NETWORK RINGS AND ALGEBRAS

by

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I. INTRODUCTION

Since 1908, when Wedderburn initiated the structure theory of algebras, certain aspects of ring theory have reached a fairly definitive state. Thus the structure of simple and semisimple rings with minimum conditions has, in a sense, been completely determined. However, our knowledge of non-semisimple rings and algebras is far from complete.

The central problem in general ring theory seems to be that of determining how to build rings by combining radical rings and semisimple rings. The aspect of this problem attacked in the present work is rather restrictive. The rings under scrutiny are assumed to be generated by semisimple subrings. Such rings will be labeled "network".

After providing the background of necessary definitions and theorems in Chapter II, examples are provided in Chapter III to show that network rings are fairly abundant. Thus for every positive integer n, there exists a corresponding network ring generated by n semisimple subrings. The next sequence of theorems, of Chapter IV, provide a rationalization for studying only the simplest type of network ring -- those that are division rings modulo the radical. After some rather general theorems in Chapter V concerning such rings, we provide necessary and sufficient conditions for rings generated by two division rings to be network. At this stage representation theory plays a leading role.

The results obtained in this thesis may be interpreted as a contribution to the theory of ring composites. A ring composite of two given rings is merely a ring which contains isomorphic images of the two rings and yet no proper subring which contains the two images. One then speaks of the composite as being generated by the two rings. Composites of fields have been studied extensively, and in general the ring composite of two fields is no longer a field. Therefore, it is not surprising that the composites of semisimple rings are not necessarily semisimple, let alone unique.

We shall confine ourselves at all times to associative rings and, more particularly, to rings containing an identity. Except in Chapter IV all rings are assumed to have the minimum condition on left ideals and are often restricted to be algebras of finite dimension over a base field.

II. DEFINITIONS AND PRELIMINARY THEOREMS

Throughout this thesis certain definitions and theorems will be used repeatedly. For convenience they are collected here.

Definition 1. A ring A with radical N is said to be cleft if there exists a subring S of A such that A = S + N as a group direct sum.

Definition 2. In the decomposition A = S + N, of a cleft ring, S is called a Wedderburn factor.

Definition 3. An algebra A over a field F is said to be separable if A_K is semisimple for any extension K of the base field F.

The classical result on cleavage is the Wedderburn principal theorem. The following is a somewhat restricted form of the theorem, but will be sufficient for the purposes of the present investigation. For a more general statement see Curtis (4, p. 79).

Theorem 1. If an algebra A of finite dimension over a field F is separable modulo its radical, then it is cleft.

The Wedderburn principal theorem says nothing about the relation between the possible Wedderburn factors of a cleft ring. A theorem of Malcev (7) describes the separable case for algebras.

Theorem 2. If A/N is separable and $A = S_1 + N = S_2 + N$ are two decompositions of A, then the isomorphism $S_1 \cong S_2$ is given by $S_2 = (1 - n)^{-1} S_1(1 - n)$, where n is a radical element.

Theorem 2 is often stated in terms of "quasi-regularity". Thus, if $1 + n'$ is the inverse of $1 + n$, then $(1 + n')(1 + n) = 1 + n' + n + n'n = 1$. Consequently, $n' + n + n'n = 0$, which is a valid relation independent of the existence of an identity in the ring. One says that the elements n and n' are quasi-regular with n and n' the quasi-inverses of one another. In the Malcev theorem the isomorphism, $a \to \rho(a)$, is then given by $\rho(a) = a - n'a - an + n'an$. However, since our rings will contain identities, this form of the theorem is inessential.

An immediate consequence of the Malcev theorem is the fact that every commutative cleft algebra which is separable modulo the radical must have a unique Wedderburn factor. We will generalize this result further in Chapter IV.

The principal purpose of this thesis is to study those rings which are generated by their Wedderburn factors.

Definition 4. A cleft ring A with Wedderburn factors S_i is called a network ring if the S_i generate A.

Definition 5. A ring A with radical N is called semi-primary if A/N is semisimple, primary if A/N is simple, and completely primary if A/N is a division ring.

III. EXISTENCE OF RINGS WITH A FINITE NUMBER OF WEDDERBURN FACTORS

In this chapter we shall first outline a construction for obtaining cleft rings, of a very simple nature, with a given integral number of Wedderburn factors. Although we begin the chapter with algebras which are separable modulo the radical, we conclude with some examples which do not satisfy the hypothesis of the Malcev theorem. These last examples also serve to highlight the failure of the Malcev theorem to resolve the question of the relations between the Wedderburn factors of cleft algebras whose residue class algebras are inseparable. Before proceeding to the construction it will be useful to note a general relation that must exist between any two Wedderburn factors.

Theorem 3. If $\rho(a) = a + n(a)$, where a and $\rho(a)$ are corresponding elements in two Wedderburn factors, then the mapping n satisfies the relation:

$$n(ab) = an(b) + n(a)b + n(a)n(b).$$

Proof. Since the two Wedderburn factors are to be isomorphic, $\rho(ab) = \rho(a)\rho(b)$. By definition, $\rho(ab) = ab + n(ab)$ and $\rho(a)\rho(b) = (a + n(a))(b + n(b))$. Hence,

$$ab + n(ab) = ab + an(b) + n(a)b + n(a)n(b).$$

Corollary. If the radical is nilpotent of index two ($N^2 = 0$), then n is a derivation of S into N.

It is easily seen that the set of all matrices of the form:

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix},$$

with a in any ring R, is a nilpotent ring with index of nilpotence two. Likewise, the set of all matrices of the form:

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix},$$

constitutes a semisimple ring if $x, y \varepsilon F$, a field. Consequently, let us consider the example of the ring A of all two by two upper triangular matrices,

$$\begin{bmatrix} x & z \\ 0 & y \end{bmatrix},$$

where $x, y, z \varepsilon F = I/(2)$.

An enumeration of the elements of A gives:

$$a_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$a_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad a_5 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad a_6 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad a_7 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The radical of A is easily seen to be $N = \{a_0, a_6\}$. Since the example is a finite dimensional algebra over a separable field, the Wedderburn principal theorem is applicable. If the subalgebras S_1 and S_2 are:

$$S_1 = \{a_0, a_1, a_2, a_3\}, \quad S_2 = \{a_0, a_4, a_5, a_3\},$$

then $A = S_1 + N$, where $S_1 \cong A/N = S$, and $A = S_2 + N$, where

$S_2 \cong A/N = S.$

Lemma. The set of all two by two upper triangular matrices with elements from $I/(2)$ is a nonsemisimple ring with two Wedderburn factors.

Now consider the case where the elements of the upper triangular matrices are from an arbitrary field. If

$$A = \left\{\begin{bmatrix} x & z \\ 0 & y \end{bmatrix}\right\},$$

it is easily shown that the radical of A is $N = \left\{\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}\right\}$. Furthermore,

$$A/N \cong \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right\} = S_1$$

If there are to be other subalgebras of A isomorphic to A/N then they must be of the form:

$$S_i = \left\{\begin{bmatrix} x & f(x,y) \\ 0 & y \end{bmatrix}\right\}.$$

Setting $a = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, $b = \begin{bmatrix} x' & 0 \\ 0 & y' \end{bmatrix}$, $\rho(a) = \begin{bmatrix} x & f(x,y) \\ 0 & y \end{bmatrix}$,

and $\rho(b) = \begin{bmatrix} x' & f(x',y') \\ 0 & y' \end{bmatrix}$,

we must have $\rho(a + b) = \rho(a) + \rho(b)$ and $\rho(ab) = \rho(a)\rho(b)$.

$$\rho(a + b) = \begin{bmatrix} x + x' & f(x+x',y+y') \\ 0 & y + y' \end{bmatrix} = \begin{bmatrix} x & f(x,y) \\ 0 & y \end{bmatrix} + \begin{bmatrix} x' & f(x',y') \\ 0 & y' \end{bmatrix}.$$

$$\rho(ab) = \begin{bmatrix} xx' & f(xx',yy') \\ 0 & yy' \end{bmatrix} = \begin{bmatrix} x & f(x,y) \\ 0 & y \end{bmatrix}\begin{bmatrix} x' & f(x',y') \\ 0 & y' \end{bmatrix} = \begin{bmatrix} xx' & xf' + fy' \\ 0 & yy' \end{bmatrix}.$$

Hence, $f(x+x',y+y') = f(x,y) + f(x',y')$ and $f(xx',yy') = xf(x',y') + f(x,y)y'$.

Since $n(a) = \begin{bmatrix} 0 & f(x,y) \\ 0 & 0 \end{bmatrix}$, $n(b) = \begin{bmatrix} 0 & f(x',y') \\ 0 & 0 \end{bmatrix}$, $n(ab) = \begin{bmatrix} 0 & xf'+fy' \\ 0 & 0 \end{bmatrix}$, these two equations give: $n(a + b) = n(a) + n(b)$ and $n(ab) = an(b) + n(a)b$. This result is, of course, also a consequence of the Corollary to Theorem 3.

Lemma. The solution of the functional equations: $f(x+x',y+y') = f(x,y) + f(x',y')$ and $f(xx',yy') = xf(x',y') + f(x,y)y'$ is $f(x,y) = \alpha(x - y)$, where $\alpha \in F$.

Proof. f must be homogeneous and linear in both variables. For $f(x,y) = f(x+0,y+0) = f(x,y) + f(0,0)$, which implies that $f(0,0) = 0$. Now, $f(x,y) = f(x+0,0+y) = f(x,0) + f(0,y)$, but $f(x,0) = f(x\cdot 1,0) = xf(1,0) + 0 = xf(1,0)$. Likewise, $f(0,y) = yf(0,1)$. This implies that $f(x,y) = xf(1,0)+yf(0,1)$. Therefore, $f(x,y) = \alpha x + \beta y$, where $\alpha,\beta \in F$. Furthermore, $f(xx',yy') = \alpha xx' + \beta yy' = xf(x',y') + f(x,y)y' = x(\alpha x' + \beta y') + y'(\alpha x + \beta y)$. Hence, $\alpha xx' + \beta yy' = \alpha xx' + \beta xy' + \alpha xy' + \beta yy'$, and $\alpha + \beta = 0$ implies that $\alpha = -\beta$.

Theorem 4. If the base field, F, has n elements then the algebra, A, of upper triangular matrices has n subalgebras isomorphic to A/N.

Proof. The α in $f(x,y) = \alpha(x - y)$ can be chosen in n different ways.

Corollary. Given an integer of the form $n = p^k$ there exists a nonsemisimple algebra with n Wedderburn factors.

Proof. For every prime, p, and integer, k, there exists a Galois field with p^k elements.

The above results could have been obtained by applying Theorem 2. For if n is a radical element, then -n is the quasi-inverse of n. From $\rho(a) = a - an - n'a + n'an = a - an + na$, we have:

$$\begin{bmatrix} x & f(x,y) \\ 0 & y \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} =$$

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} - \begin{bmatrix} 0 & zx \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & zy \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & -z(x-y) \\ 0 & y \end{bmatrix}.$$

There are several ways of obtaining rings with an arbitrary integral number of Wedderburn factors. However, the most convenient device seems to be that of a direct sum. Consider the set of all four by four matrices of the form:

$$\begin{bmatrix} x & z & & \\ 0 & y & \multicolumn{2}{c}{\theta} \\ & & u & w \\ \multicolumn{2}{c}{\theta} & 0 & v \end{bmatrix},$$

where $x, y, z, u, v, w \in I/(p)$. The choice of $GF(p^n)$ could be made, but results in no simplification. The above set of matrices is the direct sum of the algebra constructed previously with an isomorphic image. The radical of this algebra is:

$$N(A) = \left\{\begin{bmatrix} 0 & z & & \\ 0 & 0 & \Theta & \\ & & 0 & w \\ \Theta & & 0 & 0 \end{bmatrix}\right\},$$

and A/N(A) is isomorphic to the algebra of diagonal matrices. By an argument exactly like that given above, it can be shown that if

$$a = \begin{bmatrix} x & 0 & & \\ 0 & y & \Theta & \\ & & u & 0 \\ \Theta & & 0 & v \end{bmatrix}, \text{ and } \rho\,(a) = \begin{bmatrix} x & f(x,y) & & \\ 0 & y & \Theta & \\ & & u & g(u,v) \\ \Theta & & 0 & v \end{bmatrix}$$

are to represent corresponding elements in isomorphic subalgebras, then $f(x,y) = \alpha\,(x - y)$ and $g(u,v) = \beta\,(u - v)$. Therefore, the resulting algebra has 2p Wedderburn factors. An analogous argument shows that if A has p Wedderburn factors then $\sum_{i=1}^{r} \oplus A_i$, where each $A_i = A$, has rp Wedderburn factors. Thus, given any integer n, we write $n = rp$, where p is a prime factor, construct the two by two upper triangular matrices and form $\sum_{i=1}^{r} \oplus A_i$. This proves

Theorem 5. Given any positive integer n, there exists an algebra with n Wedderburn factors.

An alternate construction using Kronecker products is possible, but has the difficulty of being less visual. It should be noted that in the above examples the rings are generated by the semisimple components. By Definition 4 they are network rings. This allows an alternate formulation of Theorem 5.

Theorem 5'. Given any positive integer n, there exists a network algebra with n Wedderburn factors.

In order to construct an example of a network algebra which does not come within the domain of the Malcev theorem, we will consider inseparable field extensions. Let K be a pure inseparable extension, of finite degree, of a base field F. When the direct product of K is taken with itself, the resulting commutative algebra, $K \times_F K$, is cleft and K x 1 may be used as a Wedderburn factor. Furthermore, there are an infinite number of Wedderburn factors, but because of the commutativity of K x K there can be no inner automorphism carrying one into another. In this case K x 1 and 1 x K generate the product. Since the direct product of K can be taken with itself any number of times with a similar result, we have

Theorem 6. There exist commutative network rings with any given integral number of generators, and an infinite number of Wedderburn factors, for which Malcev's theorem does not apply.

The examples leading to Theorem 6 are completely primary network rings, since (KxKx..xK)/N is isomorphic to the field K. If we consider the complete matric algebra$((KxK...K))_n$, where K is crossed with itself r times, then the resulting algebra is non-commutative. By Theorem 10 of Chapter IV, the result is a network algebra. We thus have:

Theorem 7. There exist non-commutative network rings with any given number of generators, and an infinite number of Wedderburn factors, for which Malcev's theorem does not apply.

IV. REDUCTION THEOREMS

The present chapter begins with an application of two classical theorems of ring theory to network rings. After using these to reduce the class of rings to be investigated we proceed to develop an imbedding technique for primary network rings. This provides us with a ring element for carrying one Wedderburn factor into another. We conclude the chapter with several results concerning this transition element and an important theorem relating cleft rings and network rings.

The following two theorems will be useful in simplifying the study of network rings. Roughly speaking, they allow one to study rings of a more complicated structure in terms of their simpler parts. The rings under consideration will be SBI rings, which include the class of rings with minimum condition. For definitions and general properties of SBI rings, together with proofs of Theorem 8 and Theorem 9, see Jacobson (6). For the proofs of these theorems when the rings satisfy the minimum condition, see Artin _et al_. (2).

Theorem 8. A ring P is primary if and only if $P = C_n$ where C is completely primary.

Theorem 9. If A is semi-primary with radical $\mathfrak{N}$ then $A = P_1 + P_2 + \ldots + P_n + n$, where the P_i are primary and n is an additive subgroup of the radical.

It is now possible to give the reduction theorems for network rings. The first concerns primary rings.

Theorem 10. A primary ring is a network ring if and only if the associated completely primary ring of Theorem 8 is a network ring.

Proof. Let $P = C_n$. Suppose that P is network, that is $P = S + N$, and the Wedderburn factors generate P. However, the radical of P is $N(C)_n$. Since $P = S + N$ it follows that $C_n = S + N = S + N(C)_n = D_n + N(C)_n$. Thus C cleaves as $D + N$. Assume that P is generated by its Wedderburn factors. Then these factors must be of the form $(D_i)_n$, because they are simple. The D_i must be Wedderburn factors of C, and must generate C. For if the D_i do not generate C, then it would be impossible to generate P with the $(D_i)_n$. If C is network, then $C = D + N$ and $C_n = D_n + N_n$ will provide a cleavage for P. Again, since the Wedderburn factors of C generate C, the corresponding components of P must generate P.

Corollary. $A = B_n$ is a network ring if B is a network ring. On the other hand, if A is a network ring and B is cleft, then B is a network ring.

After obtaining the reduction from primary to completely primary the next step is to use Theorem 9 to progress from semi-primary to primary. A preliminary lemma will prove useful for the reduction.

Lemma. If $A = A_1 + A_2$ is a direct sum decomposition of

A, then A is cleft if and only if A_1 and A_2 are cleft.

Proof. If A_1 and A_2 are cleft then it is immediate that A is cleft. For if $A_1 = S_1 + N_1$, $A_2 = S_2 + N_2$, then $A = (S_1 + N_1) + (S_2 + N_2) = (S_1 + S_2) + (N_1 + N_2)$. The radical of A_i is $N(A) \cap A_i$ and thus $N_1 + N_2$ is the radical of A. Now, suppose that A is cleft; $A = S + N$. Then every $s \in S$ has a representation $s = s_1 + s_2$ where $s_i \in A_i$, and $n \in N$ has a representation $n = n_1 + n_2$, where $n_i \in A_i$. Therefore, $a = s + n = (s_1 + s_2) + (n_1 + n_2) = (s_1 + n_1) + (s_2 + n_2)$. Since $N_i < N \cap A_i$, the s_i will give a set of representatives for A_i / N_i and A_1 and A_2 must be cleft.

Theorem 11. If a semi-primary ring A is network, then in the decomposition $A = P_1 + \ldots + P_t + n$, the P_i are primary network rings.

Proof. Cleavage is a consequence of the preceding lemma. Since A is cleft, $A = S + N = S(P_1) + N(P_1) + \ldots + S(P_t) + N(P_t) + n = (S(P_1) + S(P_2) + \ldots + S(P_t)) + (N(P_1) + \ldots + N(P_t) + n)$. Because the Wedderburn factors of A generate A, we have that $(S(P_1) + \ldots + S(P_t))_i$ generate A. Each $(S(P_1) + \ldots + S(P_t))_i$ has a representation as $(S_i(P_1) + \ldots + S_i(P_t))$. If the set of $S_i(P_j)$ do not generate P_j, then since S_i is the sum of such rings, it would be impossible to generate all of A with the S_i.

Theorem 12. In the decomposition $A = P_1 + \ldots + P_t + N$, if $n = 0$, then A is network if and only if the P_i are network.

Proof. The "only if" portion of the theorem is a special case of Theorem 11. Assume, then, that the P_i are network. Hence, every $P_i = S_i + N_i$ and the set of all S_i^j of Wedderburn factors of P_i generate P_i. Consider the entire set of S_i^j. Since every $a \in A$ has a representation as $a = p_1 + p_2 + \dots + p_t$, and each p_i is generated by the corresponding Wedderburn factors, A must be generated.

The direct sum of a primary ring that is network with a radical ring provides an easy counter-example to the converse of Theorem 11. For if $A = P \oplus N$, then it is impossible to generate N by means of the simple Wedderburn factors of P. Theorem 12, by excluding such situations, guarantees that the entire semi-primary ring will be generated.

It is interesting to observe the decomposition, provided by Theorem 9, of the first examples of Chapter III. The primary rings, P_i, arise from the primitive orthogonal idempotents of the ring. In particular, $P_i = e_iAe_i$ if the e_i are primitive orthogonal idempotents. In the ring of two by two upper triangular matrices a set of such idempotents is

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$e_1Ae_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix},$$

$$e_2Ae_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} .$$

Consequently, the P_i are isomorphic to the base field and n is the entire radical.

As a result of the preceding sequence of theorems, the study of network rings can be reduced to the study of completely primary network rings. Although the Malcev theorem says nothing about network rings in particular, it does say that in the separable case every pair of Wedderburn factors are conjugated in the ring itself. In order to have an analogous situation for network algebras the following classical theorem will be used. For a proof, see Albert (1) or Deuring (5).

Theorem 13. If S_1 and S_2 are isomorphic simple subalgebras of a central simple algebra A which have the same identities as A, then there exists an inner automorphism of A carrying S_1 into S_2.

For use here, the central simple algebra A will be chosen as a complete matrix algebra. This choice is a consequence of the ease of obtaining matric representations and the extensive theory of such representations.

Suppose that A is a primary network algebra. Then if A is imbedded in a complete matric algebra, the isomorphism of the Wedderburn factors is given by an inner automorphism of the complete matric algebra. In the Introduction a statement was made concerning the uniqueness of the Wedderburn factors in the separable case. The following theorem can be considered as an extension of that result.

Theorem 14. Given a cleft primary algebra, $A = S + N$, S is unique if and only if $a^{-1}Sa < S$ for all a such that $a^{-1}Aa = A$, where $a \varepsilon M_n > A$.

In the following S_1 and S_2 will be isomorphic (over a field F) simple algebras, M will be a total matric algebra over F, and the identities of M, S_1, and S_2 will be the same. Then there exists a regular element $a \varepsilon M$ such that $a^{-1}S_1a=S_2$. S_1 and S_2 will generate a subalgebra of M, called the composite of S_1 and S_2 and denoted by $[S_1, S_2]$.

Lemma. Assume that $a \varepsilon [S_1, S_2]$ which is cleft with S_i as Wedderburn components. Then in the decompositions $a = s + n$, $a^{-1} = s' + n'$, one has $s' = s^{-1}$.

Proof. If $a \varepsilon [S_1, S_2]$, then $a^{-1} \varepsilon [S_1, S_2]$. $1 = 1 + 0 = aa^{-1} = (s + n)(s' + n') = ss' + ns' + sn' + nn'$. Since $sn' + ns' + nn' \varepsilon N$, $ss' \varepsilon S_1$ implies that $ss' = 1$, $sn' + ns' + nn' = 0$. Therefore, $s' = s^{-1}$.

Lemma. Suppose S_1, S_2, S_3 are three Wedderburn factors such that $A = [S_1, S_2, S_3]$ and $a^{-1}S_1a = S_2$, where $a \varepsilon S_3$. Then $A = [S_1, S_2]$.

Proof. $a^{-1}S_1a = S_2$, $b^{-1}S_1b = S_3$ imply that $S_1 = aS_2a^{-1} = bS_3b^{-1}$ and $S_3 = (b^{-1}a)S_2(ba^{-1})$.

Lemma. If $A = [S_1, S_2]$ is a network algebra where $a^{-1}S_1a = S_2$

and $a \varepsilon A$, then in the representations $a = a_1 + n_1 = a_2 + n_2$, where $a_i \varepsilon S_i$, $n_2 = a^{-1}n_1a$.

Proof. $a = a_1 + n_1 = a_2 + n_2$ implies that $a^{-1}aa = a = a^{-1}(a_1 + n_1)\,a = a^{-1}(a_2 + n_2)\,a = a^{-1}a_1a + a^{-1}n_1a = a^{-1}a_2a + a^{-1}n_2a = a_1 + a^{-1}n_2a$.

The relationship between cleft rings and network rings is given by the following theorem. Although perhaps an over-simplification, it might be said that a cleft r' g differs from a network ring by a superfluous portion of the fadical.

Theorem 15. Every cleft ring A contains a maximal network ring A'.

Proof. Let A' be the subring of A generated by the semisimple Wedderburn factors of A. If A' is cleft with the same Wedderburn factors as A then it must be network. Let S_1 be one of the generating subrings. From the cleavage of A, $A = S_1 + N$. Now choose an element $a' \varepsilon A'$. In A, a' can be expressed as $a' = s_1 + n$. Since a' and s_1 are in A', it follows that n is also. If b' is an arbitrary element of A', then $b'a' = b's_1 + b'n$, and $a'b' = s_1b' + nb'$, and as a consequence, b'n and nb' belong to A'. Therefore, every element in A' can be expressed as $a' = s_1 + n$, where the set of all the n will form a nilpotent ideal. This implies that A' is cleft, and must be network.

V. STRUCTURE OF COMPLETELY PRIMARY NETWORK ALGEBRAS

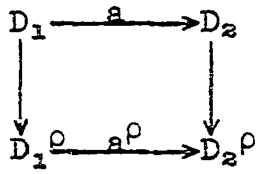
As a consequence of Theorem 11 and Theorem 12, the study of a wide class of network algebras can be reduced to the study of completely primary network algebras. We now consider network algebras generated by two division algebras, and prove some elementary theorems concerning them.

Let $A = [D_1, D_2] \in M_n$ be the composite of the division algebras D_1 and D_2 in M_n. Assume that there exists an $a \in A$ such that $D_2 = a^{-1}D_1a$. Consider any automorphism ρ of A.

Theorem 16. $A^\rho = [D_1{}^\rho, D_2{}^\rho]$.

Proof. We have that $A = A^\rho$. However, $A = [D_1, D_2]$, and if we apply the automorphism ρ to $[D_1, D_2]$, the theorem will follow if $[D_1, D_2] = [D_1{}^\rho, D_2{}^\rho]$. Every element in $[D_1, D_2]$ is a polynomial in d_1 and d_2, where $d_1 \in D_1$ and $d_2 \in D_2$. If we apply ρ to these polynomials, the result must be the same polynomial in $d_1{}^\rho$ and $d_2{}^\rho$, which in turn must generate the algebra.

Theorem 17. In the diagram:



$D_2{}^\rho = (a^\rho)^{-1}D_1{}^\rho(a^\rho)$, where $a^\rho \in A$.

Proof. $D_2{}^\rho = (a^{-1}D_1a)^\rho$. Therefore, every element in $D_2{}^\rho$ is of the form $(a^{-1}d_1a)^\rho$. Since ρ is an automorphism, $(a^{-1}d_1a)^\rho = (a^{-1})^\rho d_1{}^\rho a^\rho = (a^\rho)^{-1}d_1{}^\rho(a^\rho)$.

Theorem 18. If b is an element which induces the automorphisms, $D_1 \longrightarrow D_1^\rho$ and $D_2 \longrightarrow D_2^\rho$, then the entire automorphism $\rho: A \longrightarrow A^\rho$ is induced by b, independent of a belonging to A.

Proof. $A^\rho = [D_1^\rho, D_2^\rho] = [b^{-1}D_1b, b^{-1}D_2b] = b^{-1}[D_1, D_2]b = b^{-1}Ab.$

Corollary. $a^\rho = b^{-1}ab.$

Our inability, in this chapter, to give an abstract criterion for the automorphism which carries the generators of a network ring into one another to be inner, is unfortunate. However, the difficulties involved in providing such a characterization may be appreciated by recalling the two extreme cases. For separable algebras every two generators are related by an inner automorphism of the algebra, while in the case of commutative rings no two generators can be related in such a manner.

VI. REPRESENTATION OF NETWORK ALGEBRAS

In order to analyze more extensively the algebra $[D_1, D_2]$ introduced in the previous chapter, we now turn to representation theory. We begin with a discussion of general representation theory of rings and then specialize to the case of finite dimensional algebras over a field F. Our first result is a characterization of cleft algebras in terms of their representations. Next, necessary and sufficient conditions are provided for $[D_1, D_2]$ to be network and for the inner automorphism connecting the two generators to actually belong to the algebra. An important theorem then provides a construction, in terms of representations, for obtaining any network algebra $[D_1, D_2]$. The chapter concludes with a study of the regular representations of these network algebras.

Let V be a right vector space of finite dimension, n, over a division ring D. Suppose further that V admits the ring R as left operators and that $a(vd) = (av)d$ for all $a \varepsilon R$, $v \varepsilon V$, $d \varepsilon D$. As a consequence, the left multiplications $V \longrightarrow aV$, $a \varepsilon R$ are D-homomorphisms of V into itself.

Let the set $x_i \varepsilon V$ be a basis for V over D. Then if a is applied to the set of x_i, the result must be a linear combination of the x_i:

$$ax_i = x_j a_{ij} = x_i M_a.$$

It can be shown that the mapping $a \longrightarrow M_a$ is a homomorphism of R onto the ring of matrices M_a.

Definition 6. Any homomorphic mapping of R onto a ring of matrices is called a representation of R, and the associated space V is called a representation space.

Definition 7. If a representation is an isomorphism then it is said to be faithful.

The choice of another basis, x_i', instead of x_i yields the representation $a \longrightarrow T^{-1}M_aT$, if T is the matrix carrying the basis elements x_i into x_i'. The resulting representation is said to be equivalent to the initial representation. Equivalent representations will always be identified, which means that a representation is actually a homomorphism onto a ring of linear transformations.

If A is a finite dimensional algebra over a field F, then a particular representation of importance for us is the regular representation. Let e_i be a basis for A over F. If a is an arbitrary element in A, then:

$$ae_i = e_jf_{ji} = e_iM_a$$

is a representation by matrices with coefficients in the field F. Furthermore, if the algebra has an identity, then the regular representation is faithful.

Definition 8. A representation space, V, and the corresponding representation are said to be reducible if V has a proper (R-left, D-right) subspace V_1. If there exists no proper subspace then V is said to be irreducible.

Definition 9. A representation space, V, and the corresponding representation are said to be completely reducible if V is the direct sum of irreducible subspaces.

Let V_1 be a subspace of V, and choose a basis for V (over D) by choosing $x_1,\ldots,x_r$ as a basis for V_1 and completing it with $x_{r+1},\ldots,x_n$. Then,

$$a(x_1,\ldots,x_r,x_{r+1},\ldots,x_n) = (x_1,\ldots,x_r,x_{r+1},\ldots,x_n)\begin{bmatrix} M_{11}(a) & 0 \\ M_{21}(a) & M_{22}(a) \end{bmatrix}.$$

The matrices $M_{11}(a)$ give a representation corresponding to V_1 while the matrices M_{22} give a representation corresponding to the space V/V_1.

The following theorem is due to Brauer (3, p. 505).

Theorem 19. If $V > V_1 > V_2 > \ldots > V_{r-1} > V_r = (0)$

and $V > V_1' > V_2' > \ldots > V'_{r-1} > V'_r = (0)$

are two composition series for V, then bases for the factor spaces can be chosen so that they can be used in both composition series (in a different arrangement).

Brauer (3) considers sets of matrices with coefficients in a division ring and n-tuples constitute the representation space. However, his results will be applicable to the present situation. The following theorem of Brauer will be essential for further development:

Theorem 20. Let A and B be two sets of similar square matrices which are split into irreducible constituents. Let P be the matrix carrying A into B; $P^{-1}AP = B$. If A and B are lower triangular, then the matrix $P = P_1P_2$, where P_1 is lower triangular with identity matrices on the diagonal, and P_2 is a block permutation matrix.

Assume that we are given a semisimple ring of matrices with coefficients in a division ring D. Then the representation is completely reducible. If we have another representation in reduced form, then we can go from one to the other by means of a matrix having zeros above the diagonal and identity matrices on the diagonal.

We shall now restrict ourselves to algebras of finite dimension over a base field F.

Definition 10. A representation of an algebra A is called a cleft representation if

$$M(A) = \begin{bmatrix} M_1(A) & & \Theta \\ & M_2(A) & \\ \Theta & & \ddots \; M_n(A) \end{bmatrix} + \begin{bmatrix} \Theta_1 & & \Theta \\ & \Theta_2 & \\ * & \ddots & \Theta_n \end{bmatrix}$$

where each block in the first term is completely reducible and represents A/N.

Theorem 21. An algebra is cleft if and only if every representation of it is cleft.

Proof. Assume that A is cleft and that V is a representation space. Let S be a Wedderburn factor of A: A = S + N. Since S is semisimple, V is completely reducible relative to S. Let M(A) be a reduced representation of A corresponding to the composition series:

$$V > V_1 > \dots > V_r = (0).$$

Let $V > V_1' > \dots > V_r' = (0)$ be the composition series with respect to which V is completely reduced. Both give reduced representations of S. Using the result above of Brauer, the two representations are equivalent by a matrix of the form:

$$\begin{bmatrix} I_1 & & \theta \\ & I_2 & \\ * & & \ddots \; I_n \end{bmatrix}$$

except for a permutation of the irreducible factors. This permutation does nothing to the irreducible factors except perform the interchange. Therefore, the representation is cleft. On the other hand, if it is assumed that A has a faithful cleft representation, it is a cleft algebra by definition.

We are now ready to characterize the network algebras $[D_1, D_2]$.

Theorem 22. $A = [D_1, D_2]$ is a network algebra over the field, F, if and only if there exists a faithful representation, M(A), over F such that:

$$D_1 \to M(D_1) = \begin{bmatrix} D & & \theta \\ & D & \\ & & \ddots \\ \theta & & D \end{bmatrix}, \quad D_2 \to M(D_2) = \begin{bmatrix} D & & \theta \\ & D & \\ & & \ddots \\ f(D) & & D \end{bmatrix}, \quad N \to M(N) = M(D_1) \begin{bmatrix} \theta & & \theta \\ & \theta & \\ & & \ddots \\ f(D) & & \theta \end{bmatrix} M(D_1)$$

where D is the unique irreducible representation of D_1.

Proof. Since A is cleft with Wedderburn factor D_1, by Theorem 21 a faithful representation of A appears as:

$$M(A) = \text{diagonal}(D, D, \dots, D) + \text{lower triangular},$$

where D is the unique irreducible representation of D_1. If D_2 is another Wedderburn factor of A, then it must be represented in M(A) by matrices having the same irreducible constituents down the diagonal as $M(D_1)$. Because $M(D_1)$ and $M(D_2)$ are isomorphic, the entries below the diagonal are determined by D. Therefore, we express these entries as f(D). Since the radical of the representation is generated by $M(D_1)$ and $M(D_2)$, it must be strictly lower triangular with entries that can be obtained from polynomials with elements in $M(D_1)$ and $M(D_2)$. This leads to elements having f(D) below the diagonal, and multiples of this by $M(D_1)$ and $M(D_2)$ on either side. However, multiplication by $M(D_2)$ is equivalent to multiplication by $M(D_1)$. This leads to the form of M(N) given in the theorem.

Theorem 23. $A = [D_1, D_2]$ is network with $D_2 = a^{-1}D_1a$ if and only if there is an $a \in M(A)$,

$$a \longrightarrow \begin{bmatrix} I & & & \\ & I & & \Theta \\ & & \ddots & \\ g(I) & & & I \end{bmatrix} = P,$$

such that $P^{-1}M(D_1)P = M(D_2)$.

Proof. From Theorem 20, $P = P_1P_2$ where P_1 is a matrix of the above form, and P_2 is a permutation matrix carrying corresponding irreducible constituents into one another. However, the irreducible constituents of a completely primary algebra are isomorphic. Therefore, P_2 is the identity matrix and $P = P_1$.

Theorem 24. $a \varepsilon [D_1, D_2]$ if and only if $g(I) \varepsilon M(N)$.

Proof. If $g(I)$ belongs to $M(N)$, then it is clear that a is contained in the network algebra -- since the sum of $g(I)$ and the identity matrix, which belongs to $M(D)$, must be in the algebra. On the other hand, if $a \varepsilon [D_1, D_2]$ it has a representation as an identity matrix plus a radical element. This radical element must be $g(I)$.

Theorem 25. Any network algebra $[D_1, D_2]$ can be constructed by taking $M(D_1) = \mathrm{Diag}\,(D, D, \ldots, D)$ where D is the unique irreducible representation of D_1, choosing P as:

$$P = \begin{bmatrix} I & & & \\ & I & \theta & \\ & & \cdot & \\ g(I) & & \cdot & I \end{bmatrix},$$

forming $P^{-1}M(D_1)P$, and considering the resulting algebra generated by $M(D_1)$ and $P^{-1}M(D_1)P$.

Proof. Let $M(D_2) = P^{-1}M(D_1)P$. Then $M(D_2)$ is lower triangular, and by its construction isomorphic to $M(D_1)$. The resulting algebra generated by $M(D_1)$ and $M(D_2)$ must be cleft since every element of it is of the form of Definition 9. Conversely, by Theorem 22 any network algebra is generated by two such matric sets.

It is possible to generalize Theorem 25 by choosing a P' such that $P'^{-1}M(D_1)P' \neq M(D_1) \neq M(D_2)$. The result is a network algebra generated by three division algebras. Obviously this can be extended to any number of generators.

Corollary. Any primary network algebra can be constructed by taking complete matric algebras with entries from the completely primary network algebras constructed above.

Proof. Theorem 10 of Chapter IV.

We shall now look at some constructions which are motivated by the preceding theorems. Let a given network algebra have generators:

$$M(D_1) = \begin{bmatrix} D & \Theta \\ \Theta & D \end{bmatrix}, \quad \text{and } M(D_2) = \begin{bmatrix} D & \Theta \\ F(D) & D \end{bmatrix} .$$

The question of the existence of an element $P \in [M(D_1), M(D_2)]$ such that $P^{-1}M(D_1)P = M(D_2)$ is equivalent to solvability of a system of linear equations. For,

$$P^{-1}M(D_1)P = \begin{bmatrix} I & \Theta \\ -g(I) & I \end{bmatrix} \begin{bmatrix} D & \Theta \\ \Theta & D \end{bmatrix} \begin{bmatrix} I & \Theta \\ g(I) & I \end{bmatrix} = \begin{bmatrix} D & \Theta \\ Dg(I)-g(I)D & D \end{bmatrix} .$$

This gives

Theorem 26. The equation $Dg(I) - g(I)D = f(D)$ can be solved for $g(I)$ in $[M(D_1), M(D_2)]$ if and only if the network algebra contains an element conjugating the two generators.

If we progress to more complicated representations of $[D_1, D_2]$, similar results can be obtained. Let

$$M(D_1) = \begin{bmatrix} D & & & \\ & D & \Theta & \\ \Theta & & \ddots & \\ & & & D \end{bmatrix}, \quad M(D_2) = \begin{bmatrix} D & & & \\ f(D) & D & \Theta & \\ \vdots & & \ddots & \\ f(D) & * & & D \end{bmatrix} .$$

Take

$$P = \begin{bmatrix} I & & & \\ g(I) & I & \Theta & \\ \vdots & & \ddots & \\ g(I) & * & & I \end{bmatrix} .$$

Then, $P^{-1}M(D_1)P = M(D_2)$ implies again that $Dg(I) - g(I)D = f(D)$, giving the same result as Theorem 26.

If P is as above, except that there are different $g_i(I)$ down the first column, then the existence of P requires that the equations $Dg_i(I) - g_i(I)D = f(D)$, shall have a solution.

Another construction for network algebras can be obtained by beginning with a particular P and investigating the algebra $[M(D_1), P^{-1}M(D_1)P]$. Let

$$P = \begin{bmatrix} I & & \\ E_2 I & & \\ E_3 & \cdot & \theta \\ \cdot & & \cdot \\ E_n & \theta & I \end{bmatrix}, \text{ where } E_i \in D,$$

then

$$P^{-1}M(D_1)P = \begin{bmatrix} D & & & \\ E_2D-DE_2, D & & \theta & \\ E_3D-DE_3 & D & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ E_nD-DE_n & & & D \end{bmatrix}.$$

If we multiply the corresponding radical elements by $M(D_1)$, we have the radical of the network ring. Therefore:

Theorem 27. A necessary and sufficient condition for $P \in [M(D_1), P^{-1}M(D_1)]$ is that the equations $\Delta'(E_i\Delta - \Delta E_i) = E_i$ have a solution for some pair of elements Δ', Δ in D.

Corollary. If the E_i are all equal in the above theorem, then there always exists a solution of $\Delta'(E\Delta - \Delta E) = E$ when $E\Delta - \Delta E \neq 0$.

Proof. For if Δ is given, then $\Delta' = E(E\Delta - \Delta E)^{-1}$.

In this last portion of the chapter we shall use information available, concerning the regular representation to give a necessary and sufficient condition that an element can conjugate the completely reduced representation of D_1 to give a network algebra.

The following theorem is due to Vinograde (8, p. 506).

Theorem 28. Let R be a completely primary ring with minimum condition on left ideals. If R is cleft: R = K + N, then the reduced form of the regular representation of R over K is:

$$M_K(R) = \begin{bmatrix} K & & & & \Theta \\ K_2 & \boxed{} & & & \\ K_3 & & & & \\ \cdot & & \boxed{} & & \\ \cdot & * & & \ddots & \\ K_n & & & & \end{bmatrix},$$

where the blocks are irreducible representations of K, but not all are algebras over K, and where K_i range independently over K and the entire matrix is determined when the first column is specified.

If R is an algebra over the field F, then a replacement of K by its irreducible representation over F leads to

$$M_F(R) = \begin{bmatrix} M(K) & & & & \Theta \\ M_2(K) & \boxed{} & & & \\ M_3(K) & & & & \\ \cdot & & \boxed{} & & \\ \cdot & * & & \ddots & \\ M_n(K) & & & & \end{bmatrix},$$

where the $M_i(K)$ vary independently of one another.

This representation can be further reduced by a matrix of diagonal blocks corresponding to the blocks of $M_F(R)$. This reduction cannot change the number of independent matrices in the first column of blocks.

Now let P be the element transforming the irreducible representation of K into the above representation. Set

$$P = \begin{bmatrix} I & & & & \\ E_2 I & & & \Theta & \\ E_3 & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & * & & \cdot & \\ E_n & & & & I \end{bmatrix} .$$

Then, upon transforming by P we obtain a matrix whose entries in the first column are the matrices $DE_i - E_iD$. This allows us to state the final

Theorem 29. A necessary and sufficient condition on n for $P^{-1}M(D_1)P$ to be another generator of the network algebra is that

$$\begin{bmatrix} D & & & \\ & D & & \Theta \\ & & \cdot & \\ \Theta & & & \cdot \\ & & & D \end{bmatrix} \begin{bmatrix} \Theta \\ DE_2-E_2D \\ DE_3-E_3D \\ \vdots \\ DE_n-E_nD \end{bmatrix}$$

contain n independent matrices.

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