Cointegration and Threshold Adjustment

by

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ABSTRACT

Cointegration among interest rates for instruments with different maturities has been widely tested with mixed results. This paper proposes an extension to the Engle-Granger testing strategy by permitting asymmetry in the adjustment toward equilibrium in two different ways. We demonstrate that our test has good power and size properties over the Engle-Granger test when there are asymmetric departures from equilibrium. Empirical tests using US yields confirm the asymmetric nature of error correction among interest rates of different maturities.

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1. Introduction

One important development in the recent time-series literature is the examination of non-linear adjustment mechanisms. Much of the impetus for this interest stems from a large number of studies showing that key macroeconomic variables such as real GDP, unemployment, and industrial production display asymmetric adjustment over the course of the business cycle. For example, Neftci (1984), Falk (1986), DeLong and Summers (1988), Teräsvirta and Anderson (1992), Sichel (1993), Beaudry and Koop (1993), Potter (1995), Ramsey and Rothman (1996) and Bradley and Jensen (1997) all support various forms of asymmetric adjustment in one or more of these variables.

A natural extension to these univariate findings is to examine the possibility of non-linear adjustment in a multivariate context. Towards that end, Granger and Lee (1989) find that U.S. sales, production and inventories display asymmetric error-correction towards a long-run multi-cointegrating relationship. Balke and Fomby (1997) and Enders and Granger (1998) provide strong evidence that short-term and long-term interest rates display asymmetric adjustment towards the long-run equilibrium relationship suggested by the theory of the term-structure.

The aim of this paper is to introduce and develop an explicit test for cointegration that recognizes the possibility of asymmetric error-correction. In particular, we generalize the Enders and Granger (1998) threshold-autoregressive (TAR) and momentum-TAR tests for unit-roots to a multivariate context. The basic TAR model, developed by Tong (1983), allows the degree of autoregressive decay to depend on state of the variable of interest. The M-TAR model, introduced by Enders and Granger (1998), allows a variable to display differing amounts of autoregressive decay depending on whether it is increasing or decreasing. This is in contrast to

the Engle-Granger (1987) and Johansen (1996) tests that implicitly assume a linear adjustment mechanism. The distinction is important since Pippenger and Goering (1993), Balke and Fomby (1997) and Enders and Granger (1998) show that tests for unit-roots and cointegration all have low power in the presence of asymmetric adjustment. Our M-TAR modification of the Engle-Granger (1987) testing strategy has good power and size properties relative to the alternative assumption of symmetric adjustment. In fact, the Engle-Granger test emerges as a special case of our testing procedure.[1]

The paper is organized as follows. Section 2 describes a class of models that can capture asymmetric adjustment towards a long-run cointegrating relationship. Section 3 develops a testing methodology and analyzes the power of the two tests. Section 4 illustrates the appropriate use of the tests using U.S. short-term and long-term interest rates. It is shown that an M-TAR adjustment mechanism best describes the behavior of the interest rates. Section 5 contains our concluding remarks.

2. Asymmetric Time-Series Models

Standard models of cointegrated variables assume linearity and symmetric adjustment. Consider the simple linear relationship used as the basis for the many cointegration tests:

$$\Delta x_t = \pi x_{t-1} + v_t \tag{1}$$

where: x_t is an (n x 1) vector of random variables all integrated of degree 1,

π is an (n x n) matrix,

and: v_t is an (n x 1) vector of the normally distributed disturbances v_{it} that may be contemporaneously correlated.

For example, the methodologies developed by Johansen (1996) and Stock and Watson (1988) entail the estimation of π and determining its rank. Equation (1) can be modified in many different ways including the introduction of deterministic regressors, the addition of lagged changes in Δx_t, and allowing the components of x_t to be integrated of various orders. Nevertheless, if rank(π) $\neq 0$, the implicit assumption is that the system exhibits symmetric adjustment around $x_t = 0$ in that for any $x_t \neq 0$, Δx_{t+1} always equals πx_t. Thus, πx_t can be viewed as an attractor such that its pull is strictly proportional to $\| x_t \|$.

Similarly, the alternative hypothesis in the Engle and Granger (1987) test assumes symmetric adjustment. In the simplest case, the two-step methodology entails using OLS to estimate the long-run equilibrium relationship as:

$$x_{1t} = \beta_0 + \beta_2 x_{2t} + \beta_3 x_{3t} + \ldots + \beta_n x_{nt} + \mu_t. \tag{2}$$

where: x_{it} are the individual I(1) components of x_t, β_i are the estimated parameters, and μ_t is the disturbance term which may be serially correlated.

The second-step focuses on the OLS estimate of ρ in the regression equation:

$$\Delta\mu_t = \rho\mu_{t-1} + \epsilon_t \tag{3}$$

where: ϵ_t is a white noise disturbance, and the residuals from (2) are used to estimate (3).

Rejecting the null hypothesis of no cointegration (i.e., accepting the alternative hypothesis $-2 < \rho < 0$) implies that the residuals in (2) are stationary with mean zero. As such, (2) is an attractor such that its pull is strictly proportional to the absolute value of μ_t. The Granger-representation theorem guarantees that if $\rho \neq 0$, (2) and (3) jointly imply the existence of an error-correction representation of the variables in the form:

$$\Delta x_{it} = \alpha_i(x_{1t-1} - \beta_0 - \beta_2 x_{2t-1} - \ldots - \beta_{nt} x_{nt-1}) + \ldots + \nu_{it} \tag{4}$$

The point is that these cointegration tests and their extensions are misspecified if adjustment is asymmetric. Consider an alternative specification of the error-correction model, called the threshold autoregressive (TAR) model, such that (3) can be written as:

$$\Delta \mu_t = \begin{cases} \rho_1 \mu_{t-1} + \epsilon_t & \text{if } \mu_{t-1} \geq 0 \\ \rho_2 \mu_{t-1} + \epsilon_t & \text{if } \mu_{t-1} < 0 \end{cases} \tag{5}$$

A sufficient condition for the stationarity of $\{\mu_t\}$ is: $-2 < (\rho_1, \rho_2) < 0$. Moreover, if the sequence is stationary, the least squares estimates of ρ_1 and ρ_2 have an asymptotic multivariate normal distribution.[2] A formal way to quantify the adjustment process is to write:

$$\Delta\mu_t = I_t \rho_1 \mu_{t-1} + (1 - I_t)\rho_2 \mu_{t-1} + \epsilon_t \tag{6}$$

where: I_t is the Heaviside indicator function such that:

$$I_t = \begin{cases} 1 & \text{if } \mu_{t-1} \geq 0 \\ 0 & \text{if } \mu_{t-1} < 0 \end{cases} \tag{7}$$

If the system is convergent, $\mu_t = 0$ can be considered the long-run equilibrium value of the system in the sense that $x_{1t} = \beta_0 + \beta_2 x_{2t} + \beta_3 x_{3t} + \ldots + \beta_n x_{nt}$. If μ_t is above its long-run equilibrium value, the adjustment is $\rho_1 \mu_t$ and if μ_t is below long-run equilibrium, the adjustment is $\rho_2 \mu_t$. Since adjustment is symmetric if $\rho_1 = \rho_2$, the Engle-Granger test is a special case of (6) and (7).

Equations (2), (6) and (7) are consistent with a wide variety of error-correcting models. Given the existence of a single cointegrating vector in the form of (2), the error-correcting model for any variable x_{it} can be written in the form:

$$\Delta x_{it} = \rho_{1.i} I_t \mu_{t-1} + \rho_{2.i}(1 - I_t)\mu_{t-1} + \ldots + v_{it} \tag{8}$$

where: $\rho_{1,i}$ and $\rho_{2,i}$, are the adjustment coefficients for positive and and negative discrepancies, respectively.

Figure 1 shows the time paths of two I(1) variables–say x_{1t} and x_{2t}--exhibiting threshold cointegration. For simplicity, the cointegrating vector is such that the system is in long-run equilibrium whenever $x_{1t} = x_{2t}$. Next, two sets of five hundred normally distributed and serially uncorrelated pseudo-random numbers with standard deviations equal to unity were drawn to represent the $\{\epsilon_{1t}\}$ and $\{\epsilon_{2t}\}$ sequences. Setting $\rho_{1.1} = -\rho_{1.2} = -0.05$ and $\rho_{2.1} = -\rho_{2.2} = -0.25$, and initializing the initial values of the sequences equal to zero, the next 500 values of $\{x_{1t}\}$ and $\{x_{2t}\}$ were generated as in (8). Notice that the variables do not wander "too far" from each other in that positive and negative departures from long-run equilibrium are eventually eliminated. On inspection, it is clear that positive discrepancies persist for substantially longer periods than negative ones.

Further insight into the asymmetric nature of the adjustment process can be obtained using the specific numerical values for $\rho_{i,j}$ and subtracting Δx_{2t} from Δx_{1t}, so that (6) becomes:

$$\Delta\mu_t = -0.1I_t(x_{1t-1} - x_{2t-1}) - 0.5(1 - I_t)(x_{1t-1} - x_{2t-1}) + v_{1t} - v_{2t} \tag{9}$$

Notice that the line $x_{1t} - x_{2t} = 0$ is a more powerful attractor for negative values of the μ_{t-1} sequence than for positive values. On average, 90% of a positive discrepancy persists from one period to the next while only 50% of a negative discrepancy persists. As such, near random-walk behaviour occurs for positive values of $\{\mu_t\}$ whereas there is rapid convergence when $\{\mu_t\}$ is negative. Clearly, the opposite case can also be constructed and economic theory can provide a guide as in, for example, when policy makers are more tolerant of falling interest rates or of an appreciating exchange rate than a depreciating one.

There are two important ways to modify the basic threshold cointegration model:

1. *Higher-order Processes*: Equation (6) can be augmented with lagged changes in the $\{\mu_t\}$ sequence such that it becomes the p-th order process:[3]

$$\Delta\mu_t = I_t\rho_1\mu_{t-1} + (1-I_t)\rho_2\mu_{t-1} + \sum_{i=1}^{p-1}\gamma_i\Delta\mu_{t-1} + \epsilon_t \quad (10)$$

In working with specifications such as (10), diagnostic checks of the residuals (such as the autocorrelogram of the residuals and Ljung-Box tests) and various model selection criteria (such as the AIC or BIC) can be used to determine the appropriate lag length [see Tong (1983)]. To ensure there is no more than a single unit-root, all the values of r satisfying the inverse characteristic equation $1 - \gamma_1 r - \gamma_2 r^2 + \ldots + \gamma_{p-1} r^{p-1} = 0$ must lie outside the unit circle.

2. *Alternative Adjustment Specifications*: In (7), the Heaviside indicator depends on the *level* of μ_{t-1}. A useful alternative to allow the decay to depend on the previous period's *change* in μ_{t-1}. Consider setting the Heaviside indicator according to the following rule:

$$I_t = \begin{cases} 1 & \text{if } \Delta\mu_{t-1} \geq 0 \\ 0 & \text{if } \Delta\mu_{t-1} < 0 \end{cases} \quad (11)$$

Replacing (7) by (11) is especially valuable when adjustment is asymmetric such that the series exhibits more "momentum" in one direction than the other. Models constructed using (2), (6), and (11) can be called momentum-threshold autoregressive (M-TAR) models. Note that it is possible to use the Heaviside indicator of (11) in a dynamic model augmented by lagged changes in $\Delta\mu_t$. Again, such models are useful in economics when it is hypothesized that policy makers

wait for an accumulation of evidence, say on inflation, before acting to influence short-term interest rates.

The two series shown in Figure 2 were constructed using the identical values of ρ_{ij} and the same two sets of 500 pseudo-random numbers used to construct Figure 1. The sole difference is that the M-TAR sequence is constructed using (11) instead of (7). Although $x_{1t} - x_{2t} = 0$ remains the attractor, the attraction is more powerful for negative values of $\Delta\mu_{t-1}$ than for positive values. Comparing Figures 1 and 2, it is clear that the overall time paths follow each other reasonably well. However, the positive discrepancies from long-run equilibrium are shorter-lived in the M-TAR model than in the TAR model. In the M-TAR model, a negative realization of $\Delta\mu_t$ decays at a 50% rate and a positive realization decays at a 10% rate. Given the distributions of v_{1t} and v_{2t}, this occurs 50% of the time. In the TAR model, decay occurs at a 10% rate for as long as the discrepancy from long-run equilibrium is positive and at a 50% rate for as long as the discrepancy is negative.

Figure 3 shows the deviations from long-run equilibrium derived from Figures 1 and 2. The solid line is the deviation from long-run equilibrium in the TAR model plotted in Figure 1 and the dashed line is the deviation from the M-TAR model plotted in Figure 2. Notice that positive discrepancies are more persistent for the TAR model. Moreover, a major difference concerns the nature of the of the spikes in that decreases are sharper and more pronounced in the M-TAR model.[4] Intuitively, the M-TAR model exhibits little decay for positive $\Delta\mu_{t-1}$ but substantial decay for negative $\Delta\mu_{t-1}$. Thus, a negative realization in the quantity $(v_{1t} - v_{2t})$ results in a decline in μ_t that tends to be reversed in the next period. In a sense, increases tend to persist but decreases tend to revert quickly toward the attractor.

3. Testing for Cointegration With TAR and M-TAR Adjustment

In order to conduct a Monte Carlo experiment that can be used to test the null hypothesis of a unit-root against the alternative of a TAR or an M-TAR model, 30,000 random-walk processes of the following form were generated:

$$x_{1t} = x_{1t-1} + v_{1t}, \quad t = 1, \ldots, T \tag{12}$$

$$x_{2t} = x_{2t-1} + v_{2t}, \quad t = 1, \ldots, T. \tag{13}$$

For $T = 100$ and 500, two sets of T normally distributed and uncorrelated pseudo-random numbers with standard deviation equal to unity were drawn to represent the $\{v_{1t}\}$ and $\{v_{2t}\}$ sequences. Setting the initial values of x_{1t} and x_{2t} equal to zero, the next T values of each were generated using (12) and (13). For each of the 30,000 series, the TAR model given by (2), (6) and (7) was estimated and three different test-statistics were tabulated. The t-statistics for the null hypotheses $\rho_1 = 0$, and $\rho_2 = 0$ were recorded along with the F-statistic for the null hypothesis $\rho_1 = \rho_2 = 0$. The most significant of the t-statistics is called *T-Max*, the least significant of the t-statistics is called *T-Min*, and the F-statistic is called ϕ. Since the F-statistic has the greatest power, it is the only test statistic reported here.[5] As reported in top portion of Table 1, for $T = 100$, the ϕ-statistic for the null $\rho_1 = \rho_2 = 0$ exceeds 6.07 in approximately 5% of the 30,000 trials. Notice that these statistics can be used as critical values to test the null hypothesis of a unit-root process against the alternative of a TAR model.

Suppose that the process used to generate the data used in Figure 1 was unknown. Using realizations 201 - 300 of the TAR series shown in the figure, the estimated model and t-statistics are:

$$x_{1t} = \underset{(-5.88)}{-1.34} + \underset{(5.06)}{0.498}\, x_{2t} + \mu_t \tag{14}$$

$$\Delta\mu_t = \underset{(-2.48)}{-0.226 I_t}\, \mu_{t-1} - \underset{(-3.33)}{0.354}(1 - I_t)\mu_{t-1} + \epsilon_t \tag{15}$$

The F-statistic for the null hypothesis $\rho_1 = \rho_2 = 0$ is 8.63. As shown in the top portion of Table 1, such a value will occur in less than 1% of the trials when the data generating process is a random-walk (the critical value at the 1% level is 8.20). As such it is possible to conclude that the data is not generated from bivariate random-walk processes.

The distribution of the ϕ-statistic depends on sample size, the number of lagged changes included in the dynamic adjustment equation (i.e., equation 10), and the number of variables included in the cointegrating relationship. Table 1 reports the critical values of ϕ for sample sizes of 100 and 500, for lag lengths of zero, one, and four, and for cointegrating vectors containing two and three variables. The Monte Carlo experiment was repeated for an M-TAR model using the indicator function given by (11). The corresponding test statistics-called ϕ(M)-are reported in Table 2. It is interesting to note that the critical values for the ϕ-statistic are always larger than the corresponding ϕ(M) values.

To use the statistics, perform the following 3 steps:

Step 1: Regress one of the variables on a constant and the other variable(s) and save the residuals in the sequence $\{\hat{\mu}_t\}$. Next, depending on the type of asymmetry under consideration, set the indicator function I_t according to (7) or (11). Estimate a regression equation in the form of (6) and obtain the F-statistic for the null hypothesis $\rho_1 = \rho_2 = 0$. Compare these sample statistics with the appropriate critical value shown in Table 1 or 2.

Step 2: If the alternative hypothesis is accepted, it is possible to test for symmetric versus asymmetric adjustment since ρ_1 and ρ_2 converge to multivariate normal distributions. As such, the restriction that adjustment is symmetric (i.e., the null hypothesis: $\rho_1 = \rho_2$) can be tested using the usual F-statistic.

Step 3: Diagnostic checking of the residuals should be undertaken to ascertain whether the $\{\hat{\epsilon}_t\}$ series can reasonably be characterized by a white-noise process. If the residuals are correlated, return to Step 2 and re-estimate the model in the form:

$$\Delta\hat{\mu}_t = I_t\rho_1\hat{\mu}_{t-1} + (1 - I_t)\rho_2\hat{\mu}_{t-1} + \gamma_1\Delta\hat{\mu}_{t-1} + \ldots + \gamma_p\Delta\hat{\mu}_{t-p} + \epsilon_t \tag{16}$$

Lag lengths can be determined by an analysis of the regression residuals and/or using a number of widely used model selection criteria such as the AIC or BIC.

Step 4: If the series are cointegrated, model building can be conducted using the methodologies suggested by Chan (1993) and Tsay (1989). If adjustment is asymmetric, Tong (1983) demonstrates that the sample mean is a biased estimate of the attractor. For example, in a TAR model such that $1 > |\rho_1| > |\rho_2|$, the $\{\hat{\mu}_t\}$ sequence will exhibit relatively more persistence when $\hat{\mu}_{t-1} > 0$. As such, the sample mean of the sequence will exceed that of the attractor. Chan (1993) shows that searching over all values so as to minimize the sum of squared errors from the fitted model yields a super-consistent estimate of the threshold.

Power Tests

Since unit-root tests suffer from low power, it is of interest to compare the power of the ϕ and ϕ(M) test statistics to the power of the more traditional Engle-Granger test. Toward this end, two sets of 100 normally distributed random numbers were drawn to represent the $\{v_{1t}\}$ and $\{v_{2t}\}$ sequences. For various values of ρ_1 and ρ_2, these random numbers were used to generate

the basic 2-variable TAR model given by (2), (6), and (7) for $T = 100$. Following Steps 1 and 2 above, x_{1t} was regressed on x_{2t} and a constant and an equation in the form of (6) was estimated and the sample ϕ-statistic was calculated. This process was replicated 500 times and the number of instances in which the null hypothesis of no cointegration was correctly rejected was tabulated and recorded in column 2 of Table 3. Notice that the table reports results power for test sizes of 10%, 5%, and 1%. For comparison purposes, the Engle-Granger method (assuming symmetric adjustment) was applied to the same data. Thus, x_{1t} was regressed on x_{2t} and a constant and the residuals were used to estimate an equation in the form of (3). The estimated value of ρ was compared to the critical values reported by Engle and Granger (1987). The number of times that the null hypothesis was correctly rejected is recorded in column 3 of Table 3.

The overwhelming impression of the is that the power of the Engle-Granger test usually exceeds that of the ϕ-statistic. For example, if the true adjustment parameters are $\rho_1 = -0.10$ and $\rho_2 = -0.25$, at the 10% significance level the ϕ-statistic correctly identified the model as stationary in 310 out of the 500 trials. However, for the same sized test, the Engle-Granger correctly identified the model as stationary in 351 out of the 500 trials. Restricting the size to 1% improves the relative performance of the ϕ-statistic. For these same values of ρ_1 and ρ_2, at the 1% level the ϕ-statistic correctly identified stationarity in 3 more cases than the Engle-Granger test (i.e., 22 versus 19 instances). This pattern carries over to the other values of ρ_1 and ρ_2 reported in Table 3. The disappointingly low power of the ϕ-statistic may seem surprising since the true data-generating process displays asymmetric adjustment to long-run equilibrium. The explanation lies in the fact that the TAR model entails the estimation of an additional coefficient with a consequent loss of power. For the degree of asymmetry shown in the table, the gain in power resulting from

estimating the correctly specified model does not outweigh the loss from the additional coefficient.

The situation is quite different for the M-TAR model. The identical random numbers used for the power tests above were used to generate the basic 2-variable M-TAR model given by (2), (6), and (11) for $T = 100$. Inspection of Table 4 shows:

1. All of the tests for the M-TAR model have at least as much power than those for the corresponding TAR model.

2. If adjustment is truly symmetric (such that $\rho_1 = \rho_2$), the power of the Engle-Granger test exceeds that of the ϕ and $\phi(M)$ statistics. When adjustment is truly symmetric, the assumption of asymmetric adjustment entails the needless estimation of an additional coefficient with a consequent loss of power.

3. Increasing the degree of asymmetry increases the relative power of the ϕ and $\phi(M)$ tests over the Engle-Granger test. As such, the relative power of the $\phi(M)$-statistic is greatest when one of the adjustment coefficients is very small. For example, if $\rho_1 = -0.05$ and $\rho_2 = -0.25$, the $\phi(M)$-test correctly indicated a stationary process 3 times more often at the 5% level and about 19 times more often at the 1% level. As the asymmetry is increased, the power of both tests increases, but the power of the $\phi(M)$-statistic increases relative to that of the Engle-Granger test.

4. Empirical Results

In order to illustrate the appropriate use of the testing procedure, we obtained monthly logarithmic values of the federal funds rate (r_s) and the 10-year yield on federal government securities (r_{10}) from the CD-ROM version of the *International Financial Statistics*. Due to changes in Federal Reserve operating procedures, we begin our sample in January, 1981 and use all 196 observations through April, 1997. Figure 4 shows the time path of the two interest rate series. It is generally agreed, see Stock and Watson (1988), that interest rates series are I(1) variables that should be cointegrated.

The estimated long-run equilibrium relationship (with t-statistics in parenthesis) is:

$$r_{st} = \underset{(-9.93)}{-1.189} + \underset{(25.99)}{1.447} r_{10t} + \hat{\mu}_t \tag{17}$$

As reported in the first column of Table 5, the Engle-Granger procedure indicates that the series are **not** cointegrated. Following the Engle-Granger methodology, we used the residuals of (17) to estimate:

$$\Delta\hat{\mu}_t = \rho_1\hat{\mu}_{t-1} + \gamma_1\Delta\hat{\mu}_{t-1} + \epsilon_t \tag{18}$$

The estimated value of $\rho_1 = -0.0553$ and the t-statistic for the null hypothesis $\rho_1 = 0$ is -2.577. The Engle-Granger critical values at the 10%, 5% and 1% significance levels are -3.03, -3.37 and -4.07, respectively. Hence, at conventional significance levels, the Engle-Granger test indicates that the two interest rate series are not cointegrated. Diagnostic checking indicates that the model with 1-lag is appropriate. Both the AIC and BIC select the model using 1-lag of $\{\Delta\hat{\mu}_t\}$ in (18). Moreover, the Ljung-Box Q-statistics all indicate that the residuals of (18) are not significantly autocorrelated.

Next, we estimated TAR models in the form of (16) for various lag lengths. Both the AIC and BIC selected a model augmented by one lagged change in $\{\Delta\hat{\mu}_t\}$. As shown in the second column of Table 5, the point estimates for $\rho_1 = -0.0524$ and $\rho_2 = -0.0581$ suggest convergence. However, the sample value of $\phi = 3.31$ is less than any of the critical values shown in Table 1. Hence, at conventional significance levels, it is not possible to reject the null hypothesis $\rho_1 = \rho_2 = 0$. Diagnostic checking of the residuals indicates that the model with 1-lagged change is appropriate. Hence, both the Engle-Granger technique and the TAR model indicate that the interest rate series are not cointegrated.

The third column of Table 5 reports the estimated M-TAR model. As in the previous two cases, both the AIC and BIC selected a model augmented by one lagged change in $\{\Delta\hat{\mu}_t\}$. The sample value of the ϕ-statistic = 7.96 indicates that the null hypothesis $\rho_1 = \rho_2 = 0$ can be rejected at the 5% significance level (As shown in Table 1, the critical value is 5.98). Given that the interest rates are cointegrated, the null hypothesis of symmetric adjustment (i.e., $\rho_1 = \rho_2$) can be tested using a standard F-distribution. The sample value of $F = 9.00$ has a p-value of 0.003 so that it is possible to reject the null hypothesis of symmetric adjustment.

It is somewhat troublesome that the point estimate of $\rho_1 = 0.0086$ suggests that positive realizations of $\Delta\hat{\mu}_t$ are explosive. Recall that the point estimates of ρ_1 and ρ_2 obtained by OLS are biased. Chan's (1993) method indicated that the consistent estimate of the threshold is -0.10. As reported in the last column of Table 5, the M-TAR model using the consistent estimate of the threshold (with t-statistics in parentheses) is:

$$\Delta\hat{\mu}_t = \underset{(-1.707)}{-0.0361 I_t\hat{\mu}_{t-1}} \underset{(-4.629)}{- 0.4226(1-I_t)\hat{\mu}_{t-1}} + \underset{(2.811)}{0.1830\Delta\hat{\mu}_{t-1}} + \epsilon_t \tag{19}$$

where:

$$I_t = \begin{cases} 1 & \text{if } \Delta\mu_{t-1} \geq -0.10 \\ 0 & \text{if } \Delta\mu_{t-1} < -0.10 \end{cases} \tag{20}$$

The residuals of (19) show no evidence of serial correlation and adding additional lagged changes of $\{\Delta\hat{\mu}_t\}$ increase the AIC and BIC. Now, the point estimates of ρ_1 and ρ_2 suggest convergence. Moreover, the sample value ϕ is 12.12 and the F-test for symmetric adjustment can be rejected at any conventional significance level. Hence, (19) strongly suggests that the two interest rates are cointegrated and, in addition, that the adjustment mechanism is asymmetric.

Notice that the M-TAR model and the M-TAR model with the consistent estimate of the threshold fit substantially better than the other models. It is striking that the AIC and BIC for the two M-TAR models are substantially below those found for the models that assumed symmetric or TAR adjustment.

The positive finding of cointegration with M-TAR adjustment justifies estimation of the following error-correction model (with t-statistics in parentheses):

$$\Delta r_{st} = \underset{(-1.110)}{-0.0036} - \underset{(-1.458)}{0.0234 I_t \hat{\mu}_{t-1}} - \underset{(-2.539)}{0.1782(1-I_t)\hat{\mu}_{t-1}} + \underset{(4.843)}{0.3255\Delta r_{st-1}} + \underset{(2.865)}{0.2436\Delta r_{10t-1}} + v_{1t} \qquad (21)$$

$$\Delta r_{10t} = \underset{(0.7273)}{-0.0020} + \underset{(1.208)}{0.0165 I_t \hat{\mu}_{t-1}} + \underset{(2.888)}{0.1721(1-I_t)\hat{\mu}_{t-1}} + \underset{(0.237)}{0.0135\Delta r_{st-1}} + \underset{(3.526)}{0.2544\Delta r_{10t-1}} + v_{2t} \qquad (22)$$

where: $\hat{\mu}_{t-1} = r_{st-1} + 1.189 - 1.447 r_{10t-1}$ and the Heaviside indicator is set in accord with (20).

The point estimates of the error-correction terms indicate that both interest rates adjust strongly to negative changes in $\hat{\mu}_{t-1}$ but weakly to negative changes. Within any month, the two rates each adjust so as to eliminate approximately 17% of a 1-unit deviation from the long-run equilibrium when the discrepancy is decreasing. When the discrepancy is increasing, each rate adjusts to eliminate about 2% of the gap. Thus, adjustment to the long-run equilibrium relationship tends to be rapid when the federal funds rate is decreasing and/or the 10-year rate is increasing.

In contrast, the error-correction model assuming symmetric adjustment is:

$$\Delta r_{st} = \underset{(-1.072)}{-0.0035} - \underset{(-1.958)}{0.0310 \hat{\mu}_{t-1}} + \underset{(4.714)}{0.3195\Delta r_{st-1}} + \underset{(2.842)}{0.2439\Delta r_{10t-1}} + v_{1t} \qquad (23)$$

$$\Delta r_{10t} = \underset{(0.7494)}{-0.0021} + \underset{(1.784)}{0.0241\hat{\mu}_{t-1}} + \underset{(0.338)}{0.0195\Delta r_{st-1}} + \underset{(3.472)}{0.2542\Delta r_{10t-1}} + v_{2t} \qquad (24)$$

Except for the error-correction terms, the coefficient estimates in (23) and (24) are all similar to those in (21) and (22). The key difference is that the symmetric adjustment assumption implies that there is always slow convergence toward the long-run equilibrium. In response to a 1-unit gap, the short-rate and the long-rate adjust by approximately 3% and 2.4%, respectively.

In spite of the fact that the M-TAR model contains an additional two coefficients, the multivariate values of AIC and BIC are lower than those for (23) and (24). Respectively, the multivariate AIC and BIC are -2491.13 and -2458.40 for the consistent threshold M-TAR model and -2479.78 and -2453.58 for the system given by (23) and (24).

6. Conclusions

The standard tests for cointegration implicitly assume that adjustment to the long-run equilibrium relationship is symmetric. The paper developed a generalization of the Engle-Granger (1987) procedure that allows for TAR adjustment in the error-correction model. In addition, we introduced the concept of Momentum-TAR adjustment that allows for possible asymmetrically "sharp" adjustments. The power of the test for TAR adjustment is poor compared to that of the Engle-Granger test. However, for a plausible range of the adjustment parameters, the power of the M-TAR test can be many times that of the Engle-Granger test.

We chose to illustrate the appropriate use of the tests using short-term and long-term interest rates. It is generally agreed that interest rates are cointegrated I(1) variables. Balke and Fomby (1997) and Enders and Granger (1998) also report evidence of asymmetric adjustment in the term structure. In the empirical example reported int he paper, the Engle-Granger and TAR tests indicated that the Federal Funds rate and the 10-year yiled on government bonds are *not* cointegrated. However, models which permit M-TAR adjustment indicate that the two rates are

indeed cointegrated. Both interest rates respond strongly to a *negative*, but not positive, discrepancy from the long-run equilibrium relationship. Moreover, the M-TAR models fit the data substantially better than that assume either symmetric or TAR adjustment.

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Endnotes

1. Balke and Fomby (1997) posit a framework in which adjustment towards long-run equilibrium occurs when a shock creates a disequilibrium error exceeding some threshold. However, if the extent of the disequilibrium is "small", there may be no convergence towards the long-run relationship. Thus, in a 2-variable system, the long-run relationship is given by a band instead of a line. The intuition is that adjustment costs prevent complete convergence to the long-run relationship.

2. Tong (1983) contains the proof that the least squares estimates of ρ_1 and ρ_2 have an asymptotic multivariate normal distribution. This result easily generalizes to higher-order autoregressive processes. Tong (1990) also develops many of the properties of the TAR model.

3. Of course, it is possible to allow Δy_t to display asymmetric adjustment to its lagged changes. For example, the magnitude of each γ_i could depend on whether Δy_{t-1} was positive or negative. Moreover, as discussed in Granger and Teräsvirta (1993), the values of ρ_1 and ρ_2 can be allowed to smoothly adjust over time.

4. This pattern would be reversed if $|\rho_1| > |\rho_2|$.

5. This result is consistent with Enders and Granger (1998)

Table 1: Distribution of Φ

Two Variable Case

	No Lagged Changes			One Lagged Change			Four Lagged Changes		
#	90%	95%	99%	90%	95%	99%	90%	95%	99%
100	5.04	6.07	8.20	4.99	5.98	8.21	4.94	5.91	8.22
500	4.87	5.80	7.89	4.89	5.84	8.04	4.91	5.82	7.95

Three Variable Case

	No Lagged Changes			One Lagged Change			Four Lagged Changes		
#	90%	95%	99%	90%	95%	99%	90%	95%	99%
100	6.35	7.53	9.94	6.34	7.47	10.00	6.24	7.36	9.79
500	6.24	7.32	9.70	6.23	7.32	9.64	6.20	7.30	9.64

Table 2: Distribution of Φ(M)

Two Variable Case

	No Lagged Changes			One Lagged Change			Four Lagged Changes		
#	90%	95%	99%	90%	95%	99%	90%	95%	99%
100	5.52	6.57	9.04	5.43	6.45	8.75	5.34	6.35	8.73
500	5.29	6.34	8.54	5.35	6.33	8.61	5.32	6.30	8.53

Three Variable Case

	No Lagged Changes			One Lagged Change			Four Lagged Changes		
#	90%	95%	99%	90%	95%	99%	90%	95%	99%
100	6.85	8.03	10.56	6.82	7.97	10.60	6.69	7.86	10.35
500	6.67	7.80	10.22	6.67	7.81	10.14	6.64	7.75	10.15

Table 3: Power Tests for the TAR Model

ρ_1, ρ_2	ϕ			Engle-Granger		
	10%	5%	1%	10%	5%	1%
-0.05, -0.05	8	4	2	11	3	0
-0.05, -0.10	17	7	2	24	8	1
-0.05, -0.25	66	27	3	84	28	2
-0.10, -0.10	59	22	3	90	30	3
-0.10, -0.25	310	151	22	351	191	19
-0.25, -0.25	500	497	387	500	499	382

Table 4: Power Tests for the M-TAR Model

ρ_1, ρ_2	ϕ(M)			Engle-Granger		
	10%	5%	1%	10%	5%	1%
-0.05, -0.05	14	5	0	11	3	0
-0.05, -0.10	57	22	4	32	8	1
-0.05, -0.25	439	362	208	274	113	11
-0.10, -0.025	79	42	6	11	4	0
-0.10, -0.10	64	25	2	90	30	3
-0.10, -0.25	465	398	177	439	306	42
-0.25, -0.25	500	495	328	500	499	382

TABLE 5: Estimates of the Interest Rate Differential

	Engle-Granger	Threshold	Momentum	Momentum-Consistent
ρ_1	-0.0553 (-2.577)[a]	-0.0524 (-1.702)	0.0086 (0.2884)	-0.0361 (-1.707)
ρ_2	NA	-0.0581 (-1.942)[b]	-0.1167 (-3.977)	-0.4226 (-4.629)
AIC [c] BIC	-62.17 -55.64	-60.19 -50.39	-69.10 -59.30	-76.76 -66.96
ϕ_μ [d]	NA	3.31	7.96	12.12
$\rho_1 = \rho_2$ [e]	NA	0.018 (0.894)	9.00 (0.003)	17.05 (0.000)
Q(4) [f] Q(8) Q(12)	0.77 0.89 0.95	0.77 0.89 0.95	0.58 0.82 0.92	0.58 0.82 0.91

Notes

a. Entries in this row are the *t*-statistic for the null hypothesis $\rho_1 = 0$.

b. Entries in this row are the *t*-statistic for the null hypothesis $\rho_2 = 0$.

c. The *AIC* is calculated as: $T*\log(SSR) + 2*n$ where: T = number of usable observations, SSR = sum of squared residuals and n = number of regressors. The *BIC* is calculated as: $T*\log(SSR) + n*\log(T)$. Since the Dickey-Fuller tests were performed on the residuals of the interest differential regressed on a constant, the *AIC* and *BIC* for the Dickey-Fuller tests are directly comparable to the other values in the table.

d. Entries in this row are the sample values of ϕ_μ or ϕ_μ*.

e. Entries in this row are the sample *F*-statistic for the null hypothesis that the adjustment coefficients are equal. Significance levels are in parenthesis below.

f. $Q(p)$ is the significance level of the Ljung-Box statistic that the first p of the residual autocorrelations are jointly equal to zero.

Figure 1: Threshold Cointegration

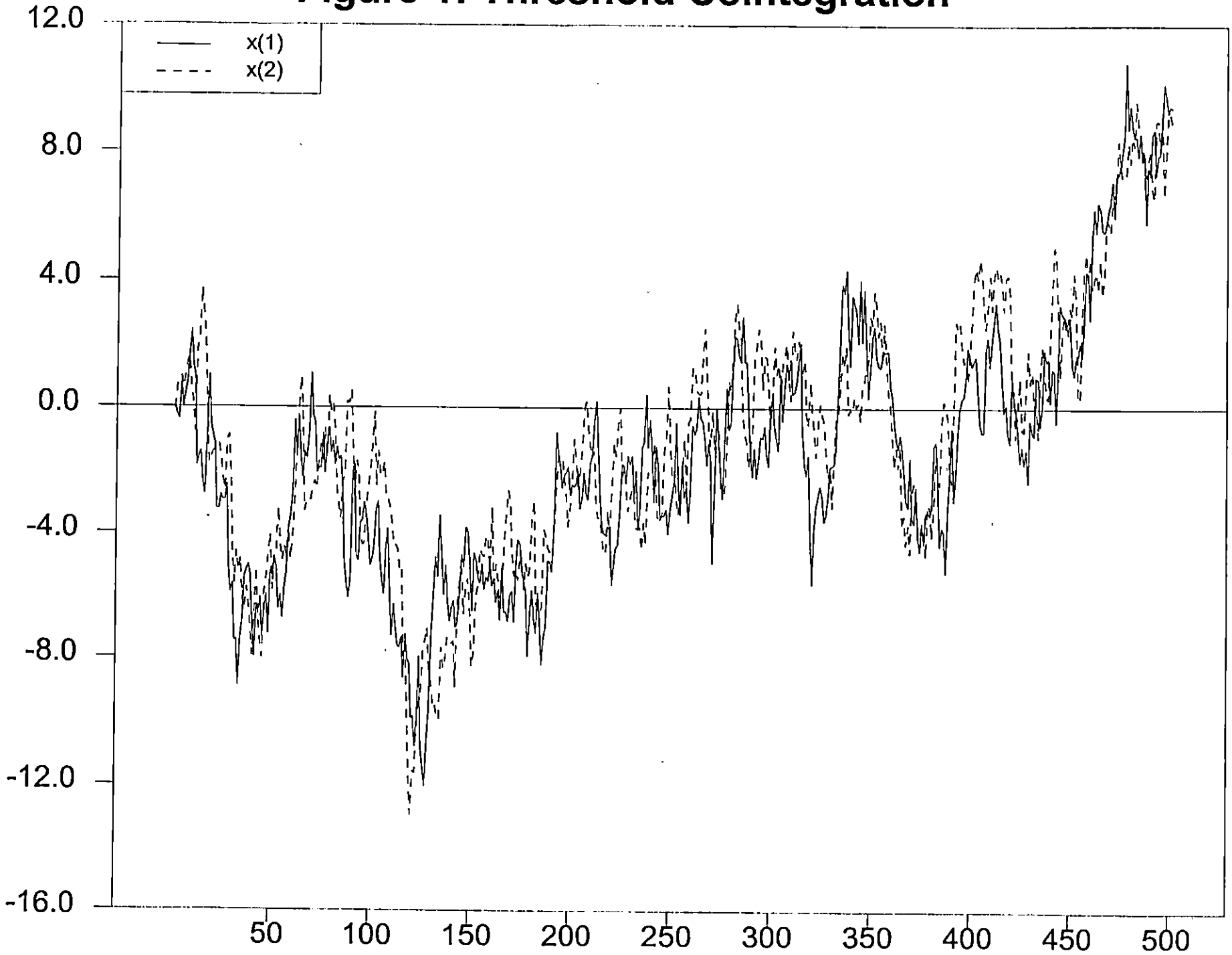

Figure 2: Momentum Cointegration

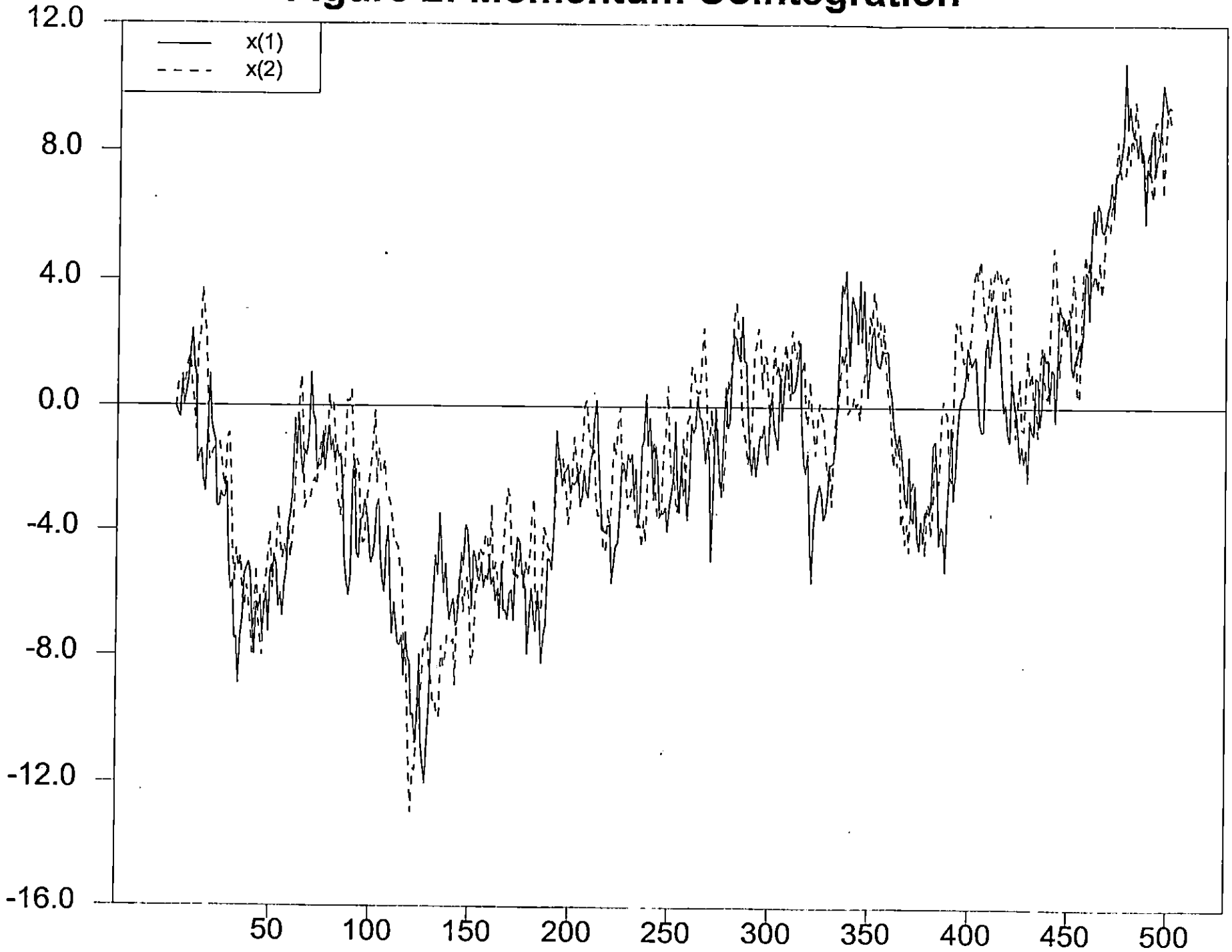

Figure 3: The Error Correction Terms

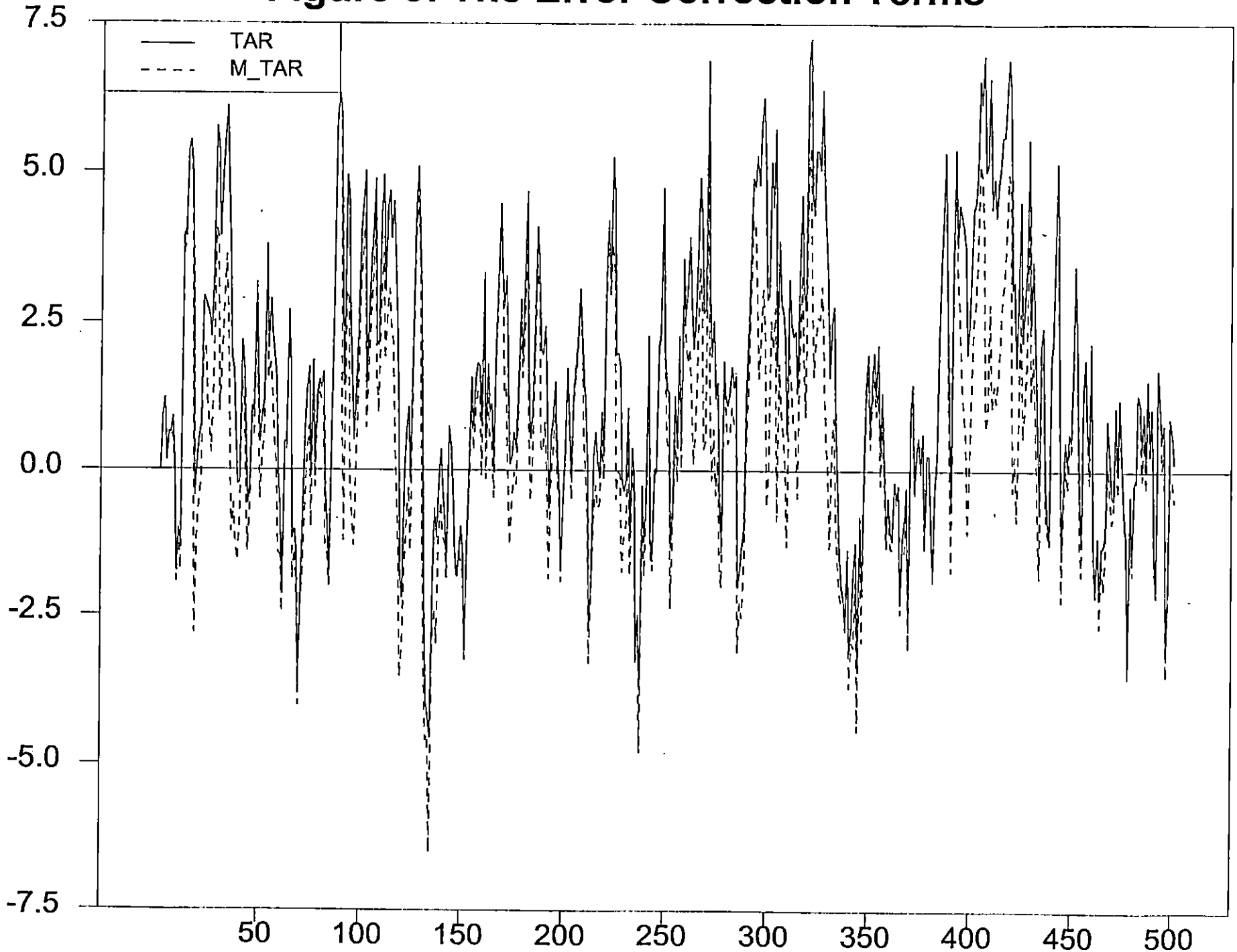

Figure 4: Short-term and Long-Term Interest Rates