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Land Allocation in the Presence of Estimation Risk

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Estimation risk occurs when parameters relevant for decision making are uncertain. Bayes' criterion is consistent with expected-utility maximization in the presence of estimation risk. This article examines optimal (Bayes') land allocations and land allocations obtained using the traditional plug-in approach and two alternative decision rules. Bayes' allocations are much better economically than the other allocations when there are few sample observations relative to activities. Calculation of certainty equivalent returns (CERs) with estimation risk is also discussed and illustrated. CERs are typically (and incorrectly) calculated with the plug-in approach. Plug-in CERs may be extremely misleading.

Key words: Bayes' decision criterion, certainty equivalent, estimation risk, expected utility, uncertainty, risk

Introduction

Whenever economic analysis involves incorporating estimated parameters into theoretically derived decision rules, the optimal outcome depends on the estimation procedure. This problem is called estimation risk (Bawa, Brown, and Klein). Typically, estimation risk is ignored and sample parameter estimates are directly substituted for the true but unknown parameters in theoretical decision rules. This method is called the plug-in approach. The plug-in approach has no axiomatic foundations and is not consistent with the expected-utility-maximization paradigm (Klein et al.; Bawa, Brown, and Klein). In the presence of estimation risk, Bayes' decision rule is consistent with expected-utility maximization (DeGroot; Berger).

Estimation risk is ever-present in economic problems; for example, it appears in decisions regarding optimal levels of export taxes, quotas, output, resource allocation, and research and development expenditures. Furthermore, some problems (e.g., the ex ante value of information) cannot be solved properly without accounting for estimation risk. Several studies have analyzed the effects of estimation risk in the financial literature (e.g., Brown; Boyle and Ananthanarayanan; Bawa, Brown, and Klein; Coles and Loewenstein; Chen and Brown; Jorion; Frost and Savarino 1986a; Lence and Hayes 1994a). But despite the pervasiveness of estimation risk, the problem has been largely ignored in agricultural economics until recently. Exceptions are Chalfant, Collender, and Subramanian (CCS), Collender, and Lence and Hayes (1994b).

This article has three interrelated objectives. The first objective is to show, by means of an everyday example, how important estimation risk can be in the analysis of applied decision making in agriculture. The second objective is to determine the differences among land allocations obtained from alternative decision rules in the presence of estimation risk.

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The third objective is to discuss and illustrate the calculation of certainty equivalent returns (CERs) with estimation risk.

The first section of this article reviews briefly the theory of optimal decision making in the presence of estimation risk, followed by a typical application in agricultural economics. The example is that of a farmer deciding how to allocate his land to various enterprises based on the risk-and-return information that is typically available to producers at planting time. This particular example was chosen because CCS recently developed a non-Bayesian solution to the land-allocation problem in the presence of estimation risk. Also, the example allows us to compare the allocations from (*a*) the traditional plug-in approach, (*b*) the CCS rule, (*c*) the Bayes' criterion, and (*d*) an approximate Bayes' decision rule used by Brown.

The most important features of the CCS solution are that it is unbiased and that it is no more difficult to implement than the plug-in approach. The approximate Bayes' decision rule presented by Brown is of interest because it relies on Bayesian principles, and like the CCS approach, it is no more difficult to use than the plug-in decision rule. Therefore, the approximate Bayes' decision rule also eliminates the main advantage of the plug-in approach over Bayes' criterion, that is, simplicity.

The example presented reveals that the plug-in, CCS, and approximate Bayes' land allocations may be quite different from the Bayes' land allocations under common agricultural situations. Important differences in allocations, however, need not be economically significant. To assess their economic significance, it is necessary to use some measure of the ex ante utility arising out of each decision criterion. Therefore, the third section of this article discusses and illustrates the computation of CERs in the presence of estimation risk. CERs reveal that the plug-in, CCS, and approximate Bayes' allocations may be economically much inferior to the Bayes' allocations. Given the pervasiveness of estimation risk in agriculture and the typical (but usually unacknowledged) use of the plug-in approach, our results imply that CER estimates commonly reported in the literature are often unreliable.

Decision Making in the Presence of Estimation Risk

Consider a decision maker characterized by a von Neumann-Morgenstern utility function of terminal wealth $[U(\pi),\ U' > 0,\ U'' \leq 0]$. Let wealth be a function of a vector of random variables x and a decision vector l [i.e., $\pi(x, l)$]. According to the expected-utility paradigm, the optimal decision vector l^* is the solution to

$$\text{(1)} \qquad \max_{l\in\Lambda} E_{x|\theta}(U) \equiv \max_{l\in\Lambda} \int_X U[\pi(x,l)] f_{x|\theta}(x|\theta)dx,$$

where Λ denotes the set of all possible decisions, $E_{x|}\theta(\cdot)$ represents the expectation with respect to x given θ, X is the domain of x, $f_{x|}\theta(x|\theta)$ is the probability density function (pdf) of x given θ, and θ is a known vector of parameters that characterizes the pdf.

As long as the parameter vector (θ) is known, the decision vector that maximizes expected utility (l^*) can be computed from (1). If θ is unknown, however, the optimization stated in (1) cannot be performed because, generally, there is no decision vector that maximizes the expected-utility function for all possible values of θ. In this situation, there is estimation risk.

In the presence of estimation risk,[1] the standard approach to optimization is the plug-in, by which sample estimates are substituted for the true but unknown parameters in the objective function, that is:

$$\text{(2)} \qquad \max_{l\in\Lambda} E_{x|\theta=\hat{\theta}}(U) \equiv \max_{l\in\Lambda} \int_X U[\pi(x,l)] f_{x|\theta}(x|\theta=\hat{\theta})dx,$$

where $\hat{\theta}$ is the vector of sample parameter estimates. The plug-in objective function (2) yields $l^{PI} \equiv l^*(\theta = \hat{\theta})$, as the decision vector under estimation risk. This approach has the advantage of simplicity, but it ignores estimation risk and is not consistent with expected-utility maximization (Klein et al.; Bawa, Brown, and Klein).

With estimation risk, the method by which the optimal decision can be obtained in a manner consistent with expected-utility maximization is Bayes' decision criterion (Klein et al.; DeGroot; Berger). Bayes' decision criterion can be summarized as follows. Let $y \equiv (y_1, \ldots, y_n)$ denote a sample of size n that is generated by the same process that generates x and is available at the time of decision making. Let $f_{p(\theta|y)}(\theta|y)$ be the agent's believed pdf regarding the parameter vector (θ), after seeing the sample data (y) but before making the decision. Then, Bayes' decision vector (l^B) is the solution to the objective function:

$$\text{(3)} \qquad \max_{l\in\Lambda} E_{p(\theta|y)}[E_{x|\theta}(U)] \equiv \max_{l\in\Lambda} \int_\Theta \{\int_X U[\pi(x,l)] f_{x|\theta}(x|\theta)dx\} f_{p(\theta|y)}(\theta|y)d\theta,$$

where Θ is the domain of θ. Comparison of (1) and (3) reveals that they are entirely analogous. In (1) expectations are taken to eliminate the random vector x. Whereas in (3), expectations are additionally taken to eliminate the agent's uncertainty regarding the true but unknown parameter vector θ. The Bayesian approach is appealing relative to the plug-in and to other methods that do not explicitly model the producer's decision through expected-utility maximization.

In a recent article, CCS showed that the plug-in decision vector (l^{PI}) is a biased estimator of the optimal decision vector *without* estimation risk (l^*). Hence, they proposed the unbiased decision vector

$$\text{(4)} \qquad l^{CCS} \equiv E_{p(\theta|y)}[l^*(\theta)],$$

as an alternative to l^{PI}. CCS proved that l^{CCS} yields greater expected utility than does l^{PI} and showed that l^{CCS} has a closed-form solution for the standard land-allocation problem which is as easy to compute as l^{PI}.[2] These two properties make l^{CCS} more desirable than l^{PI}. However, CCS decision rule shares with the plug-in approach the undesirable characteristic of not being consistent with expected-utility maximization.

The main disadvantage of Bayes' criterion is its computational complexity. To overcome this problem, Brown derived an approximate Bayes' decision vector (l^{AB}) for the special case of negative exponential utility and random variables (x) following a multivariate normal pdf.

[1] As pointed out by an anonymous reviewer, another way to view estimation risk is that the decision maker (farmer) faces an uncertain (unknown distribution) situation while the analyst incorrectly assumes that decision is risky (known distribution).

[2] Closed-form solutions for l^{PI} and l^{CCS} are reported in the appendix.

His approach consists of approximating the Bayesian objective function (3) so as to obtain a closed-form solution for the decision vector l^{AB}.[3] Therefore, this method is meant to combine the nice properties of the plug-in and Bayes' approaches (i.e., simplicity and consistency with expected-utility maximization, respectively).

Land Allocation in the Presence of Estimation Risk

The purpose of this section is to apply the decision criterion that is consistent with expected-utility maximization (i.e., Bayes' criterion) to the standard land-allocation problem and to compare it with the solutions from the plug-in, CCS, and approximate Bayes' procedures.

In the standard land-allocation problem, it is assumed that the decision maker has a negative exponential utility function, $U[\pi(x,l)] = -\exp[-rB(x,l)]$, where r denotes the Arrow-Pratt coefficient of absolute-risk aversion. Terminal wealth $[\pi(x,l) = l]$ equals the sum of the product of returns per acre times the corresponding acres planted with each crop. The random vector of next period's returns per acre $[x = (x_1, \ldots, x_k)']$ is assumed to follow a k-variate normal distribution with mean vector μ and covariance matrix Σ $[f_{x|\theta}(x|\theta) = N_k(x|\mu, \Sigma)]$. The decision vector $[l = (l_1, \ldots, l_k)']$ is composed of the land allocated to each crop. The set of feasible land allocations is $\Lambda = \{l\colon Al \leq b\}$, where A is an $(m \times k)$ matrix and b is an m-vector of constraints. Typical restrictions on the decision vector are (a) the total number of acres planted cannot exceed the total farm acreage (L), and (b) the number of acres planted with each crop cannot be negative. In such a case, $b = (L, 0, \ldots,)'$, and

$$A = \begin{bmatrix} \iota_k' \\ -I_k \end{bmatrix}, \tag{5}$$

where ι_k is a k-vector of ones, and I_k is a $(k \times k)$ identity matrix. Finally, the $(k \times n)$ matrix $y = (y_1, \ldots, y_n)'$ contains n past observations on the vector of crop returns.

Under the stated assumptions, the optimization problem (1) takes the following form:

$$\max_{l \in \Lambda} E_{x|\theta}(U) = \max_{l \in \Lambda = \{l:\, Al \leq b\}} \int_X [-\exp(-rx'l)] N_k(x|\mu, \Sigma)\, dx \tag{6a}$$

$$= \max_{l \in \Lambda = \{l:\, Al \leq b\}} [-\exp(-r\mu' l + \frac{1}{2} r^2 l' \Sigma l)], \tag{6b}$$

where expression (6b) is derived from (6a) by completing squares in the exponent (Freund). The land-allocation vector that maximizes the objective function in (6b) (l^*) has a closed-form solution. Therefore, the plug-in land allocation (l^{PI}) is obtained by replacing the true (but unknown) mean vector (μ) and covariance matrix (Σ) with their sample estimates ($\hat{\mu}$ and $\hat{\Sigma}$, respectively) in the expression for l^*. Similarly, CCS land allocation (l^{CCS}) is derived as the unbiased estimate of l^*. The closed-form solutions for l^*, l^{PI}, and l^{CCS} are reproduced in the appendix.

[3]The closed-form solution for l^{AB} is reproduced in the appendix.

Bayes' criterion requires us to postulate a prior pdf for the unknown parameters. To simplify the exposition and to avoid criticisms regarding the reasonability and subjectivity of any particular informative prior, a "noninformative" or "diffuse" prior for μ and Σ is assumed throughout.[4] This is equivalent to hypothesizing that the decision maker has no information (or beliefs) about the parameters, other than that provided by the n past observations on the vector of crop returns (y).[5] Under the stated assumptions, the optimization problem (3) becomes

$$(7)\qquad \max_{l\in\Lambda} E_{p(\theta|y)}[E_{(x|\theta)}(U)] = \max_{l\in\Lambda=\{l:\,Al\leq b\}} \int_X [-\exp(-rx'l)] S_k(x|\hat{\mu}, \Sigma^{\circ}, n-k)\, dx,$$

where $S_k(x|\hat{\mu}, \Sigma^{\circ}, n-k)$ is the k-variate Student-t pdf, and $\Sigma^{\circ} \equiv (1+1/n)(n-1)/(n-k)\hat{\Sigma}$.[6] The decision vector that solves (7) is Bayes' land allocation (l^B). Bayes' land allocation has no closed-form solution and must be solved numerically.

The approximate Bayes' land allocation (l^{AB}) is obtained by approximating the k-variate Student-t pdf $S_k(x|\hat{\mu}, \Sigma^{\circ}, n-k)$ with the k-variate normal pdf $N_k[x|\hat{\mu}, (n-k)/(n-k-2)\Sigma^{\circ}]$. Doing so yields an approximation to Bayes' objective function (7) that has the same form as (6b), in which case the optimal land allocation (l^{AB}) has a closed-form solution (see appendix).

Simulation Procedures

In order to illustrate the impact of estimation risk on land allocations and the differences among the four alternative solutions, simulations were performed using two well-known data sets. One data set is the following mean vector and covariance matrix estimates used by Freund in his classical paper on quadratic programming:

$$(8)\qquad \hat{\mu} = [83.403,\ 72.359,\ 36.023,\ 207.469]' \text{ and}$$

$$(9)\qquad \hat{\Sigma} = \begin{bmatrix} 5081.166 & 545.492 & -206.924 & -3221.997 \\ 545.492 & 324.704 & -122.807 & 165.780 \\ -206.924 & -122.807 & 145.940 & 561.040 \\ -3221.997 & 165.780 & 561.040 & 15880.968 \end{bmatrix}.$$

Estimates (8) and (9) are meant to be representative of an eastern North Carolina farm where possible activities are potatoes, corn, beef, and cabbage, respectively. Freund provided almost no detail about the estimation, but he mentioned that state averages were used to obtain (8) and (9) due to lack of data for individual farms.

[4]Implicitly, CCS criterion also assumes a (diffuse) prior.

[5]The inclusion of any nondiffuse prior would serve only to improve the relative performance of the Bayesian solution (as mesured by CERs). This assertion is true as long as (a) the nondiffuse prior is true, or (b) performance is measured by subjective performance.

[6]Expression (7) is obtained from (3) by noting that

$$\int_{\Theta} [\int_X U(\cdot) f_{x|\theta}(x|\theta)dx] f_{p(\theta|y)}(\theta|y)d\theta = \int_X U(\cdot)[\int_{\Theta} f_{x|\theta}(x|\theta) f_{p(\theta|y)}(\theta|y)d\theta]\,dx = \int_X U(\cdot) f_{x|y}(x|y)dx,$$

where $f_{x|y}(x|y)$ is the predictive pdf of x given y.

The other data set is Hazell's mean vector and covariance matrix estimates:

$$\hat{\mu} = [253,\ 443,\ 284,\ 516]', \text{ and} \tag{10}$$

$$\hat{\Sigma} = \begin{bmatrix} 11264 & -20548 & 1424 & -15627 \\ -20548 & 125145 & -27305 & 29297 \\ 1424 & -27305 & 10585 & -10984 \\ -15627 & 29297 & -10984 & 93652 \end{bmatrix}. \tag{11}$$

Hazell obtained (10) and (11) using six annual observations on gross margins of an actual fresh market vegetable farm in Florida. The respective activities are carrots, celery, cucumbers, and peppers.

Simulations were performed for four levels of estimation risk involving 7, 10, 15, and 25 annual observations (n) for each of the activities. The allocation vectors for the plug-in, CCS, and approximate Bayes' criteria were obtained from their respective closed-form solutions (see appendix). The allocation vectors for Bayes' criterion were calculated with Monte Carlo integration because the objective function (7) involves integration over four variables and exceeded the computing capacity available. More explicitly, Bayes' allocations were obtained as the solution to the objective function:

$$\max_{l \in \Lambda = \{l:\ Al \le b\}} \frac{1}{25000} \sum_{i=1}^{25000} [-\exp(-r x_i' l)], \tag{12}$$

where x_i is a 4-vector obtained from the ith random draw from the 4-variate Student-t pdf $S_4(x|\hat{\mu}, \Sigma^{o}, n-4)$. The 25,000 random draws from the 4-variate Student-t pdf were generated with the method described by van Dijk and Kloek (p. 315), modified to include antithetic replications in order to improve convergence (Geweke).[7]

Coefficients of absolute-risk aversion were standardized to achieve consistency with the analyses of Kallberg and Ziemba and Markowitz, Reid, and Tew. The argument of the exponential utility function can be rewritten as $rx'l = -aR$, where $a \equiv rw$, $R \equiv x'l / w$, and w is initial wealth. Kallberg and Ziemba show that if gross returns per dollar of initial wealth (R) are around unity, then moderate risk aversion corresponds to a values between 2 and 4. In the present simulations, w was set equal to the simple average of the corresponding sample means, and risk-aversion levels were defined as moderate-to-low ($a = 1$) and moderate-to-high ($a = 3$).

Simulation Results

Simulation results are summarized in tables 1 and 2 for the Freund and Hazell data sets, respectively. Crop allocations are reported as percentages of total acreage or, alternatively, as number of acres per crop assuming that a total of 100 acres is being allocated.

Tables 1 and 2 show that the plug-in allocations are not affected by the number of observations available to the decision maker. In contrast, the other three decision rules yield allocations that change with the number of sample observations. This result highlights a

[7]Antithetic replication means that a second draw is obtained from each original random draw by assigning an opposite value. For example, if an original random draw from the standard univariate normal pdf equals 0.52, its antithetic replication equals −0.52.

Table 1. Land Allocations for Alternative Decision Rules—Freund Data

Number of Observations	Risk Aversion	Activity	Land Allocation (percentage of total acreage) Bayes	Approx. Bayes	CCS	Plug-in
7	Mod./Low[a]	Potatoes	14.7	49.9	39.6	24.5
7	Mod./Low	Corn	71.1	0.0	0.0	0.0
7	Mod./Low	Beef	0.0	0.0	0.0	0.0
7	Mod./Low	Fall Cabb.	14.2	50.1	60.4	75.5
10	Mod./Low	Potatoes	54.1	42.3	34.6	24.5
10	Mod./Low	Corn	0.0	0.0	0.0	0.0
10	Mod./Low	Beef	0.0	0.0	0.0	0.0
10	Mod./Low	Fall Cabb.	45.9	57.7	65.4	75.5
15	Mod./Low	Potatoes	36.3	36.4	31.0	24.5
15	Mod./Low	Corn	0.0	0.0	0.0	0.0
15	Mod./Low	Beef	0.0	0.0	0.0	0.0
15	Mod./Low	Fall Cabb.	63.7	63.6	69.0	75.5
25	Mod./Low	Potatoes	31.0	31.7	28.3	24.5
25	Mod./Low	Corn	0.0	0.0	0.0	0.0
25	Mod./Low	Beef	0.0	0.0	0.0	0.0
25	Mod./Low	Fall Cabb.	69.0	68.3	71.7	75.5
7	Mod./High[b]	Potatoes	0.0	0.0	0.0	34.5
7	Mod./High	Corn	58.4	86.6	80.1	28.4
7	Mod./High	Beef	41.6	0.0	0.0	0.0
7	Mod./High	Fall Cabb.	0.0	13.4	19.9	37.2
10	Mod./High	Potatoes	10.1	0.0	21.2	34.5
10	Mod./High	Corn	82.7	81.8	54.1	28.4
10	Mod./High	Beef	0.0	0.0	0.0	0.0
10	Mod./High	Fall Cabb.	7.2	18.2	24.7	37.2
15	Mod./High	Potatoes	22.4	21.3	26.0	34.5
15	Mod./High	Corn	52.9	53.8	44.9	28.4
15	Mod./High	Beef	0.0	0.0	0.0	0.0
15	Mod./High	Fall Cabb.	24.7	24.8	29.2	37.2
25	Mod./High	Potatoes	30.2	26.6	29.5	34.5
25	Mod./High	Corn	39.3	43.7	38.0	28.4
25	Mod./High	Beef	0.0	0.0	0.0	0.0
25	Mod./High	Fall Cabb.	30.5	29.7	32.5	37.2

[a]Mod./Low risk aversion corresponds to $r = 0.0100187$.
[b]Mod./High risk aversion corresponds to $r = 0.0300561$.

Table 2. Land Allocations for Alternative Decision Rules—Hazell Data

Number of Observations	Risk Aversion	Activity	Land Allocation (percentage of total acreage): Bayes	Approx. Bayes	CCS	Plug-in
7	Mod./Low[a]	Carrots	20.3	0.0	0.0	0.0
7	Mod./Low	Celery	16.9	18.5	19.3	21.3
7	Mod./Low	Cucumbers	38.3	55.8	41.2	6.0
7	Mod./Low	Peppers	24.5	25.7	39.5	72.6
10	Mod./Low	Carrots	0.0	0.0	0.0	0.0
10	Mod./Low	Celery	18.4	19.3	26.9	21.3
10	Mod./Low	Cucumbers	36.6	40.8	0.0	0.0
10	Mod./Low	Peppers	45.0	39.9	73.1	72.6
15	Mod./Low	Carrots	0.0	0.0	0.0	0.0
15	Mod./Low	Celery	19.3	20.0	25.6	21.3
15	Mod./Low	Cucumbers	19.6	29.3	0.0	6.0
15	Mod./Low	Peppers	61.1	50.8	74.4	72.6
25	Mod./Low	Carrots	0.0	0.0	0.0	0.0
25	Mod./Low	Celery	19.5	20.5	24.6	21.3
25	Mod./Low	Cucumbers	12.1	20.0	0.0	6.0
25	Mod./Low	Peppers	68.4	59.5	75.4	72.6
7	Mod./High[b]	Carrots	41.5	30.5	0.0	5.9
7	Mod./High	Celery	17.8	14.7	18.0	18.1
7	Mod./High	Cucumbers	30.9	44.6	64.6	47.4
7	Mod./High	Peppers	9.8	10.3	17.4	28.6
10	Mod./High	Carrots	19.9	23.1	0.0	5.9
10	Mod./High	Celery	15.9	15.7	18.2	18.1
10	Mod./High	Cucumbers	44.3	45.4	60.7	47.4
10	Mod./High	Peppers	19.9	15.8	21.1	28.6
15	Mod./High	Carrots	12.3	0.0	0.0	5.9
15	Mod./High	Celery	16.7	18.2	18.3	18.1
15	Mod./High	Cucumbers	46.1	60.6	57.9	47.4
15	Mod./High	Peppers	24.9	21.2	23.7	28.6
25	Mod./High	Carrots	9.0	0.0	0.0	5.9
25	Mod./High	Celery	17.1	18.4	18.5	18.1
25	Mod./High	Cucumbers	46.6	57.6	55.8	47.4
25	Mod./High	Peppers	27.3	24.1	25.7	28.6

[a]Mod./Low risk aversion corresponds to $r = 0.0026738$.
[b]Mod./High risk aversion corresponds to $r = 0.0080214$.

allocations that change with the number of sample observations. This result highlights a basic problem with the plug-in approach, which is that it ignores the degree of confidence in the available data. Intuitively, it is unreasonable to ignore the accuracy of the parameter estimates.

The figures reported reveal striking differences among the allocations obtained from the four alternative decision rules when the number of observations is small (i.e., when there is low confidence in the information available). For example, table 1 indicates that for moderate-to-low risk aversion Bayes' criterion allocates 71.1% of the total acreage to corn. In contrast, all three of the other decision criteria indicate that corn should not be grown at all. Similarly, for moderate-to-high risk aversion, Bayes' criterion assigns 41.6% of the land to beef and 0% to cabbage, as opposed to the other three criteria that allot 0% to beef and sizable percentages to cabbage.

The reason for the allocation differences can be understood by noting that cabbage and potatoes have a relatively larger expected return per unit, but also are considerably more risky than corn and beef. In the traditional plug-in approach, it is assumed that the producer uses seven years of data to derive the exact first and second moments of returns, that is, he "plugs in" the sample estimates into the first-order conditions as if they were the true parameters. In the Bayes' solution, the producer realizes that with only seven years of data the estimates of the mean, variances, and covariances themselves are quite uncertain. This additional source of uncertainty causes the producer to grow much more corn and beef and much less cabbage and potatoes than the levels prescribed by the plug-in approach. Like Bayes' criterion, both the CCS and the approximate Bayes' approaches also account for the additional uncertainty, but neither of them seem to stress it enough.

These results suggest a possible explanation of an apparent contradiction in the original data.[8] If farmers' decisions were reasonably modeled by the plug-in, CCS, or approximate Bayes' criteria, potatoes and cabbage production (corn and beef) should have been much more (less) popular than actually observed. But potatoes and cabbage have high returns *because* few farmers are prepared to accept the risk involved in growing them. Had farmers responded according to the approximate Bayes', CCS, or plug-in approaches by reducing beef or corn production, returns on these activities should have been higher than observed (i.e., the low returns in beef and corn would not have been in equilibrium).[9]

This finding suggests that, in the presence of scarce information about the mean vector and the covariance matrix, implications about land allocations derived from the plug-in approach or from CCS and approximate Bayes' criterion are likely distorted. This observation is of particular relevance in agriculture. In contrast to the large data sets available for the analysis of portfolios of stocks and bonds, real-world farming decisions may often be based on seven annual observations or fewer for each activity. For example, farmers may forget how prices covaried eight to ten years ago, or the farm may change hands, or the farmer may believe that the process generating the revenue distribution has changed. This latter effect could be due to changes in government policy, trade distortions, technology, or consumer tastes.

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[8]Given the simplifications made in the simulations to focus on estimation risk, this dicussion should not be construed as the only possible, or even the most important, explanation of the apparent contradictions in the original data.

[9]It is tempting to stretch this analogy further and to attempt to infer the number of annual observations used by an average producer when making planting decisions. However, doing so would overestimate the completeness of the information set used in the model. For example, it would be necessary to assume that the producer indeed maximizes expected utility, that there are no other important financial or technological constraints, and that the model captures all of the uncertainties of the system (such as the shape of the statistical distribution, the form of the utility function, and the values of other relevant parameters).

Tables 1 and 2 also show that the alternative allocations tend to converge relatively fast as the number of observations increases. Although this finding weakens the case for explicitly accounting for estimation risk, two important qualifications should be mentioned. First, the results reported reveal that even with twenty-five annual observations—a situation not frequent in agriculture—there are some noticeable differences among allocations. Second and more important, the simulations were performed for the choice among only four activities because of the nature of the Freund and Hazell data sets. Frost and Savarino (1986b) have found that the problem of estimation risk depends on both the number of observations and the number of activities, and that the impact of estimation risk is about the same for all cases in which the ratio of the sample observations to the number of activities is constant. Although space prevents us from analyzing this issue more in depth, increasing the number of activities while holding the number of observations constant will clearly worsen the problem of estimation risk. This assertion is true because doing so reduces the degrees of freedom of the k-variate Student-t pdf in (7), thereby increasing the uncertainty faced by the decision maker.

Certainty Equivalent Returns

A value often computed in studies regarding uncertainty is the *CER* of a decision l [$CER(l)$]. $CER(l)$ is the return on a risk-free investment that leaves the decision maker indifferent between selecting the risky payoff from decision l and accepting the riskless payoff $CER(l)$. CERs bear a monotonically increasing relationship with expected utility, that is, $CER(l^{\prime}) > CER(l^{\prime\prime})$, if and only if, $l^{\prime}$ yields greater expected utility than does $l^{\prime\prime}$. Hence, CERs allow us to draw inferences about the expected utility of alternative decisions. Furthermore, CERs are measured in monetary units and therefore indicate the economic significance of the differences among alternative allocations.

In the absence of estimation risk, $CER(l)$ is obtained as the root of

$$(13) \qquad U[CER^n(l;\theta)] = E_{x|\theta}\{U[\pi(x,l)]\},$$

which we will call $CER^n(l;\theta)$ (the superscript n standing for "no" estimation risk). For the land-allocation example analyzed in the previous section, we have

$$(14) \qquad CER^n(l^*;\mu,\Sigma) = \mu' l^* - \frac{1}{2} r l^{*\prime} \Sigma l^*,$$

as the CER corresponding to the optimal decision vector in the absence of estimation risk (l^*).

In the presence of estimation risk, however, CERs cannot be calculated from expression (13) because such an expression depends on the true but unknown parameter vector θ. If $CER^n(l;\theta)$ were the CER in the presence of estimation risk, it would imply that θ is also known with certainty, thereby contradicting the definition of estimation risk. CER in the presence of estimation risk [$CER^e(l;y)$] is the root of

$$(15) \qquad U[CER^e(l;y)] = E_{p(\theta|y)}\{E_{x|\theta}[U(\pi(x,l))]\},$$

as opposed to (13).

Simulation Procedures

CERs were calculated for each of the four alternative approaches used to solve the standard land-allocation problem. Simulations were performed for the same data sets, estimation risk levels (n), and risk-aversion levels (r) as in the previous section. Monte Carlo integration was used in all instances because (15) has no closed-form solution. More specifically, CERs were calculated as:

$$CER^{e}(l;y) = -\frac{1}{r}\log\{-\frac{1}{25000}\sum_{i=1}^{25000}[-\exp(-rx_i'l)]\}, \tag{16}$$

where log(·) denotes the natural logarithm operator, and x_i is a 4-vector obtained from the ith random draw from the 4-variate Student-t pdf $S_4(x|\hat{\mu}, \Sigma^{o}, n-4)$.

Simulation Results

CER results are reported in tables 3 and 4 for the Freund and the Hazell data sets, respectively. The first thing to note from both tables is that the highest CERs correspond to Bayes' allocations. For example, for moderate-to-low risk aversion and seven observations, table 3 shows that Bayes' allocation has a CER of \$79,667 compared to CERs of –\$359,257; –\$1,041,400; and –\$2,048,093 for the approximate Bayes', CCS, and plug-in approaches, respectively.[10] The superiority of Bayes' criterion in terms of CERs is not surprising because it is the optimal decision rule by construction.

What is surprising, however, is the large difference between the CERs of Bayes' allocations and the CERs of the suboptimal decision criteria when the number of observations is small. In this scenario, Bayes' criterion outperforms the approximate Bayes' allocation, the CCS rule, and the plug-in approach. The approximate Bayes' allocation and the CCS rule substantially outperform the plug-in approach. These results suggest the use of either the approximate Bayes' criterion or the CCS rule over the plug-in approach when there are computing limitations. But the land allocations from the approximate Bayes' criterion or the CCS rule are so inferior relative to the Bayes' allocations that it seems wise to always use the latter when information is scarce.

The differences among CERs for alternative allocations tend to disappear rapidly as the number of sample observations increases. For example, for twenty-five annual observations the largest difference between Bayes' CER and any of the suboptimal rules' CERs is less than 2%. In assessing the potential relevance of estimation risk for a particular situation, however, it is important to consider the results by Frost and Savarino (1986b) mentioned in the previous section. That is, increasing the number of activities while holding the number of observations constant will clearly worsen the problem of estimation risk.

[10]The negative CERs of the suboptimal allocations should be interpreted with caution. In this example, negative CERs arise because suboptimal allocations are extremely risky, and agents with exponential utility put great weight on the potential negative outcomes. A literal interpretation of a negative CER is that the decision maker is willing to pay (rather than charge) a certain amount to give up the risky payoff. This situation can only occur if the agent is forced to adopt the suboptimal allocation; otherwise, he would be better off by leaving the land idle. Negative CERs should therefore be seen as demonstrating the suboptimality of the alternative allocations rather than as accurate estimates of their value. Ingersoll (pp. 102–4) discusses this issue more in depth, and Tew and Reid present other relevant arguments.

Table 3. Certainty Equivalent Returns for Alternative Decision Rules — Freund Data

Number of Observations	Risk Aversion	Certainty Equivalent Return ($)			
		Bayes	Approx. Bayes	CCS	Plug-in
7	Mod./Low[a]	79,667	– 359,257	–1,041,400	–2,048,093
10	Mod./Low	111,968	100,425	71,821	–5,304
15	Mod./Low	127,696	127,695	127,058	124,509
25	Mod./Low	132,486	132,478	132,353	131,729
∞	Mod./Low	136,422	136,422	136,422	136,422
7	Mod./High[b]	48,770	–219,271	– 482,575	– 684,096
10	Mod./High	70,373	14,077	–30,211	–153,783
15	Mod./High	86,199	86,187	85,321	77,752
25	Mod./High	91,007	90,920	90,842	89,464
∞	Mod./High	94,403	94,403	94,403	94,403

[a]Mod./Low risk aversion corresponds to $r = 0.0100187$.
[b]Mod./High risk aversion corresponds to $r = 0.0300561$.

Table 4. Certainty Equivalent Returns for Alternative Decision Rules — Hazell Data

Number of Observations	Risk Aversion	Certainty Equivalent Return ($)			
		Bayes	Approx. Bayes	CCS	Plug-in
7	Mod./Low[a]	90,370	87,748	62,844	– 695,284
10	Mod./Low	99,304	99,041	69,526	80,086
15	Mod./Low	103,972	103,377	101,748	102,892
25	Mod./Low	106,233	105,875	105,511	106,057
∞	Mod./Low	107,700	107,700	107,700	107,700
7	Mod./High[b]	78,983	7,122	–124,306	–144,237
10	Mod./High	88,083	87,514	85,497	76,230
15	Mod./High	89,977	89,398	89,570	89,476
25	Mod./High	90,921	90,616	90,715	90,838
∞	Mod./High	91,595	91,595	91,595	91,595

[a]Mod./Low risk aversion corresponds to $r = 0.0026738$.
[b]Mod./High risk aversion corresponds to $r = 0.0080214$.

As discussed earlier, in the presence of estimation risk CERs must be obtained from (15) as opposed to (13). Yet, estimation risk is typically not acknowledged and the plug-in approach is generally used to calculate CERs along with decision vectors. CERs are usually obtained by plugging-in the vector of sample estimates ($\hat{\theta}$) for the true but unknown parameter vector (θ) in (13), which yields $CER^{PI}(l; \hat{\theta}) \equiv \text{CER}^{n}(l; \theta = \hat{\theta})$. For example, the CER of the plug-in land allocation is typically calculated as

$$(17) \qquad CER^{PI}(l^{PI};\hat{\mu},\hat{\Sigma}) = \hat{\mu}'l^{PI} - \frac{1}{2}rl^{PI'}\hat{\Sigma}l^{PI}.$$

The row corresponding to infinite observations in tables 3 and 4 reveals how misleading the plug-in approach to CER calculation can be. For example, table 3 shows that the plug-in CER would erroneously lead one to conclude that the CER of the plug-in allocation for seven observations and moderate-to-low risk aversion is \$136,422, when in fact its actual CER is −\$2,048,093. Furthermore, in this scenario the plug-in CER approach indicates that the maximum attainable CER is \$136,422 (obtained from the plug-in land allocation). Such a figure grossly overestimates the actual maximum attainable CER, which is only \$79,667 (obtained from Bayes' land allocation).

Consistent with the definition of CERs, Bayes' CERs decrease monotonically as risk aversion increases. In contrast, for the other three decision criteria, CERs may go up as risk aversion rises. As an illustration, table 4 shows that for seven observations increasing risk aversion causes CER to fall from \$90,370 to \$78,983 for Bayes' allocation, but causes CER to increase from −\$695,284 to −\$144,237 for the plug-in allocation. This result can occur only because the decision vectors from the plug-in, CCS, and approximate Bayes' approaches are suboptimal.

Summary and Conclusions

Estimation risk exists whenever parameters of importance for decision making must be estimated. In the presence of estimation risk, decisions are subject to an additional source of risk related to the accuracy with which parameters are estimated. Bayes' criterion is the decision procedure consistent with expected-utility maximization in the presence of estimation risk. Typically, however, estimation risk is neglected and the plug-in decision criterion is used. The plug-in approach is suboptimal but requires considerably less computation than Bayes' criterion. There are also two alternative decision rules advocated in the literature that have the same computational requirements as the plug-in approach and in addition take estimation risk into account.

This article reexamines the land-allocation problem in the presence of estimation risk. Land allocations obtained with the optimal (Bayes') decision rule are strikingly different from land allocations obtained from the plug-in and the other two alternative decision criteria. As with land allocations, CERs are typically (and incorrectly) calculated with the plug-in approach. Plug-in CERs may be extremely misleading when there are few annual observations relative to land activities. First, plug-in CERs greatly overestimate the true value of plug-in land allocations. Second, plug-in CERs exaggerate the actual maximum attainable CER.

Estimation risk is relevant in situations characterized by few sample observations relative to the number of activities that the decision maker must choose from. Such situations are common in agriculture, which indicates that (*a*) results and conclusions of the previous literature in this area should be qualified, and that (*b*) future research should either account explicitly for estimation risk or assess its potential impact.

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Appendix

The term within brackets in (6b) is monotonically increasing in $(u'l - rl'\Sigma l / 2)$. Therefore, the optimal land vector obtained from (6b) is the same as the optimal land vector obtained from

(18) $$\max_{l,\lambda\geq 0}[\mu' l - \frac{1}{2} rl'\Sigma l + \lambda(Al - b)],$$

where λ is the m-vector of Lagrangian multipliers corresponding to the m acreage restrictions. Following Best and Grauer, the land-allocation vector that solves (18) is

(19) $$l^* = \frac{1}{r} C^* \mu + D^* b^*,$$

where $C^* \equiv \Sigma^{-1} - D^* A^* \Sigma^{-1}, D^* \equiv \Sigma^{-1} A^{*'} (A^* \Sigma^{-1} A^{*'})^{-1}, A^*$ is a $(j \times k)$ matrix whose j rows are the rows of matrix A pertaining to the j binding constraints, and b^* is a j-vector with the corresponding elements of vector b.

The closed-form solutions for the plug-in, CCS,[11] and approximate Bayes' land allocations are, respectively:

(20) $$l^{PI} = \frac{1}{r} C^{PI} \hat{\mu} + D^{PI} b^{PI},$$

(21) $$l^{CCS} = \frac{1}{r} \frac{(n-k-1)}{(n-1)} C^{CCS} \hat{\mu} + D^{CCS} b^{CCS}, \text{ and}$$

(22) $$l^{AB} = \frac{1}{r} \frac{(n-k-2)}{(1+1/n)(n-1)} C^{AB} \hat{\mu} + D^{AB} b^{AB},$$

where $\hat{u} \equiv yl_n / n, \hat{\Sigma} \equiv (y - \hat{\mu} l_n')'(y - \hat{\mu} l_n') / (n-1), C^i \equiv \hat{\Sigma}^{-1} - D^i A^i \hat{\Sigma}^{-1}, D^i \equiv \hat{\Sigma}^{-1} A^{i'} (A^i \hat{\Sigma}^{-1} A^{i'})^{-1}$, and matrix A^i and vector b^i defined analogous to A^* and b^* in (19) for $i = PI$, CCS, and AB.

[11] Strictly speaking, (21) is an approximation because in general CCS allocation has no closed-form solution when there are nonnegativity constraints on acreage. Expression (21) is employed because it is the solution advanced by CCS.