ON CRITICAL EXPONENTS FOR A SEMILINEAR PARABOLIC SYSTEM COUPLED IN AN EQUATION AND A BOUNDARY CONDITION

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Abstract. In this paper, we consider the system

$$\begin{array}{llll} u_t = \Delta u + v^p, & v_t = \Delta v & x \in \mathbb{R}^N_+, & t > 0, \\ -\dfrac{\partial u}{\partial x_1} = 0, & -\dfrac{\partial v}{\partial x_1} = u^q & x_1 = 0, & t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x) & x \in \mathbb{R}^N_+, & \end{array}$$

where $\mathbb{R}^N_+ = \{(x_1, x') | x' \in \mathbb{R}^{N-1}, x_1 > 0\}$, $p, q > 0$, and u_0, v_0 are nonnegative and bounded. We prove that if $pq \leq 1$ every nonnegative solution is global. When $pq > 1$ we let $\alpha = \frac{p+2}{2(pq-1)}$, $\beta = \frac{2q+1}{2(pq-1)}$. We show that if $\max(\alpha, \beta) > \frac{N}{2}$ or $\max(\alpha, \beta) = \frac{N}{2}$ and $p, q \geq 1$, then all nontrivial nonnegative solutions are nonglobal; whereas if $\max(\alpha, \beta) < \frac{N}{2}$ there exist both global and nonglobal nonnegative solutions.

1. Introduction.

In this paper we study the large time behavior of nonnegative solutions of a system as follows:

$$
(1.1)\qquad
\begin{aligned}
&u_t = \Delta u + v^p, && v_t = \Delta v && x \in \mathbb{R}^N_+, \quad t > 0,\\
&-\frac{\partial u}{\partial x_1} = 0, && -\frac{\partial v}{\partial x_1} = u^q && x_1 = 0, \qquad t > 0,\\
&u(x,0) = u_0(x), && v(x,0) = v_0(x) && x \in \mathbb{R}^N_+,
\end{aligned}
$$

where $\mathbb{R}^N_+ = \{(x_1, x') | x' \in \mathbb{R}^{N-1}, x_1 > 0\} (N \geq 1)$, $p, q > 0$, and both $u_0(x)$ and $v_0(x)$ are nonnegative bounded functions satisfying the compatibility condition

$$
(1.2)\qquad -\frac{\partial u_0}{\partial x_1} = 0 \quad \text{and} \quad -\frac{\partial v_0}{\partial x_1} = u_0^q \qquad \text{at} \quad x_1 = 0.
$$

In order to motivate our results for the above system, we recall a classical result of Fujita [F] for the problem

$$
(1.3)\qquad
\begin{aligned}
&u_t = \Delta u + u^p && x \in \mathbb{R}^N,\ t > 0,\\
&u(x,0) = u_0(x) && x \in \mathbb{R}^N,
\end{aligned}
$$

with nonnegative initial data u_0. He showed that (i) if $1 < p < 1 + 2/N$, then (1.3) possesses no global nonnegative solutions while (ii) if $p > 1 + 2/N$, both global and nonglobal nonnegative solutions exist. The number $1 + 2/N$ is called the critical exponent which turns out to belong to case (i). See [W] for an elegant proof by Weissler as well as references to earlier proofs of this result.

Over the past a few years there have been a number of extensions of Fujita's result in various directions. We refer the reader to the survey paper by Levine [L1].

Recently, Escobedo and Herrero [EH] investigated the initial value problem for a weakly coupled system

$$
(1.4)\qquad
\begin{aligned}
&u_t = \Delta u + v^p, && v_t = \Delta v + u^q && x \in \mathbb{R}^N,\ t > 0,\\
&u(x,0) = u_0(x) \geq 0, && v(x,0) = v_0(x) \geq 0 && x \in \mathbb{R}^N.
\end{aligned}
$$

Set, when $pq \neq 1$,

$$
\alpha_1 = \frac{p+1}{pq-1}, \quad \beta_1 = \frac{q+1}{pq-1}.
$$

The results of [EH] for (1.4) take the following form. If $\max(\alpha_1, \beta_1) \geq \frac{N}{2}$ then all nontrivial solutions are nonglobal. If $\max(\alpha_1, \beta_1) < \frac{N}{2}$ then there are global and nonglobal solutions. When $\max(\alpha_1, \beta_1)$ is negative or not defined, all solutions with L^∞ initial values are global.

Galaktionov and Levine [GL] considered the boundary–value problem:

$$
\begin{aligned}
&u_t = u_{xx} && x > 0,\ t > 0,\\
&-u_x = u^p && x = 0,\ t > 0,\\
&u(x,0) = u_0(x) \geq 0 && x > 0;\\
&-u_0'(0) = u_0^p(0).
\end{aligned}
\tag{1.5}
$$

They showed that if $1 < p \leq 2$, then $u(x,t)$ blows up in a finite time for all nontrivial u_0; whereas if $p > 2$, then $u(x,t)$ becomes unbounded in a finite time for large u_0 and $u(x,t)$ exists globally for small initial data. Their result was later extended in [DFL] to the problem

$$
\begin{aligned}
&u_t = \Delta u, && v_t = \Delta v && x \in \mathbb{R}^N_+, \quad t > 0,\\
&-\frac{\partial u}{\partial x_1} = v^p, && -\frac{\partial v}{\partial x_1} = u^q && x_1 = 0, \quad t > 0,\\
&u(x,0) = u_0(x) \geq 0, && v(x,0) = v_0(x) \geq 0 && x \in \mathbb{R}^N_+.
\end{aligned}
\tag{1.6}
$$

It was shown in [DFL] that for (1.6) the result takes the same form as in [EH] for (1.4) if we replace $\alpha_1,\ \beta_1$ by $\alpha_2 = \frac{\alpha_1}{2},\ \beta_2 = \frac{\beta_1}{2}$.

Problem (1.1) is "intermediate" between Problems (1.4) and (1.6) so we expect the same result but for the "intermediate" powers

$$\alpha = \frac{p+2}{2(pq-1)}, \qquad \beta = \frac{2q+1}{2(pq-1)}.$$

Obviously,

$$\alpha_1 > \alpha > \alpha_2, \qquad \beta_1 > \beta > \beta_2.$$

We prove the following:

Theorem. *If $pq \leq 1$ all nonnegative solutions of (1.1) are global. If $pq > 1$ then there are no nontrivial global nonnegative solutions of (1.1) if $\max(\alpha, \beta) > \frac{N}{2}$ or if $\max(\alpha, \beta) = \frac{N}{2}$ and $p, q \geq 1$. Both nonnegative global nontrivial and nonglobal solutions exist if $pq > 1$ and $\max(\alpha, \beta) < \frac{N}{2}$.*

In the non–Lipschitz case $\min(p, q) < 1$ we do not expect uniqueness to hold in general. We restrict our discussion to maximal solutions in that case.

Problems (1.4) and (1.6) are symmetric in the sense that we may always assume that $p \leq q$. We cannot do this for (1.1). Also, the representation formulae (or "variation of constants" formulae) have the same form for both components u, v of solutions of (1.4) and (1.6). But for (1.1) they are different. This is reflected in the fact that there are significant differences at the technical level between proofs in [EH], [DFL] and in the present paper.

The plan of the paper is as follows: Section 2 contains global existence results, Section 3 is devoted to global nonexistence in the Lipschitz case $\min(p, q) \geq 1$ and Section 4 to global nonexistence in the non–Lipschitz case $\min(p, q) < 1$.

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2. Global existence

Let

$$G_N(x, y, t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \mathbb{R}^N, \ \alpha > 0,$$

and for $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N_+)$ define

$$\mathcal{S}_{N-1}(t) w(\cdot, x') = \int_{\mathbb{R}^{N-1}} G_{N-1}(x', y', t) w(\cdot, y') dy',$$

$$H(x_1, y_1, t) = G_1(x_1, y_1, t) + G_1(x_1, -y_1, t),$$

and

$$\mathcal{T}(t) w(x_1, \cdot) = \int_0^\infty H(x_1, y_1, t) w(y_1, \cdot) dy_1.$$

Notice that

$$\mathcal{T}(t)\mathcal{S}_{N-1}(\tau)w = \mathcal{S}_{N-1}(\tau)\mathcal{T}(t)w.$$

Define further

$$\mathcal{R}(t)w(x_1,\cdot) = H(x_1,0,t)\mathcal{S}_{N-1}(t)w(0,\cdot),$$

$$\mathcal{S}(t)w = \mathcal{T}(t)\mathcal{S}_{N-1}(t)w.$$

Then we have the following representation formulae for the solution of (1.1):

$$u(\cdot,t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-\eta)v^p(\cdot,\eta)d\eta, \tag{2.1}$$

$$v(\cdot,t) = \mathcal{S}(t)v_0 + \int_0^t \mathcal{R}(t-\eta)u^q(\cdot,\eta)d\eta. \tag{2.2}$$

As in [EH] it is possible to prove local (in time) existence of solutions for given L^∞ initial values using the representation formulae (2.1), (2.2) and the contraction mapping principle. The details are rather standard and are therefore omitted.

Lemma 2.1. *If* $0 < pq \leq 1$ *then every solution of* (1.1) *is global.*

Proof. Let $K = \|v_0\|_\infty$ and

$$C = \max\left\{\|u_0\|_\infty, \left(\frac{1}{p}(K+1)^p\right)^{\frac{1}{2q+1}}\right\}.$$

Define

$$\bar{u}(x,t) = Ce^{\sigma t}, \quad \sigma = pc^{2q},$$

and

$$\bar{v}(x,t) = e^{\sigma^2 t}(K + e^{-\rho x_1}), \quad \rho = c^q.$$

Then it is easy to verify that

$$\begin{aligned}
\bar{u}_t &\geq \triangle\bar{u} + \bar{v}^p, & \bar{v}_t &\geq \triangle\bar{v} & & x\in\mathbb{R}^N_+,\ t>0,\\
-\frac{\partial\bar{u}}{\partial x_1} &= 0, & -\frac{\partial\bar{v}}{\partial x_1} &\geq \bar{u}^q & & x_1 = 0,\ t>0,\\
\bar{u}(x,0) &\geq u_0(x), & \bar{v}(x,0) &\geq v_0(x) & & x\in\mathbb{R}^N_+.
\end{aligned}$$

Therefore we obtain that $u \leq \bar{u}$ and $v \leq \bar{v}$. □

Lemma 2.2. *If $pq > 1$ and $\max(\alpha,\beta) < \frac{N}{2}$ then there are global solutions.*

Proof. We look for a supersolution of the self-similar type:

$$\bar{u}(x,t) = (t_0+t)^{-\alpha} f(y), \quad \bar{v}(x,t) = (t_0+t)^{-\beta} g(y), \quad y = \frac{x}{\sqrt{t_0+t}},$$

where f, g satisfy

$$\triangle f + \frac{1}{2} y \cdot \nabla f + \alpha f + g^p \le 0, \quad \triangle g + \frac{1}{2} y \cdot \nabla g + \beta g \le 0, \quad y \in \mathbb{R}^N_+, \tag{2.3}$$

$$-\frac{\partial f}{\partial y_1} \ge 0, \qquad -\frac{\partial g}{\partial y_1} \ge f^q, \quad y_1 = 0. \tag{2.4}$$

We set

$$f(y) = Ae^{-\rho|y|^2}, \quad g(y) = Be^{-\sigma(|y'|^2+(y_1+\delta)^2)}$$

where A, B, ρ, σ and δ are positive constants. With this choice of f and g the inequalities (2.3) read as follows:

$$Ae^{-\rho|y|^2}((4\rho^2-\rho)|y|^2 + \alpha - 2N\rho) + B^p e^{-p\sigma(|y'|^2+(y_1+\delta)^2)} \le 0, \tag{2.5}$$

$$(\beta - 2N\sigma + 4\sigma^2\delta^2) + \delta\sigma(8\sigma-1)y_1 + \sigma(4\sigma-1)|y|^2 \le 0. \tag{2.6}$$

Obviously, $\frac{\partial f}{\partial y_1} = 0$ and the second inequality in (2.4) is satisfied if

$$2\sigma\delta Be^{-\sigma|y'|^2} \ge A^q e^{-\rho q|y'|^2}. \tag{2.7}$$

Consider three cases.

(i) $p, q \ge 1$. Choose $\rho = \sigma < \frac{1}{4}$ such that $\alpha - 2N\rho < 0$ and $\beta - 2N\sigma < 0$. Then (2.6) is satisfied if $\delta > 0$ is such that

$$\delta^2\sigma^2(8\sigma-1)^2 - 4\sigma(4\sigma-1)(\beta - 2N\sigma + 4\sigma^2\delta^2) < 0.$$

If we now set

$$A = B^p(2N\rho-\alpha)^{-1}, \; B = (2\sigma\delta(2N\rho-\alpha)^q)^{\frac{1}{pq-1}},$$

then (2.5) and (2.7) hold.

(ii) $p < 1 < q$. Choose $\sigma < \frac{1}{4}$ such that $\beta - 2N\sigma < 0, \rho = p\sigma$ and δ, A, B as before. Since

$$\alpha = \frac{p+2}{2(pq-1)} < \frac{p+2pq}{2(pq-1)} = p\beta < 2Np\sigma = 2N\rho,$$

we see that A and B are well defined and positive and (2.5) holds. Then (2.7) is also satisfied because $pq > 1$.

(iii) $q < 1 < p$. Now choose $\rho = \frac{1}{4}$ and $\sigma = \frac{q}{4}$. Since

$$\beta = \frac{2q+1}{2(pq-1)} < \frac{2q+pq}{2(pq-1)} = p\beta < 2Np\sigma = 2N\rho,$$

we can proceed as before. □

3. Global nonexistence in the Lipschitz case

Lemma 3.1. *If $\frac{\partial u_o}{\partial x_1} \leq 0$ and $\frac{\partial v_0}{\partial x_1} \leq 0$ then $\frac{\partial u}{\partial x_1} \leq 0$ and $\frac{\partial v}{\partial x_1} \leq 0$ as long as the solution (u, v) exists.*

Proof. Let $w = u_{x_1}$ and $z = v_{x_1}$. Then

$$\begin{aligned} w_t &= \triangle w + pv^{p-1}z, & z_t &= \triangle z & & x \in \mathbb{R}^N_+, \quad 0 < t < T, \\ w &= 0, & z &\leq 0 & & x_1 = 0, \quad 0 < t < T, \\ w(x,0) &\leq 0, & z(x,0) &\leq 0 & & x \in \mathbb{R}^N_+, \end{aligned}$$

so $z \leq 0$ in $\mathbb{R}^N_+ \times (0,T)$. But then $w_t \leq \triangle w$ in $\mathbb{R}^N_+ \times (0,T)$ hence also $w \leq 0$ in $\mathbb{R}^N_+ \times (0,T)$. □

Proposition 3.2. *Suppose $p, q \geq 1$ and $pq > 1$. Then there are initial values such that the corresponding solutions are nonglobal.*

Proof. Assume $\frac{\partial u_0}{\partial x_1} \leq 0$ and $\frac{\partial v_0}{\partial x_1} \leq 0$. Define

$$\varphi_k(x) = \left(\frac{k}{\pi}\right)^{\frac{N}{2}} e^{-k|x|^2}, \quad k > 0.$$

Then $\int\limits_{\mathbb{R}^N_+} \varphi_k(x)dx = 1$, $\frac{\partial \varphi_k}{\partial x_i} = -2kx_i\varphi_k$, $\triangle\varphi_k \geq -2kN\varphi_k$. Define further

$$F(t) = \int\limits_{\mathbb{R}^N_+} \varphi_k(x)u(x,t)dx, \quad G(t) = \int\limits_{\mathbb{R}^N_+} \varphi_k(x)v(x,t)dx.$$

Then integration by parts and Jensen's inequality yield

$$F'(t) \geq -2kNF(t) + G^p(x,t).$$

On the other hand,

$$G'(t) \geq -2NG(t) + \int_{\mathbb{R}^{N-1}} u^q(0,x',t)\varphi_k(0,x')dx'.$$

Since $\frac{\partial u}{\partial x_1} \leq 0$, we obtain that

$$\left(\frac{\pi}{k}\right)^{\frac{1}{2}} \int_{\mathbb{R}^{N-1}} \varphi_k(0,x')u^q(0,x',t)dx' \geq \int_{\mathbb{R}^N_+} \varphi_k(x)u^q(x)dx \geq F^q(t).$$

So we have

$$F'(t) \geq -2kNF(t) + G^p(t) =: \phi_k(F(t),G(t)), \tag{3.1}$$

$$G'(t) \geq -2kNG(t) + \sqrt{\frac{k}{\pi}}F^q(t) =: \psi_k(F(t),G(t)), \tag{3.2}$$

as long as the solution exists. As in [L2, Theorem 2.5] we conclude that solutions are nonglobal if

$$\frac{1}{2N\sqrt{k\pi}}F^q(0) > G(0) > (2kNF(0))^{\frac{1}{p}},$$

and as in [FLU, Lemma 2.1] one can see that there are u_0, v_0 and k such that this inequality holds. □

Proposition 3.3. *Suppose $p,q \geq 1$, $pq > 1$ and $\max(\alpha,\beta) > \frac{N}{2}$. Then all nontrivial solutions are nonglobal.*

Proof. The proof is almost the same as in [FLU, Lemma 2.1]. It is based on the observation that if $f(t), g(t)$ solves

$$f' = \phi_1(f,g), \qquad g' = \psi_1(f,g), \tag{3.3}$$

then $k^\alpha f(kt)$, $k^\beta g(kt)$ solves (3.1)-(3.2). □

Now we turn to the case $\max(\alpha,\beta) = \frac{N}{2}$. The basic idea is the same as in [MS].

Lemma 3.4. *If u_0, v_0 have compact support then $u(\cdot,t), v(\cdot,t) \in L^1(\mathbb{R}^N_+)$ for $0 < t < T$.*

Proof. Choose $T' \in (t,T)$ and define

$$c = \sup_{\mathbb{R}^N_+ \times (0,T')} v^{p-1}, \quad k = \sup_{\mathbb{R}^N_+ \times (0,T')} u^{q-1}.$$

Then $w = u + v$ satisfies

$$\begin{aligned} w_t &\leq \triangle w + cw & x &\in \mathbb{R}^N_+, \quad 0 < t < T', \\ -\frac{\partial w}{\partial x_1} &\leq kw & x_1 &= 0, \quad 0 < t < T', \\ w(x,0) &= u_0(x) + v_0(x) & x &\in \mathbb{R}^N_+. \end{aligned}$$

Therefore

$$w(x,t) \leq M e^{(k^2+c)t - kx_1} \big(4\pi(t+1)\big)^{-\frac{N-1}{2}} e^{-\frac{|x'|^2}{4(t+1)}},$$

provided M is such that

$$u_0(x) + v_0(x) \leq M e^{-kx_1} (4\pi)^{-\frac{N-1}{2}} e^{-\frac{|x'|^2}{4}}. \qquad \square$$

Lemma 3.5. *Suppose $p, q \geq 1$, $pq > 1$ and $\max(\alpha,\beta) = \frac{N}{2}$. Suppose (u,v) is a global solution and u_0, v_0 have compact support. Then*

$$\int_0^\infty \int_{\mathbb{R}^N_+} v^p(x,t)\,dx\,dt < \infty \quad \text{if} \quad \alpha = \frac{N}{2}, \tag{3.4}$$

$$\int_0^\infty \int_{\mathbb{R}^{N-1}} u^q(x,t)\,dx\,dt < \infty \quad \text{if} \quad \beta = \frac{N}{2}. \tag{3.5}$$

Proof. The flow of (3.3) in the positive quadrant looks as follows (cf. [L2]). There is a unique critical point $(\bar F(k), \bar G(k))$ in the interior,

$$(\bar F(k), \bar G(k)) = (c_1 k^\alpha, c_2 k^\beta),$$

c_1 and c_2 depend only on p, q and N. There is a unique separatrix starting on the positive f-axis at the point $(F_0(k), 0)$ and terminating at $(\bar F(k), \bar G(k))$ and a unique separatrix starting on the positive g-axis at the point $(0, G_0(k))$ and terminating at $(\bar{F}(k), \bar{G}(k))$. The inequalities (3.1), (3.2) imply that if (u, v) is global then we must have for every $t > 0$:

$$F(t) \leq F_0(k) = k^{\alpha} F_0(1),$$
$$G(t) \leq G_0(k) = k^{\beta} G_0(1).$$

In other words, we obtain:

$$\int_{\mathbb{R}^N_+} e^{-k|x|^2} u(x,t)dx \leq \pi^{\frac{N}{2}} F_0(1) k^{\alpha - \frac{N}{2}},$$
$$\int_{\mathbb{R}^N_+} e^{-k|x|^2} v(x,t)dx \leq \pi^{\frac{N}{2}} G_0(1) k^{\beta - \frac{N}{2}}.$$

As $k \to 0$, the Lebesgue dominated convergence theorem and Lemma 3.4 yield:

$$\int_{\mathbb{R}^N_+} u(x,t)dx \leq \pi^{\frac{N}{2}} F_0(1) \quad \text{if} \quad \alpha = \frac{N}{2}, \tag{3.6}$$
$$\int_{\mathbb{R}^N_+} v(x,t)dx \leq \pi^{\frac{N}{2}} G_0(1) \quad \text{if} \quad \beta = \frac{N}{2}. \tag{3.7}$$

Integrating the equation

$$u_t = \triangle u + v^p$$

over $\mathbb{R}^N_+ \times [0, \tau]$ we obtain

$$\int_{\mathbb{R}^N_+} u(x,\tau)dx - \int_{\mathbb{R}^N_+} u_0(x)dx = \int_0^{\tau} \int_{\mathbb{R}^N_+} v^p(x,t)dxdt.$$

Hence, (3.4) follows from (3.6). Analogously, integrating

$$v_t = \triangle v$$

one obtains (3.5) from (3.7). $\square$

Proposition 3.6. *Suppose $p, q \geq 1$, $pq > 1$ and $\max(\alpha, \beta) = \frac{N}{2}$. Then all nontrivial solutions are nonglobal.*

Proof. We proceed by contradiction. Suppose (u, v) is a global nontrivial solution. Obviously, we may assume that u_0 and v_0 have compact support. The representation formulae (2.1), (2.2) allow application of a standard argument (cf. e.g. [EH], [DFL]) to show that for every $\tau > 0$ there are $c, \sigma > 0$ such that

$$\min(u(x,\tau), v(x,\tau)) \geq c e^{-\sigma|x|^2}, \quad x \in \mathbb{R}^N_+. \tag{3.8}$$

Assuming $\beta = \frac{N}{2}$ we now construct a subsolution that violates (3.5). Take

$$\underline{u}(x,t) = a(t+t_0)^{-\alpha} e^{-\frac{p|x|^2}{4(t+t_0)}}, \quad \underline{v}(x,t) = b(t+t_0)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4(t+t_0)}},$$

where a, b and t_0 are positive constants such that

$$t_0 \leq \frac{1}{4\sigma}, \quad a t_o^{-\alpha} \leq c \quad \text{and} \quad b t_0^{-\beta} \leq c.$$

Then

$$\underline{u}(x,0) \leq u(x,\tau) \quad \text{and} \quad \underline{v}(x,0) \leq v(x,\tau).$$

The inequality

$$\underline{u}_t \leq \triangle \underline{u} + \underline{v}^p \qquad x \in \mathbb{R}^N_+,\ t > 0, \tag{3.9}$$

holds if we choose a and b such that

$$a\left(\frac{pN}{2} - \alpha\right) \leq b^p.$$

We have

$$\underline{v}_t = \triangle \underline{v} \qquad x \in \mathbb{R}^N_+,\ t > 0,$$

and

$$-\frac{\partial \underline{u}}{\partial x_1} = 0, \quad -\frac{\partial \underline{v}}{\partial x_1} = 0 \leq \underline{u}^q \qquad x_1 = 0,\ t > 0.$$

Hence,

$$\underline{u}(x,t) \leq u(x, t+\tau), \quad \underline{v}(x,t) \leq v(x, t+\tau) \qquad x \in \mathbb{R}^N_+,\ t > 0. \tag{3.10}$$

But

$$\int_0^\infty \int_{\mathbb{R}^N_+} \underline{u}^q(x,t)dx\,dt = a\int_0^\infty (t+t_0)^{-\alpha q+\frac{N}{2}}dt \int_{\mathbb{R}^{N-1}} e^{-\frac{pq}{4}|y|^2}dy = \infty$$

since $-\alpha q + \frac{N}{2} = -1$. This contradicts (3.5).

If $\alpha = \frac{N}{2}$ we take

$$\underline{u}(x,t) = a(t+t_0)^{-\frac{N}{2}} e^{-\frac{1}{4}|y|^2}, \qquad y = \frac{x}{\sqrt{t+t_0}},$$
$$\underline{v}(x,t) = b(t+t_0)^{-\beta} e^{-\frac{q}{4}((y_1+\delta)^2+|y'|^2)}.$$

Now, (3.9) is obviously satisfied and also $-\frac{\partial \underline{u}}{\partial x_1} = 0$ if $x_1 = 0$. The inequality $\underline{v}_t \leq \triangle \underline{v}$ is valid if

$$\beta - \frac{1}{2}Nq + \frac{1}{4}q^2\delta^2 + \frac{1}{4}q\delta(2q-1)y_1 + \frac{1}{4}q(q-1)|y|^2 \geq 0.$$

Therefore we choose

$$\delta \geq \frac{2}{q}\left(\frac{Nq}{2} - \beta\right)^{\frac{1}{2}}.$$

We also find that

$$-\frac{\partial \underline{v}}{\partial x_1} \leq \underline{u}^q \qquad \text{for } x_1 = 0$$

if a and b are such that

$$a\left(\frac{Nq}{2} - \beta\right)^{\frac{1}{2}} \leq b^q.$$

Hence, with a suitable choice of the constants t_0, a and b we obtain (3.10) again. But

$$\int_0^\infty \int_{\mathbb{R}^N_+} \underline{v}^p(x,t)dx\,dt = b\int_0^\infty (t+t_0)^{-\beta p+\frac{N}{2}}dt \int_{\mathbb{R}^N_+} e^{-\frac{q}{4}((y_1+\delta)^2+|y'|^2)}dy = \infty$$

since $-\beta p + \frac{N}{2} = -1$. This contradicts (3.4). □

4. Global nonexistence in the non-Lipschitz case

In what follows we will use following lemmas.

Lemma 4.1. *If* $\frac{\partial u_0}{\partial x_1} \le 0$, *then* $W(x',t) = \mathcal{T}(t)u_0(0,x')$ *is decreasing in* t.

Proof. We have

$$W(x',t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{y_1^2}{4t}} u_0(y_1,x')dy_1$$
$$= 2\int_0^\infty e^{-y_1^2} u_0(2\sqrt{t}y_1,x')dy_1. \qquad \square$$

Lemma 4.2. *Let* $t > \eta > \sigma$. *Then*

$$\int_0^\infty H(x_1,y_1,t-\eta)H(y_1,0,\eta-\sigma)dy_1 = H(x_1,0,t-\sigma). \tag{4.1}$$

Proof. If w solves the problem

$$\begin{aligned} w_t &= w_{x_1x_1} && x_1 > 0,\ t > 0, \\ w_{x_1}(0,t) &= 0 && t > 0, \\ w(x_1,t) &= H(x_1,0,\eta-\sigma) && x_1 > 0, \end{aligned}$$

then

$$w(x_1,t) = \int_0^\infty H(x_1,y_1,t)H(y_1,0,\eta-\sigma)dy_1 = \mathcal{T}(t)H(x_1,0,\eta-\sigma). \tag{4.2}$$

If we set

$$z(x_1,t) = H(x_1,0,t+\eta-\sigma), \qquad x_1 > 0,\ t > 0,$$

then z solves the same problem as w, therefore $w(x_1,t) = z(x_1,t)$. Hence,

$$\mathcal{T}(t-\eta)H(x_1,0,\eta-\sigma) = w(x_1,t-\eta) = z(x_1,t-\eta) = H(x_1,0,t-\sigma). \qquad \square$$

Proposition 4.3. *Assume* $p > 1 > q$, $pq > 1$ *and* $\alpha(= \max(\alpha,\beta)) > \frac{N}{2}$. *Then all nontrivial solutions are nonglobal.*

Proof. We use an iteration technique established in [AW] for (1.3) and modified in [EH] for (1.4) and more recently in [DFL] for (1.6).

We proceed by contradiction. Suppose (u,v) is a global nontrivial solution. Without loss of generality we may assume that $\frac{\partial u_0}{\partial x_1} \le 0$.

Our next aim is to show that there is a constant $c>0$ such that

$$\mathcal{S}(t)u_0^q(0,\cdot) \le ct^{-\alpha q}. \tag{4.3}$$

Using (2.1) and Jensen's inequality we obtain

$$u(\cdot,t) \ge \int_0^t \mathcal{S}(t-\eta)v^p(\cdot,\eta)d\eta \ge t^{1-p}\left(\int_0^t \mathcal{S}(t-\eta)v(\cdot,\eta)d\eta\right)^p. \tag{4.4}$$

From (2.2) it follows that

$$\begin{aligned}
\mathcal{S}(t-\eta)v(x,\eta) &\ge \int_0^\eta \mathcal{T}(t-\eta)\mathcal{S}_{N-1}(t-\eta)\mathcal{R}(\eta-\sigma)u^q(x,\sigma)d\sigma \\
&= \int_0^\eta \mathcal{T}(t-\eta)\mathcal{S}_{N-1}(t-\eta)H(x_1,0,\eta-\sigma)\mathcal{S}_{N-1}(\eta-\sigma)u^q(0,x',\sigma)d\sigma \\
&= \int_0^\eta \mathcal{T}(t-\eta)H(x_1,0,\eta-\sigma)\mathcal{S}_{N-1}(t-\sigma)u^q(0,x',\sigma)d\sigma.
\end{aligned}$$

Lemma 4.2 yields now that

$$\mathcal{S}(t-\eta)v(\cdot,\eta) \ge \int_0^\eta \mathcal{R}(t-\sigma)u^q(\cdot,\sigma)d\sigma,$$

and using this inequality in (4.4) we have

$$\begin{aligned}
u(x,t) &\ge t^{1-p}\left(\int_0^t\int_0^\eta \mathcal{R}(t-\sigma)u^q(x,\sigma)d\sigma d\eta\right)^p \\
&= t^{1-p}\left(\int_0^t (t-\sigma)H(x_1,0,t-\sigma)\mathcal{S}_{N-1}(t-\sigma u^q(0,x',\sigma)d\sigma\right)^p.
\end{aligned} \tag{4.5}$$

Now,

$$u^q(\cdot,\sigma) \ge (\mathcal{S}(\sigma)u_0)^q \ge \mathcal{S}(\sigma)u_0^q$$

therefore

$$\mathcal{S}_{N-1}(t-\sigma)u^q(0,x',\sigma) \ge \mathcal{T}(\sigma)\mathcal{S}_{N-1}(t)u_0^q(0,x'), \tag{4.6}$$

Lemma 4.1 implies that

$$\mathcal{T}(\sigma)\mathcal{S}_{N-1}(t)u_0^q(0,x') \geq \mathcal{S}(t)u_0^q(0,x'), \tag{4.7}$$

and combining (4.5)-(4.7) we obtain

$$u(x_1,x',t) \geq t^{1-p}\left(\int_0^t (t-\sigma)H(x_1,0,t-\sigma)d\sigma\right)^p (\mathcal{S}(t)u_0^q(0,x'))^p. \tag{4.8}$$

Thus

$$u(0,x',t) \geq t^{1+\frac{p}{2}}\left(\frac{1}{\sqrt{\pi}}\int_0^1 (1-\sigma)^{\frac{1}{2}}d\sigma\right)^p (\mathcal{S}(t)u_0^q(0,x'))^p,$$

that is

$$u(0,x',t) \geq c_1 t^{\pi_1}(\mathcal{S}(t)u_0^q(0,x'))^{\frac{\theta_1}{q}}, \tag{4.9}$$

where

$$\theta_1 = pq,\ \pi_1 = 1+\frac{p}{2},\ c_1 = \left(\frac{1}{\sqrt{\pi}}\int_0^1 (1-\sigma)^{\frac{1}{2}}d\sigma\right)^p.$$

If we now assume that

$$u(0,x',t) \geq c_k t^{\pi_k}(\mathcal{S}(t)u_0^q(0,x'))^{\frac{\theta_k}{q}},$$

then

$$\begin{aligned}\mathcal{S}_{N-1}(t-\sigma)u^q(0,x',\sigma) &\geq c_k^q\sigma^{\pi_k q}(\mathcal{S}_{N-1}(t-\sigma)\mathcal{S}(\sigma)u^q(0,x'))^{\theta_k}\\ &= c_k^q\sigma^{\pi_k q}(\mathcal{T}(\sigma)\mathcal{S}_{N-1}(t)u_0^q(0,x'))^{\theta_k}\\ &\geq c_k^q\sigma^{\pi_k q}(\mathcal{S}(t)u_0^q(0,x'))^{\theta_k},\end{aligned}$$

the last inequality follows from Lemma 4.1. Instead of (4.8) we obtain now

$$u(x_1,x',t) \geq c_k^{pq}t^{1-p}\left(\int_0^t (t-\sigma)H(x_1,0,t-\sigma)\sigma^{\pi_k q}d\sigma\right)^p (\mathcal{S}(t)u_0^q(0,x'))^{\frac{pq\theta_k}{q}},$$

and instead of (4.9) we have

$$u(0,x',t) \geq c_{k+1}t^{\pi_{k+1}}(\mathcal{S}(t)u_0^q(0,x'))^{\frac{\theta_{k-1}}{q}} \tag{4.10}$$

with

$$c_{k+1} = \left(\frac{c_k^q}{\sqrt{\pi}} \int_0^1 (1-\sigma)^{\frac{1}{2}} \sigma^{\pi_k q} d\sigma \right)^p ,$$

$$\pi_{k+1} = 1 + \frac{p}{2} + \pi_k pq = \alpha((pq)^{k+1} - 1) \quad \text{and} \quad \theta_{k+1} = (pq)^{k+1}.$$

If we now raise (4.10) to the power $(pq)^{-k-1}$ and let $k \to \infty$ then we arrive at (4.3) provided

$$\liminf_{k\to\infty} c_{k+1}^{(pq)^{-k-1}} > 0. \tag{4.11}$$

We shall prove (4.11) in Lemma 4.4.

From (4.3) and the autonomous nature of the problem it follows that

$$\mathcal{S}(t)u^q(0,\cdot,t) \leq ct^{-\alpha q}. \tag{4.12}$$

After shifting (if necessary) the origin of time, we may assume (cf.(3.8)) that

$$u_0(x) \geq c_0 e^{-\sigma_0 |x|^2}.$$

Note that

$$u(0,x',t) \geq \mathcal{S}(t)u_0(0,x') \geq c_0(1+4\sigma_0 t)^{-\frac{N}{2}} \exp\left(-\frac{\sigma_0 |x'|^2}{1+4\sigma_0 t}\right),$$

hence

$$\mathcal{S}(t)u^q(0,x',t) \geq c_0^q (1+4\sigma_0 t)^{-\frac{N}{2}q} \left(\frac{4q\sigma_0 t}{1+4\sigma_0 t}\right)^{-\frac{N}{2}} \exp\left(-\frac{q\sigma_0 |x'|^2}{1+4(q+1)\sigma_0 t}\right),$$

and in particular

$$\mathcal{S}(t)u^q(0,t) \geq c_0^q (1+4\sigma_0 t)^{-\frac{N}{2}q}.$$

This is a contradiction with (4.12). □

Lemma 4.4. *If* $pq > 1$ *then* (4.11) *holds.*

Proof. If we set

$$I_k = \frac{1}{\sqrt{\pi}} \int_0^1 (1-\sigma)^{\frac{1}{2}} \sigma^{q\pi_k} d\sigma,$$

then

$$I_k = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(q\pi_k + 1)}{\Gamma(q\pi_k + \frac{5}{2})}$$
$$= \frac{1}{2} \frac{\Gamma(q\pi_k + 1)}{(q\pi_k + \frac{3}{2})(q\pi_k + \frac{1}{2})\Gamma(q\pi_k + \frac{1}{2})}.$$

Therefore,

$$I_k \geq \frac{1}{2(q\pi_k + 1)^2} \quad \text{if } k \text{ is large enough.}$$

So

$$c_{k+1} \geq c_k^{pq} \frac{1}{2^p(q\pi_k + 1)^{2p}}.$$

Recall that

$$\pi_k = \alpha((pq)^k - 1).$$

Thus

$$c_{k+1} \geq c_k^{pq}(\gamma(pq)^{-k})^{2p}$$

for some γ which depends on p and q. If we set $B_k = \ln c_k$, then the last inequality yields

$$B_{k+1} \geq pqB_k - (ak + b),$$

where a and b depend only on p and q. Applying the last inequality repeatedly we obtain

$$B_{k+1} \geq (pq)^2 B_{k-1} - pq(a(k-1) + b) - (ak + b)$$
$$\geq (pq)^k B_1 - \sum_{j=0}^{k-1} (pq)^j (a(k-j) + b),$$

so

$$\frac{B_{k+1}}{(pq)^k} \geq B_1 - a\sum_{j=0}^{k-1} \frac{k-j}{(pq)^{k-j}} - b\sum_{j=0}^{k-1} \frac{1}{(pq)^{k-j}}.$$

Since $pq > 1$, the sums on the right-hand side are partial sums of convergent series. Therefore, there is a constant $K > 0$ depending on p and q such that

$$B_{k+1} \geq -K(pq)^{k+1},$$

hence

$$\liminf_{k\to\infty} c_{k+1}^{(pq)^{-k-1}} \geq e^{-K}. \qquad \square$$

Proposition 4.5. *Assume $q > 1 > p$, $pq > 1$ and $\beta(= \max(\alpha, \beta)) > \frac{N}{2}$. Then all nontrivial solutions are nonglobal.*

Proof. As in the proof of Proposition 4.3 we proceed by contradiction and use iterations. We use Jensen's inequality and Lemma 4.1 without referring to them.

Our first aim is to show that there is a constant $c > 0$ such that

$$\mathcal{S}(t)v_0^p \le ct^{-\beta p}. \tag{4.13}$$

After shifting (if necessary) the origin of time, we may assume (cf. 3.8) that

$$v_0(x) \ge c_0 e^{-\sigma_0 |x|^2}$$

and in particular we may assume that

$$v_0(x) \ge f(x_1)g(x')$$

for some nonnegative functions f and g. Then

$$v^p(\cdot, \sigma) \ge (\mathcal{S}(\sigma)v_0)^p \ge \mathcal{S}(\sigma)v_0^p,$$

and

$$\mathcal{S}(\eta - \sigma)v^p(x, \sigma) \ge \mathcal{T}(\eta)f^p(x_1)\mathcal{S}_{N-1}(\eta)g^p(x'). \tag{4.14}$$

From (2.1) we have

$$u(0, x', \eta) \ge \int_0^\eta \mathcal{S}(\eta - \sigma)v^p(0, x', \sigma)d\sigma.$$

Thus (4.14) yields

$$u(0, x', \eta) \ge \int_0^\eta \mathcal{T}(\eta)f^p(0)\mathcal{S}_{N-1}(\eta)g^p(x')d\eta.$$

Now

$$\mathcal{S}_{N-1}(t - \eta)u(0, x', \eta) \ge \eta \mathcal{T}(\eta)f^p(0)\mathcal{S}_{N-1}(t)g^p(x').$$

From (2.2) we obtain

$$v(x_1, x', t) \ge \int_0^t H(x_1, 0, t - \eta)(\mathcal{S}_{N-1}(t - \eta)u(0, x', \eta))^q d\eta$$

$$\ge \int_0^t H(x_1, 0, t - \eta)\eta^q(\mathcal{T}(\eta)f^p(0))^q d\eta (\mathcal{S}_{N-1}(t)g^p(x'))^q.$$

the last inequality implies

$$v(x_1,x',t) \geq c_1 \int_0^t H(x_1,0,t-\eta)\eta^{\pi_1} d\eta \Big(\mathcal{T}(t) f^p(0) \mathcal{S}_{N-1}(t) g^p(x')\Big)^{\frac{\theta_1}{p}} \tag{4.15}$$

where $c_1 = 1$, $\pi_1 = q$ and $\theta_1 = pq$. Assume now that

$$v(x_1,x',t) \geq c_k \int_0^t H(x_1,0,t-\eta)\eta^{\pi_k} d\eta \Big(\mathcal{T}(t) f^p(0) \mathcal{S}_{N-1}(t) g^p(x')\Big)^{\frac{\theta_k}{p}}. \tag{4.16}$$

We wish to derive recurrence relations for c_k, π_k and θ_k. To do this we first need a suitable lower bound for $\mathcal{S}(\eta-\sigma)v^p(\cdot,\sigma)$. We observe that

$$\begin{aligned} \mathcal{S}(\eta-\sigma)v^p(x_1,x',\sigma) &= \mathcal{T}(\eta-\sigma)\mathcal{S}_{N-1}(\eta-\sigma)v^p(x_1,x',\sigma) \\ \geq c_k^p \mathcal{T}(\eta-\sigma) &\left(\int_0^\sigma H(x_1,0,\sigma-\rho)\rho^{\pi_k} d\rho \right)^p \Big(\mathcal{T}(\sigma) f^p(0) \mathcal{S}_{N-1}(\eta) g^p(x')\Big)^{\theta_k}. \end{aligned}$$

To bound

$$\begin{aligned} J &= \mathcal{T}(\eta-\sigma) \left(\int_0^\sigma H(x_1,0,\sigma-\rho)\rho^{\pi_k} d\rho \right)^p \\ &= \int_0^\infty H(x_1,y_1,\eta-\sigma) \left(\int_0^\sigma H(y_1,0,\sigma-\rho)\rho^{\pi_k} d\rho \right)^p dy_1 \end{aligned}$$

we argue as follows:

$$\int_0^\sigma H\rho^{\pi_k} d\rho = \int_0^\sigma ((H\rho^{\pi_k})^p)^{\frac{1}{p}} d\rho \geq \sigma^{1-\frac{1}{p}} \left(\int_0^\sigma (H\rho^{\pi_k})^p \right)^{\frac{1}{p}},$$

so

$$\begin{aligned} J &\geq \int_0^\infty \sigma^{p-1} H(x_1,y_1,\eta-\sigma) \int_0^\sigma H^p(y_1,0,\sigma-\rho)\rho^{p\pi_k} d\rho dy_1 \\ &= \sigma^{p-1} \int_0^\sigma \rho^{p\pi_k} \left(\int_0^\infty H(x_1,y_1,\eta-\sigma) H^p(y_1,0,\sigma-\rho) dy_1 \right) d\rho. \end{aligned}$$

Since

$$H^p(y_1,0,\sigma-\rho) \geq (\pi(\sigma-\rho))^{\frac{1-p}{2}} H(y_1,0,\sigma-\rho),$$

Lemma 4.2 yields that

$$J \geq \sigma^{p-1} \int_0^\sigma \rho^{p\pi_k} (\pi(\sigma-\rho))^{\frac{1-p}{2}} H(x_1, 0, \eta-\rho) d\rho.$$

Therefore

$$\mathcal{S}(\eta-\sigma) v^p(0, x', \sigma)$$

$$\geq c_k^p \sigma^{p-1} \int_0^\sigma \rho^{p\pi_k} (\pi(\sigma-\rho))^{\frac{1-p}{2}} (\pi(\eta-\rho))^{-\frac{1}{2}} d\rho \Big(\mathcal{T}(\sigma) f^p(0) \mathcal{S}_{N-1}(\eta) g^p(x')\Big)^{\theta_k}.$$

Then

$$u(0, x', \eta) \geq \int_0^\eta \mathcal{S}(\eta-\sigma) v^p(0, x', \sigma) d\sigma$$

$$\geq c_k^p \int_0^\eta \int_0^\sigma \sigma^{p-1} \rho^{\pi_k p} \pi^{-\frac{p}{2}} (\sigma-\rho)^{\frac{1-p}{2}} (\eta-\rho)^{-\frac{1}{2}} d\rho d\sigma \Big(\mathcal{T}(\eta) f^p(0) \mathcal{S}_{N-1}(\eta) g^p(x')\Big)^{\theta_k}.$$

Next we derive a suitable lower bound for

$$I = \int_0^\eta \int_0^\sigma \sigma^{p-1} \rho^{\pi_k p} \pi^{-\frac{p}{2}} (\sigma-\rho)^{\frac{1-p}{2}} (\eta-\rho)^{-\frac{1}{2}} d\rho d\sigma.$$

Changing the order of integration we obtain

$$I = \pi^{-\frac{p}{2}} \int_0^\eta \rho^{\pi_k p} (\eta-\rho)^{-\frac{1}{2}} \int_\rho^\eta (\sigma-\rho)^{\frac{1-p}{2}} \sigma^{p-1} d\sigma d\rho.$$

Since $p < 1$, we have $\sigma^{p-1} \geq \eta^{p-1}$ for $\sigma \leq \eta$. Hence,

$$I \geq \pi^{-\frac{p}{2}} \int_0^\eta \rho^{\pi_k p} (\eta-\rho)^{-\frac{1}{2}} \eta^{p-1} \int_\rho^\eta (\sigma-\rho)^{\frac{1-p}{2}} d\sigma d\rho$$

$$= \pi^{-\frac{p}{2}} \eta^{p-1} \frac{2}{3-p} \int_0^\eta \rho^{\pi_k p} (\eta-\rho)^{\frac{2-p}{2}} d\rho$$

$$> \pi^{-\frac{p}{2}} \eta^{1+\frac{p}{2}+\pi_k p} \int_0^1 (1-\rho)^{\frac{2-p}{2}} \rho^{\pi_k p} d\rho.$$

Now

$$\Big(\mathcal{S}_{N-1}(t-\eta) u(0, x', \eta)\Big)^q$$

$$\geq c_k^{pq}\eta^{q+(\pi_k+\frac{1}{2})}\left(\pi^{-\frac{p}{2}}\int_0^1(1-\rho)^{1-\frac{p}{2}}\rho^{\pi_k p}\right)^q\left(\mathcal{T}(\eta)f^p(0)\mathcal{S}_{N-1}(t)g^p(x')\right)^{\theta_k q},$$

and finally

$$v(x_1,x',t)\geq c_k^{pq}\left(\pi^{-\frac{p}{2}}\int_0^1(1-\rho)^{1-\frac{p}{2}}\rho^{\pi_k p}\right)^q\int_0^t H(x_1,0,t-\eta)\eta^{q+(\pi_k+\frac{1}{2})pq}d\eta$$
$$\cdot\left(\mathcal{T}(t)f^p(0)\mathcal{S}_{N-1}(t)g^p(x')\right)^{\frac{pq\theta_k}{p}}.$$

The recurrence relations read now as follows:

$$\theta_{k+1}=\theta_k pq,$$
$$c_{k+1}=\left(c_k\pi^{-\frac{1}{2}}\left(\int_0^1(1-\rho)^{1-\frac{p}{2}}\rho^{\pi_k p}d\rho\right)^{\frac{1}{p}}\right)^{pq},$$
$$\pi_{k+1}=q+(\pi_k+\frac{1}{2})pq.$$

Therefore

$$\theta_k=(pq)^k,\quad \pi_k+\frac{1}{2}=\left(q+\frac{1}{2}\right)\frac{(pq)^k-1}{pq-1}.$$

From (4.16) we obtain

$$v(0,x',t)\geq\frac{c_k}{\sqrt{\pi}}\int_0^t(t-\eta)^{-\frac{1}{2}}\eta^{\pi_k}d\eta\left(\mathcal{T}(t)f^p(0)\mathcal{S}_{N-1}(t)g^p(x')\right)^{\frac{\theta_k}{p}}$$
$$=\frac{c_k}{\sqrt{\pi}}t^{\pi_k+\frac{1}{2}}\int_0^1(1-\eta)^{-\frac{1}{2}}\eta^{\pi_k}d\eta\left(\mathcal{T}(t)f^p(0)\mathcal{S}_{N-1}(t)g^p(x')\right)^{\frac{\theta_k}{p}}.$$

Raising this to the power $(pq)^{-k}$ and letting $k\to\infty$, we arrive at (4.13) provided

$$\liminf_{k\to\infty}D_k^{(pq)^{-k}}>0,\quad D_k=\frac{c_k}{\sqrt{\pi}}\int_0^1(1-\eta)^{-\frac{1}{2}}\eta^{\pi_k}d\eta. \tag{4.17}$$

We shall prove (4.17) in Lemma 4.6. The rest of the proof is now analogous as the corresponding part of the proof of Proposition 4.3 ((4.12) and below). □

Lemma 4.6. *Assume $p < 1 < q$ and $pq > 1$. Then (4.17) holds.*

Proof. Denote

$$I_k = \pi^{-\frac{p}{2}} \int_0^1 (1-\rho)^{1-\frac{p}{2}} \rho^{\pi_k p} d\rho,$$

$$J_k = \pi^{-\frac{1}{2}} \int_0^1 (1-\rho)^{-\frac{1}{2}} \rho^{\pi_k} d\rho.$$

Then

$$c_{k+1} = c_k^{pq} I_k^q, \quad D_{k+1} = c_{k+1} J_{k+1} = c_k^{pq} I_k^q J_{k+1}.$$

Recall that

$$\pi_k + \frac{1}{2} = \left(q + \frac{1}{2}\right) \frac{(pq)^k - 1}{pq - 1} > \frac{3}{2}.$$

Since $\Gamma'(a) > 0$ for $a > \frac{3}{2}$, we have

$$J_k = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2})\Gamma(\pi_k + 1)}{\Gamma(\pi_k + \frac{3}{2})} \geq \frac{\Gamma(\pi_k + \frac{1}{2})}{(\pi_k + \frac{1}{2})\Gamma(\pi_k + \frac{1}{2})} = \frac{1}{\pi_k + \frac{1}{2}}.$$

Therefore there is a constant $c > 0$ such that

$$J_k \geq c(pq)^{-k}.$$

Now

$$I_k = \pi^{-\frac{p}{2}} \frac{\Gamma(2 - \frac{p}{2})\Gamma(p\pi_k + 1)}{\Gamma(3 - \frac{p}{2} + p\pi_k)},$$

and

$$\begin{aligned}\Gamma(3 - \frac{p}{2} + p\pi_k) &\leq \Gamma(3 + p\pi_k) = (2 + p\pi_k)(1 + p\pi_k)\Gamma(1 + p\pi_k) \\ &\leq (2 + \pi_k)^2 \Gamma(1 + p\pi_k).\end{aligned}$$

Therefore, there is a constant $\tilde{c} > 0$ such that

$$I_k \geq \tilde{c}(pq)^{-2k}.$$

Hence

$$D_{k+1} \geq c^* c_k^{pq} (pq)^{-2kq-k-1}$$

for some constant c^*. Thus we have

$$\frac{\ln D_{k+1}}{(pq)^{k+1}} \geq \frac{\ln c_k}{(pq)^k} - \frac{(2kq+k+1)\ln(pq)}{(pq)^{k+1}} + \frac{\ln c^*}{(pq)^{k+1}},$$

and it is obvious that it is sufficient to show the existence of a constant $K > 0$ such that

$$\frac{\ln c_k}{(pq)^k} \geq -K.$$

Set $B_k = \ln c_k$. Then

$$B_{k+1} = pqB_k + q\ln I_k \geq pqB_k - 2kq\ln(pq) + q\ln\tilde{c}.$$

Hence

$$B_{k+1} \geq pqB_k - (ak+b)$$

for some constants a and b that depend only on p and q. This is exactly the same inequality as we had in the proof of Lemma 4.4 so the rest of the proof is as before. □

Proposition 4.7. *Assume $q < 1 < p$, $pq > 1$ and $\alpha(= \max(\alpha,\beta)) < \frac{N}{2}$. Then there are initial data such that the corresponding solutions are nonglobal.*

Proof. We shall use Lemma 4.1 several times therefore we assume that $\frac{\partial v_0}{\partial x_1} \leq 0$.

Let us write (2.2) in the form

$$v(x_1,x',t) = \mathcal{S}(t)v_0(x_1,x') + \int_0^t H(x_1,0,t-\eta)\mathcal{S}_{N-1}(t-\eta)u^q(0,x',\eta)d\eta, \quad (4.18)$$

and apply Jensen's inequality in (2.1) to obtain

$$u(x_1,x',\eta) \geq \eta^{1-p}\left(\int_0^\eta \mathcal{S}(\eta-\sigma)v(x_1,x',\sigma)d\sigma\right)^p.$$

Raising this to the power q, applying $\mathcal{S}_{N-1}(t-\eta)$ and using Jensen's inequality again, we have

$$\mathcal{S}_{N-1}(t-\eta)u^q(0,x',\eta) \geq \eta^{q(1-p)}\left(\int_0^\eta \mathcal{T}(\eta-\sigma)\mathcal{S}_{N-1}(t-\sigma)v(0,x',\sigma)d\sigma\right)^{pq}. \quad (4.19)$$

Lemma 4.1 yields

$$\mathcal{T}(\eta-\sigma)\mathcal{S}_{N-1}(t-\sigma)v(x_1,x',\sigma)\geq\mathcal{T}(\eta)\mathcal{S}_{N-1}(t)v_0(x_1,x'). \tag{4.20}$$

Combining (4.18) - (4.20) and then using Lemma 4.1 again, we obtain

$$\begin{aligned}v(x_1,x',t)&\geq\mathcal{S}(t)v_0(x_1,x')+\int_0^t\eta^q H(x_1,0,t-\eta)d\eta\Big(\mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(0,x')\Big)^{pq}\\&=I_0(x_1,x',t)+I_1(x_1,x',t).\end{aligned}$$

Now replace $v(0,x',\sigma)$ in (4.19) by $I_0(0,x',\sigma)+I_1(0,x',\sigma)$ and proced as before. This yields

$$v(x_1,x',t)\geq I_0(x_1,x',t)+I_1(x_1,x',t)+\tilde{I}_2(x_1,x',t),$$

here

$$\tilde{I}_2(x_1,x',t)=\int_0^t H(x_1,0,t-\eta)\eta^{q(1-p)}\left(\int_0^\eta\mathcal{T}(\eta-\sigma)\mathcal{S}_{N-1}(t-\sigma)I_1(0,x',\sigma)d\sigma\right)^{pq}d\eta.$$

We first find a suitable lower bound for

$$J=\int_0^\eta\mathcal{T}(\eta-\sigma)\mathcal{S}_{N-1}(t-\sigma)I_1(x_1,x',\sigma)d\sigma.$$

By definition

$$J=\int_0^\eta\int_0^\infty H(x_1,y_1,\eta-\sigma)\int_0^\sigma H(y_1,0,\sigma-\rho)\rho^q d\rho dy_1\mathcal{S}_{N-1}(t-\sigma)(\mathcal{S}(\sigma)v_0)^{pq}(0,x')d\sigma.$$

Changing the order of integration and Lemma 4.2 one has

$$\mathcal{S}_{N-1}(t-\sigma)(\mathcal{S}(\sigma)v_0)^{pq}\geq(\mathcal{S}(t)v_0)^{pq}.$$

Therefore

$$J\geq\int_0^\eta\int_0^\sigma H(x_1,0,\eta-\rho)\rho^q d\rho(\mathcal{S}(t)v_0)^{pq}(0,x'),$$

and

$$J|_{x_1=0}\geq\int_0^\eta\int_0^\sigma(\pi(\eta-\rho)^{\frac{1}{2}}\rho^q d\rho d\sigma(\mathcal{S}(t)v_0)^{pq}(0,x').$$

A straightforward computation yields

$$\int_0^\eta \int_0^\sigma (\pi(\eta-\rho))^{-\frac{1}{2}} \rho^q d\rho d\sigma = \frac{2}{\sqrt{\pi}} \eta^{q+\frac{3}{2}} \int_0^1 (1-\sigma)^{\frac{1}{2}} \sigma^q d\sigma.$$

So

$$\tilde{I}_2(x_1, x', t) \geq c_2 \int_0^t H(x_1, 0, t-\eta) \eta^{\pi_2} d\eta (\mathcal{S}(t) v_0)^{\theta_2}(0, x') = I_2(x_1, x', t),$$

here

$$c_2 = \left(\frac{2}{\sqrt{\pi}} \int_0^1 \eta^q (1-\eta)^{\frac{1}{2}} d\eta \right)^{pq}, \quad \pi_2 = q + pq(q + \frac{1}{2}), \quad \theta_2 = (pq)^2.$$

By induction we obtain

$$v(x_1, x', t) \geq \sum_{k=0}^m I_k(x_1, x', t), \qquad m = 1, 2, \ldots \tag{4.21}$$

where

$$I_k(x_1, x', t) = c_k \int_0^t \sigma^{\pi_k} H(x_1, 0, t-\sigma) d\sigma (\mathcal{S}(t) v_0)^{\theta_k}(0, x'),$$

$$c_{k+1} = \left(\frac{2}{\sqrt{\pi}} c_k \int_0^1 \eta^{qp(\pi_k + \frac{1}{2}) + q} (1-\eta)^{\frac{1}{2}} d\eta \right)^{pq}, \qquad c_1 = 1,$$

$$\pi_k + \frac{1}{2} = \left(q + \frac{1}{2} \right) \frac{(pq)^k - 1}{pq - 1}, \qquad \theta_k = (pq)^k.$$

Next we show that

$$c_k \geq \delta^{(pq)^k} \qquad \text{for some } \delta > 0. \tag{4.22}$$

Define

$$\rho_k = pq \left(\pi_k + \frac{1}{2} \right) + q, \qquad J_k = \frac{2}{\sqrt{\pi}} \int_0^1 \eta^{\rho_k} (1-\eta)^{\frac{1}{2}} d\eta.$$

Then

$$J_k = \frac{\Gamma(\rho_k + 1)}{(\rho_k + \frac{3}{2})(\rho_k + \frac{1}{2}) \Gamma(\rho_k + \frac{1}{2})} \geq \frac{1}{(\rho_k + 1)^2} \geq \gamma (pq)^{-2k}$$

for some $\gamma > 0$ that depends only on p and q. Hence

$$c_{k+1} \geq c_k^{pq} \gamma^{pq} (pq)^{-2kpq},$$

and setting $B_k = \ln c_k$ we arrive at

$$B_{k+1} \geq pqB_k - (ak+b)$$

which is the same inequality as in the proof of Lemma 4.4. Therefore, (4.22) follows.

A straightforward simple computation yields

$$\int_0^t H(0,0,t-\eta)\eta^{\pi_k}\,d\eta \geq \varkappa t^{\pi_k+\frac{1}{2}}$$

for some constant $\varkappa > 0$.

So if we take

$$v_0(x) = Ae^{-|x|^2},$$

then we obtain

$$I_k(0,0,t) \geq \varkappa \left(A\delta t^{\beta(1-(pq)^{-k})}(1+4t)^{-\frac{N}{2}}\right)^{(pq)^k}.$$

If $t \geq 1$, then

$$t^{\beta(1-(pq)^{-k})} \geq t^{\beta(1-(pq)^{-1})}, \qquad k = 1,2,\dots,$$

so if we fix $t_0 \geq 1$ and choose A such that

$$A\delta t_0^{\beta(1-(pq)^{-1})}(1+4t_0)^{-\frac{N}{2}} = 1,$$

then

$$I_k(0,0,t_0) \geq \varkappa.$$

Combining this with (4.21) we see that

$$v(0,0,t_0) \geq m\varkappa$$

for every integer $m > 0$. This implies that v cannot be global. □

Proposition 4.8. *Assume $p < 1 < q$, $pq > 1$ and $\beta(= \max(\alpha,\beta)) < \frac{N}{2}$. Then there are initial data such that the corresponding solutions are nonglobal.*

Proof. The argument is similar to the preceeding one. We take

$$v_0(x_1, x') = f(x_1)g(x').$$

One starts with

$$v(x_1,x',t) \ge \mathcal{S}(t)v_0(x_1,x') = I_0(x_1,x',t).$$

Now we have

$$v(x_1,x',t) \ge \mathcal{S}(t)v_0(x_1,x') + \int_0^t H(x_1,0,t-\eta)\Big(\mathcal{S}_{N-1}(t-\eta)u(0,x',\eta)\Big)^q d\eta,$$

but

$$u(0,x',\eta) \ge \int_0^\eta \mathcal{S}(\eta-\sigma)v^p(0,x',\sigma)d\sigma.$$

Then, exactly as in (4.15) (where I_0 was ommited), we get

$$\begin{aligned} v(x_1,x',t) &\ge I_0(x_1,x',t) + \int_0^t H(x_1,0,t-\eta)\eta^q d\eta \Big(\mathcal{T}(t)f^p(0)\mathcal{S}_{N-1}(t)g^p(x')\Big)^{\frac{pq}{p}} \\ &= I_0(x_1,x',t) + I_1(x_1,x',t). \end{aligned}$$

The only difference will be to note that

$$v^p(x_1,x',t) \ge (I_0+I_1)^p \ge 2^{p-1}(I_0^p + I_1^p),$$

and then preceed as before. We omit the details. One picks up an extra factor of

$$2^{(p-1)(pq)^k}$$

in front of $I_k(x_1,x',t)$ but this causes no extra difficulties. □

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