

AN EXACT THEORY FOR COHERENT NONDESTRUCTIVE EVALUATION:  
THE APPLICATION OF THE BOJARSKI EXACT INVERSE SCATTERING THEORY TO  
THE REMOTE PROBING OF INHOMOGENEOUS MEDIA

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INTRODUCTION AND SYNOPSIS

In an inverse scattering problem, the fields in the inhomogeneous wave equation are known, and it is desired to solve for the source term. N. N. Bojarski has recently derived an Exact Inverse Scattering Theory for such "inverse source" problems. The problem of determining the generalized refractive index (i.e., the complex permeability and dielectric constant for an electromagnetic problem, or the velocity and absorption for an acoustic problem) distribution of an inhomogeneous medium from measurements of the fields scattered by the medium can be treated using this theory. This solution is applicable to all remote probing problems, and in particular, to nondestructive evaluation (NDE) using coherent radiation.

Although this paper uses scalar notation, all of the results have been shown to apply to the general, full vector field and tensor medium quantities. The equations applicable to the electromagnetic cases are used; however, the theory and results apply equally well to the acoustic equations.

THE BOJARSKI EXACT INVERSE SCATTERING THEORY

To provide the basis for the treatment of the inverse medium problem, this section presents a derivation of N. N. Bojarski's (Refs. 1-4) "Exact Inverse Scattering Theory." Consider a source  $\rho(\underline{x})$  in a domain  $D$  bounded by a surface  $S$ . Then the time harmonic field,  $\phi(\underline{x})$ , due to  $\rho(\underline{x})$  is the solution to the inhomogeneous wave equation

$$\nabla^2 \phi(\underline{x}) + k^2 \phi(\underline{x}) = -\rho(\underline{x}), \quad \underline{x} \in D \quad (1)$$

where  $k = 2\pi/\lambda$ . A direct scattering problem is one in which  $\rho(\underline{x})$  is known or specified, and a solution for  $\phi(\underline{x})$  is sought. The inverse scattering problem is one in which  $\phi(\underline{x})$  is known, and  $\rho(\underline{x})$  is sought. For the inverse source problem,  $\phi(\underline{x})$  is measured over some surface, and the object is to determine  $\rho(\underline{x})$ . In general,  $\rho(\underline{x}) = \rho_m(\underline{x}) + \rho_s(\underline{x})$ , where  $\rho_m$  is due to interaction with the medium, and  $\rho_s$  is due to actual sources. If  $n(\underline{x})$  is the complex refractive index of the medium, then

$$\rho_m(\underline{x}) = k^2 [n^2(\underline{x}) - 1] \phi(\underline{x}) \quad (2)$$

In most remote probing problems,  $\rho_s(\underline{x})$  is known, and  $\rho_m(\underline{x})$  is sought to yield  $n(\underline{x})$ . This is termed the inverse medium problem. If  $\phi(\underline{x})$  is specified (as the desired field) and  $\rho(\underline{x})$  or  $n(\underline{x})$  is sought so as to produce that  $\phi(\underline{x})$ , the problem is termed an inverse synthesis problem.

Let the following field quantity,  $\phi_H(\underline{x})$ , be defined:

$$\phi_H(\underline{x}) = \oint \left[ g^*(\underline{x}-\underline{x}') \nabla \phi(\underline{x}') - \phi(\underline{x}') \nabla g^*(\underline{x}-\underline{x}') \right] d\underline{S}' \quad (3)$$

where  $g(\underline{x})$  is the free space Green's function and the asterisk denotes complex conjugation.  $g$  satisfies Equation 1 with  $\rho(\underline{x}) = \delta(\underline{x})$ .  $\phi_H$  is in the form of the Kirchoff integral with  $g$  complex conjugated. Note that if the Kirchoff integral is applied to the field  $\phi(\underline{x})$  on  $S$  and evaluated at any point  $\underline{x}$  inside  $D$ , it is identically zero: The Kirchoff integral is nonzero only for points outside  $D$ . Conversely,  $\phi_H(\underline{x})$  is nonzero only for points inside  $D$ . Points outside of  $D$  are associated with the direct scattering problem; points inside  $D$  are of interest for the inverse scattering problem. This topological difference is the reason why direct scattering solutions are mathematically ill-posed when applied to the inverse scattering problem.

It should also be noted that  $\phi_H$  is the mathematical expression for the reconstruction obtained from a hologram ( $\phi$  in Equation 3) recorded on  $S$ . The relationship between holography and inverse scattering, along with an analysis of the consequences for remote probing and coherent imaging applications, has been presented by Stone (Ref. 5).  $\phi_H$  is, in general, known for inverse problems, since  $\phi$  is known over  $S$ .  $\phi$  is measured over  $S$  for the inverse source and medium problems, or specified over  $S$  for the inverse synthesis problem.

Applying Gauss' theorem to Equation 3 converts the surface integral into a volume integral:

$$\phi_H = \int dV (g^* \nabla^2 \phi - \phi \nabla^2 g^*) \quad (4)$$

From Equation 1,

$$\nabla^2 \phi = -k^2 \phi - \rho \quad (5)$$

and, by complex conjugation of Equation 1 for  $g$ ,

$$\nabla^2 g^* = -k^2 g^* - \delta \quad (6)$$

Substitution of Equations 5 and 6 into Equation 4 gives

$$\begin{aligned}\phi_H &= \int dV [g^*(-k^2\phi - \rho) - \phi(-k^2g^* - \delta)] \\ &= \int dV (\phi\delta - g^*\rho)\end{aligned}\quad (7)$$

and, carrying out the integration over the delta function,

$$\phi_H = \phi - \int dV g^*\rho \quad (8)$$

Direct scattering theory gives the result that

$$\begin{aligned}\phi &= \int dV g\rho + \int dS(g\nabla\phi - \phi\nabla g) \\ &= \int dV g\rho + \phi_i\end{aligned}\quad (9)$$

In Equation 8, the first integral is just the superposition integral over the sources. The second term is the Kirchoff integral, and is associated with the incident field,  $\phi_i$ . For the inverse scattering problem,  $\phi_i$  can be assumed to be known without loss of generality (e.g., it is the known probing field for the inverse medium case, or the specified incident field in the inverse synthesis case).

Equations 8 and 9 are two independent simultaneous equations in two unknowns,  $\phi$  and  $\rho$ . Substitution of Equation 9 into Equation 8 yields

$$\begin{aligned}\phi_H &= \int dV g\rho - \int dV g^*\rho + \phi_i \\ &= \int dV (g - g^*)\rho + \phi_i\end{aligned}\quad (10)$$

or

$$\phi_H(\underline{x}) = 2i \int dV' \text{Im}g(\underline{x}-\underline{x}') \rho(\underline{x}') + \phi_i(\underline{x}) \quad (11)$$

where Im denotes the imaginary part. Equation 11 is the basic equation of the Exact Inverse Scattering Theory. It is an integral, convolution equation for the single unknown,  $\rho(\underline{x})$ . It can be solved by standard deconvolution techniques. Quite recently, Bojarski (Ref. 6) has presented a closed-form solution to Equation 11.

#### UNIQUENESS OF THE SOLUTION

The uniqueness of the solution to Equation 11 was first deduced by Bojarski (Refs. 2,4) and later proven more rigorously by Bleistein and Cohen (Ref. 7). A simpler and more physically understandable proof was presented by Stone (Ref. 8). The result is that the solution to Equation 11 for the source,  $\rho(\underline{x})$  is unique if  $\rho(\underline{x})$  is identically zero outside some finite domain (i.e., is of bounded support), has finite energy, and does not contain any nonradiating components. A nonradiating source is a source component which produces a field which is identically zero outside a finite region. Although the "nonuniqueness" associated with nonradiating sources has been somewhat troublesome from a mathematical standpoint, it does not affect the uniqueness of results for practical applications (Bojarski

Ref. 9). Quite recently, Stone (Ref. 10) has proven that a conjecture by Bleistein and Bojarski (Ref. 4) that nonradiating sources are nonphysical is true.

#### INCOMPLETE KNOWLEDGE OF $\phi(x)$

Practical inverse scattering problem measurements almost always involve discrete measurements over a limited aperture, as opposed to the continuous measurements over a closed surface used in the above theory. Mager and Bleistein (Ref. 11) have shown that, in the physical optics limit, the spatial bandwidth over which measured data is known is the spatial bandwidth over which the source term can be determined. A similar but more general result follows directly from analysis of the three dimensional spatial Fourier transform of Equations 3 and 11. Let  $\underline{\nu}$  be the spatial frequency variable, and let capital letters denote the transformed functions. For the general case,  $\Phi_H(\underline{\nu})$  is known over the whole surface S, and thus for all  $0 \leq \underline{\nu} \leq \underline{\nu}_0$  (the upper bound is  $\underline{\nu}_0$  rather than  $\infty$ , since D, and thus S, are of finite size). Since the left side of the transformed version of Equation 11 is known for all  $0 \leq \underline{\nu} \leq \underline{\nu}_0$ , it follows that  $P(\underline{\nu})$  is determined over this range. Now let  $\phi(\underline{x})$  be measured at discrete points over a limited aperture. Then  $\Phi(\underline{\nu})$  is determined for  $\underline{\nu}_1 \leq \underline{\nu} \leq \underline{\nu}_2$ , where these spatial bandlimits are determined by the aperture size and sample spacing. It follows from the transform of Equation 3 that  $\phi_H(\underline{\nu})$  is similarly bandlimited, and from the transform of Equation 11 that  $p(\underline{\nu})$  can be determined over this band of spatial frequencies. It has been shown by Stone (Ref. 12) that, for coherent inverse scattering, the spatial resolution, obtainable with a measurement aperture of given size, may be significantly greater than that predicted by classical incoherent diffraction theory.

#### THE EFFECTS OF NOISY MEASUREMENTS (REF. 13)

The effects of measurement noise on the reconstructed refractive index can be seen by writing  $\phi_H(\underline{x}) = \phi_H(\underline{x}) + \phi_N(\underline{x})$ , where  $\phi_N(\underline{x})$  contains a contribution due to noise. It follows, using Equation 11, that this is equivalent to a source term  $\rho(\underline{x}) + \rho_N(\underline{x})$ , where  $\rho(\underline{x})$  is the true source and  $\rho_N(\underline{x})$  contains the effect of the noise. From this it can be seen that the signal-to-noise ratio of the solution is the signal-to-noise ratio of  $\phi_H(\underline{x})$ . Since  $\phi_H(\underline{x})$  depends on the integral over the measurement surface of the measured field values (Equation 3), the signal-to-noise ratio of  $\phi_H(\underline{x})$  [and thus of  $\rho(\underline{x})$ ] is not greater than (and may be less than) the signal-to-noise ratio of the measured data. This has been confirmed by numerical experiments. It should also be mentioned that in addition to being numerically well-posed, the solution can be implemented in an extremely efficient form. This permits addressing problems heretofore impractical because of computational effort or storage limitations.

#### THE INVERSE MEDIUM PROBLEM

Based on the above theory, the NDE inverse medium problem can be solved by the following steps:

- A. Compute  $\phi_H(\underline{x})$ , using the measured field values in Equation 3 (note that the surface of integration,  $S$ , is the measurement surface).
- B. Solve Equation 11 for  $\rho(\underline{x})$ , using  $\phi_H(\underline{x})$  from A and the known incident field,  $\phi_i(\underline{x})$ .
- C. Compute the total field,  $\phi(\underline{x})$ , from the direct scattering result, Equation 9, using  $\rho(\underline{x})$  from B.
- D. Solve Equation 2 for the desired complex refractive index,  $n(\underline{x})$ , using  $\rho(\underline{x})$  from B and  $\phi(\underline{x})$  from C.

Note that for the inverse source problem, only steps A and B are required. However, for the inverse medium problem it is necessary to carry out steps C and D in addition. The solution to the direct scattering problem (step C) is a necessary step in solving the inverse medium problem. It is important to emphasize that the computations involved in steps A through C are all convolution integrals: They can be carried out using fast Fourier transform techniques. As a result, computation time and storage requirements are proportional to  $N \log_2 N$ , where  $N$  is the number of data points. Step D is an algebraic operation.

#### THE SYNTHESIS PROBLEM

There is very close relationship between the synthesis problem and the inverse medium problem. In the inverse medium problem a known probing (incident) field is used, and the scattered field is measured. This data is sufficient to obtain a unique solution for  $n(\underline{x})$ , using the four steps in the previous section. In the synthesis problem, a specified incident field and a desired scattered field are chosen, and the  $n(\underline{x})$  required to produce this scattered field is sought. If there are no constraints (other than physical realizability) on the desired  $n(\underline{x})$ , the same four steps in the previous section will solve the synthesis problem. If there are constraints (e.g., a desired range of values for  $n(\underline{x})$ , etc.), it is necessary to regularize the solution to these constraints.

#### A CLOSED-FORM SOLUTION TO THE SYNTHESIS PROBLEM

The author has carried the solution to the synthesis problem one step further. The result is a closed-form, exact solution for the desired refractive index distribution. Let  $T$  denote the desired relationship between the incident field,  $\phi_i(\underline{x})$ , and the scattered field,  $\phi_s(\underline{x})$ .  $T$  can be a function, an operator, or, in the most general case, any desired algorithm. The only requirement is that the operation of  $T$  on  $\phi_i$  (denoted  $T\phi_i$ ) result in a field which is a valid solution of the inhomogeneous wave equation. Thus,

$$\phi_s = T\phi_i \quad (12)$$

By definition, the total field,  $\phi$ , is the sum of the incident and scattered fields:

$$\phi = \phi_i + \phi_s \quad (13)$$

Substituting Equation 12 into Equation 13,

$$\phi = (1 + T) \phi_i \quad (14)$$

Using convolution notation, Equation 9, the direct scattering result, gives

$$\phi = \rho * g + \phi_i \quad (15)$$

The source term,  $\rho$ , is related to  $\phi$  and  $n(\underline{x})$  by the constitutive Equation 2. Substituting Equation 2 for  $\rho$  and Equation 14 for  $\phi$  into Equation 15 yields

$$(1 + T)\phi_i = [k^2(n^2 - 1)(1 + T)\phi_i] * g + \phi_i \quad (16)$$

After some algebra, Equation 16 can be solved for  $n$ :

$$n^2(\underline{x}) = \left\{ k^2 [1 + T] \phi_i \right\}^{-1} \left[ \mathcal{F}^{-1} \left\{ \tilde{T} \tilde{\phi}_i / \tilde{g} \right\} \right] + 1 \quad (17)$$

where the tilde indicates the three dimensional spatial Fourier transform, and  $\mathcal{F}^{-1}$  denotes the inverse transform.

Equation 17 is a closed-form solution to the synthesis problem. Furthermore, it is very attractive from the system designer's standpoint. The designer need only specify the desired input field to output field transformation,  $T$ , and Equation 17 provides the complex refractive index distribution which will produce that transformation. From the Exact Inverse Scattering Theory it can readily be shown that Equation 17 is numerically stable. Furthermore, it can be evaluated with great efficiency using fast Fourier transform techniques. It is also readily amenable to regularization for the purpose of incorporating design constraints. Finally, the solution of Equation 17 is unique, and has the same behavior with respect to noise and incomplete measurements as discussed in the sections above.

#### A CLOSED-FORM SOLUTION TO THE NDE REMOTE PROBING PROBLEM

Let  $\phi_i$  be the known incident field in a remote probing problem, and let  $\phi_s$  be the measured, scattered field. Then the operator  $T$ , defined in Equation 12, can be determined, and Equation 17 is a closed-form solution to the remote probing problem.

Unfortunately, this involves a hidden approximation. The field measured is usually the total field,  $\phi$ , — not just the scattered field,  $\phi_s$  — over some surface. Obtaining  $\phi_s$  throughout the volume from  $\phi$  over a surface can be as complex as solving the inverse medium problem. However, under certain conditions, the approximation of  $\phi_s$  throughout the volume by  $\phi_H$ , as determined by Equation 3, may be adequate. Where such an approximation is good, Equation 17 provides a closed-form solution to the remote probing problem. A discussion of the conditions and implications associated with such an approximation

has been given by Stone (Ref. 5). Note that no approximation is involved in the closed-form solution of the above section for the synthesis problem: A designer has the freedom to specify  $\phi_s$ . Indeed, this is usually the desired quantity for specification.

#### SOME COMMENTS ABOUT "STANDARD" APPROACHES TO THE SYNTHESIS PROBLEM

The synthesis problem is usually approached using direct scattering techniques. As discussed in the second session, this approach is inherently ill-posed. An initial guess at the solution  $[n(x)]$  is made, a direct scattering analysis is carried out to obtain the scattered field, and this sequence is iterated, changing the  $n(x)$  in an attempt to minimize the difference between the computed and desired scattered fields. Ray tracing is the most common direct scattering technique employed. There are many very important reasons for not using such synthesis methods: They are iterative, with no guarantee of the nature or rate of convergence; they are mathematically and numerically ill-posed; they require the designer to specify an optimization criterion which is usually not related to desired design requirements; and (in the case of ray tracing) they are only applicable where the geometrical optics approximation is valid. The previous two sections present two solutions which eliminate all of these objections. However, it is also important to realize that the inverse scattering approaches are many powers of 10 more efficient than standard techniques. One example from optical system synthesis will suffice to demonstrate this. Using the inverse scattering techniques, computation of  $n(x)$  for 200,000 complex values requires of the order of one second using a 10 year old minicomputer with an FFT processing board. A state-of-the-art ray trace design program can, at best, compute 4,000 field values through one element of a guessed  $n(x)$  per iteration in one second, using state-of-the-art, special purpose hardware (faster than a CDC 7600 or an IBM 360/195) — and several thousand iterations are commonly required.

#### CONCLUSIONS

The following conclusions can be drawn from this work:

- A. Inverse scattering problems fall into three classes: Inverse source, inverse medium, and synthesis problems. NDE is an inverse medium problem.
- B. The Bojarski Exact Inverse Scattering Theory provides solutions to all three of these problems, and in particular, to the inverse medium and synthesis problems.
- C. An exact, closed-form solution to the synthesis problem has been presented in this paper.
- D. As shown elsewhere, the solutions of B and C are unique, well-posed, insensitive to noisy measurements, applicable with incomplete data, and computationally efficient.
- E. In addition, the closed-form solution to the synthesis problem, presented in this paper, provides a closed-form solution to the NDE remote probing (inverse medium) problem. However, the application to the

remote probing problem involves an approximation, the effects of which have been treated in detail elsewhere.

- F. The approaches to the synthesis problem presented here are both exact and many powers of 10 more efficient than standard ray tracing techniques.

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## SUMMARY DISCUSSION

James Gubernatis (Los Alamos Scientific Laboratory): Where are your calculations?

Ross Stone (IRT Corporation): You're going to see some calculations in the two papers that follow. I don't have any with me here. There are several applications that this has been applied to. The ones I have been most interested in have had to do with inhomogeneous media. In particular, I have several calculations that show the reconstruction of lens-like deformities. I don't have them with me. I will send you a copy.

A. Nayfeh (Systems Research Laboratories): How would you know the shape and make-up of the difference? There is something there, but the make-up -

Ross Stone: The data appears as a surface integral, a Kirchhoff-like integral. In fact, it involves convolution with the conjugate of the Green's function. The defect itself appears exclusively in the source term  $RHO$  and you are able to recover that source term by solving the convolution equation. So, no, you do not recover the integral of that source term  $RHO$ . You recover  $RHO$  and from that you can solve, algebraically, for the defect.

Volker Schmitz (Battelle Northwest): Does the theory you developed still work when the wavelength is comparable to the size of the defects?

Ross Stone: Thank you. The name of this session is "Short Wavelength," and that is a very good point. Yes, it does still work. The effect of wavelength in comparison to the size of the defect is to determine resolution in the reconstruction, but it does not eliminate the possibility of reconstruction. And indeed, you will get useful information out of reconstructions when the wavelength is on the order of the size of the defect.

Gordon Kino (Stanford University): You referred to the fact you could get super resolution, essentially by a definition of phase, which I would agree. But in practice doesn't this really mean that you are back to saying, "I must have a continuous reading over the region of interest." Otherwise, you have a finite sampling, and essentially you're going to get -

Ross Stone: If you know what your sampling interval is, you can remove that effect. First of all, I have recorded and reconstructed three-dimensional images of the ionosphere at radio wavelengths. I noticed in the reconstruction that my resolution was considerably better than what would have been predicted based on a Rayleigh criteria and my sampling. I then went back to the theory and the computer, and I was able to derive analytically and simulate on the computer precisely the result I alluded to there, that the signal-to-noise ratio determines your ability to measure phase. If you can measure phase significantly better than within two  $\pi$  radians, which likewise implies you have a relatively high signal-to-noise ratio, then you can indeed achieve a resolution significantly better than the Rayleigh resolution in spite of your sampling.

As one example, I plugged in a 15 db signal-to-noise ratio, reconstructed the impulse response on the aperture, where the Rayleigh criterion said the impulse response, for the distance over which I was doing the reconstruction should have been the size of the aperture and, in fact, the response half width was 1/20th of the size of the aperture, exactly as predicted by the theory.

J.D. Achenbach, Chairman: Thank you. We now have to move on to the next talk.