution of the integral equation,} \]

\[ y_{n+1}(t) = \int_0^T G(t,s)\{A(s;y_n)y_n' + B(s;y_n)y_n + h(s;y_n)\}ds. \]

Conditions imposed insure that this integral equation has a unique solution, and that the sequence so defined converges to a function which solves the problem.
We have used the particular Green's function, $G_0(t,s)I$ only because of our familiarity with it. How much more appealing it would be to define, at each step, $G_n$ to be the Green's function for the equation

$$y'' = A(t;y_n)y' + B(t;y_n)y,$$

then let $y_{n+1}(t) = \int_0^T G_n(t,s)h(s;y_n)ds$. If $A$ and $B$ are chosen judiciously, it seems probable that $h$ would be near zero, and the convergence or nonconvergence of the sequence to a solution might be determined more tightly than is presently possible. Indeed, the reader could likely state and prove a theorem which follows this line, by applying the arguments of Chapters 3 and 4. The reason such a theorem is not stated here is the difficulty of applying it. To do so, one would require information regarding the existence of, and bounds for, if not explicit representations for, infinitely many Green's functions, $G_n$. Such information is almost never available.

A somewhat more suitable approach is that of choosing a quasilinearization (5.2) in which $A$ and $B$ are not
dependent upon $y$. This has the positive effect of requiring only one Green's function $G$, at the probable expense of a uniformly good approximation of $f$ by $Ay' + By$. The resulting quasilinearization is better written as

\[(5.3) \quad y'' = A(t)y' + B(t)y + h(t;y), \quad \text{where} \]

$h(t;y) = f(t;y,y') - A(t)y' - B(t)y$ measures the quality of the linear approximation of $f$. There remains the non-trivial task of determining whether $G$ exists and, if so, its properties. In this regard, some results are available; see Reid [13], Heimes [6].

We conclude the paper by offering an explicit representation of $G$ in a very special case. We let $X$ be finite dimensional and consider the equation $y'' = A^2y$, where the constant matrix $A$ is nonsingular and similar to a diagonal matrix. We are able to arrive at our representation by appealing to a theorem of Heimes [6]. That theorem is stated here without proof.
Theorem 5.2: Suppose $A, B$ are continuous functions from the compact interval $[0,T]$ to the algebra of bounded linear operators $[X]$, and suppose $U(t), V(t)$ are functions from $[0,T]$ into $[X]$ solving $y'' = Ay' + By$ with initial conditions $U(0) = V'(0) = 0, U'(0) = V(0) = I$.

Suppose that $U(t)$ is invertible on $(0,T]$. For $0 < t \leq T$, define $K(t), H(t)$ by

\[
\begin{align*}
K(t) &= U^{-1}(t)V(t) \\
H(t) &= U'(t)K(t) - V'(t).
\end{align*}
\]

Then the Green's function for the equation (5.3) is given by

\[
G(t,s) = \begin{cases} 
0 & 0 = s \leq t \\
U(t)[K(T) - K(t)]H^{-1}(s) & 0 < s \leq t \\
U(t)[K(T) - K(s)]H^{-1}(s) & t \leq s \leq T.
\end{cases}
\]

Let $A$ be an $n \times n$ constant matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $x_1, x_2, \ldots, x_n$. We assume that $A$ is nonsingular and that $x_1, \ldots, x_n$ form a basis for complex $n$-space. We now define some matrices
that will prove useful.

**Definition 5.3**: For $A$ above, let

- $X$ be the matrix which has $x_j$ as its $j^{th}$ column.
- $D$ be the diagonal matrix which has $\lambda_j$ as the element of row $j$, column $j$.
- $I_j$ be the matrix which has a 1 in row $j$, column $j$, and zeros elsewhere.

$D$ and $X$ are invertible by assumption, and

$$\sum_{j=1}^{n} I_j = I,$$

the multiplicative identity. Let $C = (XD)^{-1}$, and, for $j = 1, 2, \ldots, n$, let $P_j = XI_jC$.

**Lemma 5.4**: The matrices defined above satisfy the following:

(i) $AP_j = \lambda_j P_j$, $j = 1, 2, \ldots, n$

(ii) $\sum_{k=1}^{n} P_k = A^{-1}$

(iii) $P_j P_k = \begin{cases} 0 & j \neq k \\ \frac{1}{\lambda_j} P_j & j = k \end{cases}$

(iv) $AP_j = P_j A$, $j = 1, 2, \ldots$. 
Proof: The columns of $P_j$ are multiples of the eigenvector $x_j$. Since $Ax_j = \lambda_j x_j$, the definition of matrix multiplication gives (i). To obtain (ii) we observe that $AX = XD$. Hence

$$\sum_{k=1}^{n} P_k = \sum_{k=1}^{n} XI_k C = X \left( \sum_{k=1}^{n} I_k \right) C = XIC$$

$$= XC = X(XD)^{-1} = (A^{-1}A)X(XD)^{-1} = A^{-1}(XD)(XD)^{-1},$$

$$\sum_{k=1}^{n} P_k = A^{-1}.$$

We arrive at (iii) directly;

$$P_k P_j = (XI_k C)(XI_j C) = X(I_k (CX) I_j) C = X(I_k D^{-1} I_j) C \quad \text{since}$$

$$CX = D^{-1}. \text{ If } k \neq j, I_k D^{-1} I_j = 0, \text{ so } P_k P_j = 0. \text{ If } k = j,$$

we have

$$P_k P_j = P_j^2 = X(I_j D^{-1} I_j) C = X(I_j D^{-1} I_j) C = \frac{1}{\lambda_j} X I_j C = \frac{1}{\lambda_j} P_j.$$

Now consider any $k = 1, 2, \ldots, n$. To show that $AP_k = P_k A$ we will establish that the matrix $AP_k - P_k A$ satisfies

$$(AP_k - P_k A) A^{-1} = 0.$$

Postmultiplication by $A$ will then give (iv). For each $j = 1, 2, \ldots, n$,

$$(AP_k - P_k A) P_j = AP_k P_j - P_k A P_j = AP_k P_j - P_k \lambda_j P_j = (A - \lambda_j I) P_k P_j.$$
When $k \neq j$, we have $P_k P_j = 0$, so $(AP_k - P_k A)P_j = 0$.

For $k = j$, we have $(AP_k - P_k A)P_k = (A - \lambda_k I)P_k^2 = (A - \lambda_k I)\frac{1}{\lambda_k}P_k$

$= \frac{1}{\lambda_k}(AP_k - \lambda_k P_k) = \frac{1}{\lambda_k}0 = 0$. Hence,

$$(AP_k - P_k A)A^{-1} = (AP_k - P_k A)\sum_{j=1}^{n} P_j = \sum_{j=1}^{n} (AP_k - P_k A)P_j = 0. \quad Q.E.D.$$ 

We now consider the matrix differential equation,

\begin{equation}
Y'' = A^2 Y, \tag{5.6}
\end{equation}

where $A$ has eigenvalues $\lambda_j$, $j = 1, \ldots, n$, none of which are integral multiples of $i \frac{\pi}{T}$ (i represents the imaginary unit), and corresponding eigenvectors $x_j$ which span complex $n$-space. We define $X, D, C, I_j, P_j$ as in Definition 5.3. For any real number $t$, we define $U(t)$ by

$$U(t) = \sum_{j=1}^{n} \sinh \lambda_j t P_j.$$ 

Further, define $V(t)$ by
\[ V(t) = \sum_{j=1}^{n} \lambda_j \cosh \lambda_j t P_j. \]

\[ U(t), V(t) \text{ have the following easily checked properties:} \]

(i) \( U(t), V(t) \) solve the matrix differential equation

(5.6)

(ii) \( V(t) = U'(t) \)

(iii) \( U(0) = V'(0) = 0, U'(0) = V(0) = I \)

We now proceed to use (5.5) to find the Green's function \( G(t,s) \). For \( t \in [0,T] \) \( \exists \sinh \lambda_j t \neq 0 \) for \( j = 1,2,...,n \), we define

\[ J(t) = \sum_{j=1}^{n} \lambda_j^2 \csch \lambda_j t P_j. \]

Then

\[ J(t)U(t) = \left( \sum_{j=1}^{n} \lambda_j^2 \csch \lambda_j t P_j \right) \left( \sum_{k=1}^{n} \sinh \lambda_k t P_k \right). \]

By property (iii), established in Lemma 5.4, we see that the only non-zero contributions that occur are those for which the subscripts are the same, i.e. no nontrivial "cross products" occur. Hence
\[ J(t)U(t) = \sum_{j=1}^{n} \lambda_j^2 \sinh \lambda_j t \csc h \lambda_j t P_j = \sum_{j=1}^{n} \lambda_j^2 P_j. \]

Again from (iii) of Lemma 5.4 we find we can simplify \( \lambda_j^2 P_j \) to \( P_j \), so that \( J(t)U(t) = \sum_{j=1}^{n} \lambda_j P_j \). But we recall that the columns of \( P_j \) are scalar multiples of the eigenvectors \( x_j \) for each \( j \); thus, \( \lambda_j P_j = \lambda P_j \).

\[ J(t)U(t) = \sum_{j=1}^{n} \lambda P_j = A \sum_{j=1}^{n} P_j = AA^{-1} \quad \text{(see (ii), Lemma 5.)} \]

\[ J(t)U(t) = I. \] We see that \( J(t) = U^{-1}(t) \), or

\[ U^{-1}(t) = \sum_{j=1}^{n} \lambda_j^2 \csc h \lambda_j t P_j, \quad \text{for } \sinh \lambda_j t \neq 0. \] It follows from (5.4) that \( K(t) = \left( \sum_{j=1}^{n} \lambda_j^2 \csc h \lambda_j t P_j \right) \left( \sum_{k=1}^{n} \lambda_k \csc h \lambda_k t \right) P_k. \)

Invocation again of Lemma 5.4 gives \( K(t) = \sum_{j=1}^{n} \lambda_j^2 \coth \lambda_j t P_j, \) for \( t \) such that \( \sinh \lambda_j t \neq 0. \) We proceed to get an expression for \( H^{-1} \), where \( H \) is given in (5.4).
\[ H^{-1}(t) = [U'(t)K(t) - V'(t)]^{-1} = [U'(t)U^{-1}(t)V(t) - V'(t)]^{-1} \]
\[ = [U'(t)U^{-1}(t)V(t) - U(t)U^{-1}(t)V'(t)]^{-1}. \]

We choose this point to mention that conditions (iii) and (iv) of Lemma 5.4 insure that \( U(t), V(t), U^{-1}(t) \) and their derivatives commute with one another and with \( A \).

Since \( U'(t) = V(t) \), and \( V'(t) = U''(t) \), we write
\[ H^{-1}(t) = [V(t)U^{-1}(t)V(t) - U(t)U^{-1}(t)U''(t)]^{-1}. \]
Applying the commutativity and the fact that \( U''(t) = A^2 U(t) \),
\[ H^{-1}(t) = [U^{-1}(t)V^2(t) - U^{-1}(t)A^2 U^2(t)]^{-1} \]
\[ = [U^{-1}(t)(V^2(t) - A^2 U^2(t))]^{-1} \]
\[ H^{-1}(t) = (V^2(t) - A^2 U^2(t))^{-1} U(t). \]
Now
\[ v^2(t) = \left( \sum_{j=1}^{n} \lambda_j \cosh \lambda_j t \ p_j \right)^2. \]
Once again exploiting the lack of cross products, we write
\[ V^2(t) = \sum_{j=1}^{n} \lambda_j^2 \cosh^2 \lambda_j t \ P_j^2 = \sum_{j=1}^{n} \lambda_j \cosh^2 \lambda_j t \ \lambda_j P_j^2 \]

\[ = \sum_{j=1}^{n} \lambda_j \cosh^2 \lambda_j t \ P_j, \text{ since } \lambda_j P_j^2 = P_j \]

\[ = A \sum_{j=1}^{n} \cosh^2 \lambda_j t \ P_j, \text{ since } \lambda_j P_j = \lambda P_j. \]

\[ A^2 U^2(t) = \sum_{j=1}^{n} A^2 \sinh^2 \lambda_j t \ P_j^2 = \sum_{j=1}^{n} A(\sinh^2 \lambda_j t) \lambda_j P_j^2 \]

\[ = \sum_{j=1}^{n} A \sinh^2 \lambda_j t \ P_j = A \sum_{j=1}^{n} \sinh^2 \lambda_j t \ P_j. \]

Therefore,

\[ V^2(t) - A^2 U^2(t) = A \sum_{j=1}^{n} (\cosh \lambda^2 t - \sinh \lambda_j t) P_j = A \sum_{j=1}^{n} P_j. \]

But (ii) of the lemma gives \( \sum_{j=1}^{n} P_j = A^{-1} \), so

\[ V^2(t) - A^2 U^2(t) = I. \text{ The conclusion is } H^{-1}(t) = U(t). \]

We are now prepared to obtain an explicit expression for \( G(t,s) \). From (5.5), with \( 0 < s \leq t \),
\[ G(t,s) = U(t)[K(T) - K(t)]H^{-1}(s) = U(t)U(s)[K(T) - K(t)] \]

\[ = \sum_{j=1}^{n} \sinh \lambda^j \cdot \text{P}_j \sum_{k=1}^{n} \sinh \lambda^k \cdot \text{P}_k \left[ \sum_{m=1}^{n} \lambda^2 (\coth \lambda^m T - \coth \lambda^m t) \cdot \text{P}_m \right]. \]

Once again we recall that \( p_k^j P_k = 0 \) if \( k \neq j \), so

\[ G(t,s) = \sum_{j=1}^{n} \lambda^2 \sinh \lambda^j t \sinh \lambda^j s (\coth \lambda^j T - \coth \lambda^j t) \cdot \text{P}_j. \]

Since \( \lambda^2 p_j^3 = \text{P}_j \), we write

\[ G(t,s) = \sum_{j=1}^{n} \frac{\sinh \lambda^j s \sinh \lambda^j (t-T)}{\sinh \lambda^j T} \cdot \text{P}_j, \quad 0 < s \leq t. \]

A similar derivation gives \( G(t,s) \) for \( t \leq s \leq T \). Hence

\[ \begin{align*}
G(t,s) &= \begin{cases}
\sum_{j=1}^{n} \frac{\sinh \lambda^j s \sinh \lambda^j (t-T)}{\sinh \lambda^j T} \cdot \text{P}_j, & 0 \leq s \leq t \\
\sum_{j=1}^{n} \frac{\sinh \lambda^j t \sinh \lambda^j (s-T)}{\sinh \lambda^j T} \cdot \text{P}_j, & t \leq s \leq T.
\end{cases}
\end{align*} \]
In order to invoke Theorem 5.2, and in order to perform several of the calculations, it has been necessary to require that \( \sinh \lambda_j^t \) be nonzero for \( j = 1,2,\ldots,n \) and \( 0 \leq t \leq T \). However, \( G(t,s) \) is clearly defined for all \( t,s \) with \( 0 \leq t,s \leq T \), as long as \( \sinh \lambda_j^T \) is nonzero for \( j = 1,2,\ldots,n \). Furthermore, it is straightforward to verify that \( G \) does serve as a Green's function for the problem.

We summarize the above discussion with

**Theorem 5.5:** Let \( A \) be an \( n \times n \) matrix which has eigenvalues \( \lambda_j \) and corresponding eigenvectors \( x_j \) for \( j = 1,2,\ldots,n \). Suppose that \( \lambda_j \) is not an integral multiple of the pure imaginary, \( \frac{T_i}{T} \), for each \( j \), and suppose that \( \{x_1,x_2,\ldots,x_n\} \) forms a basis for complex \( n \)-space. Then the Green's function for the problem \( z'' = A^2 z + g(t), \ z(0) = z(T) = 0 \) is given by (5.7), where \( P_j, j = 1,\ldots,n \) is defined in Definition 5.3.

**Example 5.6:** Let

\[
A = \begin{pmatrix}
1 & -1 & -1 \\
1 & -1 & 0 \\
1 & 0 & -1
\end{pmatrix}
\]

Then the eigenvalues and corresponding eigenvectors are
\( \lambda_1 = -1, x_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}; \lambda_2 = i, x_2 = \begin{pmatrix} 1 + i \\ 1 \\ 1 \end{pmatrix}; \lambda_3 = i, x_3 = \begin{pmatrix} 1 - i \\ 1 \\ 1 \end{pmatrix} \)

\[
X = \begin{bmatrix}
0 & 1 + i & 1 - i \\
1 & 1 & 1 \\
-1 & 1 & 1
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{bmatrix}.
\]

Then

\[
C = (XD)^{-1} = \frac{1}{4} \begin{bmatrix}
0 & -2 & 2 \\
-2 & 1 - i & 1 - i \\
-2 & 1 + i & 1 + i
\end{bmatrix}, \quad \text{so}
\]

\[
P_1 = XI_1C = \frac{1}{4} \begin{bmatrix}
0 & 1 + i & 1 - i \\
1 & 1 & 1 \\
-1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & -2 & -2 \\
-2 & 1 - i & 1 - i \\
-2 & 1 + i & 1 + i
\end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & 1 \\
0 & 1 & -1
\end{bmatrix}.
\]

Similarly, we get

\[
P_2 = \frac{1}{4} \begin{bmatrix}
-2 - 2i & 2 & 2 \\
-2 & 1 - i & 1 - i \\
-2 & 1 - i & 1 - i
\end{bmatrix}, \quad P_3 = \frac{1}{4} \begin{bmatrix}
-2 + 2i & 2 & 2 \\
-2 & 1 + i & 1 + i \\
-2 & 1 + i & 1 + i
\end{bmatrix}.
\]

We recall that if \( x \) and \( y \) are real numbers, then

\[
\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y.
\]

Hence, for
t \in [0,T], \quad \sinh \lambda_1 t = \sinh t, \quad \sinh \lambda_2 t = \sin t, \\
\sinh \lambda_3 t = -i \sin t. \quad \text{Then for } s \leq t.

\[ G(t,s) = -\frac{\sinh s \sinh(t-T)}{2 \sinh T} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} + \frac{i \sin s (i \sin(t-T))}{4i \sin T} Z + \]

\[ + \frac{-i \sin s (-i \sin(t-T))}{-4i \sin T} \overline{Z}, \quad \text{where} \]

\[ Z = \begin{pmatrix} -2-2i & 2 & 2 \\ -2 & 1-i & 1-i \\ -2 & 1-i & 1-i \end{pmatrix}, \quad \text{and } \overline{Z} \text{ is the complex conjugate of } Z. \]

We notice that the second and third terms are complex conjugates of one another, so their sum is twice the real part of either. Hence, for \( s \leq t, \)

\[ G(t,s) = -\frac{\sinh s \sinh(t-T)}{2 \sinh T} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \frac{\sin s \sin(t-T)}{2 \sin T} \Re iZ. \]

For \( t \leq s \) we simply interchange \( s \) and \( t \) above.

While it seems reasonable that Green's functions should be obtainable explicitly for a larger class of equations involving constant coefficients, we have not yet
pursued the matter. Judging by the scalar case, if $A$ and $B$ are not constant a representation more explicit than that of Heimes in Theorem 5.2 may not be possible.
VI. BIBLIOGRAPHY


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