SURFACE WAVE PROPAGATION IN FIBER COMPOSITE LAMINATES

W. Anthony Green

Department of Theoretical Mechanics
University of Nottingham
Nottingham, NG7 2RD
U.K.

INTRODUCTION

The use of surface acoustic waves to study surface and sub-surface defects is illustrated in the papers by Hsieh et al [1] and by Bashyan [2], contained in Volume 10B of the Review of Progress in QNDE. In the same volume, this present author [3] shows that for multi-layered fiber composite laminates, the existence of surface waves is in some cases associated with the short wavelength behaviour of the fundamental mode of the dispersion equation and in other cases it is associated with the higher harmonics. In the latter event, surface wave propagation occurs only over a restricted range of wavelengths or frequency for each harmonic and in this paper we examine the conditions which determine the extent of this range.

A comprehensive account of surface waves in elastic media with cubic symmetry has been presented by Chadwick and Smith [4], whilst Farnell and Adler [5] have carried out an extensive investigation of waves in a thin layer overlying a half space. Reports of more recent work on this problem are to be found in [6].

DISPERSION EQUATION

Consider a laminated plate made up of layers of a fiber composite material with the fibers in each layer being parallel to the plane of the layer and having a specified orientation relative to some reference direction. Choosing a system of cartesian coordinates $Ox_1x_2x_3$ with Ox_1 as the reference direction and Ox_3 as the direction of the normal to the plate, we let ψ_m denote the angle that the fiber direction in layer m makes with a line in the layer parallel to the x_1–axis. The material in each layer is modelled as a transversely isotropic elastic continuum with the axis of transverse isotropy being parallel to the fiber direction.

We are concerned with time harmonic waves of wavelength Λ and of circular frequency ω propagating parallel to the x_1x_2-plane at an angle γ to the reference direction Ox_1, so that the displacements and stress components have the form

$$u_i = U_i(x_3)e^{i\phi}, \quad t_{ij} = T_{ij}(x_3)e^{i\phi}, \tag{1}$$

where $\phi = k(cx_1+sx_2) - \omega t$, t is the time, $c = cos\gamma$, $s = sin\gamma$ and the wave number k is related to the wavelength Λ by the expression $k = 2\pi/\Lambda$. It is convenient to introduce the six vector of displacements and traction components $\underset{\sim}{Z}(x_3)$ defined by

$$\underset{\sim}{Z}(x_3) = \Big\{U_1(x_3)\ U_2(x_3)\ U_3(x_3)\quad T_{13}(x_3)\ T_{23}(x_3)\ T_{33}(x_3)\Big\}^{\mathrm{T}},$$

where T denotes the transpose. Then the equations of motion and the stress relations in layer m take

Review of Progress in Quantitative Nondestructive Evaluation, Vol. 11
Edited by D.O. Thompson and D.E. Chimenti, Plenum Press, New York, 1992

the form

$$\frac{d\underset{\sim}{Z}(x_3)}{dx_3} = \underset{\sim}{A}_m \underset{\sim}{Z}(x_3) , \tag{2}$$

$$\underset{\sim}{T}(x_3) = \underset{\sim}{B}_m \underset{\sim}{Z}(x_3) , \tag{3}$$

where $\underset{\sim}{T}(x_3) = \left(T_{11}(x_3)\ T_{12}(x_3)\ T_{22}(x_3)\right)^T$

and the elements of the 6×6 matrix $\underset{\sim}{A}_m$ and of the 3×6 matrix $\underset{\sim}{B}_m$ depend on k, ω, γ, ψ_m, the density ρ and the five elastic moduli. Equations (2) have the general solution (see Mal [7], Green [8])

$$\underset{\sim}{Z}(x_3) = \underset{\sim}{L}(m)\ \underset{\sim}{E}(x_3 - \bar{x}_3, m)\ \bar{\underset{\sim}{C}}(m) , \tag{4}$$

where $\bar{\underset{\sim}{C}}(m)$ is a six-vector of arbitrary constant, $\underset{\sim}{L}(m)$ is a 6×6 matrix and $\underset{\sim}{E}(x_3 - \bar{x}_3, m)$ is the diagonal matrix

$$\underset{\sim}{E}(x_3 - \bar{x}_3, m) = Diag\Big[exp\, ikp_1(m)\ (x_3 - \bar{x}_3)\ ,\ exp\, ikp_2(m)\ (x_3 - \bar{x}_3)\ ,$$

$$exp\, ikp_3(m)\ (x_3 - \bar{x}_3),\ exp - ikp_1(m)\ (x_3 - \bar{x}_3),\ exp - ikp_2(m)\ (x_3 - \bar{x}_3),\ exp - ikp_3(m)\ (x_3 - \bar{x}_3)\Big],$$

with $\bar{x}_3$ being some specified value of x_3 within layer m. The parameters $\pm ikp_\alpha(m)$ $(\alpha=1,2,3)$ are the eigenvalues of the matrix $\underset{\sim}{A}_m$ and the six columns of $\underset{\sim}{L}(m)$ are the eigenvectors of $\underset{\sim}{A}_m$ associated with the eigenvalues $+ikp_\alpha(m)$ $(\alpha=1,2,3)$ and $-ikp_\alpha(m)$ $(\alpha=1,2,3)$ in that order. Then the first three columns of $\underset{\sim}{L}(m)$ represent waves propagating through the layer in a direction with a component along the positive x_3-axis (upward propagating waves) and the last three columns represent waves propagating in the layer in a direction with a component along the negative x_3-axis (downward propagating waves). For the material considered here, the parameters p_α are either real or pure imaginary. In the latter case the term "upward propagating wave" is associated with a wave travelling in the x_1x_2-plane with amplitude decaying in the positive x_3-direction whilst the term "downward propagating wave" indicates a wave travelling in the x_1x_2-plane with amplitude decaying in the negative x_3-direction. For both real and imaginary values of p_α, the elements of the vector $\bar{\underset{\sim}{C}}(m)$ represent the amplitudes of the six waves at the level $x_3 = \bar{x}_3$ and the amplitude vector $\underset{\sim}{C}(m)$ at any other level is given by

$$\underset{\sim}{C}(m) = \underset{\sim}{E}(x_3 - \bar{x}_3, m)\ \bar{\underset{\sim}{C}}(m) . \tag{5}$$

Introducing a partitioning of the amplitude six-vector $\underset{\sim}{C}(m)$ into the two three-vectors $\underset{\sim}{C}_u(m)$ and $\underset{\sim}{C}_D(m)$ associated with upgoing and downgoing waves respectively, equation (5) yields in particular the expressions for the amplitudes $\hat{\underset{\sim}{C}}_u(m)$ and $\hat{\underset{\sim}{C}}_D(m)$ at the top of layer m in terms of the amplitudes $\bar{\underset{\sim}{C}}_u(m)$ and $\bar{\underset{\sim}{C}}_D(m)$ at the bottom of the layer. These are given by

$$\hat{\underset{\sim}{C}}_u(m) = \underset{\sim}{E}_m \bar{\underset{\sim}{C}}_u(m) ,\quad \hat{\underset{\sim}{C}}_D = \underset{\sim}{E}_m^{-1}\, \bar{\underset{\sim}{C}}_D(m) , \tag{6}$$

where $\underset{\sim}{E}_m$ is the 3×3 diagonal matrix

$$\underset{\sim}{E}_m = diag\Big[exp\ ikp_1(m)h_m\ ,\ exp\ ikp_2(m)h_m\ ,\ exp\, ikp_3(m)h_m\Big] ,$$

and h_m is the depth of the layer. At the interface between layer m and layer $(m+1)$, continuity of displacement and traction implies equality of the six-vectors $\underset{\sim}{Z}$ on each side and on using equations (4) and (5) this yields

$$\bar{\underset{\sim}{C}}(m+1) = \underset{\sim}{H}(m+1,m)\ \hat{\underset{\sim}{C}}(m) , \tag{7}$$

where the 6×6 matrix $\underset{\sim}{H}(m+1,m)$ is defined by

$$\underset{\sim}{H}(m+1,m) = \underset{\sim}{L}^{-1}(m+1)\ \underset{\sim}{L}\ (m) .$$

In the sequel we shall be concerned with a symmetric laminate of $2n$ plies, each of thickness h, for which the wave motion separates into disturbances which are symmetric relative to the mid-plane of the laminate and those which are anti-symmetric relative to the mid-plane. The solutions may then be expressed in terms of conditions in the upper half plate only and the repeated application of equations (5), (6) and (7) lead to the relation between the emplitude six-vector $\hat{C}(n)$ at the top of the laminate and the amplitude six-vector $\bar{C}(1)$ at the mid-plane. This may be written as

$$\hat{C}(n) = D(n,1)\,\bar{C}(1)\,, \tag{8}$$

where

$$D(n,1) = E(h,n)\,H(n,n-1)\,E(h,n-1)\,H(n-1,n)\ldots H(2,1)\,E(h,1)\,. \tag{9}$$

It may be shown that the wave number k and the circular frequency ω occur in the elements of the matrices $H(m,m-1)$ and in the parameters $p_\alpha(m)$ $(\alpha=1,2,3)$ only in the combination $v = \omega/k$, which is the phase velocity for propagation at the angle γ. Thus the direct dependence of $D(n,1)$ on wavenumber k (or equivalently frequency ω) arises only through the term kh which occurs in every element of the diagonal matrices $E(h,m)$. It is convenient to partition the matrices $L(m)$, $E(h,m)$, $H(m+1,m)$, $D(n,1)$ into sub-matrices of order 3 typified by the equation

$$L(m) = \begin{pmatrix} L_{11}(m) & L_{12}(m) \\ L_{21}(m) & L_{22}(m) \end{pmatrix}.$$

The condition that the upper surface of the laminate be traction free may then be written as

$$L_{21}(n)\,\hat{C}_u(n) + L_{22}(n)\,\hat{C}_D(n) = 0\,. \tag{10}$$

Choosing the mid-plane of the laminate at $x_3 = 0$, the conditions to be satisfied by a symmetric wave motion are

$$U_3(0) = 0\,,\quad T_{13}(0) = 0\,,\quad T_{23}(0) = 0\,,$$

whilst the conditions for anti-symmetric disturbances are

$$U_1(0) = 0\,,\quad U_2(0) = 0\,,\quad T_{33}(0) = 0\,.$$

Each of these sets of conditions may be written as a relation between the upgoing and downgoing wave amplitudes at the mid-plane, in the form

$$\bar{C}_n(1) + G\,\bar{C}_D(1) = 0\,, \tag{11}$$

where the 3×3 matrix G has different forms G_s and G_a for the symmetric and anti-symmetric cases respectively. Making use of equation (8) it is possible to replace (11) by a second condition relating the upgoing and downgoing waves at the top surface, namely

$$(F_{11}+G\,F_{21})\,\hat{C}_u(n) + (F_{12}+G\,F_{22})\,\hat{C}_D(n) = 0\,, \tag{12}$$

where the 3×3 matrices F_{ij} $(i,j=1,2)$ are the partitioned elements of the inverse matrix $D^{-1}(n,1)$. Eliminating the upgoing vector $\hat{C}_u(n)$ between equations (10) and (12) leads to the condition for wave motion in the form

$$K\,\hat{C}_D(n) = 0\,, \tag{13}$$

where

$$K = L_{22}(n) - L_{21}(n)\,(F_{11}+G\,F_{21})^{-1}\,(F_{12}+G\,F_{22})\,. \tag{14}$$

Equation (13) has non-trivial solution for the vector $\hat{C}_D(n)$ provided $det\,K = 0$ and this condition yields the dispersion relation for harmonic wave propagation in the laminate.

We are concerned here with surface wave propagation in the outer layer of the laminate. For a half-space of the material of the outer layer, a necessary condition for surface wave propagation is that $p_\alpha(n)$ be pure imaginary for all three values of α. A surface wave then consists of a linear combination of the three downward propagating waves only, with the amplitudes being such that the surface is traction free. The condition for surface waves in a half space is thus seen to be

$$\underset{\sim}{L}_{22}(n)\,\hat{\underset{\sim}{C}}(n) = \underset{\sim}{0}\,, \tag{15}$$

which has non-trivial solutions provided $\det \underset{\sim}{L}_{22}(n) = 0$. It can be shown [4] that this determinantal condition involves ω and k only through the ratio $v = \omega/k$ and that there exists one root v_R which satisfies the conditions that $p_\alpha(n)$ are pure imaginary for $\alpha = 1,2,3$. Thus a surface wave always exists for a half space of the material of the upper layer. In order to examine the existence of surface waves on the laminate, it is necessary to consider the details of the second term on the right hand side of equation (14) for values of ω and k such that $v = \omega/k$ is close to or equal to the value v_R. Here we consider only two possible cases. In our first case, we assume that for values of v close to or equal to v_R the parameters $p_\alpha(m)$ are pure imaginary for all values of m and α. In the second case it is assumed that there exists one value of m $(=s)$ for which $p_3(s)$ is real whilst $p_1(s)$ and $p_2(s)$ and all other $p_\alpha(m)$ are pure imaginary.

Case 1. All waves evanescent

Writing $p_\alpha(m) = i r_\alpha(m)$ $(\alpha=1,2,3,\ m=1,2,\ldots n)$ it may be seen that the elements of the diagonal matrices $\underset{\sim}{E}_m$ consist of decaying exponentials whereas those of the inverse matrix $\underset{\sim}{E}_m^{-1}$ are growing exponentials. For sufficiently large values of kh the elements of $\underset{\sim}{E}_m$ may then be neglected in comparison with those of $\underset{\sim}{E}_m^{-1}$ and the equation (14) for $\underset{\sim}{K}$ yields the approximate expression

$$\underset{\sim}{K} \approx \underset{\sim}{L}_{22}(n) - \underset{\sim}{L}_{21}(n)\,\underset{\sim}{E}_n\,\underset{\sim}{J}_{11}^{-1}(n,n-1)\,\underset{\sim}{J}_{12}(n,n-1)\,\underset{\sim}{E}_n\,, \tag{16}$$

where $\underset{\sim}{J}_{rs}(m,m-1)$, $(r,s=1,2)$ are the 3×3 partitioned submatrices of the 6×6 inverse matrix $\underset{\sim}{H}^{-1}(m,m-1)$. Since the second term on the right hand side of this equation involves the decaying exponential matrix $\underset{\sim}{E}_n$ as a repeated factor, then for large values of kh this term is negligible in comparison with the first term unless the 3×3 matrix $\underset{\sim}{J}_{11}(n,n-1)$ is singular. The matrix $\underset{\sim}{J}_{11}(n,n-1)$ is dependent on ω and k through ratio $v = \omega/k$ only and it may be shown to be non-singular in the region of $v = v_R$ provided the fiber angles ψ_n and ψ_{n-1} are different. Hence provided there exists a solution for which all the $p_\alpha(m)$ are pure imaginary, the harmonic wave dispersion equation $\det \underset{\sim}{K} = 0$ becomes, in the limit as $kh \to \infty$, identical with the condition $\det \underset{\sim}{L}_{22}(n) = 0$ for the surface wave propagation on a half space of the outer material.

Case 2. One interior wave propagating

We assume the values of kh are sufficiently large for the matrix $\underset{\sim}{E}_m$ to be neglected in comparison with $\underset{\sim}{E}_m^{-1}$ for all values of m except $m = s$. *For* $m = s$, $p_3(s)$ is real *and* $\underset{\sim}{E}_s$ *and* $\underset{\sim}{E}_s^{-1}$ involve terms in $\cos khp_3(s)$ and $\sin khp_3(s)$. The corresponding approximate expression for $\underset{\sim}{K}$ is then

$$\underset{\sim}{K} \approx \underset{\sim}{L}_{22}(n) - \underset{\sim}{L}_{21}(n)\,\underset{\sim}{E}_n\,\underset{\sim}{J}_{11}^{-1}(n,n-1)\,\underset{\sim}{R}^{-1}\,\underset{\sim}{T}_{11}^{-1}(s)\,\underset{\sim}{T}_{12}(s)\,\underset{\sim}{R}\,\underset{\sim}{J}_{12}(n,n-1)\,\underset{\sim}{E}_n\,, \tag{17}$$

where $\underset{\sim}{R}$ is a product of matrices $\underset{\sim}{J}_{11}(m+1,m)$ and $\underset{\sim}{E}_m^{-1}$ for $m = (s+1),(s+2),\ldots,(n-1)$ and the matrices $T_{11}(s)$, $T_{12}(s)$ are defined by

$$\begin{aligned}\underset{\sim}{T}_{11}(s) &= \underset{\sim}{J}_{11}(s,s-1)\,\underset{\sim}{E}_s^{-1}\,\underset{\sim}{J}_{11}(s+1,s) + \underset{\sim}{J}_{12}(s,s-1)\,\underset{\sim}{E}_s\,\underset{\sim}{J}_{21}(s+1,s)\,,\\ \underset{\sim}{T}_{12}(s) &= \underset{\sim}{J}_{11}(s,s-1)\,\underset{\sim}{E}_s^{-1}\,\underset{\sim}{J}_{12}(s+1,s) + \underset{\sim}{J}_{12}(s,s-1)\,\underset{\sim}{E}_s\,\underset{\sim}{J}_{22}(s+1,s)\,,\end{aligned} \tag{18}$$

For values of v close to v_R, the second term on the right hand side of (17) involves the decaying exponential matrix $\underset{\sim}{E}_n$ as a repeated factor so that for sufficiently large values of kh, this term is again small compared with the first term except at the values of kh for which $\underset{\sim}{T}_{11}(s)$ is singular. The equation $\det \underset{\sim}{T}_{11}(s) = 0$ is the condition for the existence of harmonic waves propagating in layer s when it is bounded above and below by semi-infinite regions of the material of layers $(s+1)$ and

(s—1) respectively, under the condition that the disturbance in each of these regions decays exponentially with distance from the interfaces. For isotropic elastic materials, conditions for the existence of such waves have been examined by Baylis [9]. If we assume the existence of such waves here then it is clear from the definition of $\underset{\sim}{T}_{11}(s)$ in equation (18) that the equation $det\,\underset{\sim}{T}_{11}(s) = 0$ involves a relation between v and kh which for a specified value of v will yield a set of discrete values of kh. The equation $det\,\underset{\sim}{L}_{22}(n) = 0$, however, involves v only and is satisfied by $v = v_R$ for all values of kh. Thus the solution of the equation $det\,\underset{\sim}{K} = 0$ is then given approximately by $v = v_R$ except close to those values of kh for which $det\,\underset{\sim}{T}_{11}(s) = 0$ at $v = v_R$.

RESULTS

We choose as an example the six-ply plate with symmetric lay-up $(0/+60^0/-60^0)_S$ and employ the elasticity moduli used by Green [8]. For waves travelling parallel to the core fiber direction (the Ox_1 axis) the quantities $p_\alpha(1)$ are all pure imaginary for values of the scaled wave speed $v/c_1 < 0.732$. In both layers 2 and 3, on the other hand, the condition for all the p_α to be pure imaginary is that $v/c_1 < 0.555$ and the scaled value of the surface wave speed in the outer material is $v_R/c_1 = 0.535$. This situation thus corresponds to case 1 of the previous section, with all waves being evanescent if $v/c_1 < 0.555$. Figure 1 shows plots of the scaled phase velocity v/c_1 versus kh for all the first seven modes of anti-symmetric motion and it may be seen that only the fundamental mode has the scaled speed v_R/c_1, as the limiting value as $kh \rightarrow \infty$. The limiting speed $v/c_1 = 0.555$ of all other modes is that associated with the slowest quasi-shear speed in the outer layer.

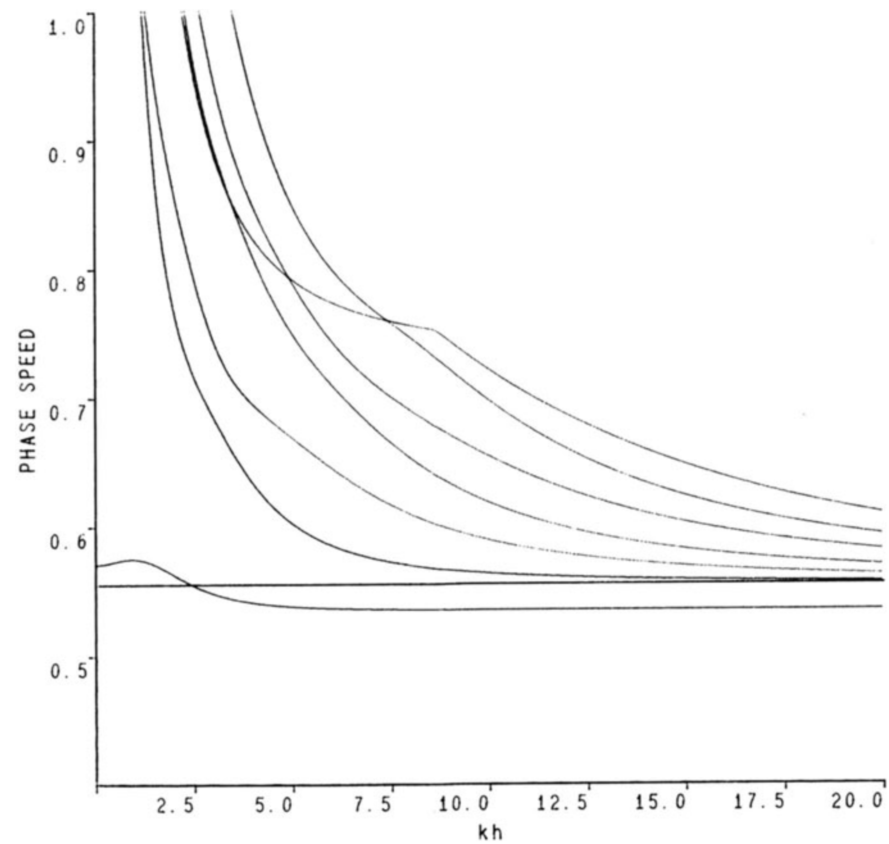

Figure 1. Dispersion curves for first seven antisymmetric modes in 6-ply at $\gamma = 0°$. The horizontal line is $v/c_1 = 0.555$. (Note that apparent intersections are osculation points of adjacent modes)

When the disturbance is set up by waves travelling at right angles to the core fiber direction, on the other hand, the waves in both layers 2 and 3 are all evanescent for values $v/c_1 < 0.678$. For the core, the condition for all $p_\alpha(1)$ to be pure imaginary is that $v/c_1 < 0.482$. This situation thus corresponds to case 2 of the previous section and the surface wave speed in the outer material is now $v_R/c_1 = 0.673$. Figure 2 shows the plots of phase velocity versus kh for the first seven modes of antisymmetric motion in this configuration. The long plateau regions of speed approximately equal to the surface wave speed v_R are clearly evident in the higher harmonics. Figure 3 shows an expanded version of Figure 2 in the neighbourhood of the surface wave speed $v_R/c_1 = 0.673$ and also includes portions of the dispersion curves obtained from solving the equation $det\, \mathcal{T}_{11}(1) = 0$ in the same neighbourhood. This clearly shows that the breakaway points of the plateau regions of Figure 2 are virtually coincident with the intersection points of the roots of $det\ \mathcal{T}_{11}(1) = 0$ with the surface wave line $v = v_R$. Farnell and Adler [5] have demonstrated a similar phenomenon in the case of a two layer plate.

ACKNOWLEDGEMENT

This work is supported by the U.S.A.F. Office of Scientific Research under grant number AFOSR-88-0353.

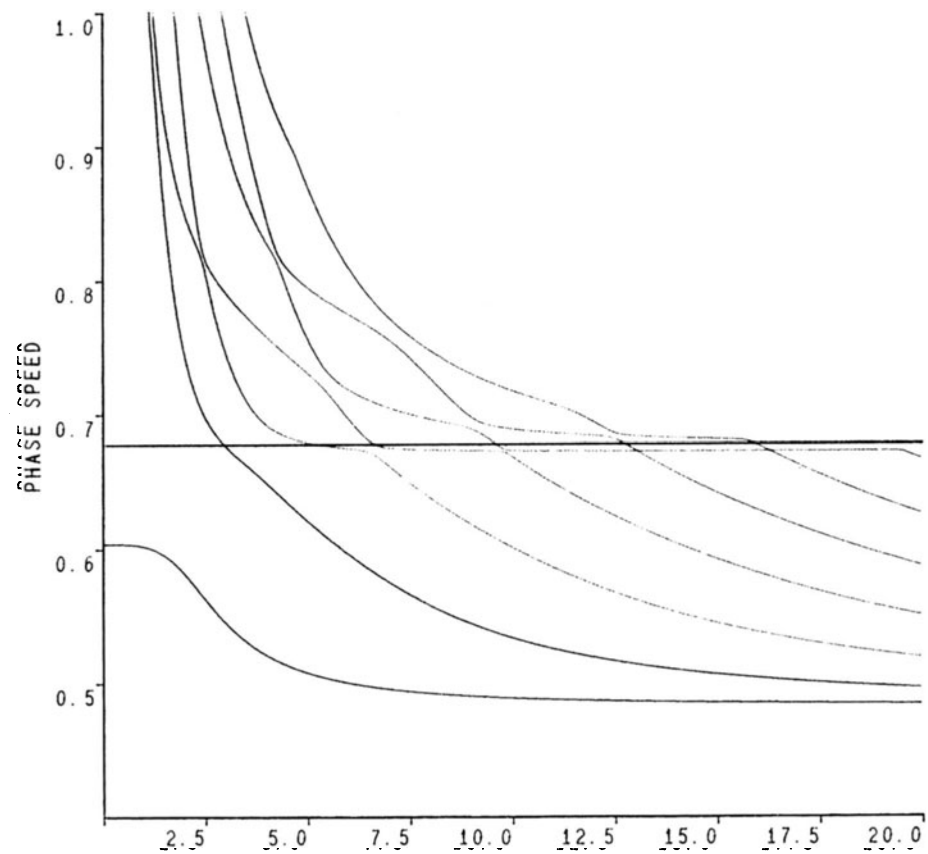

Figure 2. Dispersion curves for first seven antisymmetric modes in 6-ply plate at $\gamma = 90°$. The horizontal line in $v/c_1 = 0.678$.

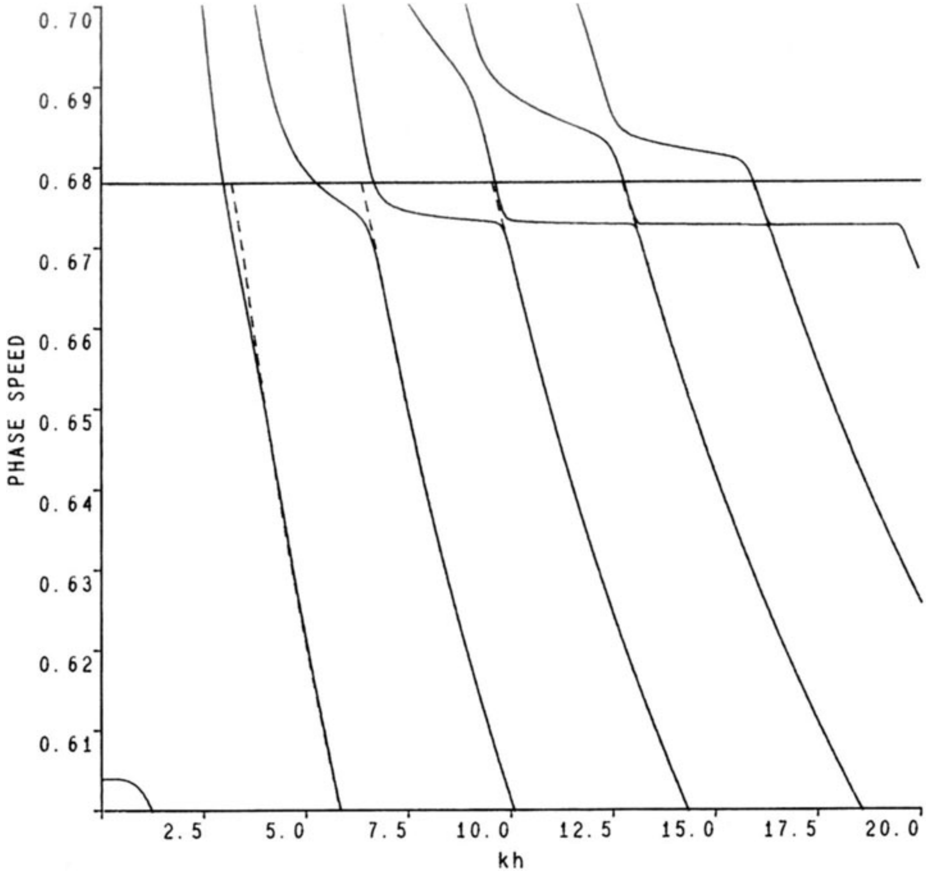

Figure 3. Expanded portion of dispersion curves for first seven antisymmetric modes in 6-ply plat at $\gamma = 90°$, solid lines.
Dispersion curves for first five modes of antisymmetric motion in core layer bounded by semi-infinite regions, dotted lines.

REFERENCES

1. C.P. Hsieh, C-H Chou and B.T. Khuri-Yakub in Review of Progress in Quantitative NDE edited by D.O. Thompson and D.E. Chimenti (Plenum Press, New York, 1991) Vol 10B, pp.1223-1230.
2. M. Bashyam, Ibid. pp.1423-1430.
3. W. A. Green, Ibid. pp.1407-1414.
4. P. Chadwick and G.D.Smith in Mechanics of Solids edited by H.G. Hopkins and M.J. Sewell (Pergamon Press, Oxford, 1982), pp.47-100.
5. G.W. Farnell and E.L. Adler in Physical Acoustics edited by W.P. Mason and R.N. Thurston (Academic Press, New York, 1972), Vol. 9, pp.35-127.
6. D.F. Parker and G.A. Maugin (editors) Recent Developments in Surface Acoustic Waves (Springer-Verlag, Berlin, 1988).
7. A.K. Mal, Wave propagation in layered composite laminates under periodic surface loads. Wave Motion **10**, 1988, 257-266.
8. E. Rhian Green, Transient impact response of a fibre composite laminate. Acta Mechanica **86**, 1991, 154-165.
9. E. Rhian Baylis. Flexural elastic waves in an isotropic internal stratum. Quart. J. Mech. Appl. Math. **39**, 1986, 99-110.