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AN OPTIMUM DOPPLER-SATELLITE INERTIAL NAVIGATION SYSTEM

by

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I. INERTIAL NAVIGATION MECHANIZATION EQUATIONS

The primary purpose of inertial navigation is to indicate the vehicle's position, velocity, and attitude. It accomplishes this purpose through the use of accelerometers and gyroscopes which are mounted on stable platforms. The inertial system is completely self-contained; thus, if there were no errors in the system, the exact position of the vehicle could be determined at any point in time by simply integrating the indicated acceleration to obtain velocity and then position. However, due to such errors as gyro drift, accelerometer bias, platform tilt, etc., the error in one's knowledge of position and azimuth tends to grow without bound. Therefore, means must be found that will cause the state of the system (which will include position and azimuth errors) to be bounded. The purpose of this paper is to present just such a means. It will be shown that by augmenting the inertial system with a measurement of the doppler frequency from an orbital satellite the errors in the inertial system can be corrected and bounded.

One must first consider the mechanization and error equations that will model the inertial system. It has been assumed that the reader is sufficiently acquainted with the physical aspects of accelerometers, gyroscopes, and level platforms such that there is no need to describe their characteristics. Otherwise, he is referred to Slater (12) which has an excellent discussion on the principles of these instruments. The following discussion on the mechanization equations can be found in more detail in Brown (2) and Pinson (11).

The accelerometers sense the second derivative of the position vector, as viewed by an observer in inertial space, minus the mass attraction gravity vector which is a vector pointing downward. Thus the basic equation which will be used to determine the mechanization equations is given by

$$\underline{A} = [\frac{d^2\underline{R}}{dt^2}]_I - \underline{g}_m \tag{1.1}$$

where "I" denotes "with respect to the inertial coordinate system", $\underline{R}$ is the position vector of the vehicle, and $\underline{g}_m$ is the mass attraction gravity vector.

By applying the Coriolis theorem that relates the derivatives of a quantity in two different coordinate systems which are rotating with respect to one another, one can write Equation 1.1 in a rotating computer coordinate system as

$$[\frac{d^2\underline{R}}{dt^2}]_c + 2\underline{\omega}_c \times [\dot{\underline{R}}]_c + [\dot{\underline{\omega}}_c]_c \times \underline{R} + \underline{\omega}_c \times \underline{\omega}_c \times \underline{R} = \underline{A} + \underline{g}_m \tag{1.2}$$

Now replace the mass attraction gravity vector, $\underline{g}_m$, by the local or "plumb-bob" gravity vector, $\underline{g}$, where $\underline{g}$ is given by

$$\underline{g} = \underline{g}_m - \underline{\Omega} \times \underline{\Omega} \times \underline{R} \tag{1.3}$$

Thus Equation 1.2 can be written as

$$[\frac{d^2\underline{R}}{dt^2}]_c + 2\underline{\omega}_c \times [\dot{\underline{R}}]_c + [\dot{\underline{\omega}}_c]_c \times \underline{R} + \underline{\omega}_c \times \underline{\omega}_c \times \underline{R} - \underline{\Omega} \times \underline{\Omega} \times \underline{R} = \underline{A} + \underline{g} \tag{1.4}$$

The basic equation which must be solved by the computer for $\underline{R}$ has now been found. The next step is to determine the error in the solution. First, however, the platform error equation will be derived since its solution is a driving function for the position error equation. Before proceeding it should be emphasized that Equation 1.4 is valid in any coordinate system which is rotating in inertial space even though some coordinate systems may be easier and more convenient than others to implement.

II. ERROR EQUATIONS

Since there are errors in the system, the solution to Equation 1.4 as implemented by the system will be in error (11). In order to determine the magnitude of this error one must go through an error analysis. It will be assumed that all errors are small so only first order effects need to be considered.

The following three coordinate systems will be considered:

x_p, y_p, z_p — Actual platform coordinate system

x, y, z — Ideal platform coordinate system

x_c, y_c, z_c — Computed platform coordinate system

Additional definitions that need to be made are:

$\underline{\phi}$ — Vector angle between the platform and ideal coordinate systems

$\underline{\psi}$ — Vector angle between the platform and computer coordinate systems

$\underline{\delta\theta}$ — Vector angle between the computed and ideal coordinate systems

where

$$\underline{\phi} = \underline{\psi} + \underline{\delta\theta} \tag{2.1}$$

Since $\underline{\psi}$ depends only on the gyro drift it will be determined first. This error arises because the computer tries to precess the gyros at the computer rate $\underline{\omega}_c$ instead of the ideal rate $\underline{\omega}$. The gyros then fail to precess at $\underline{\omega}_c$ because of gyro drift $\underline{\varepsilon}$ and because the platform and computer axes are misaligned by the angle $\underline{\psi}$; thus

$$\underline{\omega}_p = \underline{\omega}_c + \underline{\psi} \times \underline{\omega}_c + \underline{\varepsilon} \tag{2.2}$$

Now it is known that

$$[\dot{\underline{\psi}}]_c = \underline{\omega}_p - \underline{\omega}_c \quad (2.3)$$

Therefore

$$[\dot{\underline{\psi}}]_c + \underline{\omega}_c \times \psi = \underline{\varepsilon} \quad (2.4)$$

which in the inertial coordinate system is

$$[\dot{\underline{\psi}}]_I = \underline{\varepsilon} \quad (2.5)$$

or in the ideal system

$$[\dot{\underline{\psi}}] + \omega \times \underline{\psi} = \underline{\varepsilon} \quad (2.6)$$

Equation 2.6 is the so-called $\underline{\psi}$ equation. It is dependent on gyro drift and initial alignment errors and is uncoupled from the variables $\underline{\delta\theta}$ and $\underline{\phi}$. Now that the platform error equation has been obtained only the position error equation is needed to complete the error model for the inertial system.

Since the sensed acceleration in Equation 1.4 is in error due to accelerometer bias error, $\underline{\Delta}$, and is measured in one coordinate system (platform) and operated on in another (computer), $\underline{A}$ must be replaced by $\underline{A} - \underline{\psi} \times \underline{A} + \underline{\Delta}$. Another error in Equation 1.4 is in the position vector $\underline{R}$; thus, it will be replaced by $\underline{R} + \underline{\delta R}$. As a result of the error in position $\underline{g}$ must be replaced by $\underline{g} - \omega_o^2\, \underline{\delta R}$ where ω_o^2 is g/R. Replacing the appropriate variables in Equation 1.4 by the above quantities one finds that

$$[\ddot{\underline{\delta R}}]_c + 2\underline{\omega}_c \times [\dot{\underline{\delta R}}]_c + [\dot{\underline{\omega}}_c] \times \underline{\delta R} + \underline{\omega}_c \times \underline{\omega}_c \times \underline{\delta R} - \underline{\Omega} \times \underline{\Omega} \times \underline{\delta R} = \underline{\Delta} - \underline{\psi} \times \underline{A} - \omega_o^2\, \underline{\delta R} \quad (2.7)$$

Again it should be pointed out that, as in the case of Equation 1.4, the platform error equation and the position error equation given by

Equations 2.6 and 2.7 are valid in any rotating coordinate system. At this point, however, since a marine system will be used as an example for the proposed doppler-satellite inertial navigation system, it will be convenient to make several assumptions. First, it will be assumed that the marine system operates in a latitude-longitude coordinate system which has the accelerometers and gyros on a locally level plane with the sensitive axes of the accelerometers pointing north and west and the sensitive axes of the gyros pointing north, west, and up as shown in Figure 2.1. Second, the slow-moving vehicle assumption ($\omega \approx \Omega$) will be made.

Since a slow-moving vehicle was assumed, Equation 2.7 can be written as

$$[\ddot{\underline{\delta R}}]_c + 2\underline{\Omega} \times [\dot{\underline{\delta R}}]_c + \omega_o^2 \underline{\delta R} = \underline{\Delta} - \underline{\psi} \times \underline{g} \qquad (2.8)$$

It is evident that the above equation can be referred back to the inertial system and then back to the ideal system where it can be written as

$$\ddot{\underline{\delta R}} + 2\underline{\Omega} \times \dot{\underline{\delta R}} + \omega_o^2 \underline{\delta R} = \underline{\Delta} - \underline{\psi} \times \underline{g} \qquad (2.9)$$

In a locally level system the following relationship exists between the position error and the elements of $\underline{\delta\theta}$:

$$\begin{aligned} \delta R_x &= R\delta\theta_y \\ \delta R_y &= -R\delta\theta_x \end{aligned} \qquad (2.10)$$

so Equation 2.9 can be written in terms of its components as

$$\begin{aligned} R\ddot{\delta\theta}_y + 2\Omega_z R\dot{\delta\theta}_x + \omega_o^2 R\delta\theta_y &= \Delta_x - \psi_y g \\ -R\ddot{\delta\theta}_x + 2\Omega_z R\dot{\delta\theta}_y - \omega_o^2 R\delta\theta_x &= \Delta_y + \psi_x g \end{aligned} \qquad (2.11)$$

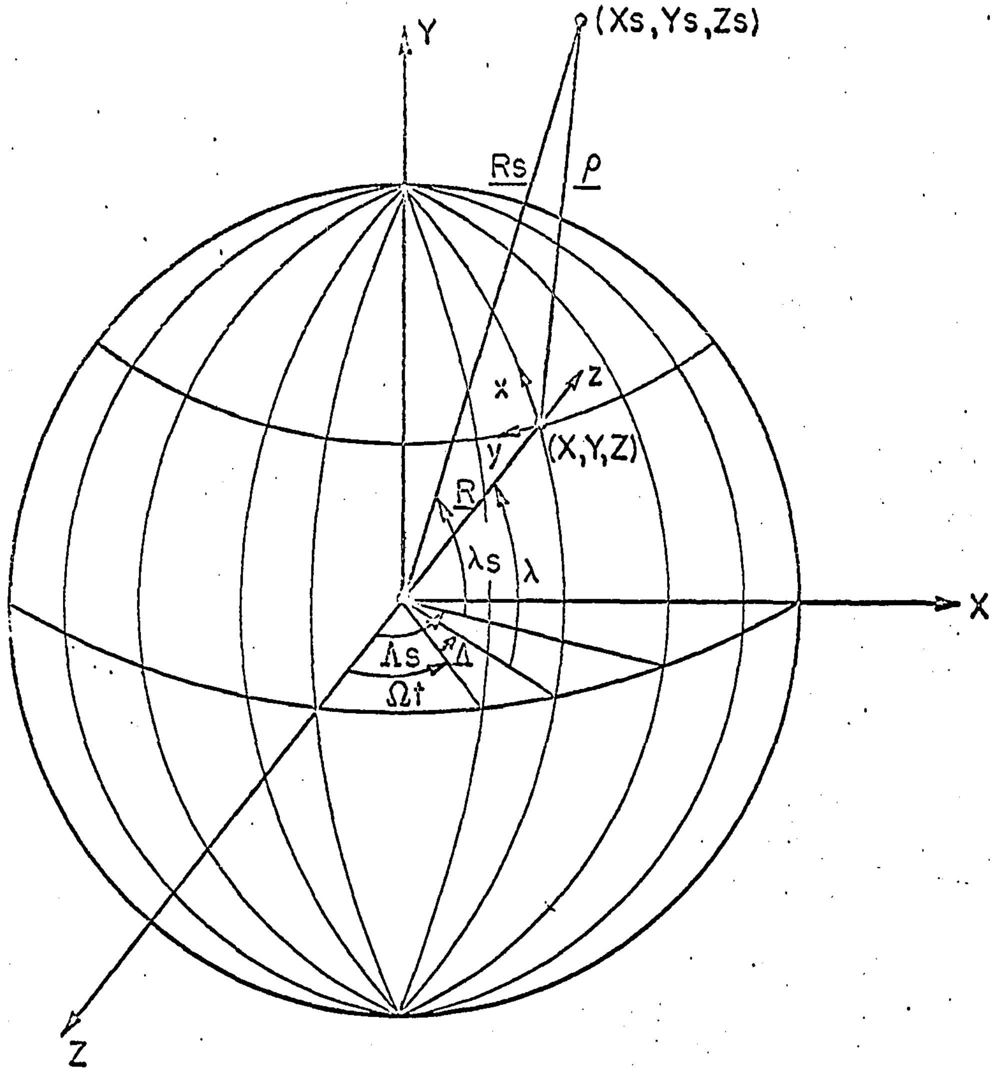

Figure 2.1. Latitude-longitude coordinate system where (X, Y, Z) and (X_s, Y_s, Z_s) are respectively the inertial coordinates of the vehicle and satellite

Therefore using the components of Equation 2.11 and Equation 2.6 one has the mathematical model of the platform and position errors in the ideal system as given by Equations 2.12 and 2.13

$$\dot{\psi}_x - \Omega_z \psi_y = \varepsilon_x$$
$$\dot{\psi}_y + \Omega_z \psi_x - \Omega_x \psi_z = \epsilon_y \qquad (2.12)$$
$$\dot{\psi}_z + \Omega_x \psi_y = \epsilon_z$$

$$\delta\ddot{\theta}_y + 2\Omega_z \ \delta\dot{\theta}_x + \omega_o^2(\delta\theta_y + \psi_y) = \Delta_x/R$$
$$\delta\ddot{\theta}_x - 2\Omega_z \ \delta\dot{\theta}_y + \omega_o^2(\delta\theta_x + \psi_x) = -\Delta_y/R \qquad (2.13)$$

It now becomes obvious that some sort of damping would be helpful to the system since the position error will tend to oscillate with an 84-minute period.[1] This can be readily seen by considering the x-channel in Equations 2.12 and 2.13 and for the moment neglecting all cross-coupling terms. The resulting equations are

$$\dot{\psi}_y = \epsilon_y$$
$$\delta\ddot{R}_x = \Delta_x - \omega_o^2(\delta R_x + R\psi_y) \qquad (2.14)$$

which corresponds to a position error in the s-domain of

$$\delta R_x(s) = \frac{\Delta_x(s)}{s^2 + \omega_o^2} - \frac{g\epsilon_y(s)}{s(s^2 + \omega_o^2)} \qquad (2.15)$$

In considering Equation 2.15 it can be determined that $\overline{\delta R_x^2}$ will tend to grow with time if Δ_x is random. This behavior is attributed to the

[1]The angular velocity ω_o corresponds to approximately an 84-minute period.

complex poles on the $j\omega$ axis in the transfer function between δR_x and Δ_x. These poles can be shifted into the left-half plane with damping. The conventional means of damping the system consists of adding a term proportional to the difference between the inertial velocity and a non-inertial reference velocity. Note that velocity damping will not cause the position error due to ϵ_y to become bounded since the transfer function in question has a pole at the origin which damping cannot remove.

It can be seen by considering the characteristic equation of Equation 2.12 that the platform error equations will have poles at the origin and $\pm j\Omega$. Thus gyro drifts will cause oscillations with 24-hour periods in the platform error and in the position error.[1] If the gyro drift is random, the modes with 24-hour periods will then have errors associated with them which are unbounded. Several systems have been devised to bound these errors. One such system uses the fact that the tilt angles ϕ_x and ϕ_y can be neglected after the alignment procedure if a velocity reference is present (2). Therefore

$$\begin{aligned} \delta\theta_x &\approx -\psi_x \\ \delta\theta_y &\approx -\psi_y \end{aligned} \qquad (2.16)$$

where it can be seen that by measuring latitude and longitude errors one can obtain an indication of the components of $\underline{\psi}$. However this method requires landmarks to be present which is a luxury that one seldom has.

Another method is to track the stars or the sun with a star-tracker. Unfortunately in the case where stars are tracked, clear weather is

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[1]The angular velocity Ω corresponds to a 24-hour period.

needed and in either case the equipment required is quite expensive.

A system that has come under intensive investigation in the last few years is one which uses a satellite to provide a non-inertial position reference and thus damp the $\underline{\psi}$ equations. The most obvious advantage of such a system is that a position fix can be obtained every $1\frac{1}{2}$ to 3 hours; whereas, in the star tracker system there are prolonged periods when the vehicle is operating either pure-inertial or damped-inertial. Some satellite systems use angle tracking while other satellite systems measure the frequency transmitted by the satellite and determine a non-inertial measurement of position from the resulting doppler shift. It is these systems with which we will be concerned throughout the rest of the paper.

III. NAVY NAVIGATION SATELLITE SYSTEM

In the late 1950's it was shown that using only the doppler shift of the radio signals coming from a satellite during one pass, one could completely determine the six parameters needed to describe the orbit of a satellite (6). If all one wished to do was to determine the satellite orbit, using one orbit is obviously too large a constraint. However, if one turns the problem around and assumes that the satellite position is known and the navigator's position is desired, the use of a single pass becomes highly significant. Each time a satellite passes within range one could determine his position and reset the inertial system; because, even though ship inertial systems may be extremely accurate for periods of up to 24 hours, the system must be periodically furnished with a new position fix from which to begin its dead reckoning calculation since its accuracy deteriorates due to long-term drift. It was from this concept that the Transit system, or what is now known as the Navy Navigation Satellite System (NNSS), originated.

The NNSS is an all-weather satellite navigational aid that provides around-the-clock fixes of extremely high accuracy. It consists of four tracking stations which determine the orbits of the satellites in the system and inject the predicted position of the satellite into the satellite memory every twelve hours. Every two minutes each satellite tells the navigator which satellite it is, what time it is according to the satellite "clock", and where the satellite is at the present time. Information is also included that enables the navigator to calculate when and where the other satellites will be available.

The satellites which make up this system weigh about 140 pounds each and are launched into a 600 nautical-mile-altitude, circular, polar orbit. The orbits are circular to attenuate acceleration and deceleration characteristics of elliptical orbits and polar to negate the precession of orbital planes toward an eventual overlap. The plan was to have four satellites with each satellite transmitting two frequencies. It is desirable to transmit two frequencies so as to remove the error due to refraction effects. This is accomplished in the following manner (6):

$$[f_D]_1 = - [\dot{\rho}/c]f_1 + A/f_1 + 0(1/f_1^2) \tag{3.1}$$

$$[f_D]_2 = - [\dot{\rho}/c]f_2 + A/f_2 + 0(1/f_2^2) \tag{3.2}$$

where

$\dot{\rho}$ — Range rate

c — Speed of light

f_1, f_2 — Transmitted frequencies

$[f_D]_1$, $[f_D]_2$ — Doppler shift corresponding to f_1 and f_2

A/f_1, A/f_2 — First order effects of refraction

Now when one multiplies Equation 3.1 by f_1 and Equation 3.2 by f_2 and subtracts, the effect of refraction is eliminated.

The NNSS uses two different systems to determine a navigation fix. The first, which is used aboard the Polaris submarine, is fully automatic. The navigator has only to read his position. This system determines from its computed position the doppler curve that it expects to obtain during the next pass. The navigator's position after the next pass is found by comparing the pre-computed doppler curve in a "least-squares sense" to the one actually obtained. This is accomplished by adjusting the computed position of the ship until the closest "fit" is found.

The second system which is used for surface vessels is less sophisticated. In order to smooth the noise and use range differences instead of range rate, the navigator actually counts the cycles of the beat frequency over a certain time interval. The beat frequency is the sum of the measured doppler frequency and the unknown but constant difference "δ" between the navigator's frequency and the satellite's transmitted frequency; thus,

$$[f_D]_b = - [f/c]\dot{\rho} + \delta \qquad (3.3)$$

Since counting the number of cycles of the beat frequency is the same as integrating the beat frequency, the number of cycles is

$$N_b = \frac{f}{c}[\rho_1 - \rho_2] + \delta[t_2 - t_1] \qquad (3.4)$$

Therefore in order to determine his position, assuming that the satellite position is known exactly, the navigator need only determine three values of N_b during the three two-minute intervals.

The three values of N_b determine three intersecting hyperboloids of revolution. The navigator's computed position is the point on the earth where these three hyperboloids intersect. It should be noted that the vehicle's inertial velocity is needed in order to compensate for the navigator's motion during the six minute interval.

There are other ways of reducing the data to an acceptable form but they all fall into the same category in that they use the NNSS together with the velocity from a good inertial system or some other source to obtain a position fix (10).

IV. DOPPLER-SATELLITE INERTIAL NAVIGATION SYSTEM

If one were to consider the problem from a slightly different viewpoint by thinking of the satellite fixes as a means of correcting for the long-term drift in the inertial system, one would realize that the previous conventional system fails to utilize fully the available information in an optimum manner. The conventional system resets the position errors but fails to correct the slowly varying random driving functions in the inertial system such as accelerometer bias and gyro drift. However, in 1965 Bona and Hutchinson (1) presented a model of a doppler-satellite inertial navigation system which used all of the available information in an optimum fashion in the sense that the mean square error of each state was minimized. Bona and Hutchinson's system was modeled with a Kalman filter in order to obtain an optimum estimate of the inertial errors using a "position fix" from an orbital satellite as the observation instead of using the "position fix" to damp the system in the usual manner as in the NNSS. Position fixes were obtained in the same fashion as were those of the NNSS. (For a brief discussion of the filter equations the reader is referred to Appendix A.)

Bona and Hutchinson assumed that a slow-moving marine system was being operated in the damped-inertial mode so that tilts could be neglected. Therefore, the solution to the $\underline{\psi}$ equation (Equation 2.6) determines the position error. The dynamics of the system are given by

$$\dot{\underline{\psi}} = V\,\underline{\psi} + \underline{\varepsilon} \qquad (4.1)$$

with $-\,\underline{\Omega} \times \underline{\psi}$ replaced by $V\underline{\psi}$ in order to use the notation of Bona and Hutchinson.

Since $\underline{\varepsilon}$ is not a white noise driving function it was assumed to consist of two parts, $\underline{\varepsilon}_c$ and $\underline{\varepsilon}_r$, which satisfy the differential equations

$$\dot{\underline{\varepsilon}}_c = \underline{0}$$

$$\dot{\underline{\varepsilon}}_r = H\underline{\varepsilon}_r + \underline{w} \tag{4.2}$$

where $\underline{w}$ is a white noise driving function. Thus the dynamics of the system can be put into the proper format for the Kalman filter by considering the state $\underline{x}$ to be

$$\underline{x} = \begin{bmatrix} \underline{\psi} \\ \hline \underline{\varepsilon}_r \\ \hline \underline{\varepsilon}_c \end{bmatrix} \tag{4.3}$$

The state equation is

$$\begin{bmatrix} \dot{\underline{\psi}} \\ \hline \dot{\underline{\varepsilon}}_r \\ \hline \dot{\underline{\varepsilon}}_c \end{bmatrix} = \left[\begin{array}{c|c|c} V & I & I \\ \hline 0 & H & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \underline{\psi} \\ \hline \underline{\varepsilon}_r \\ \hline \underline{\varepsilon}_c \end{bmatrix} + \begin{bmatrix} 0 \\ \hline \underline{w} \\ \hline 0 \end{bmatrix} \tag{4.4}$$

which is in the form of

$$\dot{\underline{x}} = F\underline{x} + G\underline{w} \tag{4.5}$$

where

$$G = \begin{bmatrix} 0 \\ \hline I \\ \hline 0 \end{bmatrix} \tag{4.6}$$

To determine the output equation it was assumed that the observation was the difference in the position indicated by the inertial system and the satellite. The measurement vector is given by

$$\underline{y} = \begin{bmatrix} \Delta\lambda \cos L \\ \\ \Delta L \end{bmatrix} = \underline{y}_s + \underline{\delta y} \tag{4.7}$$

where $\Delta\lambda$ is the observed longitude error, ΔL is the observed latitude error, and $\underline{\delta y}$ is the observation noise. The position error vector $\underline{y}_s$ is given by

$$\underline{y}_s = \begin{bmatrix} \Delta\lambda_s \cos L \\ \\ \Delta L_s \end{bmatrix} = M_1 \underline{\psi} + \underline{p} \tag{4.8}$$

where $\Delta\lambda_s$ is the system longitude error and ΔL_s is the system latitude error. The vector $\underline{p}$ is given by

$$\underline{p} = \begin{bmatrix} \phi_x \\ \\ \phi_y \end{bmatrix} \tag{4.9}$$

where ϕ_x and ϕ_y are platform misalignment angles. The matrix M_1 is

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.10}$$

Now let

$$\underline{v} = \underline{p} + \underline{\delta y} \tag{4.11}$$

and assume that the total observation noise, $\underline{v}$, is a white noise process. The output equation is therefore in the format required for the Kalman filter, that is,

$$\underline{y} = M\underline{x} + \underline{v} \tag{4.12}$$

where

$$M = [M_1 \mid 0 \mid 0] \tag{4.13}$$

The augmented system as modeled by Equations 4.5 and 4.12 would allow one to correct the inertial system in an optimum fashion if it truly modeled the physical situation. If one considers the method used to obtain the measurements of position for the observation vector $\underline{y}$, it can be seen that the model as presented by Bona and Hutchinson violates an assumption used in the derivation of the filter equations. Previously it was mentioned that in order to determine position with the NNSS one needed to know inertial velocity which is one of the states in the model. Thus if there is an error in the measurement of position it will be correlated with the velocity states of the system. One cannot even assume that the correlation can be neglected since according to Kershner (7)

> ". . . as a rough rule of thumb one can expect about 0.2 of a mile error for every knot error in the velocity measurement."

This figure is on the same order of magnitude as the total accuracy of the navigation system that is being developed.

Therefore, it will be the purpose of the following sections to present a doppler-satellite inertial navigation system that properly accounts for this correlation and satisfies the Kalman filter model.

A. State Space Formulation

The mathematical model used for this system will include the Schuler loop equations since the system will be implemented without the velocity reference. Without a velocity reference solving the $\underline{\psi}$ equations to determine position error would not be valid since the tilts would no longer be negligible. Equations 2.12 and 2.13 can be put into state space form be defining the following states:

$$x_1 = \psi_x,\ x_2 = \psi_y,\ x_3 = \psi_z,\ x_4 = \delta\theta_x$$

$$x_5 = \delta\dot{\theta}_x,\ x_6 = \delta\theta_y,\ x_7 = \delta\dot{\theta}_y \tag{4.14}$$

Therefore the model can be expressed in the form of

$$\dot{\underline{x}} = A\underline{x} + \underline{f} \tag{4.15}$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix}
=
\begin{bmatrix}
0 & \Omega_z & 0 & 0 & 0 & 0 & 0 \\
-\Omega_z & 0 & \Omega_x & 0 & 0 & 0 & 0 \\
0 & -\Omega_x & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-\omega_o^2 & 0 & 0 & -\omega_o^2 & 0 & 0 & -2\Omega_z \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -\omega_o^2 & 0 & 0 & -2\Omega_z & -\omega_o^2 & 0
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}
+
\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ 0 \\ -\Delta_y/R \\ 0 \\ \Delta_x/R \end{bmatrix}
$$

Figure 4.1. State equation for the inertial system in matrix form

$$
M = \begin{bmatrix}
0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & u_2 & u_3 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\delta \underline{y}_n = \begin{bmatrix} \delta f_{1_i} \\ \delta V_{x_2} \\ \delta V_{y_2} \end{bmatrix}
$$

Figure 4.2. Measurement matrix and measurement error for the output equation $\underline{y}_n = M_n \underline{x}_n + \delta \underline{y}_n$

where A and $\underline{f}$ are defined in Figure 4.1.

The system modeled by Equation 4.15 can be placed into the correct format for use in the Kalman filter if $\underline{f}$ is composed of independent white noise driving functions. However it would be unreasonable to assume that gyro drift and accelerometer bias are not self-correlated from one point in time to another. Therefore it will be assumed that they are random with autocorrelation functions of the form $R(\tau) = \sigma^2 e^{-\beta|\tau|}$. Using the standard technique of determining the shaping filter whose output has an autocorrelation function of the form given above when a white noise input is present, the following relationship between the state and a white noise driving function can be determined:

$$\dot{\epsilon}_x + \beta_1 \epsilon_x = f_1 \tag{4.16}$$

where the gyro drift has been chosen as an example (3). Thus five more states must be added to our model

$$\begin{aligned} &x_8 = \epsilon_x, \; x_9 = \epsilon_y, \; x_{10} = \epsilon_z, \\ &x_{11} = -\Delta_y/R, \; x_{12} = \Delta_x/R \end{aligned} \tag{4.17}$$

The output equation which will consist of the observations of velocity and frequency error must now be considered. The error in the computed value of doppler frequency will be

$$\delta f_D = f_{D_c} - f_D \tag{4.18}$$

where f_D is the true doppler frequency and f_{D_c} is the computed doppler frequency. There is also a non-inertial measurement of doppler frequency f_{D_m}. An observation will be defined to be the frequency difference measurement

$$[\delta f_D]_{meas} = f_{D_c} - f_{D_m} = \delta f_D + \delta f_{1_i} + \delta f_{2_i} \tag{4.19}$$

which implies that

$$f_{D_m} = f_D - \delta f_{1_i} - \delta f_{2_i} \tag{4.20}$$

where δf_{1_i} is the correlated part of the measured frequency error from the i^{th} satellite and δf_{2_i} is the uncorrelated part. In a similar fashion the observations for the velocity difference measurements can be written as

$$\begin{aligned} [\delta\dot{\theta}_x]_{meas} &= \delta\dot{\theta}_x + \delta V_{x_1} + \delta V_{x_2} \\ [\delta\dot{\theta}_y]_{meas} &= \delta\dot{\theta}_y + \delta V_{y_1} + \delta V_{y_2} \end{aligned} \tag{4.21}$$

As in the preceding section the correlated functions will need to be defined as states of the system and the associated shaping filters will be of a similar form since the same type of autocorrelation function will be assumed for these states. Since three satellites will be used in the example system, three frequency states will be required. The additional states are

$$\begin{aligned} &x_{13} = \delta V_{x_1},\ x_{14} = \delta V_{y_1},\ x_{15} = \delta f_{1_1}, \\ &x_{16} = \delta f_{1_2},\ x_{17} = \delta f_{1_3} \end{aligned} \tag{4.22}$$

The output equation and the state equation are now in the format needed to implement the filter equations. These equations are given in Figures 4.2 and 4.3 in matrix form where $\underline{f}$ is a vector composed of independent white noise driving functions and u_i is equal to 1 if the frequency of the i^{th} satellite is being observed and zero otherwise.

Since Kalman's model consists of the two equations

$$
A = \begin{bmatrix}
0 & \Omega_z & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\Omega_z & 0 & \Omega_x & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\Omega_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\omega_o^2 & 0 & 0 & -\omega_o^2 & 0 & 0 & 2\Omega_z & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\omega_o^2 & 0 & 0 & -2\Omega_z & -\omega_o^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_{10}
\end{bmatrix}
\qquad
f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \end{bmatrix}
$$

Figure 4.3. System matrix and driving function for the state equation $\dot{\underline{x}} = A\underline{x} + \underline{f}$

$$\underline{x}_{n+1} = \phi_n \underline{x}_n + \underline{g}_n \tag{4.23}$$

$$\underline{y}_n = M_n \underline{x}_n + \delta \underline{y}_n \tag{4.24}$$

Equation 4.15 will need to be expressed in the form of Equation 4.23. This involves the determination of the transistion matrix ϕ_n. There are several ways to determine ϕ_n but since ϕ_n is a 17 x 17 matrix hand calculation is too time consuming. However if the time increments $t_{n+1} - t_n$ are the same for all n, ϕ_n can be expressed as

$$\phi_n = e^{A\Delta t} \tag{4.25}$$

where $\Delta t = t_{n+1} - t_n$. By expressing ϕ_n as $\sum_{i=o}^{\infty} \frac{(A\Delta t)^i}{i!}$, its value can be easily found on the computer.

B. Relation Between Doppler Frequency and States

The model is now complete except for the determination of the coefficients a_1, a_2, a_3, and a_4 in the measurement matrix of Figure 4.2 which relate the states of position error and velocity error to the frequency difference measurement in a linear fashion.

The particular situation that will be modeled assumes that the satellite position and velocity are known perfectly. The satellites will be assumed to be moving in a circular polar orbit at a constant altitude of approximately 500 nautical miles above a spherical earth. The vehicle is moving at some velocity which is small compared to the angular velocity of the earth ($\underline{\omega} \approx \underline{\Omega}$).

From the geometry of Figure 2.1 it can be seen that the position of the vehicle in the inertial coordinate system can be given by

$$\underline{R} = \underline{i}\, X + \underline{j}\, Y + \underline{k}\, Z \tag{4.26}$$

where

$$X = R\cos\lambda\sin(\Lambda + \Omega t)$$
$$Y = R\sin\lambda \tag{4.27}$$
$$Z = R\cos\lambda\cos(\Lambda + \Omega t)$$

In a similar fashion one can see that the position of the satellite in the inertial coordinate system is given by

$$\underline{R}_s = \underline{i}\,X_s + \underline{j}\,\underline{Y}_s + \underline{k}\,Z_s \tag{4.28}$$

where

$$C_1 = X_s = R_s\cos\lambda_s\sin\Lambda_s$$
$$C_2 = Y_s = R_s\sin\lambda_s \tag{4.29}$$
$$C_3 = Z_s = R_s\cos\lambda_s\cos\Lambda_s$$

Therefore, the square of the radial distance between the satellite and the vehicle is given by

$$\rho^2 = (X - X_s)^2 + (Y - Y_s)^2 + (Z - Z_s)^2 \tag{4.30}$$

which can be expressed as

$$\rho^2 = R^2 + R_s^2 - 2(XX_s + YY_s + ZZ_s) \tag{4.31}$$

Since the doppler frequency is proportional to range rate, $\dot{\rho}$ will need to be determined from Equation 4.31. It may be written as

$$\dot{\rho} = -\rho^{-1}(X\dot{X}_s + X_s\dot{X} + Y\dot{Y}_s + Y_s\dot{Y} + Z\dot{Z}_s + Z_s\dot{Z}) \tag{4.32}$$

where

$$\dot{X} = R\{\cos\lambda\cos(\Lambda + \Omega t)[\dot{\Lambda} + \Omega] - \sin(\Lambda + \Omega t)\sin\lambda[\dot{\lambda}]\}$$
$$\dot{Y} = R\dot{\lambda}\cos\lambda \tag{4.33}$$
$$\dot{Z} = R\{-\cos\lambda\sin(\Lambda + \Omega t)[\dot{\Lambda} + \Omega] - \cos(\Lambda + \Omega t)\sin\lambda[\dot{\lambda}]\}$$

and

$$C_4 = \dot{X}_s = R_s\{\cos\lambda_s \cos\Lambda_s[\dot{\Lambda}_s] - \sin\Lambda_s \sin\lambda_s[\dot{\lambda}_s]\}$$

$$C_5 = \dot{Y}_s = R_s \dot{\lambda}_s \cos\lambda_s \tag{4.33}$$

$$C_6 = \dot{Z}_s = R_s\{-\cos\lambda_s \sin\Lambda_s[\dot{\Lambda}_s] - \cos\Lambda_s \sin\lambda_s[\dot{\lambda}_s]\}$$

Since satellite position and velocity are assumed to be known they will be constants in the subsequent equations. In this case it is assumed that an error in the doppler frequency is a function of only the errors in the navigator's position and velocity. It should be noted that one does not have to use a latitude-longitude coordinate system to find the range rate error; however it lends itself quite easily to the problem.

The various partial derivatives which relate the error in range rate to errors in position and velocity are derived in Appendix B and are given by Equations B.4, B.6, B.7 and B.8. To relate these partials to the mathematical model one must consider the basic equation

$$f_D = - f/c\ \dot{\rho} \tag{4.34}$$

Since it is assumed that the error in the doppler frequency and thus the error in $\dot{\rho}$ is a function only of the error in the vehicle's position and velocity, the doppler frequency error is given by

$$\delta f_D = f_{D_c} - f_D = - \frac{f\delta\dot{\rho}}{c} \tag{4.35}$$

where

$$\delta\dot{\rho} = \frac{\partial\dot{\rho}}{\partial\lambda}\delta\lambda + \frac{\partial\dot{\rho}}{\partial\Lambda}\delta\Lambda + \frac{\partial\dot{\rho}}{\partial\dot{\lambda}}\delta\dot{\lambda} + \frac{\partial\dot{\rho}}{\partial\dot{\Lambda}}\delta\dot{\Lambda} \tag{4.36}$$

In a locally level system where $\underline{\omega} \approx \underline{\Omega}$ the latitude and longitude errors in position and velocity are related to the states by

$$\delta\theta_x = \delta\Lambda \cos\lambda \qquad \delta\dot{\theta}_x = \delta\dot{\Lambda} \cos\lambda$$
$$\delta\theta_y = \delta\lambda \qquad \delta\dot{\theta}_y = \delta\dot{\lambda} \tag{4.37}$$

one can write Equation 4.36 as

$$\delta\dot{\rho} = [\frac{\partial\dot{\rho}}{\partial\Lambda}\frac{1}{\cos\lambda}]\delta\theta_x + [\frac{\partial\dot{\rho}}{\partial\dot{\Lambda}}\frac{1}{\cos\lambda}]\delta\dot{\theta}_x + [\frac{\partial\dot{\rho}}{\partial\lambda}]\delta\theta_y + [\frac{\partial\dot{\rho}}{\partial\dot{\lambda}}]\delta\dot{\theta}_y \tag{4.38}$$

Choosing

$$a_1 = -\frac{f}{c\cos\lambda}\frac{\partial\dot{\rho}}{\partial\Lambda}$$
$$a_2 = -\frac{f}{c\cos\lambda}\frac{\partial\dot{\rho}}{\partial\dot{\Lambda}} \tag{4.39}$$
$$a_3 = -\frac{f}{c}\frac{\partial\dot{\rho}}{\partial\lambda}$$
$$a_4 = -\frac{f}{c}\frac{\partial\dot{\rho}}{\partial\dot{\lambda}}$$

one can write δf_D as a linear combination of the position and velocity states as follows:

$$\delta f_D = a_1\,\delta\theta_x + a_2\,\delta\dot{\theta}_x + a_3\delta\theta_y + a_4\delta\dot{\theta}_y \tag{4.40}$$

where a_1, a_2, a_3, and a_4 will be evaluated at the vehicle's computed position.

C. Computer Simulation

Before discussing the results of the computer simulation of the example system, the problem of computation time should be mentioned. In implementing the Kalman recursive equations matrices on the order of 17 x 17 must be multiplied and, because of the large number of iterations required before the P_n matrix tends to converge to its steady state value, a great deal of computer time can be used. However the constant matrices A, ϕ_n, and H_n peculiar to this problem lend themselves well to matrix partitioning. If one considers the A matrix in Figure 4.3, one can see that the partition corresponding to the augmented states is a diagonal matrix. Thus A can be partitioned in the following manner

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{4.41}$$

where A_{11} is a 7 x 7 matrix, A_{12} is a 7 x 10 matrix, A_{21} is a 10 x 7 null matrix, and A_{22} is a 10 x 10 diagonal matrix. Since

$$\phi_n = \sum_{i=o}^{\infty} \frac{(A\Delta t)^i}{i!} \tag{4.42}$$

one can see that the term that is needed to add to the expansion on the i^{th} step will be of the form

$$\begin{aligned} (A^i)_{11} &= (A^{i-1})_{11}A_{11} \\ (A^i)_{12} &= (A^{i-1})_{11}A_{12} + (A^i)_{12}A_{22} \quad i = 1, \cdots, \infty \\ (A^i)_{21} &= 0 \\ (A^i)_{22} &= (A^{i-1})_{22}A_{22} \end{aligned} \tag{4.43}$$

It can be seen that $(A^i)_{22}$ is always a diagonal matrix and that multiplying by A_{22} consists of either a scaling of the rows or the columns. Therefore ϕ_n will have the form

$$\phi_n = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{4.44}$$

where ϕ_{21} is a null 10 x 7 matrix and ϕ_{22} is a diagonal 10 x 10 matrix.

By partitioning the other matrices in the recursive equations in a compatible fashion and using the symmetry in P_n^* and H_n considerable computer time can be saved. It can be seen that augmenting the system will not increase drastically the computer time if partitioning is used since the augmentation adds non-zero terms only to the diagonal of ϕ_{22}.

D. Results

The following two systems were simulated on the Iowa State University IBM 360/50 computer:

(1) Pure inertial with measurements of doppler frequency and non-inertial reference velocity.

(2) Pure inertial with measurements of doppler frequency.

The physical situation which was simulated corresponded to three satellites in constant altitude, circular, polar orbits spaced 60° apart in inertial space. A marine application was assumed with a slow-moving vehicle at a constant 45° latitude. The following driving functions were assumed:

(1) A rms accelerometer error in each channel of ten seconds of arc with a time constant of ten hours.

(2) A rms gyro drift error in each axis of 0.01 degree per hour with a time constant of ten hours.

(3) A rms velocity reference error in each channel of $1/\sqrt{2}$ knot random with a time constant of one hour plus a $1/\sqrt{2}$ knot white noise.

(4) A rms reference frequency error with each satellite of 2 Hz random with a one-hour time constant plus .04472 Hz white noise.

It can be seen from Figures 4.4, 4.5, 4.6, and 4.7 that in both systems the errors which are of interest — position, azimuth, and east-west gyro drift — are bounded. It is interesting to note that when the velocity reference is removed from the system which uses doppler frequency as an observation that the latitude error tends to grow faster between satellite passes than previously. This can be seen from Figures 4.4 and 4.6. This agrees with our knowledge of inertial navigation systems since the latitude channel error is bounded whereas the longitude channel error is unbounded in a damped inertial system. Therefore one would expect that removing the velocity reference would have more effect on latitude error than on longitude error. Another aspect which will be of interest later in this paper is that on passes close to the navigator's zenith there is an excellent reset on latitude error — sometimes to within .08 nautical mile (480 ft.).

The greatest interest, however, lies in the comparison of the two systems. With the proposed method of using doppler frequency as the measurement there seems to be little difference in the system errors whether a velocity reference is present or absent. Even though between satellite passes the two systems act differently, in that the

Figure 4.4. Position error when doppler frequency and non-inertial reference velocity are the measurements

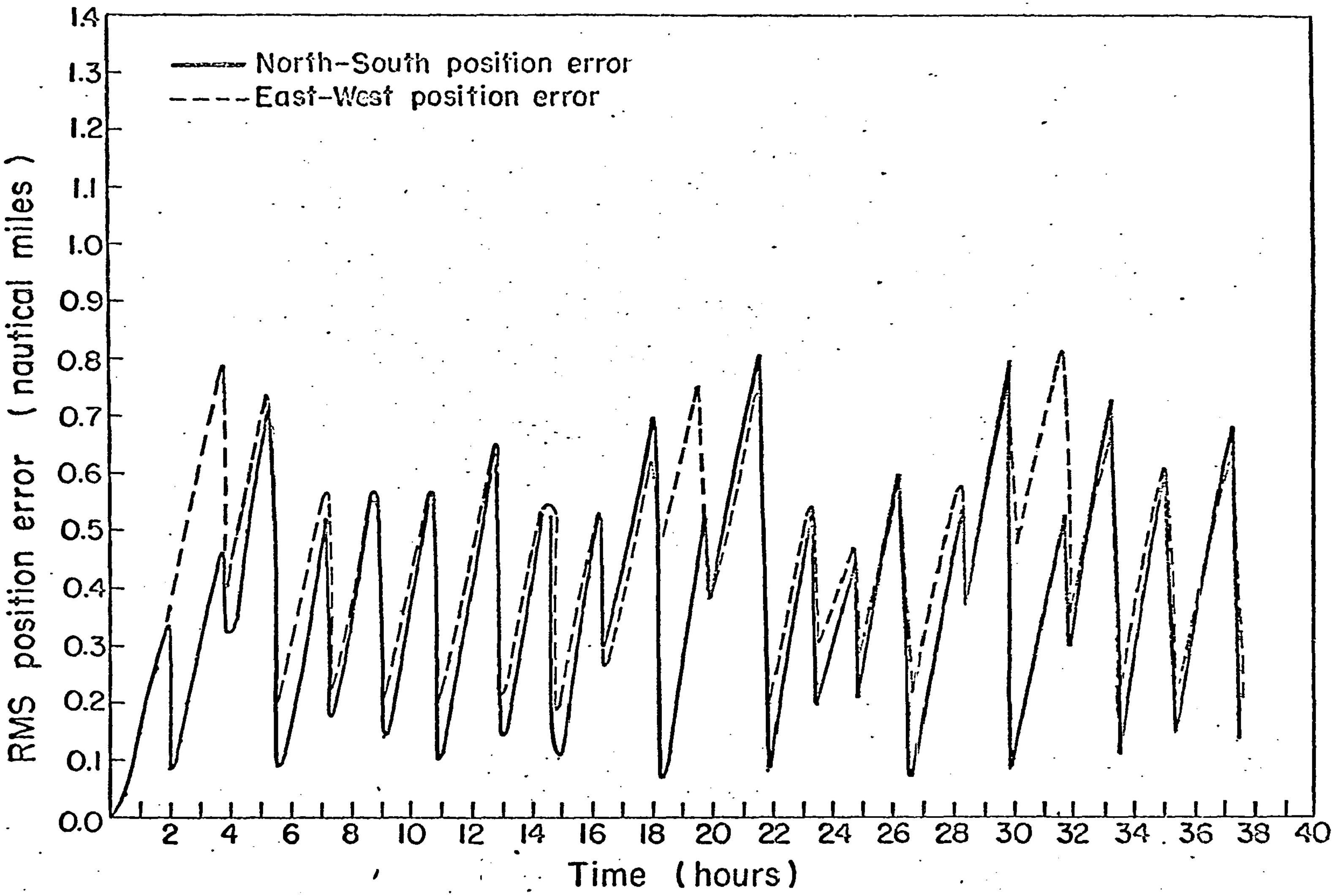

North-South position error
East-West position error
RMS position error (nautical miles)
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
Time (hours)

Figure 4.5. Azimuth error and east-west gyro drift when doppler frequency and non-inertial reference velocity are the measurements

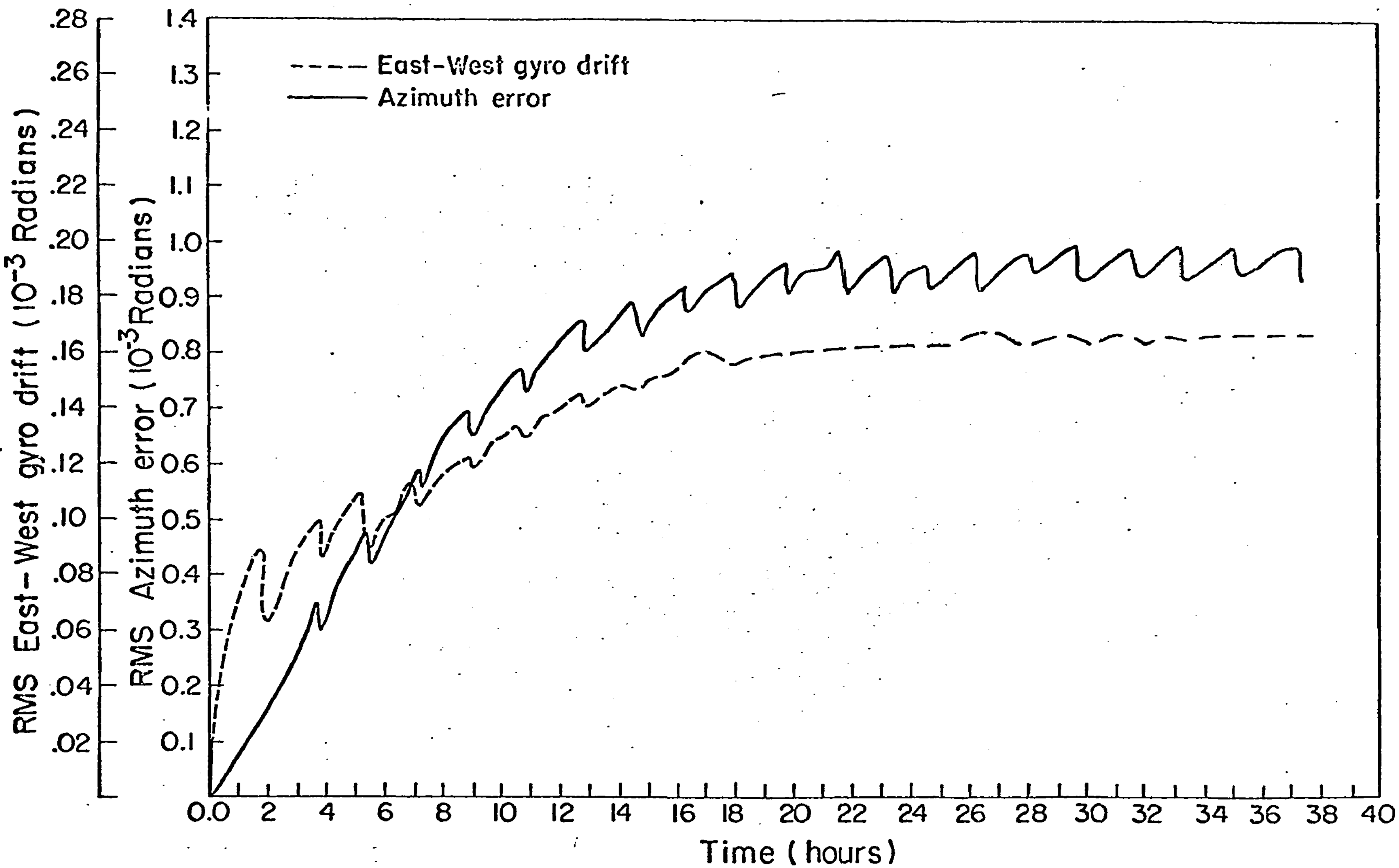

East-West gyro drift
Azimuth error
RMS East-West gyro drift (10^{-3} Radians)
RMS Azimuth error (10^{-3} Radians)
Time (hours)
.02 .04 .06 .08 .10 .12 .14 .16 .18 .20 .22 .24 .26 .28
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4
0.0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40

Figure 4.6. Position error when doppler frequency is the measurement

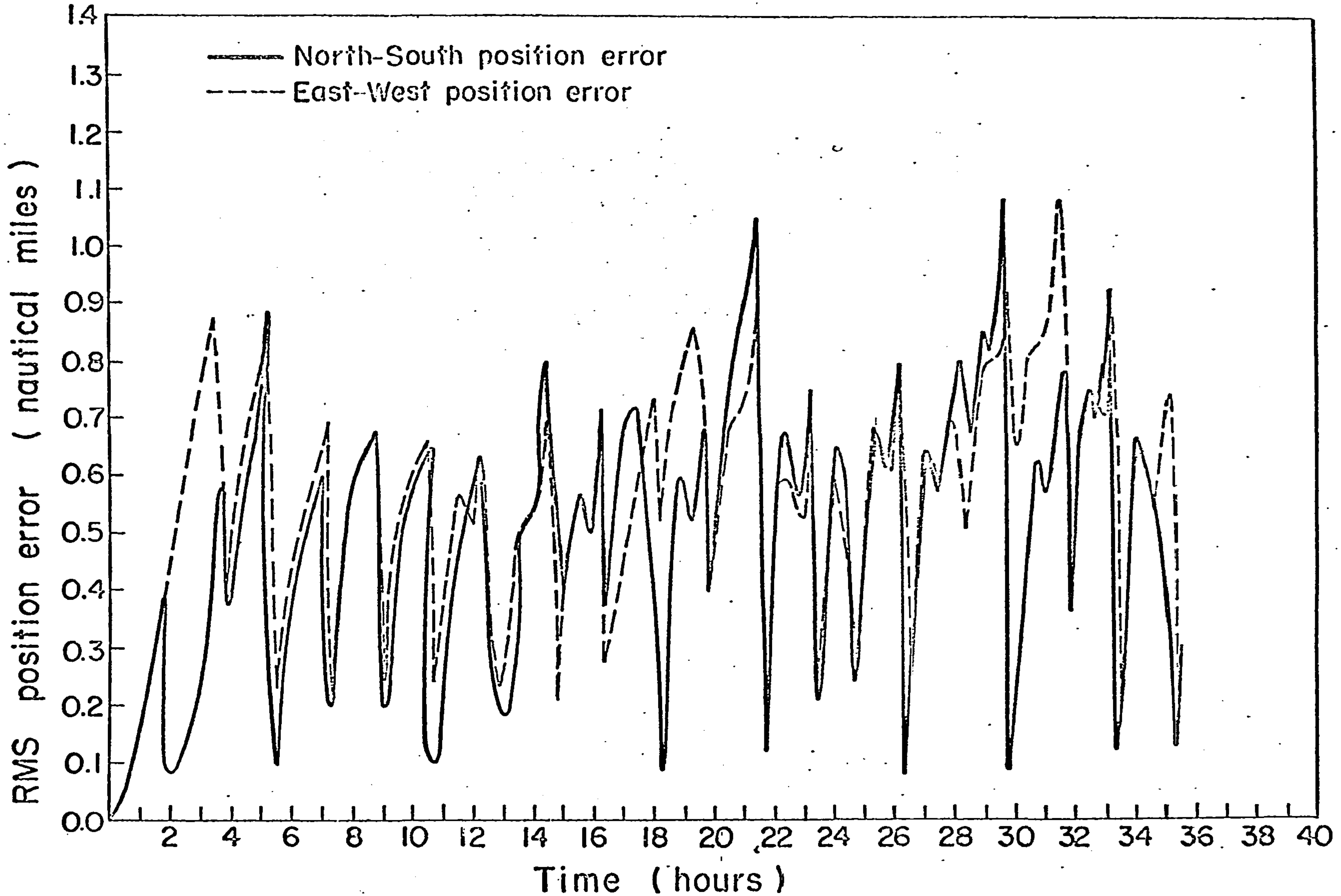
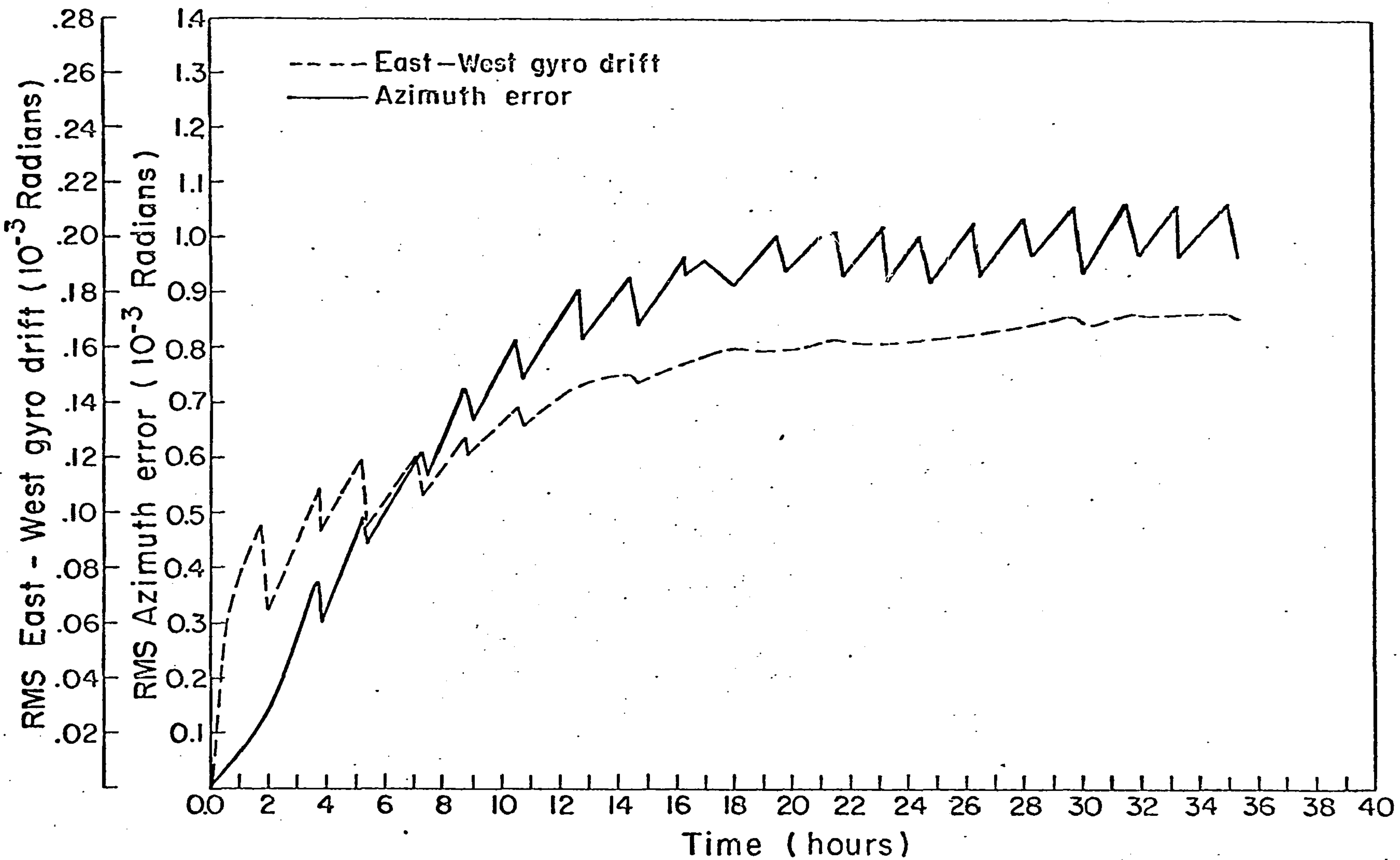

North-South position error
East-West position error
RMS position error (nautical miles)
Time (hours)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40

Figure 4.7. Azimuth error and east-west gyro drift when doppler frequency is the measurement



RMS East - West gyro drift (10^-3 Radians)
RMS Azimuth error (10^-3 Radians)
---- East—West gyro drift
—— Azimuth error
.02 .04 .06 .08 .10 .12 .14 .16 .18 .20 .22 .24 .26 .28
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4
0.0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
Time (hours)

one without a velocity reference has errors which grow a little larger and a little faster, one finds that there is little difference in their steady state errors particularly with regard to azimuth error and east-west gyro drift.

V. SUMMARY AND CONCLUSIONS

This paper has shown that an optimum doppler-satellite inertial navigation system which uses frequency as the observation instead of position is feasible and produces reasonable results. It should be emphasized again that the assumptions in the example used to illustrate the proposed system were not necessary for the development of the navigation system. The assumptions were made only to simplify the error equations used in the computer simulation of the example. Although previous systems have been developed that use an orbital satellite, they either are not optimum in that they do not use all of the available information to estimate the slowly-varying driving functions such as gyro-drifts and accelerometer biases; or, as in the system developed by Bona and Hutchinson, they do not correctly model the physical situation. Another disadvantage of the systems that implement the doppler measurement as a position fix is that a period of at least six minutes is needed in order to obtain a position fix. The satellite pass may last five minutes and forty seconds but these systems cannot use the information which is available. Also, because of the methods used to convert doppler frequency information into discrete position fixes, the present systems are not able to accept passes within about ten degrees of the zenith since longitude cannot be obtained accurately even though latitude can be accurately measured. As a matter of fact it is on these passes that the best measurement of latitude can be made.

Even though the model that Bona and Hutchinson used was not realistic since the inertial velocity was correlated with the measurement error, one should realize that vehicle velocity as such is not detrimental. It is one's lack of knowledge about the exact vehicle velocity over the earth's surface, as provided by the velocity reference during the required six minute interval, that can impair the accuracy of the NNSS system.

As can be seen from the results, not only is the proposed system feasible but it does not have many of the disadvantages of the present NNSS. First, as in the Bona and Hutchinson case it is an optimum system; but, in this case, it correctly models the physical system since the measurement error in our observation of the doppler frequency is in no way correlated to the states of the system. It can also accept information from overhead passes and, as can be seen from the curves, the latitude channel is reset quite well on these passes. The restriction on the period of the pass has been removed and the navigator gets an essentially continuous estimate of the errors as long as the doppler frequency can be obtained.

However, even though these are important advantages over the present NNSS, it would appear from the example chosen that the result which has the most important implications is that a non-inertial velocity reference may no longer be needed. The system response (bounded azimuth errors, east-west gyro drift, and position error) is about the same for the two systems. The position error in a marine system due to non-inertial referency velocity is not so large that the NNSS should be replaced by the proposed system for the above reason alone. However in aircraft systems one runs into quite a different situation. It was shown in a previous section for a simplified error model that position error depends on velocity reference error and gyro drift. It has also been shown by Doty and Nease (4) that azimuth error is also a function of velocity reference error and gyro drift. Therefore at the present time many aircraft systems operate in the pure inertial mode because of the difficulty in obtaining measurements of velocity due to the large uncertainties in wind speed. At the present time this is a serious problem in applying the NNSS as Kershner (8) states:

> "The corresponding problem for aircraft, where unknown winds can be very large, is, of course, much more serious and becomes one of the most difficult problems in extending the usefulness of the Navy Navigational Satellite System."

Therefore in supersonic jet airliners such as the SST or conventional jets on long flights which will have computers available, the doppler system which uses a "frequency fix" as opposed to a "position fix" becomes quite attractive.

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VIII. APPENDIX A

The Kalman technique is one by which an optimum estimate of the state can be obtained at discrete points of time[1]. One assumes that the mathematical model of the random process is of the form

$$\underline{x}_{n+1} = \phi_n \underline{x}_n + \underline{g}_n$$
$$\underline{y}_n = M_n \underline{x}_n + \delta \underline{y}_n \qquad (A.1)$$

The Kalman filter is the discrete analogue to the continuous Weiner filter and in the steady state mode both techniques produce identical results in that the optimum estimates and associated covariance estimates are the same in the limit as the sampling interval approaches zero. The purpose of using the Kalman filter is that one is allowed to base the estimate of the state on a sequence of discrete measurements rather than on the entire, continuous past history of the measurements. Because the Kalman technique is recursive, one does not have to contend with a "growing memory" problem in the computer. It is assumed that

$$E[\underline{x}_k \, \underline{g}_j^T] = 0 \text{ for all } k, j \qquad (A.2)$$

$$E[\underline{x}_k \, \delta\underline{y}_j^T] = 0 \text{ for all } k, j \qquad (A.3)$$

and in this case

$$E[\underline{g}_k \, \delta\underline{y}_j^T] = 0 \text{ for all } k, j \qquad (A.4)$$

The following definitions will be made:

---

[1]Indexes denote discrete times and the word "optimum" is used in the sense that the mean square error of each state is minimized.

Δt is the time increment between t_n and t_{n+1}

ϕ_n is the transition matrix which is constant since Δt is fixed

$\underline{x}_n$ is the true state at time t_n

$\hat{\underline{x}}_n$ is the optimum estimate of $\underline{x}$ after using all of the measured data through $\underline{y}_n$

$\hat{\underline{x}}_n'$ is the optimum estimate of $\underline{x}$ after using all of the measured data through $\underline{y}_{n-1}$

b_n is the weighting matrix

$\underline{y}_n$ is the observation at t_n

$\underline{g}_n$ is the response of the states to all white noise driving functions

P_n is the covariance matrix of the error $(\hat{x}_n - x_n)$ in our optimum estimate

P_n^* is the covariance matrix of the error $(\hat{\underline{x}}_n' - \underline{x}_n)$

M_n is the output matrix which determines the linear relationship between the state and the observation at t_n

V_n is the covariance matrix of the measurement error $(\delta \underline{y}_n)$

H_n is the covariance matrix of the response to the states to all white noise driving functions $(\underline{g}_n)$.

The problem can now be stated as: Using the above mathematical model, choose b_n such that an optimum estimate of $\underline{x}_n$ is obtained where $\hat{\underline{x}}_n$ is a linear combination of the a priori estimate at time t_n and the observations $\underline{y}_n$. The optimum estimate is that which minimizes the loss function

$$E[(\hat{\underline{x}}_n - x_n)^T (\hat{\underline{x}}_n - x_n)] \qquad (A.5)$$

The recursive equations which result are given by

$$\hat{\underline{x}}_n = \hat{\underline{x}}_n' + b_n(\underline{y}_n - M_n \hat{\underline{x}}_n') \qquad (A.6)$$

$$P_n = P_n^* - b_n(M_n P_n^* M_n^T + V_n) b_n^T \tag{A.7}$$

$$P_{n+1}^* = \phi_n P_n \phi_n^T + H_n \tag{A.8}$$

$$b_n = P_n^* M_n^T (M_n P_n^* M_n^T + V_n)^{-1} \tag{A.9}$$

$$\hat{\underline{x}}'_{n+1} = \phi_n \hat{\underline{x}}_n \tag{A.10}$$

where it is assumed that $\hat{\underline{x}}_o$ and P_o are known so the recursive process can be started. It should be noted that for an error analysis only Equations A.7, A.8, and A.9 need be implemented since one is interested only in the rms values of the errors in the estimates and not in the estimates themselves.

Since it has been the purpose of this section to describe briefly the mathematical model and the recursive equations needed for the Kalman filter the reader is referred to Sorenson (13) for a more complete discussion.

IX. APPENDIX B

In order to determine the coefficients a_1, a_2, a_3, and a_4 the partials of $\dot{\rho}$ with respect to λ, Λ, $\dot{\lambda}$, and Λ must be found. From Equations 4.29, 4.32, and 4.33 it can be seen that the radial velocity between the navigator and the satellite is given by

$$\dot{\rho} = -\rho(C_4X + C_1\dot{X} + C_5Y + C_2\dot{Y} + C_6Z + C_3\dot{Z}) \tag{B.1}$$

Thus in a latitude-longitude coordinate system[1]

$$-\frac{\dot{\rho}}{R} = \frac{\begin{bmatrix} C_4C_\lambda S_{\Lambda+\beta} + C_1[C_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) - \dot{\lambda}S_\lambda S_{\Lambda+\beta}] + C_5S_\lambda + C_2\dot{\lambda}C_\lambda \\ + C_6C_\lambda C_{\Lambda+\beta} - C_3[C_\lambda S_{\Lambda+\beta}(\dot{\Lambda}+\Omega) + \dot{\lambda}S_\lambda C_{\Lambda+\beta}] \end{bmatrix}}{\{R^2 + R_s^2 - 2R[C_1C_\lambda S_{\Lambda+\beta} + C_2S_\lambda + C_3C_\lambda C_{\Lambda+\beta}]\}^{\frac{1}{2}}} \tag{B.2}$$

Taking the partial with respect to λ one finds that

$$-\frac{1}{R}\frac{\partial\dot{\rho}}{\partial\lambda} = \frac{\begin{bmatrix} -C_4S_\lambda S_{\Lambda+\beta} - C_1[S_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ +\dot{\lambda}C_\lambda S_{\Lambda+\beta}] + C_5C_\lambda - C_2\dot{\lambda}S_\lambda - C_6S_\lambda C_{\Lambda+\beta} \\ +C_3[S_\lambda S_{\Lambda+\beta}(\dot{\Lambda}+\Omega) - \dot{\lambda}C_\lambda C_{\Lambda+\beta}] \end{bmatrix} \begin{bmatrix} R^2 + R_s^2 \\ -2R[C_1C_\lambda S_{\Lambda+\beta} + C_2S_\lambda \\ +C_3C_\lambda C_{\Lambda+\beta} \end{bmatrix}^{\frac{1}{2}}}{\{R^2 + R_s^2 - 2R[C_1C_\lambda S_{\Lambda+\beta} + C_2S_\lambda + C_3C_\lambda C_{\Lambda+\beta}]\}} \tag{B.3}$$

$$-\frac{1}{2}\,\frac{\begin{bmatrix} C_4C_\lambda S_{\Lambda+\beta} + C_1[C_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ -\dot{\lambda}S_\lambda S_{\Lambda+\beta}] + C_5S_\lambda + C_2\dot{\lambda}C_\lambda + C_6C_\lambda C_{\Lambda+\beta} \\ -C_3[C_\lambda S_{\Lambda+\beta}(\dot{\Lambda}+\Omega) + \dot{\lambda}S_\lambda C_{\Lambda+\beta}] \end{bmatrix} \begin{bmatrix} R^2 + R_s^2 \\ -2R[C_1C_\lambda S_{\Lambda+\beta} \\ +C_2S_\lambda + C_3C_\lambda C_{\Lambda+\beta}] \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} -2R[-C_1S_\lambda S_{\Lambda+\beta} \\ +C_2C_\lambda \\ -C_3S_\lambda C_{\Lambda+\beta} \end{bmatrix}}{\{R^2 + R_s^2 - 2R[C_1C_\lambda S_{\Lambda+\beta} + C_2S_\lambda + C_3C_\lambda C_{\Lambda+\beta}]\}}$$

Thus

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[1] In order to save space the terms "sine", "cosine", and "Ωt" will be denoted by S, C, and β.

$$-\frac{\partial\dot{\rho}}{\partial\lambda} = R\,\frac{\left[\begin{array}{l} -\,C_4 s_\lambda s_{\Lambda+\beta} - C_1[s_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ +\,\dot{\lambda} C_\lambda s_{\Lambda+\beta}] + C_5 C_\lambda - C_2\dot{\lambda} s_\lambda - C_6 s_\lambda C_{\Lambda+\beta} \\ +\,C_3[s_\lambda s_{\Lambda+\beta}(\dot{\Lambda}+\Omega) - \dot{\lambda} C_\lambda C_{\Lambda+\beta}] \end{array}\right]}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda s_{\Lambda+\beta} + C_2 s_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}^{\frac{1}{2}}}$$

$$+\,R^2\,\frac{\left[\begin{array}{l} C_4 C_\lambda s_{\Lambda+\beta} + C_1[C_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ -\,\dot{\lambda} s_\lambda s_{\Lambda+\beta}] + C_5 s_\lambda + C_2\dot{\lambda} C_\lambda + C_6 C_\lambda C_{\Lambda+\beta} \\ -\,C_3[C_\lambda s_{\Lambda+\beta}(\dot{\Lambda}+\Omega) + \dot{\lambda} s_\lambda C_{\Lambda+\beta}] \end{array}\right]\left[\begin{array}{l} -\,C_1 s_\lambda s_{\Lambda+\beta} \\ +\,C_2 C_\lambda \\ -\,C_3 s_\lambda C_{\Lambda+\beta} \end{array}\right]}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda s_{\Lambda+\beta} + C_2 s_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}^{3/2}} \qquad \text{(B.4)}$$

Taking the partial with respect to Λ one finds that

$$-\frac{1}{R}\frac{\partial\dot{\rho}}{\partial\Lambda} = \frac{\left[\begin{array}{l} C_4 C_\lambda C_{\Lambda+\beta} + C_1[-\,C_\lambda s_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ -\,\dot{\lambda} s_\lambda C_{\Lambda+\beta}] - C_6 C_\lambda s_{\Lambda+\beta} \\ -\,C_3[C_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) - \dot{\lambda} s_\lambda s_{\Lambda+\beta} \end{array}\right]\left[\begin{array}{l} R^2 + R_s^2 \\ -\,2R[C_1 C_\lambda s_{\Lambda+\beta} \\ +\,C_2 s_\lambda + C_3 C_\lambda C_{\Lambda+\beta}] \end{array}\right]^{\frac{1}{2}}}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda s_{\Lambda+\beta} + C_2 s_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}} \qquad \text{(B.5)}$$

$$-\,\frac{1}{2}\,\frac{\left[\begin{array}{l} C_4 C_\lambda s_{\Lambda+\beta} + C_1[C_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ -\,\dot{\lambda} s_\lambda s_{\Lambda+\beta}] + C_5 s_\lambda + C_2\dot{\lambda} C_\lambda + C_6 C_\lambda C_{\Lambda+\beta} \\ -\,C_3[C_\lambda s_{\Lambda+\beta}(\dot{\Lambda}+\Omega) + \dot{\lambda} s_\lambda C_{\Lambda+\beta}] \end{array}\right]\left[\begin{array}{l} R^2+R_s^2 \\ -2R[C_1 C_\lambda s_{\Lambda+\beta} \\ +C_2 C_\lambda + C_3 C_\lambda C_{\Lambda+\beta}] \end{array}\right]^{-\frac{1}{2}}\left[\begin{array}{l} -2R[C_1 C_\lambda C_{\Lambda+\beta} \\ -C_3 C_\lambda s_{\Lambda+\beta}] \\ \end{array}\right]}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda s_{\Lambda+\beta} + C_2 s_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}}$$

Therefore

$$-\frac{\partial\dot{\rho}}{\partial\Lambda} = R\frac{\begin{bmatrix} C_4 C_\lambda C_{\Lambda+\beta} + C_1[-C_\lambda S_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ -\dot{\lambda}S_\lambda C_{\Lambda+\beta}] - C_6 C_\lambda S_{\Lambda+\beta} \\ -C_3[C_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) - \dot{\lambda}S_\lambda S_{\Lambda+\beta}\end{bmatrix}}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda S_{\Lambda+\beta} + C_2 S_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}^{\frac{1}{2}}}$$

$$+ R^2\frac{\begin{bmatrix} C_4 C_\lambda S_{\Lambda+\beta} + C_1[C_\lambda C_{\Lambda+\beta}(\dot{\Lambda}+\Omega) \\ -\dot{\lambda}S_\lambda S_{\Lambda+\beta}] + C_5 S_\lambda + C_2\dot{\lambda}C_\lambda + C_6 C_\lambda C_{\Lambda+\beta} \\ -C_3[C_\lambda S_{\Lambda+\beta}(\dot{\Lambda}+\Omega) + \dot{\lambda}S_\lambda C_{\Lambda+\beta}]\end{bmatrix}\begin{bmatrix} C_1 C_\lambda C_{\Lambda+\beta} \\ -C_3 C_\lambda S_{\Lambda+\beta} \\ \end{bmatrix}}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda S_{\Lambda+\beta} + C_2 S_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}^{3/2}} \tag{B.6}$$

The partials with respect to $\dot{\lambda}$ and $\dot{\Lambda}$ are clearly

$$-\frac{\partial\dot{\rho}}{\partial\dot{\lambda}} = \frac{R\{-C_1 S_\lambda S_{\Lambda+\beta} + C_2 C_\lambda - C_3 S_\lambda C_{\Lambda+\beta}\}}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda S_{\Lambda+\beta} + C_2 S_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}^{\frac{1}{2}}} \tag{B.7}$$

and

$$-\frac{\partial\dot{\rho}}{\partial\dot{\Lambda}} = \frac{R\{C_1 C_\lambda C_{\Lambda+\beta} - C_3 C_\lambda S_{\Lambda+\beta}\}}{\{R^2 + R_s^2 - 2R[C_1 C_\lambda S_{\Lambda+\beta} + C_2 S_\lambda + C_3 C_\lambda C_{\Lambda+\beta}]\}^{\frac{1}{2}}} \tag{B.8}$$